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Analysis of Reaction-Diffusion Equations on a Time-Dependent Domain

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Abstract

We consider non-negative solutions to reaction-diffusion equations on time-dependent domains with zero Dirichlet boundary conditions. The reaction term is either linear or else monostable of the KPP type. For a range of different forms of the boundary motion, we use changes of variables, exact solutions, supersolutions and subsolutions to study the long-time behaviour.

For a linear equation on $(A(t), A(t) + L(t))$, we prove that the solution can be found exactly by a separation of variables method when $\ddot{L}L^3$ and $\ddot{A}L^3$ are constants. In these cases $L(t)$ has the form $L(t) = \sqrt{at^2 + 2bt + l^2}$.

We also analyse the linear problem near the boundary, deriving conditions on $L(t)$ such that the gradient at the boundary remains bounded above and below away from zero. Interesting links are observed between this 'critical' boundary motion and Bramson's logarithmic term for the nonlinear KPP equation.

The exact solutions and investigation of behaviour near the boundary are also extended to a ball with time-dependent radius, and a time-dependent box.

We then consider time-periodic bounded domains $\Omega(t)$. The long-time behaviour is determined by a principal periodic eigenvalue μ, for which we derive several bounds and also consider the large and small frequency limits. For the nonlinear problem, we prove that the solution converges to either zero or a unique positive periodic solution u^*.

The nonlinear problem is also studied on a bounded domain in $\mathbb{R}^N$ moving at constant velocity c, and we derive several properties of the positive stationary limit U_c.

Results describing long-time behaviour for the nonlinear equation are then extended to certain other types of time-dependent domain that have non-constant velocity and non-constant length, and to time-dependent cylinders.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed............[redacted]..

Date............22 September 2022..

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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Contents

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Notation

O	big O: $F_1 = O(F_2)$ means $\exists$ a constant B such that $\lvert\frac{F_1}{F_2}\rvert \leq B$.
o	little o: $F_1 = o(F_2)$ in a specified limit means that $\frac{F_1}{F_2} \to 0$.
$\ll$	same as o: $F_1 \ll F_2$ means $F_1 = o(F_2)$ in the specified limit.
$\underline{\overline{O}}$	$F_1 = \underline{\overline{O}}(F_2)$ means that both $F_1 = O(F_2)$ and $F_2 = O(F_1)$.
$\sim$	asymptotic to: $F_1 \sim F_2$ in a specified limit means that $\frac{F_1}{F_2} \to 1$.
$\frac{\partial}{\partial \nu}$	$\nu \cdot \nabla$ where ν is the outward unit normal
$\dot{(\ )}$	$\frac{d}{dt}(\ )$
$[\ ]^+$	positive part: $[F(t)]^+ = \max(F(t), 0) \geq 0$
$[\ ]^-$	negative part: $[F(t)]^- = -\min(F(t), 0) \geq 0$
$\lambda(\Omega_0)$	principal Dirichlet eigenvalue of $-\nabla^2$ on bounded domain Ω_0
x $[x_j]$	space variable in the time-dependent domain [jth co-ordinate]
ξ $[\xi_j]$	space variable in fixed reference domain [jth co-ordinate]
$\psi(x,t)$	solution on the time-dependent domain
$u(\xi,t)$	$u(\xi,t) = \psi(x,t)$, with transformed variable ξ in a fixed domain
U_c, U_{c,Ω_0}, U_{c,L_0}	unique positive solution to (6.6), (6.7), on Ω_0 or $(0, L_0)$
c_*	'critical speed' $c_* = 2\sqrt{Df'(0)}$, or in the linear case, $c_* = 2\sqrt{Df_0}$
D	the diffusion coefficient (positive and constant)
f_0	coefficient of the linear reaction term
f	the reaction term: either linear or satisfying (2.20)
K	the positive zero of f, in the nonlinear case
$L(t)$ $[L_j(t)]$	length of time-dependent interval [or in dimension j]
$A(t)$ $[A_j(t)]$	left hand end of time-dependent interval [or in dimension j]
γ_0 $[\gamma_0^{(j)}]$	either $\ddot{L}L^3 \equiv \gamma_0$ [or in dimension j], otherwise any constant
γ_1 $[\gamma_1^{(j)}]$	either $\ddot{A}L^3 \equiv \gamma_1$ [or in dimension j], otherwise any constant
σ_1 $[\sigma_{1,j}]$	principal Sturm-Liouville eigenvalue of (3.19), (3.20)
g_1 $[g_{1,j}]$	principal Sturm-Liouville eigenfunction of (3.19), (3.20)
$\overline{Q}(t)$, $\overline{Q}_j(t)$	defined in (3.82) and (3.137)
$\underline{Q}(t)$, $\underline{Q}_j(t)$	defined in (3.83) and (3.138)
$H(\xi,t)$, $H_j(\xi_j,t)$	defined in (3.6) and (3.131)

Ai	the Airy function
κ_1	the largest real zero of Ai
L_{crit}	'critical length' $L_{crit} = \pi\sqrt{\frac{D}{f'(0)}}$
$L_{crit}(c)$	for $c \in (-c_*, c_*)$, $L_{crit}(c) = \pi\sqrt{\frac{D}{f'(0)-\frac{c^2}{4D}}}$
L_0	any positive length
t_0, t_1	any positive time
T	in Chapter 5 only: the period of a periodic domain $\Omega(t)$
ω	in Chapter 5 only: the frequency $\omega = \frac{2\pi}{T}$
μ, $\mu(\omega)$	in Chapter 5 only: principal periodic eigenvalue of problem (5.3), (5.4), (5.5), (5.6) associated with periodic domain $\Omega(t)$
P_T	in Chapter 5 only: Poincaré map: $u(\cdot, nT) \mapsto u(\cdot,(n+1)T)$.

For functions on a bounded domain Ω, and for $k \in \mathbb{N}$, $0 < \gamma < 1$, and $1 < p < \infty$:

$$[u]_\gamma = \sup_{x\neq y\in\overline{\Omega}} \left(\frac{|u(x)-u(y)|}{|x-y|^\gamma}\right).$$

$$||u||_{C^\gamma(\overline{\Omega})} = \sup_{\overline{\Omega}} |u| + [u]_\gamma.$$

$$||u||_{C^{0,1}(\overline{\Omega})} = \sup_{\overline{\Omega}} |u| + \sup_{x\neq y\in\overline{\Omega}} \left(\frac{|u(x)-u(y)|}{|x-y|}\right).$$

$$||u||_{C^{k+\gamma}(\overline{\Omega})} = \sup_{\overline{\Omega}} |u| + \textstyle\sum_{|\beta|\leq k} \sup_{\overline{\Omega}} |\partial_x^\beta u| + \sum_{|\beta|=k} [\partial_x^\beta u]_\gamma.$$

$$||u||_{W_p^2(\Omega)} = \left(\int_\Omega (|u|^p + |D_x u|^p + |D_x^2 u|^p)\, dx\right)^{\frac{1}{p}}.$$

For functions on a bounded space-time domain $\Omega_T = \Omega \times (0,T)$, and for $k, l \in \mathbb{N}$, $0 < \alpha < 1$, and $1 < p < \infty$:

$$[u]_{\alpha,\frac{\alpha}{2}} = \sup_{(x,t)\neq(y,s)\in\overline{\Omega_T}} \left(\frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}}\right).$$

$$||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega_T})} = \sup_{\overline{\Omega_T}} |u| + [u]_{\alpha,\frac{\alpha}{2}}.$$

$$||u||_{C^{k,l}(\overline{\Omega_T})} = \sup_{\overline{\Omega_T}} |u| + \textstyle\sum_{|\beta|\leq k} \sup_{\overline{\Omega_T}} |\partial_x^\beta u| + \sum_{n\leq l} \sup_{\overline{\Omega_T}} |\partial_t^n u|.$$

$$||u||_{C^{k+\alpha,l+\frac{\alpha}{2}}(\overline{\Omega_T})} = ||u||_{C^{k,l}(\overline{\Omega_T})} + \textstyle\sum_{|\beta|=k} [\partial_x^\beta u]_{\alpha,\frac{\alpha}{2}} + [\partial_t^l u]_{\alpha,\frac{\alpha}{2}}.$$

$$\begin{aligned} ||u||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega_T})} = \sup_{\overline{\Omega_T}} |u| &+ \textstyle\sum_{|\beta|=1} \sup_{\overline{\Omega_T}} |\partial_x^\beta u| + \sum_{|\beta|=1} [\partial_x^\beta u]_{\alpha,\frac{\alpha}{2}} \\ &+ \sup_{(x,t)\neq(x,s)\in\overline{\Omega_T}} \left(\frac{|u(x,t)-u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}}\right). \end{aligned}$$

$$||u||_{W_p^{2,1}(\Omega_T)} = \left(\int_0^T \int_\Omega (|u|^p + |D_x u|^p + |D_x^2 u|^p + |D_t u|^p)\, dxdt\right)^{\frac{1}{p}}.$$

Chapter 1

Introduction

In this thesis we consider the reaction-diffusion problem

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f(\psi) \qquad \text{for } x \in \Omega(t) \tag{1.1}$$

$$\psi(x,t) = 0 \qquad \text{for } x \in \partial\Omega(t) \tag{1.2}$$

on time-dependent domains $\Omega(t) \subset \mathbb{R}^N$. A typical example is an interval with moving endpoints: $A(t) < x < A(t) + L(t)$. Here $D > 0$, $\psi \geq 0$ and we consider both the linear problem where

$$f(\psi) = f_0\psi \quad \text{with} \quad f_0 > 0, \tag{1.3}$$

and a nonlinear problem where f satisfies certain conditions (2.20) such that

$$f(0) = f(K) = 0, \quad f > 0 \text{ on } (0, K), \quad \text{and} \quad f(k) \leq f'(0)k. \tag{1.4}$$

Such nonlinearities are said to be of 'KPP' type, named after the authors Kolmogorov, Petrowsky and Piskunov of [41]. We consider both the linear and nonlinear reaction terms on a range of time-dependent domains. We seek to understand the behaviour of $\psi(x,t)$ — how it depends on x, t, and the form of the boundary motion — and to describe the asymptotic long-time behaviour both overall and in a neighbourhood of the boundary. For a range of different forms of the boundary motion, we prove results about exact solutions, upper and lower bounds on the long-time behaviour, and convergence to stable limits.

Reaction-diffusion problems with $\psi \geq 0$ are used to model population dynamics, chemical diffusion, and several other biological, ecological and physical processes. The work in this thesis is relevant when these occur in a domain whose boundaries move due to an external influence. In the context of population dynamics, this could represent a habitat that changes over time due to factors such as flooding, climate change, habitat destruction, forest fire, melting ice, loss of snow cover, or 're-wilding' areas of land. The domain's size, as well as location, can change with time. For further applications involving time-dependent domains see [40]; see also the introduction to [18].

In the mathematical literature on population dynamics, typical boundary conditions include Dirichlet, Neumann or Robin. Examples of ecological scenarios where each of these is appropriate are discussed in [29]. Here, we always assume zero Dirichlet conditions on the boundary of the domain.

The domains that we consider will vary 'sufficiently smoothly' with time; there is always a parametrisation that is continuously differentiable (or in some cases twice differentiable) with respect to t. It is possible to study parabolic equations on time-dependent domains that have lower regularity with respect to time; see for example [18] which uses a time-slicing strategy to prove existence of weak solutions. However such cases are not the focus of this thesis; instead we assume sufficient smoothness to guarantee a classical solution, and our aim is to understand how this solution behaves. Our approach generally involves transforming the problem onto a fixed domain, making changes of variables, deriving solutions, supersolutions and subsolutions, and then interpreting the results in terms of the original co-ordinates in the time-dependent domain.

Some of the results will include parameters that are quite standard on fixed domains, such as the 'critical speed'

$$c_* = 2\sqrt{Df'(0)} \tag{1.5}$$

which is important for the spreading of the solution on the whole space, and the 'critical length' $L_{crit} = \pi\sqrt{\frac{D}{f'(0)}}$ which is important for the problem on a fixed interval of constant length.

This speed $c_* = 2\sqrt{Df'(0)}$ is the asymptotic spreading speed for a solution to

$$\frac{\partial u}{\partial t} = D\nabla^2 u + f(u) \qquad \text{on } \mathbb{R}^N \tag{1.6}$$

with compactly supported initial conditions, in the following sense. For the linear equation, where $f(u) = f_0 u$, the solution on the whole space $\mathbb{R}^N$ is

$$u(x,t) = \frac{e^{f_0 t}}{(4\pi Dt)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(y,0) e^{-\frac{|x-y|^2}{4Dt}} dy, \tag{1.7}$$

so as $t \to \infty$, $u(x,t) \to \infty$ in $\{|x| \leq ct\}$ for each $0 \leq c < c_*$, and $u(x,t) \to 0$ uniformly in the region $\{|x| \geq c_* t\}$. Moreover $u(x,t)$ takes order one values at the positions $|x| = c_* t - \frac{ND}{c_*} \log \frac{t}{t_0} + O(1)$ as $t \to \infty$. For the nonlinear problem, with f sufficiently smooth and satisfying (1.4), Aronson and Weinberger [5, 4] show that $\max_{|x| \geq c_* t} u(x,t) \to 0$, and that for each $0 \leq c < c_*$ it holds that $\max_{|x| \leq ct} |u(x,t) - K| \to 0$ as $t \to \infty$.

The same speed c_* is also the minimal travelling wave speed for the one-dimensional nonlinear problem. For f satisfying (1.4), it is shown in [41] and [5] that for each $c \geq c_*$ there exist travelling wave solutions $u(x,t) = \tilde{U}_c(x - ct)$ to the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \qquad \text{for } -\infty < x < \infty \tag{1.8}$$

that are monotonically decreasing in x from K to zero. Such solutions do not exist for $c < c_*$. The monotone travelling wave $\tilde{U}_{c_*}$ of minimal speed c_* satisfies

$$0 = D\tilde{U}''_{c_*}(z) + c_* \tilde{U}'_{c_*}(z) + f(\tilde{U}_{c_*}(z)) \qquad \text{for } -\infty < z < \infty \tag{1.9}$$

with $\tilde{U}_{c_*}(-\infty) = K$, $\tilde{U}_{c_*}(+\infty) = 0$ and $\tilde{U}_{c_*}(z) \sim Bze^{-\frac{c_* z}{2D}}$ as $z \to \infty$. Moreover, Kolmogorov, Petrowsky and Piskunov prove in [41] that for a solution to the 'KPP equation' (1.8) with Heaviside initial conditions ($u(x,0) = K$ for $x < 0$, and $u(x,0) = 0$ for $x > 0$), there is a function $\varphi(t)$ with $\frac{d\varphi}{dt} \to c_*$ as $t \to \infty$ such that

$$\sup_{x \in \mathbb{R}} \left| u(x,t) - \tilde{U}_{c_*}(x - \varphi(t)) \right| \to 0 \qquad \text{as } t \to \infty. \tag{1.10}$$

This function $\varphi(t)$ is the asymptotic front position, at which the solution $u(x,t)$ takes on some value strictly between K and zero. Uchiyama proves in [62] that $\varphi(t) = c_* t - \frac{3D}{c_*}\log\frac{t}{t_0} + O\left(\log\log\frac{t}{t_0}\right)$ for large t, and Bramson [16, 15] derives the asymptotic behaviour of $\varphi(t)$ up to order one:

$$\varphi(t) = c_* t - \frac{3D}{c_*}\log\frac{t}{t_0} + \text{constant} + o(1) \qquad \text{as } t \to \infty. \tag{1.11}$$

This has become known as Bramson's logarithmic correction and is proved in [16, 15] using probabilistic arguments. An alternative proof of the fact that $\varphi(t) = c_* t - \frac{3D}{c_*}\log\frac{t}{t_0} + O(1)$ as $t \to \infty$, using PDE methods, is given by Hamel, Nolen, Roquejoffre and Ryzhik in [36]. A similar result, with locally uniform convergence, also holds for compactly supported initial conditions. In [33] and [27] the result is extended to the multi-dimensional case. Namely, for (1.6) on $\mathbb{R}^N$ with compactly supported initial conditions, there is a function $\hat{\varphi}$ such that

$$\sup_{|x|\geq r_0}\left|u(x,t) - \tilde{U}_{c_*}\left(|x| - \hat{\varphi}(t; x/|x|)\right)\right| \to 0 \qquad \text{as } t \to \infty \tag{1.12}$$

(for any $r_0 > 0$), and the wave front is uniformly at the radial position

$$|x| = c_* t - \frac{(2+N)D}{c_*}\log\frac{t}{t_0} + O(1) \qquad \text{as } t \to \infty. \tag{1.13}$$

[Note the difference to the linear case, where the solution is of order one at $|x| = c_* t - \frac{ND}{c_*}\log\frac{t}{t_0} + O(1)$ as $t \to \infty$.]

The speed c_* and these same logarithmic correction terms also appear in this thesis, but now in the context of a critical boundary motion for the linear equation on a finite, time-dependent domain. For the linear equation on the interval $\frac{-L(t)}{2} < x < \frac{L(t)}{2}$, we show that if the endpoints move apart 'too fast' then the solution tends to zero in a neighbourhood of the boundary, whereas if they move apart 'too slowly' then the solution tends to infinity at order one distances from the boundary. Using precise bounds on this behaviour, we derive conditions under which the solution remains exactly of order one (bounded above and bounded below away from zero) at an order one distance from the boundary. This 'critical' boundary motion satisfies $\frac{L(t)}{2} = c_* t - \frac{3D}{c_*}\log\frac{t}{t_0} + O(1)$ as $t \to \infty$

(see Section 4.2). Similarly, for a time-dependent ball in $\mathbb{R}^N$, we show that the 'critical' radius $R(t)$ in dimension $N \leq 3$ has a logarithmic term that matches (1.13) (see Section 4.3).

The $\frac{3D}{c_*} \log \frac{t}{t_0}$ term also arises in the paper [12], where J. Berestycki, Brunet and Derrida analyse a linear reaction-diffusion equation on a semi-infinite, time-dependent interval $\mu(t) < x < \infty$. They prescribe constant values of the solution h and its gradient $\frac{\partial h}{\partial x}$ at the boundary $x = \mu(t)$, and seek an asymptotic expansion as $t \to \infty$ for the boundary position $\mu(t)$. They use a clever method based on an integral transform $g(r,t) = \int_0^\infty h(\mu(t)+z, t)e^{rz}dz$ for r in a suitable range, and a singularity analysis in a small parameter ε, where $r = r_{crit} - \varepsilon$ for some critical value r_{crit}. Initial conditions $h(\cdot, 0)$ with different rates of exponential decay as $x \to \infty$ are considered, and for each of these they deduce an asymptotic form of the boundary position $\mu(t)$ as $t \to \infty$, by matching singularities in ε. For initial conditions with sufficiently fast decay, including those with compact support, $\mu(t) = c_* t - \frac{3D}{c_*} \log \frac{t}{t_0} + \text{constant} + o(1)$ as $t \to \infty$. This is the same as the front position (1.11) for the nonlinear KPP equation on $\mathbb{R}$. Several subsequent terms in the expansion are calculated in [12]. The same authors also apply their method to the nonlinear KPP problem in [11].

In the current thesis, the method from [12] will be applied when we investigate the gradient of the solution at the boundary of a time-dependent domain: see Section 4.5 for the linear problem, and Proposition 7.21 and Theorem 7.23 for the nonlinear problem. Such results relate to time-dependent boundaries moving with asymptotic speed c_* and with domain length becoming infinitely large as $t \to \infty$.

We are also interested in time-dependent intervals with $L(t)$ varying close to the 'critical length' $L_{crit} = \pi\sqrt{\frac{D}{f'(0)}}$. We know that for the linear equation (1.1), (1.2) on a fixed interval $0 < x < l$, the solution can be expressed as a Fourier series using the separation of variables. Namely, in this case we have

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \exp\left(\left(f_0 - \frac{Dn^2\pi^2}{l^2}\right)t\right) \tag{1.14}$$

where $a_n = \frac{2}{l}\int_0^l \psi(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$. For positive solutions, $a_1 > 0$ and it is then clear that for $l > L_{crit}$ (i.e. $f_0 > \frac{D\pi^2}{l^2}$) the solution tends to infinity as $t \to \infty$, for $l < L_{crit}$ (i.e. $f_0 < \frac{D\pi^2}{l^2}$) the solution tends to zero, and for $l = L_{crit}$ (i.e. $f_0 = \frac{D\pi^2}{l^2}$) there is a positive stationary solution.

For the problem on a time-dependent interval,

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + f'(0)\psi \qquad \text{for } A(t) < x < A(t) + L(t) \tag{1.15}$$

$$\psi = 0 \qquad \text{at } x = A(t) \text{ and } x = A(t) + L(t), \tag{1.16}$$

we cannot in general write the solution in a separated form and make such conclusions. Here, however, we use changes of variables to prove that when $\ddot{L}L^3$ and $\ddot{A}L^3$ are constants then we can obtain exact solutions by a separation of variables method. In these cases, $L(t)$ has the form $L(t) = \sqrt{at^2 + 2bt + l^2}$, and we present formulae for the separable solutions in terms of a, b, and $al^2 - b^2$. We also extend this result to a time-dependent ball and a time-dependent box. For cases that are not separable, we prove comparison results which bound the long-time behaviour. These allow us to analyse the role of L_{crit} when $L(t)$ depends on time, including some cases where $L(t)$ converges to L_{crit} as $t \to \infty$.

We also give bounds on the solution when $L(t)$ varies periodically with time, and consider more general bounded time-periodic domains $\Omega(t) \subset \mathbb{R}^N$. For such domains we convert the problem into a periodic-parabolic equation on a fixed domain; the results of [20] then show that the long-time behaviour is determined by a principal periodic eigenvalue, μ. Although periodic-parabolic problems have been studied by several authors, and we make use of results from Castro and Lazer [20], Hess [37] and Liu, Lou, Peng and Zhou [46], none of these have worked specifically on time-periodic domains or the periodic-parabolic equations that arise from them. Here, we derive bounds on the principal periodic eigenvalue μ associated to a time-periodic domain $\Omega(t)$.

As shown by the work [20] of Castro and Lazer, there are certain aspects in the theory of periodic-parabolic problems that correspond to those in the theory of elliptic problems. The principal periodic eigenvalue, which is key to determining the long-time behaviour, is in some ways analogous to the principal eigenvalue in the elliptic case. There are also certain similarities for the nonlinear problem. Here, for the nonlinear equation on a time-periodic domain, we prove convergence to either zero or a unique positive periodic solution u^*. Similarly, when $\Omega(t)$ is a bounded domain Ω_0 translating at constant velocity c, we prove convergence to either zero or a unique positive solution U_c to an elliptic equation. In both cases, the proof uses a monotonic sub- and supersolution argument.

We also give results about long-time solution behaviour for the nonlinear equation in certain other cases, where the domain and its velocity are neither periodic nor constant. This includes some results on an interval whose length tends to infinity and whose endpoints travel at asymptotically constant speeds less than c_*. Such results can be related to the moving boundary problem that is introduced by Du and Lin in [24] and considered with co-authors in [23, 17, 26]. There, they fix a constant $\mu > 0$ and consider the free boundary problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \qquad \text{for } g(t) < x < h(t) \tag{1.17}$$

$$u(g(t),t) = u(h(t),t) = 0 \tag{1.18}$$

$$\dot{g}(t) = -\mu\frac{\partial u}{\partial x}(g(t),t), \qquad \dot{h}(t) = -\mu\frac{\partial u}{\partial x}(h(t),t) \tag{1.19}$$

with some given initial values $h(0) = h_0 = -g(0)$, $u(x,0) = u_0(x)$, and where $f(0) = f(K) = 0$, $f > 0$ on $(0,K)$, $f \in C^1([0,K])$, and $f'(K) < 0 < f'(0)$. They prove a 'spreading/vanishing dichotomy': as $t \to \infty$ either $g(t) \to -\infty$, $h(t) \to +\infty$ and u spreads at an asymptotically constant speed $\hat{c} = \hat{c}(\mu)$ in both directions, or else $g(t) \to g_\infty$, $h(t) \to h_\infty$ with $h_\infty - g_\infty \le \pi\sqrt{D/f'(0)}$ and there is 'vanishing', i.e. $u \to 0$. In the case of spreading, the speed $\hat{c}$ is strictly between 0 and c_*. Theorem 1.2 of [26] is that: $\dot{g}(t) \to -\hat{c}$ and $\dot{h}(t) \to \hat{c}$; there are constants $\hat{G}$, $\hat{H}$ such that $g(t) + \hat{c}t \to \hat{G}$ and $h(t) - \hat{c}t \to \hat{H}$; and

$$0 = \lim_{t\to\infty} \sup_{x\in[g(t),0]} |u(x,t) - q_{\hat{c}}(x - g(t))| = \lim_{t\to\infty} \sup_{x\in[0,h(t)]} |u(x,t) - q_{\hat{c}}(h(t) - x)|, \tag{1.20}$$

where for $c \in [0, c_*)$, q_c denotes the unique semi-wave satisfying

$$Dq'' - cq' + f(q) = 0 \qquad \text{for } 0 < x < \infty \tag{1.21}$$

$$q(0) = 0, \qquad \lim_{x\to\infty} q(x) = K, \qquad q > 0 \text{ on } (0, \infty). \tag{1.22}$$

The spreading speed $\hat{c}$ is uniquely determined by μ, via the relation $\mu q'_{\hat{c}}(0) = \hat{c}$. (See [24, 23, 17, 26] for full statements and proofs of these facts.) This free boundary problem has since been extended to a wider range of reaction terms and scenarios; see in particular the paper [25]. It clearly has some similarities with our own problem, however a key difference is that our boundary motion is prescribed: the boundaries move due to some external influence, rather than being determined as part of the solution. The scopes of the two problems therefore differ, however it can be instructive to consider what may be learned about one problem based on the solution to the other (see Section 7.5.2).

Time-dependent domains also arise in modelling living tissue and developing organisms. Such domains are actually physically growing, or expanding, and applications include pattern formation on growing organisms [21, 35, 42]. In [60], a reaction-diffusion problem on an isotropically growing domain is used to model insect dispersal on a growing leaf, with reaction term $f(u) = u(a - bu^q)$. A linear growth-diffusion equation on an expanding domain is analysed by Simpson in [56], with an extension to coupled systems in [57]. The domain is itself expanding at each position x, to model the uniform growth of living tissue. In both [60] and [56] the method is to transform the problem onto a fixed domain, with the domain growth leading to advection and dilution terms. On the fixed domain, the equation takes the form

$$\frac{\partial u}{\partial t} = \frac{D}{L(t)^2}\frac{\partial^2 u}{\partial \xi^2} - \sigma(t)u(\xi, t) + f(u(\xi, t)) \qquad \text{for } 0 < \xi < 1 \tag{1.23}$$

where $\sigma(t)$ depends on the growth of the domain, which is assumed uniform and isotropic. In [56], Simpson derives exact separable solutions to the linear problem, while in [60], Tang and Lin make further assumptions on the domain growth that allow them to derive the asymptotic long-time behaviour of the solution to the nonlinear problem. This ‘growing domain’ model differs to the case that we consider in this thesis, since in our problem the physical points inside the domain are not themselves being expanded or stretched, but rather the boundary of the domain is moving. (In this respect, our case is more similar to the moving boundary problem in [24, 23, 17, 26].) In our model, therefore, the problem that results from transforming onto a fixed domain differs from (1.23). The corresponding equation, if we transform $(A(t), A(t)+L(t))$ to $(0,1)$, is now

$$\frac{\partial u}{\partial t} = \frac{D}{L(t)^2}\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\dot{A}(t)}{L(t)} + \frac{\dot{L}(t)}{L(t)}\xi\right)\frac{\partial u}{\partial \xi} + f(u(\xi,t)) \qquad \text{for } 0 < \xi < 1. \quad (1.24)$$

Compared to the analysis in [56], some further changes of variables are required in order to put the linear equation into a separable form and derive exact solutions, but the principle is the same.

We also note that the linear Schrödinger equation has been studied on a time-varying interval $(0, L(t))$ in [22] and [39]. In [22] it is assumed that $\frac{dL}{dt}$ is constant, while [39] derives necessary conditions on $L(t)$ in order to solve the problem by the separation of variables.

It seems that our exact separable solutions for a linear reaction-diffusion equation on a time-dependent interval were previously unknown. The only other explicit representations of a solution to a linear problem of this type are those of Xia, Fokas and Pelloni in [66, 31] which are of a very different style. Their solutions to the heat equation on a time-dependent interval $l_1(t) < x < l_2(t)$ are given in terms of contour integrals in the complex plane involving transforms of the known data (initial and boundary values), the domain (l_1, l_2 and their derivatives), and the solution to a pair of coupled Volterra integral equations. These formulae are derived by applying the transform method for evolution equations that was introduced by Fokas in [30], to the heat equation on a time-dependent interval. This has also been applied to a semi-infinite time-dependent interval $l(t) < x < \infty$ in [45]. Unlike the exact solutions presented in this thesis, which rely on the zero Dirichlet boundary values and certain special forms of the domain length $L(t)$, the analysis of Xia, Fokas and Pelloni allows for more general boundary conditions and more generality in l_1, l_2. However, in order to apply their formulae it is necessary to solve a system of integral equations, as well as perform a number of integral transforms, inverse transforms, and contour integrals in the complex plane. Overall the complex representation of the solution means that the formulae in [66, 31] are not in such a convenient form for readily understanding the solution's dependence on x and t, or its behaviour as $t \to \infty$, as our separable solutions. So the two, very different, ways of representing exact solutions will have their own uses and applications.

Since the approach in this thesis is based on space-and-time-dependent changes of variables in a diffusion equation, let us mention the works [58, 59] by Suazo, Suslov and Vega-Guzmán, which are also based on such methods. They use carefully chosen transformations of variables to convert between (i) a diffusion-type equation with variable coefficients and (ii) the heat equation. They pose their problem on the real line, and derive the fundamental solution for their non-autonomous class of diffusion-type equation. This fundamental solution is given in terms of the solution $\mu(t)$ to a second order ODE, and a set of six coefficients defined by integrals involving $\mu(t)$, $\dot{\mu}(t)$, and the time-dependent coefficients of the parabolic equation. They give several applications, to a range of PDEs, but always on $\mathbb{R}$ and not on a domain with a boundary.

Our own problem, with moving boundaries, has potential application to a species population subjected to habitat movement. Due to the importance of climate change — and its consequences for the migration, survival or extinction of species — this theme has been considered by several authors; but typically these incorporate the shifting habitat into the reaction term and not the domain boundaries themselves. Let us note in particular the paper by Potapov and Lewis [54] on a two-species competition, and the paper of H. Berestycki, Diekmann, Nagelkerke and Zegeling [6] for a single species. They consider the nonlinear problem on the whole real line, with a reaction term that leads to growth in a favourable region and decay elsewhere. The favourable region has a fixed length L and moves at a constant speed c. In general, therefore, in their model it is not the domain itself whose boundaries move, but a subdomain. Nevertheless, the case of a finite interval moving at constant speed c with Dirichlet boundary conditions is included as a limiting case of their problem. Several results are proved in [54, 6] regarding the dynamics on a moving interval as opposed to a stationary one. Both papers prove the existence of a minimal domain length L needed for survival, and express this as a function of c. If c is greater than the critical value c_* (see equation (1.5)) then the solution decays exponentially to zero regardless of the domain length. The implication is that if the climate changes too rapidly then the species is unable to keep up, and goes extinct. The critical speed c_* features in our solutions in a similar manner. In the case of the two-species competition, Potapov and Lewis [54] give some interesting results regarding invasibility in a domain moving at constant speed c. In particular, the outcome of the competition may change depending on c. The paper [7] by H. Berestycki, Desvillettes and Diekmann also considers a two-species competition in an environment that shifts at constant speed c. They prove that as $t \to \infty$ a 'gap' may form between the two species if the climate shift forces one to retreat faster than the other species can invade the territory. Another piece of work on a shifting climate is [55] on a single species in a two-dimensional domain. Their reaction term is favourable for growth only in a finite 'climate envelope' of constant length L which moves 'north' at a constant speed. They study the effects of different boundary geometries (domain shapes) on the population, as well as considering reaction terms with an Allee effect. Many other studies have been carried out into population dynamics with a shifting climate, including higher dimensional problems on the whole space or a cylinder [9, 10, 13], other types of reaction terms [14], and nonlocal equations [1, 65]. All of these include the climate shift as part of the reaction term, and all make the mathematically convenient assumption that climate change translates the favourable habitat at a constant speed c. Here, we instead consider a domain whose boundaries are moving, and we consider not only the case of a fixed length L and constant speed c, but also some other much more general moving boundaries.

The outline of the thesis is as follows. In Chapter 2 we introduce the problem and assumptions in more detail. We define the types of time-dependent domain that will be considered, which come under the headings ‘smooth and bounded’, ‘box-like’, and ‘cylinder-like’. We also give some preliminary results on a time-dependent domain including existence and uniqueness of a classical solution, and interior estimates relating to higher order derivatives. Chapter 2 also includes certain conditions that guarantee convergence to zero in an L^2 norm.

Chapter 3 is about the solution to the linear equation. We begin by considering the one-dimensional problem on the interval $A(t) < x < A(t) + L(t)$ and by transforming onto a fixed spatial domain. We let $\xi = \left(\frac{x-A(t)}{L(t)}\right) L_0$, and $u(\xi,t) = \psi(x,t)$, and obtain the equation

$$\frac{\partial u}{\partial t} = D\frac{L_0^2}{L(t)^2}\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\dot{A}(t)L_0 + \xi\dot{L}(t)}{L(t)}\right)\frac{\partial u}{\partial \xi} + f_0 u \qquad \text{for } 0 < \xi < L_0. \quad (1.25)$$

We introduce a further change of variables, $w(\xi,t) = u(\xi,t)H(\xi,t)e^{-f_0 t}$ with $H(\xi,t)$ given in (3.6), to transform the problem into an equation for $w(\xi,t)$ which has a particularly convenient form. Thus we deduce conditions, namely that $\ddot{L}L^3$ and $\ddot{A}L^3$ are constants, under which the transformed equation can be solved exactly by separation of variables. This separability condition means that $L(t)$ has the form $L(t) = \sqrt{at^2 + 2bt + l^2}$. The form of $A(t)$ and of the exact solutions then depend on whether a and b are zero or non-zero, and the sign of $al^2 - b^2$. We give the explicit expressions for $u(\xi,t)$ in each exactly-solvable case (Section 3.2), and discuss several implications. The main result is contained in Theorem 3.2 which gives the formula for the exact solutions for $u(\xi,t)$:

$$\begin{aligned} u_n(\xi,t) = &\exp\left(\sigma_n \int_0^t \frac{L_0^2}{L(\zeta)^2}d\zeta\right) g_n(\xi)\left(\frac{L(0)}{L(t)}\right)^{1/2} \\ &\times \exp\left(f_0 t - \int_0^t \frac{\dot{A}(\zeta)^2}{4D}d\zeta - \frac{\xi^2\dot{L}(t)L(t)}{4DL_0^2} - \frac{\xi\dot{A}(t)L(t)}{2DL_0}\right) \end{aligned} \quad (1.26)$$

where $g_n(\xi)$ satisfies the Sturm-Liouville problem (3.19), (3.20) with $\gamma_0 = al^2 - b^2$

and eigenvalue σ_n. We also apply the separation of variables method to a time-dependent box, cylinder-like domain, and ball in $\mathbb{R}^N$.

In Section 3.5 we prove upper and lower bounds on the solution for forms of $L(t)$ and $A(t)$ which do not satisfy the separability condition. These bounds can, in many cases, give the exact order of the time dependence of the solution as $t \to \infty$. Using these comparison results, we study the role of the 'critical length' $L_{crit} = \pi\sqrt{\frac{D}{f_0}}$ which is more complicated on $0 < x < L(t)$ than on a fixed domain. We present cases for which $L(t)$ is strictly less than L_{crit} for all $t \geq 0$, and yet the solution does not tend to zero as $t \to \infty$. We see in Example 3.20 that if $L(t) = L_{crit}(1 - \varepsilon(t + t_0)^{-k})$ then the outcome depends on k. If $0 < k \leq 1$ then $\psi(x,t) \to 0$, whereas if $k > 1$ then $\psi(x,t)$ has a non-trivial lower bound.

In Chapter 4 we continue to study the linear equation, but now focusing on the behaviour near the boundaries. There are zero Dirichlet conditions on the boundary itself, but we are interested in understanding how the solution behaves in a neighbourhood of the boundary and how this depends on the boundary motion. Broadly speaking, if the endpoints move apart too fast then the solution tends to zero in a neighbourhood of the boundary, whereas if they move apart too slowly then the solution tends to infinity at order one distances from the boundary. We give precise bounds on this, which involve the critical speed c_*. In particular, if the endpoints of the interval are $\pm\frac{L(t)}{2} = \pm(c_*t - \delta(t))$ with $\delta(t) \ll t$ as $t \to \infty$ and satisfying the conditions (4.29) and (4.31), then $\psi\left(\frac{-L(t)}{2} + y, t\right)$ is exactly of order $yt^{-\frac{3}{2}}\exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D}d\zeta\right)$ as $t \to \infty$, when $0 < y = O(1)$. If $\frac{L(t)}{2} = c_*t - \alpha\log(\frac{t}{t_0} + 1) - \theta(t)$ where $\alpha > 0$ and $\theta(t) = O(1)$ satisfies (4.38) then $\psi\left(\frac{-L(t)}{2} + y, t\right)$ is of order $yt^{-\frac{3}{2}+\frac{\alpha c_*}{2D}}$ as $t \to \infty$, and so it remains exactly of order one precisely when $\alpha = \frac{3D}{c_*}$. This 'critical' choice $\frac{L(t)}{2} = c_*t - \frac{3D}{c_*}\log(\frac{t}{t_0}+1) + O(1)$ for the linear equation on $\frac{-L(t)}{2} < x < \frac{L(t)}{2}$ matches the logarithmic term in both the front position for the nonlinear KPP problem on $\mathbb{R}$, and the boundary position $\mu(t)$ in [12], starting from compactly supported initial conditions.

In Section 4.3, the behaviour near the boundary is studied on a ball in $\mathbb{R}^N$ with time-dependent radius $R(t)$. For $N \leq 3$ we show that the 'critical' choice of $R(t)$, so that the solution remains exactly of order one at an order one distance from the boundary, satisfies $R(t) = c_* t - \frac{(2+N)D}{c_*} \log(\frac{t}{t_0} + 1) + O(1)$. Again, this matches the coefficient of the logarithmic term in the front position (1.13) for the nonlinear KPP problem on $\mathbb{R}^N$. Note that the restriction $N \leq 3$ on the dimension in the case of the ball is only needed for our proof of the subsolution, and it is possible that the 'critical' $R(t)$ may have the same form also for $N > 3$.

These results on the interval and the ball suggest that there is a correspondence between (i) the 'critical' choice of boundary motion, such that the solution to the linear equation on a symmetric time-dependent domain with zero Dirichlet conditions remains of order one at an order one distance from the boundary, and (ii) the front positions for the solution to the nonlinear KPP problem on the unbounded domain with compactly supported initial conditions.

In Section 4.4 we pose the linear problem on a box in $\mathbb{R}^{N+1}$ with cross-section $A_j(t) < x_j < A_j(t) + L_j(t)$ $(1 \leq j \leq N)$, and with $\frac{-L_{N+1}(t)}{2} < x_{N+1} < \frac{L_{N+1}(t)}{2}$. We consider the following problem. Given $A_j(t)$, $L_j(t)$ for $1 \leq j \leq N$, we would like to choose $L_{N+1}(t)$ so that, at some given positions $x_j(t)$ in the 'cross-section', ψ remains exactly of order one when x_{N+1} is an order one distance from the boundary $x_{N+1} = \frac{-L_{N+1}(t)}{2}$. Our approach combines the separable sub- and supersolutions on the box from Chapter 3 with our results about the behaviour near the endpoints of an interval. In several cases we can indeed find a 'critical' choice of $L_{N+1}(t)$ satisfying the required property. It is not yet known whether there is any correspondence between this special choice of $L_{N+1}(t)$ and the nonlinear KPP problem on the associated unbounded time-dependent domain, with $A_j(t) < x_j < A_j(t) + L_j(t)$ for $1 \leq j \leq N$ and $-\infty < x_{N+1} < \infty$.

In Chapter 5 we consider domains $\Omega(t) \subset \mathbb{R}^N$ that are bounded and T-periodic. After transforming the problem onto a fixed domain, we obtain a parabolic equation whose coefficients are T-periodic in t. The results of [20] for periodic-parabolic problems mean that the long-time behaviour is determined by a principal periodic eigenvalue, μ. We derive bounds on this eigenvalue for our equation, under a range of different assumptions on $\Omega(t)$, and apply these bounds to some illustrative examples. In Section 5.3 we then consider μ as a function of the frequency $\omega = \frac{2\pi}{T}$ and prove results concerning the small and large frequency limits, as well as a monotonicity property. Where indicated, some of our proofs are based on adapting results of Liu, Lou, Peng and Zhou from [46], where they also study the dependence of a principal periodic eigenvalue on the frequency for a different problem. We identify $\lim_{\omega \to 0} \mu(\omega)$ for a $\frac{2\pi}{\omega}$-periodic domain in any dimension, and we show that very different types of asymptotic behaviour of $\mu(\omega)$ are possible as $\omega \to \infty$. Indeed, let $L(t) = L_0 l\left(\frac{\omega t}{2\pi}\right)$ and $A(t) = A_0 a\left(\frac{\omega t}{2\pi}\right)$ where $L_0 > 0$, $A_0 \geq 0$, $\omega > 0$, and where $l(\cdot)$ and $a(\cdot)$ are 1-periodic functions with $l \geq 1$. Let $\mu(\omega)$ be the principal periodic eigenvalue associated with $\Omega(t) = (A(t), A(t)+L(t))$. We show that if $a(\cdot)$ is constant then $\mu(\omega) = O(1)$ as $\omega \to \infty$, but if $a(\cdot)$ is non-constant then there exist C_1, C_2 such that: if $\frac{A_0}{L_0} < C_1$ then $\mu(\omega) = O(1)$ as $\omega \to \infty$, and if $\frac{A_0}{L_0} > C_2$ then $\mu(\omega) \to \infty$ as order ω^2 as $\omega \to \infty$ (see Theorem 5.12). This raises several questions, for example: is there a threshold value of $\frac{A_0}{L_0}$ at which $\mu(\omega)$ stops being $O(1)$? If so, what is it? Are there values of $\frac{A_0}{L_0}$ such that $\mu(\omega) \to \infty$ at a different rate as $\omega \to \infty$? Different methods may be required to answer such questions.

We conclude Chapter 5 with the nonlinear problem on a periodic domain. Using a result of Hess [37] and methods involving the Poincaré map, we prove convergence to either zero or a unique positive periodic solution u^*.

In Chapter 6 we consider the nonlinear reaction-diffusion problem on a bounded domain $\Omega_0 \subset \mathbb{R}^N$ translating at constant velocity $c \in \mathbb{R}^N$. Depending on the sign of $f'(0) - D\lambda(\Omega_0) - \frac{|c|^2}{4D}$, where $\lambda(\Omega_0)$ is the principal eigenvalue of $-\nabla^2$ on Ω_0 with zero Dirichlet boundary conditions, we prove that the solution converges to either zero or a positive stationary limit U_c which (when it exists) is unique. In Section 6.2 we prove several properties of U_c, in particular with respect to different velocity vectors c, different domains Ω_0, and the asymptotic behaviour of U_c as the domain length tends to either a critical length or infinity.

In Chapter 7 we extend results about the long-time behaviour for the nonlinear equation to other cases, where the domain's size and velocity are not necessarily constant. This includes, for example, cases in which the domain and its velocity converge as $t \to \infty$, and also cases such that the side lengths $L_j(t)$ tend to ∞ as $t \to \infty$, with $\dot{L}_j(t) \to \alpha_j \geq 0$ and $\dot{A}_j(t) \to c_j$. Many of these proofs rely heavily on the properties that were proven in Chapter 6 for the constant velocity case. Solutions to the nonlinear problem on a cylinder-like time-dependent domain are also discussed in Chapter 7. Finally, the solution to the nonlinear problem on the interval is analysed in more detail, and we again consider the behaviour near the boundary. Consider a symmetric interval with $\frac{L(t)}{2} = c_* t - \alpha \log(\frac{t}{t_0} + 1) + \frac{l_0}{2}$ and $\alpha > 0$. We know from Chapter 4 that for the linear problem, $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right)$ behaves like a multiple of $t^{-\frac{3}{2}+\frac{\alpha c_*}{2D}}$ as $t \to \infty$. For the nonlinear equation, in contrast, we show that $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right)$ decays faster than every power of t, in the sense that it cannot be bounded below by any power (see Theorem 7.23).

Chapter 8 briefly summarises the main conclusions of the thesis and recommends a number of related mathematical questions for further work.

The publication [2] is based on certain parts of this PhD work: the exact solutions on an interval and a ball (Sections 3.1, 3.2 and 3.8) and results about critical boundary motion for the interval and ball (Sections 4.2 and 4.3). Also, [3] contains the comparison result of Theorem 3.13 and work from Section 3.5.2.

Chapter 2

Time-dependent domains, assumptions, and preliminary results

2.1 Equation and change of variables

We consider the problem (1.1), (1.2) for $\psi(x,t)$; that is:

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f(\psi) \qquad \text{for } x \in \Omega(t)$$

$$\psi(x,t) = 0 \qquad \text{for } x \in \partial\Omega(t),$$

where $t \geq 0$, $\Omega(t) \subset \mathbb{R}^N$ is a time-dependent domain, and f is a given function. We shall assume that $\Omega(t)$ has one of three general types, each of which will allow us to change variables from $x \in \Omega(t)$ to $\xi \in \Omega_0$ for some fixed domain $\Omega_0 \subset \mathbb{R}^N$, such that the function $u(\xi,t) = \psi(x,t)$ satisfies a parabolic problem of the form

$$\frac{\partial u}{\partial t} = \mathcal{L}(\xi,t)u + f(u) \qquad \text{for } \xi \in \Omega_0 \tag{2.1}$$

$$u(\xi,t) = 0 \qquad \text{for } \xi \in \partial\Omega_0. \tag{2.2}$$

We shall say that $\Omega(t)$ is of Type 1 if it is bounded, connected, and there is a one-to-one mapping $h(\cdot,t) : \overline{\Omega(t)} \to \overline{\Omega_0}$ that transforms $\Omega(t)$ into a bounded, connected reference domain Ω_0 with sufficiently smooth boundary (at least $C^{2+\varepsilon}$ with $\varepsilon > 0$), and satisfies the following conditions. The mapping $h(x,t)$ should be at least twice differentiable in $x \in \overline{\Omega(t)}$ and once differentiable in t, and these derivatives should be Hölder continuous in both variables. For $T > 0$, the matrix $\tilde{a}_{ij}(x,t) = \sum_k \frac{\partial h_i}{\partial x_k}(x,t)\frac{\partial h_j}{\partial x_k}(x,t)$ should be uniformly positive definite on $\{0 \le t \le T,\ x \in \overline{\Omega(t)}\}$.

If $\Omega(t)$ is of Type 1 then we can change variables onto the fixed domain Ω_0 by letting $\xi = h(x,t)$ and $u(\xi,t) = \psi(x,t)$. Then u satisfies (2.1), (2.2) where

$$\mathcal{L}(\xi,t)u = \sum_{i,j,k} D\left(\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right)\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_j \left(D\nabla^2 h_j - \frac{\partial h_j}{\partial t}\right)\frac{\partial u}{\partial \xi_j} \qquad \text{for } \xi \in \Omega_0. \tag{2.3}$$

These coefficients are locally Hölder continuous in both variables, and the condition that the matrix $\tilde{a}_{ij}$ is uniformly positive definite ensures that equation (2.1) is parabolic on $\Omega_0 \times (0,T]$.

Next, we shall say that $\Omega(t)$ is of Type 2 if it can be separated as the Cartesian product $\Omega(t) = \prod_{j=1}^n \Omega^{(j)}(t)$ where each $\Omega^{(j)}(t)$ is a Type 1 domain, i.e. smooth and bounded. By changing variables from $x^{(j)} \in \Omega^{(j)}(t)$ to $\xi^{(j)} \in \Omega_0^{(j)}$, the equation becomes one of the form

$$\frac{\partial u}{\partial t} = \sum_{j=1}^n \mathcal{L}_j(\xi^{(j)},t)u + f(u) \qquad \text{for } \xi \in \Omega_0 = \prod_{j=1}^n \Omega_0^{(j)} \tag{2.4}$$

where $\mathcal{L}_j(\xi^{(j)},t)$ are the transformed operators corresponding to each of the Type 1 domains. Thus, $\mathcal{L}_j$ contains the spatial derivatives with respect to $\xi^{(j)}$, and has coefficients depending on $\xi^{(j)}$ and t.

Finally, we shall say that $\Omega(t)$ is of Type 3 if it can be separated as the Cartesian product $\Omega(t) = \tilde{\Omega}(t) \times \mathbb{R}$ where $\tilde{\Omega}(t)$ is a domain of Type 1 or 2. This is a time-dependent strip, or cylinder. By changing variables from $x \in \tilde{\Omega}(t)$ to $\xi \in \tilde{\Omega}_0$ and leaving $y \in \mathbb{R}$ unchanged, the equation for $u(\xi,y,t)$ becomes

$$\frac{\partial u}{\partial t} = \mathcal{L}_{\tilde{\Omega}}(\xi,t)u + D\frac{\partial u}{\partial y^2} + f(u) \qquad \text{for } (\xi,y) \in \Omega_0 = \tilde{\Omega}_0 \times \mathbb{R} \tag{2.5}$$

where the $\mathcal{L}_{\tilde{\Omega}}$ is the transformed operator for the Type 1 or Type 2 domain.

We shall always assume, even when not stated explicitly, that the domain is of one of these three types, which we refer to as 'smooth and bounded', 'box-like', and 'cylinder-like'. For the domains $\Omega(t)$ which we shall focus on, there is a natural choice of the map $h(x,t)$. Let us give some examples.

Example 2.1.

$$\Omega(t) = \Omega_0 + A(t) = \big\{x \in \mathbb{R}^N : x - A(t) \in \Omega_0\big\} \tag{2.6}$$

for a smooth bounded domain $\Omega_0 \subset \mathbb{R}^N$ and twice differentiable vector $A(t)$ in $\mathbb{R}^N$. Letting $\xi = x - A(t)$, the equation for $u(\xi,t)$ becomes

$$\frac{\partial u}{\partial t} = D\nabla^2 u + \dot{A}(t)\cdot\nabla u + f(u) \qquad \text{for } \xi \in \Omega_0. \tag{2.7}$$

Example 2.2. *A time-dependent interval, $\Omega(t) = (A(t), A(t)+L(t))$ for some $A(t)$ and $L(t) > 0$, both twice differentiable. We change variables from x to $\xi = \left(\frac{x-A(t)}{L(t)}\right)L_0$ (for some $L_0 > 0$), and let $\psi(x,t) = u(\xi,t)$, so that the problem*

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + f(\psi) \qquad \text{on } A(t) < x < A(t)+L(t) \tag{2.8}$$

$$\psi(x,t) = 0 \qquad \text{at } x = A(t) \text{ and } x = A(t)+L(t) \tag{2.9}$$

becomes

$$\frac{\partial u}{\partial t} = D\frac{L_0^2}{L(t)^2}\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\xi\dot{L}(t) + L_0\dot{A}(t)}{L(t)}\right)\frac{\partial u}{\partial \xi} + f(u) \qquad \text{for } 0 < \xi < L_0 \tag{2.10}$$

$$u(\xi,t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{2.11}$$

Example 2.3. *Let $\Omega(t)$ be any smooth, connected, bounded domain in $\mathbb{R}^2$, smoothly varying in t. Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, via*

$$z = x_1 + ix_2, \qquad w = \xi_1 + i\xi_2, \tag{2.12}$$

and find a conformal mapping $w = p^{(t)}(z)$ that maps $z \in \overline{\Omega(t)}$ to $w \in \overline{\Omega_0}$ for some fixed reference domain $\Omega_0 \subset \mathbb{C}$. (For example if $\Omega(t)$ is simply-connected then *we can take Ω_0 to be the unit disk $|w| < 1$.) This conformal map corresponds to a map $\xi = h(x,t)$ on $\overline{\Omega(t)} \subset \mathbb{R}^2$, and by the Cauchy-Riemann equations,*

$$\sum_k \frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_k} = |(p^{(t)})'(z)|^2 \delta_{ij}, \qquad \nabla^2 h_j \equiv 0. \tag{2.13}$$

So the second order term in the transformed equation (2.1) is $D|(p^{(t)})'(z)|^2\nabla^2 u$ (see equation (2.3)). Since $p^{(t)}$ is a conformal mapping, $(p^{(t)})'(z) \neq 0$ and the equation is parabolic in $\xi \in \Omega_0$.

Example 2.3 allows us to analyse a range of unusually shaped domains in $\mathbb{R}^2$, whose time dependence may be completely non-isotropic. We are also interested in the following box-like domains, which can have different time dependence in each direction $1 \leq j \leq N$.

Example 2.4. *A time-dependent box:*

$$\Omega(t) = \big\{x \in \mathbb{R}^N : A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N\big\} \tag{2.14}$$

for some $A_j(t) \in \mathbb{R}$, and $L_j(t) > 0$, each twice differentiable. We change variables to $\xi_j = \left(\frac{x_j - A_j(t)}{L_j(t)}\right) L_0$ and let $\psi(x,t) = u(\xi,t)$. The equation becomes

$$\frac{\partial u}{\partial t} = D\sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2}\frac{\partial^2 u}{\partial \xi_j^2} + \sum_{j=1}^{N}\left(\frac{\xi_j \dot{L}_j(t) + L_0 \dot{A}_j(t)}{L_j(t)}\right)\frac{\partial u}{\partial \xi_j} + f(u) \quad \text{for } 0 < \xi_j < L_0 \tag{2.15}$$

$$u(\xi,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{2.16}$$

Example 2.5.

$$\Omega(t) = \big\{(x_0,x) \in \mathbb{R}^{m+N} : x_0 - A_0(t) \in \omega_0 \subset \mathbb{R}^m, \\ A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N\big\} \tag{2.17}$$

for some smooth bounded domain $\omega_0 \subset \mathbb{R}^m$, and twice differentiable $A_0(t) \in \mathbb{R}^m$, $A_j(t) \in \mathbb{R}$, and $L_j(t) > 0$. Letting $\xi_0 = x_0 - A_0(t)$ and $\xi_j = \left(\frac{x_j - A_j(t)}{L_j(t)}\right) L_0$, the equation for $u(\xi_0, \xi, t)$ on $\Omega_0 := \omega_0 \times (0, L_0)^N$ becomes:

$$\frac{\partial u}{\partial t} =D\nabla^2_{\xi_0} u + \sum_{j=1}^{N} D\frac{L_0^2}{L_j(t)^2}\frac{\partial^2 u}{\partial \xi_j^2} + A_0(t)\cdot\nabla_{\xi_0} u$$

$$+\sum_{j=1}^{N}\left(\frac{\xi_j \dot{L}_j(t) + L_0 \dot{A}_j(t)}{L_j(t)}\right)\frac{\partial u}{\partial \xi_j} + f(u) \qquad \text{for } (\xi_0,\xi)\in\Omega_0 \qquad (2.18)$$

$$u(\xi_0,\xi,t) = 0 \qquad \text{for } (\xi_0,\xi)\in\partial\Omega_0. \qquad (2.19)$$

The function f in our problem is assumed to be either linear: $f(\psi) = f_0\psi$ with $f_0 > 0$, or nonlinear and such that for some $K > 0$,

$$f(0) = f(K) = 0, \quad f \text{ is Lipschitz continuous}, \quad f'(0) \text{ exists and } > 0,$$
$$\frac{f(u)}{u} \text{ is non-increasing on } u > 0. \qquad (2.20)$$

Example 2.6. *Examples of f satisfying these conditions include:*

1. $f(u) = au(1-(\frac{u}{K})^{\beta})$ *for any* $a > 0$, $\beta > 0$;

2. $f(u) = a\sin(\frac{\pi u}{K})$ *for any* $a > 0$;

3. $f(u) = \frac{f'(0)}{2}(K - |2u - K|)$.

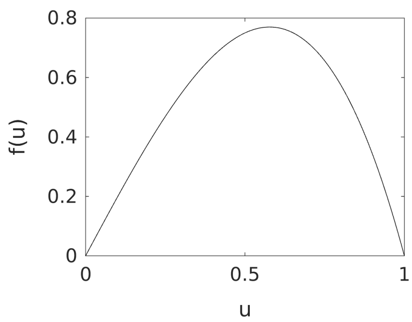

Figure 2.1: Sketch of a typical nonlinearity f satisfying (2.20).

The problem on a time-dependent domain has been transformed into a non-autonomous parabolic problem (2.1), (2.2) on a fixed domain Ω_0. The standard parabolic maximum principles, comparison principles, and uniqueness theorems can be applied to $u(\xi,t)$ (see [32, chapter 2] and [64, Section 3.2]). In particular, we observe the following:

1. $u \equiv 0$ is a solution to (2.1), (2.2). If $u(\cdot, 0) \geq 0$ then by the comparison principle $u(\cdot, t) \geq 0$ for all $t \geq 0$. For the nonlinear problem, $u \equiv K$ is a supersolution. If $u(\cdot, 0) \leq K$ then $u(\cdot, t) \leq K$ for all $t \geq 0$.

2. If $u(\cdot, 0) \geq 0$ and is not identically zero, then for every $t_0 > 0$ the solution $u(\cdot, t_0)$ is strictly positive on the interior of Ω_0, satisfies the zero Dirichlet boundary conditions, and has finite and non-zero normal derivative on $\partial\Omega_0$ (or $\partial\Omega_0$ minus the corners for domains of Type 2 or 3). This follows from the strong maximum principle and Hopf's Lemma in the parabolic case; see [51, chapter 2, Theorem 1.4].

3. Assumptions (2.20) on f imply that $\frac{f(u)}{u} \leq f'(0)$ for $u > 0$. In particular, the solution u_{lin} to the linear problem — with $f(u)$ replaced by $f'(0)u$ — is a supersolution to the nonlinear equation, and so $u \leq u_{lin}$.

We shall assume initial conditions $u(\cdot, 0) = u_0$ that satisfy $u_0 \geq 0$, u_0 is not identically zero, and (for the nonlinear case) $u_0 \leq K$. Due to the observations made above, we may also assume without loss of generality (by relabelling $t = 0$) that $u_0 > 0$ in Ω_0, that it satisfies the zero Dirichlet boundary conditions, and that it has a finite and non-zero normal derivative on $\partial\Omega_0$ (or $\partial\Omega_0$ minus the corners for domains of Type 2 or 3). We shall always make these assumptions unless stated otherwise.

2.2 Existence, uniqueness, and comparison

Here we show that a solution to the problem (2.1), (2.2) exists and that, given some initial conditions, the solution is unique.

Lemma 2.7. *If $\Omega(t)$ is of Type 1, 2 or 3, then the linear operator $\frac{\partial}{\partial t} - \mathcal{L}$ in equations* (2.1), (2.2) *has a Dirichlet Green's function $G(\xi, z, t, \sigma) > 0$.*

This means that the linear initial-boundary value problem

$$\frac{\partial u}{\partial t} - \mathcal{L}u = F(\xi, t) \text{ in } \Omega_0, \qquad u(\xi, t) = 0 \text{ on } \partial\Omega_0, \qquad u(\xi, 0) = u_0(\xi), \quad (2.21)$$

for a locally Hölder continuous function $F(\xi,t)$ and continuous u_0 satisfying the boundary conditions, has a unique classical solution which can be expressed as

$$u(\xi,t)=\int_{\Omega_0} G(\xi,z,t,0)u_0(z)dz+\int_0^t\int_{\Omega_0} G(\xi,z,t,\sigma)F(z,\sigma)dzd\sigma. \qquad (2.22)$$

Proof. Suppose first that $\Omega(t)$ is of Type 1, so Ω_0 is smooth and bounded. The existence of a unique, classical, solution to the linear initial-boundary value problem is guaranteed by [32, chapter 3, Theorem 9 and Corollary 2]. Moreover [51, chapter 2, Theorem 1.1] and [32, chapter 3, section 7] show that this solution can be expressed in the form (2.22), and $G(\xi,z,t,\sigma)>0$.

Suppose next that $\Omega(t)$ is of Type 2 (box-like), and that $G_j(\xi^{(j)},z^{(j)},t,\sigma)$ are the Dirichlet Green's functions for $\frac{\partial}{\partial t}-\mathcal{L}_j$ on each $\Omega_0^{(j)}$. Then the Dirichlet Green's function for $\frac{\partial}{\partial t}-\mathcal{L}$ on Ω_0 is $G(\xi,z,t,\sigma)=\prod_{j=1}^n G_j(\xi^{(j)},z^{(j)},t,\sigma)$.

Finally, suppose that $\Omega(t)=\tilde{\Omega}(t)\times\mathbb{R}$ is of Type 3 (cylinder-like), and that $G_{\tilde{\Omega}}(\xi,z,t,\sigma)$ is the Dirichlet Green's function for $\frac{\partial}{\partial t}-\mathcal{L}_{\tilde{\Omega}}$ on $\tilde{\Omega}_0$. It can be seen that $G((\xi,y),(z,y'),t,\sigma)=G_{\tilde{\Omega}}(\xi,z,t,\sigma)\frac{1}{\sqrt{4\pi D(t-\sigma)}}e^{-\frac{(y-y')^2}{4D(t-\sigma)}}$ is the Dirichlet Green's function for $\frac{\partial}{\partial t}-\mathcal{L}$ on $\Omega_0=\tilde{\Omega}_0\times\mathbb{R}$. □

We conclude that for the linear reaction term f_0u, the problem (2.1), (2.2) with $u(\cdot,0)=u_0$ has a unique solution: $u(\xi,t)=e^{f_0t}\int_{\Omega_0}G(\xi,z,t,0)u_0(z)dz$. In Proposition 2.8 we treat the nonlinear case. Existence of a solution for the nonlinear problem (2.1), (2.2) can be proved using the monotone iteration method of Pao [51, chapter 2, Theorem 4.1]. However, here we shall give an alternative proof which again uses the Green's function $G(\xi,z,t,\sigma)$, and which is adapted from the proof of [63, Theorem 5.1].

We shall use the fact that $\int_{\Omega_0}G(\xi,z,t,\sigma)dz\le 1$ for all $\xi\in\Omega_0$ and $0\le\sigma<t$. To see why this is true, define $U_\sigma(\xi,t)=\int_{\Omega_0}G(\xi,z,t,\sigma)dz$ for $0\le\sigma<t$. Then U_σ is the solution to

$$\frac{\partial U_\sigma}{\partial t}=\mathcal{L}U_\sigma \qquad \text{for } \xi\in\Omega_0,\ t>\sigma \qquad (2.23)$$

with $U_\sigma(\xi,t)=0$ for $\xi\in\partial\Omega_0$ and $U_\sigma(\xi,\sigma)\equiv 1$ on Ω_0. It therefore corresponds, via $U_\sigma(\xi,t) = \Psi_\sigma(x,t)$, to the solution to

$$\frac{\partial \Psi_\sigma}{\partial t} = D\nabla^2 \Psi_\sigma \qquad \text{for } x \in \Omega(t),\ t > \sigma \tag{2.24}$$

with $\Psi_\sigma(x,t) = 0$ on $\partial\Omega(t)$ and $\Psi_\sigma(x,\sigma) \equiv 1$ on $\Omega(\sigma)$. By the comparison principle, this is less than the solution $\overline{\Psi}$ to the same problem on $\mathbb{R}^N$ with $\overline{\Psi}(x,\sigma) \equiv 1$ on $\Omega(\sigma)$ and zero elsewhere. In particular, $U_\sigma(\xi,t) = \Psi_\sigma(x,t) \leq \overline{\Psi}(x,t) \leq 1$ for all $t \geq \sigma$.

Proposition 2.8. *For $\Omega(t)$ of Type 1, 2, or 3, there exists a unique solution to the nonlinear problem in equations* (2.1), (2.2) *with initial conditions u_0.*

Proof. Let $G(\xi,z,t,\sigma)$ be the Dirichlet Green's function for the linear parabolic operator in (2.1), (2.2). Let

$$u^{(0)}(\xi,t) = \int_{\Omega_0} G(\xi,z,t,0)u_0(z)dz \tag{2.25}$$

and iteratively define

$$u^{(n+1)}(\xi,t) = u^{(0)}(\xi,t) + \int_0^t \int_{\Omega_0} G(\xi,z,t,\sigma) f(u^{(n)}(z,\sigma))dzd\sigma. \tag{2.26}$$

Note that $0 \leq u^{(0)}(\xi,t) \leq \int_{\Omega_0} G(\xi,z,t,0)Kdz \leq K$, and that if f_∞ is some constant such that $0 \leq f \leq f_\infty$ on $[0,K]$ then

$$\begin{aligned} |u^{(1)}(\xi,t) - u^{(0)}(\xi,t)| &= \int_0^t \int_{\Omega_0} G(\xi,z,t,\sigma) f(u^{(0)}(z,\sigma))dzd\sigma \\ &\leq \int_0^t \int_{\Omega_0} G(\xi,z,t,\sigma) f_\infty dzd\sigma \\ &\leq \int_0^t f_\infty d\sigma = f_\infty t. \end{aligned} \tag{2.27}$$

Also define

$$M_{n+1}(t) = \sup_{\xi \in \Omega_0, 0 \leq \tau \leq t} |u^{(n+1)}(\xi,\tau) - u^{(n)}(\xi,\tau)|. \tag{2.28}$$

If C is the Lipschitz constant of f, then for $n \geq 1$

$$\begin{aligned} &|u^{(n+1)}(\xi,t) - u^{(n)}(\xi,t)| \\ &\leq \int_0^t \int_{\Omega_0} G(\xi,z,t,\sigma) \left| f(u^{(n)}(z,\sigma)) - f(u^{(n-1)}(z,\sigma)) \right| dzd\sigma \end{aligned}$$

$$\leq \int_0^t \int_{\Omega_0} G(\xi, z, t, \sigma) C \left| u^{(n)}(z, \sigma) - u^{(n-1)}(z, \sigma) \right| dz d\sigma$$
$$\leq \int_0^t \int_{\Omega_0} G(\xi, z, t, \sigma) C M_n(\sigma) dz d\sigma$$
$$\leq \int_0^t C M_n(\sigma) d\sigma. \quad (2.29)$$

Therefore, we have shown that

$$M_1(t) \leq f_\infty t \qquad \text{and} \qquad M_{n+1}(t) \leq \int_0^t C M_n(\sigma) d\sigma \quad (2.30)$$

which gives, by induction on n,

$$M_{n+1}(t) \leq \frac{f_\infty C^n t^{n+1}}{(n+1)!} \,. \quad (2.31)$$

Therefore, for each fixed $t > 0$ and for $m \geq n$, $\sum_{k=n}^m M_k(t) \to 0$ as $m, n \to \infty$ and so $u^{(n)}$ is Cauchy on $\overline{\Omega_0} \times [0, t]$. It therefore converges uniformly on $\overline{\Omega_0} \times [0, t]$ to a limit function u which must satisfy the integral equation

$$u(\xi, t) = \int_{\Omega_0} G(\xi, z, t, 0) u_0(z) dz + \int_0^t \int_{\Omega_0} G(\xi, z, t, \sigma) f(u(z, \sigma)) dz d\sigma. \quad (2.32)$$

By applying [32, chapter 1, Theorem 9], we deduce that u is in fact a classical solution to the nonlinear problem.

Uniqueness is proved as follows: suppose u_1, u_2 are both solutions and let $w = u_1 - u_2$. Then $w(\xi, t)$ satisfies $\frac{\partial w}{\partial t} = \mathcal{L} w + c(\xi, t) w$ for $\xi \in \Omega_0$, where $c(\xi, t)$ is the (bounded since f is Lipschitz continuous) function

$$c(\xi, t) = \frac{f(u_1(\xi, t)) - f(u_2(\xi, t))}{u_1(\xi, t) - u_2(\xi, t)}. \quad (2.33)$$

Also $w(\xi, t) = 0$ on $\partial \Omega_0$ and $w(\xi, 0) \equiv 0$ at $t = 0$. The parabolic maximum principle then implies that $w(\xi, t) \equiv 0$. □

Remark 2.9. *For initial conditions u_0 that do not satisfy the boundary conditions or are only piecewise continuous, the existence and uniqueness of a classical solution still applies, subject to understanding the sense in which the initial conditions are satisfied. As $t \to 0$, $u(\xi, t) \to u_0(\xi)$ for each point of continuity of u_0. For the limits as $t \to 0$ at the points of discontinuity, see the statement at the bottom of page 40 of [32, chapter 2, section 3]. See also the paper [49].*

From the uniqueness, we deduce a reflection property on $\Omega_0 + A(t)$.

Lemma 2.10. *Let* $\Omega(t) = \Omega_0 + A(t)$ *be as in* (2.6) *and let* $\tilde{\Omega}(t) = \tilde{\Omega}_0 + \tilde{A}(t)$ *where* $\tilde{\Omega}_0 = \{\xi \in \mathbb{R}^N : (-\xi_1, \xi_2, \ldots, \xi_N) \in \Omega_0\}$ *and* $\tilde{A}(t) = (-A_1(t), A_2(t), \ldots, A_N(t))$. *Let* ψ, $\tilde{\psi}$ *be the solutions on* $\Omega(t)$, $\tilde{\Omega}(t)$ *respectively. If* $\psi(x_1, x_2, \ldots, x_N, 0) \equiv \tilde{\psi}(-x_1, x_2, \ldots, x_N, 0)$ *then* $\psi(x_1, x_2, \ldots, x_N, t) \equiv \tilde{\psi}(-x_1, x_2, \ldots, x_N, t)$ *for* $t \geq 0$.

Proof. Both $\psi(x, t)$ and $\Psi(x, t) := \tilde{\psi}(-x_1, x_2, \ldots, x_N, t)$ satisfy the same equation, initial conditions, and zero Dirichlet boundary conditions. The uniqueness result from Proposition 2.8 implies that they must be equal. □

We shall also need the following comparison principle for enclosed domains.

Lemma 2.11. *Suppose that* $\Omega(t) \subset \hat{\Omega}(t)$ *for all* $t \geq 0$. *Let* ψ *and* $\hat{\psi}$ *be the solutions to* (1.1), (1.2) *on* $\Omega(t)$ *and* $\hat{\Omega}(t)$ *respectively. If* $\psi(x, 0) \leq \hat{\psi}(x, 0)$ *on* $\overline{\Omega(0)}$, *then* $\psi(x, t) \leq \hat{\psi}(x, t)$ *for all* $x \in \Omega(t)$, $t \geq 0$, *and there is strict inequality unless* $\psi(x, 0) \equiv \hat{\psi}(x, 0)$ *and* $\hat{\Omega}(\tau) \equiv \Omega(\tau)$ *for all* $0 \leq \tau \leq t$.

Proof. Change variables using the transformation that maps $\Omega(t)$ to a reference domain Ω_0, and for $\xi \in \Omega_0$ denote the solutions by $u(\xi, t) = \psi(x, t)$ and $\hat{u}(\xi, t) = \hat{\psi}(x, t)$. Then both $u(\xi, t)$ and $\hat{u}(\xi, t)$ satisfy the same equation (2.1) in Ω_0, and $u(\xi, 0) \leq \hat{u}(\xi, 0)$ on Ω_0. Also, $u(\xi, t) = 0 \leq \hat{u}(\xi, t)$ on $\partial\Omega_0$, with strict inequality on at least part of the boundary unless $\hat{\Omega}(t) \equiv \Omega(t)$. The result then follows by applying the parabolic comparison principle to u and $\hat{u}$ on Ω_0. □

Remark 2.12. *The solution to the linear equation on the whole space* $\mathbb{R}^N$ *(with initial conditions equal to* $\psi(x, 0)$ *on* $\Omega(0)$ *and zero elsewhere) is always a supersolution for* ψ. *Using the expression* (1.7), *we therefore find that* $\psi(x(t), t) \to 0$ *as* $t \to \infty$ *for any* $x(t) \in \Omega(t)$ *satisfying* $|x(t)| - \left(c_* t - \frac{ND}{c_*} \log \frac{t}{t_0}\right) \to \infty$.

2.3 Interior estimates

The next proposition follows from parabolic regularity estimates (Theorem A.3).

Proposition 2.13. *Suppose there is a vector $x_0(t)$ and bounded domain Ω_1 such that $x_0(t)+\Omega_1 \subset \Omega(t)$. For $y \in \Omega_1$, let $U(y,t) = \psi(x_0(t)+y,t)$ where ψ satisfies (1.1), (1.2). Assume that U is bounded on Ω_1 and that $\dot{x}_0(t)$ is bounded in a Hölder norm. Let $t_0 > 0$, and let $\Omega' \subset \Omega_1$ be such that either (i) $\Omega' \subset\subset \Omega_1$ or else (ii) there exists $\Delta \subset \partial\Omega_1$ on which $U(y,t) = 0$, and Ω' is such that $(\partial\Omega' \cap \partial\Omega_1) \subset \Delta$ and (if $\Delta \neq \partial\Omega_1$) $dist(\partial\Omega', \partial\Omega_1 \backslash \Delta) > 0$.*

Then $U(y,t)$, $\frac{\partial U}{\partial y_i}$, $\frac{\partial^2 U}{\partial y_i \partial y_j}$ and $\frac{\partial U}{\partial t}$ are all bounded in a Hölder norm on $\overline{\Omega'}$, independently of $t \geq t_0$. Moreover, if there is a continuous function U_∞ such that $U(y,t) \to U_\infty(y)$ either pointwise in y or in $L^2(\Omega_1)$ as $t \to \infty$, then there is convergence to U_∞ in $C^2(\overline{\Omega'})$ and $\frac{\partial U}{\partial t}(\cdot,t)$ converges uniformly to zero on $\overline{\Omega'}$.

Proof. Let $0 < t_n \to \infty$ and define $U_n(y,t) = U(y,t+t_n)$. Then

$$\frac{\partial U_n}{\partial t} = D\nabla^2 U_n + \dot{x}_0(t+t_n) \cdot \nabla U_n + f(U_n) \qquad \text{in } \Omega_1. \tag{2.34}$$

Given $0 < t_0 < T$, and $\dot{x}_0(t)$ bounded in a Hölder norm, Theorem A.3 ensures Hölder bounds on U_n and the relevant derivatives in $\Omega' \times [t_0, T]$. Moreover, there must be a subsequence n_k such that U_{n_k} and the relevant derivatives are uniformly convergent on $\overline{\Omega'} \times [t_0, T]$. If $U(y,t) \to U_\infty(y)$ as $t \to \infty$ (pointwise or in L^2), then the limit of this subsequence U_{n_k} must be U_∞. Then Lemma A.5 guarantees the convergence for the whole sequence U_n and its derivatives, not just a subsequence. Finally, since the Hölder bounds and the convergence are independent of the choice of t_n, the conclusions follow. □

There are several instances in Chapter 6 and Chapter 7 in which we prove pointwise convergence to some limit as $t \to \infty$, and Proposition 2.13 then guarantees uniform convergence and convergence of the derivatives. Note that if $x_0(t) + \Omega_1 = \Omega(t)$, then due to the zero Dirichlet conditions on $\partial\Omega(t)$ we can apply Proposition 2.13 with $\Omega' = \Omega_1$ and $\Delta = \partial\Omega_1$.

2.4 Convergence to zero in L^2 norm

Let $\Omega(t)$ be a time-dependent domain such that for every $0 \le t < \infty$, $\Omega(t)$ is bounded and $|\Omega(t)| > 0$. We shall give some sufficient conditions for convergence to zero in an L^2 sense. For this purpose it is enough to show that the solution to the linear problem converges to zero, so for the remainder of this section we assume that $\psi(x,t)$ satisfies (1.1), (1.2) with $f(\psi) = f'(0)\psi$. For each fixed time t, we define $\lambda(\Omega(t))$ to be the principal Dirichlet eigenvalue of $-\nabla^2$ on the bounded domain $\Omega(t)$. In Proposition 2.14 we give a sufficient condition for convergence to zero, in terms of $\lambda(\Omega(t))$. This will be used in Chapter 5, when we consider domains that vary periodically with time, to prove Proposition 5.2.

Proposition 2.14. *At each fixed time t, let $\lambda(\Omega(t))$ be the principal eigenvalue of $-\nabla^2$ on the domain $\Omega(t)$ with zero Dirichlet boundary conditions on $\partial\Omega(t)$. If $f'(0)t - \int_0^t D\lambda(\Omega(\zeta))d\zeta \to -\infty$ as $t \to \infty$ then*

$$E(t) := \frac{1}{2}\int_{\Omega(t)} \psi(x,t)^2 dx \to 0. \tag{2.35}$$

Proof.

$$\frac{dE}{dt} = \int_{\Omega(t)} \psi(D\nabla^2\psi + f'(0)\psi)dx = \int_{\Omega(t)} (-D|\nabla\psi|^2 + f'(0)\psi^2)dx \tag{2.36}$$

$$\le (-D\lambda(\Omega(t)) + f'(0)) \int_{\Omega(t)} \psi^2 dx \tag{2.37}$$

$$= 2\left(f'(0) - D\lambda(\Omega(t))\right)E(t), \tag{2.38}$$

where the inequality follows from Poincaré's inequality. So, as $t \to \infty$, we have $0 \le E(t) \le E(0)\exp\left(2\int_0^t \left(f'(0) - D\lambda(\Omega(\zeta))\right)d\zeta\right) \to 0$. □

Example 2.15. *Let $\Omega(t) = (A(t), A(t) + L(t))$. Proposition 2.14 shows that if $f'(0)t - \int_0^t \frac{D\pi^2}{L(\zeta)^2}d\zeta \to -\infty$ as $t \to \infty$, then $E(t) = \frac{1}{2}\int_{A(t)}^{A(t)+L(t)} \psi(x,t)^2 dx \to 0$.*

Example 2.15 gives a sufficient condition for convergence to zero, on an interval of time-dependent length $L(t)$. However the condition does not use any information about $A(t)$. Next we prove a result about convergence to zero on interval of fixed length L_0, and the condition (2.39) does involve $A(t)$.

Proposition 2.16. *Let $\Omega(t) = (A(t), A(t) + L_0)$, and let c be a constant. If*

$$2\left(f'(0) - \frac{D\pi^2}{L_0^2} - \frac{c^2}{4D}\right)t - \frac{c}{D}\int_0^t (\dot{A}(\zeta) - c)d\zeta \to -\infty \qquad \text{as } t \to \infty \tag{2.39}$$

then $u(\xi, t) := \psi(A(t) + \xi, t)$ converges to 0 in $L^2([0, L_0])$.

Proof. First, note that the principal eigenvalue of the problem

$$D(y'(\xi)e^{c\xi/D})' = -\mu y(\xi)e^{c\xi/D}, \qquad y(0) = y(L_0) = 0 \tag{2.40}$$

is $\mu = \frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}$. Therefore, using the Rayleigh-Ritz formula (minimisation of the Rayleigh quotient; see Theorem A.6), we have

$$\int_0^{L_0} Dv'(\xi)^2 e^{c\xi/D} d\xi \geq \left(\frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}\right)\int_0^{L_0} v(\xi)^2 e^{c\xi/D} d\xi \tag{2.41}$$

for all $v \in C^2([0, L_0])$ satisfying $v(0) = v(L_0) = 0$.

Define $E_c(t) = \frac{1}{2}\int_0^{L_0} u(\xi, t)^2 e^{c\xi/D} d\xi$. Then

$$\frac{dE_c}{dt} = \int_0^{L_0} u\left(D\frac{\partial^2 u}{\partial \xi^2} + \dot{A}(t)\frac{\partial u}{\partial \xi} + f'(0)u\right)e^{c\xi/D} d\xi \tag{2.42}$$

$$= \int_0^{L_0}\left(-D\left(\frac{\partial u}{\partial \xi}\right)^2 + (\dot{A}(t) - c)u\frac{\partial u}{\partial \xi} + f'(0)u^2\right)e^{c\xi/D} d\xi \tag{2.43}$$

$$= \int_0^{L_0}\left(-D\left(\frac{\partial u}{\partial \xi}\right)^2 e^{c\xi/D} - (\dot{A}(t) - c)\frac{u^2}{2}\frac{c}{D}e^{c\xi/D} + f'(0)u^2 e^{c\xi/D}\right)d\xi \tag{2.44}$$

$$\leq \left(-\left(\frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}\right) - \frac{c}{2D}(\dot{A}(t) - c) + f'(0)\right)\int_0^{L_0} u^2 e^{c\xi/D} d\xi \tag{2.45}$$

$$= 2\left(f'(0) - \left(\frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}\right) - \frac{c}{2D}(\dot{A}(t) - c)\right)E_c(t), \tag{2.46}$$

since in each integration by parts the boundary terms vanish, and the inequality follows from (2.41). So, as $t \to \infty$,

$$0 \leq E_c(t) \leq E_c(0)\exp\left(2\left(f'(0) - \frac{D\pi^2}{L_0^2} - \frac{c^2}{4D}\right)t - \frac{c}{D}\int_0^t (\dot{A}(\zeta) - c)d\zeta\right) \to 0. \tag{2.47}$$

Therefore also $\|u(\cdot, t)\|_{L^2([0,L_0])} \to 0$. □

We give a corollary, for cases where $\dot{A}(t)$ converges to a constant as $t \to \infty$. This will be used again in Section 7.5.1.

Corollary 2.17. *Let* $\Omega(t) = (A(t), A(t) + L_0)$*, and suppose that* $\dot{A}(t) \to \hat{c}$ *as* $t \to \infty$*, where* $f'(0) < \frac{D\pi^2}{L_0^2} + \frac{\hat{c}^2}{4D}$*. Then* $\|u(\cdot, t)\|_{L^2([0,L_0])} \to 0$*.*

Proof. If $\hat{c} = 0$ then the assumption becomes $f'(0) < \frac{D\pi^2}{L_0^2}$ and the result follows from Proposition 2.14 and Example 2.15. If $\hat{c} \neq 0$ then we can find c sufficiently close to $\hat{c}$, and with $|c| < |\hat{c}|$, such that $f'(0) < \frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}$. Then there exists T such that for all $t \geq T$ either $\dot{A}(t) \geq c \geq 0$ or $\dot{A}(t) \leq c \leq 0$. Therefore condition (2.39) holds, and applying Proposition 2.16 gives the result. □

2.5 Definition and notation

Let us introduce the following definition and the notation we use for it.

Definition 2.18. *We say that a function* F_1 *is 'exactly of order'* F_2*, and write* $F_1(t) = \underline{\overline{O}}(F_2(t))$ *as* $t \to \infty$*, to mean that* $F_1 = O(F_2)$ *and* $F_2 = O(F_1)$*.*

For example, in the statement '$w_1(\xi, t) = \underline{\overline{O}}(w_2(\xi, t))$ as $t \to \infty$', we mean that there exist $0 < \beta_0 \leq \beta_1$ such that $\beta_0|w_2(\xi, t)| \leq |w_1(\xi, t)| \leq \beta_1|w_2(\xi, t)|$ as $t \to \infty$, uniformly in ξ. This notation is non-standard, but it is helpful in Chapter 3 and Chapter 4. It is also used, with ω instead of t, in Chapter 5.

A list of notation and symbols has been included at the start of the thesis.

Chapter 3

Linear equation: exact solutions, bounds and global behaviour

In this chapter we derive exact solutions for the linear equation on an interval $A(t) < x < A(t) + L(t)$ under certain conditions on $L(t)$ and $A(t)$. In Section 3.5 we then prove upper and lower bounds on the solution for much more general $L(t)$ and $A(t)$, and discuss the implications. These include some results concerning the role of L_{crit} when the length $L(t)$ depends on time. Finally, we extend the analysis of the linear equation to multi-dimensional time-dependent domains, specifically to a box in Section 3.6, a cylinder-like domain in Section 3.7 and a ball in Section 3.8.

3.1 Introduction, change of variables, and separability condition

We begin by considering the linear problem

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + f_0\psi \qquad \text{for } A(t) < x < A(t) + L(t) \tag{3.1}$$

$$\psi(x,t) = 0 \qquad \text{at } x = A(t) \text{ and } x = A(t) + L(t) \tag{3.2}$$

on a time-dependent interval $(A(t), A(t)+L(t))$. The start of the interval, $A(t)$, and the length of the interval, $L(t) > 0$, both vary with time and are assumed to be twice differentiable. To work on a fixed domain, we change variables from x to $\xi = \left(\frac{x-A(t)}{L(t)}\right) L_0$ for some $L_0 > 0$, and let $u(\xi, t) = \psi(x, t)$. Then

$$\left.\frac{\partial}{\partial x}\right|_t = \frac{L_0}{L(t)} \left.\frac{\partial}{\partial \xi}\right|_t \quad \text{and} \quad \left.\frac{\partial}{\partial t}\right|_x = \left.\frac{\partial}{\partial t}\right|_\xi - \left(\frac{\dot{A}(t)L_0 + \xi \dot{L}(t)}{L(t)}\right) \left.\frac{\partial}{\partial \xi}\right|_t \tag{3.3}$$

so the problem (3.1), (3.2) becomes:

$$\frac{\partial u}{\partial t} = D \frac{L_0^2}{L(t)^2} \frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\dot{A}(t)L_0 + \xi \dot{L}(t)}{L(t)}\right) \frac{\partial u}{\partial \xi} + f_0 u \qquad \text{for } 0 < \xi < L_0 \tag{3.4}$$

$$u(\xi, t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{3.5}$$

Let $w(\xi, t) = u(\xi, t) H(\xi, t) e^{-f_0 t}$ where

$$H(\xi, t) = \left(\frac{L(t)}{L(0)}\right)^{1/2} \exp\left(\int_0^t \frac{\dot{A}(\zeta)^2}{4D} d\zeta + \frac{\xi^2 \dot{L}(t) L(t)}{4DL_0^2} + \frac{\xi \dot{A}(t) L(t)}{2DL_0} \right). \tag{3.6}$$

As we shall see, the purpose of this change of variables is to remove the first order $\frac{\partial}{\partial \xi}$ term, and to transform the problem into an equation for $w(\xi, t)$ which has a particularly convenient form for deriving separability conditions and applying the comparison principle. We calculate:

$$\begin{aligned}\frac{\partial w}{\partial t} =& \frac{\partial u}{\partial t} H(\xi, t) e^{-f_0 t} - f_0 w(\xi, t) + \frac{\dot{L}(t)}{2L(t)} w(\xi, t) \\ &+ \left(\frac{\dot{A}(t)^2}{4D} + \frac{\xi^2 \ddot{L}(t) L(t)}{4DL_0^2} + \frac{\xi^2 \dot{L}(t)^2}{4DL_0^2} + \frac{\xi \ddot{A}(t) L(t)}{2DL_0} + \frac{\xi \dot{A}(t) \dot{L}(t)}{2DL_0}\right) w(\xi, t),\end{aligned} \tag{3.7}$$

$$\frac{\partial w}{\partial \xi} = \frac{\partial u}{\partial \xi} H(\xi, t) e^{-f_0 t} + \left(\frac{\xi \dot{L}(t) L(t)}{2DL_0^2} + \frac{\dot{A}(t) L(t)}{2DL_0}\right) u(\xi, t) H(\xi, t) e^{-f_0 t}, \tag{3.8}$$

$$\begin{aligned}\frac{\partial^2 w}{\partial \xi^2} =& \frac{\partial^2 u}{\partial \xi^2} H(\xi, t) e^{-f_0 t} + 2\left(\frac{\xi \dot{L}(t) L(t)}{2DL_0^2} + \frac{\dot{A}(t) L(t)}{2DL_0}\right) \frac{\partial u}{\partial \xi} H(\xi, t) e^{-f_0 t} \\ &+ \frac{\dot{L}(t) L(t)}{2DL_0^2} w(\xi, t) + \left(\frac{\xi \dot{L}(t) L(t)}{2DL_0^2} + \frac{\dot{A}(t) L(t)}{2DL_0}\right)^2 w(\xi, t).\end{aligned} \tag{3.9}$$

Therefore, using the expressions (3.7) and (3.9), we find that

$$\frac{\partial w}{\partial t} - D\frac{L_0^2}{L(t)^2}\frac{\partial^2 w}{\partial \xi^2}$$
$$= \left(\frac{\partial u}{\partial t} - D\frac{L_0^2}{L(t)^2}\frac{\partial^2 u}{\partial \xi^2} - \left(\frac{\xi \dot{L}(t) + \dot{A}(t)L_0}{L(t)}\right)\frac{\partial u}{\partial \xi} - f_0 u\right) H(\xi, t)e^{-f_0 t}$$
$$+ \left(\frac{\dot{L}(t)}{2L(t)} + \frac{\dot{A}(t)^2}{4D} + \frac{\xi^2 \ddot{L}(t)L(t)}{4DL_0^2} + \frac{\xi^2 \dot{L}(t)^2}{4DL_0^2} + \frac{\xi \ddot{A}(t)L(t)}{2DL_0} + \frac{\xi \dot{A}(t)\dot{L}(t)}{2DL_0}\right) w$$
$$- \frac{\dot{L}(t)}{2L(t)}w - D\left(\frac{\xi \dot{L}(t)}{2DL_0} + \frac{\dot{A}(t)}{2D}\right)^2 w. \tag{3.10}$$

The first bracket vanishes since u satisfies equation (3.4), and (noting the cancellation among the remaining terms) we find that

$$\frac{\partial w}{\partial t} = D\frac{L_0^2}{L(t)^2}\frac{\partial^2 w}{\partial \xi^2} + \left(\frac{\xi^2 \ddot{L}(t)L(t)}{4DL_0^2} + \frac{\xi \ddot{A}(t)L(t)}{2DL_0}\right) w \qquad \text{for } 0 < \xi < L_0 \tag{3.11}$$

$$w(\xi, t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{3.12}$$

Now we change the time variable from t to s where

$$s(t) = \int_0^t \frac{L_0^2}{L(\zeta)^2} d\zeta, \tag{3.13}$$

and write $v(\xi, s) = w(\xi, t)$. Then:

$$\frac{\partial v}{\partial s} = D\frac{\partial^2 v}{\partial \xi^2} + \left(\frac{\xi^2 \ddot{L}(t(s))L(t(s))^3}{4DL_0^4} + \frac{\xi \ddot{A}(t(s))L(t(s))^3}{2DL_0^3}\right) v \qquad \text{for } 0 < \xi < L_0 \tag{3.14}$$

$$v(\xi, s) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{3.15}$$

This equation for $v(\xi, s)$ is separable if and only if

$$\ddot{L}L^3 \equiv \gamma_0 = \text{constant} \qquad \text{and} \qquad \ddot{A}L^3 \equiv \gamma_1 = \text{constant}. \tag{3.16}$$

The condition $\ddot{L}L^3 \equiv \gamma_0$ corresponds to $L(t)^2 = at^2 + 2bt + l^2$ for some a, b, $l = L(0)$, and $\gamma_0 = al^2 - b^2$. Then, given $L(t) = \sqrt{at^2 + 2bt + l^2}$, the equation $\ddot{A}(t) = \frac{\gamma_1}{L(t)^3}$ can be integrated twice to give $A(t)$.

It is worth noting that, depending on a, b and l, it is possible either that $L(t) > 0$ for all $t \geq 0$, or that there is some finite time t_* such that $L(t) > 0$

for $0 \le t < t_*$ but $L(t_*) = 0$. In the first case we consider the problem on $0 \le t < \infty$; in the second case we consider it for $0 \le t < t_*$.

The separable equation is

$$\frac{\partial v}{\partial s} = D\frac{\partial^2 v}{\partial \xi^2} + \left(\frac{\gamma_0 \xi^2}{4DL_0^4} + \frac{\gamma_1 \xi}{2DL_0^3}\right) v \qquad \text{for } 0 < \xi < L_0 \tag{3.17}$$

$$v(\xi, s) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0, \tag{3.18}$$

and the separable solutions have the form $v(\xi, s) = e^{\sigma s} g(\xi)$ where $g(\xi)$ satisfies the Sturm-Liouville problem

$$Dg''(\xi) + \left(\frac{\gamma_0 \xi^2}{4DL_0^4} + \frac{\gamma_1 \xi}{2DL_0^3}\right) g(\xi) = \sigma g(\xi) \qquad \text{for } 0 < \xi < L_0 \tag{3.19}$$

$$g(\xi) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0 \tag{3.20}$$

with eigenvalue σ. The Sturm-Liouville theory (see, for example, [61] or [53, chapter 6]) gives that there is a countably infinite set of eigenfunctions g_n with eigenvalues σ_n satisfying $\sigma_{n+1} < \sigma_n$, and $\lim_{n\to\infty} \sigma_n = -\infty$. The largest eigenvalue, σ_1, corresponds to an eigenfunction that is positive in the open interval $(0, L_0)$ and zero only at the endpoints. Each eigenfunction g_{n+1} has one more zero than g_n. Any initial condition $v(\cdot, 0)$ in $L^2([0, L_0])$ can be written as a linear expansion in the Sturm-Liouville eigenfunctions g_n, and if $v(\cdot, 0)$ is positive then the coefficient of the principal eigenfunction g_1 is positive.

Remark 3.1. *If we were to consider the same problem but with zero Neumann conditions instead of the Dirichlet conditions, then the same analysis would still hold subject to replacing* (3.20) *by Neumann conditions for $g(\xi)$.*

Summarising the above analysis, we have proved the following theorem.

Theorem 3.2. *Suppose that*

$$L(t)^2 = at^2 + 2bt + l^2 \qquad \textit{for some } a, b, \textit{ and } l = L(0) > 0, \tag{3.21}$$

$$\ddot{A}(t) = \frac{\gamma_1}{(at^2 + 2bt + l^2)^{3/2}} \qquad \textit{for some } \gamma_1. \tag{3.22}$$

Then, given initial conditions $u(\xi, 0)$ *in* $L^2([0, L_0])$*, the solution* $u(\xi, t)$ *of* (3.4), (3.5) *can be obtained exactly, as a sum of* $u_n(\xi, t)$ *with coefficients depending only on the initial conditions. The functions* u_n *are given by*

$$u_n(\xi, t) = \exp\left(\sigma_n \int_0^t \frac{L_0^2}{L(\zeta)^2} d\zeta\right) g_n(\xi) \left(\frac{L(0)}{L(t)}\right)^{1/2}$$

$$\times \exp\left(f_0 t - \int_0^t \frac{\dot{A}(\zeta)^2}{4D} d\zeta - \frac{\xi^2 \dot{L}(t) L(t)}{4DL_0^2} - \frac{\xi \dot{A}(t) L(t)}{2DL_0}\right) \qquad (3.23)$$

where $g_n(\xi)$ *satisfies the Sturm-Liouville problem in equations* (3.19), (3.20) *with* $\gamma_0 = al^2 - b^2$*, with eigenvalue* σ_n*.*

The explicit expressions for these exact solutions depend on whether a and b are zero or non-zero, and on the sign of $al^2 - b^2$ (see Section 3.2). These exact formulae determine precisely how the solution will evolve over time. This is instructive as it helps us to understand the ways in which the time dependence of the domain influences the development of the solution in both the long and the short term.

The well-known Fourier series solution on a fixed interval expresses the solution of (3.1), (3.2) on $0 < x < L_0$ as a sum of

$$\tilde{u}_n(x, t) = \exp\left(\frac{-Dn^2\pi^2}{L_0^2} t\right) \sin\left(\frac{n\pi x}{L_0}\right) \exp(f_0 t). \qquad (3.24)$$

Theorem 3.2 can be considered as the generalisation of this to the time-dependent interval whenever the condition (3.16) holds.

Remark 3.3. *Since we have not yet used the assumption that* $f_0 > 0$*, Theorem 3.2 and the formulae in Section 3.2 are valid for any* f_0*. In particular, by taking* $f_0 = 0$ *we obtain exact solutions for the heat equation on these time-dependent intervals.*

3.2 Separated solutions

In Theorem 3.2 we proved that there are exact solutions whenever $L(t)$ has the form $L(t) = \sqrt{at^2 + 2bt + l^2}$ and $A(t)$ satisfies (3.22). Now we consider the different forms of $L(t)$ that this covers, and apply equation (3.23) to produce the explicit formula for each of the separable solutions.

3.2.1 $L(t) \equiv l$

If $L(t) \equiv l$ then $\ddot{L}(t)L(t)^3 \equiv \gamma_0 = 0$, and the condition $\ddot{A}(t)L(t)^3 \equiv \gamma_1$ corresponds to

$$A(t) = \frac{\gamma_1}{2l^3}t^2 + ct + d \tag{3.25}$$

for any constants c, d. The separable solutions have the form

$$\begin{aligned} u_n(\xi,t) = & \exp(\sigma_n t) g_n(\xi) \exp\left(f_0 t - \frac{1}{4D}\left(\frac{\gamma_1^2}{3l^6}t^3 + \frac{c\gamma_1}{l^3}t^2 + c^2 t\right)\right) \\ & \times \exp\left(-\frac{\xi}{2DL_0}\left(\frac{\gamma_1}{l^2}t + cl\right)\right) \end{aligned} \tag{3.26}$$

for $0 \leq \xi \leq L_0$. This follows by calculating that $\dot{L}(t)L(t) \equiv 0$, $s(t) = t$,

$$\dot{A}(t)L(t) = \frac{\gamma_1}{l^2}t + cl, \qquad \int_0^t \dot{A}(\zeta)^2 d\zeta = \frac{\gamma_1^2 t^3}{3l^6} + \frac{c\gamma_1 t^2}{l^3} + c^2 t, \tag{3.27}$$

and by substituting these into equation (3.23).

3.2.2 $L(t) = l + \alpha t$ with $\alpha \neq 0$

If $L(t) = l + \alpha t$ with $\alpha \neq 0$, then again $\ddot{L}(t)L(t)^3 \equiv \gamma_0 = 0$, and the condition $\ddot{A}(t)L(t)^3 \equiv \gamma_1$ now corresponds to

$$A(t) = \frac{\gamma_1}{2\alpha^2(l + \alpha t)} + ct + d \tag{3.28}$$

for any constants c, d. The separable solutions have the form

$$\begin{aligned} u_n(\xi,t) = & \exp\left(\frac{\sigma_n L_0^2 t}{l(l+\alpha t)}\right) g_n(\xi) \left(\frac{l}{l+\alpha t}\right)^{1/2} \exp(f_0 t) \\ & \times \exp\left(-\frac{\gamma_1^2}{48D\alpha^3}\left(\frac{1}{l^3} - \frac{1}{(l+\alpha t)^3}\right) + \frac{c\gamma_1 t}{4D\alpha l(l+\alpha t)} - \frac{c^2}{4D}t\right) \\ & \times \exp\left(-\frac{\xi^2\alpha(l+\alpha t)}{4DL_0^2} + \frac{\xi\gamma_1}{4D\alpha L_0(l+\alpha t)} - \frac{\xi}{2DL_0}c(l+\alpha t)\right) \end{aligned} \tag{3.29}$$

for $0 \leq \xi \leq L_0$. This follows by calculating that

$$\dot{L}(t)L(t) = \alpha(l + \alpha t), \tag{3.30}$$

$$\frac{s(t)}{L_0^2} = \int_0^t \frac{1}{(l+\alpha\zeta)^2} d\zeta = \frac{t}{l(l+\alpha t)}, \tag{3.31}$$

$$\dot{A}(t)L(t) = \frac{-\gamma_1}{2\alpha(l+\alpha t)} + c(l + \alpha t), \tag{3.32}$$

$$\int_0^t \dot{A}(\zeta)^2 d\zeta = \frac{\gamma_1^2}{12\alpha^3}\left(\frac{1}{l^3} - \frac{1}{(l+\alpha t)^3}\right) - \frac{c\gamma_1}{\alpha}\frac{t}{l(l+\alpha t)} + c^2 t, \tag{3.33}$$

and by substituting these into equation (3.23).

3.2.3 $L(t) = \sqrt{l^2 + 2\rho t}$ with $\rho \neq 0$

If $L(t) = \sqrt{l^2 + 2\rho t}$ with $l = L(0) > 0$ and $\rho \neq 0$ then

$$\ddot{L}(t)L(t)^3 \equiv \gamma_0 = -\rho^2 < 0, \tag{3.34}$$

and the condition $\ddot{A}(t)L(t)^3 \equiv \gamma_1$ corresponds to

$$A(t) = \frac{-\gamma_1\sqrt{l^2 + 2\rho t}}{\rho^2} + ct + d \tag{3.35}$$

for any constants c, d. The separable solutions have the form

$$u_n(\xi, t) = \left(\frac{l^2 + 2\rho t}{l^2}\right)^{\frac{\sigma_n L_0^2}{2\rho} - \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D}} \exp\left(f_0 t + \frac{c\gamma_1}{2\rho^2 D}\left(\sqrt{l^2 + 2\rho t} - l\right)\right)$$
$$\times\, g_n(\xi) \exp\left(-\frac{c^2}{4D}t - \frac{\xi^2\rho}{4DL_0^2} + \frac{\xi\gamma_1}{2DL_0\rho} - \frac{\xi c\sqrt{l^2 + 2\rho t}}{2DL_0}\right) \tag{3.36}$$

for $0 \leq \xi \leq L_0$. This follows by calculating that

$$\dot{L}(t)L(t) = \rho, \tag{3.37}$$

$$\frac{s(t)}{L_0^2} = \int_0^t \frac{1}{l^2 + 2\rho\zeta} d\zeta = \frac{1}{2\rho}\log\left(\frac{l^2 + 2\rho t}{l^2}\right), \tag{3.38}$$

$$\dot{A}(t)L(t) = \frac{-\gamma_1}{\rho} + c\sqrt{l^2 + 2\rho t}, \tag{3.39}$$

$$\int_0^t \dot{A}(\zeta)^2 d\zeta = \frac{\gamma_1^2}{2\rho^3}\log\left(\frac{l^2 + 2\rho t}{l^2}\right) - \frac{2c\gamma_1}{\rho^2}\left(\sqrt{l^2 + 2\rho t} - l\right) + c^2 t, \tag{3.40}$$

and by substituting these into equation (3.23).

3.2.4 $L(t) = \sqrt{at^2 + 2bt + l^2}$ with $a \neq 0$ and $al^2 - b^2 < 0$

If $L(t) = \sqrt{at^2 + 2bt + l^2}$ with $a \neq 0$, $al^2 - b^2 < 0$, and $l = L(0) > 0$, then

$$\ddot{L}(t)L(t)^3 \equiv \gamma_0 = al^2 - b^2 < 0, \tag{3.41}$$

and the condition $\ddot{A}(t)L(t)^3 \equiv \gamma_1$ corresponds to

$$A(t) = \frac{-\gamma_1}{b^2 - al^2}\sqrt{at^2 + 2bt + l^2} + c(t + b/a) + d \tag{3.42}$$

for any constants c, d. The separable solutions have the form

$$\begin{aligned} u_n(\xi, t) =& \Phi_n(t) g_n(\xi) \left(\frac{l^2}{at^2 + 2bt + l^2}\right)^{1/4} \exp\left(f_0 t - \left(\frac{\gamma_1^2 a}{(b^2 - al^2)^2} + c^2\right) t\right) \\ & \times \exp\left(\frac{c\gamma_1}{2D(b^2 - al^2)}\left(\sqrt{at^2 + 2bt + l^2} - l\right)\right) \\ & \times \exp\left(-\frac{\xi^2(at + b)}{4DL_0^2} + \frac{\xi\gamma_1(at + b)}{2DL_0(b^2 - al^2)} - \frac{\xi c}{2DL_0}\sqrt{at^2 + 2bt + l^2}\right) \end{aligned} \tag{3.43}$$

for $0 \leq \xi \leq L_0$, where

$$\Phi_n(t) = \left(\frac{\left(at + b - \sqrt{b^2 - al^2}\right)\left(b + \sqrt{b^2 - al^2}\right)}{\left(b - \sqrt{b^2 - al^2}\right)\left(at + b + \sqrt{b^2 - al^2}\right)}\right)^{\frac{\sigma_n L_0^2}{2\sqrt{b^2 - al^2}} - \frac{\gamma_1^2}{8D(b^2 - al^2)^{3/2}}}. \tag{3.44}$$

This follows by calculating that

$$\dot{L}(t)L(t) = at + b, \tag{3.45}$$

$$\begin{aligned} \frac{s(t)}{L_0^2} &= \int_0^t \frac{1}{a\zeta^2 + 2b\zeta + l^2} d\zeta \\ &= \frac{1}{2\sqrt{b^2 - al^2}} \log\left(\frac{\left(at + b - \sqrt{b^2 - al^2}\right)\left(b + \sqrt{b^2 - al^2}\right)}{\left(b - \sqrt{b^2 - al^2}\right)\left(at + b + \sqrt{b^2 - al^2}\right)}\right), \end{aligned} \tag{3.46}$$

$$\dot{A}(t)L(t) = \frac{-\gamma_1}{b^2 - al^2}(at + b) + c\sqrt{at^2 + 2bt + l^2}, \tag{3.47}$$

$$\begin{aligned} \int_0^t \dot{A}(\zeta)^2 d\zeta =& \left(\frac{\gamma_1^2 a}{(b^2 - al^2)^2} + c^2\right) t - \frac{2c\gamma_1}{b^2 - al^2}\left(\sqrt{at^2 + 2bt + l^2} - l\right) \\ & + \frac{\gamma_1^2}{2(b^2 - al^2)^{3/2}} \log\left(\frac{\left(at + b - \sqrt{b^2 - al^2}\right)\left(b + \sqrt{b^2 - al^2}\right)}{\left(b - \sqrt{b^2 - al^2}\right)\left(at + b + \sqrt{b^2 - al^2}\right)}\right), \end{aligned} \tag{3.48}$$

and by substituting these into equation (3.23).

3.2.5 $L(t) = \sqrt{at^2 + 2bt + l^2}$ with $a \neq 0$ and $al^2 - b^2 > 0$

If $L(t) = \sqrt{at^2 + 2bt + l^2}$ with $al^2 - b^2 > 0$ and $l = L(0) > 0$, then

$$\ddot{L}(t)L(t)^3 \equiv \gamma_0 = al^2 - b^2 > 0, \tag{3.49}$$

and the condition $\ddot{A}(t)L(t)^3 \equiv \gamma_1$ again corresponds to $A(t)$ as in equation (3.42). For $0 \leq \xi \leq L_0$ the separable solutions have the form given in equation (3.43), but with

$$\Phi_n(t) = e^{\left(\frac{\sigma_n L_0^2}{\sqrt{al^2-b^2}} + \frac{\gamma_1^2}{4D(al^2-b^2)^{3/2}}\right)\left(\arctan\left(\frac{at+b}{\sqrt{al^2-b^2}}\right) - \arctan\left(\frac{b}{\sqrt{al^2-b^2}}\right)\right)}. \tag{3.50}$$

This follows by calculating that $\dot{L}(t)L(t)$ is again given by (3.45), $\dot{A}(t)L(t)$ is again given by (3.47), and that

$$\begin{aligned} \frac{s(t)}{L_0^2} &= \int_0^t \frac{1}{a\zeta^2 + 2b\zeta + l^2} d\zeta \\ &= \frac{1}{\sqrt{al^2 - b^2}} \left(\arctan\left(\frac{at+b}{\sqrt{al^2-b^2}}\right) - \arctan\left(\frac{b}{\sqrt{al^2-b^2}}\right)\right), \end{aligned} \tag{3.51}$$

$$\begin{aligned} \int_0^t \dot{A}(\zeta)^2 d\zeta = &\left(\frac{\gamma_1^2 a}{(al^2-b^2)^2} + c^2\right) t + \frac{2c\gamma_1}{al^2 - b^2}\left(\sqrt{at^2+2bt+l^2} - l\right) \\ &- \frac{\gamma_1^2}{(al^2-b^2)^{3/2}}\left(\arctan\left(\frac{at+b}{\sqrt{al^2-b^2}}\right) - \arctan\left(\frac{b}{\sqrt{al^2-b^2}}\right)\right), \end{aligned} \tag{3.52}$$

and by substituting these into equation (3.23).

3.3 Bounds on the principal eigenvalue, σ_1

Under the conditions of Theorem 3.2, $u(\xi, t)$ can be expressed as an expansion in $u_n(\xi, t)$ (defined in (3.23)). Since we are considering positive solutions, we know that there is a positive coefficient of the principal eigenfunction $g_1(\xi)$ and hence $u_1(\xi, t)$. In certain cases, the principal eigenvalue σ_1 of (3.19), (3.20) therefore becomes important for assessing the long-time behaviour of the solution.

When $\gamma_0 = \gamma_1 = 0$, we know that $\sigma_1 = \frac{-D\pi^2}{L_0^2}$, and the separable solutions of (3.17), (3.18) are $v(\xi, s) = e^{\sigma_n s} \sin\left(\frac{n\pi\xi}{L_0}\right)$ where $\sigma_n = \frac{-Dn^2\pi^2}{L_0^2}$ ($n \in \mathbb{N}$). In Proposition 3.4 we shall prove an upper bound on σ_1 when $\gamma_0 < 0$. This bound will be used in Section 3.4, in order to prove Corollary 3.5 about the asymptotic behaviour of the exact solutions $u(\xi, t)$. Proposition 3.4 also includes the asymptotic form of σ_1 when $\gamma_1 = -\frac{1}{2}\gamma_0 \to \infty$, which will be used in the proof of Theorem 4.35 in Chapter 4.

Proposition 3.4. *Let $\gamma_0 = -\rho^2 < 0$, and let σ_1 be the principal eigenvalue of (3.19), (3.20). Then*

$$\sigma_1 + \frac{|\rho|}{2L_0^2} - \frac{\gamma_1^2}{4D\rho^2 L_0^2} < 0. \tag{3.53}$$

Moreover, if $\gamma_1 = -\frac{1}{2}\gamma_0 = \frac{1}{2}\rho^2$ then

$$\sigma_1 + \frac{|\rho|}{2L_0^2} - \frac{\gamma_1^2}{4D\rho^2 L_0^2} \to 0 \qquad \text{as } \rho^2 \to \infty. \tag{3.54}$$

Proof. Let $g_1(\xi) > 0$ be the principal Sturm-Liouville eigenfunction, which satisfies equation (3.19) with $\gamma_0 = -\rho^2 < 0$ and has eigenvalue σ_1. Rescale ξ to $\eta = \sqrt{\frac{|\rho|}{2D}}\frac{\xi}{L_0}$, let $\tilde{g}_1(\eta) = g_1(\xi)$ and define $\eta_0 = \frac{\gamma_1}{|\rho|^{3/2}\sqrt{2D}}$; then the equation becomes

$$\tilde{g}_1''(\eta) + (-\eta^2 + 2\eta_0\eta)\tilde{g}_1(\eta) = \frac{2L_0^2}{|\rho|}\sigma_1\tilde{g}_1(\eta) \qquad \text{for } 0 < \eta < \sqrt{\frac{|\rho|}{2D}}. \tag{3.55}$$

Now let $\tilde{g}_1(\eta) = e^{-\frac{1}{2}(\eta-\eta_0)^2}y_1(\eta)$, so that y_1 satisfies

$$y_1''(\eta) - 2(\eta - \eta_0)y_1'(\eta) - \lambda y_1(\eta) = 0 \tag{3.56}$$

where

$$\lambda = 1 + \frac{2L_0^2}{|\rho|}\sigma_1 - \frac{\gamma_1^2}{2D|\rho|^3}. \tag{3.57}$$

This can be written in self-adjoint form:

$$\frac{d}{d\eta}\left(y_1'(\eta)e^{-(\eta-\eta_0)^2}\right) = \lambda y_1(\eta)e^{-(\eta-\eta_0)^2} \qquad \text{for } 0 < \eta < \sqrt{\frac{|\rho|}{2D}} \tag{3.58}$$

$$y_1(\eta) = 0 \qquad \text{at } \eta = 0 \text{ and } \eta = \sqrt{\frac{|\rho|}{2D}}. \tag{3.59}$$

By integrating equation (3.58) over the interval $0 < \eta < \sqrt{\frac{|\rho|}{2D}}$, and using that y_1 is positive and has non-zero gradient at the endpoints, it follows that $\lambda < 0$. From the definition of λ in (3.57), $\lambda < 0$ is equivalent to the bound (3.53).

For the second part of the Proposition, suppose that $\gamma_1 = -\frac{1}{2}\gamma_0 = \frac{1}{2}\rho^2$. Then

$$\sqrt{\frac{|\rho|}{2D}} = 2\eta_0 \tag{3.60}$$

so we can transform the problem (3.58) onto the symmetric interval $(-\eta_0, \eta_0)$ by letting $z = \eta - \eta_0$ and $\hat{y}_1(z) = y_1(\eta)$. The problem then becomes:

$$\frac{d}{dz}\left(\hat{y}_1'(z)e^{-z^2}\right) = \lambda \hat{y}_1(z)e^{-z^2} \qquad \text{for } -\eta_0 < z < \eta_0 \tag{3.61}$$

$$\hat{y}_1(\pm\eta_0) = 0. \tag{3.62}$$

We are interested in the limit $|\rho| \to \infty$ which is the same as $\eta_0 \to \infty$. Since λ is the principal eigenvalue, we can express it in terms of the Rayleigh quotient:

$$-\lambda = \min_{\hat{y}} \left(\frac{\int_{-\eta_0}^{\eta_0} \hat{y}'(z)^2 e^{-z^2} dz}{\int_{-\eta_0}^{\eta_0} \hat{y}(z)^2 e^{-z^2} dz} \right) \tag{3.63}$$

where the minimum is taken over C^2 functions $\hat{y} \neq 0$ satisfying the boundary conditions (see Theorem A.6). Let us choose the test function $\hat{y}(z) = 1 - \frac{z^2}{\eta_0^2}$. Then, if $I(\eta_0) = \int_{-\eta_0}^{\eta_0} e^{-z^2} dz$, we calculate that

$$\frac{\int_{-\eta_0}^{\eta_0} \hat{y}'(z)^2 e^{-z^2} dz}{\int_{-\eta_0}^{\eta_0} \hat{y}(z)^2 e^{-z^2} dz} = \frac{\frac{4}{\eta_0^4}\left(\frac{1}{2}I(\eta_0) - \eta_0 e^{-\eta_0^2}\right)}{I(\eta_0)\left(1 + O\left(\frac{1}{\eta_0^2}\right)\right) + O\left(\frac{1}{\eta_0}e^{-\eta_0^2}\right)} \sim \frac{2}{\eta_0^4} \qquad \text{as } \eta_0 \to \infty. \tag{3.64}$$

Therefore we have the bound

$$0 < -\lambda \leq \frac{\int_{-\eta_0}^{\eta_0} \hat{y}'(z)^2 e^{-z^2} dz}{\int_{-\eta_0}^{\eta_0} \hat{y}(z)^2 e^{-z^2} dz} \sim \frac{2}{\eta_0^4} \qquad \text{as } \eta_0 \to \infty, \tag{3.65}$$

and so $\lambda = O\left(\frac{1}{\eta_0^4}\right)$ as $\eta_0 \to \infty$. Since λ is given by equation (3.57) and η_0 is given by equation (3.60), this becomes

$$1 + \frac{2L_0^2}{|\rho|}\sigma_1 - \frac{\gamma_1^2}{2D|\rho|^3} = O\left(\frac{1}{\rho^2}\right) \qquad \text{as } |\rho| \to \infty. \tag{3.66}$$

It follows that

$$\frac{|\rho|}{2L_0^2} + \sigma_1 - \frac{\gamma_1^2}{4D\rho^2 L_0^2} = O\left(\frac{1}{|\rho|}\right) \to 0 \qquad \text{as } |\rho| \to \infty. \tag{3.67}$$

□

We shall make particular use of (3.53) in the following forms:

$$\text{If } \gamma_0 = -\rho^2 \text{ with } \rho < 0: \qquad \frac{\sigma_1 L_0^2}{2\rho} - \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D} > 0. \tag{3.68}$$

$$\text{If } \gamma_0 = -\rho^2 \text{ with } \rho > 0: \qquad \frac{\sigma_1 L_0^2}{2\rho} + \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D} < 0. \tag{3.69}$$

Other bounds on σ_1 can also be derived using the Rayleigh-Ritz formula (Theorem A.6) and results from Sturm-Liouville theory such as [47, Theorem 18.1, page 54]. However we omit the other bounds here, as we will not need to use them.

3.4 Properties of the exact solutions

Although the formulae in Sections 3.2.1–3.2.5 differ, we note that they share some common properties in the asymptotic large time (or finite time) limit.

Corollary 3.5. *Let $L(t)$, $A(t)$ be a separable case for which $L(t) \to 0$ as $t \to t_* < \infty$. Then the solution $u(\xi, t)$ converges to zero uniformly in ξ as $t \to t_*$.*

Proof. First take the case $L(t) = l + \alpha t$ with $\alpha < 0$. Then $L(t) \to 0$ as $t \to t_* = \frac{l}{-\alpha}$. If $\gamma_1 \neq 0$ then we see from (3.29) that the behaviour as $l + \alpha t \to 0$, is governed by $\exp\left(\frac{\gamma_1^2}{48D\alpha^3(l+\alpha t)^3}\right)$, and so $u \to 0$ since $\alpha < 0$. If $\gamma_1 = 0$ then it is governed by $\exp\left(\frac{\sigma_n L_0^2 t}{l(l+\alpha t)}\right)$ where $\sigma_n L_0^2 = -Dn^2\pi^2 < 0$ and so again, $u \to 0$.

Next suppose $L(t) = \sqrt{l^2 + 2\rho t}$ with $\rho < 0$. Now $L(t) \to 0$ as $t \to t_* = \frac{l^2}{-2\rho}$ and equation (3.36) shows that in this limit the behaviour is governed by

$$(l^2 + 2\rho t)^{\frac{\sigma_n L_0^2}{2\rho} - \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D}}. \tag{3.70}$$

Using Proposition 3.4 and the bound (3.68), it follows that $u \to 0$ as $l^2 + 2\rho t \to 0$.

Finally, consider the case $L(t) = \sqrt{at^2 + 2bt + l^2}$ with $a \neq 0$ and $al^2 - b^2 < 0$. It is possible that there is a finite time t_* such that $L(t) > 0$ for $0 \leq t < t_*$ but $L(t) \to 0$ as $t \to t_*$. If so, then

$$t_* = -\frac{1}{a}\sqrt{b^2 - al^2} - \frac{b}{a}. \tag{3.71}$$

As $t \to t_*$, equations (3.43) and (3.44) show that the behaviour is governed by

$$\left(\frac{1}{at + b + \sqrt{b^2 - al^2}}\right)^{\frac{\sigma_n L_0^2}{2\sqrt{b^2 - al^2}} - \frac{\gamma_1^2}{8D(b^2 - al^2)^{3/2}} + \frac{1}{4}}. \tag{3.72}$$

By Proposition 3.4 and the bound (3.69) with $\rho = \sqrt{b^2 - al^2}$, we have

$$\frac{\sigma_n L_0^2}{2\sqrt{b^2 - al^2}} - \frac{\gamma_1^2}{8D(b^2 - al^2)^{3/2}} + \frac{1}{4} < 0, \tag{3.73}$$

and therefore $u \to 0$. $\square$

In Corollary 3.5, the convergence to zero in each case follows from an upper bound on the principal eigenvalue σ_1. This eigenvalue may also play a key role in determining the long-time behaviour on a domain of fixed length $L(t) \equiv l$, which is the case covered in Section 3.2.1. From equation (3.26), we see that if $\gamma_1 \neq 0$ then as $t \to \infty$, the dominant term is $\exp\left(-\frac{\gamma_1^2}{12Dl^6}t^3\right)$, and so $u \to 0$. However if $\gamma_1 = 0$ then the behaviour as $t \to \infty$ is governed by $\exp\left(\sigma_n t + f_0 t - \frac{c^2}{4D}t\right)$ where $\sigma_n = -\frac{Dn^2\pi^2}{L_0^2}$. There is exponential growth or decay depending on whether $f_0 - \frac{D\pi^2}{L_0^2} - \frac{c^2}{4D} > 0$ or < 0 respectively. If $\gamma_1 = 0$ and $f_0 - \frac{D\pi^2}{L_0^2} - \frac{c^2}{4D} = 0$, then there is a stationary solution proportional to the principal eigenmode.

In contrast, in the separable cases with $L(t) \to \infty$ as $t \to \infty$ (Section 3.2.2 with $\alpha > 0$, Section 3.2.3 with $\rho > 0$, some cases of Section 3.2.4, and Section 3.2.5), the long-time behaviour does not in general depend on σ_1. In these cases $s(t) = o(t)$ as $t \to \infty$ and the term $\exp(\sigma_n s(t))$ is not of leading order. In Corollary 3.6 we shall give a property about the exponential growth or decay of these solutions, which will involve the critical speed $c_* = 2\sqrt{Df_0}$. For this, we note that in each of the separable cases with $L(t) \to \infty$ as $t \to \infty$, there are constants $\hat{\alpha} \geq 0$ and $\hat{c}$ such that $\dot{L}(t) \to \hat{\alpha}$ and $\dot{A}(t) \to \hat{c}$ as $t \to \infty$.

Corollary 3.6. *Let $L(t)$, $A(t)$ be a separable case with $L(t) > 0$ for all $t \geq 0$ and $L(t) \to \infty$, $\dot{L}(t) \to \hat{\alpha}$, and $\dot{A}(t) \to \hat{c}$ as $t \to \infty$. Then $u(\xi, t)$ grows exponentially at any $\xi \in (0, L_0)$ such that $\left(\hat{c} + \hat{\alpha}\frac{\xi}{L_0}\right)^2 < c_*^2$ and $u(\xi, t)$ decays exponentially at any $\xi \in (0, L_0)$ such that $\left(\hat{c} + \hat{\alpha}\frac{\xi}{L_0}\right)^2 > c_*^2$.*

Proof. First consider the case of Section 3.2.2, where $L(t) = l + \alpha t$ with $\alpha > 0$. As $t \to \infty$ the dominant term in (3.29) is $\exp\left(f_0 t - \frac{c^2}{4D}t - \frac{\xi^2\alpha^2}{4DL_0^2}t - \frac{\xi c\alpha}{2DL_0}t\right)$, which can be written as $\exp\left(f_0 t - \frac{1}{4D}\left(c + \frac{\xi\alpha}{L_0}\right)^2 t\right)$. This proves the claim in this case, since $\hat{\alpha} = \alpha$ and $\hat{c} = c$.

Next consider the cases of Section 3.2.4 and Section 3.2.5 such that $L(t) = \sqrt{at^2 + 2bt + l^2}$ remains positive for all $t \geq 0$ and $L(t) \to \infty$. Noting that this implies $a > 0$, we see from (3.43) that the behaviour as $t \to \infty$ is governed by

$$\exp\left(f_0 t - \frac{1}{4D}\left(c - \frac{\gamma_1\sqrt{a}}{b^2 - al^2} + \frac{\xi\sqrt{a}}{L_0}\right)^2 t\right), \tag{3.74}$$

which proves the claim in this case since $\hat{\alpha} = \sqrt{a}$ and $\hat{c} = c - \frac{\gamma_1\sqrt{a}}{b^2 - al^2}$.

Finally, consider the case of Section 3.2.3, where $L(t) = \sqrt{l^2 + 2\rho t}$ with $\rho > 0$. If $f_0 - \frac{c^2}{4D} \neq 0$, then equation (3.36) shows that the behaviour is governed by $\exp\left(f_0 t - \frac{c^2}{4D}t\right)$ as $t \to \infty$. Either $u \to \infty$ for every $\xi \in (0, L_0)$, or $u \to 0$ for every $\xi \in (0, L_0)$, depending whether $f_0 - \frac{c^2}{4D} > 0$ or < 0 respectively. Since $\hat{\alpha} = 0$ and $\hat{c} = c$, this proves the claim. □

Remark 3.7. *In Chapter 7 we study the nonlinear problem, and for each case of $L(t)$ and $A(t)$ from Sections 3.2.1–3.2.5 we consider the long-time solution behaviour for the nonlinear equation. This is summarised in Example 7.7.*

Remark 3.8. *Given $\xi \in (0, L_0)$, write $x(\xi, t) = A(t) + \frac{\xi}{L_0}L(t)$ as the original space variable. Then Corollary 3.6 says that $u(\xi, t)$ grows exponentially at any $\xi \in (0, L_0)$ such that $-c_* < \lim_{t\to\infty}\frac{x(\xi,t)}{t} < c_*$, and $u(\xi, t)$ decays exponentially at any $\xi \in (0, L_0)$ such that $\left|\lim_{t\to\infty}\frac{x(\xi,t)}{t}\right| > c_*$. This is similar to the solution on the whole real line with compactly supported initial conditions, which also spreads at the speed c_*.*

If $\hat{\alpha} > 0$ then Corollary 3.6 implies that there is exponential growth on the subinterval $(\xi_*^-, \xi_*^+) \cap (0, L_0)$ where

$$\xi_*^{\pm} = \frac{L_0}{\hat{\alpha}}(\pm c_* - \hat{c}). \tag{3.75}$$

We note that if $\xi_*^{\pm} \in (0, L_0)$, then (3.29) and (3.43) show that $u(\xi_*^{\pm}, t) = O(t^{-1/2}) \to 0$ as $t \to \infty$.

If $\hat{\alpha} = 0$ (which is when $L(t) = \sqrt{l^2 + 2\rho t}$ with $\rho > 0$) and $\hat{c} = c \in (-c_*, c_*)$, then there is exponential growth for every $\xi \in (0, L_0)$. To understand what happens if $c = \pm c_*$, we consider equation (3.36) with $\rho > 0$ and $f_0 - \frac{c^2}{4D} = 0$. In this case, the behaviour at $\xi \in (0, L_0)$ is determined by the sign of $\frac{c}{2D}\left(\frac{\gamma_1}{\rho^2} - \frac{\xi}{L_0}\right)$:

$$u_n(\xi, t) = \underline{\overline{O}}\left((l^2 + 2\rho t)^{\frac{\sigma_n L_0^2}{2\rho} - \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D}} \exp\left(\frac{c}{2D}\left(\frac{\gamma_1}{\rho^2} - \frac{\xi}{L_0}\right)\sqrt{l^2 + 2\rho t}\right)\right). \tag{3.76}$$

Moreover, if there is some $\xi_* \in (0, L_0)$ such that $\frac{\gamma_1}{\rho^2} = \frac{\xi_*}{L_0}$ then we note from Proposition 3.4 and the bound (3.69) that

$$u_n(\xi_*, t) = \underline{\overline{O}}\left((l^2 + 2\rho t)^{\frac{\sigma_n L_0^2}{2\rho} - \frac{1}{4} - \frac{\gamma_1^2}{8\rho^3 D}}\right) = o(t^{-1/2}) \to 0 \qquad \text{as } t \to \infty. \tag{3.77}$$

To conclude this section, we note that it is possible to have contrasting short-time and long-time behaviour. Even a solution proportional to the principal eigenmode does not necessarily evolve as a monotonic function of time. In the next example, the solution grows exponentially as $t \to \infty$, but is decreasing with time over an initial time period $0 \leq t < \hat{t}$.

Example 3.9. *Let $L(t) = \sqrt{1 + 6t}$, $A(t) \equiv 0$, $f_0 = 10$ and $D = L_0 = 1$. In order to apply Theorem 3.2, the relevant Sturm-Liouville equation is*

$$g''(\xi) - \frac{9\xi^2}{4} g(\xi) = \sigma g(\xi) \qquad \text{for } 0 < \xi < 1, \qquad g(0) = g(1) = 0. \tag{3.78}$$

The principal eigenfunction and eigenvalue are given by

$$g_1(\xi) = e^{-\frac{3}{4}\xi^2}(\xi - \xi^3), \qquad \sigma_1 = -\frac{21}{2}, \tag{3.79}$$

and by Theorem 3.2 and equation (3.36), *we have the exact solution*

$$u(\xi, t) = g_1(\xi) e^{10t} (1 + 6t)^{-\frac{21}{12} - \frac{1}{4}}. \tag{3.80}$$

For each $\xi \in (0,1)$*,* $u(\xi,t) \to \infty$ *as* $t \to \infty$ *because of the* e^{10t} *term. However,*

$$\frac{\partial u}{\partial t}(\xi,t) = \left(10 + \frac{\left(-\frac{21}{12} - \frac{1}{4}\right)}{\left(t + \frac{1}{6}\right)}\right) u(\xi,t) \tag{3.81}$$

and, for each $0 < \xi < 1$*, this is strictly negative for* $0 \le t < \hat{t} := \frac{1}{30}$*.*

3.5 Non-separable cases

3.5.1 Comparison theorems

In Sections 3.1 and 3.2 we gave exact solutions for $u(\xi,t)$ in the special cases where $\ddot{L}L^3 \equiv \gamma_0$ and $\ddot{A}L^3 \equiv \gamma_1$. We would also like to understand how the solution behaves for more general forms of $L(t)$ and $A(t)$. Here we shall give two main comparison results which can be applied in such cases. They are each based on somehow bounding the coefficient of w in equation (3.11), and applying the parabolic comparison principle. Proposition 3.10 gives (upper and/or lower) bounds on the solution $w(\xi,t)$ in cases such that $\ddot{L}L^3$ and $\ddot{A}L^3$ are bounded (above and/or below). Theorem 3.11 gives both upper and lower bounds on $w(\xi,t)$ under the more general condition that $A(t)$ and $L(t)$ are twice differentiable.

We shall apply Theorem 3.11 in Section 3.5.2, where it allows us to study the possible long-time behaviour of $\psi(x,t)$ if $L(t) \to L_{crit} = \pi\sqrt{\frac{D}{f_0}}$ as $t \to \infty$. In Chapter 5, we shall also apply both Proposition 3.10 and Theorem 3.11 to a periodically varying interval $(A(t), A(t)+L(t))$, to prove Proposition 5.5 and many subsequent results in Chapter 5.

Proposition 3.10. *Assume that* $\ddot{L}(t)L(t)^3 \le \gamma_0$*,* $\ddot{A}(t)L(t)^3 \le \gamma_1$*, and* $L(t) > 0$ *for* $0 \le t \le T$*. Let* g_1 *and* σ_1 *be the principal eigenfunction and eigenvalue of* (3.19), (3.20)*, let* $w(\xi,t)$ *satisfy* (3.11), (3.12) *and define* $s(t) = \int_0^t \frac{L_0^2}{L(\zeta)^2} d\zeta$*. If* $0 \le w(\xi,0) \le a g_1(\xi)$ *for some* $a > 0$*, then* $w(\xi,t) \le a g_1(\xi) e^{\sigma_1 s(t)}$ *for all* $0 \le t \le T$*.*

If instead $\gamma_0 \leq \ddot{L}(t)L(t)^3$, $\gamma_1 \leq \ddot{A}(t)L(t)^3$, *and* $L(t) > 0$ *for* $0 \leq t \leq T$, *and if* $bg_1(\xi) \leq w(\xi, 0)$ *for some* $b > 0$, *then* $bg_1(\xi)e^{\sigma_1 s(t)} \leq w(\xi, t)$ *for all* $0 \leq t \leq T$.

Proof. As in Section 3.1, let $v(\xi, s) = w(\xi, t)$, so v satisfies (3.14), (3.15). For the first part, $ag_1(\xi)e^{\sigma_1 s}$ is a supersolution for $v(\xi, s)$, and for the second part $bg_1(\xi)e^{\sigma_1 s}$ is a subsolution. So the required estimates follow by applying the comparison principle and using $w(\xi, t) = v(\xi, s(t))$. □

Next we shall derive another comparison result by treating the coefficient of w in equation (3.11) in a different way. It can be applied to any $A(t)$ and $L(t)$ that are twice differentiable. It is, however, especially useful in cases where the condition (3.85) holds (i.e. $\overline{Q}(t)$ and $\underline{Q}(t)$ are integrable). Then, the upper and lower bounds are of the same order as each other and we deduce the exact order of the solution as $t \to \infty$ (see equation (3.86)).

Theorem 3.11. *Given constants* γ_0 *and* γ_1, *let* g_1 *and* σ_1 *be the principal eigenfunction and eigenvalue of* (3.19), (3.20). *Let* w *satisfy* (3.11), (3.12) *and assume that* $C_1 g_1(\xi) \leq w(\xi, 0) \leq C_2 g_1(\xi)$ *for some constants* $0 < C_1 \leq C_2$. *Define*

$$\overline{Q}(t) = \max_{0 \leq \eta \leq 1} \left(\frac{\eta^2}{2} \left(\ddot{L}(t)L(t) - \frac{\gamma_0}{L(t)^2} \right) + \eta \left(\ddot{A}(t)L(t) - \frac{\gamma_1}{L(t)^2} \right) \right), \tag{3.82}$$

$$\underline{Q}(t) = -\min_{0 \leq \eta \leq 1} \left(\frac{\eta^2}{2} \left(\ddot{L}(t)L(t) - \frac{\gamma_0}{L(t)^2} \right) + \eta \left(\ddot{A}(t)L(t) - \frac{\gamma_1}{L(t)^2} \right) \right). \tag{3.83}$$

If $t \geq 0$ *is such that* $L(\tau) > 0$ *on* $0 \leq \tau \leq t$, *then*

$$C_1 g_1(\xi) e^{\int_0^t \left(\frac{\sigma_1 L_0^2}{L(\zeta)^2} - \frac{\underline{Q}(\zeta)}{2D} \right) d\zeta} \leq w(\xi, t) \leq C_2 g_1(\xi) e^{\int_0^t \left(\frac{\sigma_1 L_0^2}{L(\zeta)^2} + \frac{\overline{Q}(\zeta)}{2D} \right) d\zeta}. \tag{3.84}$$

In particular, if $L(t) > 0$ *for all* $0 \leq t < \infty$, *and if*

$$\int_0^\infty \frac{\overline{Q}(t)}{2D} dt < \infty \qquad \text{and} \qquad \int_0^\infty \frac{\underline{Q}(t)}{2D} dt < \infty \tag{3.85}$$

then

$$w(\xi, t) = \underline{\overline{O}} \left(g_1(\xi) e^{\int_0^t \frac{\sigma_1 L_0^2}{L(\zeta)^2} d\zeta} \right). \tag{3.86}$$

Proof. Consider the coefficient of w in equation (3.11). We can write

$$\frac{\xi^2 \ddot{L}(t)L(t)}{4DL_0^2} + \frac{\xi \ddot{A}(t)L(t)}{2DL_0} = \frac{L_0^2}{L(t)^2}\left(\frac{\xi^2\gamma_0}{4DL_0^4} + \frac{\xi\gamma_1}{2DL_0^3}\right) + r(\xi, t) \tag{3.87}$$

where

$$r(\xi, t) = \frac{\xi^2\left(\ddot{L}(t)L(t) - \frac{\gamma_0}{L(t)^2}\right)}{4DL_0^2} + \frac{\xi\left(\ddot{A}(t)L(t) - \frac{\gamma_0}{L(t)^2}\right)}{2DL_0}. \tag{3.88}$$

For $0 \le \xi \le L_0$, this satisfies

$$-\frac{\underline{Q}(t)}{2D} \le r(\xi, t) \le \frac{\overline{Q}(t)}{2D} \tag{3.89}$$

where $\overline{Q}(t)$ and $\underline{Q}(t)$ are as in equations (3.82), (3.83). Therefore, equation (3.11) implies that

$$-\frac{\underline{Q}(t)}{2D}w \le \frac{\partial w}{\partial t} - \frac{L_0^2}{L(t)^2}\left(D\frac{\partial^2 w}{\partial \xi^2} + \left(\frac{\xi^2\gamma_0}{4DL_0^4} + \frac{\xi\gamma_1}{2DL_0^3}\right)w\right) \le \frac{\overline{Q}(t)}{2D}w. \tag{3.90}$$

If $W(\xi, t)$ is the solution to

$$\frac{\partial W}{\partial t} = \frac{L_0^2}{L(t)^2}\left(D\frac{\partial^2 W}{\partial \xi^2} + \left(\frac{\xi^2\gamma_0}{4DL_0^4} + \frac{\xi\gamma_1}{2DL_0^3}\right)W\right) \qquad \text{for } 0 < \xi < L_0 \tag{3.91}$$

$$W(\xi, t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0, \tag{3.92}$$

$$W(\xi, 0) = g_1(\xi), \tag{3.93}$$

then $C_1 W(\xi, t)e^{-\int_0^t \frac{\underline{Q}(\zeta)}{2D}d\zeta}$ is a subsolution for $w(\xi, t)$ and $C_2 W(\xi, t)e^{\int_0^t \frac{\overline{Q}(\zeta)}{2D}d\zeta}$ is a supersolution. But since $W(\xi, t) = g_1(\xi)e^{\sigma_1 s(t)}$ with $s(t)$ as in (3.13), this gives the estimates in equation (3.84).

For the final part, note that the lower and upper bounds in (3.84) differ by a factor of order $\exp\left(\int_0^t \frac{\overline{Q}(\zeta) - \underline{Q}(\zeta)}{2D}d\zeta\right)$. If (3.85) holds then this converges to the finite positive value $\exp\left(\int_0^\infty \frac{\overline{Q}(\zeta) - \underline{Q}(\zeta)}{2D}d\zeta\right)$ as $t \to \infty$. □

Even when we do not have an exact solution, there are many cases where Theorem 3.11 allows us to determine the long-time behaviour of the solution w (and hence u) up to multiplication by an order one factor. We can know, for instance, not only that $u \to 0$ or $u \to \infty$ as $t \to \infty$, but the precise t dependence of the decay or growth, as in the following example.

Example 3.12. *Suppose that* $L(t) = L_0 e^{-\alpha t}$ *with* $\alpha > 0$*, and* $A(t) \equiv 0$*. Take* $\gamma_0 = \gamma_1 = 0$ *in Theorem 3.11, so that* $\sigma_1 = -\frac{D\pi^2}{L_0^2}$ *and* $g_1(\xi) = \sin\left(\frac{\pi\xi}{L_0}\right)$*. Then we calculate that* $\overline{Q}(t) = \frac{L_0^2\alpha^2}{2}e^{-2\alpha t}$ *and* $\underline{Q}(t) \equiv 0$*. So,*

$$\int_0^t \frac{\overline{Q}(\zeta)}{2D} d\zeta = \frac{L_0^2\alpha}{8D}(1 - e^{-2\alpha t}) = O(1), \qquad \int_0^t \frac{\underline{Q}(\zeta)}{2D} d\zeta = 0. \tag{3.94}$$

Also, $\dot{L}(t)L(t) = -\alpha L_0^2 e^{-2\alpha t}$ *and*

$$\int_0^t \frac{1}{L(\zeta)^2} d\zeta = \int_0^t \frac{1}{L_0^2 e^{-2\alpha\zeta}} d\zeta = \frac{e^{2\alpha t} - 1}{2\alpha L_0^2}. \tag{3.95}$$

Therefore when we apply Theorem 3.11, and then change variables back to $u(\xi, t)$ *using the definition of* $w(\xi, t)$ *from Section 3.1, we deduce that for* $0 \leq \xi \leq L_0$

$$u(\xi, t) = \underline{\overline{O}}\left(\sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(-\frac{D\pi^2}{2\alpha L_0^2}e^{2\alpha t} + \left(f_0 + \frac{\alpha}{2}\right)t\right)\right) \to 0 \qquad \text{as } t \to \infty. \tag{3.96}$$

Theorem 3.11 is valid with any choice of γ_0, γ_1, but in certain cases a particular choice of γ_0, γ_1 will lead to the best bounds on the solution. For example, if the separability condition (3.16) does hold, and we apply Theorem 3.11 with $\gamma_0 \equiv \ddot{L}(t)L(t)^3$ and $\gamma_1 \equiv \ddot{A}(t)L(t)^3$, then $\underline{Q}(t) \equiv \overline{Q}(t) \equiv 0$ and the theorem bounds $w(\xi, t)$ between constant multiples of $g_1(\xi)e^{\int_0^t \frac{\sigma_1 L_0^2}{L(\zeta)^2}d\zeta}$.

Consider next a case such that the separability condition (3.16) does not hold, but that for some $a > 0$

$$L(t) \sim at^{\frac{1}{2}}, \qquad \ddot{L}(t) \sim -\frac{1}{4}at^{-\frac{3}{2}} \qquad \text{as } t \to \infty. \tag{3.97}$$

In such cases,

$$\ddot{L}(t)L(t) \sim -\frac{a^2}{4t} \sim -\frac{a^4}{4L(t)^2} \qquad \text{as } t \to \infty. \tag{3.98}$$

Therefore if we take $\gamma_0 = -\frac{a^4}{4}$ then $\ddot{L}(t)L(t) = \frac{\gamma_0}{L(t)^2} + o\left(\frac{1}{t}\right)$ as $t \to \infty$, and so

$$\ddot{L}(t)L(t) - \frac{\gamma_0}{L(t)^2} = o\left(\ddot{L}(t)L(t)\right) \qquad \text{as } t \to \infty. \tag{3.99}$$

If, also, $A(t)$ satisfies $\ddot{A}(t) \sim bt^{-\frac{3}{2}}$ as $t \to \infty$, then the choice $\gamma_1 = ba^3$ leads to

$$\ddot{A}(t)L(t) - \frac{\gamma_1}{L(t)^2} = o\left(\ddot{A}(t)L(t)\right) \qquad \text{as } t \to \infty. \tag{3.100}$$

These observations mean that we can obtain better bounds on the solution w (and so u) by taking into account these particular values γ_0, γ_1, and the separable solutions associated to them, than we could by simply using $\gamma_0 = \gamma_1 = 0$. In particular, if $\ddot{L}(t)L(t) - \frac{\gamma_0}{L(t)^2}$ and $\ddot{A}(t)L(t) - \frac{\gamma_1}{L(t)^2}$ are integrable, then Theorem 3.11 will give the exact order of $w(\xi, t)$ as $t \to \infty$.

If there are no special choices of γ_0 or γ_1 such that (3.99) or (3.100) holds, then (as in Example 3.12) we take $\gamma_0 = \gamma_1 = 0$. This often provides useful bounds on the solution, as we shall see in Example 3.15 and Section 3.5.2. Let us therefore state Theorem 3.11 in the case $\gamma_0 = \gamma_1 = 0$ as a separate theorem.

Theorem 3.13. *Let w satisfy* (3.11), (3.12), *with $0 < C_1 \leq C_2$ such that $C_1 \sin\left(\frac{\pi\xi}{L_0}\right) \leq w(\xi, 0) \leq C_2 \sin\left(\frac{\pi\xi}{L_0}\right)$. Define*

$$\overline{Q}(t) = \max_{0\leq\eta\leq 1}\left(\frac{\eta^2 \ddot{L}(t)L(t)}{2} + \eta\ddot{A}(t)L(t)\right),$$

$$\underline{Q}(t) = -\min_{0\leq\eta\leq 1}\left(\frac{\eta^2 \ddot{L}(t)L(t)}{2} + \eta\ddot{A}(t)L(t)\right). \tag{3.101}$$

If $t \geq 0$ is such that $L(\tau) > 0$ on $0 \leq \tau \leq t$, then

$$C_1 \sin\left(\frac{\pi\xi}{L_0}\right) e^{\int_0^t \left(-\frac{D\pi^2}{L(\zeta)^2} - \frac{\underline{Q}(\zeta)}{2D}\right)d\zeta} \leq w(\xi, t) \leq C_2 \sin\left(\frac{\pi\xi}{L_0}\right) e^{\int_0^t \left(-\frac{D\pi^2}{L(\zeta)^2} + \frac{\overline{Q}(\zeta)}{2D}\right)d\zeta}. \tag{3.102}$$

In particular, if $L(t) > 0$ for all $0 \leq t < \infty$, and if

$$\int_0^\infty \frac{\overline{Q}(t)}{2D}dt < \infty \qquad \text{and} \qquad \int_0^\infty \frac{\underline{Q}(t)}{2D}dt < \infty \tag{3.103}$$

then

$$w(\xi, t) = \underline{\overline{O}}\left(\sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(\int_0^t \frac{-D\pi^2}{L(\zeta)^2}d\zeta\right)\right). \tag{3.104}$$

Remark 3.14. *Given a function $F(t)$, denote its positive and negative parts by $[F(t)]^+ \geq 0$ and $[F(t)]^- \geq 0$, so that $F(t) \equiv [F(t)]^+ - [F(t)]^-$. The functions $\overline{Q}(t)$ and $\underline{Q}(t)$ defined by* (3.101) *always satisfy*

$$0 \leq \overline{Q}(t) \leq \frac{L(t)[\ddot{L}(t)]^+}{2} + L(t)[\ddot{A}(t)]^+, \qquad 0 \leq \underline{Q}(t) \leq \frac{L(t)[\ddot{L}(t)]^-}{2} + L(t)[\ddot{A}(t)]^-. \tag{3.105}$$

Example 3.15. *Let* $\overline{Q}(t)$ *and* $\underline{Q}(t)$ *be given by* (3.101)*. Examples of* $L(t)$*,* $A(t)$ *for which the condition* (3.103) *is satisfied include:*

1. $L(t) = a(t + t_0)^k$ *for any* $k < \frac{1}{2}$*, and* $\ddot{A}(t) = O(t^{m-2})$ *with* $m + k < 1$*.*

2. $L(t) \to L_\infty > 0$ *and* $\ddot{L}(t) = O(t^{-1-\varepsilon})$*,* $\ddot{A}(t) = O(t^{-1-\delta})$ *for any* $\varepsilon > 0$*,* $\delta > 0$*, as* $t \to \infty$*.*

3. $L(t) \sim \alpha t$*,* $\alpha > 0$*, and* $\ddot{L}(t) = O(t^{-2-\varepsilon})$*,* $\ddot{A}(t) = O(t^{-2-\delta})$ *for any* $\varepsilon > 0$*,* $\delta > 0$*, as* $t \to \infty$*.*

3.5.2 $L(t)$ and L_{crit}

Here we consider the problem (3.1), (3.2) on the interval $0 < x < L(t)$. If L is a constant, then we know that the 'critical length'

$$L_{crit} = \pi\sqrt{\frac{D}{f_0}} \tag{3.106}$$

is the length for which the principal eigenmode $\sin\left(\frac{\pi x}{L}\right)$ is a stationary solution to the linear parabolic problem. If $L < L_{crit}$ then the solution tends to zero as $t \to \infty$, and if $L > L_{crit}$ then the solution tends to infinity. When $L(t)$ is not constant but varies with t, the role of L_{crit} is not so straightforward. Considering $L(t)$ and its relation to L_{crit}, we shall derive sufficient conditions such that the solution does not, or does, tend to zero as $t \to \infty$. These will be proved as two corollaries of Theorem 3.13. Then, in Examples 3.19 and 3.20, we shall demonstrate cases for which $L(t)$ is strictly less than L_{crit} for all $t \geq 0$, and yet the solution does not tend to zero as $t \to \infty$. We shall see that if $L(t)$ tends to L_{crit} from below as an inverse power of t, then the outcome depends on this power.

First, in Corollary 3.16, we give conditions under which the solution does not converge to zero but has a non-trivial lower bound.

Corollary 3.16. *Let $\psi(x,t)$ satisfy* (3.1), (3.2) *on* $0 < x < L(t)$*. Assume that* $L(t) > 0$ *for all* $0 \le t < \infty$*, and that the following conditions hold:*

$$L(t) \text{ and } \dot{L}(t)L(t) \text{ are bounded above,} \tag{3.107}$$

$$\int_0^t \left(\frac{1}{L(\zeta)^2} - \frac{1}{L_{crit}^2} \right) d\zeta \text{ is bounded above,} \tag{3.108}$$

$$\int_0^\infty L(t)[\ddot{L}(t)]^- dt < \infty. \tag{3.109}$$

Then for $\psi(x,0) \ge 0$*, and not identically zero,* $\psi(x,t)$ *does not converge to zero as* $t \to \infty$*. In particular,* $\liminf_{t\to\infty} \left(\frac{2}{L(t)} \int_0^{L(t)} \psi(x,t) \sin\left(\frac{\pi x}{L(t)}\right) dx \right) > 0$.

Proof. Let $u(\xi,t)$ and $w(\xi,t)$ be as in Section 3.1. We may assume that

$$u(\xi,0) \ge b \sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(\frac{-\xi^2 \dot{L}(0)L(0)}{4DL_0^2}\right) \tag{3.110}$$

for some $b > 0$. It follows from Theorem 3.13, and the definition $w(\xi,t) = u(\xi,t)H(\xi,t)e^{-f_0 t}$ with $H(\xi,t)$ as in (3.6), that

$$u(\xi,t) \ge b \sin\left(\frac{\pi\xi}{L_0}\right) \left(\frac{L(0)}{L(t)}\right)^{1/2} e^{f_0 t - \int_0^t \left(\frac{D\pi^2}{L(\zeta)^2} + \frac{Q(\zeta)}{2D}\right) d\zeta - \frac{\xi^2 \dot{L}(t)L(t)}{4DL_0^2}} \tag{3.111}$$

where $\underline{Q}(t) = \frac{L(t)[\ddot{L}(t)]^-}{2}$. Substituting $f_0 = \frac{D\pi^2}{L_{crit}^2}$ and using the assumptions in equation (3.107) gives that for some $b' > 0$

$$u(\xi,t) \ge b' \sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(\int_0^t \left(\frac{D\pi^2}{L_{crit}^2} - \frac{D\pi^2}{L(\zeta)^2} - \frac{L(\zeta)[\ddot{L}(\zeta)]^-}{4D}\right) d\zeta\right). \tag{3.112}$$

Now, assumptions (3.108) and (3.109) imply that there exists $B > 0$ such that $u(\xi,t) \ge B \sin\left(\frac{\pi\xi}{L_0}\right)$ for all $t \ge 0$. So $u(\xi,t)$ does not converge to zero, and

$$\frac{2}{L(t)} \int_0^{L(t)} \psi(x,t) \sin\left(\frac{\pi x}{L(t)}\right) dx = \frac{2}{L_0} \int_0^{L_0} u(\xi,t) \sin\left(\frac{\pi\xi}{L_0}\right) d\xi \ge B. \tag{3.113}$$

□

Remark 3.17. *Let $\psi(x,t)$ satisfy* (3.1), (3.2) *on* $A(t) < x < A(t) + L(t)$ *where* $L(t)$ *satisfies the conditions* (3.107), (3.108), (3.109) *and where* $A(t) = \beta(t+t_1)^m$

for some $0 < m < \frac{1}{2}$. *In this case also,* ψ *does not converge to zero but has a non-trivial lower bound. As in Corollary 3.16, this can be proved by an application of Theorem 3.13 and equation* (3.6)*, and by also using that* $\dot{A}(t)L(t)$ *is bounded above and that* $\dot{A}(t)^2$ *and* $L(t)[\ddot{A}(t)]^-$ *are integrable.*

Next we give conditions under which the solution does converge to zero.

Corollary 3.18. *Let* $\psi(x,t)$ *satisfy* (3.1), (3.2) *on* $0 < x < L(t)$. *Assume that* $L(t) > 0$ *for all* $0 \le t < \infty$, *and that the following conditions hold:*

$$\dot{L}(t)L(t) \text{ is bounded below,} \tag{3.114}$$

$$\int_0^T \left(\frac{D\pi^2}{L(t)^2} - \frac{D\pi^2}{L_{crit}^2} - \frac{L(t)[\ddot{L}(t)]^+}{4D} + \frac{\dot{L}(t)}{2L(t)} \right) dt \to \infty \qquad \text{as } T \to \infty. \tag{3.115}$$

Then $\psi(x,t)$ *converges uniformly to zero as* $t \to \infty$.

Proof. Let $u(\xi,t)$ and $w(\xi,t)$ be as in Section 3.1. We may assume that

$$u(\xi,0) \le a \sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(\frac{-\xi^2 \dot{L}(0)L(0)}{4DL_0^2}\right) \tag{3.116}$$

for some $a > 0$. It follows from Theorem 3.13, and the definition $w(\xi,t) = u(\xi,t)H(\xi,t)e^{-f_0 t}$ with $H(\xi,t)$ as in (3.6), that

$$u(\xi,t) \le a \sin\left(\frac{\pi\xi}{L_0}\right) \left(\frac{L(0)}{L(t)}\right)^{1/2} e^{f_0 t + \int_0^t \left(-\frac{D\pi^2}{L(\zeta)^2} + \frac{\overline{Q}(\zeta)}{2D}\right) d\zeta - \frac{\xi^2 \dot{L}(t)L(t)}{4DL_0^2}} \tag{3.117}$$

where $\overline{Q}(t) = \frac{L(t)[\ddot{L}(t)]^+}{2}$. Substituting $f_0 = \frac{D\pi^2}{L_{crit}^2}$ and using the assumption in equation (3.114) gives that for some $a' > 0$

$$u(\xi,t) \le a' \sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(\int_0^t \left(\frac{D\pi^2}{L_{crit}^2} - \frac{D\pi^2}{L(\zeta)^2} + \frac{L(\zeta)[\ddot{L}(\zeta)]^+}{4D} - \frac{\dot{L}(\zeta)}{2L(\zeta)} \right) d\zeta \right). \tag{3.118}$$

So, under the assumption in equation (3.115), there is uniform convergence to zero. □

Example 3.19. *Consider an interval* $(0, L(t))$ *where* $L(t) < L_{crit}$ *tends exponentially towards* L_{crit}:

$$L(t) = L_{crit}(1 - \varepsilon e^{-\alpha t}) \tag{3.119}$$

where $0 < \varepsilon < 1$ *and* $\alpha > 0$. *The conditions of Corollary 3.16 are satisfied and so we deduce that* $\psi(x,t) \geq B\sin\left(\frac{\pi x}{L(t)}\right)$ *for some* $B > 0$.

Example 3.20. *Consider the interval* $(0, L(t))$ *where* $L(t) < L_{crit}$ *is given by*

$$L(t) = L_{crit}(1 - \varepsilon(t + t_0)^{-k}) \tag{3.120}$$

where $0 < \varepsilon < 1$ *and* $k > 0$. *If* $k > 1$, *then the conditions of Corollary 3.16 are satisfied and so* $\psi(x,t) \geq B\sin\left(\frac{\pi x}{L(t)}\right)$ *for some* $B > 0$. *On the other hand, if* $0 < k \leq 1$ *then the conditions of Corollary 3.18 are satisfied and so* $\psi(x,t) \to 0$ *uniformly in* x *as* $t \to \infty$. *This example gives an indication of how fast or slowly* $L(t)$ *may be expected to converge to* L_{crit}, *to give each of the two outcomes.*

Analogous results to Corollaries 3.16 and 3.18 also hold in the case of an interval $(A_0 + ct, A_0 + ct + L(t))$ where $c \in (-c_*, c_*)$. In this case we define

$$L_{crit}(c) = \pi\sqrt{\frac{D}{f_0 - \frac{c^2}{4D}}} \tag{3.121}$$

and, exactly as above but with this new definition of L_{crit}, we obtain the following corollaries of Theorem 3.13.

Corollary 3.21. *Let* $\psi(x,t)$ *satisfy* (3.1), (3.2) *on* $(A_0 + ct, A_0 + ct + L(t))$. *Assume that* $L(t) > 0$ *for all* $0 \leq t < \infty$, *and that the following conditions hold:*

$$L(t),\ \dot{L}(t)L(t) \text{ and } cL(t) \text{ are bounded above,} \tag{3.122}$$

$$\int_0^t \left(\frac{1}{L(\zeta)^2} - \frac{1}{L_{crit}(c)^2}\right) d\zeta \text{ is bounded above,} \tag{3.123}$$

$$\int_0^\infty L(t)[\ddot{L}(t)]^- dt < \infty. \tag{3.124}$$

Then for $\psi(x,0) \geq 0$, *and not identically zero,* $\psi(x,t)$ *does not converge to zero as* $t \to \infty$.

Corollary 3.22. *Let* $\psi(x,t)$ *satisfy* (3.1), (3.2) *on* $(A_0 + ct, A_0 + ct + L(t))$. *Assume that* $L(t) > 0$ *for all* $0 \leq t < \infty$, *and that the following conditions hold:*

$$\dot{L}(t)L(t) \text{ and } cL(t) \text{ are bounded below,} \tag{3.125}$$

$$\int_0^T \left(\frac{D\pi^2}{L(t)^2} - \frac{D\pi^2}{L_{crit}(c)^2} - \frac{L(t)[\ddot{L}(t)]^+}{4D} + \frac{\dot{L}(t)}{2L(t)} \right) dt \to \infty \qquad \textit{as } T \to \infty. \tag{3.126}$$

Then $\psi(x,t)$ converges uniformly to zero as $t \to \infty$.

In Chapter 7 we shall consider the equation with a nonlinear reaction term f. Section 7.5.1 concerns the roles of L_{crit} and $L_{crit}(c)$ in the nonlinear case, and we show that if $L(t) \to L_{crit}$ as $t \to \infty$ then the long-time behaviour depends on whether or not f is linear on a neighbourhood of 0. If it is, then a result similar to Corollary 3.16 holds.

3.6 Linear problem on a time-dependent box

Here, we show that the analysis of the linear equation on the interval extends in a straightforward way to the box defined in Example 2.4 and equation (2.14); that is:

$$\Omega(t) = \left\{ x \in \mathbb{R}^N : A_j(t) < x_j < A_j(t) + L_j(t) : 1 \le j \le N \right\}$$

for some $A_j(t) \in \mathbb{R}$, and $L_j(t) > 0$, each twice differentiable. The same methods which were used to prove Theorem 3.2 and Theorem 3.11 on the interval also lead to the corresponding results — Theorem 3.23 and Theorem 3.24 — on the box. We shall use these results again in Chapter 4.

With $\Omega(t)$ given by equation (2.14), consider the problem

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f_0\psi \qquad \text{in } \Omega(t) \subset \mathbb{R}^N \tag{3.127}$$

$$\psi(x,t) = 0 \qquad \text{on } \partial\Omega(t). \tag{3.128}$$

As in Section 3.1, change variables from x_j to ξ_j and $\psi(x,t) = u(\xi,t)$ where $\xi_j = \left(\frac{x_j - A_j(t)}{L_j(t)}\right) L_0$ lies in a fixed interval. Then problem (3.127), (3.128) becomes

$$\frac{\partial u}{\partial t} = D\sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2}\frac{\partial^2 u}{\partial \xi_j^2} + \sum_{j=1}^{N} \left(\frac{\dot{A}_j(t)L_0 + \xi_j \dot{L}_j(t)}{L_j(t)} \right) \frac{\partial u}{\partial \xi_j} + f_0 u \qquad \text{for } 0 < \xi_j < L_0 \tag{3.129}$$

$$u(\xi,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{3.130}$$

Again proceeding as in Section 3.1, let $w(\xi,t) = e^{-f_0 t}u(\xi,t)\prod_{j=1}^{N} H_j(\xi_j,t)$ where

$$H_j(\xi_j,t) = \left(\frac{L_j(t)}{L_j(0)}\right)^{1/2} \exp\left(\int_0^t \frac{\dot{A}_j(\zeta)^2}{4D}d\zeta + \frac{\xi_j^2 \dot{L}_j(t)L_j(t)}{4DL_0^2} + \frac{\xi_j \dot{A}_j(t)L_j(t)}{2DL_0}\right). \tag{3.131}$$

Then for $0 < \xi_j < L_0$ we have

$$\frac{\partial w}{\partial t} = D\sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2}\frac{\partial^2 w}{\partial \xi_j^2} + \sum_{j=1}^{N}\left(\frac{\xi_j^2 \ddot{L}_j(t)L_j(t)}{4DL_0^2} + \frac{\xi_j \ddot{A}_j(t)L_j(t)}{2DL_0}\right)w \tag{3.132}$$

$$w(\xi,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0, \tag{3.133}$$

and we arrive at the following extension of Theorem 3.2.

Theorem 3.23. *Suppose that*

$$\ddot{L}_j(t)L_j(t)^3 \equiv \gamma_0^{(j)} \qquad \textit{and} \qquad \ddot{A}_j(t)L_j(t)^3 \equiv \gamma_1^{(j)} \qquad \textit{for all } 1 \le j \le N. \tag{3.134}$$

Then there exist separable solutions of (3.129), (3.130) *of the form*

$$u(\xi,t) = \exp\left(f_0 t + \int_0^t \sum_{j=1}^{N} \frac{\sigma_{j,n_j} L_0^2}{L_j(\zeta)^2}d\zeta\right)\prod_{j=1}^{N}\left(\frac{g_{j,n_j}(\xi_j)}{H_j(\xi_j,t)}\right), \tag{3.135}$$

where $H_j(\xi_j,t)$ is given by equation (3.131) *and where $g_{j,n}$ and $\sigma_{j,n}$ are the eigenfunctions and eigenvalues of the Sturm-Liouville problem* (3.19), (3.20) *with $\gamma_0 = \gamma_0^{(j)}$ and $\gamma_1 = \gamma_1^{(j)}$.*

In general, the separability condition may hold for some, all, or none of the dimensions $1 \le j \le N$. We can also extend Theorem 3.11 to the box, as follows.

Theorem 3.24. *Given constants $\gamma_0^{(j)}$ and $\gamma_1^{(j)}$ for $1 \le j \le N$, let $g_{j,1}$ and $\sigma_{j,1}$ be the principal eigenfunction and eigenvalue of* (3.19), (3.20) *with $\gamma_0 = \gamma_0^{(j)}$ and $\gamma_1 = \gamma_1^{(j)}$. Let w satisfy* (3.132), (3.133) *and assume that*

$$C_1 \prod_{j=1}^{N} g_{j,1}(\xi_j) \le w(\xi,0) \le C_2 \prod_{j=1}^{N} g_{j,1}(\xi_j), \tag{3.136}$$

for some constants $0 < C_1 \leq C_2$. *Define*

$$\overline{Q}_j(t) = \max_{0\leq\eta\leq 1}\left(\frac{\eta^2}{2}\left(\ddot{L}_j(t)L_j(t) - \frac{\gamma_0^{(j)}}{L_j(t)^2}\right) + \eta\left(\ddot{A}_j(t)L_j(t) - \frac{\gamma_1^{(j)}}{L_j(t)^2}\right)\right), \tag{3.137}$$

$$\underline{Q}_j(t) = -\min_{0\leq\eta\leq 1}\left(\frac{\eta^2}{2}\left(\ddot{L}_j(t)L_j(t) - \frac{\gamma_0^{(j)}}{L_j(t)^2}\right) + \eta\left(\ddot{A}_j(t)L_j(t) - \frac{\gamma_1^{(j)}}{L_j(t)^2}\right)\right). \tag{3.138}$$

If $t \geq 0$ *is such that* $L_j(\tau) > 0$ *on* $0 \leq \tau \leq t$ *for all* $1 \leq j \leq N$, *then*

$$C_1 \exp\left(\int_0^t \sum_{j=1}^N \left(\frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} - \frac{\underline{Q}_j(\zeta)}{2D}\right) d\zeta\right) \prod_{j=1}^N g_{j,1}(\xi_j) \leq w(\xi,t) \leq C_2 \exp\left(\int_0^t \sum_{j=1}^N \left(\frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} + \frac{\overline{Q}_j(\zeta)}{2D}\right) d\zeta\right) \prod_{j=1}^N g_{j,1}(\xi_j). \tag{3.139}$$

In particular, if $L_j(t) > 0$ *for all* $0 \leq t < \infty$, *and if*

$$\int_0^\infty \frac{\overline{Q}_j(t)}{2D} dt < \infty \qquad \text{and} \qquad \int_0^\infty \frac{\underline{Q}_j(t)}{2D} dt < \infty \qquad \text{for all } 1 \leq j \leq N, \tag{3.140}$$

then

$$w(\xi,t) = \overline{O}\left(\exp\left(\int_0^t \sum_{j=1}^N \frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} d\zeta\right) \prod_{j=1}^N g_{j,1}(\xi_j)\right). \tag{3.141}$$

The condition (3.140) will be satisfied if, for example, $L_j(t)$ and $A_j(t)$ are of the types given in Example 3.15 for each $1 \leq j \leq N$.

3.7 Linear problem on a time-dependent cylinder

The analysis of the linear equation also extends in a straightforward way to cylinder-like domains in $\mathbb{R}^{N+1}$ that are infinite in the x_{N+1} direction (denoted by y) and have time-dependent cross-section. Here we are concerned with the cylinder-like domain $\Omega^*(t) = \Omega(t) \times (-\infty, \infty)$ where $\Omega(t)$ is a time-dependent box as in equation (2.14), and we consider the problem

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f_0\psi \qquad \text{for } (x,y) \in \Omega(t) \times (-\infty,\infty) \subset \mathbb{R}^{N+1} \tag{3.142}$$

$$\psi(x,y,t) = 0 \qquad \text{for } (x,y) \in \partial\Omega(t) \times (-\infty,\infty) \tag{3.143}$$

with $\psi(x,y,0)$ non-negative and compactly supported. We use the same change of variables from x_j to ξ_j as in Section 3.6 (for $1 \leq j \leq N$), and denote the solution by $\psi(x,y,t) = u(\xi,y,t)$. Then for $0 < \xi_j < L_0$, $-\infty < y < \infty$ the problem (3.142), (3.143) becomes

$$\frac{\partial u}{\partial t} = D\sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2}\frac{\partial^2 u}{\partial \xi_j^2} + D\frac{\partial^2 u}{\partial y^2} + \sum_{j=1}^{N}\left(\frac{\dot{A}_j(t)L_0 + \xi_j\dot{L}_j(t)}{L_j(t)}\right)\frac{\partial u}{\partial \xi_j} + f_0 u \tag{3.144}$$

$$u(\xi,y,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{3.145}$$

Let $w(\xi,y,t) = e^{-f_0 t}u(\xi,y,t)\prod_{j=1}^{N} H_j(\xi_j,t)$ where $H_j(\xi_j,t)$ is given in equation (3.131). Then exactly as above,

$$\frac{\partial w}{\partial t} = D\sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2}\frac{\partial^2 w}{\partial \xi_j^2} + D\frac{\partial^2 w}{\partial y^2} + \sum_{j=1}^{N}\left(\frac{\xi_j^2\ddot{L}_j(t)L_j(t)}{4DL_0^2} + \frac{\xi_j\ddot{A}_j(t)L_j(t)}{2DL_0}\right)w \tag{3.146}$$

$$w(\xi,y,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{3.147}$$

The separation of variables method now leads to the following extension of Theorem 3.2 (and Theorem 3.23).

Theorem 3.25. *Suppose that*

$$\ddot{L}_j(t)L_j(t)^3 \equiv \gamma_0^{(j)} \qquad \textit{and} \qquad \ddot{A}_j(t)L_j(t)^3 \equiv \gamma_1^{(j)} \qquad \textit{for all } 1 \leq j \leq N. \tag{3.148}$$

Then there exist separable solutions of (3.144), (3.145) *of the form*

$$u(\xi,y,t) = \int_{-\infty}^{\infty} \frac{\overline{u}(y')e^{-\frac{(y-y')^2}{4Dt}}}{\sqrt{4\pi Dt}}dy' \exp\left(f_0 t + \int_0^t \sum_{j=1}^{N} \frac{\sigma_{j,n_j}L_0^2}{L_j(\zeta)^2}d\zeta\right)\prod_{j=1}^{N}\left(\frac{g_{j,n_j}(\xi_j)}{H_j(\xi_j,t)}\right) \tag{3.149}$$

where $\overline{u}(y)$ is any continuous, compactly supported function, where $H_j(\xi_j, t)$ is given by equation (3.131)*, and where $g_{j,n}$ and $\sigma_{j,n}$ are the eigenfunctions and eigenvalues of the Sturm-Liouville problem* (3.19)*,* (3.20) *with $\gamma_0 = \gamma_0^{(j)}$ and $\gamma_1 = \gamma_1^{(j)}$.*

Similarly, Theorem 3.11 (and Theorem 3.24) extends to the cylinder as follows.

Theorem 3.26. *Given constants $\gamma_0^{(j)}$ and $\gamma_1^{(j)}$ for $1 \le j \le N$, let $g_{j,1}$ and $\sigma_{j,1}$ be the principal eigenfunction and eigenvalue of* (3.19)*,* (3.20) *with $\gamma_0 = \gamma_0^{(j)}$ and $\gamma_1 = \gamma_1^{(j)}$. Let w satisfy* (3.146)*,* (3.147) *and assume that*

$$C_1 \overline{u}(y) \prod_{j=1}^{N} g_{j,1}(\xi_j) \le w(\xi, y, 0) \le C_2 \overline{u}(y) \prod_{j=1}^{N} g_{j,1}(\xi_j), \tag{3.150}$$

for some continuous, non-negative, compactly supported function $\overline{u}(y)$ and some constants $0 < C_1 \le C_2$. Define $\overline{Q}_j(t)$ and $\underline{Q}_j(t)$ as in (3.137)*,* (3.138)*. If $t \ge 0$ is such that $L_j(\tau) > 0$ on $0 \le \tau \le t$ for all $1 \le j \le N$, then*

$$\begin{aligned} &\frac{C_1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \overline{u}(y') e^{-\frac{(y-y')^2}{4Dt}} dy' \exp\left(\int_0^t \sum_{j=1}^{N} \left(\frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} - \frac{\underline{Q}_j(\zeta)}{2D} \right) d\zeta \right) \prod_{j=1}^{N} g_{j,1}(\xi_j) \\ &\qquad \le w(\xi, y, t) \le \\ &\frac{C_2}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \overline{u}(y') e^{-\frac{(y-y')^2}{4Dt}} dy' \exp\left(\int_0^t \sum_{j=1}^{N} \left(\frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} + \frac{\overline{Q}_j(\zeta)}{2D} \right) d\zeta \right) \prod_{j=1}^{N} g_{j,1}(\xi_j). \end{aligned} \tag{3.151}$$

In particular, if $L_j(t) > 0$ for $0 \le t < \infty$ and if (3.140) *holds, then as $t \to \infty$,*

$$w(\xi, y, t) = \underline{\overline{O}} \left(\int_{-\infty}^{\infty} \frac{\overline{u}(y')}{\sqrt{4\pi D t}} e^{-\frac{(y-y')^2}{4Dt}} dy' \exp\left(\int_0^t \sum_{j=1}^{N} \frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} d\zeta \right) \prod_{j=1}^{N} g_{j,1}(\xi_j) \right). \tag{3.152}$$

In cases such that $L_j(t) > 0$ for all $0 \le t < \infty$ and such that (3.140) holds, Theorem 3.26 gives the exact order of the solution as $t \to \infty$. The next result gives more detail about the long-time behaviour of the solution at a given position $\xi(t)$ in the cross-section. Under the condition that

$$f_0 t + \sum_{j=1}^{N}\left(\int_0^t \left(\frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} - \frac{\dot{A}_j(\zeta)^2}{4D}\right) d\zeta - \frac{\xi_j(t)^2 \dot{L}_j(t) L_j(t)}{4DL_0^2} - \frac{\xi_j(t)\dot{A}_j(t)L_j(t)}{2DL_0}\right)$$
$$+ \sum_{j=1}^{N}\left(\log\left(g_{j,1}(\xi_j(t))\right) - \frac{1}{2}\log\left(\frac{L_j(t)}{L_j(0)}\right)\right) = \frac{C^2}{4D}t - p(t) \qquad (3.153)$$

for some $C > 0$ and $p(t) = o(t)$ as $t \to \infty$, we are interested in the asymptotic y positions at which $u(\xi(t), y, t)$ takes order one values. (Note that this condition (3.153) simply says that if we take $f_0 t$ and add and subtract the terms due to the time-dependent domain, then we are left with a positive multiple of t plus smaller order terms.)

Corollary 3.27. *Assume that $L_j(t) > 0$ for all $0 \le t < \infty$ and that (3.140) holds. Let $\xi_j(t) \in (0, L_0)$ be given for $1 \le j \le N$, and suppose that $A_j(t)$, $L_j(t)$ and $\xi_j(t)$ are such that (3.153) holds for some $C > 0$ and $p(t) = o(t)$ as $t \to \infty$. Then:*

1. *$u(\xi(t), ct, t) \to \infty$ as $t \to \infty$ for every $|c| < C$, and $u(\xi(t), ct, t) \to 0$ as $t \to \infty$ for every $|c| > C$.*

2. *Let $y = y_*(t)$ denote the y positions at which $u(\xi(t), y, t)$ is equal to some positive, order one value. Then for large t,*

$$y_*(t) = \pm(Ct - \delta_*(t)), \qquad \frac{C}{2D}\delta_*(t) - \frac{\delta_*(t)^2}{4Dt} = p(t) + \frac{1}{2}\log\frac{t}{t_0} + O(1). \qquad (3.154)$$

3. *If, in addition to the above assumptions,*

$$\dot{L}_j(t)L_j(t) = O(1) \qquad \textit{and} \qquad \dot{A}_j(t)L_j(t) = O(1) \qquad (3.155)$$

for every $1 \le j \le N$, then $y_*(t)$ is given by (3.154) uniformly for $\xi(t)$ in any compact subset of $(0, L_0)^N$, and C and $p(t)$ can be found from

$$f_0 t + \sum_{j=1}^{N}\left(\int_0^t \left(\frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} - \frac{\dot{A}_j(\zeta)^2}{4D}\right) d\zeta - \frac{1}{2}\log\left(\frac{L_j(t)}{L_j(0)}\right)\right)$$
$$= \frac{C^2}{4D}t - p(t) + O(1). \qquad (3.156)$$

Proof. Under the stated assumptions, Theorem 3.26 gives the bound in equation (3.152). In terms of the original function $u(\xi, y, t)$, this becomes

$$u(\xi, y, t) = \underline{O}\left(\int_{-\infty}^{\infty} \frac{\overline{u}(y')e^{-\frac{(y-y')^2}{4Dt}}}{\sqrt{4\pi Dt}} dy' e^{f_0 t + \int_0^t \sum_{j=1}^{N} \frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} d\zeta} \prod_{j=1}^{N} \left(\frac{g_{j,1}(\xi_j)}{H_j(\xi_j, t)}\right)\right) \tag{3.157}$$

as $t \to \infty$, with $H_j(\xi_j, t)$ given by equation (3.131). Evaluating this at $\xi = \xi(t)$, and using equation (3.153), we get precisely

$$u(\xi(t), y, t) = \underline{O}\left(\frac{e^{\frac{C^2}{4D}t - p(t)}}{\sqrt{t}} \int_{-\infty}^{\infty} \overline{u}(y')e^{-\frac{(y-y')^2}{4Dt}} dy'\right) \tag{3.158}$$

as $t \to \infty$. It is clear from (3.158) that as $t \to \infty$, $u(\xi(t), ct, t) \to \infty$ for every $|c| < C$, and $u(\xi(t), ct, t) \to 0$ for every $|c| > C$. Also, writing $y = Ct - \delta_*(t)$ and equating (3.158) to some constant, we find that $\delta_*(t)$ must satisfy

$$\frac{C}{2D}\delta_*(t) - \frac{\delta_*(t)^2}{4Dt} - p(t) - \frac{1}{2}\log\frac{t}{t_0} = O(1) \qquad \text{as } t \to \infty. \tag{3.159}$$

In general C and $p(t)$ will depend on $\xi_j(t) \in (0, L_0)$. However, if the condition (3.155) holds, then we have

$$-\frac{\xi_j(t)^2 \dot{L}_j(t) L_j(t)}{4DL_0^2} - \frac{\xi_j(t)\dot{A}_j(t)L_j(t)}{2DL_0} + \log\left(g_{j,1}(\xi_j(t))\right) = O(1) \qquad \text{as } t \to \infty, \tag{3.160}$$

uniformly for $\xi_j(t)$ in any compact subset of $(0, L_0)$. This observation, together with all the above, proves the final statement. □

Example 3.28. *Examples of possible long-time behaviour of $L_j(t)$, $A_j(t)$ for which the conditions* (3.140), (3.155) *and* (3.156) *are simultaneously satisfied include:*

1. $L_j(t) = a(t + t_0)^k$, $A_j(t) = b(t + t_1)^m$ *for any* $0 < k < \frac{1}{2}$ *and* $m + k < 1$.

2. $L_j(t)$, $A_j(t)$ *such that as* $t \to \infty$,

$$L_j(t) = l_j + o(1), \quad \dot{L}_j(t) = O(t^{-\varepsilon}), \quad \ddot{L}_j(t) = O(t^{-1-\varepsilon}), \tag{3.161}$$

$$\dot{A}_j(t) = c_j + O(t^{-\delta}), \quad \ddot{A}_j(t) = O(t^{-1-\delta}), \tag{3.162}$$

where $l_j > 0$, $\varepsilon > 0$, $\delta > 0$ *and* $f_0 > \sum_{j=1}^{N}\left(\frac{D\pi^2}{l_j^2} + \frac{c_j^2}{4D}\right)$.

3.8 Linear problem on a time-dependent ball

The method of separation of variables can also be used to derive exact solutions to the linear problem on a ball in any dimension N. For some time-dependent vector $A(t) \in \mathbb{R}^N$ and time-dependent radius $R(t) > 0$, let

$$\Omega(t) = \left\{x \in \mathbb{R}^N : |x - A(t)| < R(t)\right\}, \qquad (3.163)$$

and consider the problem

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f_0\psi \qquad \text{in } \Omega(t) \subset \mathbb{R}^N \qquad (3.164)$$

$$\psi(x,t) = 0 \qquad \text{on } \partial\Omega(t). \qquad (3.165)$$

Following the approach of Section 3.1, change variables from x to $\xi = \left(\frac{x-A(t)}{R(t)}\right) R_0$ for some $R_0 > 0$ and let $u(\xi, t) = \psi(x, t)$. The problem (3.164), (3.165) becomes:

$$\frac{\partial u}{\partial t} = D\frac{R_0^2}{R(t)^2}\nabla^2 u + \left(\frac{R_0\dot{A}(t) + \dot{R}(t)\xi}{R(t)}\right) \cdot \nabla u + f_0 u \qquad \text{for } \xi \in \Omega_0 \qquad (3.166)$$

$$u(\xi, t) = 0 \qquad \text{for } \xi \in \partial\Omega_0, \qquad (3.167)$$

where $\Omega_0 = \left\{\xi \in \mathbb{R}^N : |\xi| < R_0\right\}$. Now let $w(\xi, t) = u(\xi, t)H(\xi, t)e^{-f_0 t}$ where

$$H(\xi, t) = \left(\frac{R(t)}{R(0)}\right)^{\frac{N}{2}} \exp\left(\int_0^t \frac{|\dot{A}(\zeta)|^2}{4D}d\zeta + \frac{\dot{R}(t)R(t)}{4DR_0^2}|\xi|^2 + \frac{R(t)}{2DR_0}(\xi \cdot \dot{A}(t))\right), \qquad (3.168)$$

to get the equation

$$\frac{\partial w}{\partial t} = D\frac{R_0^2}{R(t)^2}\nabla^2 w + \left(\frac{|\xi|^2\ddot{R}(t)R(t)}{4DR_0^2} + \frac{(\xi \cdot \ddot{A}(t))R(t)}{2DR_0}\right) w \qquad \text{for } \xi \in \Omega_0 \qquad (3.169)$$

$$w(\xi, t) = 0 \qquad \text{for } \xi \in \partial\Omega_0. \qquad (3.170)$$

Remark 3.29. *Notice the similarity to the one-dimensional case (see Section 3.1). The factor* $\left(\frac{R(t)}{R(0)}\right)^{\frac{N}{2}}$ *in equation* (3.168) *is the only explicit dependence on the dimension* N*, and this term is included in the change of variables in order to remove a term in the equation involving* $\nabla \cdot \xi$ *(*$= N$ *in* $\mathbb{R}^N$*).*

As in the one-dimensional case, change variables from t to $s(t) = \int_0^t \frac{R_0^2}{R(\zeta)^2} d\zeta$, and write $v(\xi, s) = w(\xi, t)$. This gives:

$$\frac{\partial v}{\partial s} = D\nabla^2 v + \left(\frac{|\xi|^2 \ddot{R}(t) R(t)^3}{4DR_0^4} + \frac{(\xi \cdot \ddot{A}(t)) R(t)^3}{2DR_0^3} \right) v \qquad \text{for } |\xi| < R_0 \quad (3.171)$$

$$v(\xi, s) = 0 \qquad \text{at } |\xi| = R_0. \quad (3.172)$$

This is separable in s, $r = |\xi|$, and θ (the angular co-ordinates) if $\ddot{R}(t)R(t)^3 \equiv \gamma_0$ is a constant and $\ddot{A}(t)R(t)^3 \equiv 0$. This corresponds to $R(t)^2 = at^2 + 2bt + r_0^2$ for some constants a, b, $r_0 = R(0) > 0$, and $\gamma_0 = ar_0^2 - b^2$; and to $A(t) = A_0 + ct$ for some constant vectors A_0 and c. We arrive at the following extension of Theorem 3.2 to the N-dimensional ball.

Theorem 3.30. *Suppose that*

$$R(t)^2 = at^2 + 2bt + r_0^2 \qquad \textit{for some } a, b, \textit{ and } r_0 = R(0) > 0, \quad (3.173)$$

$$A(t) = A_0 + ct \qquad \textit{for some } A_0, c \in \mathbb{R}^N. \quad (3.174)$$

Then the solution to (3.166), (3.167) *on* $\Omega_0 = \left\{ \xi \in \mathbb{R}^N : |\xi| < R_0 \right\}$ *can be obtained exactly, as a sum of* $u_l(\xi, t)$ *with coefficients depending only on the initial conditions* $u(\xi, 0) \in L^2(\Omega_0)$. *The functions* u_l *are given by*

$$u_l(\xi, t) = \exp\left(\hat{\sigma}_l \int_0^t \frac{R_0^2}{R(\zeta)^2} d\zeta \right) \hat{g}_l(\xi) \left(\frac{R(0)}{R(t)} \right)^{N/2} \times \exp\left(f_0 t - \int_0^t \frac{|\dot{A}(\zeta)|^2}{4D} d\zeta - \frac{\dot{R}(t)R(t)}{4DR_0^2} |\xi|^2 - \frac{R(t)}{2DR_0} (\xi \cdot \dot{A}(t)) \right) \quad (3.175)$$

where $\hat{\sigma}_l$ *and* $\hat{g}_l(\xi)$ *are the eigenvalues and eigenfunctions of*

$$\hat{\sigma}_l \hat{g}_l(\xi) = D\nabla^2 \hat{g}_l + \frac{|\xi|^2 \gamma_0}{4DR_0^4} \hat{g}_l \qquad \textit{in } \{|\xi| < R_0\} \subset \mathbb{R}^N \quad (3.176)$$

$$\hat{g}_l(\xi) = 0 \qquad \textit{at } |\xi| = R_0. \quad (3.177)$$

As in the one-dimensional case, it is possible to write out explicit formulae for u_l in terms of a, b, r_0 and the relevant eigenfunctions $\hat{g}_l$ and eigenvalues $\hat{\sigma}_l$.

Chapter 4

Linear equation: behaviour near the boundary

4.1 Introduction: behaviour near the boundary for $\frac{-L(t)}{2} < x < \frac{L(t)}{2}$

Here we consider the linear equation on a symmetric interval $(\frac{-L(t)}{2}, \frac{L(t)}{2})$:

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + f_0 \psi \qquad \text{for } \frac{-L(t)}{2} < x < \frac{L(t)}{2} \tag{4.1}$$

$$\psi(x,t) = 0 \qquad \text{at } x = \pm\frac{L(t)}{2}, \tag{4.2}$$

and we are interested in the solution near the boundaries $x = \pm\frac{L(t)}{2}$. We shall focus on understanding the behaviour near the left hand end $x = \frac{-L(t)}{2}$; the corresponding results at the other end follow by symmetry. We begin by observing the following corollary to Theorem 3.2 where, as in equation (1.5), $c_* = 2\sqrt{Df_0}$.

Corollary 4.1. *Let $\psi(x,t) \geq 0$, $\not\equiv 0$ satisfy (4.1), (4.2) where $\frac{\dot{L}(t)}{2} \equiv c > 0$ is constant, and let $0 < y = O(1)$. Then $\psi\left(\frac{-L(t)}{2} + y, t\right)$ is exponentially growing as $t \to \infty$ if $0 < c < c_*$, and it is exponentially decaying as $t \to \infty$ if $c > c_*$. If $c = c_*$ and $y = O(1)$ then $\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}(yt^{-3/2}) \to 0$ as $t \to \infty$.*

Proof. Assume $L(t) = 2\left(ct + \frac{L_0}{2}\right)$ with $c > 0$, and let $\xi = \left(\frac{x}{L(t)} + \frac{1}{2}\right) L_0$ and $u(\xi, t) = \psi(x, t)$. Since $\dot{L}(t) \equiv 2c$ and $\dot{A}(t) \equiv -c$ are constants, this is one of the separable cases from Chapter 3. By Theorem 3.2, there are exact solutions

$$u_{n,c}(\xi, t) = \exp\left(\frac{-Dn^2\pi^2 t}{L_0(L_0 + 2ct)}\right) \sin\left(\frac{n\pi\xi}{L_0}\right) \left(\frac{L_0}{L_0 + 2ct}\right)^{1/2} \times \exp\left(\left(f_0 - \frac{c^2}{4D}\right) t + \frac{c(L_0 + 2ct)}{2D} \frac{\xi}{L_0} \left(1 - \frac{\xi}{L_0}\right)\right). \tag{4.3}$$

Any positive solution $u(\xi, t)$ can be bounded above and below by multiples of $u_{1,c}(\xi, t)$. Note that the position $x = \frac{-L(t)}{2} + y$ is equivalent to $\frac{\xi}{L_0} = \frac{y}{L(t)} \in (0, 1)$. We see from (4.3) that as $t \to \infty$, and for $y = o(t)$,

$$\psi_{1,c}\left(\frac{-L(t)}{2} + y, t\right) = u_{1,c}\left(\frac{yL_0}{L_0 + 2ct}, t\right) = \underline{\overline{O}}\left(\frac{y}{t} \times \frac{1}{t^{1/2}} \times \exp\left(\left(f_0 - \frac{c^2}{4D}\right) t + \frac{cy}{2D}\right)\right). \tag{4.4}$$

Equation (4.4) shows that $\psi_{1,c}\left(\frac{-L(t)}{2} + y, t\right)$ is exponentially growing if $f_0 > \frac{c^2}{4D}$, and $\psi_{1,c}\left(\frac{-L(t)}{2} + y, t\right)$ is exponentially decaying if $f_0 < \frac{c^2}{4D}$. If $f_0 = \frac{c^2}{4D}$ (i.e. $c = c_*$) and $y = O(1)$ then $\psi_{1,c}\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}(yt^{-3/2}) \to 0$ as $t \to \infty$. □

This leads us to consider the following problem. We would like, if possible, to choose $L(t)$ in such a way that the solution neither grows nor decays but remains exactly of order one for x at an order one distance from the boundary. That is, we would like to choose $L(t)$ in such a way that

$$\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}(y) \qquad \text{for } 0 < y = O(1), \text{ as } t \to \infty. \tag{4.5}$$

This means that for $y_0 > 0$, there exist $0 < \beta_0 \leq \beta_1$ such that for all $0 \leq y \leq y_0$ and all t sufficiently large, $\beta_0 y \leq \psi\left(\frac{-L(t)}{2} + y, t\right) \leq \beta_1 y$. This will then also imply that $\beta_0 \leq \frac{\partial\psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) \leq \beta_1$, i.e. $\frac{\partial\psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{\overline{O}}(1)$ as $t \to \infty$.

Further motivation for studying this problem may be inspired by [24, 23, 17, 26, 25] and [12], all of which analyse relationships between the time-dependent motion of a boundary and the gradient of a solution there. The papers [24, 23, 17, 26, 25] concern the nonlinear equation (1.17) on $g(t) < x < h(t)$ and they impose the boundary conditions (1.18), (1.19) which relate the speed of the boundary to the gradient of the solution. We shall return to their results when we consider the nonlinear problem in Chapter 7 (see Section 7.5.2).

In [12], J. Berestycki, Brunet and Derrida consider the linear problem on $\mu(t) < x < \infty$. As discussed in Chapter 1, they derive very precise asymptotic behaviour of the boundary position $\mu(t)$ such that the solution and its gradient have constant values at $x = \mu(t)$. For initial conditions with sufficiently fast decay they show that $\mu(t) = c_* t - \frac{3D}{c_*} \log \frac{t}{t_0} + \text{constant} + o(1)$ as $t \to \infty$, which is, significantly, the same as the front position (1.11) for the nonlinear KPP equation. They also calculate several subsequent terms ('vanishing corrections') in their expansions. This shows that interesting and significant properties may be discovered by analysing the relationship between time-dependent boundary motion and the gradient of the solution at the boundary.

In the following sections we derive super- and subsolutions for the linear problem on $(\frac{-L(t)}{2}, \frac{L(t)}{2})$ under certain assumptions on $\ddot{L}(t)L(t)^3$, and thus give a form of $L(t)$ such that (4.5) holds. We also investigate similar problems on a ball and a box, and discuss links with the nonlinear KPP equation on an unbounded domain. In Section 4.5 we apply a method from [12], to derive vanishing correction terms for the critical choices of boundary motion on a symmetric interval or box.

4.2 Critical super- and subsolutions

4.2.1 Equation and change of variables

We change variables in (4.1) to $\xi = \left(\frac{x}{L(t)} + \frac{1}{2}\right) L_0$ for some $L_0 > 0$, and let $u(\xi, t) = \psi(x, t)$. The equation becomes:

$$\frac{\partial u}{\partial t} = D \frac{L_0^2}{L(t)^2} \frac{\partial^2 u}{\partial \xi^2} + \left(\xi - \frac{L_0}{2}\right) \frac{\dot{L}(t)}{L(t)} \frac{\partial u}{\partial \xi} + f_0 u \qquad \text{for } 0 < \xi < L_0 \tag{4.6}$$

$$u(\xi, t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{4.7}$$

Next let

$$w(\xi,t)=u(\xi,t)\left(\frac{L(t)}{L(0)}\right)^{1/2}\exp\left(-f_0t+\int\limits_0^t\frac{\dot{L}(\zeta)^2}{16D}d\zeta+\frac{\dot{L}(t)L(t)}{4D}\frac{\xi}{L_0}\left(\frac{\xi}{L_0}-1\right)\right). \tag{4.8}$$

This is the same change of variables as we used in Chapter 3, but now taking into account the symmetry $A(t)=\frac{-L(t)}{2}$. Therefore, w now satisfies

$$\frac{\partial w}{\partial t}=D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2 w}{\partial\xi^2}+P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0}-1\right)\frac{w}{L_0^2}\right)\qquad\text{for } 0<\xi<L_0 \tag{4.9}$$

$$w(\xi,t)=0\qquad\text{at } \xi=0 \text{ and } \xi=L_0, \tag{4.10}$$

where

$$P(t)=\frac{\ddot{L}(t)L(t)^3}{4D^2}. \tag{4.11}$$

We shall derive a super- and subsolution for (4.9), (4.10). The supersolution is valid for any function $P(t)$ which is non-negative; the subsolution is valid if $P(t)$ is large and positive with $0<P(t)\to\infty$ as $t\to\infty$ and $\dot{P}(t)\geq 0$.

4.2.2 Supersolution

Proposition 4.2. *Let $w(\xi,t)$ satisfy equations* (4.9), (4.10) *for some function $P(t)$. If $P(t)\geq 0$ then (up to multiplication by a constant) $w(\xi,t)\leq\overline{w}(\xi,t)$ where*

$$\overline{w}(\xi,t)=\sin\left(\frac{\pi\xi}{L_0}\right)\exp\left(-\int_0^t\frac{D\pi^2}{L(\zeta)^2}d\zeta\right). \tag{4.12}$$

Therefore $w(\xi,t)=O(\xi)$ independently of time as $t\to\infty$, in the sense that there exists β_1 such that

$$w(\xi,t)\leq\beta_1\xi \textit{ as } t\to\infty,\ \textit{for all } 0\leq\xi\leq L_0. \tag{4.13}$$

Proof. This $\overline{w}(\xi,t)$ is a supersolution for $w(\xi,t)$: it satisfies the boundary conditions and, since $P(t)\geq 0$, it satisfies the inequality

$$\frac{\partial\overline{w}}{\partial t}=D\frac{L_0^2}{L(t)^2}\frac{\partial^2\overline{w}}{\partial\xi^2}\geq D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2\overline{w}}{\partial\xi^2}+P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0}-1\right)\frac{\overline{w}}{L_0^2}\right). \tag{4.14}$$

Hence, up to multiplication by a constant, $w(\xi,t)\leq\overline{w}(\xi,t)$ and $\overline{w}(\xi,t)=O(\xi)$ independently of time as $t\to\infty$. □

4.2.3 Subsolution

Next we construct a subsolution. Let $\kappa_1 < 0$ be the largest real zero of the Airy function Ai. We know that $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$ (Airy's equation), and we note here for reference that $\kappa_1 < 0$, $\mathrm{Ai}'(\kappa_1) > 0$, $\mathrm{Ai}(0) > 0$, $\mathrm{Ai}'(0) < 0$, and $\mathrm{Ai}''(0) = 0$. If $P(t) > 0$ is sufficiently large, define $\underline{w}(\xi, t)$ and $a(t)$ by:

$$\underline{w}(\xi,t) = \begin{cases} \frac{1}{P(t)^{1/3}} \mathrm{Ai}\left(P(t)^{1/3}\frac{\xi}{L_0} + \kappa_1\right) \\ \qquad \text{for } 0 \le \frac{\xi}{L_0} \le -\kappa_1 P(t)^{-1/3}\text{: Region I} \\ \frac{1}{P(t)^{1/3}} \left(\mathrm{Ai}(0) + \mathrm{Ai}'(0)\left(P(t)^{1/3}\frac{\xi}{L_0} + \kappa_1\right)\right) \\ \qquad \text{for } -\kappa_1 P(t)^{-1/3} \le \frac{\xi}{L_0} \le -\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) P(t)^{-1/3}\text{: Region II} \\ 0 \qquad \text{for } -\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) P(t)^{-1/3} \le \frac{\xi}{L_0} \le 1\text{: Region III,} \end{cases} \tag{4.15}$$

and

$$a(t) = \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \frac{DP(\zeta)^{2/3}}{L(\zeta)^2} d\zeta\right). \tag{4.16}$$

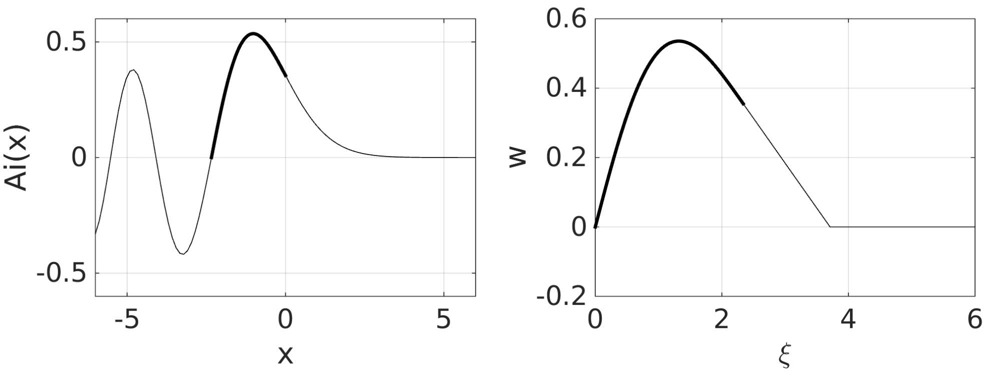

Figure 4.1: *Left:* the Airy function $\mathrm{Ai}(x)$, with the portion $\kappa_1 \le x \le 0$ highlighted; *Right:* a sketch of $\underline{w}(\xi, t)$ as a function of ξ, at a fixed t.

Proposition 4.3. *Let $w(\xi, t)$ satisfy equations* (4.9), (4.10) *for some function $P(t) > 0$. If $P(t) \to \infty$ as $t \to \infty$ and $\dot{P}(t) \ge 0$, then (up to multiplication by a constant) $w(\xi, t) \ge \tilde{w}(\xi, t) = \underline{w}(\xi, t)a(t)$ where $\underline{w}(\xi, t)$ and $a(t)$ are given by* *equations* (4.15) *and* (4.16). *Moreover, if*

$$\int_0^\infty \frac{P(\zeta)^{2/3}}{L(\zeta)^2} d\zeta < \infty \tag{4.17}$$

then for ξ *in Region I,* $w(\xi,t)$ *can be bounded below by a positive multiple of* ξ *(independently of* t*) as* $t \to \infty$*. In other words, there exists* $\beta_0 > 0$ *such that*

$$\beta_0 \xi \leq w(\xi,t) \text{ as } t \to \infty, \text{ for all } 0 \leq \xi \leq -\kappa_1 P(t)^{-1/3} L_0. \tag{4.18}$$

Proof. Note that $\underline{w}$ is continuous and non-negative on $[0, L_0]$, and satisfies the boundary conditions. Furthermore, both $\frac{\partial \underline{w}}{\partial \xi}$ and $\frac{\partial^2 \underline{w}}{\partial \xi^2}$ are continuous across Regions I–II, including at the point where Regions I and II meet, where the left and right limits both give $\frac{\partial \underline{w}}{\partial \xi} = \frac{\mathrm{Ai}'(0)}{L_0}$ and $\frac{\partial^2 \underline{w}}{\partial \xi^2} = 0$. In each Region I and Region II, $\frac{\partial \underline{w}}{\partial t}$ satisfies

$$\frac{\partial \underline{w}}{\partial t} = \frac{\dot{P}(t)}{3P(t)} \left(-\underline{w} + \xi \frac{\partial \underline{w}}{\partial \xi} \right) \tag{4.19}$$

and it follows from the continuity of each term that $\frac{\partial \underline{w}}{\partial t}$ is also continuous across Regions I–II. Therefore across Regions I–II, $\underline{w}(\xi,t)$ is C^2 in ξ and C^1 in t.

In Region I:

$$\begin{aligned}
\frac{\partial \underline{w}}{\partial t} - D\frac{L_0^2}{L(t)^2} &\left(\frac{\partial^2 \underline{w}}{\partial \xi^2} + P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0} - 1\right)\frac{\underline{w}}{L_0^2} \right) \\
&= -\frac{\dot{P}(t)}{3P(t)}\underline{w} + \frac{\dot{P}(t)}{3P(t)}\frac{\xi}{L_0}\mathrm{Ai}'\left(P(t)^{1/3}\frac{\xi}{L_0} + \kappa_1\right) \\
&\quad - \frac{DP(t)^{1/3}}{L(t)^2}\mathrm{Ai}''\left(P(t)^{1/3}\frac{\xi}{L_0} + \kappa_1\right) \\
&\quad - \frac{DP(t)}{L(t)^2}\frac{\xi^2}{L_0^2}\underline{w} + \frac{DP(t)}{L(t)^2}\frac{\xi}{L_0}\underline{w} &(4.20)\\
&= \frac{\dot{P}(t)}{3P(t)}\left(-\underline{w} + \xi\frac{\partial \underline{w}}{\partial \xi}\right) - \frac{D}{L(t)^2}P(t)^{2/3}\left(P(t)^{1/3}\frac{\xi}{L_0} + \kappa_1\right)\underline{w} \\
&\quad - \frac{DP(t)}{L(t)^2}\frac{\xi^2}{L_0^2}\underline{w} + \frac{DP(t)}{L(t)^2}\frac{\xi}{L_0}\underline{w} &(4.21)\\
&= \frac{\dot{P}(t)}{3P(t)}\left(-\underline{w} + \xi\frac{\partial \underline{w}}{\partial \xi}\right) - \frac{DP(t)^{2/3}}{L(t)^2}\kappa_1\underline{w} - \frac{DP(t)}{L(t)^2}\frac{\xi^2}{L_0^2}\underline{w}. &(4.22)
\end{aligned}$$

Note that $\frac{\partial^2 \underline{w}}{\partial \xi^2} \leq 0$ in Region I, since $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x) \leq 0$ on $[\kappa_1, 0]$. Therefore, since $\underline{w}(0,t) = 0$, we have that

$$\xi \frac{\partial \underline{w}}{\partial \xi}(\xi,t) \leq \underline{w}(\xi,t) \qquad \text{in Region I.} \tag{4.23}$$

Thus equation (4.22) together with the assumption that $P(t) \geq 0$ and $\dot{P}(t) \geq 0$ implies that, in Region I,

$$\frac{\partial \underline{w}}{\partial t} - D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2 \underline{w}}{\partial \xi^2} + P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0} - 1\right)\frac{\underline{w}}{L_0^2}\right) \leq -\kappa_1 \frac{DP(t)^{2/3}}{L(t)^2}\underline{w}. \tag{4.24}$$

In Region II, since $P(t) \geq 0$, $\dot{P}(t) \geq 0$, and $\mathrm{Ai}'(0) < 0$,

$$\begin{aligned}
&\frac{\partial \underline{w}}{\partial t} - D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2 \underline{w}}{\partial \xi^2} + P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0} - 1\right)\frac{\underline{w}}{L_0^2}\right) \\
&\qquad = -\frac{\dot{P}(t)}{3P(t)}\underline{w} + \frac{\dot{P}(t)}{3P(t)}\frac{\xi}{L_0}\mathrm{Ai}'(0) - \frac{DP(t)}{L(t)^2}\frac{\xi^2}{L_0^2}\underline{w} + \frac{DP(t)}{L(t)^2}\frac{\xi}{L_0}\underline{w} \\
&\qquad \leq \frac{DP(t)}{L(t)^2}\frac{\xi}{L_0}\underline{w} \\
&\qquad \leq \left(-\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} - \kappa_1\right)\frac{DP(t)^{2/3}}{L(t)^2}\underline{w}.
\end{aligned} \tag{4.25}$$

Let $\tilde{w}(\xi, t) = \underline{w}(\xi, t)a(t)$ where $a(t)$ is given in equation (4.16). Then $\tilde{w}(\xi, t)$ is a classical subsolution in Regions I–II, as it is C^2 in ξ, C^1 in t and satisfies

$$\frac{\partial \tilde{w}}{\partial t} - D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2 \tilde{w}}{\partial \xi^2} + P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0} - 1\right)\frac{\tilde{w}}{L_0^2}\right) \leq 0. \tag{4.26}$$

In Region III, since $\tilde{w} \equiv 0$, it is clear that

$$\frac{\partial \tilde{w}}{\partial t} - D\frac{L_0^2}{L(t)^2}\left(\frac{\partial^2 \tilde{w}}{\partial \xi^2} + P(t)\frac{\xi}{L_0}\left(\frac{\xi}{L_0} - 1\right)\frac{\tilde{w}}{L_0^2}\right) = 0. \tag{4.27}$$

So, at the point where Region II and Region III meet, $\tilde{w}$ is continuous, it is a classical subsolution on either side, and $\frac{\partial \tilde{w}}{\partial \xi}$ has a jump discontinuity from a negative value on the left (Region II) to zero on the right (Region III). It follows from Lemma A.7 that $\tilde{w}(\xi, t)$ is a weak subsolution for $w(\xi, t)$. Therefore, up to multiplication by a constant, $w(\xi, t) \geq \tilde{w}(\xi, t) = \underline{w}(\xi, t)a(t)$.

Now note that $\mathrm{Ai}(y + \kappa_1) \sim \mathrm{Ai}'(\kappa_1)y$ as $y \to 0$ and $\mathrm{Ai}(y + \kappa_1) \geq \frac{\mathrm{Ai}(0)}{-\kappa_1}y$ for all $0 \leq y \leq -\kappa_1$. Consequently we have

$$\underline{w}(\xi, t) \geq \frac{\mathrm{Ai}(0)}{-\kappa_1}\frac{\xi}{L_0} \qquad \text{for all } \xi \text{ in Region I.} \tag{4.28}$$

If (4.17) holds then $a(t)$ converges to a strictly positive value as $t \to \infty$. Then for ξ in Region I, $\tilde{w}(\xi, t)$ (and hence also $w(\xi, t)$) can be bounded below by a positive multiple of ξ, independently of time as $t \to \infty$. □

In the case where $P(t)$ is given by equation (4.11), the condition (4.17) in Proposition 4.3 becomes simply $\int_0^\infty \ddot{L}(\zeta)^{2/3} d\zeta < \infty$.

4.2.4 Behaviour near an endpoint at $\frac{-L(t)}{2} = -c_* t + o(t)$

Theorem 4.4. *Let* $L(t) = 2(c_* t - \delta(t))$ *where* $\frac{c_*^2}{4D} = f_0$ *and*

$$\delta(t) = o(t), \quad \dot{\delta}(t) = o(1), \quad \ddot{\delta}(t) = o(1),$$
$$0 < -\ddot{\delta}(t)(c_* t - \delta(t))^3 \text{ is increasing and tends to } \infty, \quad \text{as } t \to \infty. \tag{4.29}$$

Let ψ *satisfy* (4.1), (4.2). *Then for each* $y_0 > 0$ *there exist constants* $\beta_0 > 0$, $\beta_1 > 0$ *(depending on the initial conditions) such that*

$$\beta_0 y t^{-\frac{3}{2}} \exp\left(\frac{c_*}{2D}\delta(t) + \int_0^t \left(-\frac{\dot{\delta}(\zeta)^2}{4D} + \left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right)\left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}}\right) d\zeta\right)$$
$$\le \psi\left(\frac{-L(t)}{2} + y, t\right) \le \beta_1 y t^{-\frac{3}{2}} \exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right) \tag{4.30}$$

for $0 \le y \le y_0$, *as* $t \to \infty$. *In particular, if also*

$$\int_0^\infty (-\ddot{\delta}(t))^{2/3} dt < \infty \tag{4.31}$$

then

$$\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}\left(y t^{-\frac{3}{2}} \exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right)\right) \qquad \text{as } t \to \infty. \tag{4.32}$$

Proof. Assumptions (4.29) on $\delta(t)$ ensure that the function $P(t) = \dfrac{-\ddot{\delta}(t) L(t)^3}{2D^2}$ obeys $\dot{P}(t) \ge 0$ and $0 < P(t) \to \infty$ as $t \to \infty$. So applying Proposition 4.2 and Proposition 4.3 gives that there are positive constants C_1, C_2 such that

$$C_1 \xi \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}} d\zeta\right) \le w(\xi, t) \le C_2 \xi$$
$$\text{for } 0 \le \xi \le -\kappa_1 P(t)^{-1/3} L_0. \tag{4.33}$$

Consider $x = \frac{-L(t)}{2} + y$ with $0 \le y \le y_0$, which corresponds to $\xi = \frac{yL_0}{L(t)}$. Noting that $P(t) = o(t^3)$ as $t \to \infty$, we have $\xi = \frac{yL_0}{L(t)} = O\left(\frac{1}{t}\right) = o(P(t)^{-1/3})$. Therefore, by (4.33) we conclude that there are constants $\hat{C}_1$, $\hat{C}_2$ such that

$$\hat{C}_1 \frac{y}{t} \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}} d\zeta\right) \le w\left(\frac{yL_0}{L(t)}, t\right) \le \hat{C}_2 \frac{y}{t} \tag{4.34}$$

for $0 \le y \le y_0$, as $t \to \infty$. Recall that the original function $\psi(x,t) = u(\xi, t)$ is related to $w(\xi, t)$ by equation (4.8). Since $L(t) = 2(c_* t - \delta(t))$, we can calculate

$$f_0 t - \int_0^t \frac{\dot{L}(\zeta)^2}{16D} d\zeta = \frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta + O(1) \tag{4.35}$$

so that for $0 \le y \le y_0$ and t large,

$$\begin{aligned}\left(\frac{L(0)}{L(t)}\right)^{1/2} &\exp\left(f_0 t - \int_0^t \frac{\dot{L}(\zeta)^2}{16D} d\zeta - \frac{y}{L(t)}\left(\frac{y}{L(t)} - 1\right)\frac{\dot{L}(t)L(t)}{4D}\right) \\ &= \underline{O}\left(\frac{1}{t^{1/2}} \exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right)\right). \end{aligned} \tag{4.36}$$

The result (4.30) follows by putting this into equation (4.8) and combining with (4.34). Equation (4.31) then implies (4.32). □

Remark 4.5. *If* (4.32) *holds then this also implies that*

$$\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{O}\left(t^{-\frac{3}{2}} \exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right)\right) \qquad \text{as } t \to \infty. \tag{4.37}$$

The following corollary to Theorem 4.4 gives sufficient conditions on $L(t)$ such that the required property (4.5) holds.

Corollary 4.6. *Let* $L(t) = 2\left(c_* t - \alpha \log(\frac{t}{t_0} + 1) - \theta(t)\right)$ *where* $\alpha > 0$ *and*

$$\theta(t) = O(1), \quad \dot{\theta}(t) = o(1/t), \quad \ddot{\theta}(t) = o(1/t^2), \quad \dddot{\theta}(t) = o(1/t^3) \quad \text{as } t \to \infty. \tag{4.38}$$

Then as $t \to \infty$ *and for* $y = O(1)$,

$$\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{O}\left(y t^{-\frac{3}{2} + \frac{\alpha c_*}{2D}}\right), \qquad \frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{O}\left(t^{-\frac{3}{2} + \frac{\alpha c_*}{2D}}\right). \tag{4.39}$$

In particular if $\alpha = \frac{3D}{c_*}$ *then* $\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{O}(y)$ *and* $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{O}(1)$.

Proof. This $L(t)$ satisfies the assumptions (4.29) and (4.31) with

$$\delta(t) = \alpha \log\left(\frac{t}{t_0} + 1\right) + \theta(t) \tag{4.40}$$

and $\ddot{\delta}(t) \sim -\frac{\alpha}{t^2}$ as $t \to \infty$. So we conclude (4.32) and (4.37), and the claimed results follow since

$$\exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right) = \underline{\overline{O}}\left(t^{\frac{\alpha c_*}{2D}}\right) \qquad \text{as } t \to \infty. \tag{4.41}$$

□

As well as giving a form of $L(t)$ such that (4.5) holds, Corollary 4.6 shows that $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right)$ tends to zero if $0 < \alpha < \frac{3D}{c_*}$ and to infinity if $\alpha > \frac{3D}{c_*}$.

It is not clear whether the conditions (4.29), (4.31) on $\delta(t)$ or (4.38) on $\theta(t)$ are necessary in order for the conclusions to hold, or whether they can be improved. Here, these conditions arise because of our method of proof (so that the subsolution and supersolution are valid) however they may not be optimal. The conditions (4.38) on the $O(1)$ term $\theta(t)$ are satisfied by, for example: a constant, any inverse power of t, or an exponentially decaying term.

Remark 4.7. *Let $L(t)$ be as in Corollary 4.6, with $0 < \alpha \leq \frac{3D}{c_*}$. Given $y_0 > 0$, Corollary 4.6 shows that ψ is bounded on $\left(\frac{-L(t)}{2}, \frac{-L(t)}{2} + y_0\right)$. Proposition 2.13 then implies that $\frac{\partial \psi}{\partial t}$, $\frac{\partial \psi}{\partial x}$, $\frac{\partial^2 \psi}{\partial x^2}$, are all also bounded (independently of time) on this order one neighbourhood of the boundary.*

Remark 4.8. *The 'critical' choice $\frac{L(t)}{2} = c_* t - \frac{3D}{c_*}\log(\frac{t}{t_0} + 1) + O(1)$, such that $\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ and $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{\overline{O}}(1)$, has a logarithmic term which matches that from two instances mentioned in Chapter 1:*

1. *the front position $\varphi(t)$ for the nonlinear KPP problem on $\mathbb{R}$, such that $u(\varphi(t) + x, t) \to \tilde{U}_{c_*}(x)$ as $t \to \infty$ [16, 15, 36], and*

2. *the boundary position $\mu(t)$ for the linear equation on $\mu(t) < x < \infty$, such that the solution and its gradient at $x = \mu(t)$ are constants [12],*

each starting from compactly supported initial conditions. This illustrates an interesting correspondence between two (related but distinct) problems involving the linear equation on domains with moving boundaries, and the nonlinear KPP problem on $\mathbb{R}$ with compactly supported initial conditions. We shall observe a similar relationship in $\mathbb{R}^N$, in Remark 4.16.

We also give another example to which we can apply Theorem 4.4.

Example 4.9. *Let $L(t) = 2(c_* t - a(t+t_0)^k + b)$ where $0 < k < \frac{1}{2}$, $a > 0$ and $b > 0$. Then for $y = O(1)$,*

$$\psi\left(\frac{-L(t)}{2} + y, t\right) = \underline{\overline{O}}\left(yt^{-\frac{3}{2}} \exp\left(\frac{c_* a}{2D} t^k\right)\right) \qquad \text{as } t \to \infty. \tag{4.42}$$

Proof. This $L(t)$ has the form required for Theorem 4.4, with $\delta(t) = a(t+t_0)^k - b$. This satisfies all the conditions in (4.29) due to $\ddot{\delta}(t) \sim -ak(1-k)t^{k-2}$ as $t \to \infty$, and (4.31) since $\frac{2}{3}(k-2) < -1$. So we conclude the exact bound (4.32). Finally, $\int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta = O(1)$ since $2(k-1) < -1$, and so (4.42) follows. □

4.3 Critical super- and subsolutions on a ball $|x| < R(t)$ in $\mathbb{R}^N$

4.3.1 Equation and change of variables

Here we consider (3.164), (3.165) where $\Omega(t) = \{x \in \mathbb{R}^N : |x| < R(t)\}$ is the N-dimensional ball with centre 0 and radius $R(t) > 0$. We change variables to $z = \frac{x}{R(t)} R_0$ for some $R_0 > 0$, and let $u(z,t) = \psi(x,t)$. Then the equation becomes:

$$\frac{\partial u}{\partial t} = D \frac{R_0^2}{R(t)^2} \nabla^2 u + \frac{\dot{R}(t)}{R(t)} z \cdot \nabla u + f_0 u \qquad \text{for } |z| < R_0 \tag{4.43}$$

$$u(z,t) = 0 \qquad \text{at } |z| = R_0. \tag{4.44}$$

Let

$$W(z,t) = u(z,t)\left(\frac{R(t)}{R_0}\right)^{\frac{N}{2}} \exp\left(-f_0 t + \int_0^t \frac{|\dot{R}(\zeta)|^2}{4D} d\zeta + \frac{\dot{R}(t)R(t)}{4D}\left(\frac{|z|^2}{R_0^2} - 1\right)\right) \tag{4.45}$$

to get the equation

$$\frac{\partial W}{\partial t} = D\frac{R_0^2}{R(t)^2}\left(\nabla^2 W + \tilde{P}(t)\left(\frac{|z|^2}{R_0^2} - 1\right)\frac{W}{R_0^2}\right) \qquad \text{for } |z| < R_0 \tag{4.46}$$

$$W(z,t) = 0 \qquad \text{at } |z| = R_0 \tag{4.47}$$

where

$$\tilde{P}(t) = \frac{\ddot{R}(t)R(t)^3}{4D^2}. \tag{4.48}$$

We shall derive a super- and subsolution for $W(z,t)$. The supersolution is valid if $\tilde{P}(t)$ is non-negative; the subsolution is valid if $\tilde{P}(t)$ is large and positive with $\tilde{P}(t) \to \infty$ as $t \to \infty$ and $\dot{\tilde{P}}(t) \geq 0$.

4.3.2 Supersolution

Let $\phi(x) = \phi_0(|x|)$ be the radially symmetric principal eigenfunction of

$$\lambda\phi(x) = -\nabla^2\phi \quad \text{for } |x| < 1, \qquad \phi(x) = 0 \quad \text{at } |x| = 1 \tag{4.49}$$

in the N-dimensional ball, and let its eigenvalue be λ_0. For $0 \leq r \leq R_0$ define

$$\overline{W}(r,t) = \phi_0\left(\frac{r}{R_0}\right)\exp\left(\int_0^t -\frac{D\lambda_0}{R(\zeta)^2}d\zeta\right). \tag{4.50}$$

Proposition 4.10. *Let $W(z,t)$ satisfy equations* (4.46), (4.47). *If $\tilde{P}(t) \geq 0$ then (up to multiplication by a constant) $W(z,t) \leq \overline{W}(|z|,t)$.*

Proof. The radial function $\overline{W}(|z|,t)$ satisfies

$$\frac{\partial \overline{W}}{\partial t} = D\frac{R_0^2}{R(t)^2}\nabla^2\overline{W} \quad \text{for } |z| < R_0, \qquad \overline{W}(|z|,t) = 0 \quad \text{at } |z| = R_0 \tag{4.51}$$

and is therefore a supersolution for $W(z,t)$, since $\tilde{P}(t) \geq 0$. So, up to multiplication by a constant, $W(z,t) \leq \overline{W}(|z|,t)$. □

4.3.3 Subsolution

Lemma 4.11. *Let $\hat{w}(r,t)$ be a non-negative, radially symmetric function in N dimensions where $N \leq 3$. Then*

$$r^{-\left(\frac{N-1}{2}\right)} \frac{\partial^2}{\partial r^2} \left(r^{\frac{N-1}{2}} \hat{w} \right) \leq \nabla^2 \hat{w}. \tag{4.52}$$

Proof. A straightforward calculation gives

$$\frac{\partial^2}{\partial r^2} \left(r^{\frac{N-1}{2}} \hat{w} \right) \equiv r^{\frac{N-1}{2}} \left(\frac{\partial^2 \hat{w}}{\partial r^2} + \frac{(N-1)}{r} \frac{\partial \hat{w}}{\partial r} + \left(\frac{N-1}{2} \right) \left(\frac{N-3}{2} \right) \frac{\hat{w}}{r^2} \right) \tag{4.53}$$

$$\equiv r^{\frac{N-1}{2}} \left(\nabla^2 \hat{w} + \left(\frac{N-1}{2} \right) \left(\frac{N-3}{2} \right) \frac{\hat{w}}{r^2} \right) \tag{4.54}$$

where we have used the form of the Laplacian in N-dimensional polar coordinates. So for $N \leq 3$ and $\hat{w} \geq 0$, it holds that $\frac{\partial^2}{\partial r^2} \left(r^{\frac{N-1}{2}} \hat{w} \right) \leq r^{\frac{N-1}{2}} \nabla^2 \hat{w}$. □

Let us restrict to $N \leq 3$ and construct a non-trivial radial subsolution for $W(z,t)$. Set $L(t) = 2R(t)$, $L_0 = 2R_0$ and $\xi = R_0 - r \in (0, R_0) = (0, L_0/2)$, and recall from Proposition 4.3 the definition of $\tilde{w}(\xi, t) = \underline{w}(\xi, t) a(t)$ where $\underline{w}(\xi, t)$ is given by equation (4.15) and $a(t)$ by equation (4.16).

Proposition 4.12. *Assume $N \leq 3$ and let $W(z,t)$ be a non-negative, non-zero solution to* (4.46), (4.47) *for some function $\tilde{P}(t) > 0$. If $\tilde{P}(t) \to \infty$ as $t \to \infty$ and $\dot{\tilde{P}}(t) \geq 0$, then (up to multiplication by a constant) $W(z,t) \geq \hat{w}(|z|, t)$ where*

$$\hat{w}(r,t) = r^{-\left(\frac{N-1}{2}\right)} \tilde{w}(R_0 - r, t). \tag{4.55}$$

Proof. Let $w_1(r,t) = \tilde{w}(R_0 - r, t)$. Then $w_1(r,t) \geq 0$ on $0 \leq r \leq R_0$ and satisfies $w_1(R_0, t) = 0$. Furthermore, by the construction of $\tilde{w}$ and since $\tilde{P}(t) \to \infty$, it holds that for all t sufficiently large, $w_1(r,t) = 0$ on a neighbourhood of $r = 0$.

Next, note that $\frac{L_0^2}{L(t)^2} = \frac{R_0^2}{R(t)^2}$, and

$$\left(\frac{r^2}{R_0^2} - 1 \right) \frac{\tilde{P}(t)}{R_0^2} = \left(\frac{\xi - R_0}{R_0} + 1 \right) \left(\frac{\xi - R_0}{R_0} - 1 \right) \frac{1}{4 R_0^2 D^2} \ddot{R}(t) R(t)^3$$

$$= \frac{2\xi}{L_0} \left(\frac{2\xi}{L_0} - 2 \right) \frac{1}{L_0^2 D^2} \frac{\ddot{L}(t) L(t)^3}{16}$$

$$= \frac{\xi}{L_0}\left(\frac{\xi}{L_0}-1\right)\frac{1}{L_0^2}\frac{\ddot{L}(t)L(t)^3}{4D^2}. \tag{4.56}$$

Therefore since $\tilde{w}$ is a subsolution to equation (4.9), the function $w_1(r,t)$ satisfies

$$\frac{\partial w_1}{\partial t} \leq D\frac{R_0^2}{R(t)^2}\left(\frac{\partial^2 w_1}{\partial r^2}+\tilde{P}(t)\left(\frac{r^2}{R_0^2}-1\right)\frac{w_1}{R_0^2}\right) \qquad \text{for } r < R_0. \tag{4.57}$$

Finally, an application of Lemma 4.11 shows that $\hat{w}(r,t) := r^{-\left(\frac{N-1}{2}\right)}w_1(r,t)$ is a subsolution for $W(z,t)$. Indeed, by (4.57) and (4.52), we have for $r < R_0$:

$$\frac{\partial \hat{w}}{\partial t} = r^{-\left(\frac{N-1}{2}\right)}\frac{\partial w_1}{\partial t} \leq r^{-\left(\frac{N-1}{2}\right)}D\frac{R_0^2}{R(t)^2}\left(\frac{\partial^2 w_1}{\partial r^2}+\tilde{P}(t)\left(\frac{r^2}{R_0^2}-1\right)\frac{w_1}{R_0^2}\right) \tag{4.58}$$

$$= D\frac{R_0^2}{R(t)^2}\left(r^{-\left(\frac{N-1}{2}\right)}\frac{\partial^2}{\partial r^2}\left(r^{\frac{N-1}{2}}\hat{w}\right)+\tilde{P}(t)\left(\frac{r^2}{R_0^2}-1\right)\frac{\hat{w}}{R_0^2}\right) \tag{4.59}$$

$$\leq D\frac{R_0^2}{R(t)^2}\left(\nabla^2\hat{w}+\tilde{P}(t)\left(\frac{r^2}{R_0^2}-1\right)\frac{\hat{w}}{R_0^2}\right). \tag{4.60}$$

□

4.3.4 Behaviour near the boundary $R(t) = c_* t - o(t)$

Let $R(t) = c_* t - o(t)$ as $t \to \infty$. We are interested in $|x| = R(t) - y$ with $y = O(1)$, which corresponds to $|z| = R_0 - \frac{yR_0}{R(t)}$. If $\tilde{P}(t) = \frac{\ddot{R}(t)R(t)^3}{4D^2}$ satisfies the conditions of Propositions 4.10 and 4.12, and if $\int_0^\infty \ddot{R}(\zeta)^{2/3}d\zeta < \infty$, then for $y = O(1)$ the supersolution (Proposition 4.10) and the subsolution (Proposition 4.12) provide the bounds $w(z,t) = \underline{\overline{O}}(R_0 - |z|) = \underline{\overline{O}}\left(\frac{yR_0}{R(t)}\right)$, independently of t as $t \to \infty$. Therefore we obtain the following results by using the same calculations as in the one-dimensional case, but now using equation (4.45) in place of equation (4.8).

Theorem 4.13. *Assume $N \leq 3$ and $R(t) = c_* t - \delta(t)$ where $\frac{c_*^2}{4D} = f_0$ and $\delta(t)$ satisfies* (4.29) *and* (4.31)*. Let $\psi(x,t)$ satisfy* (3.164), (3.165) *on the ball $\Omega(t) = \{x \in \mathbb{R}^N : |x| < R(t)\}$. Then for $|x| = R(t) - y$ with $y = O(1)$,*

$$\psi(x,t) = \underline{\overline{O}}\left(yt^{-1-\frac{N}{2}}\exp\left(\frac{c_*}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D}d\zeta\right)\right) \qquad \textit{as } t \to \infty. \tag{4.61}$$

Corollary 4.14. *Assume $N \leq 3$ and let $R(t) = c_* t - \alpha \log(\frac{t}{t_0} + 1) - \theta(t)$ where $\alpha > 0$ and $\theta(t)$ satisfies* (4.38)*. Then for $|x| = R(t) - y$ with $y = O(1)$,*

$$\psi(x,t) = \underline{\overline{O}}\left(yt^{-1-\frac{N}{2}+\frac{\alpha c_*}{2D}}\right) \qquad \text{as } t \to \infty. \tag{4.62}$$

In particular, the 'critical value' of α, for which the solution behaves exactly as order y, is $\alpha_{crit} = \frac{(2+N)D}{c_*}$.

Remark 4.15. *Lemma 4.11, which is used in the derivation of the subsolution, only holds for $N \leq 3$, so these are the values for which Corollary 4.14 has been proved. However it is possible that the 'critical radius' $R(t)$ may have the same form as in Corollary 4.14 for all N. The supersolution is valid for all N, and shows that for $y = O(1)$ and $|x| = R(t) - y$ with $R(t)$ as in Corollary 4.14, $\psi(x,t) = O(yt^{-1-\frac{N}{2}+\frac{\alpha c_*}{2D}})$ as $t \to \infty$.*

Remark 4.16. *Observe that, as in the one-dimensional case, the logarithmic correction in the 'critical' $R(t)$ in dimension $N \leq 3$ matches the front position $|x| = c_* t - \frac{(2+N)D}{c_*} \log \frac{t}{t_0} + O(1)$ for the nonlinear KPP problem on $\mathbb{R}^N$ with compactly supported initial conditions; see equation* (1.13)*.*

4.4 Critical super- and subsolutions on a box

4.4.1 Equation and change of variables

Here we consider the linear problem on a time-dependent box in $\mathbb{R}^{N+1}$,

$$\frac{\partial \psi}{\partial t} = D\nabla^2 \psi + f_0 \psi \qquad \text{in } \Omega(t) \subset \mathbb{R}^{N+1} \tag{4.63}$$

$$\psi(x,t) = 0 \qquad \text{on } \partial\Omega(t), \tag{4.64}$$

where

$$\Omega(t) = \Big\{ x \in \mathbb{R}^{N+1} : A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N, \\ \frac{-L_{N+1}(t)}{2} < x_{N+1} < \frac{L_{N+1}(t)}{2} \Big\}. \tag{4.65}$$

The time-dependent positions $A_j(t)$ and lengths $L_j(t)$ for $1 \leq j \leq N$ are prescribed, with $L_j(t) > 0$ for all $0 \leq t < \infty$. In dimension $N+1$, the domain is symmetric: $\frac{-L_{N+1}(t)}{2} < x_{N+1} < \frac{L_{N+1}(t)}{2}$, and we consider the following problem. Given $A_j(t)$, $L_j(t)$, and a (possibly time-dependent) position $x_j(t)$ for $1 \leq j \leq N$, we would like to choose $L_{N+1}(t)$ in such a way that for $0 < y = O(1)$,

$$\psi\left(x_1(t), \ldots, x_N(t), \frac{-L_{N+1}(t)}{2} + y, t\right) = \underline{\overline{O}}(y) \text{ as } t \to \infty. \tag{4.66}$$

In other words we would like to choose $L_{N+1}(t)$ so that, at these given positions $x_j(t)$ in the 'cross-section', ψ remains exactly of order one for x_{N+1} at an order one distance from the boundary $x_{N+1} = \frac{-L_{N+1}(t)}{2}$. Our approach will be as follows. We begin by applying the results about separable sub- and supersolutions on the box, in order to reduce the problem to a one-dimensional problem in x_{N+1}. Then, we use the results from Section 4.2 about the behaviour near the boundary in the case of an interval.

First we change variables from x_j to $\xi_j = \left(\frac{x_j - A_j(t)}{L_j(t)}\right) L_0$ for $1 \leq j \leq N$, and from x_{N+1} to $\xi_{N+1} = \left(\frac{x_{N+1}}{L_{N+1}(t)} + \frac{1}{2}\right) L_0$, and write $\psi(x,t) = u(\xi, \xi_{N+1}, t)$ where $\xi = (\xi_1, \ldots, \xi_N)$. Then let $w(\xi, \xi_{N+1}, t) = e^{-f_0 t} u(\xi, \xi_{N+1}, t) \prod_{j=1}^{N+1} H_j(\xi_j, t)$ where $H_j(\xi_j, t)$ is given in equation (3.131) and, similarly,

$$H_{N+1}(\xi_{N+1}, t) = \left(\frac{L_{N+1}(t)}{L_{N+1}(0)}\right)^{1/2} \exp\left(\int_0^t \frac{\dot{L}_{N+1}(\zeta)^2}{16D} d\zeta\right) \times \exp\left(\frac{\dot{L}_{N+1}(t) L_{N+1}(t)}{4D} \frac{\xi_{N+1}}{L_0}\left(\frac{\xi_{N+1}}{L_0} - 1\right)\right). \tag{4.67}$$

We know that, for $\xi \in (0, L_0)^N$ and $\xi_{N+1} \in (0, L_0)$, $w(\xi, \xi_{N+1}, t)$ satisfies:

$$\frac{\partial w}{\partial t} = D \sum_{j=1}^{N+1} \frac{L_0^2}{L_j(t)^2} \frac{\partial^2 w}{\partial \xi_j^2} + \sum_{j=1}^{N} \left(\frac{\xi_j^2 \ddot{L}_j(t) L_j(t)}{4DL_0^2} + \frac{\xi_j \ddot{A}_j(t) L_j(t)}{2DL_0}\right) w + \frac{\ddot{L}_{N+1}(t) L_{N+1}(t)}{4D} \frac{\xi_{N+1}}{L_0}\left(\frac{\xi_{N+1}}{L_0} - 1\right) w \tag{4.68}$$

$$w(\xi, \xi_{N+1}, t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0 \quad (1 \leq j \leq N+1). \tag{4.69}$$

4.4.2 Supersolutions and subsolutions

Given constants $\gamma_0^{(j)}$ and $\gamma_1^{(j)}$ for $1 \le j \le N$, define $\overline{Q}_j(t)$ and $\underline{Q}_j(t)$ as in (3.137), (3.138), and let $g_{j,1}$ and $\sigma_{j,1}$ be the principal eigenfunction and eigenvalue of (3.19), (3.20) with $\gamma_0 = \gamma_0^{(j)}$ and $\gamma_1 = \gamma_1^{(j)}$. The following result follows from the comparison principle, in the same way as Theorem 3.24.

Proposition 4.17. *Let w satisfy (4.68), (4.69) and assume that*

$$b_1 w_0(\xi_{N+1}) \prod_{j=1}^{N} g_{j,1}(\xi_j) \le w(\xi, \xi_{N+1}, 0) \le b_2 w_0(\xi_{N+1}) \prod_{j=1}^{N} g_{j,1}(\xi_j), \tag{4.70}$$

for some $0 < b_1 \le b_2$ and non-negative function w_0. Let $w_*(\xi_{N+1}, t)$ satisfy

$$\frac{\partial w_*}{\partial t} = D\frac{L_0^2}{L_{N+1}(t)^2}\left(\frac{\partial^2 w_*}{\partial \xi_{N+1}^2} + P(t)\frac{\xi_{N+1}}{L_0}\left(\frac{\xi_{N+1}}{L_0} - 1\right)\frac{w_*}{L_0^2}\right) \quad \text{for } 0 < \xi_{N+1} < L_0 \tag{4.71}$$

$$w_*(\xi_{N+1}, t) = 0 \qquad \text{at } \xi_{N+1} = 0 \text{ and } \xi_{N+1} = L_0, \tag{4.72}$$

where

$$P(t) = \frac{\ddot{L}_{N+1}(t) L_{N+1}(t)^3}{4D^2}, \tag{4.73}$$

and with initial conditions $w_*(\xi_{N+1}, 0) = w_0(\xi_{N+1})$. Then, for all $t \ge 0$,

$$\begin{aligned} &b_1 w_*(\xi_{N+1}, t) \exp\left(\int_0^t \sum_{j=1}^{N} \left(\frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} - \frac{\underline{Q}_j(\zeta)}{2D}\right) d\zeta\right) \prod_{j=1}^{N} g_{j,1}(\xi_j) \\ &\le w(\xi, \xi_{N+1}, t) \le b_2 w_*(\xi_{N+1}, t) \exp\left(\int_0^t \sum_{j=1}^{N} \left(\frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} + \frac{\overline{Q_j}(\zeta)}{2D}\right) d\zeta\right) \prod_{j=1}^{N} g_{j,1}(\xi_j). \end{aligned} \tag{4.74}$$

In particular, if $\overline{Q_j}$, $\underline{Q_j}$ satisfy (3.140), then

$$w(\xi, \xi_{N+1}, t) = \underline{\overline{O}}\left(w_*(\xi_{N+1}, t) \exp\left(\int_0^t \sum_{j=1}^{N} \frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} d\zeta\right) \prod_{j=1}^{N} g_{j,1}(\xi_j)\right). \tag{4.75}$$

Equation (4.75) gives the exact order of the solution $w(\xi, \xi_{N+1}, t)$ in terms of the function $w_*(\xi_{N+1}, t)$. The other factors are all known, depending only on $A_j(t)$ and $L_j(t)$ in dimensions $1 \le j \le N$. We will combine this with bounds on w_*, which we have from Propositions 4.2 and 4.3 in Section 4.2, and which are summarised in the next proposition.

Proposition 4.18. *Let* $w_*(\xi_{N+1},t)$ *be as in Proposition 4.17 and* $P(t)$ *as defined in equation* (4.73). *If* $0 < P(t) \to \infty$ *as* $t \to \infty$ *and* $\dot{P}(t) \geq 0$, *then there are positive constants* $\hat{C}_1$, $\hat{C}_2$ *such that*

$$\hat{C}_1 \underline{w}(\xi_{N+1},t) \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \left(\frac{\ddot{L}_{N+1}(\zeta)^2}{16D}\right)^{1/3} d\zeta\right) \leq w_*(\xi_{N+1},t) \leq \hat{C}_2 \sin\left(\frac{\pi \xi_{N+1}}{L_0}\right) \exp\left(-\int_0^t \frac{D\pi^2}{L_{N+1}(\zeta)^2} d\zeta\right) \tag{4.76}$$

where $\underline{w}$ *is given by equation* (4.15). *Therefore there exist positive constants* b_1, b_2 *such that for* $0 \leq \xi_{N+1} \leq -\kappa_1 P(t)^{-1/3} L_0$,

$$b_1 \xi_{N+1} \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \left(\frac{\ddot{L}_{N+1}(\zeta)^2}{16D}\right)^{1/3} d\zeta\right) \leq w_*(\xi_{N+1},t) \leq b_2 \xi_{N+1} \exp\left(-\int_0^t \frac{D\pi^2}{L_{N+1}(\zeta)^2} d\zeta\right). \tag{4.77}$$

Moreover, if

$$\int_0^\infty \ddot{L}_{N+1}(\zeta)^{2/3} d\zeta < \infty \tag{4.78}$$

then $w_*(\xi_{N+1},t) = \underline{O}(\xi_{N+1})$ *as* $t \to \infty$ *for* $0 \leq \xi \leq -\kappa_1 P(t)^{-1/3} L_0$.

If (3.140) and (4.78) are satisfied simultaneously then we know the exact order of $w(\xi,\xi_{N+1},t)$ (and hence $u(\xi,\xi_{N+1},t)$) for $0 \leq \xi_{N+1} \leq -\kappa_1 P(t)^{-1/3} L_0$. We are then ready to look for $L_{N+1}(t)$ such that (4.66) holds.

4.4.3 Behaviour near the boundary $\frac{-L_{N+1}(t)}{2} = -Ct + o(t)$

Note that the given positions $x_j(t)$ in the cross-section correspond to some $\xi_j(t) \in (0, L_0)$, and we shall assume that for $1 \leq j \leq N$, $L_j(t)$ and $A_j(t)$ and $\xi_j(t)$ are such that (3.153) holds. Recall the definition of $H_j(\xi_j,t)$ in equation (3.131), and note that (3.153) means precisely that

$$\frac{e^{f_0 t} \exp\left(\int_0^t \sum_{j=1}^N \frac{\sigma_{j,1} L_0^2}{L_j(\zeta)^2} d\zeta\right) \prod_{j=1}^N g_{j,1}(\xi_j(t))}{\prod_{j=1}^N H_j(\xi_j(t),t)} = \exp\left(\frac{C^2}{4D} t - p(t)\right). \tag{4.79}$$

We now give a multi-dimensional version of Theorem 4.4.

Theorem 4.19. *Assume that for $1 \le j \le N$, $L_j(t)$ and $A_j(t)$ and $\xi_j(t)$ are such that* (3.153) *holds for some $C > 0$ and $p(t) = o(t)$ as $t \to \infty$. Let $L_{N+1}(t) = 2(Ct - \delta(t))$ where*

$$\begin{aligned}
&\delta(t) = o(t), \quad \dot{\delta}(t) = o(1), \quad \ddot{\delta}(t) = o(1), \\
&0 < -\ddot{\delta}(t)(Ct - \delta(t))^3 \text{ is increasing and tends to } \infty, \quad \text{as } t \to \infty.
\end{aligned} \tag{4.80}$$

Let ψ satisfy (4.63), (4.64)*. Then for each $y_0 > 0$ there exist $\beta_0 > 0$, $\beta_1 > 0$ (depending on the initial conditions) such that for $0 \le y \le y_0$ and $t \to \infty$,*

$$\begin{aligned}
\beta_0 y t^{-\frac{3}{2}} &\exp\left(\frac{C}{2D}\delta(t) - p(t) - \int_0^t \left(\frac{\dot{\delta}(\zeta)^2}{4D} + \sum_{j=1}^N \frac{\underline{Q_j}(\zeta)}{2D}\right) d\zeta\right) \\
&\times \exp\left(\int_0^t \left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right)\left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}} d\zeta\right) \\
&\le \psi\left(x_1(t), \ldots, x_N(t), \frac{-L_{N+1}(t)}{2} + y, t\right) \\
&\le \beta_1 y t^{-\frac{3}{2}} \exp\left(\frac{C}{2D}\delta(t) - p(t) + \int_0^t \left(-\frac{\dot{\delta}(\zeta)^2}{4D} + \sum_{j=1}^N \frac{\overline{Q_j}(\zeta)}{2D}\right) d\zeta\right).
\end{aligned} \tag{4.81}$$

In particular, if $L_j(t)$ and $A_j(t)$ are such that (3.140) *also holds, and if $\delta(t)$ is such that* (4.31) *is also satisfied, then as $t \to \infty$, for $y = O(1)$,*

$$\psi\left(x_1(t), \ldots, x_N(t), \frac{-L_{N+1}(t)}{2} + y, t\right) = \overline{\underline{O}}\left(y t^{-\frac{3}{2}} e^{\frac{C}{2D}\delta(t) - p(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta}\right). \tag{4.82}$$

Proof. Without loss of generality, we may assume that $w(\xi, \xi_{N+1}, 0)$ satisfies (4.70). Then let $w_*(\xi_{N+1}, t)$ be as in Proposition 4.17 and let $P(t)$ be as defined in equation (4.73). By Proposition 4.17, $w(\xi, \xi_{N+1}, t)$ is related to $w_*(\xi_{N+1}, t)$ by the inequalities in (4.74).

Now the assumptions on $\delta(t)$ imply that $0 < P(t) \to \infty$ as $t \to \infty$ and $\dot{P}(t) \ge 0$ so we may apply Proposition 4.18 to w_*. Consider $x_{N+1} = \frac{-L_{N+1}(t)}{2} + y$, which corresponds to $\xi_{N+1} = \frac{yL_0}{L_{N+1}(t)}$. Noting that $P(t) = o(t^3)$ as $t \to \infty$, we have for $0 \le y \le y_0$ that $\xi_{N+1} = \frac{yL_0}{L_{N+1}(t)} = \overline{\underline{O}}\left(\frac{y}{t}\right) = o(P(t)^{-1/3})$ as $t \to \infty$. We conclude from Proposition 4.18 that there are some positive constants $\hat{C}_1$, $\hat{C}_2$ such that

$$\hat{C}_1 \frac{y}{t} \exp\left(\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right) \int_0^t \left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}} d\zeta\right) \leq w_* \left(\frac{yL_0}{L(t)}, t\right) \leq \hat{C}_2 \frac{y}{t} \quad (4.83)$$

for $0 \leq y \leq y_0$, as $t \to \infty$. Also, we can use the definition of $H_{N+1}(\xi_{N+1}, t)$ in equation (4.67) to find that for $0 \leq y \leq y_0$ and $t \to \infty$,

$$\begin{aligned} H_{N+1}&\left(\frac{yL_0}{L_{N+1}(t)}, t\right) \\ &= \left(\frac{L_{N+1}(t)}{L_{N+1}(0)}\right)^{1/2} \exp\left(\int_0^t \frac{\dot{L}_{N+1}(\zeta)^2}{16D} d\zeta + \frac{\dot{L}_{N+1}(t) y}{4D}\left(\frac{y}{L_{N+1}(t)} - 1\right)\right) \end{aligned} \quad (4.84)$$

$$= \underline{\overline{O}}\left(t^{1/2} \exp\left(\int_0^t \frac{\dot{L}_{N+1}(\zeta)^2}{16D} d\zeta\right)\right) \quad (4.85)$$

$$= \underline{\overline{O}}\left(t^{1/2} \exp\left(\frac{C^2}{4D} t - \frac{C}{2D}\delta(t) + \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta\right)\right). \quad (4.86)$$

Now it just remains to combine all of these bounds. First, substitute (4.83) into (4.74) to get upper and lower bounds on $w\left(\xi_1, \ldots, \xi_N, \frac{yL_0}{L_{N+1}(t)}, t\right)$. Then combine this with equations (4.79), (4.86), and the fact that

$$u\left(\xi_1(t), \ldots, \xi_N(t), \frac{yL_0}{L_{N+1}(t)}, t\right) = \frac{e^{f_0 t} w\left(\xi_1(t), \ldots, \xi_N(t), \frac{yL_0}{L_{N+1}(t)}, t\right)}{H_{N+1}\left(\frac{yL_0}{L_{N+1}(t)}, t\right) \prod_{j=1}^N H_j(\xi_j(t), t)}. \quad (4.87)$$

□

Remark 4.20. *If* (4.82) *holds then this also implies that as* $t \to \infty$

$$\frac{\partial \psi}{\partial x_{N+1}}\left(x_1(t), \ldots, x_N(t), \frac{-L_{N+1}(t)}{2}, t\right) = \underline{\overline{O}}\left(t^{-\frac{3}{2}} e^{\frac{C}{2D}\delta(t) - p(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta}\right). \quad (4.88)$$

Remark 4.21. *Suppose that* $\delta(t)$ *satisfies* (4.80)*, but that either or both of the conditions* (4.31) *(on* $\ddot{\delta}$*) or* (3.140) *(on* $\overline{Q_j}$*,* $\underline{Q_j}$*) does not hold. Then Theorem*

4.19 still provides upper and lower bounds on the solution near the boundary, however the upper bound is bigger than the lower bound by a factor of order

$$\exp\left(\int_0^t\left(\sum_{j=1}^N \frac{\overline{Q}_j(\zeta)+\underline{Q}_j(\zeta)}{2D}-\left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)}+\kappa_1\right)\left(\frac{\ddot{\delta}(\zeta)^2}{4D}\right)^{\frac{1}{3}}\right)d\zeta\right) \tag{4.89}$$

and this is not $O(1)$ as $t\to\infty$.

The following corollary to Theorem 4.19 gives conditions on $L_{N+1}(t)$ such that (4.66) holds. We shall refer to this as the 'critical choice' of $L_{N+1}(t)$.

Corollary 4.22. *Assume that for $1\le j\le N$, $L_j(t)$, $A_j(t)$ and $\xi_j(t)$ are such that* (3.140) *and* (3.153) *hold. Suppose there exists a function $\delta(t)$ satisfying* (4.80) *and* (4.31), *and such that also*

$$\frac{C}{2D}\delta(t)-\int_0^t\frac{\dot{\delta}(\zeta)^2}{4D}d\zeta=p(t)+\frac{3}{2}\log\frac{t}{t_0}+O(1)\qquad \text{as } t\to\infty. \tag{4.90}$$

Let $L_{N+1}(t)=2(Ct-\delta(t))$. Then for $y=O(1)$ the solution ψ to (4.63), (4.64) *satisfies*

$$\psi\left(x_1(t),\dots,x_N(t),\frac{-L_{N+1}(t)}{2}+y,t\right)=\underline{O}(y)\ \text{and}$$
$$\frac{\partial\psi}{\partial x_{N+1}}\left(x_1(t),\dots,x_N(t),\frac{-L_{N+1}(t)}{2},t\right)=\underline{O}(1)\qquad \text{as } t\to\infty. \tag{4.91}$$

Remark 4.23. *In general C and $p(t)$ in equation* (3.153) *will depend on the points $\xi_j(t)\in(0,L_0)$. However, suppose* (3.155) *holds for every $1\le j\le N$, and let $C>0$ and $p(t)$ satisfy* (3.156) *with $p(t)=o(t)$ as $t\to\infty$. If there exists a function $\delta(t)$ satisfying* (4.80) *and* (4.31), *and such that also* (4.90) *holds, then the choice $L_{N+1}(t)=2(Ct-\delta(t))$ will have the required property* (4.91) *uniformly for $\xi(t)$ in any compact subset of $(0,L_0)^N$.*

In Section 4.4.4 we shall give examples of time-dependent boxes where we can apply Theorem 4.19 and Corollary 4.22.

Before moving on to examples, we recall that in the preceding sections (Section 4.2 and Remark 4.8 for the one-dimensional case, and in Section 4.3 and

Remark 4.16 for the N-dimensional case), we observed a relationship between our 'critical' choices of boundary motion and known results from [16, 15, 36, 33, 27] relating to the front position for the KPP equation. We noted that there appears to be a correspondence between (i) the 'critical' choice of boundary motion, such that the solution to the linear equation on a symmetric time-dependent domain with zero Dirichlet conditions remains of order one at an order one distance from the boundary, and (ii) the front positions for the solution to the nonlinear KPP problem on the unbounded domain with compactly supported initial conditions.

As we consider our results on the time-dependent box, it is natural to wonder whether the same correspondence will again be true. Let $A_j(t)$ and $L_j(t) > 0$ (the time-dependent 'cross-section') be prescribed for $1 \leq j \leq N$, $0 \leq t < \infty$. In several cases, using Corollary 4.22 we can give an expression for a 'critical' choice $L_{N+1}(t) = L^*(t) = 2(Ct - \delta(t))$ such that the property (4.91) holds uniformly in compact subsets of ξ_j. In such cases, one may conjecture that the solution to the nonlinear equation on the unbounded domain

$$\left\{x \in \mathbb{R}^{N+1} : A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N,\ -\infty < x_{N+1} < \infty\right\} \tag{4.92}$$

with zero Dirichlet conditions on the boundaries of the cylinder, and with compactly supported initial conditions, may converge to a travelling wave in an appropriate sense, and may have asymptotic front positions at

$$x_{N+1} = \pm\frac{L^*(t)}{2} + O(1) \text{ as } t \to \infty. \tag{4.93}$$

4.4.4 Examples

Let us now provide examples for which we can use Theorem 4.19 to derive bounds on the solution, and Corollary 4.22 to give a 'critical' choice of $L_{N+1}(t)$. To keep the notation simple, we give the following corollaries in the two-dimensional case. The higher dimensional versions use the same calculations.

Corollary 4.24. *Let $\Omega(t)$ be a two-dimensional box of the type* (4.65) *with* $L_1(t) = a(t+t_0)^k$ *and* $A_1(t) = b(t+t_1)^m$*, where* $a > 0$, $b \in \mathbb{R}$, $\frac{1}{4} < k < \frac{1}{2}$ *and* $m < 1-k$.

1. *If* $\frac{1}{4} < k < \frac{1}{2}$ *and* $m < \frac{1}{2}$*, let* $L_2(t) = 2(c_* t - \delta(t))$ *with*
$$\delta(t) = \frac{2D^2\pi^2}{c_* a^2(1-2k)}(t+t_0)^{1-2k} + \frac{(k+3)D}{c_*}\log\left(\frac{t}{t_0}+1\right) + \delta_0. \quad (4.94)$$

2. *If* $\frac{1}{4} < k < \frac{1}{2}$ *and* $m = \frac{1}{2}$*, let* $L_2(t) = 2(c_* t - \delta(t))$ *with*
$$\delta(t) = \frac{2D^2\pi^2}{c_* a^2(1-2k)}(t+t_0)^{1-2k} + \left(k+3+\frac{b^2}{8D}\right)\frac{D}{c_*}\log\left(\frac{t}{t_0}+1\right) + \delta_0. \quad (4.95)$$

3. *If* $\frac{1}{4} < k < \frac{1}{2}$ *and* $\frac{1}{2} < m < 1-k$*, let* $L_2(t) = 2(c_* t - \delta(t))$ *with*
$$\begin{aligned}\delta(t) =& \frac{2D^2\pi^2}{c_* a^2(1-2k)}(t+t_0)^{1-2k} + \frac{b^2 m^2}{2c_*(2m-1)}(t+t_0)^{2m-1}\\ &+ \frac{(k+3)D}{c_*}\log\left(\frac{t}{t_0}+1\right) + \delta_0.\end{aligned} \quad (4.96)$$

Then the solution ψ to (4.63), (4.64) *will have the required property* (4.91) *as* $t \to \infty$*, uniformly for ξ_1 in any compact subset of* $(0, L_0)$.

Proof. The conditions $k < \frac{1}{2}$ and $m < 1-k$ ensure that $\ddot{L}_1(t)L_1(t)$ and $\ddot{A}_1(t)L_1(t)$ are integrable. Hence, taking $\gamma_0^{(1)} = 0$ and $\gamma_1^{(1)} = 0$, $\underline{Q}_1(t)$ and $\overline{Q}_1(t)$ are integrable (i.e. (3.140) holds). These conditions also ensure that (3.155) holds, and so we seek a choice of $L_2(t)$ that will give the required property uniformly for ξ_1 in compact subsets. Consider the terms in (3.156). As $t \to \infty$,
$$\int_0^t \frac{1}{L_1(\zeta)^2}d\zeta = \frac{1}{a^2(1-2k)}t^{1-2k} + O(1) \quad (4.97)$$
$$\int_0^t \frac{\dot{A}_1(\zeta)^2}{4D}d\zeta = \begin{cases} O(1) & \text{if } m < \frac{1}{2} \\ \frac{b^2}{16D}\log\frac{t}{t_0} + O(1) & \text{if } m = \frac{1}{2} \\ \frac{m^2 b^2}{4D(2m-1)}t^{2m-1} + O(1) & \text{if } \frac{1}{2} < m \end{cases} \quad (4.98)$$

and $\frac{1}{2}\log\frac{L_1(t)}{L_1(0)} = \frac{k}{2}\log\frac{t}{t_0} + O(1)$. Therefore (3.156) holds with $C = c_*$ and

$$p(t) = \frac{D\pi^2}{a^2(1-2k)}t^{1-2k} + \int_0^t \frac{\dot{A}_1(\zeta)^2}{4D}d\zeta + \frac{k}{2}\log\left(\frac{t}{t_0}\right) + O(1) \quad \text{as } t \to \infty, \tag{4.99}$$

with the second term given by equation (4.98). Note that the assumptions $2m-1 < 1-2k$ and $k > \frac{1}{4}$ imply that $\int_0^t \frac{\dot{\delta}(\zeta)^2}{4D}d\zeta = O(1)$, for each of the cases in equations (4.94), (4.95), (4.96). Then it is straightforward to see that $\delta(t)$ satisfies (4.90) in each case. Finally, $\delta(t)$ also satisfies the conditions (4.80) and (4.31). So by Corollary 4.22, $L_2(t) = 2(c_* t - \delta(t))$ has the required property. □

Example 4.25. *For some $a > 0$, let $L_1(t) = a(t+t_0)^{\frac{1}{3}}$ and $A_1(t) \equiv \frac{-L_1(t)}{2}$. By Corollary 4.24, the choice*

$$L_2(t) = 2\left(c_* t - \frac{6D^2\pi^2}{c_* a^2}(t+t_0)^{\frac{1}{3}} - \frac{10D}{3c_*}\log\left(\frac{t}{t_0}+1\right) - \delta_0\right) \tag{4.100}$$

will give the property (4.91) as $t \to \infty$, uniformly for ξ_1 in any compact subset of $(0, L_0)$. Moreover, this is the same 'critical' choice of $L_2(t)$ as if $A_1(t)$ were a constant, or as if $A_1(t) = b(t+t_1)^m$ for any $b \in \mathbb{R}$ and $m < \frac{1}{2}$.

Corollary 4.24 assumes that $L_1(t)$ is of order t^k and $A_1(t)$ is of order t^m as $t \to \infty$, with $\frac{1}{4} < k < \frac{1}{2}$ and $m < 1-k$. Next we treat the case with $k = m = \frac{1}{2}$.

Corollary 4.26. *Let $\Omega(t)$ be a two-dimensional box of the type (4.65) with $L_1(t) = \sqrt{l^2 + 2\rho t}$ for some $\rho > 0$, and $A_1(t) \equiv -\frac{\gamma_1}{\rho^2}L_1(t)$. Let σ_1 be the principal eigenvalue of the Sturm-Liouville problem*

$$Dg''(\xi) + \left(-\frac{\rho^2\xi^2}{4DL_0^4} + \frac{\gamma_1\xi}{2DL_0^3}\right)g(\xi) = \sigma g(\xi), \qquad g(0) = g(L_0) = 0 \tag{4.101}$$

and let $\theta(t) = O(1)$ satisfy (4.38). Then the choice $L_2(t) = 2(c_* t - \delta(t))$ with

$$\delta(t) = \frac{2D}{c_*}\left(-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} + \frac{3}{2}\right)\log\left(\frac{t}{t_0}+1\right) + \theta(t) \tag{4.102}$$

will have the required property (4.91) uniformly for ξ_1 in any compact subset of $(0, L_0)$. The coefficient of the logarithmic term in $\delta(t)$ is greater than $\frac{4D}{c_*}$.

Proof. This is a separable case: $\underline{Q}_1(t) \equiv \overline{Q}_1(t) \equiv 0$ for $\gamma_0^{(1)} = -\rho^2$, $\gamma_1^{(1)} = \gamma_1$. Clearly also $\dot{L}_1(t)L_1(t) \equiv \rho$ and $\dot{A}_1(t)L_1(t) \equiv -\frac{\gamma_1}{\rho}$ are $O(1)$. Consider the terms in (3.156). As $t \to \infty$,

$$\int_0^t \frac{L_0^2}{L_1(\zeta)^2} d\zeta = \frac{L_0^2}{2\rho} \log\left(\frac{t}{t_0}\right) + O(1), \tag{4.103}$$

$$\int_0^t \frac{\dot{A}_1(\zeta)^2}{4D} d\zeta = \frac{\gamma_1^2}{8\rho^3 D} \log\left(\frac{t}{t_0}\right) + O(1), \tag{4.104}$$

and $\frac{1}{2}\log\frac{L_1(t)}{L_1(0)} = \frac{1}{4}\log\frac{t}{t_0} + O(1)$ for large t. So equation (3.156) holds with $C = c_*$ and

$$p(t) = \left(-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4}\right)\log\left(\frac{t}{t_0}\right) + O(1) \qquad \text{as } t \to \infty, \tag{4.105}$$

where σ_1 is the principal eigenvalue of (4.101). By Corollary 4.22, we see that the choice $L_2(t) = 2(c_*t - \delta(t))$ with $\delta(t)$ as in (4.102) will have the required property (4.91), uniformly in compact subsets of ξ_1. The fact that the coefficient of the logarithmic term is greater than $\frac{4D}{c_*}$ is due to Proposition 3.4 which implies that $-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} > \frac{1}{2}$. □

The case $k = m = 1$ is also a separable case and we have the following result.

Corollary 4.27. *Let $\Omega(t)$ be a box of the type (4.65) with $L_j(t) = l_j + \alpha_j t$ and $A_j(t) = c_j t$, for some $l_j > 0$, $\alpha_j > 0$, $c_j \in \mathbb{R}$, for each $1 \le j \le N$. Consider the solution to (4.63), (4.64) at fixed positions $\frac{\xi_j}{L_0} = \eta_j \in (0,1)$ such that $f_0 - \sum_{j=1}^N \frac{(c_j+\alpha_j\eta_j)^2}{4D} > 0$. Let C be the positive solution to*

$$f_0 - \sum_{j=1}^{N} \frac{(c_j + \alpha_j \eta_j)^2}{4D} = \frac{C^2}{4D}. \tag{4.106}$$

Let $\theta(t) = O(1)$ satisfy conditions (4.38), and let

$$\frac{L_{N+1}(t)}{2} = Ct - \frac{(N+3)D}{C}\log\left(\frac{t}{t_0} + 1\right) - \theta(t). \tag{4.107}$$

Then the required property (4.91) will hold at $\frac{\xi_j}{L_0} = \eta_j$.

Proof. This is a separable case: $\underline{Q}_j(t) \equiv \overline{Q}_j(t) \equiv 0$ for $\gamma_0^{(j)} = \gamma_1^{(j)} = 0$. At $\frac{\xi_j}{L_0} = \eta_j$, equation (3.153) holds with the speed C as in (4.106) and with

$$p(t) = \sum_{j=1}^{N} \frac{1}{2} \log \frac{L_j(t)}{L_j(0)} + O(1) = \frac{N}{2} \log \frac{t}{t_0} + O(1) \quad \text{as } t \to \infty. \tag{4.108}$$

Applying Corollary 4.22 with $\delta(t) = \frac{(N+3)D}{C} \log\left(\frac{t}{t_0} + 1\right) + \theta(t)$ shows that the required property (4.91) holds when $L_{N+1}(t)$ is as in equation (4.107). □

Remark 4.28. *In Corollary 4.27, $\frac{\xi_j}{L_0} = \eta_j \in (0,1)$ corresponds to an x_j position $x_j(t) = (c_j + \alpha_j\eta_j)t + \eta_j l_j$. A constant x_j corresponds to $c_j + \alpha_j\eta_j = 0$. So if we can apply Corollary 4.27 at constant $(x_1, \ldots, x_N)$ then $C = c_*$. In this case the 'critical' choice of $L_{N+1}(t)$ from equation (4.107) is the same as the 'critical' choice of $R(t)$ for the ball in $\mathbb{R}^{N+1}$ from Corollary 4.14 (when $N+1 = 2$ or 3).*

Next we consider some cases where $L_j(t)$ and $\dot{A}_j(t)$ converge as $t \to \infty$.

Corollary 4.29. *Let $\Omega(t)$ be a two-dimensional box of the type (4.65). Assume $L_1(t) = L_\infty(1 + \varepsilon(t))$ with $L_\infty > 0$ and $\varepsilon(t) = o(t^{-\frac{1}{2}}) \to 0$ as $t \to \infty$, and such that $\dot{\varepsilon}(t) = O(1)$ and $|\ddot{\varepsilon}|$ is integrable. Suppose that either (i) $A_1(t) = A_0 + ct$ with $f_0 > \frac{c^2}{4D} + \frac{D\pi^2}{L_\infty^2}$; or (ii) $A_1(t)$ is a constant multiple of $L_1(t)$, $\dot{\varepsilon}^2$ is integrable, and $f_0 > \frac{D\pi^2}{L_\infty^2}$. Let C be the positive solution to $\frac{C^2}{4D} = f_0 - \frac{c^2}{4D} - \frac{D\pi^2}{L_\infty^2}$ in case (i); or $\frac{C^2}{4D} = f_0 - \frac{D\pi^2}{L_\infty^2}$ in case (ii).*

1. *If $\varepsilon(t) = O(\frac{1}{t^\beta})$ as $t \to \infty$ for some $\beta > 1$, let $L_2(t) = 2(Ct - \delta(t))$ with*
$$\delta(t) = \frac{3D}{C} \log\left(\frac{t}{t_0} + 1\right) + \delta_0. \tag{4.109}$$

2. *If $\varepsilon(t) = -\frac{\alpha}{t} + O(\frac{1}{t^\beta})$ as $t \to \infty$ for some $\beta > 1$ and $\frac{3}{2} + \frac{2D\pi^2\alpha}{L_\infty^2} > 0$, let $L_2(t) = 2(Ct - \delta(t))$ with*
$$\delta(t) = \left(3 + \frac{4D\pi^2\alpha}{L_\infty^2}\right) \frac{D}{C} \log\left(\frac{t}{t_0} + 1\right) + \delta_0. \tag{4.110}$$

3. *If $\varepsilon(t) = -\frac{\alpha}{t^k} + O(\frac{1}{t^\beta})$ as $t \to \infty$ for some $\beta > 1$ with $\frac{1}{2} < k < 1$ and $\alpha > 0$, let $L_2(t) = 2(Ct - \delta(t))$ with*
$$\delta(t) = \frac{4D^2\pi^2\alpha}{C(1-k)L_\infty^2}(t + t_0)^{1-k} + \frac{3D}{C} \log\left(\frac{t}{t_0} + 1\right) + \delta_0. \tag{4.111}$$

Then the solution to (4.63), (4.64) *will have the required property* (4.91) *as* $t \to \infty$, *uniformly for* ξ_1 *in any compact subset of* $(0, L_0)$.

Proof. The conditions (3.140) and (3.155) hold, and so we seek a choice of $L_2(t)$ that will give the required property uniformly for ξ_1 in any compact subset of $(0, L_0)$. Equation (3.156) holds with speed C and we calculate that as $t \to \infty$,

$$\int_0^t \frac{1}{L_1(\zeta)^2} d\zeta = \frac{1}{L_\infty^2} \int_0^t (1+\varepsilon(\zeta))^{-2} d\zeta = \begin{cases} \frac{1}{L_\infty^2} t + O(1) & \text{in case 1} \\ \frac{1}{L_\infty^2} t + \frac{2\alpha}{L_\infty^2} \log \frac{t}{t_0} + O(1) & \text{in case 2} \\ \frac{1}{L_\infty^2} t + \frac{2\alpha}{1-k} t^{1-k} + O(1) & \text{in case 3.} \end{cases} \tag{4.112}$$

In each of the cases 1, 2, 3, it is then straightforward to see that the stated form of $\delta(t)$ satisfies equation (4.90) and also satisfies the conditions (4.80) and (4.31). Therefore by Corollary 4.22, $L_2(t) = 2(Ct - \delta(t))$ has the required property. □

Remark 4.30. *Case 1 of Corollary 4.29 shows that if* $\varepsilon(t) = O(\frac{1}{t^\beta})$ *as* $t \to \infty$ *with* $\beta > 1$, *then we can choose the same form of* $L_2(t)$ *to give the required property* (4.91) *as we would if* $L_1(t) \equiv L_\infty$.

Remark 4.31. *Similar calculations to those in Corollary 4.29 can also be used to find* C *and* $p(t)$ *in cases where, for each* $1 \leq j \leq N$, *both* $L_j(t)$ *and* $\dot{A}_j(t)$ *depend on time but converge as* $t \to \infty$. *The* $o(1)$ *corrections to* $L_j(t)$ *and* $\dot{A}_j(t)$ *do not affect the speed* C, *but they do affect the sublinear term* $p(t)$. *This follows from similar calculations to those in Corollary 4.29, together with the fact that if* $\dot{A}_1(t) = c_\infty + c\hat{\varepsilon}(t)$ *for some* $c_\infty, c \in \mathbb{R}$ *and* $\hat{\varepsilon}(t) = o(1)$ *as* $t \to \infty$, *then*

$$\int_0^t \frac{\dot{A}_1(\zeta)^2}{4D} d\zeta - \frac{c_\infty^2}{4D} t = \frac{c_\infty c}{2D} \int_0^t \hat{\varepsilon}(\zeta) d\zeta + \frac{c^2}{4D} \int_0^t \hat{\varepsilon}(\zeta)^2 d\zeta = o(t). \tag{4.113}$$

Example 4.32. *Let* $\Omega(t) \subset \mathbb{R}^2$ *be a box of the type* (4.65) *with* $L_1(t) \equiv L_0$, $A_1(t) = b\sqrt{t + t_0}$. *Let* $C > 0$ *be the positive solution to* $\frac{C^2}{4D} = f_0 - \frac{D\pi^2}{L_0^2}$. *Using* (4.113) *and Corollary 4.22 we deduce that the solution will have the required property* (4.91) *as* $t \to \infty$, *uniformly for* ξ_1 *in any compact subset of* $(0, L_0)$, *if*

$$\frac{L_2(t)}{2} = Ct - \left(\frac{b^2}{8C} + \frac{3D}{C}\right) \log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.114}$$

Next we discuss some cases for which our upper and lower bounds on the solution near the boundary differ by a factor that is not $O(1)$. This means we cannot use the above methods to derive a form of $L_{N+1}(t)$ that gives the required property (4.91).

In Corollary 4.24 we have shown how to choose $L_2(t)$ to satisfy the property (4.91) in cases where $L_1(t)$ is of order t^k and $A_1(t)$ is of order t^m as $t \to \infty$ with $\frac{1}{4} < k < \frac{1}{2}$ and $m < 1 - k$. If instead $0 < k \leq \frac{1}{4}$, then we can still calculate C and $p(t)$ by similar calculations to those in Corollary 4.24. However, if we choose $\delta(t)$ to satisfy equation (4.90), then condition (4.31) does not hold. This means that Theorem 4.19 gives upper and lower bounds for the solution near the boundary that differ by a factor that is not $O(1)$. To illustrate this, we give the following example where $k = \frac{1}{4}$. This leads to $p(t)$ exactly of order $t^{\frac{1}{2}}$ as $t \to \infty$, and both $\int_0^t \dot{\delta}(\zeta)^2 d\zeta$ and $\int_0^t (-\ddot{\delta}(\zeta))^{2/3} d\zeta$ are of order $\log \frac{t}{t_0}$ for large t.

Example 4.33. *Let $\Omega(t)$ be a two-dimensional box of the type* (4.65). *Assume that $L_1(t) = a(t + t_0)^{\frac{1}{4}}$ and $A_1(t) = b(t + t_1)^m$ with $a > 0$, $b \in \mathbb{R}$, $m < \frac{1}{2}$. Then exactly as in Corollary 4.24, equation* (3.156) *holds with $C = c_*$ and*

$$p(t) = \frac{2D\pi^2}{a^2} t^{\frac{1}{2}} + \frac{1}{8} \log \frac{t}{t_0} + O(1) \quad \text{as } t \to \infty. \tag{4.115}$$

Let us define

$$B = \frac{4D^2\pi^2}{c_* a^2} \tag{4.116}$$

and take $L_2(t) = 2(c_* t - \delta(t))$ with

$$\delta(t) = B(t + t_0)^{\frac{1}{2}} + \alpha \log \left(\frac{t}{t_0} + 1 \right) + \delta_0 \tag{4.117}$$

for a value of α to be chosen. Then $\delta(t)$ satisfies the conditions (4.80). *We apply Theorem 4.19 to deduce that for $\xi_1(t)$ in a given compact subset of $(0, L_0)$, and for $0 \leq y \leq y_0$, there are constants $\beta_0 > 0$, $\beta_1 > 0$ such that as $t \to \infty$,*

$$\beta_0 y t^{-3/2} \exp\left(\frac{c_*\delta(t)}{2D} - p(t) + \int_0^t \left(-\frac{\dot\delta(\zeta)^2}{4D} + \left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right)\left(\frac{\ddot\delta(\zeta)^2}{4D}\right)^{\frac{1}{3}}\right) d\zeta\right)$$
$$\leq \psi\left(x_1(t), \frac{-L_2(t)}{2} + y, t\right) \leq \beta_1 y t^{-3/2} \exp\left(\frac{c_*\delta(t)}{2D} - p(t) - \int_0^t \frac{\dot\delta(\zeta)^2}{4D} d\zeta\right). \tag{4.118}$$

For large t,

$$\frac{c_*\delta(t)}{2D} - p(t) = \left(\frac{c_*\alpha}{2D} - \frac{1}{8}\right)\log\frac{t}{t_0} + O(1), \tag{4.119}$$

$$\int_0^t \frac{\dot\delta(\zeta)^2}{4D} d\zeta = \frac{B^2}{16D}\log\frac{t}{t_0} + O(1), \qquad \int_0^t (-\ddot\delta(\zeta))^{2/3} d\zeta = \frac{B^{2/3}}{4^{2/3}}\log\frac{t}{t_0} + O(1). \tag{4.120}$$

Therefore from (4.118) *we deduce that if*

$$\alpha < \alpha_1 := \left(\frac{3}{2} + \frac{1}{8} + \frac{B^2}{16D}\right)\frac{2D}{c_*} \tag{4.121}$$

then $\psi\left(x_1(t), \frac{-L_2(t)}{2} + y, t\right) \to 0$ as $t \to \infty$, and that if

$$\alpha > \alpha_0 := \left(\frac{3}{2} + \frac{1}{8} + \frac{B^2}{16D} - \left(\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} + \kappa_1\right)\frac{B^{2/3}}{4^{2/3}}\right)\frac{2D}{c_*} \tag{4.122}$$

then $\psi\left(x_1(t), \frac{-L_2(t)}{2} + y, t\right) \to \infty$ as $t \to \infty$. This suggests that if there is a choice of $L_2(t) = 2(c_*t - \delta(t))$ such that (4.91) *holds, then as $t \to \infty$ it will have $\delta(t) - Bt^{\frac{1}{2}} = O(\log\frac{t}{t_0})$ with B given in equation* (4.116)*, and that $\frac{\delta(t) - Bt^{\frac{1}{2}}}{\log\frac{t}{t_0}}$ will asymptotically lie in the range $[\alpha_1, \alpha_0]$. However the above methods are not sufficient to determine the exact asymptotic behaviour.*

Remark 4.34. *If instead $L_1(t) = a(t+t_0)^k$ with $0 < k < \frac{1}{4}$ (and, again, $A_1(t) = b(t+t_1)^m$ with $m < \frac{1}{2}$) then the gap between the upper and lower bounds is even larger, since for $\delta(t)$ satisfying equation* (4.90)*, $\int_0^t (-\ddot\delta(\zeta))^{2/3} d\zeta$ is of order $t^{\frac{1}{3}(1-4k)}$ as $t \to \infty$. Suppose $0 < k < \frac{1}{4}$ and let $\delta_1(t)$ satisfy*

$$\frac{c_*\delta_1(t)}{2D} - \int_0^t \frac{\dot\delta_1(\zeta)^2}{4D} d\zeta - \frac{D\pi^2}{a^2(1-2k)}(t+t_0)^{1-2k} = o(t^{\frac{1}{3}(1-4k)}) \qquad \text{as } t \to \infty. \tag{4.123}$$

Then let $L_2(t) = 2(c_* t - \delta(t))$ *with* $\delta(t) = \delta_1(t) + \alpha(t+t_0)^{\frac{1}{3}(1-4k)}$ *for a value of* α *to be chosen. We can apply Theorem 4.19 to deduce upper and lower bounds as in equation* (4.118)*. From this, and the fact that* $\int_0^t (-\ddot{\delta}(\zeta))^{2/3} d\zeta = \underline{\overline{O}}(t^{\frac{1}{3}(1-4k)})$ *as* $t \to \infty$*, we see that if* α *is too small then* $\psi\left(x_1(t), \frac{-L_2(t)}{2} + y, t\right) \to 0$*, and if* α *is too large then* $\psi\left(x_1(t), \frac{-L_2(t)}{2} + y, t\right) \to \infty$ *as* $t \to \infty$ *(for* $\xi_1(t)$ *in a compact subset of* $(0, L_0)$*).*

Next we briefly explain why the methods cannot give the exact behaviour when $\frac{1}{2} < k < 1$. As above, let $L_j(t) = a(t+t_0)^k$ and $A_j(t) = b(t+t_1)^m$ with $a > 0$, $b \in \mathbb{R}$. If $k \geq \frac{1}{2}$ and/or $m + k \geq 1$, then as $t \to \infty$, $|\ddot{L}_j(t) L_j(t)|$ and $|\ddot{A}_j(t) L_j(t)|$ are of order t^{2k-2} and t^{m+k-2} respectively, which are not integrable. So (except for the special case $m = k = \frac{1}{2}$ which has been covered in Corollary 4.26), we find that $\underline{Q_j}(t)$ and $\overline{Q_j}(t)$ are not integrable for any choice of $\gamma_0^{(j)}$ and $\gamma_1^{(j)}$ and so (3.140) fails to hold. Theorem 4.19 gives upper and lower bounds on the solution that differ by a factor of order

$$\exp\left(\int_0^t \sum_{j=1}^N \frac{\overline{Q}_j(\zeta) + \underline{Q}_j(\zeta)}{2D} d\zeta\right). \tag{4.124}$$

This tends to ∞ as $t \to \infty$; in fact

$$\int_0^t \sum_{j=1}^N \frac{\overline{Q}_j(\zeta) + \underline{Q}_j(\zeta)}{2D} d\zeta = \begin{cases} \underline{\overline{O}}(t^{\max(2k-1, m+k-1)}) & \text{if } k > \frac{1}{2} \text{ and/or } m + k > 1 \\ \underline{\overline{O}}\left(\log \frac{t}{t_0}\right) & \text{if } m = 1 - k > \frac{1}{2}. \end{cases} \tag{4.125}$$

Since we do not know the exact order of the solution near the boundary, we cannot determine a 'critical' choice of $L_{N+1}(t)$ such that (4.91) holds.

However, by combining the preceding results, examples and remarks, and by using the comparison principle to prove some additional bounds, we obtain the results of the following theorem.

Theorem 4.35. *Consider the box* $\{\frac{-L_j(t)}{2} < x_j < \frac{L_j(t)}{2} : j = 1, 2\}$ *in* $\mathbb{R}^2$*, where* $L_1(t) = a(t+t_0)^k$ *for some* $a > 0$ *and* $k \geq 0$*. Let* $0 < y = O(1)$ *and* $c_* = 2\sqrt{Df_0}$*.*

1. *If* $k \geq 1$*, let*

$$\frac{L_2(t)}{2} = c_* t - \frac{4D}{c_*} \log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.126}$$

Then $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ *as* $t \to \infty$.

2. *If* $\frac{1}{2} < k < 1$, *let*

$$\frac{L_2(t)}{2} = c_* t - \alpha \log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.127}$$

If $\alpha < \frac{4D}{c_*}$ *then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to 0$ *as* $t \to \infty$. *If* $\alpha > \frac{4D}{c_*}$ *then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to \infty$ *as* $t \to \infty$.

3. *If* $k = \frac{1}{2}$, *let* $\rho = \frac{a^2}{2}$, $\gamma_1 = \frac{\rho^2}{2}$, *and let* σ_1 *be the principal eigenvalue of the Sturm-Liouville problem* (4.101). *Let*

$$\frac{L_2(t)}{2} = c_* t - \frac{2D}{c_*}\left(-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} + \frac{3}{2}\right)\log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.128}$$

Then $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ *as* $t \to \infty$.

4. *If* $\frac{1}{4} < k < \frac{1}{2}$, *let*

$$\frac{L_2(t)}{2} = c_* t - \frac{2D^2\pi^2}{c_* a^2(1-2k)}(t+t_0)^{1-2k} - \frac{(k+3)D}{c_*}\log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.129}$$

Then $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ *as* $t \to \infty$.

5. *If* $k = \frac{1}{4}$, *let*

$$\frac{L_2(t)}{2} = c_* t - \frac{4D^2\pi^2}{c_* a^2}(t+t_0)^{\frac{1}{2}} - \alpha \log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.130}$$

If α *is too small then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to 0$, *and if* α *is too large then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to \infty$, *as* $t \to \infty$.

6. *If* $0 < k < \frac{1}{4}$, *let* $\delta_1(t)$ *satisfy*

$$\frac{c_*\delta_1(t)}{2D} - \int_0^t \frac{\dot{\delta}_1(\zeta)^2}{4D} d\zeta = \frac{D\pi^2}{a^2(1-2k)}(t+t_0)^{1-2k} + o(t^{\frac{1}{3}(1-4k)}) \qquad \text{as } t \to \infty, \tag{4.131}$$

and let

$$\frac{L_2(t)}{2} = c_* t - \delta_1(t) - \alpha(t+t_0)^{\frac{1}{3}(1-4k)}. \tag{4.132}$$

If α *is too small then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to 0$, *and if* α *is too large then* $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to \infty$, *as* $t \to \infty$.

7. If $k = 0$ and $f_0 > \frac{D\pi^2}{a^2}$ let C be the positive solution to $\frac{C^2}{4D} = f_0 - \frac{D\pi^2}{a^2}$. Let

$$\frac{L_2(t)}{2} = Ct - \frac{3D}{C} \log\left(\frac{t}{t_0} + 1\right) - \delta_0. \tag{4.133}$$

Then $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ as $t \to \infty$.

Proof. The case $\frac{1}{4} < k < \frac{1}{2}$ follows from Corollary 4.24 (with $m = k$); the case $k = \frac{1}{2}$ follows from Corollary 4.26; the case $k = 1$ follows from Corollary 4.27; and the case $k = 0$ follows from Corollary 4.29. The case $k = \frac{1}{4}$ follows from Example 4.33; and the claimed properties for $0 < k < \frac{1}{4}$ follow from the arguments in Remark 4.34. So it just remains to prove the claimed properties for $k > 1$ and for $\frac{1}{2} < k < 1$.

Consider the case $k > 1$. The solution on the box with $k = 1$ (i.e. with $\frac{-a(t+t_0)}{2} < x_1 < \frac{a(t+t_0)}{2}$) is then a subsolution, and the solution on the infinite strip (with $-\infty < x_1 < \infty$) is a supersolution. In both of these cases the solution has a contribution from dimension 1 which, at $x_1 = 0$, is of order $t^{-\frac{1}{2}}$ as $t \to \infty$. (For $k = 1$ this was shown in Corollary 4.27, and in the case of the infinite strip the contribution is of the form

$$\frac{1}{\sqrt{4Dt}} \int_{-\infty}^{\infty} \hat{\psi}(y, 0) e^{-\frac{(x_1 - y)^2}{4Dt}} dy \tag{4.134}$$

with $\hat{\psi}(y, 0)$ of compact support.) Let $L_2(t) = 2(c_* t - \frac{4D}{c_*} \log\left(\frac{t}{t_0} + 1\right) - \delta_0)$. By the same arguments as in Corollary 4.22, we find that our subsolution and supersolution both have the property that $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$, and the result follows by the comparison principle.

Finally, suppose $\frac{1}{2} < k < 1$, and let $L_2(t)$ be as in equation (4.127). The solution on the box with $k = 1$ is a supersolution. Since this supersolution has $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to 0$ as $t \to \infty$ if $\alpha < \frac{4D}{c_*}$, we deduce the same for the case $\frac{1}{2} < k < 1$. The solution on a box with $\frac{-\sqrt{l^2+2\rho t}}{2} < x_1 < \frac{\sqrt{l^2+2\rho t}}{2}$ is a subsolution (for t large enough), for any $\rho > 0$. We know that this subsolution has $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to \infty$ as $t \to \infty$ if $\frac{c_* \alpha}{2D} > \left(-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} + \frac{3}{2}\right)$. But since $\gamma_1 = \frac{\rho^2}{2}$, we also know that

$$-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} \to \frac{1}{2} \qquad \text{as } \rho \to \infty \tag{4.135}$$

due to (3.54) in Proposition 3.4. So, if $\alpha > \frac{4D}{c_*}$ then there exists $\rho > 0$ large enough that $\frac{c_*\alpha}{2D} > \left(-\frac{\sigma_1 L_0^2}{2\rho} + \frac{\gamma_1^2}{8\rho^3 D} + \frac{1}{4} + \frac{3}{2}\right)$, and so $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) \to \infty$. $\square$

Let us highlight some key points from Theorem 4.35. The theorem considers ψ at the symmetric position $x_1 = 0$, in a symmetric box in $\mathbb{R}^2$ with $L_1(t)$ of order t^k as $t \to \infty$. When $k \geq 1$, the choice $\frac{L_2(t)}{2} = c_*t - \frac{4D}{c_*}\log\left(\frac{t}{t_0} + 1\right) - \delta_0$ gives the 'critical' property $\psi\left(0, \frac{-L_2(t)}{2} + y, t\right) = \underline{\overline{O}}(y)$ as $t \to \infty$. This holds for every $k \geq 1$, and is also the same as the 'critical' radius for a ball in $\mathbb{R}^2$. The conclusion of the theorem for $\frac{1}{2} < k < 1$ suggests that if there is a 'critical' choice of $L_2(t)$ when $\frac{1}{2} < k < 1$ then it will have the form $\frac{L_2(t)}{2} = c_*t - \frac{4D}{c_*}\log\frac{t}{t_0} + o(\log\frac{t}{t_0})$ as $t \to \infty$. That is, it will have the same leading order logarithmic correction term as for $k \geq 1$ but the order of the next correction might no longer be $O(1)$. In the case $k = \frac{1}{2}$, which is an important threshold, the leading correction term is still logarithmic in t but its coefficient is no longer $\frac{4D}{c_*}$. Instead, its coefficient depends on the constant that multiplies $t^{\frac{1}{2}}$ in the asymptotic behaviour of $L_1(t)$. We note that, by Proposition 3.4, the coefficient of the logarithmic delay in (4.128) is $> \frac{4D}{c_*}$ and it tends to $\frac{4D}{c_*}$ as $a \to \infty$. For $0 < k < \frac{1}{2}$, the 'critical' choice of $\frac{L_2(t)}{2}$ still has the same speed c_* but the leading correction term is not logarithmic in t any more. Instead, the leading $o(t)$ correction is a power of t that depends on k. For the final case of the theorem, $k = 0$, the speed itself changes. Whereas in each of the other cases (with $k > 0$) $L_1(t) \to \infty$, in the case $k = 0$ the domain has a constant length L_1 in the x_1 direction. Consequently, the speed required in a 'critical' choice of $\frac{L_2(t)}{2}$ is strictly less than c_* and is instead determined by $\frac{C^2}{4D} = f_0 - \frac{D\pi^2}{L_1^2}$. The leading correction term in this case is logarithmic in t with coefficient $\frac{3D}{C}$. This is, in some sense, as if we had 'moved down a dimension', as well as replacing the critical speed c_* by C.

4.5 Vanishing correction terms

4.5.1 Introduction

Recall that in Corollary 4.6 we proved the following. If $\tilde{\psi}(y,t)$ satisfies

$$\frac{\partial\tilde{\psi}}{\partial t} = D\frac{\partial^2\tilde{\psi}}{\partial y^2} + \frac{C^2}{4D}\tilde{\psi} \quad \text{for } \frac{-L(t)}{2} < y < \frac{L(t)}{2}, \qquad \tilde{\psi}\left(\pm\frac{L(t)}{2}, t\right) = 0 \tag{4.136}$$

where $C > 0$, and $L(t) = 2\left(Ct - \frac{3D}{C}\log(\frac{t}{t_0}+1) - \theta(t)\right)$ with $\theta(t) = O(1)$ satisfying (4.38), then $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right) = \underline{\overline{O}}(1)$ as $t \to \infty$. Now, using a method of J. Berestycki, Brunet and Derrida from [12] (which they apply to the problem on a semi-infinite interval $\mu(t) < x < \infty$), we shall seek a particular choice of $\theta(t)$ such that not only is $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right)$ exactly of order one but it is equal to a constant for all t sufficiently large. In fact we shall consider a slightly more general problem, which also allows us to derive a similar result for special cases of the time-dependent box (4.65). We shall assume for convenience that $\tilde{\psi}(y,t)$ is symmetric in y.

4.5.2 Integral transform method

Let $\tilde{\psi}(y,t)$ satisfy (4.136) where $L(t) = 2(Ct - \delta(t))$ and where $\delta(t)$ satisfies (4.80). Recall from Proposition 4.2 and the change of variables (4.8) that

$$\tilde{\psi}\left(\frac{-L(t)}{2} + \frac{\xi}{L_0}L(t), t\right) = O\left(\sin\left(\frac{\pi\xi}{L_0}\right)\left(\frac{L(0)}{L(t)}\right)^{\frac{1}{2}} e^{\frac{C\delta(t)}{2D} - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D}d\zeta - \xi(\xi-L_0)\frac{\dot{L}(t)L(t)}{4DL_0^2}}\right) \tag{4.137}$$

and that

$$\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right) = O\left(t^{-\frac{3}{2}}e^{\frac{C}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D}d\zeta}\right) \qquad \text{as } t \to \infty. \tag{4.138}$$

Lemma 4.36. *Let $\tilde{\psi}(y,t)$ satisfy* (4.136)*, where $L(t) = 2(Ct - \delta(t))$ and where $\delta(t)$ satisfies* (4.80)*. Define*

$$g(r,t) = \int_0^{L(t)} \tilde{\psi}\left(\frac{-L(t)}{2} + z, t\right) e^{rz}dz. \tag{4.139}$$

Then for each $r < -\frac{C}{2D}$, it holds that $g(r,t)e^{-D\left(r+\frac{C}{2D}\right)^2 t + r\delta(t)} \to 0$ as $t \to \infty$.

Proof. Put $z = \frac{\xi}{L_0}L(t)$ into (4.137) and note that

$$-\xi(\xi - L_0)\frac{\dot{L}(t)L(t)}{4DL_0^2} = -\frac{z^2}{4Dt}(1+o(1)) + \frac{Cz}{2D}(1+o(1)) \quad \text{as } t \to \infty. \quad (4.140)$$

So, given $r < -\frac{C}{2D}$, we can multiply $\tilde{\psi}\left(\frac{-L(t)}{2} + z, t\right)$ by e^{rz} and integrate over z to get the following bound on $g(r,t)$:

$$g(r,t) = O\left(t^{-\frac{1}{2}} e^{\frac{C}{2D}\delta(t) - \int_0^t \frac{\dot{\delta}(\zeta)^2}{4D} d\zeta}\right). \quad (4.141)$$

Therefore $g(r,t)e^{-D\left(r+\frac{C}{2D}\right)^2 t + r\delta(t)} \to 0$ as $t \to \infty$. □

Proposition 4.37. *Let $\tilde{\psi}(y,t)$ satisfy* (4.136)*, where $L(t) = 2(Ct - \delta(t))$ and where $\delta(t)$ satisfies* (4.80)*, and assume that $\tilde{\psi}(-y,0) \equiv \tilde{\psi}(y,0)$. Then for $\varepsilon > 0$, the integral*

$$\int_0^\infty e^{-\frac{C^2}{4D}\varepsilon^2 t - \frac{C}{2D}(1+\varepsilon)\delta(t)} D\frac{\partial \tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right) dt \quad (4.142)$$

has one-sided derivatives of all orders with respect to ε at $\varepsilon = 0$.

Proof. Define $g(r,t)$ as in (4.139), and differentiate this with respect to t. Using (4.136), this gives:

$$\frac{\partial g}{\partial t}(r,t) = \int_0^{L(t)} \left(D\frac{\partial^2 \tilde{\psi}}{\partial y^2}\left(\frac{-L(t)}{2} + z, t\right) - \frac{\dot{L}(t)}{2}\frac{\partial \tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2} + z, t\right)\right) e^{rz} dz$$
$$+ \int_0^{L(t)} \frac{C^2}{4D}\tilde{\psi}\left(\frac{-L(t)}{2} + z, t\right) e^{rz} dz. \quad (4.143)$$

Integrate by parts in z, and use the boundary conditions and the symmetry ($\tilde{\psi}(-y,t) \equiv \tilde{\psi}(y,t)$ for all t), to get

$$\frac{\partial g}{\partial t}(r,t) = \left(Dr^2 + r\frac{\dot{L}(t)}{2} + \frac{C^2}{4D}\right) g(r,t) - D\frac{\partial \tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right)(1 + e^{rL(t)}). \quad (4.144)$$

Use an integrating factor of $e^{\phi(r,t)}$ where

$$\phi(r,t) = -\left(Dr^2 t + r\frac{L(t)}{2} + \frac{C^2}{4D}t\right) = -D\left(r + \frac{C}{2D}\right)^2 t + r\delta(t). \quad (4.145)$$

This leads to the following equation for every $t > 0$:

$$g(r,t)e^{-D\left(r+\frac{C}{2D}\right)^2 t + r\delta(t)} - g(r,0)e^{r\delta(0)}$$
$$= -\int_0^t e^{-D\left(r+\frac{C}{2D}\right)^2 s + r\delta(s)} D\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(s)}{2}, s\right)(1+e^{rL(s)})ds. \qquad (4.146)$$

Consider $r < -\frac{C}{2D}$, and let $t \to \infty$ in equation (4.146). Using Lemma 4.36, this gives that for each $r < -\frac{C}{2D}$,

$$g(r,0)e^{r\delta(0)} = \int_0^\infty e^{-D\left(r+\frac{C}{2D}\right)^2 s + r\delta(s)} D\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(s)}{2}, s\right)\left(1+e^{2r(Cs-\delta(s))}\right)ds. \qquad (4.147)$$

Set $r = -\frac{C}{2D}(1+\varepsilon)$ with $\varepsilon > 0$ and get

$$g\left(-\frac{C}{2D}(1+\varepsilon), 0\right)e^{-\frac{C}{2D}(1+\varepsilon)\delta(0)} = \int_0^\infty e^{-\frac{C^2}{4D}\varepsilon^2 s - \frac{C}{2D}(1+\varepsilon)\delta(s)} D\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(s)}{2}, s\right)$$
$$\times\left(1+e^{-\frac{C}{D}(1+\varepsilon)(Cs-\delta(s))}\right)ds. \qquad (4.148)$$

The left hand side (depending only on the initial conditions) is infinitely differentiable in ε. Since the equation holds for all $\varepsilon > 0$, the right hand side must have one-sided derivatives of all orders with respect to ε at $\varepsilon = 0$.

The terms on the right hand side with a factor $e^{-\frac{C}{D}(1+\varepsilon)(Cs-\delta(s))}$ all have exponential decay as $s \to \infty$ (since (4.138) bounds the other factor) and can be repeatedly differentiated through the integral with respect to ε. So (by subtracting these terms) the conclusion is that the remaining term, which is precisely the expression in (4.142), must also have one-sided derivatives of all orders with respect to ε at $\varepsilon = 0$. □

4.5.3 Finding the vanishing corrections

Now suppose that $p(t) = p_0 \log\left(\frac{t}{t_0}+1\right) + \text{constant}$ for some $p_0 > 0$ and that

$$\delta(t) = \frac{3D}{C}\log\left(\frac{t}{t_0}+1\right) + \frac{2D}{C}p(t) + \theta(t) \qquad (4.149)$$

where $\theta(t)$ satisfies (4.38). Corollary 4.6 implies that $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right)e^{-p(t)} = \underline{\overline{O}}(1)$ as $t \to \infty$. Using Proposition 4.37 and the method of [12], we shall now seek a particular choice of $\theta(t)$ such that $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2},t\right)e^{-p(t)}$ is equal to a constant for all t sufficiently large. The method will be the same as that in [12], but applied to the integral (4.142). First we calculate the leading singularity as $\varepsilon \to 0$ that results from the known terms of the $\delta(t)$ expansion. (By this, we mean the leading order term that fails to have one-sided ε-derivatives of some order at $\varepsilon = 0$.) Then we must identify the next correction in the $\delta(t)$ expansion that is required in order to cancel this singularity. Having thus reduced the ε singularity to one of lower order, we repeat the process. At each step we derive one more term in the $\delta(t)$ expansion (for large t), by removing the highest remaining singularity as $\varepsilon \to 0$. The paper [12] contains several useful formulae for the singular terms of integrals (see [12, equations (47), (51), (52), (53), (54), and (55)]; given as equations (A.41), (A.49), (A.50), (A.51), (A.52), and (A.53) here). In order to apply these formulae it is convenient to non-dimensionalise the time variable. So, let $\tau = \frac{C^2}{4D}t$, and for $\tau \geq 1$ define $\tilde{\delta}(\tau)$ and $q(\tau)$ by

$$\frac{C}{2D}\delta(t) = \tilde{\delta}(\tau), \qquad \frac{C}{2D}\delta(t) - p(t) = \frac{3}{2}\log\tau + q(\tau), \tag{4.150}$$

i.e. $q(\tau)$ is the $O(1)$ part. Then, since we are assuming $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2},t\right)e^{-p(t)}$ is a constant for all t sufficiently large, Proposition 4.37 becomes that

$$\int_1^\infty \frac{e^{-\varepsilon^2\tau-\varepsilon\tilde{\delta}(\tau)-q(\tau)}}{\tau^{\frac{3}{2}}}d\tau \tag{4.151}$$

has one-sided derivatives of all orders with respect to ε at $\varepsilon = 0$. With $\delta(t)$ as in (4.149), note that $\tilde{\delta}(\tau)$ has leading order behaviour

$$\tilde{\delta}(\tau) = \alpha\log\tau + O(1) \qquad \text{as } \tau \to \infty \tag{4.152}$$

where $\alpha = \left(\frac{3}{2} + p_0\right)$. We consider $\tilde{\delta}(\tau)$ of the form (4.152) with $\alpha > 0$, and we seek an asymptotic expansion

$$q(\tau) \sim q_0(\tau) + q_1(\tau) + q_2(\tau) + \ldots \qquad \text{as } \tau \to \infty \tag{4.153}$$

such that the integral (4.151) has one-sided derivatives of all orders with respect to ε at $\varepsilon = 0$.

Proposition 4.38. *Let* $\tilde{\psi}(y,t)$ *satisfy* (4.136) *where* $L(t) = 2(Ct - \delta(t))$*. Let* $\tilde{\delta}(\tau)$ *and* $q(\tau)$ *be as in* (4.150)*, and assume* (4.152) *holds. If* $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L(t)}{2}, t\right) e^{-p(t)}$ *is equal to a constant for* t *sufficiently large, then* (4.153) *holds with* $q_0(\tau) =$ *constant and*

$$q_1(\tau) = \frac{-2\alpha\sqrt{\pi}}{\tau^{\frac{1}{2}}}, \qquad q_2(\tau) = -\frac{3}{2}\left(\alpha + \alpha^2(1 - 2\log 2)\right)\frac{\log\tau}{\tau}. \tag{4.154}$$

Proof. First, note that there cannot be any singularity as $\varepsilon \searrow 0$ in the integral

$$\int_1^\infty \frac{e^{-\varepsilon^2\tau - q_0(\tau)}}{\tau^{\frac{3}{2}}} d\tau \tag{4.155}$$

which would contradict Proposition 4.37. Using equation (A.41) and Remark A.14, the only singularity in ε^2 coming from the integral $\int_1^\infty \frac{e^{-\varepsilon^2\tau}}{\tau^{\frac{3}{2}}} d\tau$ is a multiple of $(\varepsilon^2)^{\frac{1}{2}} = |\varepsilon|$. This is not singular in $\varepsilon > 0$ as $\varepsilon \searrow 0$, and so we find that q_0 can be any constant.

Next, note that without the $q_1(\tau)$ term — and for $\tilde{\delta}(\tau)$ given by (4.152) — the leading order singularity of the integral (4.151) would come from

$$\int_1^\infty \frac{-\alpha\varepsilon\log(\tau)e^{-\varepsilon^2\tau - q_0}}{\tau^{\frac{3}{2}}} d\tau. \tag{4.156}$$

Equation (A.50) shows that as $\varepsilon \searrow 0$, the singularity is precisely

$$-2\sqrt{\pi}\alpha e^{-q_0}\varepsilon^2\log(\varepsilon^2). \tag{4.157}$$

This must be cancelled by adding the singularity from

$$\int_1^\infty \frac{-q_1(\tau)e^{-\varepsilon^2\tau - q_0}}{\tau^{\frac{3}{2}}} d\tau, \tag{4.158}$$

which (using equation (A.41)) gives that $q_1(\tau)\tau^{-\frac{3}{2}} = -2\sqrt{\pi}\alpha\tau^{-2}$, and so

$$q_1(\tau) = \frac{-2\alpha\sqrt{\pi}}{\tau^{\frac{1}{2}}}, \tag{4.159}$$

as stated in equation (4.154). Now, without the $q_2(\tau)$ term, the leading order singularity of the integral (4.151) would come from

$$\int_1^\infty \frac{\left(-\varepsilon q_1(\tau) + \frac{1}{2}(\varepsilon\alpha\log\tau + q_1(\tau))^2\right)e^{-\varepsilon^2\tau - q_0}}{\tau^{\frac{3}{2}}} d\tau \tag{4.160}$$

where $q_1(\tau)$ is given in (4.159). Using equations (A.41), (A.52) and (A.53), we find that the leading singularity as $\varepsilon \searrow 0$ is:

$$\varepsilon^3 \log(\varepsilon^2) 2\sqrt{\pi} \left(\alpha + \alpha^2(1 - 2\log 2)\right) e^{-q_0}. \tag{4.161}$$

This must be cancelled by adding the singularity from

$$\int_1^\infty \frac{-q_2(\tau) e^{-\varepsilon^2 \tau - q_0}}{\tau^{\frac{3}{2}}} d\tau, \tag{4.162}$$

which (using equation (A.51)) gives that

$$\frac{q_2(\tau)}{\tau^{\frac{3}{2}}} = \tilde{q}_2 \frac{\log \tau}{\tau^{\frac{5}{2}}}, \quad \text{where} \quad -\frac{4}{3}\sqrt{\pi}\tilde{q}_2 = 2\sqrt{\pi}\left(\alpha + \alpha^2(1 - 2\log 2)\right). \tag{4.163}$$

That is, $q_2(\tau)$ is as stated in equation (4.154). □

4.5.4 Examples

Example 4.39. *Let $\psi(x,t)$ satisfy (4.1), (4.2) on the interval $\frac{-L(t)}{2} < x < \frac{L(t)}{2}$. From Section 4.2 we know that if $\frac{L(t)}{2} = c_* t - \frac{3D}{c_*}\log(\frac{t}{t_0} + 1) - \theta(t)$ where $\theta(t)$ satisfies (4.38), then $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right) = \underline{O}(1)$ as $t \to \infty$. Applying Proposition 4.38 with $C = c_*$ and $\alpha = \frac{3}{2}$ shows that if $\frac{\partial \psi}{\partial x}\left(\frac{-L(t)}{2}, t\right)$ is equal to a constant for t sufficiently large, then as $t \to \infty$*

$$\frac{L(t)}{2} \sim c_* t - \frac{2D}{c_*}\left(\frac{3}{2}\log\left(\frac{c_*^2 t}{4D}\right) + \delta_0 - \frac{3\sqrt{\pi}}{\sqrt{\frac{c_*^2 t}{4D}}} - \frac{9}{8}(5 - 6\log 2)\frac{\log \frac{c_*^2 t}{4D}}{\frac{c_*^2 t}{4D}} + \dots\right). \tag{4.164}$$

Remark 4.40. *This is similar to the expansion for $\mu(t)$ obtained in [12] for the problem on $\mu(t) < x < \infty$ when the initial conditions have sufficient decay. However the $\frac{1}{\sqrt{t}}$ correction has the opposite sign. This is due to the fact that in our problem the domain lies to the right of the boundary moving at $-c_* t + o(t)$ whereas in [12] the domain lies to the right of a boundary moving at $+c_* t - o(t)$. The range of r values that can be used in the integral transform (in Proposition 4.37) is therefore different in the two cases, and the $\varepsilon\delta(t)$ term in the exponent ends up with the opposite sign. The same observation is discussed in [12, section*

6] where they explain how their results would differ if $\mu(t)$ moved to the left at $-c_*t + o(t)$ instead of to the right at $+c_*t - o(t)$.

Using our separation of variables method we can also derive results of this type for certain special cases of the time-dependent box (4.65).

Example 4.41. *Consider a box of the form* (4.65)*, with a cross-section that is independent of t: $L_j(t) \equiv l_j$ and $A_j(t) \equiv -\frac{l_j}{2}$ for each $1 \le j \le N$. Assume that $f_0 - \sum_{j=1}^{N} \frac{D\pi^2}{l_j^2} > 0$ and define C as the positive solution to*

$$f_0 - \sum_{j=1}^{N} \frac{D\pi^2}{l_j^2} = \frac{C^2}{4D}. \tag{4.165}$$

Then there are positive solutions to (4.63), (4.64) *that are separable in dimensions $1 \le j \le N$, of the form*

$$\psi(x_1, \ldots, x_N, x_{N+1}, t) = \tilde{\psi}(x_{N+1}, t) \prod_{j=1}^{N} \sin\left(\frac{\pi \xi_j}{L_0}\right) \tag{4.166}$$

where $\frac{\xi_j}{L_0} = \frac{x_j}{l_j} + \frac{1}{2}$ for $1 \le j \le N$, and where $\tilde{\psi}(y,t)$ satisfies (4.136) *on $\frac{-L_{N+1}(t)}{2} < y < \frac{L_{N+1}(t)}{2}$. If $\frac{L_{N+1}(t)}{2} = Ct - \frac{3D}{C}\log(\frac{t}{t_0}+1) - \theta(t)$ where $\theta(t)$ satisfies* (4.38)*, then we know that $\frac{\partial \psi}{\partial x_{N+1}}\left(0, \ldots, 0, \frac{-L_{N+1}(t)}{2}, t\right) = \underline{\overline{O}}(1)$ as $t \to \infty$. Applying Proposition 4.38 with $\alpha = \frac{3}{2}$ shows that if $\frac{\partial \psi}{\partial x_{N+1}}\left(0, \ldots, 0, \frac{-L_{N+1}(t)}{2}, t\right)$ is equal to a constant for t sufficiently large, then*

$$\frac{L_{N+1}(t)}{2} \sim Ct - \frac{2D}{C}\left(\frac{3}{2}\log\left(\frac{C^2 t}{4D}\right) + \delta_0 - \frac{3\sqrt{\pi}}{\sqrt{\frac{C^2 t}{4D}}} - \frac{9}{8}(5 - 6\log 2)\frac{\log\frac{C^2 t}{4D}}{\frac{C^2 t}{4D}} + \ldots\right) \tag{4.167}$$

as $t \to \infty$, with C given by equation (4.165).

Example 4.42. *Consider a box of the form* (4.65) *where for each $1 \le j \le N$*

$$L_j(t) = \sqrt{l_j^2 + 2\rho_j t} \text{ with } \rho_j > 0, \quad A_j(t) \equiv -\frac{\gamma_1^{(j)}}{\rho_j^2} L_j(t) \text{ with } \frac{\gamma_1^{(j)}}{\rho_j^2} \in (0,1). \tag{4.168}$$

Since $\dot{L}_j(t) L_j(t) \equiv \rho_j$ and $\dot{A}_j(t) L_j(t) \equiv -\frac{\gamma_1^{(j)}}{\rho_j}$, the functions $H_j(\xi_j, t)$ in equation (3.131) *are now independent of t and given by*

$$H_j(\xi_j, t) = \tilde{H}_j(\xi_j) = \exp\left(\frac{\rho_j \xi_j^2}{4DL_0^2} - \frac{\gamma_1^{(j)} \xi_j}{2DL_0 \rho_j}\right) \qquad \text{for } 0 \le \xi_j \le L_0. \tag{4.169}$$

For each $1 \leq j \leq N$*, let* $g_{j,1}$ *and* $\sigma_{j,1}$ *be the principal eigenfunction and eigenvalue of* (3.19), (3.20) *with* $\gamma_0 = \gamma_0^{(j)}$ *and* $\gamma_1 = \gamma_1^{(j)}$*. Then let*

$$p(t) = \sum_{j=1}^{N}\left(\int_0^t \left(-\frac{\sigma_{j,1}L_0^2}{L_j(\zeta)^2} + \frac{\dot{A}_j(\zeta)^2}{4D}\right)d\zeta + \frac{1}{2}\log\left(\frac{L_j(t)}{L_j(0)}\right)\right) \tag{4.170}$$

$$= \sum_{j=1}^{N}\left(-\frac{\sigma_{j,1}L_0^2}{2\rho_j} + \frac{(\gamma_1^{(j)})^2}{8\rho_j^3 D} + \frac{1}{4}\right)\log\left(\frac{l_j^2 + 2\rho_j t}{l_j^2}\right) + \textit{constant}. \tag{4.171}$$

Then there are positive solutions to (4.63), (4.64) *on the box that are separable in dimensions* $1 \leq j \leq N$*, of the form*

$$\psi(x_1, \ldots, x_N, x_{N+1}, t) = \tilde{\psi}(x_{N+1}, t)e^{-p(t)}\prod_{j=1}^{N}\frac{g_{j,1}(\xi_j)}{\tilde{H}_j(\xi_j)} \tag{4.172}$$

where $\frac{\xi_j}{L_0} = \frac{x_j - A_j(t)}{L_j(t)}$ *for* $1 \leq j \leq N$*, and where* $\tilde{\psi}(y,t)$ *satisfies* (4.136) *on* $\frac{-L_{N+1}(t)}{2} < y < \frac{L_{N+1}(t)}{2}$ *with* $C = c_*$*. The results of Section 4.4 show that if* $\frac{L_{N+1}(t)}{2} = c_* t - \frac{3D}{c_*}\log\left(\frac{t}{t_0}+1\right) - \frac{2D}{c_*}p(t) - \theta(t)$*, with* $p(t)$ *given by* (4.171) *and where* $\theta(t) = O(1)$ *satisfies* (4.38)*, then* $\frac{\partial\psi}{\partial x_{N+1}}\left(0, \ldots, 0, \frac{-L_{N+1}(t)}{2}, t\right) = \overline{O}(1)$ *as* $t \to \infty$*. Here, we note that* $\frac{\partial\psi}{\partial x_{N+1}}\left(0, \ldots, 0, \frac{-L_{N+1}(t)}{2}, t\right)$ *is exactly of order one if and only if* $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L_{N+1}(t)}{2}, t\right)e^{-p(t)}$ *is exactly of order one, and it is equal to a constant if and only if* $\frac{\partial\tilde{\psi}}{\partial y}\left(\frac{-L_{N+1}(t)}{2}, t\right)e^{-p(t)}$ *is equal to a constant. Applying Proposition 4.38 with* $C = c_*$ *and*

$$\alpha = \frac{3}{2} + \sum_{j=1}^{N}\left(-\frac{\sigma_{j,1}L_0^2}{2\rho_j} + \frac{(\gamma_1^{(j)})^2}{8\rho_j^3 D} + \frac{1}{4}\right) \tag{4.173}$$

shows that if $\frac{\partial\psi}{\partial x_{N+1}}\left(0, \ldots, 0, \frac{-L_{N+1}(t)}{2}, t\right)$ *is equal to a constant for* t *sufficiently large, then*

$$\begin{aligned}\frac{L_{N+1}(t)}{2} \sim & c_* t - \frac{2D}{c_*}\left(\alpha\log\left(\frac{c_*^2 t}{4D}\right) + \delta_0 - \frac{2\alpha\sqrt{\pi}}{\sqrt{\frac{c_*^2 t}{4D}}}\right) \\ & - \frac{2D}{c_*}\left(-\frac{3}{2}(\alpha + \alpha^2(1 - 2\log 2))\frac{\log\frac{c_*^2 t}{4D}}{\frac{c_*^2 t}{4D}} + \ldots\right) \quad \textit{as } t \to \infty,\end{aligned} \tag{4.174}$$

with α *given by equation* (4.173).

Chapter 5

Periodically varying domains

5.1 Introduction and principal periodic eigenvalue μ

In this chapter we consider the problem (1.1), (1.2); that is:

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + f(\psi) \qquad \text{for } x \in \Omega(t)$$

$$\psi(x,t) = 0 \qquad \text{for } x \in \partial\Omega(t),$$

for first the linear and then the nonlinear case, on a domain $\Omega(t) \subset \mathbb{R}^N$ that is bounded, connected, and periodic in t with period T. As in Chapter 2, we assume there is a one-to-one mapping $h(\cdot,t) : \overline{\Omega(t)} \to \overline{\Omega_0}$ that transforms $\Omega(t)$ into a bounded, connected reference domain Ω_0 with sufficiently smooth boundary, and such that the change of variables $\xi = h(x,t)$ and $u(\xi,t) = \psi(x,t)$ transforms (1.1), (1.2) into a parabolic equation of the form (2.1), (2.2) where

$$\mathcal{L}(\xi,t)u = \sum_{i,j} a_{ij}(\xi,t)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j} + \sum_j \left(b_j(\xi,t) + c_j(\xi,t)\right)\frac{\partial u}{\partial \xi_j} \qquad \text{for } \xi \in \Omega_0, \tag{5.1}$$

$$a_{ij}(\xi,t) = \sum_k D\left(\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right), \qquad b_j(\xi,t) = -\frac{\partial h_j}{\partial t}, \qquad c_j(\xi,t) = D\nabla^2 h_j. \tag{5.2}$$

Since $\Omega(t)$ is periodic with period T, the map h and the coefficients of $\mathcal{L}$ are also T-periodic in t. We assume the coefficients a_{ij}, b_j, c_j belong to $C^{\alpha,\alpha/2}(\overline{\Omega_0}\times[0,T])$ for some $\alpha > 0$ and that a_{ij} is uniformly elliptic.

By Theorem 1 of Castro and Lazer [20], there exists a value μ and a function $\phi(\xi,t)$ such that

$$\frac{\partial\phi}{\partial t} - \mathcal{L}\phi = \mu\phi \qquad \text{for } \xi \in \Omega_0,\ t \in \mathbb{R} \tag{5.3}$$

$$\phi(\xi,t) = 0 \qquad \text{for } \xi \in \partial\Omega_0 \tag{5.4}$$

$$\phi(\xi,t) > 0 \qquad \text{for } \xi \in \Omega_0 \tag{5.5}$$

$$\phi(\xi,t) \equiv \phi(\xi,t+T). \tag{5.6}$$

This function ϕ is unique up to scaling [20, Theorem 1], and is called the principal periodic eigenfunction, while μ is called the principal periodic eigenvalue. Here, we shall say that μ is 'the principal periodic eigenvalue of $\Omega(t)$' to mean that it is the principal periodic eigenvalue of (5.3), (5.4), (5.5), (5.6), when $\mathcal{L}$ is defined by (5.1), (5.2). Throughout this chapter, unless stated otherwise, $\Omega(t)$ is a T-periodic domain and μ denotes the principal periodic eigenvalue of $\Omega(t)$.

Consider the linear reaction-diffusion equation (2.1), (2.2) with this periodic parabolic operator $\mathcal{L}$ and with $f(u) = f_0 u$. The function $\phi(x,t)e^{(f_0-\mu)t}$ is a solution, so if the initial conditions satisfy $b\phi(\xi,0) \le u(\xi,0) \le a\phi(\xi,0)$ for some $0 < b \le a$ then by the comparison principle

$$b\phi(\xi,t)e^{(f_0-\mu)t} \le u(\xi,t) \le a\phi(\xi,t)e^{(f_0-\mu)t} \qquad \text{for all } t \ge 0. \tag{5.7}$$

The principal periodic eigenvalue is therefore a threshold such that if $f_0 > \mu$ then $u(\xi,t) \to \infty$ as $t \to \infty$, whereas if $f_0 < \mu$ then $u(\xi,t) \to 0$. See also [19, page 192]. This means that the principal periodic eigenvalue μ of $\Omega(t)$ is, in some sense, the key to understanding the long-time behaviour of $u(\xi,t)$.

In Section 5.2 we derive upper and lower bounds on μ under a range of assumptions on $\Omega(t)$ and give some illustrative examples. Then in Section 5.3 we consider μ as a function of the frequency $\omega = \frac{2\pi}{T}$ and prove results concerning the limits $\omega \to 0$ and $\omega \to \infty$, as well as a monotonicity result. Proposition 5.8 and

Theorem 5.15 are based on related theorems from [46], in which the dependence of a principal periodic eigenvalue on the frequency is studied in a different context. In Section 5.4 we consider the nonlinear problem on a periodically varying domain, and we use results of Hess [37] in order to understand the long-time behaviour of the solution.

5.2 Bounds on μ and examples

To prove bounds on μ, we often use the following lemma and the property (5.8).

Lemma 5.1. *Let $k_1(t)$ and $k_2(t)$ be continuous and T-periodic, let m_1 and m_2 be constants, and let $u(\xi, t)$ satisfy* (2.1), (2.2) *with $f(u) = f_0 u$. If the condition $f_0 t - m_1 t - \int_0^t k_1(\zeta)d\zeta \to -\infty$ as $t \to \infty$ implies that $u(\xi, t) \to 0$ as $t \to \infty$, then $\mu \geq m_1 + \frac{1}{T}\int_0^T k_1(t)dt$. If the condition $f_0 t - m_2 t - \int_0^t k_2(\zeta)d\zeta \to +\infty$ implies that $u(\xi, t) \to \infty$ as $t \to \infty$, then $\mu \leq m_2 + \frac{1}{T}\int_0^T k_2(t)dt$.*

Proof. For any continuous T-periodic function $k(t)$,

$$\int_0^t k(\zeta)d\zeta = \frac{t}{T}\int_0^T k(\zeta)d\zeta + O(1) \qquad \text{as } t \to \infty. \tag{5.8}$$

So the assumptions become that the inequality $f_0 < m_1 + \frac{1}{T}\int_0^T k_1(t)dt$ implies $u(\xi, t) \to 0$, and that the inequality $f_0 > m_2 + \frac{1}{T}\int_0^T k_2(t)dt$ implies $u(\xi, t) \to \infty$. By (5.7) we deduce that $\mu \geq m_1 + \frac{1}{T}\int_0^T k_1(t)dt$ and $\mu \leq m_2 + \frac{1}{T}\int_0^T k_2(t)dt$. □

Proposition 5.2. *1. At each fixed time $0 \leq t \leq T$, let $\lambda(\Omega(t))$ be the principal Dirichlet eigenvalue of $-\nabla^2$ on the domain $\Omega(t)$. Then*

$$\mu \geq \frac{1}{T}\int_0^T D\lambda(\Omega(t))dt. \tag{5.9}$$

2. Let Ω_1 be a bounded domain such that $\Omega_1 \subset \Omega(t)$ for all t, and let $\lambda(\Omega_1)$ be the principal Dirichlet eigenvalue of $-\nabla^2$ on Ω_1. Then

$$\mu \leq D\lambda(\Omega_1). \tag{5.10}$$

Proof. Consider a solution ψ to the problem (1.1), (1.2) with $f(\psi) = f_0\psi$. Since $\lambda(\Omega(t))$ is T-periodic, the first part follows from Proposition 2.14 and Lemma 5.1.

For the second part, let ψ_1 be the solution to (1.1), (1.2) but on the fixed domain Ω_1, and with $0 \le \psi_1(x,0) \le \psi(x,0)$. By the comparison principle, $0 \le \psi_1(x,t) \le \psi(x,t)$ for all $x \in \Omega_1$ and $t \ge 0$. If $f_0 < \mu$ then $\psi \to 0$ as $t \to \infty$, which implies that $\psi_1 \to 0$ and so that $f_0 < D\lambda(\Omega_1)$. Therefore we conclude that $\mu \le D\lambda(\Omega_1)$. □

Example 5.3. *Let $\Omega(t) = (A(t), A(t) + L(t))$ where $L(t) > 0$ and $A(t)$ are both T-periodic. Proposition 5.2 implies that*

$$\mu \ge \frac{1}{T}\int_0^T \frac{D\pi^2}{L(t)^2}dt. \tag{5.11}$$

In particular whenever $L(t) \equiv l > 0$ and $A(t)$ is periodic then we have the lower bound $\mu \ge \frac{D\pi^2}{l^2}$. This means that if the solution on the fixed interval $(0,l)$ tends to zero (i.e. $f_0 < \frac{D\pi^2}{l^2}$), then the solution on a periodic interval $(A(t), A(t)+l)$ also tends to zero (i.e. $f_0 < \mu$) for every choice of the periodic function $A(t)$.

Example 5.4. *Let $L(t) > 0$ and $A(t)$ be T-periodic functions satisfying*

$$\max_{[0,T]} A < \min_{[0,T]}(A + L). \tag{5.12}$$

The fixed interval $\Omega_1 := (\max(A), \min(A + L))$ is always contained within $\Omega(t) := (A(t), A(t) + L(t))$. Proposition 5.2 implies the upper bound

$$\mu \le \frac{D\pi^2}{(\min(A + L) - \max A)^2}. \tag{5.13}$$

In particular, if $A(t)$ is constant and $L(t) > 0$ is periodic, then $\mu \le \frac{D\pi^2}{(\min L)^2}$.

Following these examples, let us continue to consider the linear problem on the interval $A(t) < x < A(t)+L(t)$ where $L(t) > 0$ and $A(t)$ are both T-periodic and belong to $C^{2+\alpha}([0,T])$ for some $\alpha > 0$. We change variables to the reference domain $0 < \xi < L_0$, and let $u(\xi,t)$ and $w(\xi,t)$ be as in Section 3.1. Then $u(\xi,t)$ satisfies (3.4), (3.5) and $w(\xi,t)$ satisfies (3.11), (3.12). Both of these are now linear, T-periodic, parabolic problems. Let $\mu_u = \mu$ denote the principal periodic eigenvalue of the operator that acts on u in equation (3.4), and let μ_w denote the principal periodic eigenvalue of the operator that acts on w in equation (3.11). Then as in equation (5.7) we know that

$$u(\xi,t) = \underline{\overline{O}}(\phi_u(\xi,t)e^{(f_0-\mu_u)t}), \qquad w(\xi,t) = \underline{\overline{O}}(\phi_w(\xi,t)e^{-\mu_w t}) \tag{5.14}$$

where ϕ_u and ϕ_w are the principal periodic eigenfunctions associated with μ_u and μ_w. Next we give the relation between μ_u and μ_w, and we bound μ_w.

Proposition 5.5. *1.*

$$\mu_u = \mu_w + \frac{1}{T}\int_0^T \frac{\dot{A}(t)^2}{4D}dt. \tag{5.15}$$

2. Let

$$\gamma_0^+ := \max_{[0,T]}(\ddot{L}L^3), \quad \gamma_1^+ := \max_{[0,T]}(\ddot{A}L^3), \tag{5.16}$$

$$\gamma_0^- := \min_{[0,T]}(\ddot{L}L^3), \quad \gamma_1^- := \min_{[0,T]}(\ddot{A}L^3). \tag{5.17}$$

If σ_1^+ is the principal eigenvalue of the Sturm-Liouville problem (3.19), (3.20) *with $\gamma_0 = \gamma_0^+$ and $\gamma_1 = \gamma_1^+$, and if σ_1^- is the principal eigenvalue of* (3.19), (3.20) *with $\gamma_0 = \gamma_0^-$ and $\gamma_1 = \gamma_1^-$, then*

$$\frac{-\sigma_1^+}{T}\int_0^T \frac{L_0^2}{L(t)^2}dt \le \mu_w \le \frac{-\sigma_1^-}{T}\int_0^T \frac{L_0^2}{L(t)^2}dt. \tag{5.18}$$

3. Let $\overline{Q}(t)$, $\underline{Q}(t)$ be given by equation (3.101)*. Then*

$$\frac{1}{T}\int_0^T \left(\frac{D\pi^2}{L(t)^2} - \frac{\overline{Q}(t)}{2D}\right)dt \le \mu_w \le \frac{1}{T}\int_0^T \left(\frac{D\pi^2}{L(t)^2} + \frac{\underline{Q}(t)}{2D}\right)dt. \tag{5.19}$$

Proof. Consider the function $H(\xi,t)$ given by (3.6), which occurs in the change of variables from u to w. Since $L(t) > 0$ and $A(t)$ are both periodic, note that

$$\left(\frac{L(t)}{L(0)}\right)^{1/2} \exp\left(\frac{\xi^2\dot{L}(t)L(t)}{4DL_0^2} + \frac{\xi\dot{A}(t)L(t)}{2DL_0}\right) = \underline{\overline{O}}(1) \tag{5.20}$$

in the sense that the left hand side has a finite upper bound a positive lower bound, uniformly in $t \geq 0$, $0 \leq \xi \leq L_0$. Therefore from the change of variables $w(\xi,t) = u(\xi,t)H(\xi,t)e^{-f_0 t}$ and the periodicity of $\dot{A}(t)$, we have

$$u(\xi,t) = \underline{O}\left(w(\xi,t)e^{f_0 t - \int_0^t \frac{\dot{A}(\zeta)^2}{4D} d\zeta}\right) = \underline{O}\left(w(\xi,t)e^{f_0 t - \frac{t}{T}\int_0^T \frac{\dot{A}(\zeta)^2}{4D} d\zeta}\right). \tag{5.21}$$

Part 1 of the proposition follows by combining (5.21) with (5.14). Part 2 follows from combining Lemma 5.1 with Proposition 3.10, and part 3 from combining Lemma 5.1 with Theorem 3.13. □

Example 5.6. *We shall apply Propositions 5.2 and 5.5 to* $\Omega(t) = (0, L(t))$ *where* $L(t) = L_0(1 + \varepsilon\sin(\omega t))$ *with* $\omega > 0$ *and* $0 < \varepsilon < 1$. *We must consider*

$$s(t) := \int_0^t \frac{L_0^2}{L(\zeta)^2} d\zeta = \int_0^t \frac{1}{(1 + \varepsilon\sin(\omega\zeta))^2} d\zeta. \tag{5.22}$$

The integral (5.22) *can be calculated exactly. For* $-\frac{\pi}{\omega} < t < \frac{\pi}{\omega}$,

$$\begin{aligned} s(t) =& \frac{2}{\omega(1-\varepsilon^2)^{3/2}}\left(\arctan\left(\frac{\tan(\frac{\omega t}{2}) + \varepsilon}{\sqrt{1-\varepsilon^2}}\right) - \arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)\right) - \frac{2\varepsilon}{\omega(1-\varepsilon^2)} \\ &+ \frac{2\varepsilon^2\tan(\frac{\omega t}{2}) + 2\varepsilon}{\omega(1-\varepsilon^2)\left((\tan(\frac{\omega t}{2}) + \varepsilon)^2 + 1 - \varepsilon^2\right)}. \end{aligned} \tag{5.23}$$

Then $s\left(\pm\frac{\pi}{\omega}\right) = \frac{2}{\omega(1-\varepsilon^2)^{3/2}}\left(\pm\frac{\pi}{2} - \arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)\right) - \frac{2\varepsilon}{\omega(1-\varepsilon^2)}$, *and for* $t > \frac{\pi}{\omega}$ *we use the fact that the integrand of* $s(t)$ *is* $\frac{2\pi}{\omega}$*-periodic. We find that* $s\left(\frac{2\pi}{\omega}\right) = \frac{2\pi}{\omega}\frac{1}{(1-\varepsilon^2)^{3/2}}$ *and so, by the* $\frac{2\pi}{\omega}$*-periodicity,*

$$s(t) = \int_0^t \frac{L_0^2}{L(\zeta)^2} d\zeta = \frac{t}{(1-\varepsilon^2)^{3/2}} + O(1) \qquad \text{as } t \to \infty. \tag{5.24}$$

Therefore we conclude from Proposition 5.2 that

$$\frac{D\pi^2}{L_0^2(1-\varepsilon^2)^{3/2}} \leq \mu \leq \frac{D\pi^2}{L_0^2(1-\varepsilon)^2}. \tag{5.25}$$

To apply Proposition 5.5, we calculate $\gamma_0^- := \min(\ddot{L}L^3)$ *and* $\gamma_0^+ := \max(\ddot{L}L^3)$:

$$\gamma_0^- = -L_0^4\varepsilon\omega^2(1+\varepsilon)^3 \quad \text{and} \quad \gamma_0^+ = \begin{cases} L_0^4\varepsilon\omega^2(1-\varepsilon)^3 & \text{if } 0 < \varepsilon \leq \frac{1}{4} \\ \frac{3^3}{4^4}L_0^4\omega^2 & \text{if } \frac{1}{4} < \varepsilon < 1. \end{cases} \tag{5.26}$$

Let σ_1^+ and σ_1^- be the principal eigenvalues of (3.19), (3.20) *with* $\gamma_1 = 0$, *and with* $\gamma_0 = \gamma_0^+$ *and* γ_0^- *respectively. Part 2 of Proposition 5.5 together with* (5.24) *then implies that*

$$\frac{-\sigma_1^+}{(1-\varepsilon^2)^{3/2}} \le \mu \le \frac{-\sigma_1^-}{(1-\varepsilon^2)^{3/2}}. \tag{5.27}$$

To apply part 3 of Proposition 5.5, we calculate the $\frac{2\pi}{\omega}$*-periodic functions* $\overline{Q}(t)$ *and* $\underline{Q}(t)$ *defined in* (3.101)*:*

$$\overline{Q}(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{\pi}{\omega} \\ -\frac{L_0^2 \varepsilon \omega^2}{2} \sin(\omega t)(1+\varepsilon \sin(\omega t)) & \text{for } \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases} \tag{5.28}$$

$$\underline{Q}(t) = \begin{cases} \frac{L_0^2 \varepsilon \omega^2}{2} \sin(\omega t)(1+\varepsilon \sin(\omega t)) & \text{for } 0 \le t \le \frac{\pi}{\omega} \\ 0 & \text{for } \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases} \tag{5.29}$$

and therefore

$$\int_0^{\frac{2\pi}{\omega}} \frac{\overline{Q}(\zeta)}{2D} d\zeta = \frac{L_0^2 \varepsilon \omega}{2D}\left(1 - \frac{\varepsilon\pi}{4}\right), \qquad \int_0^{\frac{2\pi}{\omega}} \frac{\underline{Q}(\zeta)}{2D} d\zeta = \frac{L_0^2 \varepsilon \omega}{2D}\left(1 + \frac{\varepsilon\pi}{4}\right). \tag{5.30}$$

By equation (5.19) *together with* (5.24) *we deduce that*

$$\frac{D\pi^2}{L_0^2(1-\varepsilon^2)^{3/2}} - \frac{L_0^2 \varepsilon \omega^2}{4\pi D}\left(1 - \frac{\varepsilon\pi}{4}\right) \le \mu \le \frac{D\pi^2}{L_0^2(1-\varepsilon^2)^{3/2}} + \frac{L_0^2 \varepsilon \omega^2}{4\pi D}\left(1 + \frac{\varepsilon\pi}{4}\right). \tag{5.31}$$

Remark 5.7. *Let* $\mu(\omega)$ *be the principal periodic eigenvalue of* $\Omega(t) = (0, L(t))$ *with* $L(t)$ *as in Example 5.6 for a fixed* $\varepsilon \in (0,1)$. *The bounds* (5.31) *imply that*

$$\mu(\omega) = \frac{D\pi^2}{L_0^2(1-\varepsilon^2)^{3/2}} + O(\omega^2) = \frac{1}{T}\int_0^T \frac{D\pi^2}{L(t)^2} dt + O(\omega^2) \qquad \text{as } \omega \to 0. \tag{5.32}$$

In the next section we shall see that (5.32) is an instance of a more general property, valid for $\frac{2\pi}{\omega}$-periodic domains $\Omega(t)$ in any dimension, as $\omega \to 0$.

5.3 Dependence of μ on the frequency ω

5.3.1 Converting to a 1-periodic problem in $s = \frac{\omega t}{2\pi}$

In this section we consider the principal periodic eigenvalue μ as a function of the frequency $\omega = \frac{2\pi}{T}$. We consider a 1-periodic domain $\tilde{\Omega}(s)$ and let $\mu = \mu(\omega)$

be the principal periodic eigenvalue associated with the domain

$$\Omega(t) = \tilde{\Omega}\left(\frac{\omega t}{2\pi}\right). \tag{5.33}$$

We shall give the limit $\lim_{\omega\to 0}\mu(\omega)$ for a $\frac{2\pi}{\omega}$-periodic domain in any dimension. For one-dimensional cases, we show that different types of asymptotic behaviour of $\mu(\omega)$ are possible as $\omega \to \infty$, and we also prove a result concerning monotonicity of $\mu(\omega)$ with respect to $\omega > 0$, under certain conditions.

Note that the map $h(\cdot, t) : \Omega(t) \to \Omega_0$ which we used in the change of variables can now be expressed as $h(\cdot, t) = \tilde{h}(\cdot, \frac{\omega t}{2\pi})$, for a 1-periodic map $\tilde{h}(\cdot, s)$: $\tilde{\Omega}(s) \to \Omega_0$. If we change variables from t to $s = \frac{\omega t}{2\pi}$ in (2.1), (2.2), then the operator $\frac{\partial}{\partial t} - \mathcal{L}(\xi, t)$ becomes an operator of the form $\frac{\omega}{2\pi}\frac{\partial}{\partial s} - \mathcal{L}_\omega(\xi, s)$ where

$$\mathcal{L}_\omega(\xi, s) = \sum_{i,j} \tilde{a}_{ij}(\xi, s)\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_j \left(\frac{\omega}{2\pi}\tilde{b}_j(\xi, s) + \tilde{c}_j(\xi, s)\right)\frac{\partial}{\partial \xi_j}, \tag{5.34}$$

$$\tilde{a}_{ij}(\xi, s) = \sum_k D\left(\frac{\partial \tilde{h}_i}{\partial x_k}\frac{\partial \tilde{h}_j}{\partial x_k}\right), \quad \tilde{b}_j(\xi, s) = -\frac{\partial \tilde{h}_j}{\partial s}, \quad \tilde{c}_j(\xi, s) = D\nabla^2 \tilde{h}_j. \tag{5.35}$$

So the principal periodic eigenvalue of $\Omega(t) = \tilde{\Omega}\left(\frac{\omega t}{2\pi}\right)$ is the same as the principal periodic eigenvalue $\mu(\omega)$ of the problem

$$\frac{\omega}{2\pi}\frac{\partial \phi}{\partial s} - \mathcal{L}_\omega(\xi, s)\phi = \mu(\omega)\phi(\xi, s) \qquad \xi \in \Omega_0, s \in [0, 1] \tag{5.36}$$

$$\phi(\xi, s) = 0 \qquad \xi \in \partial\Omega_0, s \in [0, 1] \tag{5.37}$$

$$\phi(\xi, s) \equiv \phi(\xi, s + 1) \qquad \xi \in \Omega_0. \tag{5.38}$$

For each $\omega > 0$, the coefficients of $\mathcal{L}_\omega(\xi, s)$ are 1-periodic in s. However, we note that the term $\frac{\omega}{2\pi}\tilde{b}_j(\xi, s)\frac{\partial}{\partial \xi_j}$ in (5.34) still depends on, and scales with, ω.

In the paper [46], Liu, Lou, Peng and Zhou consider parabolic equations with periodic coefficients, and they investigate how the principal periodic eigenvalue varies with respect to the frequency. However, the coefficients in their equation are independent of the frequency ω except where it appears inside the periodic functions as ωt. Therefore, after the change of time variables to give an operator with 1-periodic coefficients, the problems they consider have the form

$$\frac{\omega}{2\pi}\frac{\partial \hat{\phi}}{\partial s} - \hat{\mathcal{L}}(\xi, s)\hat{\phi} = \hat{\lambda}(\omega)\hat{\phi}(\xi, s) \qquad \xi \in \Omega_0, s \in [0, 1] \tag{5.39}$$

$$\hat{\phi}(\xi, s) = 0 \qquad \xi \in \partial\Omega_0, s \in [0, 1] \tag{5.40}$$

$$\hat{\phi}(\xi, s) \equiv \hat{\phi}(\xi, s+1) \qquad \xi \in \Omega_0, \tag{5.41}$$

where the coefficients of the operator $\hat{\mathcal{L}}(\xi, s)$ are 1-periodic in s and do not depend on ω. Theorem 1.3 in [46] gives results about the limits of the principal periodic eigenvalue $\hat{\lambda}(\omega)$ as $\omega \to 0$ and as $\omega \to \infty$, and Theorem 1.1 in [46] is a monotonicity property. Our proofs of the limit $\lim_{\omega\to 0}\mu(\omega)$, and of a monotonicity result, will be based on ideas from [46] but have to be adapted to our own case since $\mathcal{L}_\omega(\xi, s)$ depends on ω through the $\frac{\omega}{2\pi}\tilde{b}_j(\xi, s)\frac{\partial}{\partial \xi_j}$ term.

5.3.2 Asymptotic behaviour of $\mu(\omega)$ as $\omega \to 0$

Consider the limit $\omega \to 0$. The result of [46, Theorem 1.3(i)] for $\hat{\lambda}(\omega)$ (the principal periodic eigenvalue of (5.39), (5.40), (5.41)) is that

$$\lim_{\omega\to 0}\hat{\lambda}(\omega) = \int_0^1 \lambda^0(s)ds \tag{5.42}$$

where for each $0 \le s \le 1$, $\lambda^0(s)$ is the principal Dirichlet eigenvalue of the elliptic operator $-\hat{\mathcal{L}}(\xi, s)$ on Ω_0. The result can be extended in a natural way to the operator $-\mathcal{L}_\omega(\xi, s)$, and we deduce the following result for $\mu(\omega)$.

Proposition 5.8. *Let Ω_0 be a smooth bounded domain, and for each $s \in [0, 1]$ and $\omega \ge 0$ let $\mathcal{L}_\omega(\xi, s)$ be as defined in equation* (5.34), (5.35)*. Assume the coefficients $\tilde{a}_{ij}$, $\tilde{b}_j$, $\tilde{c}_j$ belong to $C^{1+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega_0} \times [0, 1])$ for some $\alpha > 0$. For each $s \in [0, 1]$ and $\omega \ge 0$, let $\lambda^0(s, \omega)$ be the principal eigenvalue of the operator $-\mathcal{L}_\omega(\xi, s)$ on Ω_0, with zero Dirichlet conditions on $\partial\Omega_0$. For $\omega > 0$, let $\mu(\omega)$ be the principal periodic eigenvalue of* (5.36), (5.37), (5.38)*. Then*

$$\lim_{\omega\to 0}\mu(\omega) = \int_0^1 \lambda^0(s, 0)ds. \tag{5.43}$$

The proof is essentially the same as that used in [46, Theorem 1.3(i)] to prove (5.42), however a slight generalisation is needed to allow for the ω-dependence of the coefficients in $\mathcal{L}_\omega(\xi, s)$. For completeness, we give the proof here.

Proof. For each $s \in [0,1]$ and $\omega \geq 0$, let $\lambda^0(s,\omega)$ and $\phi^0(\xi; s,\omega)$ be the principal eigenvalue and eigenfunction of the operator $-\mathcal{L}_\omega(\xi, s)$ on Ω_0 with zero Dirichlet conditions on $\partial\Omega_0$, and normalised to $||\phi^0(\cdot\ ; s,\omega)||_{L^2(\Omega_0)} = 1$. As in [46], note that for every $\xi \in \overline{\Omega_0}$ and $\omega \geq 0$, both $\phi^0(\xi; s,\omega)$ and $\nabla\phi^0(\xi; s,\omega)$ are C^1 and 1-periodic in s. We also note that they, and $\lambda^0(s,\omega)$, depend continuously on ω.

For $\omega > 0$ and $s \geq 0$, define

$$\rho_\omega(s) = \exp\left(\frac{2\pi}{\omega}\left(s\int_0^1 \lambda^0(\tau,\omega)d\tau - \int_0^s \lambda^0(\tau,\omega)d\tau\right)\right), \tag{5.44}$$

$$\overline{\phi}_\omega(\xi, s) = \phi^0(\xi; s,\omega)\rho_\omega(s). \tag{5.45}$$

Note that $\rho_\omega > 0$, ρ_ω is periodic with period 1, and it satisfies

$$\frac{\omega}{2\pi}\frac{d\rho_\omega}{ds} = \left(\int_0^1 \lambda^0(\tau,\omega)d\tau - \lambda^0(s,\omega)\right)\rho_\omega(s). \tag{5.46}$$

We shall show that given $\varepsilon > 0$, $\omega_\varepsilon > 0$ can be chosen small enough such that

$$\left(\int_0^1 \lambda^0(\tau,\omega)d\tau - \varepsilon\right)\overline{\phi}_\omega \leq \frac{\omega}{2\pi}\frac{\partial\overline{\phi}_\omega}{\partial s} - \mathcal{L}_\omega(\xi, s)\overline{\phi}_\omega \leq \left(\int_0^1 \lambda^0(\tau,\omega)d\tau + \varepsilon\right)\overline{\phi}_\omega \quad \text{for all } 0 < \omega \leq \omega_\varepsilon. \tag{5.47}$$

Then, since $\overline{\phi}_\omega$ is positive, 1-periodic in s, and satisfies the Dirichlet boundary conditions on $\partial\Omega_0$, it follows from [52, Proposition 2.1] that

$$\int_0^1 \lambda^0(\tau,\omega)d\tau - \varepsilon \leq \mu(\omega) \leq \int_0^1 \lambda^0(\tau,\omega)d\tau + \varepsilon \qquad \text{for all } 0 < \omega \leq \omega_\varepsilon, \tag{5.48}$$

and so we reach the conclusion

$$\lim_{\omega\to 0}\left(\mu(\omega) - \int_0^1 \lambda^0(s,\omega)ds\right) = 0. \tag{5.49}$$

Finally, since $\lambda^0(s,\omega)$ depends continuously on ω, (5.49) implies (5.43).

It remains to show that $\omega_\varepsilon > 0$ can be chosen such that (5.47) holds. Using (5.45), (5.46), and the fact that $\phi^0(\xi; s,\omega)$ is an eigenfunction of $-\mathcal{L}_\omega(\xi, s)$ with eigenvalue $\lambda^0(s,\omega)$, we calculate:

$$\frac{\omega}{2\pi}\frac{\partial\overline{\phi}_\omega}{\partial s} - \mathcal{L}_\omega(\xi, s)\overline{\phi}_\omega = \frac{\omega}{2\pi}\frac{\partial\phi^0(\xi; s, \omega)}{\partial s}\rho_\omega + \frac{\omega}{2\pi}\frac{d\rho_\omega}{ds}\phi^0(\xi; s, \omega) + \lambda^0(s, \omega)\phi^0(\xi; s, \omega)\rho_\omega \tag{5.50}$$

$$= \left(\frac{\omega}{2\pi}\frac{\partial\phi^0(\xi; s, \omega)}{\partial s} + \int_0^1 \lambda^0(\tau, \omega)d\tau\ \phi^0(\xi; s, \omega)\right)\rho_\omega. \tag{5.51}$$

Therefore (5.47) will hold provided we can choose $\omega_\varepsilon > 0$ such that

$$\frac{\omega}{2\pi}\left|\frac{\partial\phi^0(\xi; s, \omega)}{\partial s}\right| \leq \varepsilon\phi^0(\xi; s, \omega) \quad \text{for all } \xi \in \Omega_0,\ s \in [0, 1],\ 0 < \omega \leq \omega_\varepsilon. \tag{5.52}$$

Since $\phi^0(\xi; s, \omega)$ is positive in Ω_0, we know that $\frac{\frac{\partial\phi^0}{\partial s}(\xi;s,\omega)}{\phi^0(\xi;s,\omega)}$ is finite for each ξ in Ω_0. Given $\xi_0 \in \partial\Omega_0$ with outward normal ν, consider a sequence $\xi \in \Omega_0$, $\xi \to \xi_0$ with $\frac{\xi-\xi_0}{|\xi-\xi_0|} \cdot \nu \nrightarrow 0$. For any $s \in [0, 1]$ and $\omega \geq 0$ we have

$$\lim_{\xi\to\xi_0} \frac{\frac{\partial\phi^0}{\partial s}(\xi; s, \omega)}{\phi^0(\xi; s, \omega)} = \lim_{\xi\to\xi_0} \frac{\nabla\frac{\partial\phi^0}{\partial s}(\xi; s, \omega) \cdot \nu}{\nabla\phi^0(\xi; s, \omega) \cdot \nu} = \frac{\nabla\frac{\partial\phi^0}{\partial s}(\xi_0; s, \omega) \cdot \nu}{\nabla\phi^0(\xi_0; s, \omega) \cdot \nu} = O(1) \tag{5.53}$$

since Hopf's Lemma in the elliptic case [51, chapter 3, Theorem 1.1] implies that the normal derivative $\nabla\phi^0(\xi_0; s, \omega) \cdot \nu \neq 0$. Then by continuity, and the compactness of $\overline{\Omega_0} \times [0, 1] \times [0, 1]$, it follows that $\frac{\frac{\partial\phi^0}{\partial s}(\xi;s,\omega)}{\phi^0(\xi;s,\omega)}$ is bounded uniformly with respect to $(\xi_0, s, \omega) \in \Omega_0 \times [0, 1] \times [0, 1]$. Therefore $\omega_\varepsilon > 0$ can be chosen to satisfy (5.52). □

This leads to the following theorem. We recall also that $\int_0^1 D\lambda(\tilde{\Omega}(s))ds$ is a lower bound for $\mu(\omega)$ for every $\omega > 0$ (see Proposition 5.2).

Theorem 5.9. *Let $\tilde{\Omega}(s)$ be a smooth bounded domain that varies smoothly and 1-periodically with s, and for each $0 \leq s \leq 1$ let $\lambda(\tilde{\Omega}(s))$ be the principal Dirichlet eigenvalue of $-\nabla^2$ on $\tilde{\Omega}(s)$. Let $\mu(\omega)$ be the principal periodic eigenvalue associated with $\Omega(t) = \tilde{\Omega}\left(\frac{\omega t}{2\pi}\right)$. Then*

$$\lim_{\omega\to 0} \mu(\omega) = \int_0^1 D\lambda(\tilde{\Omega}(s))ds. \tag{5.54}$$

Proof. For $0 \le s \le 1$ and $\omega \ge 0$, let $\lambda^0(s,\omega)$ be as in Proposition 5.8. Then $\lim_{\omega\to 0}\mu(\omega)$ is given by equation (5.43). Now the change of variables $\tilde{h}$ from $\tilde{\Omega}(s)$ to Ω_0 transforms the operator $D\nabla^2$ on $\tilde{\Omega}(s)$ to

$$\sum_{i,j,k} D\left(\frac{\partial \tilde{h}_i}{\partial x_k}\frac{\partial \tilde{h}_j}{\partial x_k}\right)\frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_j D\nabla^2 \tilde{h}_j \frac{\partial}{\partial \xi_j} \quad \text{on } \Omega_0. \tag{5.55}$$

By equations (5.34) and (5.35), this is precisely $\mathcal{L}_0(\xi,s)$ (i.e. $\mathcal{L}_\omega(\xi,s)$ with $\omega = 0$). So we have

$$D\lambda(\tilde{\Omega}(s)) = \lambda^0(s,0) \tag{5.56}$$

and then (5.43) is equivalent to (5.54). □

Example 5.10. *Let* $\Omega(t) = (A(t), A(t)+L(t))$ *where* $A(t) = A_0 a\left(\frac{\omega t}{2\pi}\right)$ *and* $L(t) = L_0 l\left(\frac{\omega t}{2\pi}\right)$ *for some smooth and* 1*-periodic functions* $l(\cdot) > 0$ *and* $a(\cdot)$*. By Theorem 5.9, as* $\omega \to 0$ *the principal periodic eigenvalue* μ *converges to* $\int_0^1 \frac{D\pi^2}{L_0^2 l(s)^2} ds$*.*

In fact, since $\underline{Q}$ *and* $\overline{Q}$ *as defined in* (3.101) *are both* $O(\omega^2)$ *as* $\omega \to 0$*, we can conclude from the bounds* (5.19) *that*

$$\mu(\omega) = \int_0^1 \frac{D\pi^2}{L_0^2 l(s)^2} ds + O(\omega^2) \qquad \text{as } \omega \to 0. \tag{5.57}$$

Theorem 5.9 gives a way to numerically compute a value for $\lim_{\omega\to 0}\mu(\omega)$, which is also a lower bound on $\mu(\omega)$ for every $\omega > 0$, provided that we can compute the principal eigenvalue of the Laplacian on each domain $\tilde{\Omega}(s)$. The following corollary gives an alternative approach, for cases where $\tilde{\Omega}(s)$ is obtained by applying a known conformal mapping to some fixed domain Ω_0, for example a disk or an annulus. We can instead consider a set of weighted eigenvalue problems on this Ω_0, where just the weight function depends on s.

Corollary 5.11. *For* $0 \le s \le 1$ *let* $\tilde{\Omega}(s) \subset \mathbb{R}^2$ *be a smooth, connected, bounded domain, and assume the dependence on s is smooth and* 1*-periodic. Let* $\mu(\omega)$ *be the principal periodic eigenvalue associated with the domain* $\Omega(t) = \tilde{\Omega}(\frac{\omega t}{2\pi})$*.*

Identify $\mathbb{R}^2$ *with the complex plane* $\mathbb{C}$*, via*

$$z = x_1 + ix_2, \qquad w = \xi_1 + i\xi_2, \tag{5.58}$$

and let $w \mapsto q^{(s)}(w) = z$ *be a conformal mapping that maps* $\overline{\Omega_0}$ *to* $\overline{\tilde{\Omega}(s)}$ *for some smooth bounded reference domain* Ω_0*. For each* $0 \le s \le 1$ *let* $\lambda_q(s)$ *be the minimum eigenvalue of the following weighted eigenvalue problem on* Ω_0*:*

$$-D\nabla^2\phi(\xi) = \lambda|(q^{(s)})'(\xi_1 + i\xi_2)|^2\phi(\xi) \quad \text{in } \Omega_0, \qquad \phi(\xi) = 0 \quad \text{on } \partial\Omega_0. \tag{5.59}$$

Then $\mu(\omega) \ge \int_0^1 \lambda_q(s)ds$ *for every* $\omega > 0$*, and* $\lim_{\omega\to 0}\mu(\omega) = \int_0^1 \lambda_q(s)ds$*.*

Proof. Write $w = p^{(s)}(z)$ for the inverse of $z = q^{(s)}(w)$. Then the mapping $\xi_1 + i\xi_2 = p^{(s)}(x_1 + ix_2)$ corresponds to a change of variables $\xi = h(x, s)$ taking $x \in \overline{\tilde{\Omega}(s)}$ to $\xi \in \overline{\Omega_0}$. The Cauchy-Riemann equations imply that

$$\sum_k \frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_k} = |(p^{(s)})'(z)|^2\delta_{ij} = \frac{1}{|(q^{(s)})'(w)|^2}\delta_{ij}, \qquad \nabla^2 h_j \equiv 0. \tag{5.60}$$

So, by equations (5.34), (5.35) and (5.60), the Dirichlet eigenvalue problem of $-\mathcal{L}_0(\xi, s)$ on Ω_0 is equivalent to the following weighted eigenvalue problem:

$$\frac{-D}{|(q^{(s)})'(\xi_1 + i\xi_2)|^2}\nabla^2\phi(\xi) = \lambda\phi(\xi) \quad \text{in } \Omega_0, \qquad \phi(\xi) = 0 \quad \text{on } \partial\Omega_0. \tag{5.61}$$

In particular, if $\lambda^0(s, 0)$ denotes the principal Dirichlet eigenvalue of $-\mathcal{L}_0(\xi, s)$ on Ω_0, then $\lambda^0(s, 0) = \lambda_q(s)$. Then the claimed results follow from Proposition 5.2 and Theorem 5.9 since these show that for each $\omega > 0$,

$$\mu(\omega) \ge \lim_{\omega\to 0}\mu(\omega) = \int_0^1 D\lambda(\tilde{\Omega}(s))ds = \int_0^1 \lambda^0(s, 0)ds. \tag{5.62}$$

□

5.3.3 Asymptotic behaviour of $\mu(\omega)$ as $\omega \to \infty$

Consider next the limit $\omega \to \infty$. The large frequency limit of $\hat{\lambda}(\omega)$ (the principal periodic eigenvalue of (5.39), (5.40), (5.41)) is given in [46, Theorem 1.3(ii)]. They prove, by adapting an argument from [50, Theorem 3.10], that $\lim_{\omega\to\infty}\hat{\lambda}(\omega) = \lambda_\infty$ where λ_∞ is the principal eigenvalue of the elliptic operator whose coefficients are equal to the time-averages of those of $-\hat{\mathcal{L}}(\xi, s)$. Neither this result nor the analysis of [50] applies in cases such as ours when some of the coefficients depend on ω and become unbounded as $\omega \to \infty$.

As we shall see, in fact the behaviour of $\mu(\omega)$ in the limit $\omega \to \infty$ depends on the detail of the problem and there is no 'general' expression for the large ω limit. For one-dimensional domains $\Omega(t) = (A(t), A(t) + L(t))$, we shall give conditions under which $\mu(\omega)$ does and does not remain bounded as $\omega \to \infty$.

Theorem 5.12. *Let $l(\cdot)$ and $a(\cdot)$ be 1-periodic functions, belonging to $C^{2+\alpha}([0,1])$ for some $\alpha > 0$, and with $\min_{[0,1]} l = 1$, $\max_{[0,1]} |a| = 1$. For some $L_0 > 0$, $A_0 \geq 0$, $\omega > 0$, let*

$$L(t) = L_0 l\left(\frac{\omega t}{2\pi}\right), \qquad A(t) = A_0 a\left(\frac{\omega t}{2\pi}\right), \tag{5.63}$$

and let $\mu(\omega) = \mu_u(\omega)$ be the principal periodic eigenvalue associated with the domain $\Omega(t) = (A(t), A(t) + L(t))$. Then $\mu(\omega) = O(\omega^2)$ as $\omega \to \infty$, and if $a(\cdot)$ is constant, then $\mu(\omega) = O(1)$ as $\omega \to \infty$. If $a(\cdot)$ is non-constant, then there exist constants C_1, C_2 depending only on the functions l and a such that:

1. *If $\frac{A_0}{L_0} < C_1$ then $\mu(\omega) = O(1)$ as $\omega \to \infty$.*
2. *If $\frac{A_0}{L_0} > C_2$ then $\mu(\omega) = \underline{\overline{O}}(\omega^2)$ as $\omega \to \infty$.*

Proof. If $\max_{s\in[0,1]}(A_0 a(s)) < \min_{s\in[0,1]}(A_0 a(s) + L_0 l(s))$ then by Proposition 5.2 and Example 5.4, we have upper and lower bounds on $\mu(\omega)$ that are independent of $\omega > 0$:

$$\frac{D\pi^2}{L_0^2}\int_0^1 \frac{1}{l(s)^2} ds \leq \mu(\omega) \leq \frac{D\pi^2}{(\min_{[0,1]}(A_0 a + L_0 l) - \max_{[0,1]}(A_0 a))^2}. \tag{5.64}$$

So, $\mu(\omega) = O(1)$ as $\omega \to \infty$ as long as $\max_{[0,1]}(A_0 a) < \min_{[0,1]}(A_0 a + L_0 l)$. If $a(\cdot)$ is constant then this will be satisfied because, by assumption, $\min_{[0,1]}(L_0 l) > 0$. If $a(\cdot)$ is non-constant, then a sufficient condition is that

$$\frac{A_0}{L_0} < \frac{\min \tilde{l}}{\max \tilde{a} - \min \tilde{a}}. \tag{5.65}$$

Next, in order to prove the other claimed properties, we shall consider the bounds (5.19) that were proved in Proposition 5.5. Define non-negative constants c_1, c_2, c_3, c_4, c_5, c_6 in terms of the functions l and a as follows:

$$c_1 = \int_0^1 \frac{1}{l(s)^2} ds, \qquad c_2 = \int_0^1 a'(s)^2 ds, \qquad c_3 = \int_0^1 l(s)[a''(s)]^+ ds,$$
$$c_4 = \int_0^1 l(s)[l''(s)]^+ ds, \qquad c_5 = \int_0^1 l(s)[a''(s)]^- ds, \qquad c_6 = \int_0^1 l(s)[l''(s)]^- ds.$$

Then note that

$$\frac{1}{T}\int_0^T \frac{D\pi^2}{L(t)^2} dt = \frac{D\pi^2}{L_0^2} c_1, \qquad \frac{1}{T}\int_0^T \frac{\dot{A}(t)^2}{4D} dt = \left(\frac{\omega}{2\pi}\right)^2 \frac{A_0^2}{4D} c_2, \tag{5.66}$$

$$0 \leq \frac{1}{T}\int_0^T \frac{\overline{Q}(t)}{2D} dt \leq \left(\frac{\omega}{2\pi}\right)^2 \left(\frac{A_0 L_0}{2D} c_3 + \frac{L_0^2}{4D} c_4\right), \tag{5.67}$$

$$0 \leq \frac{1}{T}\int_0^T \frac{\underline{Q}(t)}{2D} dt \leq \left(\frac{\omega}{2\pi}\right)^2 \left(\frac{A_0 L_0}{2D} c_5 + \frac{L_0^2}{4D} c_6\right). \tag{5.68}$$

Therefore parts 1 and 3 of Proposition 5.5 imply that

$$\begin{aligned} &\frac{D\pi^2}{L_0^2} c_1 + \left(\frac{\omega}{2\pi}\right)^2 \left(\frac{A_0^2}{4D} c_2 - \frac{A_0 L_0}{2D} c_3 - \frac{L_0^2}{4D} c_4\right) \\ &\quad \leq \mu(\omega) \leq \frac{D\pi^2}{L_0^2} c_1 + \left(\frac{\omega}{2\pi}\right)^2 \left(\frac{A_0^2}{4D} c_2 + \frac{A_0 L_0}{2D} c_5 + \frac{L_0^2}{4D} c_6\right), \end{aligned} \tag{5.69}$$

which proves that $\mu(\omega) = O(\omega^2)$ as $\omega \to \infty$. Moreover, $\mu(\omega) = \overline{\underline{O}}(\omega^2)$ as $\omega \to \infty$ if $\frac{A_0^2}{4D} c_2 - \frac{A_0 L_0}{2D} c_3 - \frac{L_0^2}{4D} c_4 > 0$. If $a(\cdot)$ is non-constant then $c_2 \neq 0$, so this inequality will hold for $\frac{A_0}{L_0}$ large enough (depending on c_2, c_3, c_4). □

Remark 5.13. *Theorem 5.12 leads to questions about what happens in the $\omega \to \infty$ limit when $\frac{A_0}{L_0}$ is in the intermediate parameter range. For example, is there a threshold value of $\frac{A_0}{L_0}$ at which $\mu(\omega)$ stops being $O(1)$, and if so, what is it? To answer such questions may require different methods to those presented here.*

In the following example we give the estimates of Theorem 5.12 explicitly.

Example 5.14. *Let $L_0 > 0$ be constant and let $A(t) = A_0 \sin(\omega t)$ with $\omega > 0$, $A_0 > 0$. Consider the $\frac{2\pi}{\omega}$-periodic domain $\Omega(t) = (A(t), A(t) + L_0)$ and let*

$\mu(\omega) = \mu_u(\omega)$ *be the principal periodic eigenvalue of* $\Omega(t)$*. By Proposition 5.2 we conclude that*

$$\frac{D\pi^2}{L_0^2} \leq \mu(\omega) \qquad \text{for every } \omega > 0, \tag{5.70}$$

$$\text{and if } 2A_0 < L_0, \text{ then} \qquad \mu(\omega) \leq \frac{D\pi^2}{(L_0 - 2A_0)^2} \qquad \text{for every } \omega > 0. \tag{5.71}$$

To apply the bounds from Proposition 5.5, we calculate

$$\int_0^t \frac{\dot{A}(\zeta)^2}{4D} d\zeta = \frac{A_0^2\omega^2}{4D}\left(\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}\right). \tag{5.72}$$

We also calculate the $\frac{2\pi}{\omega}$*-periodic functions* $\overline{Q}(t)$ *and* $\underline{Q}(t)$ *as defined in* (3.101)*:*

$$\overline{Q}(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{\pi}{\omega} \\ -A_0L_0\omega^2\sin(\omega t) & \text{for } \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega}, \end{cases} \tag{5.73}$$

$$\underline{Q}(t) = \begin{cases} A_0L_0\omega^2\sin(\omega t) & \text{for } 0 \leq t \leq \frac{\pi}{\omega} \\ 0 & \text{for } \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega}, \end{cases} \tag{5.74}$$

and so

$$\int_0^{\frac{2\pi}{\omega}} \frac{\overline{Q}(\zeta)}{2D} d\zeta = \int_0^{\frac{2\pi}{\omega}} \frac{\underline{Q}(\zeta)}{2D} d\zeta = \frac{A_0L_0\omega}{D}. \tag{5.75}$$

By parts 1 and 3 of Proposition 5.5 we deduce that

$$\frac{D\pi^2}{L_0^2} + \frac{A_0^2\omega^2}{8D} - \frac{A_0L_0\omega^2}{2\pi D} \leq \mu(\omega) \leq \frac{D\pi^2}{L_0^2} + \frac{A_0^2\omega^2}{8D} + \frac{A_0L_0\omega^2}{2\pi D}, \tag{5.76}$$

which is the bound (5.69) *for this example. In agreement with Theorem 5.9 and Theorem 5.12, the bounds* (5.70), (5.71) *and* (5.76) *show that:*

$$\mu = \frac{D\pi^2}{L_0^2} + O(\omega^2) \text{ as } \omega \to 0. \tag{5.77}$$

$$\mu(\omega) = O(\omega^2) \text{ as } \omega \to \infty. \tag{5.78}$$

$$\text{If} \quad \frac{A_0}{L_0} < \frac{1}{2} \quad \text{then} \quad \mu(\omega) = O(1) \text{ as } \omega \to \infty. \tag{5.79}$$

$$\text{If} \quad \frac{A_0}{L_0} > \frac{4}{\pi} \quad \text{then} \quad \mu(\omega) = \underline{O}(\omega^2) \text{ as } \omega \to \infty. \tag{5.80}$$

It would be very interesting to understand the $\omega \to \infty$ *limit in the intermediate parameter range* $\frac{1}{2} \leq \frac{A_0}{L_0} \leq \frac{4}{\pi}$.

5.3.4 Monotonicity of $\mu(\omega)$ with respect to $\omega > 0$

It is proven in [46, Theorem 1.1] that, under the extra condition that their parabolic operator has no advection term (i.e. $\frac{\partial}{\partial \xi}$ term), the principal periodic eigenvalue $\hat{\lambda}(\omega)$ of (5.39), (5.40), (5.41) is non-decreasing with respect to $\omega > 0$: $\frac{d\hat{\lambda}(\omega)}{d\omega} \geq 0$. In this section, we shall consider the principal periodic eigenvalue, $\mu(\omega)$, for a periodic interval $\Omega(t) = (A(t), A(t) + L(t))$. This is the eigenvalue of the operator $\mathcal{L}$, which does have an advection term. However, we can convert our problem to one with no advection term and which has principal periodic eigenvalue $\mu_w(\omega)$. We shall use certain aspects of the proof of [46, Theorem 1.1] to derive a lower bound for $\frac{d\mu_w(\omega)}{d\omega}$. A modification to their proof is needed in order to account for the explicit dependence on ω in the coefficients. Finally, the conversion (5.94) between $\mu_w(\omega)$ and $\mu_u(\omega)$ (i.e. $\mu(\omega)$) involves adding a term which is quadratic in ω — this is related to the fact that the advection term in our operator $\mathcal{L}$ is proportional to ω, which was not the case in [46]. The consequent (linear in ω) term in $\frac{d\mu(\omega)}{d\omega}$, together with the bound on $\frac{d\mu_w(\omega)}{d\omega}$, is what allows us to prove a monotonicity result for $\mu(\omega)$ under certain conditions.

We consider the interval $\Omega(t) = (A(t), A(t) + L(t))$ where, as in Theorem 5.12, $L(t)$ and $A(t)$ are given by equation (5.63) for 1-periodic functions $l(\cdot) > 0$ and $a(\cdot)$ belonging to $C^{2+\alpha}([0, 1])$ for some $\alpha > 0$. We shall show in Theorem 5.15 that $\mu(\omega)$ is increasing in $\omega > 0$ provided that $a(\cdot)$ is non-constant and $\frac{A_0}{L_0}$ is sufficiently large.

Since we shall need to work with a parabolic operator with no advection term, we change variables to $w(\xi, t) = u(\xi, t)H(\xi, t)e^{-f_0 t}$ where (as before) $H(\xi, t)$ is given by equation (3.6), and we consider the operator that acts on $w(\xi, t)$. This is given in equation (3.11). Then we change variables to $s = \frac{\omega t}{2\pi}$, and this leads us to consider the operator

$$\mathcal{P}_\omega = \frac{\omega}{2\pi}\frac{\partial}{\partial s} - \frac{D}{l(s)^2}\frac{\partial^2}{\partial \xi^2} - \left(\frac{\omega}{2\pi}\right)^2 \left(\left(\frac{\xi}{L_0}\right) \frac{A_0 L_0}{2D} a''(s)l(s) + \left(\frac{\xi}{L_0}\right)^2 \frac{L_0^2}{4D} l''(s)l(s) \right) \tag{5.81}$$

on $0 < \xi < L_0$. This has the form

$$\mathcal{P}_\omega = \frac{\omega}{2\pi}\frac{\partial}{\partial s} - \tilde{D}(s)\frac{\partial^2}{\partial \xi^2} + V(\xi, s, \omega) \tag{5.82}$$

where $\tilde{D}(s) > 0$ and $V(\xi, s, \omega)$ are both periodic in s with period 1.

As in [46], let $u_\omega(\xi, s) > 0$ be the principal periodic eigenfunction of the operator $\mathcal{P}_\omega = \frac{\omega}{2\pi}\frac{\partial}{\partial s} - \tilde{D}(s)\frac{\partial^2}{\partial \xi^2} + V(\xi, s, \omega)$, and $v_\omega(\xi, s) > 0$ the principal periodic eigenfunction of its adjoint operator $\mathcal{P}^*_\omega = -\frac{\omega}{2\pi}\frac{\partial}{\partial s} - \tilde{D}(s)\frac{\partial^2}{\partial \xi^2} + V(\xi, s, \omega)$ (see [20]), with zero Dirichlet boundary conditions and with $u_\omega(\xi, s)$ and $v_\omega(\xi, s)$ normalised so that

$$\int_0^1 \int_0^{L_0} u_\omega(\xi, s)^2 d\xi ds = \int_0^1 \int_0^{L_0} u_\omega(\xi, s) v_\omega(\xi, s) d\xi ds = 1. \tag{5.83}$$

In [46, Lemma 2.1], given as Theorem A.9 here, Liu, Lou, Peng and Zhou prove that for every $\omega > 0$,

$$\int_0^1 \int_{\Omega_0} (v_\omega \mathcal{P}_\omega u_\omega - u_\omega \mathcal{P}_\omega v_\omega) d\xi ds \geq 0. \tag{5.84}$$

Consequently (see [46])

$$\begin{aligned} \int_0^1 \int_{\Omega_0} v_\omega \frac{\partial u_\omega}{\partial s} d\xi ds &= \frac{\pi}{\omega} \int_0^1 \int_{\Omega_0} v_\omega (\mathcal{P}_\omega - \mathcal{P}^*_\omega) u_\omega d\xi ds \\ &= \frac{\pi}{\omega} \int_0^1 \int_{\Omega_0} (v_\omega \mathcal{P}_\omega u_\omega - u_\omega \mathcal{P}_\omega v_\omega) d\xi ds \\ &\geq 0. \end{aligned} \tag{5.85}$$

When the coefficient V does not depend on ω (i.e. $\frac{\partial V}{\partial \omega}(\xi, s, \omega) \equiv 0$), it is proved in [46, Theorem 1.1] that the principal periodic eigenvalue $\hat{\lambda}(\omega)$ of their operator $\mathcal{P}_\omega$ satisfies

$$\frac{d\hat{\lambda}}{d\omega}(\omega) = \int_0^1 \int_0^{L_0} v_\omega(\xi, s) \frac{\partial u_\omega(\xi, s)}{\partial s} d\xi ds. \tag{5.86}$$

Combining this with (5.85) immediately gives $\frac{d\hat{\lambda}}{d\omega}(\omega) \geq 0$.

For our own case where $V(\xi, s, \omega)$ depends on ω, this argument must be modified. In Theorem 5.15 we prove a monotonicity result for $\mu(\omega)$ under certain conditions, with a proof adapted from that of [46, Theorem 1.1].

Theorem 5.15. *Let $L(t)$ and $A(t)$ be given by equation* (5.63) *for* 1*-periodic functions $l(\cdot) > 0$ and $a(\cdot)$ belonging to $C^{2+\alpha}([0,1])$ for some $\alpha > 0$, and assume $a(\cdot)$ is non-constant. Let $\mu(\omega) = \mu_u(\omega)$ be the principal periodic eigenvalue on $\Omega(t) = (A(t), A(t) + L(t))$. Then there exist constants C, $\beta > 0$ depending only on the functions l and a, such that if $\frac{A_0}{L_0} > C$ then $\frac{d\mu(\omega)}{d\omega} > \frac{\beta}{D}\omega$ for all $\omega > 0$.*

Proof. Let $u_\omega(\xi, s)$ and $v_\omega(\xi, s)$ be as above, for the operator $\mathcal{P}_\omega$ in (5.81). From [46, Lemma 2.1], or Theorem A.9 here, we know that (5.85) holds for each $\omega > 0$. (Although [46] assumes that V does not depend on ω, [46, Lemma 2.1] is a pointwise-in-ω result, and its proof is unchanged if the coefficient $V(\xi, s, \omega)$ depends on ω.)

As in the proof of [46, Theorem 1.1], we take the equation $\mathcal{P}_\omega u_\omega = \mu_w(\omega) u_\omega$ and differentiate the whole equation with respect to ω. Writing u'_ω for $\frac{\partial u_\omega}{\partial \omega}$ this becomes:

$$\mathcal{P}_\omega u'_\omega + \frac{1}{2\pi}\frac{\partial u_\omega(\xi, s)}{\partial s} + \frac{\partial V}{\partial \omega}(\xi, s, \omega) u_\omega = \mu_w(\omega) u'_\omega + \frac{d\mu_w(\omega)}{d\omega} u_\omega. \tag{5.87}$$

Since the coefficient $V(\xi, s, \omega)$ now depends on ω, the term $\frac{\partial V}{\partial \omega} u_\omega$ is new compared to those in [46].

Next, we multiply equation (5.87) by v_ω and integrate, using the fact that $\mathcal{P}^*_\omega v_\omega = \mu_w(\omega) v_\omega$ and the normalisation (5.83). We obtain the equation

$$\int_0^1 \int_0^{L_0} \left(\frac{v_\omega(\xi, s)}{2\pi} \frac{\partial u_\omega(\xi, s)}{\partial s} + \frac{\partial V}{\partial \omega}(\xi, s, \omega) u_\omega(\xi, s) v_\omega(\xi, s) \right) d\xi ds = \frac{d\mu_w}{d\omega}(\omega). \tag{5.88}$$

Upon combining this with (5.85), we get

$$\frac{d\mu_w}{d\omega}(\omega) \geq \int_0^1 \int_0^{L_0} \frac{\partial V}{\partial \omega}(\xi, s, \omega) u_\omega(\xi, s) v_\omega(\xi, s) d\xi ds, \tag{5.89}$$

and recalling the positivity of u_ω and v_ω, and the normalisation (5.83), this gives the lower bound

$$\frac{d\mu_w}{d\omega}(\omega) \geq \min_{0 \leq \xi \leq L_0, 0 \leq s \leq 1} \frac{\partial V}{\partial \omega}(\xi, s, \omega). \tag{5.90}$$

Now consider the form of the coefficient $V(\xi, s, \omega)$ in our operator. Define

$$b_1 = \max_{s \in [0,1]} |a''(s) l(s)|, \qquad b_2 = \max_{s \in [0,1]} |l''(s) l(s)|. \tag{5.91}$$

Then, uniformly in $0 \le \xi \le L_0$ and $0 \le s \le 1$, we have the bound

$$\left| \left(\frac{\xi}{L_0}\right) \frac{A_0 L_0}{2D} a''(s) l(s) + \left(\frac{\xi}{L_0}\right)^2 \frac{L_0^2}{4D} l''(s) l(s) \right| \le \frac{A_0 L_0}{2D} b_1 + \frac{L_0^2}{4D} b_2, \tag{5.92}$$

and so the coefficient $V(\xi, s, \omega)$ in our operator (see equation (5.81)) satisfies

$$\frac{\partial V}{\partial \omega}(\xi, s, \omega) \ge -\frac{2\omega}{(2\pi)^2} \left(\frac{A_0 L_0}{2D} b_1 + \frac{L_0^2}{4D} b_2 \right) \qquad \text{for all } 0 \le \xi \le L_0,\ 0 \le s \le 1. \tag{5.93}$$

Also, if $c_2 = \int_0^1 a'(s)^2 ds$ then we know from part 1 of Proposition 5.5 that the eigenvalues $\mu_u(\omega)$ and $\mu_w(\omega)$ are related by

$$\mu_u(\omega) = \mu_w(\omega) + \left(\frac{\omega}{2\pi}\right)^2 \frac{A_0^2}{4D} c_2. \tag{5.94}$$

By combining this with (5.90) and (5.93), we have

$$\frac{d\mu_u}{d\omega} = \frac{d\mu_w}{d\omega} + \frac{2\omega}{(2\pi)^2} \frac{A_0^2}{4D} c_2 \ge \frac{2\omega}{(2\pi)^2} \left(-\frac{A_0 L_0}{2D} b_1 - \frac{L_0^2}{4D} b_2 + \frac{A_0^2}{4D} c_2 \right) \quad \text{for all } \omega > 0. \tag{5.95}$$

Therefore if

$$A_0^2 c_2 - 2 A_0 L_0 b_1 - L_0^2 b_2 > 0 \tag{5.96}$$

then there exists $\beta > 0$ such that $\frac{d\mu_u}{d\omega}(\omega) \ge \frac{\beta}{D}\omega$ for all $\omega > 0$. Since by assumption $c_2 \neq 0$, the inequality (5.96) will be satisfied whenever $\frac{A_0}{L_0}$ is sufficiently large (depending on c_2, b_1, b_2). □

Example 5.16. *Let $\Omega(t) = (A(t), A(t) + L_0)$ with $A(t)$ as in Example 5.14. Then $b_1 = 4\pi^2$, $b_2 = 0$ and $c_2 = 2\pi^2$, and so we find that* (5.96) *holds and $\mu_u(\omega)$ is monotonically increasing in $\omega > 0$ provided that $\frac{A_0}{L_0} > 4$.*

5.4 Nonlinear equation on a T-periodic domain

In this section, we consider the nonlinear periodic parabolic problem (2.1), (2.2) where f is assumed to satisfy the conditions (2.20); that is, for some $K > 0$,

$$f(0) = f(K) = 0, \quad f \text{ is Lipschitz continuous}, \quad f'(0) \text{ exists and } > 0,$$

$\frac{f(u)}{u}$ is non-increasing on $u > 0$.

As above, let μ and $\phi(\xi, t)$ be the principal periodic eigenvalue and eigenfunction satisfying equations (5.3), (5.4), (5.5), (5.6), and normalised so that $||\phi||_\infty = 1$. Now the solution to the linear equation (with $f_0 = f'(0)$) is a supersolution to the nonlinear problem, so if $f'(0) < \mu$ then $u \to 0$ as $t \to \infty$.

From now on, assume $f'(0) > \mu$. Fix any $\gamma \in (0, f'(0) - \mu)$. Then since $f(u) = f'(0)u + o(u)$ as $u \to 0$, there exists $\varepsilon > 0$ (depending on γ) such that for all $0 \le u \le \varepsilon$,

$$(\gamma - f'(0) + \mu)u + (f'(0)u - f(u)) \le 0. \tag{5.97}$$

Now, for every $0 < \delta \le \varepsilon e^{-\gamma T}$, the function $\hat{u}(\xi, t) = \delta\phi(\xi, t)e^{\gamma t}$ is a subsolution for $u(\xi, t)$ over $0 \le t \le T$:

$$\frac{\partial \hat{u}}{\partial t} - \mathcal{L}\hat{u} - f(\hat{u}) = \gamma\hat{u} + \mu\hat{u} - f(\hat{u}) \tag{5.98}$$

$$= (\gamma - f'(0) + \mu)\hat{u} + (f'(0)\hat{u} - f(\hat{u})) \ \le 0 \tag{5.99}$$

since $\hat{u}(\xi, t) \le \varepsilon$ for $0 \le t \le T$. The function $\hat{u}$ also satisfies $\hat{u}(\xi, t) = 0$ on $\partial\Omega_0$, and $\hat{u}(\xi, 0) \le \hat{u}(\xi, T)$, and so it is a subsolution to the periodic problem (2.1), (2.2) in the sense of Hess [37, chapter III Definition 21.1]. Moreover the constant K is a supersolution. By applying [37, Theorem 22.3, chapter III], there exists a stable periodic solution $u^*(\xi, t)$ to

$$\frac{\partial u^*}{\partial t} = \mathcal{L}u^* + f(u^*) \qquad \text{for } \xi \in \Omega_0,\ t \in \mathbb{R} \tag{5.100}$$

$$u^*(\xi, t) = 0 \qquad \text{for } \xi \in \partial\Omega_0 \tag{5.101}$$

$$u^*(\xi, t) \equiv u^*(\xi, t + T) \tag{5.102}$$

such that

$$\varepsilon\phi(\xi, t)e^{\gamma(t-T)} \le u^*(\xi, t) \le K \qquad \text{for } \xi \in \Omega_0,\ 0 \le t \le T. \tag{5.103}$$

We now wish to prove that the periodic solution u^* is unique and that for every non-zero initial condition $0 \le u(\cdot, 0) \le K$, the solution u to (2.1), (2.2) converges to u^*.

Remark 5.17. *It is straightforward to derive a lower bound on* $u(\xi, t)$ *showing that it does not converge to zero. We can assume without loss of generality that there exists* $0 < \delta \leq \varepsilon e^{-\gamma T}$ *such that* $\delta\phi(\xi, 0) \leq u(\xi, 0)$*. Then since* $\delta\phi(\xi, t)e^{\gamma t}$ *is a subsolution on* $0 \leq t \leq T$*, we have*

$$\delta\phi(\xi, t')e^{\gamma t'} \leq u(\xi, t') \qquad \text{for all } 0 \leq t' \leq T. \tag{5.104}$$

Then $u(\xi, T) \geq \delta\phi(\xi, T)e^{\gamma T} = \delta\phi(\xi, 0)e^{\gamma T} \geq \delta\phi(\xi, 0)$ *and, by applying this argument repeatedly, we can conclude that*

$$\delta\phi(\xi, t')e^{\gamma t'} \leq u(\xi, t' + nT) \qquad \text{for all } 0 \leq t' \leq T,\ n \in \mathbb{N}. \tag{5.105}$$

Therefore,

$$\liminf_{t\to\infty} u(\xi, t) \geq \delta \min_{0\leq t'\leq T}(\phi(\xi, t')e^{\gamma t'}). \tag{5.106}$$

To prove the convergence to u^*, we shall use the Poincaré map P_T. For each $\tau > 0$, define P_τ to be the map $P_\tau(u_0) = u(\cdot, \tau)$ where $u(\xi, t)$ is the solution to the problem (2.1), (2.2) with initial conditions $u(\cdot, 0) = u_0(\cdot)$. Since the coefficients are periodic, this is the same as the map taking $u(\cdot, nT)$ to $u(\cdot, nT+\tau)$ for every $n \in \mathbb{N}$. The Poincaré map is P_T, which takes the solution at time nT to the solution at time $(n + 1)T$. Note that if u^* is any T-periodic solution satisfying equations (5.100), (5.101), (5.102) then $u^*(\cdot, 0)$ is a fixed point of the Poincaré map P_T. We shall use the following two properties of P_τ.

Lemma 5.18. *For each* $\tau > 0$*, the map* P_τ *is monotonic, in the sense that if* $u_0 \leq v_0$ *then* $P_\tau(u_0) \leq P_\tau(v_0)$*. Moreover, either* $u_0 \equiv v_0$ *or there is strict inequality* $P_\tau(u_0) < P_\tau(v_0)$ *on* Ω_0 *and* $\frac{\partial}{\partial\nu}P_\tau(u_0) \neq \frac{\partial}{\partial\nu}P_\tau(v_0)$ *on* $\partial\Omega_0$*.*

Proof. This is a consequence of the parabolic comparison principle, strong maximum principle, and Hopf's Lemma (see [51, chapter 2, Theorem 1.4]). □

Lemma 5.19. *Let* f *satisfy* (2.20)*. Then for each* $\tau > 0$*, the map* P_τ *is sublinear, in the following sense. Let* $0 \leq \alpha \leq 1$*, and let* $u_0 > 0$ *on* Ω_0 *with* $u_0 = 0$ *and* $\frac{\partial u_0}{\partial\nu} \neq 0$ *on* $\partial\Omega_0$*. Then*

$$\alpha P_\tau(u_0) \leq P_\tau(\alpha u_0). \tag{5.107}$$

Proof. If $\alpha = 0$ or 1 then it is obvious, so assume $0 < \alpha < 1$. Let $u(\xi, t)$ be the solution to (2.1), (2.2) with initial conditions $u(\xi, 0) = u_0(\xi)$ and $v(\xi, t)$ the solution with initial conditions $v(\xi, 0) = \alpha u_0(\xi)$. We need to show that $v(\xi, t) \geq \alpha u(\xi, t)$ for all $t \geq 0$.

By the assumption that $\frac{f(k)}{k}$ is non-increasing on $k > 0$, we have that $f(\alpha u_0) \geq \alpha f(u_0)$. For $\varepsilon > 0$ small, define $f_\varepsilon(k) = f(k) - \varepsilon k^2$, so that $\frac{f_\varepsilon(k)}{k}$ is strictly decreasing in $k > 0$ and

$$f_\varepsilon(\alpha u_0) - \alpha f_\varepsilon(u_0) \geq \varepsilon\alpha(1-\alpha)u_0^2 \ > 0 \quad \text{in } \Omega_0. \tag{5.108}$$

Let v_ε, u_ε be the corresponding solutions to the problem with f replaced by f_ε:

$$\frac{\partial u_\varepsilon}{\partial t} = \mathcal{L}u_\varepsilon + f_\varepsilon(u_\varepsilon), \qquad \frac{\partial v_\varepsilon}{\partial t} = \mathcal{L}v_\varepsilon + f_\varepsilon(v_\varepsilon) \tag{5.109}$$

with $v_\varepsilon(\xi, 0) = \alpha u_\varepsilon(\xi, 0) = \alpha u_0(\xi)$. We shall show that $v_\varepsilon(\xi, t) \geq \alpha u_\varepsilon(\xi, t)$ for every $t \geq 0$. Then by taking $\varepsilon \to 0$ we conclude that the same inequality holds for the solutions v, u with the original reaction function f.

At $t = 0$ we have

$$\begin{aligned}\frac{\partial}{\partial t}(v_\varepsilon - \alpha u_\varepsilon)|_{t=0} =& \mathcal{L}(\alpha u_0) + f_\varepsilon(\alpha u_0) - \alpha\mathcal{L}(u_0) - \alpha f_\varepsilon(u_0)\\ =& f_\varepsilon(\alpha u_0) - \alpha f_\varepsilon(u_0)\\ \geq& \varepsilon\alpha(1-\alpha)u_0^2.\end{aligned} \tag{5.110}$$

Therefore, there exists $\hat{t} > 0$ such that $v_\varepsilon(\xi, t) \geq \alpha u_\varepsilon(\xi, t)$ for $0 \leq t \leq \hat{t}$. We claim that $\hat{t}$ can be taken as large as we like. Suppose not, and let t^* be the maximal such that $v_\varepsilon(\xi, t) \geq \alpha u_\varepsilon(\xi, t)$ for $0 \leq t \leq t^*$. Let $\tilde{v}_\varepsilon$ be the solution on $t \geq t^*$ with $\tilde{v}_\varepsilon(\xi, t^*) = \alpha u_\varepsilon(\xi, t^*)$. Then by applying the same argument as above, to the function $\tilde{v}_\varepsilon$ at time t^*, we deduce that

$$\frac{\partial}{\partial t}(\tilde{v}_\varepsilon - \alpha u_\varepsilon)|_{t=t^*} \geq \varepsilon\alpha(1-\alpha)u_\varepsilon(\cdot, t^*)^2, \tag{5.111}$$

and so there exists $\Delta t > 0$ such that $\tilde{v}_\varepsilon \geq \alpha u_\varepsilon$ for $t^* \leq t \leq t^* + \Delta t$. But the comparison principle gives $v_\varepsilon \geq \tilde{v}_\varepsilon$ for all $t \geq t^*$, and so $v_\varepsilon \geq \alpha u_\varepsilon$ for $t^* \leq t \leq t^* + \Delta t$, contradicting the maximality of t^*. Therefore, we do have $v_\varepsilon(\xi, t) \geq \alpha u_\varepsilon(\xi, t)$ for all $t \geq 0$, as required. □

Now using the monotonicity and sublinearity properties of P_T, we shall prove the uniqueness of a periodic solution, given ordering.

Theorem 5.20. *Suppose $f'(0) > \mu$, and suppose that $\underline{U}(\xi, t)$, $\overline{U}(\xi, t)$ are both positive, T-periodic solutions to the problem* (5.100), (5.101), (5.102), *and such that* $0 \le \underline{U}(\xi, 0) \le \overline{U}(\xi, 0)$ *for all* $\xi \in \Omega_0$. *Then* $\underline{U}(\xi, t) \equiv \overline{U}(\xi, t)$.

Proof. By the strong maximum principle and Hopf's Lemma [51, chapter 2, Theorem 1.4], we know that $0 < \underline{U}(\xi, 0) \le \overline{U}(\xi, 0)$ for all $\xi \in \Omega_0$, and that $\underline{U}$ and $\overline{U}$ have non-zero normal derivatives on $\partial\Omega_0$. Therefore for $r > 0$ small enough we have $r\overline{U}(\xi, 0) \le \underline{U}(\xi, 0)$ for all $\xi \in \Omega_0$. On the other hand this does not hold for any $r > 1$. Let

$$\hat{r} = \sup\{r \in (0, 1) : r\overline{U}(\xi, 0) \le \underline{U}(\xi, 0) \text{ for all } \xi \in \Omega_0\}. \tag{5.112}$$

Then we know that

$$\hat{r}\overline{U}(\xi, 0) \le \underline{U}(\xi, 0) \text{ for all } \xi \in \Omega_0 \tag{5.113}$$

and by maximality of $\hat{r}$ there exists some

$$\begin{aligned} &\xi_0 \in \Omega_0 \quad \text{such that} \quad \hat{r}\overline{U}(\xi_0, 0) = \underline{U}(\xi_0, 0) \\ \text{or} \qquad &\xi_0 \in \partial\Omega_0 \quad \text{such that} \quad \hat{r}\frac{\partial \overline{U}}{\partial \nu}(\xi_0, 0) = \frac{\partial \underline{U}}{\partial \nu}(\xi_0, 0). \end{aligned} \tag{5.114}$$

Now we apply the Poincaré map, P_T. By the monotonicity (Lemma 5.18) we have

$$P_T(\hat{r}\overline{U}(\cdot, 0)) \le P_T(\underline{U}(\cdot, 0)) \tag{5.115}$$

with either $\hat{r}\overline{U} \equiv \underline{U}$ or else strict inequality

$$P_T(\hat{r}\overline{U}(\cdot, 0)) < P_T(\underline{U}(\cdot, 0)) \qquad \text{on } \Omega_0 \tag{5.116}$$

$$\text{and} \qquad \frac{\partial}{\partial \nu}P_T(\hat{r}\overline{U}(\cdot, 0)) \ne \frac{\partial}{\partial \nu}P_T(\underline{U}(\cdot, 0)) \qquad \text{on } \partial\Omega_0. \tag{5.117}$$

Combining this with the sublinearity property (Lemma 5.19) and the fact that $\overline{U}$ and $\underline{U}$ are fixed points of P_T, we find that

$$\hat{r}\overline{U}(\cdot, 0) = \hat{r}P_T(\overline{U}(\cdot, 0)) \le P_T(\hat{r}\overline{U}(\cdot, 0)) \le P_T(\underline{U}(\cdot, 0)) = \underline{U}(\cdot, 0) \tag{5.118}$$

and that either $\hat{r}\overline{U} \equiv \underline{U}$ or else equations (5.116) and (5.117) hold. Incorporating these strict inequalities into equation (5.118) would contradict the existence of ξ_0 as in equation (5.114). Therefore, in fact

$$\hat{r}\overline{U} \equiv \underline{U} \qquad \text{on } \overline{\Omega_0} \times [0, T]. \tag{5.119}$$

This shows that $\overline{U}$ and $\hat{r}\overline{U}$ are both solutions to (5.100), (5.101), (5.102), and hence $\hat{r}f(\overline{U}) \equiv f(\hat{r}\overline{U})$. By the assumption that $\frac{f(u)}{u}$ is non-increasing on $u > 0$, this implies that either $\hat{r} = 1$ or else $f(\overline{U}) \equiv f'(0)\overline{U}$. But we know that $\overline{U}$ does not satisfy the linear equation because that would contradict the fact that $f'(0) > \mu$. Therefore, it must be that $\hat{r} = 1$ and $\underline{U} \equiv \overline{U}$. □

Next we prove convergence to $u^*(\xi, t)$ (the positive T-periodic solution to (5.100), (5.101), (5.102) whose existence is guaranteed by [37, Theorem 22.3, chapter III]).

Theorem 5.21. *Assume that f satisfies* (2.20) *and $f'(0) > \mu$, and let $u^*(\xi, t)$ be a positive T-periodic solution to* (5.100), (5.101), (5.102). *Given non-zero initial conditions $0 \le u(\xi, 0) \le K$, let $u(\xi, t)$ be the solution to the nonlinear problem* (2.1), (2.2), *and for $n \in \mathbb{N}$ define $u_n(\xi, t) = u(\xi, nT + t)$. Then as $n \to \infty$, u_n converges to u^* in $C^{2,1}(\overline{\Omega_0} \times [0, T])$. In particular, u^* is unique.*

Proof. Without loss of generality we can assume that

$$\delta u^*(\xi, 0) \le u(\xi, 0) \le B u^*(\xi, 0) \tag{5.120}$$

for some $0 < \delta \le 1$ and $B \ge 1$. Let $\underline{u}(\xi, t)$ and $\overline{u}(\xi, t)$ be the solutions to (2.1), (2.2) with initial conditions $\underline{u}(\xi, 0) = \delta u^*(\xi, 0)$ and $\overline{u}(\xi, 0) = B u^*(\xi, 0)$. By the comparison principle,

$$\underline{u}(\cdot, t) \le u(\cdot, t) \le \overline{u}(\cdot, t) \quad \text{and} \quad \underline{u}(\cdot, t) \le u^*(\cdot, t) \le \overline{u}(\cdot, t) \tag{5.121}$$

for all $t \ge 0$. For $n \in \mathbb{N}$ define $u_n(\xi, t) = u(\xi, nT + t)$; also $\underline{u}_n(\xi, t) = \underline{u}(\xi, nT + t)$ and $\overline{u}_n(\xi, t) = \overline{u}(\xi, nT + t)$. Then

$$\underline{u}_n(\cdot, t) \le u_n(\cdot, t) \le \overline{u}_n(\cdot, t) \qquad \text{and} \qquad \underline{u}_n(\cdot, t) \le u^*(\cdot, t) \le \overline{u}_n(\cdot, t) \tag{5.122}$$

for all $0 \le t \le T$, $n \in \mathbb{N}$. Using the fact that $u^*(\cdot, 0)$ is a fixed point of the Poincaré map P_T, together with the sublinearity (Lemma 5.19), we get that

$$\underline{u}(\xi,0) = \delta u^*(\xi,0) = \delta P_T(u^*(\xi,0)) \le P_T(\delta u^*(\xi,0)) = P_T(\underline{u}(\xi,0)) = \underline{u}(\xi,T) \tag{5.123}$$

and

$$\frac{1}{B}\overline{u}(\xi,0) = u^*(\xi,0) = P_T(u^*(\xi,0)) \\ \ge \frac{1}{B}P_T(Bu^*(\xi,0)) = \frac{1}{B}P_T(\overline{u}(\xi,0)) = \frac{1}{B}\overline{u}(\xi,T). \tag{5.124}$$

Therefore, $\underline{u}(\xi,0) \le \underline{u}(\xi,T)$ and $\overline{u}(\xi,T) \le \overline{u}(\xi,0)$. By applying P_T again and using the monotonicity property (Lemma 5.18) and the ordering (5.121), we deduce that

$$\underline{u}(\xi,nT) \le \underline{u}(\xi,(n+1)T) \le u^*(\xi,0) \le \overline{u}(\xi,(n+1)T) \le \overline{u}(\xi,nT) \tag{5.125}$$

for all $\xi \in \Omega_0$, $n \in \mathbb{N}$. Therefore, pointwise limits $\underline{v}(\xi) \le \overline{v}(\xi)$ exist such that $\underline{v}(\xi) \le u^*(\xi,0) \le \overline{v}(\xi)$ and

$$\underline{u}(\xi,nT) \to \underline{v}(\xi), \qquad \overline{u}(\xi,nT) \to \overline{v}(\xi) \qquad \text{as } n \to \infty. \tag{5.126}$$

We can apply Theorem A.3 to $\underline{u}_n$ to deduce that there is a subsequence $\underline{u}_{n_k}$ that converges in $C^{2,1}(\overline{\Omega_0} \times [0,T])$ to a solution $\underline{U}(\xi,t)$ of the nonlinear parabolic problem (2.1), (2.2). By equating this to the pointwise limit at times 0 and T, we have that $\underline{U}(\xi,0) = \underline{U}(\xi,T) = \underline{v}(\xi)$. Likewise, there is a subsequence $\overline{u}_{n_r}$ of $\overline{u}_n$ that converges in $C^{2,1}(\overline{\Omega_0} \times [0,T])$ to a solution $\overline{U}(\xi,t)$ of (2.1), (2.2), with $\overline{U}(\xi,0) = \overline{U}(\xi,T) = \overline{v}(\xi)$.

Recall that $\underline{U}(\xi,0) = \underline{v}(\xi) \le u^*(\xi,0) \le \overline{v}(\xi) = \overline{U}(\xi,0)$ and so by the comparison principle, $\underline{U}(\xi,t) \le u^*(\xi,t) \le \overline{U}(\xi,t)$ for all $t \ge 0$. Therefore $\underline{U}$ and $\overline{U}$ satisfy the conditions of Theorem 5.20, and we conclude that

$$\underline{U} \equiv \overline{U} \equiv u^*. \tag{5.127}$$

Since the limit is uniquely identified, Lemma A.5 implies that actually the whole sequences $\underline{u}_n$ and $\overline{u}_n$ converge to u^* as $n \to \infty$ and the convergence is in $C^{2,1}(\overline{\Omega_0} \times [0, T])$. But since u_n satisfies (5.122), it must also converge uniformly to u^* as $n \to \infty$, and by the same argument as above (applying Theorem A.3 and Lemma A.5) the convergence is in $C^{2,1}(\overline{\Omega_0} \times [0, T])$. $\square$

The convergence of $u(\xi, nT + t)$ to a unique positive T-periodic solution $u^*(\xi, t)$ on $\Omega_0 \times [0, T]$ can now be interpreted in terms of the original problem for $\psi(x, t)$ on the T-periodic domain $\Omega(t)$. The function $u^*(\xi, t)$ for $\xi \in \Omega_0$ corresponds to a positive solution $\psi^*(x, t)$ to (1.1), (1.2) that satisfies $\psi(x, t) \equiv \psi(x, t + T)$ for all $x \in \Omega(t)$, $t \in \mathbb{R}$. Theorem 5.21 means that $\psi(x, nT + t)$ converges uniformly to $\psi^*(x, t)$ as $n \to \infty$.

Chapter 6

Nonlinear equation on a bounded domain Ω_0 moving at a constant velocity c

We assume throughout this chapter that f is a nonlinear function satisfying assumptions (2.20); that is, for some $K > 0$,

$$f(0) = f(K) = 0, \quad f \text{ is Lipschitz continuous}, \quad f'(0) \text{ exists and } > 0,$$
$$\frac{f(u)}{u} \text{ is non-increasing on } u > 0,$$

and that the domain has the form $\Omega(t) = \Omega_0 + ct$ where Ω_0 is bounded (either smooth and bounded or box-like) and c is a constant vector. We prove convergence to either zero or a positive stationary limit U_c which is unique. We also derive a number of properties of this positive limit, many of which will subsequently be used in Chapter 7.

6.1 Convergence to $U_c(\xi)$ or zero

Let $\xi = x - ct$ and $u(\xi, t) = \psi(x, t)$. Then we have

$$\frac{\partial u}{\partial t} = D\nabla^2 u + c \cdot \nabla u + f(u) \qquad \text{for } \xi \in \Omega_0 \tag{6.1}$$

$$u(\xi, t) = 0 \qquad \text{on } \partial\Omega_0. \tag{6.2}$$

Let $\lambda(\Omega_0)$ and y_1 denote the principal eigenvalue and eigenfunction of $-\nabla^2$ on the bounded domain Ω_0 with zero Dirichlet boundary conditions:

$$\nabla^2 y_1 = -\lambda(\Omega_0) y_1 \quad \text{in } \Omega_0, \qquad y_1 = 0 \text{ on } \partial\Omega_0, \qquad y_1 > 0 \text{ in } \Omega_0. \tag{6.3}$$

Define $\phi_1^{(c)}(\xi) = y_1(\xi) e^{-\frac{c\cdot\xi}{2D}}$ and normalise to $\max_{\overline{\Omega_0}} \phi_1^{(c)} = 1$. This is the principal eigenfunction of

$$D\nabla^2\phi + c \cdot \nabla\phi = -\mu\phi \quad \text{in } \Omega_0, \qquad \phi = 0 \text{ on } \partial\Omega_0, \tag{6.4}$$

and has principal eigenvalue $\mu = D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. We shall see that the long-time behaviour of $u(\xi, t)$ depends on whether $f'(0) - D\lambda(\Omega_0) - \frac{|c|^2}{4D}$ is < 0 or > 0. First we show that there is convergence to zero if $f'(0) < D\lambda(\Omega_0) + \frac{|c|^2}{4D}$.

Proposition 6.1. *If $f'(0) < D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ then as $t \to \infty$, $u(\xi, t) \to 0$ in $C^2(\overline{\Omega_0})$, and $\frac{\partial u}{\partial t}(\xi, t) \to 0$ uniformly in $\overline{\Omega_0}$.*

Proof. For $B > 0$ large enough, $u(\cdot, 0) \le B\phi_1^{(c)}$ on Ω_0. Then the function

$$\overline{u}(\xi, t) = B\phi_1^{(c)}(\xi) \exp\left(\left(f'(0) - D\lambda(\Omega_0) - \frac{|c|^2}{4D}\right) t\right) \tag{6.5}$$

is a supersolution for u, and tends to zero uniformly on Ω_0. Proposition 2.13 then implies the claimed space and time derivatives also converge to zero. □

Next we consider the case $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$.

Lemma 6.2. *Suppose $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. There exists $\delta_0 > 0$ such that for $0 < \delta \le \delta_0$, the function $\hat{u}(\xi) = \delta\phi_1^{(c)}(\xi)$ is a subsolution to the elliptic problem*

$$D\nabla^2 U + c \cdot \nabla U + f(U(\xi)) = 0 \qquad \textit{in } \Omega_0 \tag{6.6}$$

$$U(\xi) = 0 \qquad \textit{on } \partial\Omega_0. \tag{6.7}$$

Proof.

$$D\nabla^2\hat{u} + c \cdot \nabla\hat{u} + f(\hat{u}) = \left(-D\lambda(\Omega_0) - \frac{|c|^2}{4D} + f'(0)\right)\hat{u} + f(\hat{u}) - f'(0)\hat{u} \tag{6.8}$$

$$= \left(f'(0) - D\lambda(\Omega_0) - \frac{|c|^2}{4D}\right)\hat{u} + o(\hat{u}) \quad \text{as } \delta \to 0. \tag{6.9}$$

There exists $\delta_0 > 0$ such that the right hand side is ≥ 0 for all $0 < \delta \le \delta_0$. □

In Theorem 6.4 we shall show that if $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ then $u(\xi, t)$ converges as $t \to \infty$ to a positive solution to the elliptic problem (6.6), (6.7). First, we show in Theorem 6.3 that such a solution is unique.

Theorem 6.3. *Suppose $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. There exists a unique positive solution $0 < U_c(\xi) \le K$ to the elliptic problem (6.6), (6.7).*

Proof. Fix any $0 < \delta \le \delta_0$ and let $\hat{u}(\xi) = \delta\phi_1^{(c)}(\xi)$. This is a subsolution for (6.6), (6.7) by Lemma 6.2, and the function $\tilde{u} = K$ is a supersolution. By applying the monotone iteration scheme results of [51, section 3.2], starting with initial iterations $\hat{u}$ and $\tilde{u}$, we deduce the existence of solutions $\underline{u}$ and $\overline{u}$ such that $\hat{u} \le \underline{u} \le \overline{u} \le \tilde{u}$ and that every solution u with $\hat{u} \le u \le \tilde{u}$ satisfies $\underline{u} \le u \le \overline{u}$. We shall show that in fact $\underline{u} \equiv \overline{u}$. The uniqueness result will then follow, since every non-negative, non-zero solution U of the elliptic problem (6.6), (6.7) must be strictly positive (by the elliptic strong maximum principle) and have a non-zero normal derivative on $\partial\Omega_0$ (by Hopf's Lemma in the elliptic case [51, chapter 3, Theorem 1.1]), and so $\delta\phi_1^{(c)} \le U$ for some $0 < \delta \le \delta_0$.

We multiply the $\overline{u}$ equation by $\underline{u}e^{c\cdot\xi/D}$ and vice-versa, subtract, and integrate (by parts) over Ω_0. This yields that

$$\begin{aligned} 0 &= \int_{\Omega_0} e^{c\cdot\xi/D}\left(\underline{u}(D\nabla^2\overline{u} + c\cdot\nabla\overline{u}) - \overline{u}(D\nabla^2\underline{u} + c\cdot\nabla\underline{u}) + \underline{u}f(\overline{u}) - \overline{u}f(\underline{u})\right) d\xi \\ &= \int_{\Omega_0} e^{c\cdot\xi/D}\underline{u}\,\overline{u}\left(\frac{f(\overline{u})}{\overline{u}} - \frac{f(\underline{u})}{\underline{u}}\right) d\xi \le 0 \end{aligned} \tag{6.10}$$

since $\frac{f(u)}{u}$ is a non-increasing function and since $\overline{u} \ge \underline{u} > 0$ in Ω_0. Therefore there must be equality in (6.10) and

$$\frac{f(\overline{u})}{\overline{u}} \equiv \frac{f(\underline{u})}{\underline{u}} \qquad \text{in } \Omega_0. \tag{6.11}$$

Then the function $\overline{u} - \underline{u} \ge 0$ satisfies $\overline{u} - \underline{u} = 0$ on $\partial\Omega_0$ and

$$D\nabla^2(\overline{u} - \underline{u}) + c\cdot\nabla(\overline{u} - \underline{u}) + \frac{f(\overline{u})}{\overline{u}}(\overline{u} - \underline{u}) = 0. \tag{6.12}$$

If there is any interior point at which $\overline{u} - \underline{u} = 0$, then by the strong maximum principle, $\overline{u} - \underline{u} \equiv 0$. If there is no such interior point, then $\overline{u} - \underline{u} > 0$ inside Ω_0.

But then, since $\frac{f(u)}{u}$ is non-increasing and $\frac{f(\overline{u})}{\overline{u}} \equiv \frac{f(\underline{u})}{\underline{u}}$, it must be that

$$\frac{f(u)}{u} = \text{ constant} \qquad \text{for } u \in (\min(\underline{u}), \max(\overline{u})) = (0, \max(\overline{u})), \tag{6.13}$$

and so the constant must be equal to $\lim_{u\to 0} \frac{f(u)}{u} = f'(0)$. But then $\overline{u}$ and $\underline{u}$ are positive solutions to the linear problem

$$D\nabla^2 u + c \cdot \nabla u + f'(0)u = 0 \quad \text{in } \Omega_0, \qquad u = 0 \quad \text{on } \partial\Omega_0, \tag{6.14}$$

which is a contradiction since $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ (the principal eigenvalue). This proves that $\underline{u} \equiv \overline{u}$. □

Now we consider the parabolic problem (6.1), (6.2) with initial conditions u_0, and we fix some $0 < \delta < \delta_0$ such that

$$\delta\phi_1^{(c)}(\xi) \leq u_0(\xi) \leq K \qquad \text{in } \Omega_0. \tag{6.15}$$

Theorem 6.4. *Suppose $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ and let $u(\xi, t)$ satisfy* (6.1), (6.2) *with $u(\cdot, 0) = u_0$ satisfying* (6.15). *Then as $t \to \infty$, u converges in $C^2(\overline{\Omega_0})$ to the unique positive solution U_c to* (6.6), (6.7), *and $\frac{\partial u}{\partial t} \to 0$ uniformly in $\overline{\Omega_0}$.*

Proof. Let $\underline{u}$ be the solution with initial conditions $\underline{u}(\xi, 0) = \delta\phi_1^{(c)}(\xi)$, and let $\overline{u}$ be the solution with $\overline{u}(\xi, 0) = K$. Since $\delta\phi_1^{(c)}(\xi) \leq u_0(\xi, 0) \leq K$, it follows from the comparison principle that

$$\underline{u}(\xi, t) \leq u(\xi, t) \leq \overline{u}(\xi, t) \qquad \text{for all } t \geq 0. \tag{6.16}$$

Since $\underline{u}(\xi, 0)$ is a subsolution to (6.6), (6.7), we have $\frac{\partial u}{\partial t} \geq 0$ at time $t = 0$. By applying the parabolic maximum principle to $\underline{v}(x, t) := \underline{u}(x, t + \delta t) - \underline{u}(x, t)$ for $\delta t > 0$, we deduce that $\underline{u}(\xi, t)$ is an increasing function of t. Since $\underline{u}(\xi, t)$ is also bounded above by K, it converges to some limit $\underline{U}(\xi)$ as $t \to \infty$. Similarly, since $K = \overline{u}(\xi, 0)$ is a supersolution, we can deduce that $\overline{u}(\xi, t)$ is a non-increasing function of t, bounded below by 0, and converges pointwise to some limit $\overline{U}(\xi)$. Proposition 2.13 implies that $\underline{U}(\xi)$ and $\overline{U}(\xi)$ are limits in $C^2(\overline{\Omega_0})$ and must satisfy (6.6), (6.7). The inequality $\underline{u}(\xi, t) \leq \overline{u}(\xi, t)$ also implies that $\underline{U}(\xi) \leq$

$\overline{U}(\xi)$. So by Theorem 6.3, $\underline{U} \equiv \overline{U} \equiv U_c$. Finally, since $u(\xi, t)$ lies between $\underline{u}(\xi, t)$ and $\overline{u}(\xi, t)$, it must also converge pointwise to $U_c(\xi)$. It then follows from Proposition 2.13 that there is convergence in $C^2(\overline{\Omega_0})$ and $\frac{\partial u}{\partial t}(\xi, t)$ converges uniformly to zero. □

We shall often use Theorem 6.4 in the following form.

Corollary 6.5. *Let $v(\xi, t) \geq 0$ satisfy*

$$\frac{\partial v}{\partial t} = D\nabla^2 v + f\left(ve^{\frac{-c\cdot\xi}{2D}}\right)e^{\frac{c\cdot\xi}{2D}} - \frac{|c|^2}{4D}v \qquad \text{in } \Omega_0 \tag{6.17}$$

$$v(\xi, t) = 0 \qquad \text{on } \partial\Omega_0 \tag{6.18}$$

with $0 \leq v(\xi, 0) \leq Ke^{\frac{c\cdot\xi}{2D}}$ not identically zero. If $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ then as $t \to \infty$, $v(\xi, t) \to U_c(\xi)e^{\frac{c\cdot\xi}{2D}}$ in $C^2(\overline{\Omega_0})$, and $\frac{\partial v}{\partial t} \to 0$ uniformly in $\overline{\Omega_0}$, where U_c is the unique positive solution to (6.6), (6.7).

Proof. Let $u(\xi, t) = v(\xi, t)e^{\frac{-c\cdot\xi}{2D}}$. Then u satisfies (6.1), (6.2) and the result follows from Theorem 6.4. □

Remark 6.6. *Consider an interval of the form $\Omega(t) = (A(t), A(t) + L_0)$ with $\dot{A}(t) = c$, and let $\xi = x - A(t)$. The equation becomes*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial \xi^2} + c\frac{\partial u}{\partial \xi} + f(u) \qquad \text{for } 0 < \xi < L_0 \tag{6.19}$$

$$u(\xi, t) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{6.20}$$

Theorems 6.3 and 6.4 then become as follows.

If $f'(0) > \frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}$ then there exists a unique positive solution $0 < U_{c,L_0} \leq K$ to the nonlinear ordinary differential equation

$$DU''(\xi) + cU'(\xi) + f(U(\xi)) = 0 \qquad \text{for } 0 < \xi < L_0 \tag{6.21}$$

$$U(\xi) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = L_0. \tag{6.22}$$

As $t \to \infty$, $u(\cdot, t)$ converges in $C^2([0, L_0])$ to U_{c,L_0} and $\frac{\partial u}{\partial t} \to 0$ uniformly.

6.2 Properties of U_c

In this section we prove certain properties of the unique positive solution U_c to (6.6), (6.7), which features in Theorem 6.3 and Theorem 6.4. Many of these properties will be needed in Chapter 7 in order to prove results on more general time-dependent domains.

6.2.1 General properties

Lemma 6.7. *Let U be a solution of equations* (6.6), (6.7). *Then*

$$\int_{\Omega_0} D|\nabla U(\xi)|^2 d\xi = \int_{\Omega_0} U(\xi)f(U(\xi))d\xi \tag{6.23}$$

and

$$\int_{\Omega_0} D|\nabla U(\xi)|^2 e^{c\cdot\xi/D} d\xi = \int_{\Omega_0} U(\xi)f(U(\xi))e^{c\cdot\xi/D} d\xi. \tag{6.24}$$

Proof. These follow by multiplying equation (6.6) by $U(\xi)$, or $U(\xi)e^{c\cdot\xi/D}$, and integrating by parts. □

Next we give three results which all concern relationships between the solutions U_{c,Ω_0} for different vectors c and/or different domains Ω_0. We begin with a reflection property which is a result of Lemma 2.10.

Lemma 6.8. *Given a bounded domain $\Omega_0 \subset \mathbb{R}^N$ and a vector $c \in \mathbb{R}^N$ such that $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$, define also $\tilde{\Omega}_0 = \{\xi \in \mathbb{R}^N : (-\xi_1, \xi_2, \dots, \xi_N) \in \Omega_0\}$ and $\tilde{c} = (-c_1, c_2, \dots, c_N)$. Then $U_{\tilde{c},\tilde{\Omega}_0}(\xi_1, \xi_2, \dots, \xi_N) \equiv U_{c,\Omega_0}(-\xi_1, \xi_2, \dots, \xi_N)$.*

Proof. This follows by applying Lemma 2.10 to the domains $\Omega_0 + ct$ and $\tilde{\Omega}_0 + \tilde{c}t$, taking the limit $t \to \infty$ and applying Theorem 6.4. □

Next we give a comparison result for enclosed domains.

Lemma 6.9. *If $\Omega_1 + \xi_0 \subset \Omega_2$ then $U_{c,\Omega_1}(\xi) \leq U_{c,\Omega_2}(\xi + \xi_0)$ for all $\xi \in \Omega_1$ and c such that $f'(0) > D\lambda(\Omega_1) + \frac{|c|^2}{4D}$.*

Proof. Consider the solutions ψ, $\hat{\psi}$ on $\Omega(t) = \Omega_1 + \xi_0 + ct$ and $\hat{\Omega}(t) = \Omega_2 + ct$. Since $\Omega(t) \subset \hat{\Omega}(t)$ for all $t \geq 0$, the comparison principle (Lemma 2.11) gives $\psi(x,t) \leq \hat{\psi}(x,t)$ for all $x \in \Omega(t)$, $t \geq 0$. Equivalently,

$$\psi(\xi + \xi_0 + ct, t) \leq \hat{\psi}(\xi + \xi_0 + ct, t) \qquad \text{for all } \xi \in \Omega_1. \tag{6.25}$$

Let $t \to \infty$. By Theorem 6.4, the limit of the left hand side is $U_{c,\Omega_1}(\xi)$ and that of the right hand side is $U_{c,\Omega_2}(\xi + \xi_0)$. □

In general the solutions U_c are not ordered with respect to c, but we shall show next that $U_c(\xi)e^{\frac{c\cdot\xi}{2D}}$ does have an ordering property. Note that by a translation of the bounded domain, we can always assume that $\xi_1 \geq 0$ in Ω_0.

Proposition 6.10. *Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded domain, and with $\xi_1 \geq 0$ for all $\xi \in \Omega_0$. Let $c \in \mathbb{R}^N$ such that $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$, and let $\overline{c} = (\overline{c}_1, c_2, \ldots, c_N)$ where $c_1 \leq \overline{c}_1 \leq -c_1$. Then $U_c(\xi)e^{\frac{c\cdot\xi}{2D}} \leq U_{\overline{c}}(\xi)e^{\frac{\overline{c}\cdot\xi}{2D}}$ for all $\xi \in \Omega_0$.*

Proof. Suppose that $v(\xi,t)$ and $\overline{v}(\xi,t)$ satisfy

$$\frac{\partial v}{\partial t} = D\nabla^2 v + f\left(ve^{\frac{-c\cdot\xi}{2D}}\right)e^{\frac{c\cdot\xi}{2D}} - \frac{|c|^2}{4D}v \qquad \text{in } \Omega_0 \tag{6.26}$$

$$\frac{\partial \overline{v}}{\partial t} = D\nabla^2 \overline{v} + f\left(\overline{v}e^{\frac{-\overline{c}\cdot\xi}{2D}}\right)e^{\frac{\overline{c}\cdot\xi}{2D}} - \frac{|\overline{c}|^2}{4D}\overline{v} \qquad \text{in } \Omega_0 \tag{6.27}$$

with $v(\xi,t) = \overline{v}(\xi,t) = 0$ on $\partial\Omega_0$, and with $v(\cdot,0) \equiv \overline{v}(\cdot,0)$. Since $c_1 \leq \overline{c}_1 \leq -c_1$ and $\xi_1 \geq 0$, we have the inequalities $-|c|^2 \leq -|\overline{c}|^2$ and $e^{\frac{-c\cdot\xi}{2D}} \geq e^{\frac{-\overline{c}\cdot\xi}{2D}}$ for all $\xi \in \Omega_0$. Using the assumptions (2.20) on f, it follows that

$$f\left(ue^{\frac{-c\cdot\xi}{2D}}\right)e^{\frac{c\cdot\xi}{2D}} \leq f\left(ue^{\frac{-\overline{c}\cdot\xi}{2D}}\right)e^{\frac{\overline{c}\cdot\xi}{2D}} \qquad \text{for all } u > 0 \text{ and } \xi \in \Omega_0. \tag{6.28}$$

Therefore $\overline{v}$ is a supersolution for v, and so $v(\xi,t) \leq \overline{v}(\xi,t)$ for all $t \geq 0$. The result follows by letting $t \to \infty$ and applying Corollary 6.5. □

6.2.2 Continuity properties

The next result describes the continuity of U_{c,Ω_0} with respect to c and with respect to scalings of the domain Ω_0.

Proposition 6.11. *(Continuity with respect to Ω_0 and c.)*
Consider sequences of positive numbers $R_1^{(n)} > 0, \ldots, R_N^{(n)} > 0$, and vectors $c^{(n)}$ in $\mathbb{R}^N$. For some fixed bounded domain $\Omega_0 \subset \mathbb{R}^N$, let

$$\Omega_n = \left\{ \xi \in \mathbb{R}^N : \left(\frac{\xi_1}{R_1^{(n)}}, \ldots, \frac{\xi_N}{R_N^{(n)}} \right) \in \Omega_0 \right\}. \tag{6.29}$$

Assume that f satisfies assumptions (2.20), and $f'(0) > D\lambda(\Omega_n) + \frac{|c^{(n)}|^2}{4D}$ for each n, and let U_n be the unique positive solution to

$$D\nabla^2 U_n(\xi) + c^{(n)} \cdot \nabla U_n(\xi) + f(U_n(\xi)) = 0 \qquad \text{in } \Omega_n \tag{6.30}$$

$$U_n(\xi) = 0 \qquad \text{on } \partial\Omega_n. \tag{6.31}$$

Assume that $c_j^{(n)} \to c_j$ and $R_j^{(n)} \to 1$ as $n \to \infty$, for all $1 \le j \le N$. Then $\hat{U}_n(X) := U_n(R_1^{(n)} X_1, \ldots, R_N^{(n)} X_N)$ is convergent in $C^2(\overline{\Omega_0})$ to a non-negative solution of

$$D\nabla^2 U + c \cdot \nabla U + f(U) = 0 \qquad \text{in } \Omega_0 \tag{6.32}$$

$$U = 0 \qquad \text{on } \partial\Omega_0. \tag{6.33}$$

Moreover:

1. *If $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$, then $\hat{U}_n$ converges to the unique positive solution of (6.32), (6.33).*

2. *Suppose $f'(0) = D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. If f is not linear on any neighbourhood $[0, s_0)$ of 0, then $\hat{U}_n$ converges to zero. If f is linear on some neighbourhood $[0, s_0)$ of 0, with s_0 defined as the maximum such, then $\hat{U}_n$ converges to $s_0 \phi_1^{(c)}$.*

Proof. The change of variables to $X_j = \frac{\xi_j}{R_j^{(n)}}$ and $\hat{U}_n(X) = U_n(\xi)$ puts the problem (6.30), (6.31) into the form

$$\sum_{j=1}^{N} \frac{D}{{R_j^{(n)}}^2} \frac{\partial^2 \hat{U}_n}{\partial X_j^2} + \sum_{j=1}^{N} \frac{c_j^{(n)}}{R_j^{(n)}} \frac{\partial \hat{U}_n}{\partial X_j} + f(\hat{U}_n(X)) = 0 \qquad \text{in } \Omega_0 \tag{6.34}$$

$$\hat{U}_n(X) = 0 \qquad \text{on } \partial\Omega_0. \tag{6.35}$$

By Theorem A.1, there is a subsequence $\hat{U}_{n_k}$ that is convergent in $C^2(\overline{\Omega_0})$ to a solution of (6.32), (6.33). Denote this solution by U.

1. Suppose that $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. Then we know U must be either zero or the unique positive solution. So we just need to prove that $U \not\equiv 0$. Without loss of generality (by taking a further subsequence, and/or by using the reflection property in Lemma 6.8 if needed) we can assume that $c_j^{(n_k)} \leq 0$ for all $1 \leq j \leq N$ and all $k \in \mathbb{N}$. Then we can find $k^* \in \mathbb{N}$, a vector $\hat{c}$, and a domain $\hat{\Omega}$ such that

$$f'(0) > D\lambda(\hat{\Omega}) + \frac{|\hat{c}|^2}{4D} \tag{6.36}$$

and that for all $k \geq k^*$,

$$\hat{c}_j \leq c_j^{(n_k)} \leq 0 \qquad \text{and} \qquad \hat{\Omega} \subset \Omega_{n_k}. \tag{6.37}$$

Define $\hat{U}$ to be the unique positive solution to

$$D\nabla^2\hat{U}(\xi) + \hat{c}\cdot\nabla\hat{U}(\xi) + f(\hat{U}(\xi)) = 0 \qquad \text{in } \hat{\Omega} \tag{6.38}$$

$$\hat{U}(\xi) = 0 \qquad \text{on } \partial\hat{\Omega}. \tag{6.39}$$

Assuming (without loss of generality, by translating the domain if necessary) that $\xi_j \geq 0$ for all $1 \leq j \leq N$ and $\xi \in \hat{\Omega}$, Proposition 6.10 gives a positive lower bound:

$$U_{n_k}(\xi)e^{\frac{c^{(n_k)}\cdot\xi}{2D}} \geq \hat{U}(\xi)e^{\frac{\hat{c}\cdot\xi}{2D}} \qquad \text{for all } \xi \in \hat{\Omega},\ k \geq k^*. \tag{6.40}$$

Therefore we also deduce a lower bound for $\hat{U}_{n_k}$, and hence U, which rules out $U \equiv 0$. So U must be the unique positive solution to (6.32), (6.33).

2. Suppose that $f'(0) = D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. By equation (6.24),

$$\begin{aligned}\int_{\Omega_0} D|\nabla U(\xi)|^2 e^{c\cdot\xi/D}d\xi &= \int_{\Omega_0} U(\xi)f(U(\xi))e^{c\cdot\xi/D}d\xi \\ &\leq f'(0)\int_{\Omega_0} U(\xi)^2 e^{c\cdot\xi/D}d\xi\end{aligned}$$

$$= \left(D\lambda(\Omega_0) + \frac{|c|^2}{4D} \right) \int_{\Omega_0} U(\xi)^2 e^{c\cdot\xi/D} d\xi \qquad (6.41)$$

where the inequality follows from $f(U) \leq f'(0)U$. However, the characterisation of the principal eigenvalue as the minimiser of the Rayleigh quotient gives that for all $u \in C^2(\Omega_0)$ with $u = 0$ on $\partial\Omega_0$,

$$\left(D\lambda(\Omega_0) + \frac{|c|^2}{4D} \right) \int_{\Omega_0} u(\xi)^2 e^{c\cdot\xi/D} d\xi \leq \int_{\Omega_0} D|\nabla u(\xi)|^2 e^{c\cdot\xi/D} d\xi. \qquad (6.42)$$

It follows that there is equality throughout (6.41) and that the limit U satisfies $f(U) \equiv f'(0)U$ on $\overline{\Omega_0}$. If f is not linear on any neighbourhood $[0, s_0)$ of 0, then the only non-negative solution to this is zero, so $U \equiv 0$.

Suppose instead that f is linear on some neighbourhood $[0, s_0)$ of 0, with s_0 the maximal such. Then $f(U) \equiv f'(0)U$ implies that $||U||_\infty \leq s_0$ and (since $f'(0) = D\lambda(\Omega_0) + \frac{|c|^2}{4D}$) that U satisfies the linear elliptic problem

$$D\nabla^2 U + c \cdot \nabla U = -\left(D\lambda(\Omega_0) + \frac{|c|^2}{4D} \right) U \qquad \text{in } \Omega_0 \qquad (6.43)$$

$$U(\xi) = 0 \qquad \text{on } \partial\Omega_0. \qquad (6.44)$$

Therefore U must be a multiple of the principal eigenfunction $\phi_1^{(c)}$ with $||U||_\infty \leq s_0$. However $||\hat{U}_n||_\infty = ||U_n||_\infty > s_0$ for each n, as otherwise U_n would satisfy the linear equation, contradicting $f'(0) > D\lambda(\Omega_n) + \frac{|c^{(n)}|^2}{4D}$. Therefore the limit must satisfy $||U||_\infty \geq s_0$. So, $U = s_0\phi_1^{(c)}$.

In all cases, the limit is uniquely determined and so by Lemma A.5 the whole sequence $\hat{U}_n$ must converge, not just a subsequence. □

A similar approach is used to prove continuity with respect to f.

Proposition 6.12. *Let Ω_0 be a bounded domain and $c \in \mathbb{R}^N$. Let f_n be functions satisfying conditions of type* (2.20)*, and $||f_n - f||_{C^{0,1}([0,K+\varepsilon])} \to 0$ as $n \to \infty$ (for some $\varepsilon > 0$). Assume that $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ and that $f_n'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ for each n, and let U_n be the unique positive solution to*

$$D\nabla^2 U_n(\xi) + c \cdot \nabla U_n(\xi) + f_n(U_n(\xi)) = 0 \qquad \text{in } \Omega_0 \qquad (6.45)$$

$$U_n(\xi) = 0 \qquad on\ \partial\Omega_0. \tag{6.46}$$

Then U_n converges in $C^2(\overline{\Omega_0})$ to the unique positive solution U of (6.32), (6.33).

Proof. The proof is similar to part 1 of Proposition 6.11. Again, Theorem A.1 implies that there is a subsequence U_{n_k} that is convergent in $C^2(\overline{\Omega_0})$ to a solution U of (6.32), (6.33). We just need to prove that $U \not\equiv 0$. We are assuming that $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$ and $f_n \to f$ in $C^{0,1}([0, K+\varepsilon])$, so in particular $f_n'(0) \to f'(0)$. Therefore there exists a function $\hat{f}$ and $k^* \in \mathbb{N}$ such that $\hat{f}$ satisfies conditions of the type (2.20), $\hat{f}'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$, and $\hat{f} \leq f_{n_k}$ for all $k \geq k^*$. Let $\hat{U}$ denote the unique positive solution with reaction term $\hat{f}$. Then $\hat{U} \leq U_{n_k}$ for all $k \geq k^*$ (which follows by considering the solutions $\hat{u} \leq u_{n_k}$ to the parabolic problems with reaction terms $\hat{f}$ and f_{n_k} respectively). Hence also $\hat{U} \leq U$. This shows that $U \not\equiv 0$, and so U must be the unique positive solution to (6.32), (6.33). Finally, by Lemma A.5 the whole sequence U_n must converge, not just a subsequence. □

Example 6.13. *Let $f_{\alpha,\beta,\gamma}(u) = \frac{f((1+\alpha)u)}{(1+\alpha)} + \beta u - \gamma u^2$ for sufficiently small α, β, $\gamma \geq 0$, and let $U_{\alpha,\beta,\gamma}$ be the unique positive solution to the problem* (6.32), (6.33) *with reaction term $f_{\alpha,\beta,\gamma}$. Proposition 6.12 shows that as α, β and γ all tend to* 0, *$U_{\alpha,\beta,\gamma}$ converges in C^2 to the unique positive solution U with reaction term f.*

6.2.3 Asymptotic properties as the domain gets large

The next lemma is a Liouville-type property. It follows from Theorem A.10, which is a special case of [8, Theorem 3.7].

Lemma 6.14. *Let $c \in \mathbb{R}^N$ and $f'(0) > \frac{|c|^2}{4D}$. Then $U(y) \equiv 0$ and $U(y) \equiv K$ are the only solutions (with $0 \leq U \leq K$) to*

$$D\nabla^2 U + c \cdot \nabla U + f(U) = 0 \qquad y \in \mathbb{R}^N. \tag{6.47}$$

Lemma 6.14 shows that the constant K is the unique non-zero solution to (6.47) on the whole space $\mathbb{R}^N$. We now use this to prove that U_c converges locally uniformly to K as the domain $\Omega_0 \subset \mathbb{R}^N$ gets large in all directions.

Proposition 6.15. *Fix $c \in \mathbb{R}^N$ with $f'(0) > \frac{|c|^2}{4D}$. Let Ω_1 be either a ball $\Omega_1 = \{\xi \in \mathbb{R}^N : |\xi| < r_0\}$, or a box $\Omega_1 = \{\xi \in \mathbb{R}^N : -l_j < \xi_j < l_j,\ 1 \le j \le N\}$. For $R > 0$ let $\Omega_R = R\Omega_1 = \left\{\xi \in \mathbb{R}^N : \frac{\xi}{R} \in \Omega_1\right\}$ and for R large enough let $U_R(\xi)$ be the unique positive solution to (6.6), (6.7) on Ω_R. As $R \to \infty$, $U_R \to K$ in $C^2_{loc}(\mathbb{R}^N)$. In other words, for every compact set $V \subset \mathbb{R}^N$,*

$$||U_R - K||_{C^2(V)} \to 0 \qquad \text{as } R \to \infty. \tag{6.48}$$

Proof. Let $R_n < R_{n+1} \to \infty$ and $u_n = U_{R_n}$. By Lemma 6.9, u_n is increasing in n, and since it is bounded above by K there is some pointwise limit $\overline{U}(\xi)$, defined on the whole of $\mathbb{R}^N$, such that $u_n(\xi) \to \overline{U}(\xi)$ as $n \to \infty$. Theorem A.1 and Lemma A.5 then imply that the convergence is in fact in $C^2(V)$ for every compact subset $V \subset \mathbb{R}^N$. So $\overline{U}(\xi)$ satisfies (6.47) on $\mathbb{R}^N$, and $\overline{U} \neq 0$ since it is the limit of the increasing sequence u_n. By Lemma 6.14, $\overline{U} \equiv K$. □

Next we would like to consider domains of the type

$$\Omega_L = \left\{(\xi_0, \xi) :\ \xi_0 \in \omega_0,\ \frac{-L_j}{2} < \xi_j < \frac{L_j}{2},\ 1 \le j \le N\right\} \tag{6.49}$$

where $\omega_0 \subset \mathbb{R}^m$ is bounded, and to understand the asymptotic behaviour of U_c on such domains as $L_j \to \infty$. The limiting domain is now $\omega_0 \times \mathbb{R}^N$ instead of the whole space, and we begin by proving a uniqueness result for solutions to

$$D\nabla^2\tilde{U} + (c_0, \hat{c}) \cdot \nabla\tilde{U} + f(\tilde{U}) = 0 \qquad \text{in } \omega_0 \times \mathbb{R}^N \tag{6.50}$$

$$\tilde{U}(\xi_0, y) = 0 \qquad \text{on } \partial\omega_0 \times \mathbb{R}^N. \tag{6.51}$$

If $U_{c_0}(\xi_0)$ is the unique positive solution to

$$D\nabla^2 U(\xi_0) + c_0 \cdot \nabla U(\xi_0) + f(U(\xi_0)) = 0 \qquad \text{in } \omega_0 \tag{6.52}$$

$$U(\xi_0) = 0 \qquad \text{for } \xi_0 \in \partial\omega_0, \tag{6.53}$$

then we show in Theorem 6.16 that the only solution to (6.50), (6.51) that also satisfies the bounds (6.54) is $U_{c_0}(\xi_0)$ itself. The proof uses a contradiction argument based on that of H. Berestycki, Hamel and Rossi in [8, Theorem 3.7], and also their strong maximum principle for strict super-solutions in unbounded domains [8, Lemma 2.1(iii)], which is stated as Theorem A.11 here.

Theorem 6.16. *Let $\omega_0 \subset \mathbb{R}^m$ be a bounded domain and let $c_0 \in \mathbb{R}^m$, $\hat{c} \in \mathbb{R}^N$. Let $f'(0) > D\lambda(\omega_0) + \frac{|c_0|^2}{4D}$ where f satisfies assumptions* (2.20), *and assume that $\frac{f(u)}{u}$ is a strictly decreasing, uniformly continuous function of $u > 0$. Let $U_{c_0}(\xi_0)$ be the unique positive solution to* (6.52), (6.53), *and let $\tilde{U}(\xi_0, y)$ be a solution to* (6.50), (6.51). *Suppose there is some $a > 0$ such that*

$$aU_{c_0}(\xi_0) \leq \tilde{U}(\xi_0, y) \leq U_{c_0}(\xi_0) \qquad \textit{for all } \xi_0 \in \omega_0,\ y \in \mathbb{R}^N. \tag{6.54}$$

Then $\tilde{U}(\xi_0, y) \equiv U_{c_0}(\xi_0)$.

Proof. For $y \in \mathbb{R}^N$, let

$$V(y) = \int_{\omega_0} \tilde{U}(\xi_0, y) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0. \tag{6.55}$$

Then V satisfies

$$\begin{aligned} D\nabla_y^2 V + \hat{c} \cdot \nabla_y V &= \int_{\omega_0} (D\nabla_y^2 \tilde{U} + \hat{c} \cdot \nabla_y \tilde{U}) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \\ &= \int_{\omega_0} (-D\nabla_{\xi_0}^2 \tilde{U} - c_0 \cdot \nabla_{\xi_0} \tilde{U} - f(\tilde{U})) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \\ &= \int_{\omega_0} \left(-\tilde{U}(D\nabla_{\xi_0}^2 U_{c_0} + c_0 \cdot \nabla_{\xi_0} U_{c_0}) - f(\tilde{U}) U_{c_0}\right) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \\ &= \int_{\omega_0} \left(\tilde{U} f(U_{c_0}) - f(\tilde{U}) U_{c_0}\right) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \leq 0, \end{aligned} \tag{6.56}$$

which follows by using the equations satisfied by $\tilde{U}$ and U_{c_0}, and integrating by parts with respect to ξ_0. The right hand side is ≤ 0 due to $0 < \tilde{U} \leq U_{c_0}$ and the assumption that $\frac{f(u)}{u}$ is a decreasing function. Define

$$V_0 = \int_{\omega_0} U_{c_0}(\xi_0)^2 e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \qquad \text{and} \qquad \mu = \frac{\inf_{y \in \mathbb{R}^N} V(y)}{V_0}. \tag{6.57}$$

By (6.54), we know that $0 < a \leq \mu \leq 1$. In order to prove the proposition we just need to show that $\mu = 1$, since this holds if and only if $\tilde{U}(\xi_0, y) \equiv U_{c_0}(\xi_0)$. We shall follow, where possible, the proof of [8, Theorem 3.7]. Suppose (for a contradiction) that $\mu < 1$, and let $\mu < \rho < 1$. Choose $y_0 \in \mathbb{R}^N$ such that $V(y_0) < \rho V_0$, and let $\Omega_\rho \subset \mathbb{R}^N$ be the connected component of $V^{-1}(-\infty,\ \rho V_0)$ that contains y_0. If Ω_ρ were bounded then the elliptic maximum principle would imply that $V(y_0) \geq \inf_{\overline{\Omega_\rho}} V = \min_{\partial\Omega_\rho} V = \rho V_0$, which is not true. So Ω_ρ is unbounded, and for all $y \in \Omega_\rho$,

$$\mu V_0 \leq V(y) \leq \rho V_0. \tag{6.58}$$

We claim that there exists $\varepsilon > 0$ such that

$$D\nabla_y^2 V + \hat{c} \cdot \nabla_y V \leq -\varepsilon \qquad \text{for } y \in \Omega_\rho. \tag{6.59}$$

Let us assume this for now. Then in the unbounded domain $\Omega_\rho \subset \mathbb{R}^N$,

$$-(D\nabla_y^2 + \hat{c} \cdot \nabla_y)V(y) \geq \varepsilon > 0, \qquad -(D\nabla_y^2 + \hat{c} \cdot \nabla_y)(\rho V_0) = 0 \tag{6.60}$$

and $V(y) = \rho V_0 > 0$ for $y \in \partial\Omega_\rho$. By applying [8, Lemma 2.1(iii)] (given as Theorem A.11 here) to $V(y)$ on Ω_ρ, we deduce that $V(y) \geq \rho V_0$ in Ω_ρ, which contradicts $V(y_0) < \rho V_0$. So, in fact, $\mu = 1$ and $\tilde{U}(\xi_0, y) \equiv U_{c_0}(\xi_0)$.

So to complete the proof we just need to find $\varepsilon > 0$ such that (6.59) holds.

Step 1: There is a constant K_0 such that $|\nabla\tilde{U}(\xi_0, y)| \leq K_0$ (uniformly in $\omega_0 \times \mathbb{R}^N$) for every function $0 \leq \tilde{U}(\xi_0, y) \leq K$ that satisfies (6.50), (6.51). Indeed, by Theorem A.1 we find that (for any $l > 0$) the functions $\tilde{U}(\xi_0, \xi + y')$ are bounded in $C^{2+\gamma}(\overline{\omega_0} \times [-l, +l]^N)$, independently of $y' \in \mathbb{R}^N$.

Step 2: Define $I(y)$ by

$$I(y) = \int_{\omega_0} \tilde{U}(\xi_0, y) U_{c_0}(\xi_0) \left(\frac{f(U_{c_0}(\xi_0))}{U_{c_0}(\xi_0)} - \frac{f(\tilde{U}(\xi_0, y))}{\tilde{U}(\xi_0, y)} \right) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0. \tag{6.61}$$

Then equation (6.56) gives

$$D\nabla_y^2 V + \hat{c} \cdot \nabla_y V = I(y) \leq \sup_{\Omega_\rho} I \qquad \text{for } y \in \Omega_\rho. \tag{6.62}$$

We know this is ≤ 0, but we need to show the strict inequality $\sup_{\Omega_\rho} I < 0$. Let

$$\delta = \frac{(1-\rho)V_0}{\int_{\omega_0} U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0}. \tag{6.63}$$

Then for each $y \in \Omega_\rho$ there must exist a subset $\omega_y' \subset \omega_0$ such that

$$\tilde{U}(\xi_0, y) \leq U_{c_0}(\xi_0) - \delta \qquad \text{for } \xi_0 \in \omega_y'. \tag{6.64}$$

Indeed, if not, then there exists $y \in \Omega_\rho$ such that $\tilde{U}(\xi_0, y) > U_{c_0}(\xi_0) - \delta$ everywhere in ω_0. Then

$$\begin{aligned} V(y) = \int_{\omega_0} \tilde{U}(\xi_0, \xi) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 &> \int_{\omega_0} (U_{c_0}(\xi_0) - \delta) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \\ &= V_0 - \delta \int_{\omega_0} U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 = \rho V_0 \end{aligned} \tag{6.65}$$

(by definition of δ), which contradicts $y \in \Omega_\rho$.

Now using Step 1, $|\nabla \tilde{U}| \leq K_0$ everywhere, and so in fact for each $y \in \Omega_\rho$ there must be a subset $\omega_y'' \subset \omega_0$ such that

$$\tilde{U}(\xi_0, y) \leq U_{c_0}(\xi_0) - \frac{\delta}{2} \qquad \text{for } \xi_0 \in \omega_y'' \qquad \text{and} \qquad |\omega_y''| \geq \left(\frac{\delta}{2K_0}\right)^m. \tag{6.66}$$

Step 3: Since $\frac{f(u)}{u}$ is a strictly decreasing, uniformly continuous function, there is an increasing function θ_f with $\theta_f(u) > 0$ for $u > 0$, such that

$$\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \leq -\theta_f(u_2 - u_1) \qquad \text{for all } 0 \leq u_1 \leq u_2 \leq K. \tag{6.67}$$

Then, for $y \in \Omega_\rho$ and $\xi_0 \in \omega_y''$,

$$\frac{f(U_{c_0}(\xi_0))}{U_{c_0}(\xi_0)} - \frac{f(\tilde{U}(\xi_0, y))}{\tilde{U}(\xi_0, y)} \leq -\theta_f\left(\frac{\delta}{2}\right). \tag{6.68}$$

Also, for $\xi_0 \in \omega_y''$, (6.66) implies $U_{c_0}(\xi_0) \geq \frac{\delta}{2}$ and then $\tilde{U}(\xi_0, y) \geq a\frac{\delta}{2}$ by (6.54). So for $y \in \Omega_\rho$,

$$\begin{aligned} I(y) &= \int_{\omega_0} \tilde{U}(\xi_0, y) U_{c_0}(\xi_0) \left(\frac{f(U_{c_0}(\xi_0))}{U_{c_0}(\xi_0)} - \frac{f(\tilde{U}(\xi_0, y))}{\tilde{U}(\xi_0, y)} \right) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \\ &\leq -\theta_f\left(\frac{\delta}{2}\right) \int_{\omega_y''} \tilde{U}(\xi_0, y) U_{c_0}(\xi_0) e^{\frac{c_0 \cdot \xi_0}{D}} d\xi_0 \end{aligned} \tag{6.69}$$

$$\leq -\theta_f \left(\frac{\delta}{2}\right) \frac{a\delta^2}{4} \inf_{\xi_0 \in \omega_0} \left(e^{\frac{c_0 \cdot \xi_0}{D}}\right) \left(\frac{\delta}{2K_0}\right)^m . \tag{6.70}$$

This bound is independent of $y \in \Omega_\rho$, so $\sup_{\Omega_\rho} I < 0$ and (6.59) holds with $\varepsilon = \theta_f \left(\frac{\delta}{2}\right) \frac{a\delta^2}{4} \inf_{\xi_0 \in \omega_0} \left(e^{\frac{c_0 \cdot \xi_0}{D}}\right) \left(\frac{\delta}{2K_0}\right)^m$. $\square$

Having proved Theorem 6.16, we now use it to consider the asymptotic behaviour of U_c on the domain Ω_L as $L \to \infty$. Here we use the notation $L \to \infty$ to mean that $L_j \to \infty$ for all $1 \leq j \leq N$. Similarly, when we use inequalities involving L, ξ, y, we mean that each component satisfies the inequality.

Theorem 6.17. *Let $\omega_0 \subset \mathbb{R}^m$ be a bounded domain, and for positive vectors $L \in \mathbb{R}^N$ ($L_j > 0$ for $1 \leq j \leq N$), let Ω_L be given by* (6.49). *Let f satisfy* (2.20), *and assume that $\frac{f(u)}{u}$ is a strictly decreasing, uniformly continuous function of $u > 0$. Let $c_0 \in \mathbb{R}^m$, $\hat{c} \in \mathbb{R}^N$ be such that $f'(0) > D\lambda(\omega_0) + \frac{|c_0|^2}{4D} + \frac{|\hat{c}|^2}{4D}$, and for L large enough let $U_{c_0,\hat{c},L}(\xi_0, \xi)$ be the unique positive solution to*

$$D\nabla^2 U + (c_0, \hat{c}) \cdot \nabla U + f(U) = 0 \qquad \textit{in } \Omega_L \tag{6.71}$$

$$U(\xi_0, \xi) = 0 \qquad \textit{on } \partial\Omega_L. \tag{6.72}$$

Also, let $U_{c_0}(\xi_0)$ be the unique positive solution to (6.52), (6.53) *on $\omega_0 \subset \mathbb{R}^m$. Then as $L \to \infty$,*

$$\sup_{\frac{-L}{2} \leq \xi \leq \frac{L}{2}} U_{c_0,\hat{c},L}(\xi_0, \xi) \to U_{c_0}(\xi_0) \qquad \textit{uniformly in } \xi_0 \in \overline{\omega_0}. \tag{6.73}$$

Proof. For L large, let $u_L(\xi_0, \xi, t)$ be the solution to

$$\frac{\partial u_L}{\partial t} = D\nabla^2 u_L + (c_0, \hat{c}) \cdot \nabla u_L + f(u_L) \qquad \text{in } \Omega_L \tag{6.74}$$

$$u_L(\xi_0, \xi, t) = 0 \qquad \text{on } \partial\Omega_L \tag{6.75}$$

with initial conditions $u_0(\xi_0, \xi) \geq 0$, $\not\equiv 0$. Also, let $u_\infty(\xi_0, t)$ be the solution to

$$\frac{\partial u_\infty}{\partial t} = D\nabla^2 u_\infty + c_0 \cdot \nabla u_\infty + f(u_\infty) \qquad \text{in } \omega_0 \tag{6.76}$$

$$u_\infty(\xi_0, t) = 0 \qquad \text{on } \partial\omega_0 \tag{6.77}$$

with initial conditions $u_\infty(\xi_0, 0) \equiv ||u_0||_\infty$. The comparison principle implies that $u_L(\xi_0, \xi, t) \le u_\infty(\xi_0, t)$ for all $(\xi_0, \xi) \in \Omega_L$ and $t \ge 0$. We also know that $u_L(\xi_0, \xi, t) \to U_{c_0,\hat{c},L}(\xi_0, \xi)$ and $u_\infty(\xi_0, t) \to U_{c_0}(\xi_0)$ as $t \to \infty$. This gives the upper bound

$$U_{c_0,\hat{c},L}(\xi_0, \xi) \le U_{c_0}(\xi_0) \tag{6.78}$$

which holds for all $\frac{-L}{2} \le \xi \le \frac{L}{2}$, and all L large enough that $U_{c_0,\hat{c},L}$ exists. Lemma 6.9 implies that $U_{c_0,\hat{c},L}$ is a non-decreasing function of L. Since it is bounded above it must converge (pointwise) as $L \to \infty$: there is some function $\tilde{U}_{c_0,\hat{c}}$ defined on $\omega_0 \times \mathbb{R}^N$ such that

$$U_{c_0,\hat{c},L}(\xi_0, y) \to \tilde{U}_{c_0,\hat{c}}(\xi_0, y) \qquad \text{(pointwise in } \xi_0,\ y) \text{ as } L \to \infty. \tag{6.79}$$

Theorem A.1 implies that the convergence is in fact in $C^2(\overline{\omega_0} \times V)$ for every compact set $V \subset \mathbb{R}^N$, and so the limit function $\tilde{U}_{c_0,\hat{c}}$ satisfies (6.50), (6.51) on $\omega_0 \times \mathbb{R}^N$. In order to prove (6.73) it just remains to show that $\tilde{U}_{c_0,\hat{c}}(\xi_0, y) \equiv U_{c_0}(\xi_0)$. Since (6.78) holds for all sufficiently large L, we have

$$\tilde{U}_{c_0,\hat{c}}(\xi_0, y) \le U_{c_0}(\xi_0) \qquad \text{for all } y \in \mathbb{R}^N. \tag{6.80}$$

Next we shall prove that there is some $a > 0$ such that $aU_{c_0}(\xi_0) \le \tilde{U}_{c_0,\hat{c}}(\xi_0, y)$ for all $\xi_0 \in \omega_0$ and $y \in \mathbb{R}^N$. To get this, we consider some fixed (and sufficiently large) $\hat{L} < \frac{L}{2}$. Lemma 6.9 implies that

$$U_{c_0,\hat{c},\hat{L}}(\xi_0, y) \le U_{c_0,\hat{c},L}(\xi_0, y + \hat{y}) \tag{6.81}$$

for all $\xi_0 \in \omega_0$, $-\frac{\hat{L}}{2} \le y \le \frac{\hat{L}}{2}$ and $-\left(\frac{L-\hat{L}}{2}\right) \le \hat{y} \le \left(\frac{L-\hat{L}}{2}\right)$. This leads to

$$\sup_y U_{c_0,\hat{c},\hat{L}}(\xi_0, y) \le U_{c_0,\hat{c},L}(\xi_0, Y) \tag{6.82}$$

for every $\xi_0 \in \omega_0$ and $-\left(\frac{L}{2} - \hat{L}\right) \le Y \le \left(\frac{L}{2} - \hat{L}\right)$. Fixing $\hat{L}$ and letting $L \to \infty$ gives

$$\tilde{U}_{c_0,\hat{c}}(\xi_0, Y) \ge \sup_y U_{c_0,\hat{c},\hat{L}}(\xi_0, y) \ge U_{c_0,\hat{c},\hat{L}}(\xi_0, 0) \qquad \text{for all } \xi_0 \in \omega_0,\ Y \in \mathbb{R}^N. \tag{6.83}$$

But $U_{c_0,\hat{c},\hat{L}}(\xi_0, 0) > 0$ for $\xi_0 \in \omega_0$ and the normal derivative is non-zero for $\xi_0 \in \partial\omega_0$. So, there must exist some $a > 0$ such that $U_{c_0,\hat{c},\hat{L}}(\xi_0, 0) \geq aU_{c_0}(\xi_0)$ and hence we get

$$\tilde{U}_{c_0,\hat{c}}(\xi_0, Y) \geq U_{c_0,\hat{c},\hat{L}}(\xi_0, 0) \geq aU_{c_0}(\xi_0) \qquad \text{for all } \xi_0 \in \omega_0,\ Y \in \mathbb{R}^N. \tag{6.84}$$

It then follows from Theorem 6.16 that $\tilde{U}_{c_0,\hat{c}}(\xi_0, y) \equiv U_{c_0}(\xi_0)$. □

6.2.4 Properties of U_{c,L_0} on the interval

Given $0 < L_0 < \infty$ and $c \in \mathbb{R}$ such that $f'(0) > \frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}$, let U_{c,L_0} denote the unique positive solution to (6.21), (6.22). Several properties of U_{c,L_0} can be proven in the one-dimensional case, in addition to those from Section 6.2.

Proposition 6.18. *1. U_{c,L_0} has a single maximum point ξ^*, such that*

$$U'_{c,L_0} > 0 \text{ on } [0, \xi^*), \qquad U'_{c,L_0}(\xi^*) = 0, \qquad U'_{c,L_0} < 0 \text{ on } (\xi^*, L_0]. \tag{6.85}$$

2. $|U'_{c,L_0}(L_0)| < |U'_{c,L_0}(0)|$ if $c > 0$, or the opposite inequality if $c < 0$. Also $U''_{c,L_0}(0)$ has the opposite sign to c, and $U''_{c,L_0}(L_0)$ has the same sign as c.

Proof. 1. There must be an interior maximum since by Hopf's Lemma [51, chapter 3, Theorem 1.1] we know that $U'_{c,L_0}(0) > 0$ but $U'_{c,L_0}(L_0) < 0$. But at any interior stationary point ξ^*, $DU''_{c,L_0}(\xi^*) = -f(U_{c,L_0}(\xi^*)) < 0$. So U'_{c,L_0} can only change sign once, and ξ^* is unique.

2. By multiplying equation (6.21) by U'_{c,L_0} and integrating by parts, we get

$$c \int_0^{L_0} (U'_{c,L_0}(\xi))^2 d\xi = \frac{D}{2} \left(U'_{c,L_0}(0)^2 - U'_{c,L_0}(L_0)^2 \right), \tag{6.86}$$

which proves that $|U'_{c,L_0}(L_0)| < |U'_{c,L_0}(0)|$ if $c > 0$, or the opposite inequality if $c < 0$. Finally, note that at both endpoints, $DU''_{c,L_0} = -cU'_{c,L_0}$ which can be used to deduce the sign of U''_{c,L_0} at the endpoints.

□

In this one-dimensional case, the equation (6.21) for U_{c,L_0} is an ordinary differential equation and we may consider the problem in the phase plane:

$$U' = V, \qquad V' = -\frac{c}{D}V - \frac{f(U)}{D}, \tag{6.87}$$

where $U = U_{c,L_0}$ and $V = U'$. In the phase plane (see Figure 6.1) there are two fixed points: a saddle point at $(K, 0)$ and a spiral or centre at $(0, 0)$.

Next we define the semi-wave of speed c. For $c \in (-c_*, c_*)$, the semi-wave $\hat{U}_c$ is the function satisfying

$$D\hat{U}_c'' + c\hat{U}_c' + f(\hat{U}_c) = 0 \qquad \text{for } 0 < x < \infty \tag{6.88}$$

$$\hat{U}_c(0) = 0, \qquad \lim_{x\to\infty} \hat{U}_c(x) = K, \qquad \hat{U}_c > 0 \text{ on } (0, \infty). \tag{6.89}$$

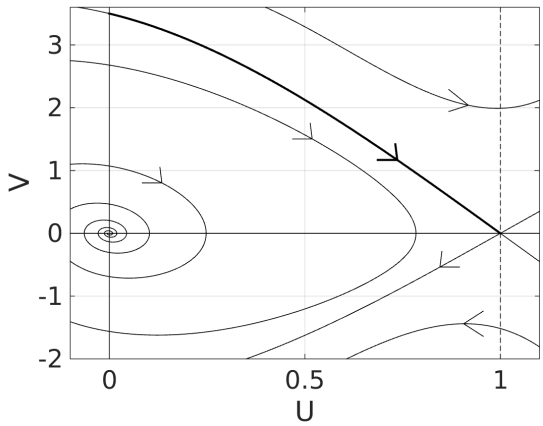

Figure 6.1: Sketch of a typical phase plane for (6.87), with the semi-wave trajectory highlighted.

In the phase plane (Figure 6.1), $\hat{U}_c$ corresponds to the trajectory that tends towards $(K, 0)$ from the region $U < K, V > 0$. Along this trajectory U is monotonically increasing, and following it backwards it crossed the $U = 0$ axis at some point $(0, \hat{U}_c'(0))$ with $\hat{U}_c'(0) > 0$. We shall show that $\lim_{L_0\to\infty} U_{c,L_0}'(0) = \hat{U}_c'(0)$.

Proposition 6.19. *Fix $c \in (-c_*, c_*)$, and let $\hat{U}_c$ be the semi-wave satisfying (6.88), (6.89). Then $U_{c,L_0}'(0) \to \hat{U}_c'(0)$ as $L_0 \to \infty$.*

Proof. If $L_1 \leq L_2$ then by Lemma 6.9, $U_{c,L_1}(y) \leq U_{c,L_2}(y)$ for all $0 \leq y \leq L_1$ and consequently also $U'_{c,L_1}(0) \leq U'_{c,L_2}(0)$. So $U'_{c,L_0}(0)$ is monotonic non-decreasing with respect to L_0. By considering the phase plane, it must also be bounded above by $\hat{U}'_c(0)$ since the trajectories cannot cross. Therefore as $L_0 \to \infty$, $U'_{c,L_0}(0)$ converges to some finite value $\hat{v} \leq \hat{U}'_c(0)$. If $\hat{v} < \hat{U}'_c(0)$ then we can choose $V_0 \in (\hat{v}, \hat{U}'_c(0))$ and consider the trajectory starting from $U = 0$, $V = V_0$ at $\xi = 0$. This must return to the $U = 0$ axis after some finite distance $\overline{L}$ in ξ, and so it corresponds to $U_{c,\overline{L}}$. But then $U'_{c,\overline{L}}(0) = V_0$, and this contradicts the fact that $U'_{c,L}(0) \leq \hat{v}$ for all L. Thus, $\hat{v} = \hat{U}'_c(0)$. □

The following result relates to the case $c = 0$ only.

Proposition 6.20. *Let* $L_2 \geq L_1 > \pi\sqrt{\frac{D}{f'(0)}}$. *Then*

$$U_{0,L_1}(\eta L_1) \leq U_{0,L_2}(\eta L_2) \qquad \text{for all } 0 \leq \eta \leq 1. \tag{6.90}$$

Proof. Let $u_1(\xi, t)$ and $u_2(\xi, t)$ satisfy

$$\frac{\partial u_1}{\partial t} = D\frac{\partial^2 u_1}{\partial \xi^2} + f(u_1) \qquad \text{for } 0 < \xi < L_1 \tag{6.91}$$

$$\frac{\partial u_2}{\partial t} = D\frac{\partial^2 u_2}{\partial \xi^2} + \frac{L_2^2}{L_1^2}f(u_2) \qquad \text{for } 0 < \xi < L_1 \tag{6.92}$$

with $u_1(\xi, t) = u_2(\xi, t) = 0$ at $\xi = 0$ and $\xi = L_1$, and with $u_1(\xi, 0) \equiv u_2(\xi, 0) \equiv u_0(\xi)$. Since $L_2 > L_1$ and $f \geq 0$, it follows that u_2 is a supersolution for u_1, and so $u_1(\xi, t) \leq u_2(\xi, t)$ for all $0 \leq \xi \leq L_1$ and $t \geq 0$. Change variables in the u_2 equation to $z = \frac{L_2}{L_1}\xi \in (0, L_2)$ and $s = \frac{L_2^2}{L_1^2}t$, and write $\overline{u}_2(z, s) = u_2(\xi, t)$. This satisfies

$$\frac{\partial \overline{u}_2}{\partial s} = D\frac{\partial^2 \overline{u}_2}{\partial z^2} + f(\overline{u}_2) \qquad \text{for } 0 < z < L_2 \tag{6.93}$$

with $\overline{u}_2(\xi, t) = 0$ at $\xi = 0$ and $\xi = L_2$. As $t \to \infty$ (and $s \to \infty$), we have uniform convergence $u_1(\xi, t) \to U_{0,L_1}(\xi)$ and $\overline{u}_2(z, s) \to U_{0,L_2}(z)$. Then the inequality

$$u_1(\xi, t) \leq u_2(\xi, t) = \overline{u}_2\left(\frac{L_2}{L_1}\xi, \frac{L_2^2}{L_1^2}t\right) \qquad \text{for all } 0 \leq \xi \leq L_1,\ t \geq 0 \tag{6.94}$$

leads to $U_{0,L_1}(\xi) \leq U_{0,L_2}\left(\frac{L_2}{L_1}\xi\right)$ for $0 \leq \xi \leq L_1$, which completes the proof. □

6.3 Convergence is locally uniform in c

In Theorem 6.4 we showed that $\sup_{\xi\in\Omega_0}|u(\xi,t)-U_c(\xi)| \to 0$ as $t\to\infty$ if $f'(0) > D\lambda(\Omega_0)+\frac{|c|^2}{4D}$. Next we shall prove that this convergence is uniform in compact subsets of c. This will be used in the proofs of Theorems 7.5 and 7.8.

Theorem 6.21. *Given some $c'_j \le c''_j$, suppose that $f'(0) > D\lambda(\Omega_0)+\frac{|c|^2}{4D}$ for all $c\in\Delta := \{c\in\mathbb{R}^N : c'_j \le c_j \le c''_j : 1\le j\le N\}$. Let $u_c(\xi,t)$ satisfy* (6.1), (6.2)*, with initial conditions $u_c(\xi,0)=u_0(\xi)$. Then the convergence to $U_c(\xi)$ as $t\to\infty$ (given by Theorem 6.4 for each $c\in\Delta$) is uniform with respect to $c\in\Delta$:*

$$\sup_{c\in\Delta}\sup_{\xi\in\Omega_0}|u_c(\xi,t)-U_c(\xi)|\to 0 \qquad \text{as } t\to\infty. \tag{6.95}$$

Proof. First, consider the solutions $\underline{u}_c$ with initial conditions $\underline{u}_c(\xi,0)=\delta\phi_1^{(c)}(\xi)$. As in Lemma 6.2 and Theorem 6.4, for δ small enough ($0<\delta\le\delta_0$ which can be chosen independently of $c\in\Delta$), $\underline{u}_c(\xi,0)$ is a subsolution to (6.6), (6.7). So, as before, $\underline{u}_c(\xi,t)$ is an increasing function of t and it converges uniformly to $U_c(\xi)$ as $t\to\infty$. Take any sequence $t_n\to\infty$, and for $c\in\Delta$ define

$$\Phi_n(c)=\sup_{\xi\in\Omega_0}|\underline{u}_c(\xi,t_n)-U_c(\xi)|. \tag{6.96}$$

Then Φ_n is continuous with respect to $c\in\Delta$ for each fixed n (by Proposition 6.11 and Lemma A.8); Φ_n is monotonic decreasing in n for each fixed c; and as $n\to\infty$, Φ_n converges pointwise in c to 0 (by Theorem 6.4). Hence, by Dini's Theorem [48, Example 5.4], Φ_n converges to 0 uniformly in $c\in\Delta$. Since the sequence $t_n\to\infty$ was arbitrary, we have $\sup_{\xi\in\Omega_0}|\underline{u}_c(\xi,t)-U_c(\xi)|\to 0$ uniformly in $c\in\Delta$ as $t\to\infty$.

Next consider the solutions $\overline{u}_c$ with initial conditions $\overline{u}_c(\xi,0)\equiv K$. Each $\overline{u}_c(\xi,t)$ is a decreasing function of t, and converges uniformly to $U_c(\xi)$ as $t\to\infty$. We apply Dini's Theorem to functions Φ_n defined as in (6.96) but with $\overline{u}_c$ instead of $\underline{u}_c$, and find that $\sup_{\xi\in\Omega_0}|\overline{u}_c(\xi,t)-U_c(\xi)|\to 0$ uniformly in $c\in\Delta$ as $t\to\infty$.

Finally, $\delta\phi_1^{(c)}(\xi)\le u_0(\xi)\le K$ implies that $\underline{u}_c(\xi,t)\le u_c(\xi,t)\le\overline{u}_c(\xi,t)$ for all $t\ge 0$. Therefore also $\sup_{\xi\in\Omega_0}|u_c(\xi,t)-U_c(\xi)|\to 0$ uniformly in $c\in\Delta$ as $t\to\infty$. □

Chapter 7

Nonlinear equation on other time-dependent domains $\Omega(t)$

In Chapter 6 the domain had the form Ω_0+ct. Now, we consider several domains $\Omega(t)$ that are not of this form. Sections 7.1–7.3 concern domains of Types 1 and 2 whose size and velocity are not constant but satisfy certain limiting behaviour as $t \to \infty$. We prove results about convergence to either positive stationary solutions or K. We extend the analysis to cylinder-like (Type 3) domains in Section 7.4. In the final part of the chapter, we consider the nonlinear equation on an interval $(A(t), A(t)+L(t))$. We discuss the role of $L_{crit}(c)$ in Section 7.5.1, and investigate the long-time behaviour of the gradient at the boundary in Section 7.5.2. Throughout the chapter, ψ will always denote a solution (≥ 0 and $\not\equiv 0$) to the nonlinear problem (1.1), (1.2) on the specified domain $\Omega(t)$.

We begin with a corollary to Theorem 6.4.

Corollary 7.1. *Assume that $f'(0) > D\lambda(\Omega_1) + \frac{|c|^2}{4D}$ for some bounded domain Ω_1 and vector c.*

1. *Suppose $\Omega(t)$ is such that for all t sufficiently large $\Omega_1 + ct \subset \Omega(t)$. Then $\liminf_{t\to\infty} \inf_{\xi\in\Omega_1} (\psi(\xi+ct,t) - U_{c,\Omega_1}(\xi)) \geq 0$.*
2. *Suppose instead that for all t sufficiently large, $\Omega(t) \subset \Omega_1 + ct$. Then $\limsup_{t\to\infty} \sup_{x\in\Omega(t)} (\psi(x,t) - U_{c,\Omega_1}(x-ct)) \leq 0$.*

Proof. These results follow by applying the comparison principle (Lemma 2.11), then letting $t \to \infty$ and using Theorem 6.4. □

7.1 A bounded domain moving at $\dot{A}(t) \to c$

We extend the result of Theorem 6.4 to $\Omega(t) = \Omega_0 + A(t)$ with $\dot{A}(t) \to c$.

Theorem 7.2. *Let* $\Omega(t) = \Omega_0 + A(t)$ *be as in equation* (2.6)*, and* $u(\xi, t) = \psi(x, t)$ *where* $\xi = x - A(t)$*. Suppose that* $\dot{A}_j(t) \to c_j$ *and* $\ddot{A}_j(t) \to 0$ *as* $t \to \infty$ *(for each* $1 \le j \le N$*), and suppose* $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$*. Let* U_c *be the unique positive solution to* (6.6), (6.7)*. Then* $\lim_{t\to\infty} \sup_{\xi\in\Omega_0} |u(\xi, t) - U_c(\xi)| = 0$.

Proof. Without loss of generality (by using the reflection properties of Lemmas 2.10 and 6.8 if necessary) we may assume $c_j \le 0$ for each $1 \le j \le N$. Furthermore (by reordering co-ordinates) we can assume that in fact $c_j = 0$ for $j \le m$, and $c_j < 0$ for $j > m$ (where m is some number between 0 and N, and where $m = 0$ if $c_j < 0$ for all j, or $m = N$ if $c_j = 0$ for all j). Finally (by translating the domain if necessary) we can assume $\xi_j \ge 0$ for each $1 \le j \le N$ and all $\xi \in \Omega_0$.

Let $\underline{c}$, $\overline{c} \in \mathbb{R}^N$ be such that $\underline{c}_j = \overline{c}_j = 0$ for each $j \le m$, and $\underline{c}_j < c_j < \overline{c}_j < 0$ for each $m < j \le N$. Assume $\underline{c}_j$ and $\overline{c}_j$ are sufficiently close to c_j and $\varepsilon_0 > 0$ is sufficiently small that for $0 < \varepsilon \le \varepsilon_0$

$$f'(0)(1 - m\varepsilon) > D\lambda(\Omega_0) + \frac{|\underline{c}|^2}{4D}. \tag{7.1}$$

For each $0 < \varepsilon \le \varepsilon_0$, there exists T such that for all $t \ge T$, $\xi \in \Omega_0$, $1 \le j \le N$:

$$1 - \varepsilon \le e^{\frac{-\dot{A}_j(t)\xi_j}{2D}} \le 1 + \varepsilon \quad \text{and} \quad \left| \frac{\ddot{A}_j(t)\xi_j}{2D} - \frac{\dot{A}_j(t)^2}{4D} \right| \le \varepsilon f'(0) \quad \text{if } j \le m, \tag{7.2}$$

$$-\overline{c}_j \le -\dot{A}_j(t) \le -\underline{c}_j \quad \text{and} \quad -\underline{c}_j^2 \le 2\ddot{A}_j(t)\xi_j - \dot{A}_j(t)^2 \le -\overline{c}_j^2 \quad \text{if } j > m. \tag{7.3}$$

Let $v(\xi, t) = u(\xi, t)e^{\frac{\dot{A}(t)\cdot\xi}{2D}}$. Then for $\xi \in \Omega_0$,

$$\frac{\partial v}{\partial t} = D\nabla^2 v + f\left(v(\xi, t)e^{\frac{-\dot{A}(t)\cdot\xi}{2D}}\right)e^{\frac{\dot{A}(t)\cdot\xi}{2D}} + \sum_{j=1}^{N}\left(\frac{\ddot{A}_j(t)\xi_j}{2D} - \frac{\dot{A}_j(t)^2}{4D}\right)v(\xi, t) \tag{7.4}$$

with $v(\xi,t)=0$ on $\partial\Omega_0$. For $\xi\in\Omega_0$, $t\geq T$, let $\underline{v}$ and $\overline{v}$ be the solutions to

$$\frac{\partial \underline{v}}{\partial t}=D\nabla^2\underline{v}+f\left((1+\varepsilon)^m\underline{v}(\xi,t)e^{\frac{-\underline{c}\cdot\xi}{2D}}\right)(1+\varepsilon)^{-m}e^{\frac{\underline{c}\cdot\xi}{2D}}-m\varepsilon f'(0)\underline{v}-\frac{|\underline{c}|^2}{4D}\underline{v} \quad (7.5)$$

$$\frac{\partial \overline{v}}{\partial t}=D\nabla^2\overline{v}+f\left((1-\varepsilon)^m\overline{v}(\xi,t)e^{\frac{-\overline{c}\cdot\xi}{2D}}\right)(1-\varepsilon)^{-m}e^{\frac{\overline{c}\cdot\xi}{2D}}+m\varepsilon f'(0)\overline{v}-\frac{|\overline{c}|^2}{4D}\overline{v} \quad (7.6)$$

with $\underline{v}(\xi,t)=\overline{v}(\xi,t)=0$ on $\partial\Omega_0$, and with $\underline{v}(\xi,T)\equiv\overline{v}(\xi,T)\equiv v(\xi,T)$. Then, using (7.2), (7.3) and the fact that $\frac{f(u)}{u}$ is a non-increasing function of $u>0$, $\underline{v}(\xi,t)$ will be a subsolution for v and $\overline{v}(\xi,t)$ will be a supersolution. So,

$$\underline{v}(\xi,t)\leq v(\xi,t)\leq\overline{v}(\xi,t) \qquad \text{for all } t\geq T. \quad (7.7)$$

Now, $\underline{v}$ and $\overline{v}$ each satisfy problems of the form (6.17), (6.18) with (respectively) constant velocities $\underline{c}$, $\overline{c}$ and reaction terms

$$\underline{f}_\varepsilon(u)=f\left((1+\varepsilon)^m u\right)(1+\varepsilon)^{-m}-m\varepsilon f'(0)u, \quad (7.8)$$

$$\overline{f}_\varepsilon(u)=f\left((1-\varepsilon)^m u\right)(1-\varepsilon)^{-m}+m\varepsilon f'(0)u. \quad (7.9)$$

These satisfy conditions of the type (2.20), with $\underline{f}'_\varepsilon(0)=(1-m\varepsilon)f'(0)$ and $\overline{f}'_\varepsilon(0)=(1+m\varepsilon)f'(0)$. So for ε small enough and $\underline{c}_j$, $\overline{c}_j$ such that (7.1) is satisfied, Corollary 6.5 implies that $\underline{v}(\xi,t)\to\underline{U}_{\underline{c}}(\xi)e^{\frac{\underline{c}\cdot\xi}{2D}}$ and $\overline{v}(\xi,t)\to\overline{U}_{\overline{c}}(\xi)e^{\frac{\overline{c}\cdot\xi}{2D}}$ uniformly in ξ as $t\to\infty$, where these are the unique positive solutions to

$$D\nabla^2\underline{U}_{\underline{c}}(\xi)+\underline{c}\cdot\nabla\underline{U}_{\underline{c}}+\underline{f}_\varepsilon(\underline{U}_{\underline{c}}(\xi))=0 \qquad \text{for } \xi\in\Omega_0, \quad (7.10)$$

$$D\nabla^2\overline{U}_{\overline{c}}(\xi)+\overline{c}\cdot\nabla\overline{U}_{\overline{c}}+\overline{f}_\varepsilon(\overline{U}_{\overline{c}}(\xi))=0 \qquad \text{for } \xi\in\Omega_0, \quad (7.11)$$

with $\underline{U}_{\underline{c}}(\xi)=\overline{U}_{\overline{c}}(\xi)=0$ on $\partial\Omega_0$. Therefore we deduce from (7.7) and the definition of v that

$$0\leq\liminf_{t\to\infty}\inf_{\xi\in\Omega_0}\left(u(\xi,t)e^{\frac{c\cdot\xi}{2D}}-\underline{U}_{\underline{c}}(\xi)e^{\frac{\underline{c}\cdot\xi}{2D}}\right), \quad (7.12)$$

$$\limsup_{t\to\infty}\sup_{\xi\in\Omega_0}\left(u(\xi,t)e^{\frac{c\cdot\xi}{2D}}-\overline{U}_{\overline{c}}(\xi)e^{\frac{\overline{c}\cdot\xi}{2D}}\right)\leq 0. \quad (7.13)$$

As $\varepsilon\to 0$ both $\underline{f}_\varepsilon$ and $\overline{f}_\varepsilon$ converge to f in $C^{0,1}([0,K(1-\varepsilon_0)^{-m}])$. So letting $\varepsilon\to 0$, $\underline{c}_j\to c_j$ and $\overline{c}_j\to c_j$, the continuous dependence (Propositions 6.11 and 6.12) means that $\underline{U}_{\underline{c}}$ and $\overline{U}_{\overline{c}}$ both converge uniformly to U_c on $\overline{\Omega_0}$. Therefore, we conclude that $\lim_{t\to\infty}\sup_{\xi\in\Omega_0}|u(\xi,t)-U_c(\xi)|=0$. □

We can also extend the result of Theorem 6.4 to domains whose size and velocity are both non-constant but convergent as $t \to \infty$.

Theorem 7.3. *Let Ω_0 be either a ball, $\Omega_0 = \{\xi \in \mathbb{R}^N : |\xi| < R_0\}$, or a box $\Omega_0 = \{\xi \in \mathbb{R}^N : 0 < \xi_j < l_j,\ 1 \le j \le N\}$. For a given $A(t) \in \mathbb{R}^N$, and $R_1(t) > 0$, ..., $R_N(t) > 0$, let*

$$\Omega(t) = \left\{ x \in \mathbb{R}^N : \left(\frac{x_1 - A_1(t)}{R_1(t)}, \ldots, \frac{x_N - A_N(t)}{R_N(t)} \right) \in \Omega_0 \right\}. \tag{7.14}$$

Assume that $R_j(t) \to 1$, $\dot{A}_j(t) \to c_j$, and $\ddot{A}_j(t) \to 0$ as $t \to \infty$ for $1 \le j \le N$, where $f'(0) > D\lambda(\Omega_0) + \frac{|c|^2}{4D}$. If $\xi_j = \frac{x_j - A_j(t)}{R_j(t)}$ and $u(\xi, t) = \psi(x, t)$ then

$$\lim_{t\to\infty} \sup_{\xi \in \Omega_0} |u(\xi, t) - U_{c,\Omega_0}(\xi)| = 0. \tag{7.15}$$

Proof. Let $\varepsilon > 0$ be small enough that $f'(0) > \frac{D\lambda(\Omega_0)}{(1-\varepsilon)^2} + \frac{|c|^2}{4D}$. Then there exists T such that $1 - \varepsilon \le R_j(t) \le 1 + \varepsilon$ for all $t \ge T$ and $1 \le j \le N$. Let

$$\Omega_\varepsilon^\pm(t) = \left\{ x \in \mathbb{R}^N : \left(\frac{x_1 - A_1(t)}{1 \pm \varepsilon}, \ldots, \frac{x_N - A_N(t)}{1 \pm \varepsilon} \right) \in \Omega_0 \right\}. \tag{7.16}$$

Then $\Omega_\varepsilon^-(t) \subset \Omega(t) \subset \Omega_\varepsilon^+(t)$ for all $t \ge T$, and so the solutions $\psi^\pm(x, t)$ to

$$\frac{\partial \psi^\pm}{\partial t} = D\nabla^2 \psi^\pm + f(\psi^\pm) \qquad \text{in } \Omega_\varepsilon^\pm(t),\ t \ge T \tag{7.17}$$

$$\psi^\pm(x, t) = 0 \qquad \text{on } \partial\Omega_\varepsilon^\pm(t) \tag{7.18}$$

with $\psi^+(x, T) = \psi^-(x, T) = \psi(x, T)$, are a supersolution and subsolution for ψ. Using the supersolution we get

$$\psi(x, t) - \psi^+(x, t) \le 0 \qquad \text{for all } x \in \Omega(t),\ t \ge T. \tag{7.19}$$

Equivalently,

$$u(\xi, t) - \psi^+(R_1(t)\xi_1 + A_1(t), \ldots, R_N(t)\xi_N + A_N(t), t) \le 0 \quad \text{for all } \xi \in \Omega_0,\ t \ge T. \tag{7.20}$$

But by Theorem 7.2

$$\sup_{y \in (1+\varepsilon)\Omega_0} |\psi^+(y + A(t), t) - U_{c,(1+\varepsilon)\Omega_0}(y)| \to 0 \qquad \text{as } t \to \infty. \tag{7.21}$$

So we deduce from (7.20) and (7.21) that

$$\limsup_{t\to\infty}\left(\sup_{\xi\in\Omega_0}\left(u(\xi,t)-U_{c,(1+\varepsilon)\Omega_0}(R_1(t)\xi_1,\ldots,R_N(t)\xi_N)\right)\right)\le 0, \tag{7.22}$$

and thus

$$\limsup_{t\to\infty}\left(\sup_{\xi\in\Omega_0}\left(u(\xi,t)-U_{c,(1+\varepsilon)\Omega_0}(\xi_1,\ldots,\xi_N)\right)\right)\le 0. \tag{7.23}$$

Similarly, by applying Theorem 7.2 to the subsolution $\psi^-(x,t)$, we find that

$$\liminf_{t\to\infty}\left(\inf_{\xi\in(1-\varepsilon)\Omega_0}\left(u(\xi,t)-U_{c,(1-\varepsilon)\Omega_0}(\xi)\right)\right)\ge 0. \tag{7.24}$$

Let $\varepsilon\to 0$ in (7.23) and (7.24), and use the continuity (Proposition 6.11), to conclude that $\lim_{t\to\infty}\sup_{\xi\in\Omega_0}|u(\xi,t)-U_{c,\Omega_0}(\xi)|=0$. □

Remark 7.4. *Having obtained the results of Theorems 7.2 and 7.3, it then follows from Proposition 2.13 that in each case there is convergence not only uniformly in ξ but in $C^2(\overline{\Omega_0})$, and that $\frac{\partial u}{\partial t}$ converges uniformly to zero.*

7.2 Time-dependent box in $\mathbb{R}^N$, as the side lengths tend to ∞

Suppose that $\Omega(t)$ is a time-dependent box of the form (2.14); that is

$$\Omega(t)=\left\{x\in\mathbb{R}^N : A_j(t)<x_j<A_j(t)+L_j(t) : 1\le j\le N\right\}$$

for some $A_j(t)\in\mathbb{R}$, and $L_j(t)>0$, each twice differentiable. Changing variables from x_j to $\xi_j=\left(\frac{x_j-A_j(t)}{L_j(t)}\right)L_0$ and $\psi(x,t)=u(\xi,t)$, the problem becomes (2.15), (2.16); that is:

$$\frac{\partial u}{\partial t}=D\sum_{j=1}^{N}\frac{L_0^2}{L_j(t)^2}\frac{\partial^2 u}{\partial\xi_j^2}+\sum_{j=1}^{N}\left(\frac{\xi_j\dot{L}_j(t)+L_0\dot{A}_j(t)}{L_j(t)}\right)\frac{\partial u}{\partial\xi_j}+f(u)\quad\text{for } 0<\xi_j<L_0$$

$$u(\xi,t)=0\qquad\text{at } \xi_j=0 \text{ and } \xi_j=L_0.$$

Let $v(\xi,t)=u(\xi,t)E(\xi,t)$ where

$$E(\xi,t)=\exp\left(\sum_{j=1}^{N}\left(\frac{\xi_j^2\dot{L}_j(t)L_j(t)}{4DL_0^2}+\frac{\xi_j\dot{A}_j(t)L_j(t)}{2DL_0}\right)\right). \tag{7.25}$$

Then for $0 < \xi_j < L_0$, v satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} =& D\sum_{j=1}^{N}\frac{L_0^2}{L_j(t)^2}\frac{\partial^2 v}{\partial \xi_j^2} + f\left(\frac{v(\xi,t)}{E(\xi,t)}\right)E(\xi,t)\\ &+\sum_{j=1}^{N}\left(\frac{\xi_j^2\ddot{L}_j(t)L_j(t)}{4DL_0^2} + \frac{\xi_j\ddot{A}_j(t)L_j(t)}{2DL_0} - \frac{\dot{A}_j(t)^2}{4D} - \frac{\dot{L}_j(t)}{2L_j(t)}\right)v \end{aligned}\tag{7.26}$$

$$v(\xi,t) = 0 \qquad \text{at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{7.27}$$

In the following theorem we consider the nonlinear equation on a box (or an interval, taking $N = 1$) such that $L_j(t) \to \infty$ as $t \to \infty$, with $\dot{L}_j(t) \to \alpha_j \geq 0$ and $\dot{A}_j(t) \to c_j$. This may be compared with Corollary 3.6 for the separable solutions to the linear equation. We make this comparison in Example 7.7.

Theorem 7.5. *Let $\Omega(t)$ be given by (2.14). Suppose there are finite constants c_j and α_j (for $1 \leq j \leq N$) such that $\dot{A}_j(t) \to c_j$, $\ddot{A}_j(t) \to 0$, $L_j(t) \to \infty$, $\dot{L}_j(t) \to \alpha_j \geq 0$, $\ddot{L}_j(t) \to 0$, as $t \to \infty$. If the set*

$$S = \left\{\xi \in \mathbb{R}^N : \xi_j \in (0, L_0),\ f'(0) > \frac{1}{4D}\sum_{j=1}^{N}\left(c_j + \alpha_j\frac{\xi_j}{L_0}\right)^2\right\} \tag{7.28}$$

is non-empty, then for every compact set $V \subset S$,

$$\sup_{\xi\in V}|u(\xi,t) - K| \to 0 \qquad \textit{as } t \to \infty. \tag{7.29}$$

Proof. Each compact set $V \subset S$ can be enclosed by a finite union of sets of the form $V' = \left\{\xi \in \mathbb{R}^N : \frac{\xi}{L_0} \in \Lambda\right\} \subset S$ where

$$\Lambda = \{\underline{\eta}_j \leq \eta_j \leq \overline{\eta}_j : 1 \leq j \leq N\} \tag{7.30}$$

$$0 < \underline{\eta}_j \leq \overline{\eta}_j < 1 \qquad \text{with} \qquad (c_j + \underline{\eta}_j\alpha_j)(c_j + \overline{\eta}_j\alpha_j) \geq 0, \tag{7.31}$$

i.e. $(c_j + \underline{\eta}_j\alpha_j)$ and $(c_j + \overline{\eta}_j\alpha_j)$ have the same sign as each other. So it is enough to prove the claim on V'. Moreover, by the reflection property (Lemma 2.10 and Lemma 6.8) it is enough to prove it for the case where $c_j + \underline{\eta}_j\alpha_j \leq c_j + \overline{\eta}_j\alpha_j \leq 0$ for every j. [If $0 \leq c_j + \underline{\eta}_j\alpha_j \leq c_j + \overline{\eta}_j\alpha_j$ for some j then we can replace $A_j(t)$

by $\hat{A}_j(t) = -(A_j(t) + L_j(t))$, and ξ_j by $\hat{\xi}_j = L_0 - \xi_j$. Then $c_j + \eta_j\alpha_j$ is replaced by $-(c_j + \alpha_j) + (1 - \eta_j)\alpha_j = -(c_j + \eta_j\alpha_j)$ which is ≤ 0.] So, we assume:

$$c_j + \underline{\eta}_j\alpha_j \leq c_j + \overline{\eta}_j\alpha_j \leq 0 \quad \text{for } 1 \leq j \leq N \quad \text{and} \quad f'(0) > \frac{1}{4D}\sum_{j=1}^{N}\Big(c_j + \alpha_j\underline{\eta}_j\Big)^2. \tag{7.32}$$

Let us write $\tilde{c}+\eta\alpha$ and $A(t)+\eta L(t)$ for the vectors with jth component $\tilde{c}_j+\eta_j\alpha_j$ and $A_j(t) + \eta_j L_j(t)$ (and likewise for the time derivatives). Then we would like to show that $|\psi(A(t) + \eta L(t), t) - K| \to 0$ uniformly in $\eta \in \Lambda$ as $t \to \infty$.

Choose $\tilde{c}_j < c_j$ close enough to c_j and L_* large enough such that

$$f'(0) > \frac{1}{4D}\sum_{j=1}^{N}\Big(\tilde{c}_j + \alpha_j\underline{\eta}_j\Big)^2 + N\frac{D\pi^2}{L_*^2}. \tag{7.33}$$

Then choose $T \geq 0$ large enough such that for all $t \geq T$ and $1 \leq j \leq N$, $\underline{\eta}_j L_j(t) \geq L_*$ and $(1 - \overline{\eta}_j)L_j(t) \geq L_*$, and also

$$\begin{aligned} \tilde{c}_j + \eta_j\alpha_j &\leq \dot{A}_j(t) + \eta_j\dot{L}_j(t), \\ -(\tilde{c}_j + \eta_j\alpha_j)^2 &\leq -(\dot{A}_j(t) + \eta_j\dot{L}_j(t))^2 + \min(0, 2L_*(\ddot{A}_j(t) + \eta_j\ddot{L}_j(t))). \end{aligned} \tag{7.34}$$

Write $\Omega_* = \{x \in \mathbb{R}^N : 0 < x_j < L_*\}$ and note that there exists some $u_* \geq 0$ not identically zero such that

$$\psi(A(T) + y + x, T) \geq u_*(x) \qquad \text{for all } x \in \Omega_*,\ 0 \leq y_j \leq L_j(T) - L_*. \tag{7.35}$$

In particular, by choice of T, this holds with $y_j = \eta_j L_j(T) - z_j$ for every $z \in \Omega_*$ and $\eta \in \Lambda$:

$$\psi(A(T) + \eta L(T) - z + x, T) \geq u_*(x) \qquad \text{for all } \eta \in \Lambda,\ z \in \Omega_*,\ x \in \Omega_*. \tag{7.36}$$

Now let u_η be the solution to

$$\frac{\partial u_\eta}{\partial t} = D\nabla^2 u_\eta + \sum_{j=1}^{N}(\dot{A}_j(t) + \eta_j\dot{L}_j(t))\frac{\partial u_\eta}{\partial x_j} + f(u_\eta) \qquad \text{in } \Omega_* \times (T, \infty) \tag{7.37}$$

with $u_\eta(x,t) = 0$ on $\partial\Omega_*$, and with $u_\eta(x, T) = u_*(x)$. Then for $\eta \in \Lambda$ and every $z \in \Omega_*$, $u_\eta(x,t)$ is a subsolution for $\psi(A(t) + \eta L(t) - z + x, t)$ on Ω_*. Therefore,

$$\psi(A(t) + \eta L(t) - z + x, t) \geq u_\eta(x,t) \qquad \text{for all } z \in \Omega_*,\ x \in \Omega_*,\ t \geq T, \tag{7.38}$$

and thus

$$\psi(A(t)+\eta L(t),t) \geq \sup_{x\in\Omega_*} u_\eta(x,t) \qquad \text{for all } t \geq T. \tag{7.39}$$

Also define

$$v_\eta(x,t) = u_\eta(x,t) \exp\left(\sum_{j=1}^{N} \frac{(\dot{A}_j(t)+\eta_j \dot{L}_j(t))x_j}{2D}\right), \tag{7.40}$$

which satisfies

$$\begin{aligned}\frac{\partial v_\eta}{\partial t} =& D\nabla^2 v_\eta + \sum_{j=1}^{N}\left(\frac{x_j(\ddot{A}_j(t)+\eta_j\ddot{L}_j(t))}{2D} - \frac{(\dot{A}_j(t)+\eta_j\dot{L}_j(t))^2}{4D}\right)v_\eta \\ &+ f\left(\frac{v_\eta(x,t)}{\exp\left(\frac{(\dot{A}(t)+\eta\dot{L}(t))\cdot x}{2D}\right)}\right)\exp\left(\frac{(\dot{A}(t)+\eta\dot{L}(t))\cdot x}{2D}\right) \qquad \text{in } \Omega_* \end{aligned} \tag{7.41}$$

$$v_\eta(x,t) = 0 \qquad \text{on } \partial\Omega_*. \tag{7.42}$$

Using (7.34) and the fact that $\frac{f(u)}{u}$ is non-increasing in u, we see that $v_\eta(x,t)$ is a supersolution for $v_{\eta,\tilde{c}}(x,t)$ where

$$\begin{aligned}\frac{\partial v_{\eta,\tilde{c}}}{\partial t} =& D\nabla^2 v_{\eta,\tilde{c}} - \sum_{j=1}^{N}\frac{(\tilde{c}_j+\eta_j\alpha_j)^2}{4D}v_{\eta,\tilde{c}} \\ &+ f\left(\frac{v_{\eta,\tilde{c}}(x,t)}{\exp\left(\frac{(\tilde{c}+\eta\alpha)\cdot x}{2D}\right)}\right)\exp\left(\frac{(\tilde{c}+\eta\alpha)\cdot x}{2D}\right) \qquad \text{in } \Omega_* \end{aligned} \tag{7.43}$$

$$v_{\eta,\tilde{c}}(x,t) = 0 \qquad \text{on } \partial\Omega_* \tag{7.44}$$

with $v_{\eta,\tilde{c}}(\cdot,T) = v_\eta(\cdot,T)$. So,

$$v_\eta(x,t) \geq v_{\eta,\tilde{c}}(x,t) \qquad \text{for all } x\in\Omega_*,\ t\geq T,\ \eta\in\Lambda. \tag{7.45}$$

By Corollary 6.5, $v_{\eta,\tilde{c}}(x,t)$ converges to $U_{\tilde{c}+\eta\alpha,\Omega_*}(x)\exp\left(\frac{(\tilde{c}+\eta\alpha)\cdot x}{2D}\right)$ uniformly on Ω_* as $t\to\infty$, and by Theorem 6.21 this convergence is uniform in $\eta\in\Lambda$. So,

$$\liminf_{t\to\infty} \inf_{x\in\Omega_*,\eta\in\Lambda}\left(v_\eta(x,t) - U_{\tilde{c}+\eta\alpha,\Omega_*}(x)\exp\left(\sum_{j=1}^{N}\frac{(\tilde{c}_j+\eta_j\alpha_j)x_j}{2D}\right)\right) \geq 0. \tag{7.46}$$

Using (7.40) and the convergence $\dot{A}_j(t)\to c_j$, $\dot{L}_j(t)\to\alpha_j$, this implies that

$$\liminf_{t\to\infty} \inf_{x\in\Omega_*,\eta\in\Lambda}\left(u_\eta(x,t) - U_{\tilde{c}+\eta\alpha,\Omega_*}(x)\exp\left(\sum_{j=1}^{N}\frac{(\tilde{c}_j-c_j)x_j}{2D}\right)\right) \geq 0. \tag{7.47}$$

Recalling (7.39), we deduce that

$$\liminf_{t\to\infty}\min_{\eta\in\Lambda}\left(\psi(A(t)+\eta L(t),t)-\sup_{x\in\Omega_*}\left[U_{\tilde{c}+\eta\alpha,\Omega_*}(x)\exp\left(\frac{(\tilde{c}-c)\cdot x}{2D}\right)\right]\right)\geq 0. \tag{7.48}$$

Now let $L_* \to \infty$ and $\tilde{c} \to c$, and use Propositions 6.15 and 6.11. This leads to

$$\liminf_{t\to\infty}\min_{\eta\in\Lambda}(\psi(A(t)+\eta L(t),t)-K)\geq 0. \tag{7.49}$$

Since K is also an upper bound for ψ, the result follows. □

Remark 7.6. *By Proposition 2.13, we then deduce that on compact subsets of S not only does $u(\xi,t)$ converge uniformly to K, but its first time derivative and its spatial derivatives of orders one and two all converge uniformly to zero.*

The following example illustrates an application of Theorem 7.5.

Example 7.7. *Consider the time-dependent intervals for which the linear equation has exact (separable) solutions $u_{lin}(\xi,t)$. Since $u \leq u_{lin}$, it is clear that $u(\xi,t) \to 0$ for any ξ such that $u_{lin}(\xi,t) \to 0$. Here, we shall consider the $L(t)$, $A(t)$ and ξ for which $u_{lin}(\xi,t) \to \infty$ as $t \to \infty$ (see Corollary 3.6).*

First consider the case $L(t) \equiv L_0$ and $A(t) = ct + A_0$ with $f'(0) > \frac{D\pi^2}{L_0^2} + \frac{c^2}{4D}$. Theorem 6.4 implies that $u(\xi,t) \to U_{c,L_0}(\xi)$, uniformly in $0 \leq \xi \leq L_0$.

Next take $L(t) = \sqrt{L_0^2 + 2\rho t}$ with $\rho > 0$, and $A(t) = \frac{-\gamma_1}{\rho^2}\sqrt{L_0^2+2\rho t} + ct + d$, and $f'(0) > \frac{c^2}{4D}$. Theorem 7.5 shows that $u(\xi,t) \to K$ as $t \to \infty$, uniformly on every compact subset of $0 < \xi < L_0$.

Finally we consider cases where either $L(t) = L_0 + \alpha t$ with $\alpha > 0$; or $L(t) = \sqrt{at^2 + 2bt + L_0^2}$ with $a \neq 0$, $aL_0^2 - b^2 \neq 0$ and $L(t) > 0$ for all $0 \leq t < \infty$; and $\ddot{A}(t)L(t)^3 \equiv \gamma_1$. There are constants $\hat{\alpha} \geq 0$ and $\hat{c}$ such that $L(t) \to \infty$, $\dot{L}(t) \to \hat{\alpha}$ and $\dot{A}(t) \to \hat{c}$ as $t \to \infty$. Corollary 3.6 shows that $u_{lin}(\xi,t) \to \infty$ at any $\xi \in (0, L_0)$ such that $\left(\hat{c} + \hat{\alpha}\frac{\xi}{L_0}\right)^2 < c_*^2$. Theorem 7.5 shows that for the nonlinear equation, $u(\xi,t) \to K$ uniformly on compact subsets in this range.

7.3 Time-dependent domain in $\mathbb{R}^{m+N}$, as the side lengths tend to ∞ in N dimensions

Suppose that $\Omega(t)$ has the form (2.17); that is:

$$\Omega(t) = \Big\{(x_0, x) \in \mathbb{R}^{m+N} : x_0 - A_0(t) \in \omega_0 \subset \mathbb{R}^m, \\ A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N\Big\}$$

for some smooth bounded domain ω_0 and vector $A_0(t)$ in $\mathbb{R}^m$, and some $A_j(t)$, $L_j(t) > 0$. As in Example 2.5, the change of variables to $\xi_0 = x_0 - A_0(t)$, $\xi_j = \left(\frac{x_j - A_j(t)}{L_j(t)}\right) L_0$, and $u(\xi_0, \xi, t) = \psi(x_0, x, t)$ leads to the problem (2.18), (2.19); namely

$$\begin{aligned}\frac{\partial u}{\partial t} =& D\nabla^2_{\xi_0} u + \sum_{j=1}^{N} D \frac{L_0^2}{L_j(t)^2} \frac{\partial^2 u}{\partial \xi_j^2} + A_0(t) \cdot \nabla_{\xi_0} u \\ &+ \sum_{j=1}^{N} \left(\frac{\xi_j \dot{L}_j(t) + L_0 \dot{A}_j(t)}{L_j(t)} \right) \frac{\partial u}{\partial \xi_j} + f(u) \qquad \text{for } (\xi_0, \xi) \in \Omega_0\end{aligned}$$

$$u(\xi_0, \xi, t) = 0 \qquad \text{for } (\xi_0, \xi) \in \partial\Omega_0.$$

Let $v(\xi_0, \xi, t) = u(\xi_0, \xi, t) E(\xi_0, \xi, t)$ where

$$E(\xi_0, \xi, t) = \exp\left(\frac{\xi_0 \cdot \dot{A}_0(t)}{2D} + \sum_{j=1}^{N} \left(\frac{\xi_j^2 \dot{L}_j(t) L_j(t)}{4DL_0^2} + \frac{\xi_j \dot{A}_j(t) L_j(t)}{2DL_0} \right) \right). \tag{7.50}$$

Then

$$\begin{aligned}\frac{\partial v}{\partial t} =& D\nabla^2_{\xi_0} v + D \sum_{j=1}^{N} \frac{L_0^2}{L_j(t)^2} \frac{\partial^2 v}{\partial \xi_j^2} + \left(\frac{\xi_0 \cdot \ddot{A}_0(t)}{2D} - \frac{|\dot{A}_0(t)|^2}{4D} \right) v(\xi_0, \xi, t) \\ &+ \sum_{j=1}^{N} \left(\frac{\xi_j^2 \ddot{L}_j(t) L_j(t)}{4DL_0^2} + \frac{\xi_j \ddot{A}_j(t) L_j(t)}{2DL_0} - \frac{\dot{A}_j(t)^2}{4D} - \frac{\dot{L}_j(t)}{2L_j(t)} \right) v(\xi_0, \xi, t) \\ &+ f\left(\frac{v(\xi_0, \xi, t)}{E(\xi_0, \xi, t)} \right) E(\xi_0, \xi, t) \qquad \text{for } \xi_0 \in \omega_0,\ 0 < \xi_j < L_0\end{aligned} \tag{7.51}$$

$$v(\xi_0, \xi, t) = 0 \qquad \text{for } \xi_0 \in \partial\omega_0, \text{ and at } \xi_j = 0 \text{ and } \xi_j = L_0. \tag{7.52}$$

The following result is the analogy of Theorem 7.5, for this type of domain.

Theorem 7.8. *Let $\Omega(t)$ be of the form (2.17), with $\dot{A}_0(t) \equiv c_0 \in \mathbb{R}^m$. Let f satisfy (2.20), and assume $\frac{f(u)}{u}$ is a strictly decreasing, uniformly continuous function of $u > 0$. Suppose there are finite constants c_j and α_j (for $1 \le j \le N$) such that $\dot{A}_j(t) \to c_j$, $\ddot{A}_j(t) \to 0$, $L_j(t) \to \infty$, $\dot{L}_j(t) \to \alpha_j \ge 0$, $\ddot{L}_j(t) \to 0$, as $t \to \infty$. If the set*

$$S = \left\{ \xi \in \mathbb{R}^N : 0 < \xi_j < L_0,\ f'(0) > D\lambda(\omega_0) + \frac{|c_0|^2}{4D} + \frac{1}{4D} \sum_{j=1}^{N} \left(c_j + \alpha_j \frac{\xi_j}{L_0} \right)^2 \right\} \tag{7.53}$$

is non-empty, then for every compact set $V \subset S$,

$$\sup_{\xi_0 \in \omega_0, \xi \in V} |u(\xi_0, \xi, t) - U_{c_0,\omega_0}(\xi_0)| \to 0 \qquad \text{as } t \to \infty, \tag{7.54}$$

where $U_{c_0,\omega_0}(\xi)$ is the unique positive solution to (6.52), (6.53).

Remark 7.9. *Theorem 7.8 also holds under more general conditions on the cross-section in $\mathbb{R}^m$: $\omega_0 + c_0 t$ can be replaced by $\omega_0 + A_0(t)$ with $\dot{A}_0(t) \to c_0$ and $\ddot{A}_0(t) \to 0$ as $t \to \infty$. Also ω_0 could be replaced by a box or ball in $\mathbb{R}^m$ satisfying the assumptions of Theorem 7.3. These generalisations can be proved by combining the proofs of Theorems 7.2 and 7.3 with that of Theorem 7.8.*

Proof. The proof is similar to that of Theorem 7.5; it only differs at the end. Subject to the inclusion of ξ_0 and c_0 in the required places, we follow exactly the same steps as led to (7.48) in the proof of Theorem 7.5. These now lead to:

$$\liminf_{t\to\infty} \inf_{\eta\in\Lambda, \xi_0\in\omega_0} \Bigg(\psi(\xi_0 + A_0(t), A(t) + \eta L(t), t) \\ - \sup_{\xi\in\Omega_*} \left[U_{c_0,\tilde{c}+\eta\alpha,\omega_0,\Omega_*}(\xi_0,\xi) \exp\left(\frac{(\tilde{c} - c)\cdot\xi}{2D} \right) \right] \Bigg) \ge 0 \tag{7.55}$$

where (as before) we write $\tilde{c} + \eta\alpha$ for the vector with jth component $\tilde{c}_j + \eta_j \alpha_j$, and where $U_{c_0,\tilde{c}+\eta\alpha,\omega_0,\Omega_*}(\xi_0,\xi)$ is the unique positive solution to

$$D\nabla^2 U(\xi_0,\xi) + (c_0, \tilde{c} + \eta\alpha)\cdot\nabla U(\xi_0,\xi) + f(U(\xi_0,\xi)) = 0 \qquad \text{for } \xi_0 \in \omega_0,\ \xi \in \Omega_* \tag{7.56}$$

$$U(\xi_0,\xi) = 0 \qquad \text{for } \xi_0 \in \partial\omega_0 \text{ or } \xi \in \partial\Omega_*. \tag{7.57}$$

Now let $L_* \to \infty$ and $\tilde{c} \to c$ in (7.55), and use the results of Theorem 6.17 and Proposition 6.11. This gives the lower bound

$$\liminf_{t\to\infty} \inf_{\eta\in\Lambda,\xi_0\in\omega_0} (\psi(\xi_0 + A_0(t), A(t) + \eta L(t), t) - U_{c_0,\omega_0}(\xi_0)) \geq 0. \tag{7.58}$$

Finally, the solution $u_\infty(\xi_0, t)$ to (6.76), (6.77) with initial conditions $u_\infty(\xi_0, 0) \equiv ||u_0||_\infty$, is a supersolution for u and so $u(\xi_0, \xi, t) \leq u_\infty(\xi_0, t)$ for all $t \geq 0$. Since $u_\infty(\xi_0, t)$ converges uniformly to $U_{c_0,\omega_0}(\xi_0)$ as $t \to \infty$, we deduce that $\limsup_{t\to\infty} u(\xi_0, \xi, t) \leq U_{c_0,\omega_0}(\xi_0)$ uniformly in ξ_0, ξ, and the result follows. □

Remark 7.10. *Proposition 2.13 then implies that for compact subsets V of S, $u(\xi_0, \xi, t)$ converges to $U_{c_0,\omega_0}(\xi_0)$ in $C^2(\overline{\omega_0} \times V)$, and $\frac{\partial u}{\partial t} \to 0$ uniformly in $\overline{\omega_0} \times V$.*

7.4 Cylinder-like domains

Here we consider domains of Type 3 (cylinder-like domains). As explained in Chapter 2, we change variables from x in the time-dependent cross-section $\tilde{\Omega}(t)$ to ξ in a fixed $\tilde{\Omega}_0$, and write the solution $\psi(x, y, t)$ to (1.1), (1.2) as $u(\xi, y, t)$. We shall use Theorems 7.5 and 7.8 to prove convergence results on sets of the form $\{\xi \in V, |y| \leq \tilde{c}t\}$ when (in the cross-section) $L_j(t) \to \infty$ and $\dot{L}_j(t) \to 0$.

Theorem 7.11. *Let $\Omega(t) = \tilde{\Omega}(t) \times \mathbb{R} \subset \mathbb{R}^{N+1}$ where $\tilde{\Omega}(t) \subset \mathbb{R}^N$ is of the form (2.14). Suppose that (for $1 \leq j \leq N$) $\dot{A}_j(t) \to c_j$, $\ddot{A}_j(t) \to 0$, $L_j(t) \to \infty$, $\dot{L}_j(t) \to 0$, $\ddot{L}_j(t) \to 0$ as $t \to \infty$. Assume that $f'(0) > \sum_{j=1}^{N} \frac{c_j^2}{4D}$ and let $C > 0$ be the positive solution to*

$$\frac{C^2}{4D} = f'(0) - \sum_{j=1}^{N} \frac{c_j^2}{4D}. \tag{7.59}$$

Then for each compact set $V \subset \tilde{\Omega}_0$ and each $0 \leq \tilde{c} < C$,

$$\sup_{\xi\in V,|y|\leq\tilde{c}t} |u(\xi, y, t) - K| \to 0 \qquad \text{as } t \to \infty. \tag{7.60}$$

Proof. Fix $L_0 > 0$ and let $\psi_1(x, y, t)$ be the solution to (1.1), (1.2) on the domain $\Omega_1(t) = \tilde{\Omega}(t) \times (-Ct - \frac{L_0}{2}, Ct + \frac{L_0}{2})$. Let $u_1(\xi, \xi_{N+1}, t)$ denote the solution ψ_1 in the transformed domain $\{\xi \in \tilde{\Omega}_0,\, 0 < \xi_{N+1} < L_0\}$, where $\xi_{N+1} = \left(\frac{y}{2Ct+L_0} + \frac{1}{2}\right) L_0$. The domain $\Omega_1(t)$ satisfies the conditions of Theorem 7.5, with $c_{N+1} = -C$ and $\alpha_{N+1} = 2C$. Note also that the inequality

$$f'(0) > \sum_{j=1}^{N} \frac{c_j^2}{4D} + \left(-C + 2C\frac{\xi_{N+1}}{L_0}\right)^2 \tag{7.61}$$

is satisfied for ξ_{N+1} in every compact subset of $(0, L_0)$. So Theorem 7.5 implies that for all compact sets $V \subset \tilde{\Omega}_0$ and $V' \subset (0, L_0)$,

$$\sup_{\xi \in V, \xi_{N+1} \in V'} |u_1(\xi, \xi_{N+1}, t) - K| \to 0 \qquad \text{as } t \to \infty. \tag{7.62}$$

But since $\psi_1(x, y, t) = u_1(\xi, \xi_{N+1}, t)$ is a subsolution for $\psi(x, y, t) = u(\xi, y, t)$,

$$u_1\left(\xi, \left(\frac{y}{2Ct + L_0} + \frac{1}{2}\right) L_0, t\right) \leq u(\xi, y, t) \leq K. \tag{7.63}$$

Also, for $0 \leq \tilde{c} < C$ the range $|y| \leq \tilde{c}t$ corresponds to

$$\left(\frac{-\tilde{c}t}{2Ct + L_0} + \frac{1}{2}\right) L_0 \leq \xi_{N+1} \leq \left(\frac{\tilde{c}t}{2Ct + L_0} + \frac{1}{2}\right) L_0 \tag{7.64}$$

which is always contained in the compact set

$$V' := \left[\frac{L_0}{2}\left(1 - \frac{\tilde{c}}{C}\right), \; \frac{L_0}{2}\left(1 + \frac{\tilde{c}}{C}\right)\right] \subset (0, L_0). \tag{7.65}$$

We apply (7.62) with this V', and combine with (7.63), to get the result. □

We also have the analogous result for domains of the type (2.17).

Theorem 7.12. *Let* $\Omega(t) = \tilde{\Omega}(t) \times \mathbb{R}$ *where* $\tilde{\Omega}(t)$ *is of the form* (2.17) *with* $\dot{A}_0(t) \equiv c_0 \in \mathbb{R}^m$. *Let* f *satisfy assumptions* (2.20), *and assume that* $\frac{f(u)}{u}$ *is a strictly decreasing, uniformly continuous function of* $u > 0$. *Suppose that (for* $1 \leq j \leq N$*)* $\dot{A}_j(t) \to c_j$, $\ddot{A}_j(t) \to 0$, $L_j(t) \to \infty$, $\dot{L}_j(t) \to 0$, $\ddot{L}_j(t) \to 0$ *as* $t \to \infty$. *Assume that* $f'(0) > D\lambda(\omega_0) + \frac{|c_0|^2}{4D} + \sum_{j=1}^{N} \frac{c_j^2}{4D}$ *and let* $C > 0$ *be the positive solution to*

$$\frac{C^2}{4D} = f'(0) - D\lambda(\omega_0) - \frac{|c_0|^2}{4D} - \sum_{j=1}^{N} \frac{c_j^2}{4D} \tag{7.66}$$

Then for each compact set $V \subset \tilde{\Omega}_0$ and each $0 \le \tilde{c} < C$,

$$\sup_{\xi_0 \in \omega_0, \xi \in V, |y| \le \tilde{c}t} |u(\xi_0, \xi, y, t) - U_{c_0,\omega_0}(\xi_0)| \to 0 \qquad \text{as } t \to \infty, \tag{7.67}$$

where $U_{c_0,\omega_0}(\xi_0)$ is the unique positive solution to (6.52), (6.53).

Proof. The proof is essentially the same as for Theorem 7.11, except now we apply Theorem 7.8 to the domain $\Omega_1(t)$ instead of Theorem 7.5. □

Remark 7.13. *As in Remark 7.9, Theorem 7.12 is also valid if we replace the cross-section $\omega_0 + c_0 t$ in $\mathbb{R}^m$ either by $\omega_0 + A_0(t)$ with $\dot{A}_0(t) \to c_0$ and $\ddot{A}_0(t) \to 0$, or by a box or ball in $\mathbb{R}^m$ satisfying the assumptions of Theorem 7.3.*

7.5 An interval $(A(t), A(t) + L(t))$

Throughout this section we assume that $\psi(x,t)$ satisfies the nonlinear problem (2.8), (2.9) on an interval $A(t) < x < A(t) + L(t)$. Under the change of variables to $\xi = \left(\frac{x - A(t)}{L(t)}\right) L_0$ and $u(\xi, t) = \psi(x,t)$, the problem becomes (2.10), (2.11). In Section 7.5.2 we consider the behaviour of the gradient at the boundary, $\frac{\partial \psi}{\partial x}(A(t), t)$, for certain forms of $A(t)$ and $L(t)$. We contrast this with our results from Chapter 4 for the linear equation. Before that, in Section 7.5.1 we discuss the role of the 'critical length' $L_{crit}(c) = \pi \sqrt{\frac{D}{f'(0) - \frac{c^2}{4D}}}$ for $c \in (-c_*, c_*)$.

7.5.1 Role of $L_{crit}(c)$ when $\dot{A}(t) \to c \in (-c_*, c_*)$

Recall that in Section 3.5.2 we considered the role of L_{crit} and $L_{crit}(c)$ for the linear equation on a time-dependent interval, and we proved Corollaries 3.16, 3.18, 3.21 and 3.22. We now turn to some related properties for the nonlinear equation.

Proposition 7.14. *Suppose $\dot{A}(t) \to c \in (-c_*, c_*)$ and $\ddot{A}(t) \to 0$ as $t \to \infty$. Let $\overline{L} = \limsup_{t\to\infty} L(t)$ and $\underline{L} = \liminf_{t\to\infty} L(t)$.*

1. *If $\overline{L} < L_{crit}(c)$, then $\psi(x,t) \to 0$ uniformly in x as $t \to \infty$.*

2. *If $\overline{L} = L_{crit}(c)$ and f is not linear on any neighbourhood $[0, s_0)$ of 0, then $\psi(x,t) \to 0$ uniformly in x as $t \to \infty$.*

3. *If f is linear on some neighbourhood $[0, s_0)$ of 0, then there exist cases where $\dot{A}(t) \equiv c \in (-c_*, c_*)$ and $L(t) < \overline{L} = L_{crit}(c)$ for all t and yet $\psi(x,t)$ does not tend to zero as $t \to \infty$ but has a non-trivial lower bound.*

4. *If $\underline{L} > L_{crit}(c)$ and $L_1 \in (L_{crit}(c), \underline{L})$, then $\liminf_{t\to\infty} \psi(A(t)+y, t) \geq U_{c,L_1}(y)$ uniformly on $0 \leq y \leq L_1$.*

Proof. 1. Suppose $\overline{L} < L_0 < L_{crit}(c)$. Then for all t large enough, $L(t) \leq L_0$ and the solution ψ_{L_0} on $A(t) < x < A(t) + L_0$ is a supersolution. Since $\dot{A}(t) \to c$ and $L_0 < L_{crit}(c)$, we know from Corollary 2.17 that $u_{L_0}(\xi, t) := \psi_{L_0}(A(t) + \xi, t)$ converges to zero in $L^2([0, L_0])$ as $t \to \infty$. Then since $\dot{A}(t) \to c$ and $\ddot{A}(t) \to 0$, Proposition 2.13 ensures that in fact u_{L_0} converges uniformly. Thus: $||\psi(\cdot, t)||_\infty \leq ||\psi_{L_0}(\cdot, t)||_\infty = ||u_{L_0}(\cdot, t)||_\infty \to 0$.

2. Next suppose that $\overline{L} = L_{crit}(c)$. For each $\varepsilon > 0$, there exists T such that $L(t) \leq L_\varepsilon := L_{crit}(c) + \varepsilon$ for all $t \geq T$. The solution $\hat{\psi}_\varepsilon(x,t)$ on the interval $A(t) < x < A(t) + L_\varepsilon$ is a supersolution for $\psi(x,t)$, and so $||\psi(\cdot,t)||_\infty \leq ||\hat{\psi}_\varepsilon(\cdot,t)||_\infty$ for all $t \geq T$. But by Theorem 7.2, $\hat{u}_\varepsilon(\xi,t) := \hat{\psi}_\varepsilon(A(t)+\xi, t)$ converges to $U_{c,L_\varepsilon}(\xi)$ uniformly in ξ as $t \to \infty$. Therefore

$$\limsup_{t\to\infty} ||\psi(\cdot,t)||_\infty \leq \lim_{t\to\infty} ||\hat{\psi}_\varepsilon(\cdot,t)||_\infty = ||U_{c,L_\varepsilon}||_\infty. \tag{7.68}$$

If f is not linear on any neighbourhood $[0, s_0)$ of 0, then Proposition 6.11 implies that $||U_{c,L_\varepsilon}||_\infty \to 0$ as $\varepsilon \to 0$. So, $\lim_{t\to\infty} ||\psi(\cdot,t)||_\infty = 0$.

3. First suppose $\dot{A}(t) \equiv 0$ and let $L_{crit} = L_{crit}(0)$. The proof will be similar to that of Corollary 3.16 for the linear equation. Suppose there exist constants m_1, m_2, M, I_1, I_2 such that $0 < m_1 \leq L(t) \leq m_2$ and $|\dot{L}(t)L(t)| \leq M$ and

$$\left| \int_0^t \left(\frac{1}{L(\zeta)^2} - \frac{1}{L_{crit}^2} \right) d\zeta \right| \leq I_1 \text{ and } \int_0^t L(\zeta)[\ddot{L}(\zeta)]^- d\zeta \leq I_2 \tag{7.69}$$

for all $t \geq 0$. (These are satisfied if, for example, $L(t) = L_{crit}(1 - \varepsilon e^{-\alpha t})$ with $0 < \varepsilon < 1$ and $\alpha > 0$.) Choose $\hat{b} > 0$ small enough such that both

$$u(\xi, 0) \geq \hat{b} \sin\left(\frac{\pi\xi}{L_0}\right) \exp\left(-\frac{\xi^2 \dot{L}(0) L(0)}{4DL_0^2}\right) \tag{7.70}$$

and

$$\hat{b}\left(\frac{L(0)}{m_1}\right)^{1/2} \exp\left(I_1 + \frac{M}{4D}\right) \leq s_0. \tag{7.71}$$

Let $\underline{Q}(t) = \frac{L(t)[\ddot{L}(t)]^-}{2}$. Then by Theorem 3.13 the function

$$\hat{u}(\xi, t) = \hat{b} \sin\left(\frac{\pi\xi}{L_0}\right) \left(\frac{L(0)}{L(t)}\right)^{1/2} e^{\left(f'(0)t + \int_0^t \left(-\frac{D\pi^2}{L(\zeta)^2} - \frac{\underline{Q}(\zeta)}{2D}\right) d\zeta - \frac{\xi^2 \dot{L}(t) L(t)}{4DL_0^2}\right)} \tag{7.72}$$

is a subsolution for the linear equation. But by choice of $\hat{b}$, we have $0 \leq \hat{u}(\xi, t) \leq s_0$ for all $0 \leq t < \infty$, and so $\hat{u}$ is also a subsolution for the nonlinear equation. Therefore, for all $t \geq 0$,

$$u(\xi, t) \geq \hat{u}(\xi, t) \geq \hat{b}\left(\frac{L(0)}{m_2}\right)^{1/2} \exp\left(-I_1 - \frac{I_2}{4D} - \frac{M}{4D}\right) \sin\left(\frac{\pi\xi}{L_0}\right). \tag{7.73}$$

The proof for $\dot{A}(t) \equiv c \neq 0$ is similar; see also Corollary 3.21.

4. Assume $\underline{L} > L_1 > L_{crit}(c)$, i.e. $f'(0) > \frac{D\pi^2}{L_1^2} + \frac{c^2}{4D}$. Then for t large enough, $L(t) \geq L_1$ and the solution ψ_{L_1} on $A(t) < x < A(t) + L_1$ is a subsolution. So $\psi(A(t) + y, t) \geq \psi_{L_1}(A(t) + y, t)$ for all $0 \leq y \leq L_1$ and t sufficiently large. Theorem 7.2 implies that as $t \to \infty$, $\psi_{L_1}(A(t) + y, t) \to U_{c,L_1}(y)$ uniformly on $[0, L_1]$.

□

7.5.2 Behaviour of $\psi(x, t)$ near the endpoints

In this final section, we consider the solution $\psi(x, t)$ to the nonlinear equation on the interval $(A(t), A(t) + L(t))$ and its behaviour in a neighbourhood of the endpoints. As in equation (1.5), $c_* = 2\sqrt{Df'(0)}$. We begin by considering cases where $L(t) \to \infty$ and $\dot{A}(t)$ converges to some $c \in (-c_*, c_*)$ as $t \to \infty$. Here, as before, $\hat{U}_c$ is the semi-wave of speed c satisfying (6.88), (6.89).

Proposition 7.15. *Suppose that $L(t) \to \infty$, $\dot{A}(t) \to c \in (-c_*, c_*)$, and $\ddot{A}(t) \to 0$ as $t \to \infty$. Then*

$$\liminf_{t\to\infty} \left(\frac{\partial \psi}{\partial x}(A(t), t) \right) \geq \hat{U}_c'(0). \tag{7.74}$$

Proof. For $L_0 > 0$, let ψ_{L_0} be the solution on the interval $(A(t), A(t) + L_0)$. For t large enough, ψ is a supersolution for ψ_{L_0}. So $\psi(A(t)+y,t) \geq \psi_{L_0}(A(t)+y,t)$ for $0 \leq y \leq L_0$, and consequently also $\frac{\partial \psi}{\partial x}(A(t),t) \geq \frac{\partial \psi_{L_0}}{\partial x}(A(t),t)$. Choose $L_0 > L_{crit}(c)$ and let $t \to \infty$. Then Theorem 7.2 and Remark 7.4 imply that $\psi_{L_0}(A(t)+y,t) \to U_{c,L_0}(y)$ uniformly in y, and $\frac{\partial \psi_{L_0}}{\partial x}(A(t),t) \to U'_{c,L_0}(0)$. Therefore for all $L_0 > L_{crit}(c)$

$$\liminf_{t\to\infty} \left(\frac{\partial \psi}{\partial x}(A(t), t) \right) \geq \lim_{t\to\infty} \left(\frac{\partial \psi_{L_0}}{\partial x}(A(t), t) \right) = U'_{c,L_0}(0). \tag{7.75}$$

The result follows by letting $L_0 \to \infty$ and using Proposition 6.19. □

Under some additional assumptions, we shall also prove upper bounds on $\liminf_{t\to\infty} \left(\frac{\partial \psi}{\partial x}(A(t),t)\right)$, in Proposition 7.18. These are proved by comparing the solution on $A(t) < x < A(t) + L(t)$ with the free boundary solution on $g(t) < x < h(t)$ of Du, Lin and co-authors [24, 23, 17, 26]. In these papers they fix $\mu > 0$ and consider the problem (1.17), (1.18), (1.19) where f satisfies

$$f(0) = f(K) = 0, \quad f > 0 \text{ on } (0,K), \quad f \in C^1([0,K]), \quad f'(K) < 0 < f'(0). \tag{7.76}$$

As discussed in Chapter 1 they prove that as $t \to \infty$ either $\dot{g}(t) \to -\hat{c}$ and $\dot{h}(t) \to \hat{c}$ for a constant speed $\hat{c} = \hat{c}(\mu) \in (0, c_*)$, or else $g(t) \to g_\infty$, $h(t) \to h_\infty$, and $u \to 0$. In the case of spreading, the speed $\hat{c}$ is determined by the property that $\mu q'_{\hat{c}}(0) = \hat{c}$ where, q_c denotes the unique semi-wave satisfying (1.21), (1.22). Note that $q'_c(0)$ is a continuous and decreasing function of $c \in [0, c_*)$, with $q'_c(0) \to 0$ as $c \to c_*$ and $q'_c(0) \to q'_0(0) \in (0,\infty)$ as $c \to 0$. The speed $\hat{c}(\mu)$ is continuous and monotonic increasing in $0 < \mu < \infty$, with $\hat{c}(\mu) \to c_*$ as $\mu \to \infty$ and $\hat{c}(\mu) = O(\mu) \to 0$ as $\mu \to 0$ [24, 23, 17, 26]. It therefore has an inverse, $\hat{\mu}(c)$ for $0 < c < c_*$. In our present notation, $q_c = \hat{U}_{-c}$ and so

$$\hat{\mu}(c) = \frac{c}{\hat{U}'_{-c}(0)} \qquad \text{for } 0 < c < c_*. \tag{7.77}$$

The paper [24] includes the following comparison result.

Lemma 7.16. *(See [24, Lemma 5.6 and Remark 5.8].) Let u be the solution to the free boundary problem* (1.17), (1.18), (1.19), *and let ψ satisfy*

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + f(\psi) \qquad \text{for } \overline{g}(t) < x < \overline{h}(t) \tag{7.78}$$

$$\psi(\overline{g}(t), t) = 0, \qquad \dot{\overline{g}}(t) \geq -\mu\frac{\partial \psi}{\partial x}(\overline{g}(t), t) \tag{7.79}$$

$$\psi(\overline{h}(t), t) = 0, \qquad \overline{h}(t) \leq h(t) \tag{7.80}$$

with some given initial values $\overline{g}(0) \geq g_0$ and $\psi(x, 0) \leq u_0(x)$ on $[\overline{g}(0), \overline{h}(0)]$. Then: $\overline{g}(t) \geq g(t)$ for all $t \geq 0$, and $\psi(x, t) \leq u(x, t)$ on $\overline{g}(t) < x < \overline{h}(t)$.

We can use Lemma 7.16 together with the known behaviour of the free boundary solution $u(x, t)$, to prove a relationship between $\dot{A}(t)$ and $\frac{\partial \psi}{\partial x}(A(t), t)$.

Lemma 7.17. *Assume that f satisfies the conditions in equations* (2.20) *and* (7.76)*. Suppose there exists $c \in (0, c_*)$ such that $\limsup_{t\to\infty}\left(\frac{A(t)+L(t)}{t}\right) < c$, and a sequence $t_n \to \infty$ such that $A(t_n) + ct_n \to -\infty$. Then*

$$\liminf_{t\to\infty}\left(\dot{A}(t) + \hat{\mu}(c)\frac{\partial \psi}{\partial x}(A(t), t)\right) \leq 0. \tag{7.81}$$

Proof. Let $\mu = \hat{\mu}(c)$, and let $\overline{g}(t) = A(t)$ and $\overline{h}(t) = A(t) + L(t)$. For large t, $A(t) + L(t) < h(t) \sim ct$. If $\dot{A}(t) \geq -\mu\frac{\partial \psi}{\partial x}(A(t), t)$ for all t sufficiently large, then Lemma 7.16 would imply that $A(t) \geq g(t) \sim -ct$ as $t \to \infty$. However this contradicts the sequence t_n. So, $\liminf_{t\to\infty}\left(\dot{A}(t) + \mu\frac{\partial \psi}{\partial x}(A(t), t)\right) \leq 0$. □

Lemma 7.17 then allows us to bound $\liminf_{t\to\infty}\left(\frac{\partial \psi}{\partial x}(A(t), t)\right)$ under certain assumptions on $A(t)$ and $L(t)$.

Proposition 7.18. *Assume that f satisfies the conditions in equations* (2.20) *and* (7.76)*. Suppose that there exist $\underline{c} > 0$ and $C < \underline{c}$ such that $\dot{A}(t) \to -\underline{c}$ and $A(t) + L(t) \sim Ct$ as $t \to \infty$.*

1. *If $0 < \underline{c} < c_*$, then $\liminf_{t\to\infty}\left(\frac{\partial \psi}{\partial x}(A(t), t)\right) \leq \hat{U}'_{-\underline{c}}(0)$.*

2. If $C < c_* \leq \underline{c}$, then $\liminf_{t\to\infty} \left(\frac{\partial\psi}{\partial x}(A(t),t)\right) = 0$.

Proof. Choose any $c \in (0, c_*) \cap (C, \underline{c})$ and apply Lemma 7.17. This gives

$$\liminf_{t\to\infty} \left(-\underline{c} + \hat{\mu}(c) \frac{\partial\psi}{\partial x}(A(t),t) \right) \leq 0, \tag{7.82}$$

or equivalently,

$$\liminf_{t\to\infty} \left(\frac{\partial\psi}{\partial x}(A(t),t) \right) \leq \frac{\underline{c}\, \hat{U}'_{-c}(0)}{c}. \tag{7.83}$$

For part 1, let $c \to \underline{c}$. For part 2, let $c \to c_*$ and use $\hat{U}'_{-c}(0) \to 0$ as $c \to c_*$. □

Another result from [24], which provides an upper bound on $\psi(x,t)$ near the boundary under the much more general condition that $\dot{A}(t)$ is bounded above, is the following adaptation of [24, Lemma 2.2].

Proposition 7.19. *Let $\psi(x,t)$ satisfy* (2.8), (2.9). *Suppose $\dot{A}(t) \leq c_0$ is bounded above and $\psi(\cdot, 0) = \psi_0 \in C^1([A(0), A(0) + L(0)])$ satisfies the boundary conditions. Then there exists $0 < m_0 < \infty$ (depending on ψ_0 and c_0) such that*

$$\psi(A(t)+y,t) \leq K(2m_0 y - m_0^2 y^2) \qquad \text{for } 0 \leq y \leq \min\left(\frac{1}{m_0}, L(t)\right), \ t \geq 0. \tag{7.84}$$

In particular, $\frac{\partial\psi}{\partial x}(A(t),t)$ is bounded.

Proof. (Based on [24, Lemma 2.2].)
Let $m > 0$ and for $t \geq 0$ define

$$\Omega_m(t) = \left(A(t), A(t) + \frac{1}{m}\right) \cap (A(t), A(t) + L(t)), \tag{7.85}$$

$$w(x,t) = K\left(2m(x - A(t)) - m^2(x - A(t))^2\right) \qquad \text{for } x \in \Omega_m(t). \tag{7.86}$$

We will show that $m = m_0$ can be chosen such that w is a supersolution for ψ on $\Omega_{m_0}(t)$. First, $w \geq \psi$ on $\partial\Omega_m(t)$, since $w(A(t),t) = \psi(A(t),t) = 0$ and

$$\text{if } \frac{1}{m} \leq L(t): \quad w\left(A(t) + \frac{1}{m}, t\right) = K > \psi\left(A(t) + \frac{1}{m}, t\right), \tag{7.87}$$

$$\text{or if } \frac{1}{m} > L(t): \ w(A(t)+L(t),t) \geq 0 = \psi(A(t)+L(t),t). \tag{7.88}$$

We also calculate

$$\frac{\partial w}{\partial x} = K\left(2m - 2m^2(x - A(t))\right), \qquad \frac{\partial^2 w}{\partial x^2} = -K2m^2, \tag{7.89}$$

$$\frac{\partial w}{\partial t} = K\dot{A}(t)\left(-2m + 2m^2(x - A(t))\right) \geq -2mKc_0 \qquad \text{on } \Omega_m(t). \tag{7.90}$$

Therefore, for $M = \sup_{[0,K]} f$ and for $m \geq \frac{c_0}{2D} + \sqrt{\frac{c_0^2}{4D^2} + \frac{M}{2DK}}$,

$$\frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} - f(w) \geq -2mKc_0 + DK2m^2 - M \geq 0 \quad \text{on } \Omega_m(t). \tag{7.91}$$

Finally, we want to choose m so that $\psi_0(x) \geq w(x, 0)$ on $\Omega_m(0)$. Suppose $m \geq \frac{4}{3K}||\psi_0||_{C^1}$ and $m \geq \frac{1}{L(0)}$. Then for $0 < x - A(0) \leq \frac{1}{2m}$,

$$\frac{\partial w}{\partial x}(x, 0) \geq Km \geq \frac{4}{3}\psi_0'(x) \quad \text{and so} \quad w(x, 0) \geq \psi_0(x), \tag{7.92}$$

and for $\frac{1}{2m} \leq x - A(0) \leq \frac{1}{m}$,

$$w(x, 0) \geq \frac{3K}{4} \geq \frac{1}{m}||\psi_0||_{C^1} \geq (x - A(0))||\psi_0||_{C^1} \geq \psi_0(x). \tag{7.93}$$

Therefore, if m_0 satisfies

$$m_0 \geq \max\left\{\frac{c_0}{2D} + \sqrt{\frac{c_0^2}{4D^2} + \frac{M}{2DK}}, \ \frac{1}{L(0)}, \ \frac{4}{3K}||\psi_0||_{C^1}\right\} \tag{7.94}$$

then w is a supersolution for ψ on $\Omega_{m_0}(t)$. So $\psi(x, t) \leq w(x, t)$ for $t \geq 0$ and $x \in \Omega_{m_0}(t)$, which is (7.84). In particular, since ψ is below w at the boundary,

$$\frac{\partial \psi}{\partial x}(A(t), 0) \leq \frac{\partial w}{\partial x}(A(t), 0) = 2Km_0. \tag{7.95}$$

□

Remark 7.20. *The symmetric result is as follows. If $\dot{A}(t) + \dot{L}(t)$ is bounded below, then there exists $m_0 > 0$ such that $\psi(A(t)+L(t)-y, t) \leq K(2m_0 y - m_0^2 y^2)$ for $0 \leq y \leq \frac{1}{m_0}$. In particular, $\frac{\partial \psi}{\partial x}(A(t) + L(t), t)$ is bounded.*

We can apply Proposition 7.19 to the case of the symmetric interval

$$-A(t) = \frac{L(t)}{2} = c_* t - \alpha \log\left(\frac{t}{t_0} + 1\right) + \frac{l_0}{2} \tag{7.96}$$

with $\alpha > 0$. For the linear case, recall from Corollary 4.6 in Chapter 4 that $\frac{\partial \psi_{lin}}{\partial x}(A(t),t)$ behaves exactly as order $t^{\frac{\alpha c_*}{2D}-\frac{3}{2}}$ as $t \to \infty$. When $\alpha > \frac{3D}{c_*}$ this means that $\frac{\partial \psi_{lin}}{\partial x}(A(t),t) \to \infty$ as $t \to \infty$. Proposition 7.19 proves that this sort of behaviour cannot happen in the nonlinear case: if $\dot{A}(t)$ is bounded above then $\frac{\partial \psi}{\partial x}(A(t),t) = O(1)$. However if $\alpha \leq \frac{3D}{c_*}$ then $\frac{\partial \psi}{\partial x}(A(t),t)$ is bounded for both the linear and nonlinear cases. Moreover $\frac{\partial \psi}{\partial x}(A(t),t) \leq \frac{\partial \psi_{lin}}{\partial x}(A(t),t)$, and one might wonder if they behave in a similar way. We know that $\frac{\partial \psi_{lin}}{\partial x}(A(t),t)$ behaves exactly as order $t^{-\beta}$ as $t \to \infty$, with $\beta = -\frac{\alpha c_*}{2D} + \frac{3}{2}$. In contrast, for the nonlinear problem we shall see in Theorem 7.23 that $\frac{\partial \psi}{\partial x}(A(t),t)$ cannot be bounded below by any power of t. This proof is based on the integral transform method of J. Berestycki, Brunet and Derrida from [12].

Write $f(k) = f'(0)k - f_1(k)$. The assumptions (2.20) on f mean that $f_1 \geq 0$ and is bounded on $[0, K]$.

Proposition 7.21. *Let $0 \leq \psi(x,t) \leq K$ be a solution to (2.8), (2.9), where*

$$A(t) = -c_* t + \delta(t), \qquad \delta(t) \to \infty, \qquad \delta(t) = o(t) \text{ as } t \to \infty \tag{7.97}$$

and $L(t) \geq ct$ for some $c > 0$. Let $X(t) \in (A(t), A(t) + L(t))$ be such that $X(t) - A(t) \geq c_0 t$ for some $c_0 > 0$, and that $\dot{X}(t)$ and $\frac{\partial \psi}{\partial x}(X(t),t)$ are bounded. Let

$$F_1(r,t) = \int_0^{X(t)-A(t)} f_1(\psi(A(t)+z,t)) e^{rz} dz. \tag{7.98}$$

Then for $0 < \varepsilon_0 < 1$ small enough, the integral

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}(1+\varepsilon)\delta(t)} \left(D \frac{\partial \psi}{\partial x}(A(t),t) + F_1\left(-\frac{c_*}{2D}(1+\varepsilon), t\right)\right) dt \tag{7.99}$$

is infinitely differentiable in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Remark 7.22. *For example, one could take $X(t)$ such that (for all t large enough) $X(t) = X_0 + (-c_* + c_1)t$ for some $c_1 > 0$. It follows from Theorem 7.5 and Remark 7.6 that $\psi(X(t),t) \to K$ and $\frac{\partial \psi}{\partial x}(X(t),t) \to 0$ as $t \to \infty$, and so $X(t)$ satisfies the required conditions. Alternatively, if $\dot{A}(t) + \dot{L}(t)$ is bounded below, then one could take $X(t) = A(t) + L(t)$. In this case $\psi(X(t),t) \equiv 0$ and, by Proposition 7.19 and Remark 7.20, $\frac{\partial \psi}{\partial x}(X(t),t)$ is bounded.*

Proof. For $r \in \mathbb{R}$, let

$$g(r,t) = \int_0^{X(t)-A(t)} \psi(A(t)+z,t)e^{rz}dz. \tag{7.100}$$

Differentiate this with respect to t and use equation (2.8), to get:

$$\begin{aligned}\frac{\partial g}{\partial t}(r,t) =& (\dot{X}(t)-\dot{A}(t))\psi(X(t),t)e^{r(X(t)-A(t))} \\ &+ \dot{A}(t)\int_0^{X(t)-A(t)} \frac{\partial \psi}{\partial x}(A(t)+z,t)e^{rz}dz \\ &+ \int_0^{X(t)-A(t)} \left(D\frac{\partial^2\psi}{\partial x^2}(A(t)+z,t) + f(\psi(A(t)+z,t))\right) e^{rz}dz.\end{aligned} \tag{7.101}$$

After using $f(k) = f'(0)k - f_1(k)$, integrating by parts in z and using the boundary condition at $A(t)$, this becomes:

$$\begin{aligned}\frac{\partial g}{\partial t}(r,t) =& \left(Dr^2 - r\dot{A}(t) + f'(0)\right) g(r,t) - D\frac{\partial \psi}{\partial x}(A(t),t) - F_1(r,t) \\ &+ \left((\dot{X}(t)-Dr)\psi(X(t),t) + D\frac{\partial \psi}{\partial x}(X(t),t)\right) e^{r(X(t)-A(t))}.\end{aligned} \tag{7.102}$$

Use an integrating factor of $e^{\phi(r,t)}$ where

$$\phi(r,t) = -\left(Dr^2t - rA(t) + f'(0)t\right) = -\left(Dr^2 + c_*r + f'(0)\right)t + r\delta(t) \tag{7.103}$$

$$= -D\left(r + \frac{c_*}{2D}\right)^2 t + r\delta(t). \tag{7.104}$$

This leads to the following equation, for every $t > 0$:

$$\begin{aligned}& g(r,t)e^{-D\left(r+\frac{c_*}{2D}\right)^2t + r\delta(t)} - g(r,0)e^{r\delta(0)} \\ &\quad = -\int_0^t e^{\phi(r,s)}\left(D\frac{\partial\psi}{\partial x}(A(s),s) + F_1(r,s)\right)ds \\ &\quad\quad + \int_0^t e^{\phi(r,s)+r(X(s)-A(s))}\left((\dot{X}(s)-Dr)\psi(X(s),s) + D\frac{\partial\psi}{\partial x}(X(s),s)\right)ds.\end{aligned} \tag{7.105}$$

Now consider the limit $t \to \infty$. If $r < 0$ then $g(r,t) \leq K\int_0^\infty e^{rz}dz = O(1)$ and therefore

$$g(r,t)e^{-D\left(r+\frac{c_*}{2D}\right)^2t+r\delta(t)} \to 0 \qquad \text{as } t \to \infty. \tag{7.106}$$

So for each $r < 0$, we let $t \to \infty$ and get

$$\int_0^\infty e^{\phi(r,s)+r(X(s)-A(s))} \left((\dot{X}(s) - Dr)\psi(X(s),s) + D\frac{\partial \psi}{\partial x}(X(s),s) \right) ds$$
$$+g(r,0)e^{r\delta(0)} = \int_0^\infty e^{\phi(r,s)} \left(D\frac{\partial \psi}{\partial x}(A(s),s) + F_1(r,s) \right) ds. \quad (7.107)$$

Take equation (7.107) and apply it at $r = -\frac{c_*}{2D}(1+\varepsilon)$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ (for some small $0 < \varepsilon_0 < 1$ to be chosen). The term $g\left(-\frac{c_*}{2D}(1+\varepsilon), 0\right) e^{-\frac{c_*}{2D}(1+\varepsilon)\delta(0)}$ is infinitely differentiable in ε. Consider the term involving $X(t)$. By choice of $X(t)$,

$$(\dot{X}(s) - Dr)\psi(X(s),s) + D\frac{\partial \psi}{\partial x}(X(s),s) = O(1)$$
$$\text{and} \qquad -\frac{c_*}{2D}(1+\varepsilon)(X(s) - A(s)) \le -\frac{c_* c_0}{2D}(1+\varepsilon)s \qquad \text{as } s \to \infty. \quad (7.108)$$

So for $\varepsilon_0 > 0$ small enough, the integrand in this term is $O\left(e^{-\frac{c_* c_0}{4D}s}\right) \times O(1)$, uniformly in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. This term is then infinitely differentiable with respect to $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, with derivatives found by differentiating through the integral.

So the remaining term in equation (7.107), which is precisely (7.99), must also be infinitely differentiable in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. □

We now use this result to prove that $\frac{\partial \psi}{\partial x}(A(t),t)$ decays faster than every power of t, in the sense that it cannot be bounded below by any power.

Theorem 7.23. *Let $A(t)$ and $L(t)$ satisfy the assumptions of Proposition 7.21. Suppose that $\delta(t) = \alpha \log(\frac{t}{t_0} + 1) + O(1)$ as $t \to \infty$, for some $\alpha > 0$. Then $\frac{\partial \psi}{\partial x}(A(t),t)$ cannot be bounded below by any power of t. That is:*

$$\liminf_{t\to\infty} \left(t^\beta \frac{\partial \psi}{\partial x}(A(t),t) \right) = 0 \qquad \text{for every } \beta \in \mathbb{R}. \quad (7.109)$$

Proof. Suppose this is false. Then we can fix some $\beta > 0$ and $a > 0$ such that

$$D\frac{\partial \psi}{\partial x}(A(t),t) \ge a(t+t_0)^{-\beta}. \quad (7.110)$$

We can also assume that $\frac{c_*\alpha}{2D} + \beta \notin \mathbb{N}$. We shall show that the integral

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}(1+\varepsilon)\delta(t)} \left(D\frac{\partial \psi}{\partial x}(A(t),t) + F_1\left(-\frac{c_*}{2D}(1+\varepsilon), t\right) \right) dt \quad (7.111)$$

(with F_1 defined as in Proposition 7.21) has a singularity at $\varepsilon = 0$, at least as bad as order $|\varepsilon|^{-2(\nu+1)}$ where $\nu = -\frac{c_*\alpha}{2D} - \beta$. This will contradict Proposition 7.21, and so prove equation (7.109).

First, note that $F_1\left(-\frac{c_*}{2D}, t\right) \geq 0$ (since $f_1 \geq 0$). It follows from (7.110) and Theorem A.12 that the leading singularity of

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}\delta(t)} \left(D\frac{\partial \psi}{\partial x}(A(t), t) + F_1\left(-\frac{c_*}{2D}, t\right)\right) dt \tag{7.112}$$

is at least as bad as that of

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}\delta(t)} a(t+t_0)^{-\beta} dt = \int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t + O(1)} a(t+t_0)^{-\frac{c_*\alpha}{2D} - \beta} dt. \tag{7.113}$$

By Theorem A.13 this is proportional to $|\varepsilon|^{-2(\nu+1)}$ where $\nu = -\frac{c_*\alpha}{2D} - \beta$. So it remains to show that the leading singular term of (7.111) is the same as that of (7.112).

Note that $F_1(r, t)$ is itself not singular at $r = -\frac{c_*}{2D}$, and for each $n \in \mathbb{N}$,

$$\frac{\partial^n F_1}{\partial r^n}\left(-\frac{c_*}{2D}, t\right) = \int_0^{X(t)-A(t)} z^n f_1(\psi(A(t)+z, t)) e^{-\frac{c_*}{2D}z} dz. \tag{7.114}$$

This can be bounded by splitting the integral as follows:

$$\begin{aligned}
&\int_0^{\frac{2D}{c_*}\beta\log(\frac{t}{t_0}+2)} z^n f_1(\psi(A(t)+z, t)) e^{-\frac{c_*}{2D}z} dz \\
&\leq \left(\frac{2D}{c_*}\beta \log\left(\frac{t}{t_0}+2\right)\right)^n \int_0^{X(t)-A(t)} f_1(\psi(A(t)+z, t)) e^{-\frac{c_*}{2D}z} dz \qquad (7.115)\\
&= O\left(\log^n\left(\frac{t}{t_0}+2\right)\right) F_1\left(-\frac{c_*}{2D}, t\right), \qquad (7.116)
\end{aligned}$$

and, if $M_1 = \sup_{[0,K]} f_1$ then

$$\begin{aligned}
\int_{\frac{2D}{c_*}\beta\log(\frac{t}{t_0}+2)}^{X(t)-A(t)} z^n f_1(\psi(A(t)+z, t)) e^{-\frac{c_*}{2D}z} dz &\leq M_1 \int_{\frac{2D}{c_*}\beta\log(\frac{t}{t_0}+2)}^{\infty} z^n e^{-\frac{c_*}{2D}z} dz \\
&= O\left(\log^n\left(\frac{t}{t_0}+2\right) t^{-\beta}\right). \qquad (7.117)
\end{aligned}$$

Adding (7.116) and (7.117), and combining this with the assumed bound (7.110), gives:

$$\frac{\partial^n F_1}{\partial r^n}\left(-\frac{c_*}{2D}, t\right) = O\left(\log^n\left(\frac{t}{t_0}+2\right)\left(F_1\left(-\frac{c_*}{2D}, t\right) + D\frac{\partial \psi}{\partial x}(A(t), t)\right)\right). \tag{7.118}$$

Therefore as $\varepsilon \to 0$ the leading singularity in (7.111) does indeed come from (7.112), since the next terms in the expansion are

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}\delta(t)} \Bigg(-\varepsilon\delta(t)\frac{c_*}{2D}\left(D\frac{\partial\psi}{\partial x}(A(t),t) + F_1\left(-\frac{c_*}{2D},t\right)\right) - \varepsilon\frac{c_*}{2D}\frac{\partial F_1}{\partial r}\left(-\frac{c_*}{2D},t\right)\Bigg) dt. \tag{7.119}$$

Using the bound (7.118), and the fact that $\delta(t) = O(\log(\frac{t}{t_0}+2))$, this is

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}\delta(t)} \left(D\frac{\partial\psi}{\partial x}(A(t),t) + F_1\left(-\frac{c_*}{2D},t\right)\right) \times O\left(\varepsilon \log\left(\frac{t}{t_0}+2\right)\right) dt \tag{7.120}$$

and subsequent terms are (similarly) of the form

$$\int_0^\infty e^{-\frac{c_*^2}{4D}\varepsilon^2 t - \frac{c_*}{2D}\delta(t)} \left(D\frac{\partial\psi}{\partial x}(A(t),t) + F_1\left(-\frac{c_*}{2D},t\right)\right) \times O\left(\varepsilon^n \log^n\left(\frac{t}{t_0}+2\right)\right) dt, \tag{7.121}$$

all of which produce singularities of strictly lower order than that of (7.112). □

Chapter 8

Conclusions and further work

This thesis has been concerned with non-negative solutions to linear and non-linear (of KPP type) reaction-diffusion equations on time-dependent domains with zero Dirichlet boundary conditions. The main example has been a time-dependent interval $A(t) < x < A(t) + L(t)$; several higher dimensional domains have also been considered.

For a linear reaction term, we have derived exact solutions when $\ddot{L}L^3$ and $\ddot{A}L^3$ are constants, using changes of variables and a separation of variables method. All of the terms in these exact solutions have been worked out explicitly, and the formulae are interesting because they show precisely how a solution develops over time, as well as how it depends on L, A, and the spatial variable. We have also proved similar results for time-dependent balls and boxes.

For more general time-dependent intervals and boxes, we have derived sub- and supersolutions to the linear equation, which bound the long-time behaviour of the solution in a useful way. By applying these results, it was shown that different outcomes are possible if $L(t) \to L_{crit} = \pi\sqrt{\frac{D}{f'(0)}}$ as $t \to \infty$, and we gave particular examples such that the solution does and does not tend to zero.

Continuing to consider the equation with a linear reaction term, we also proved detailed bounds on the solution behaviour near the moving boundaries. In several scenarios we derived a 'critical' choice of boundary movement, such that the gradient at the boundary is bounded above and below away from zero for all time. For an interval, and also for a ball in dimension $N \leq 3$, it was remarkable that this 'critical' choice of moving boundary matched the asymptotic front position for the solution to the nonlinear KPP equation on the whole space. We also considered boxes of the form

$$\Omega(t) = \Big\{ x \in \mathbb{R}^{N+1} : A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N, \\ \frac{-L_{N+1}(t)}{2} < x_{N+1} < \frac{L_{N+1}(t)}{2} \Big\}, \tag{8.1}$$

and for several cases of $A_j(t)$ and $L_j(t)$, we derived a 'critical' choice of $L_{N+1}(t)$.

For bounded time-periodic domains we derived useful upper and lower bounds on the principal periodic eigenvalue μ associated with the domain. We also proved results concerning the dependence of μ on the frequency ω, including an expression for $\lim_{\omega \to 0} \mu(\omega)$. For certain cases we proved monotonicity with respect to ω, and considered the $\omega \to \infty$ limit. In some regimes μ remains bounded as $\omega \to \infty$, whereas in others it tends to infinity at the rate ω^2.

For a nonlinear reaction term of KPP type, this thesis contains results about long-time convergence — both on bounded time-periodic domains and on a number of time-dependent domains involving constant or asymptotically constant velocities. Furthermore we proved results concerning the gradient at the moving boundary for the nonlinear equation, and how this differs significantly to the linear case.

Some further questions raised by this thesis, and recommended as directions for future work, are:

- To investigate and explain the links between the linear equation on the finite domains and the nonlinear KPP problem on unbounded domains.
- In the cases that remain unanswered (including where $L_j(t)$ is of order t^k as $t \to \infty$ with either $0 < k \leq \frac{1}{4}$ or $\frac{1}{2} < k < 1$), to derive choices of $L_{N+1}(t)$ such that the gradient at the boundary is bounded above and below away from zero, for boxes of the form (8.1).

- To discover whether the same formula for the 'critical' choice of radius $R(t)$ (for the linear equation on a ball) is also valid in dimensions $N \geq 4$.
- To continue to study the nonlinear KPP equation on the time-dependent cylindrical domain

$$\left\{ A_j(t) < x_j < A_j(t) + L_j(t) : 1 \leq j \leq N,\ -\infty < x_{N+1} < \infty \right\}$$

  with zero Dirichlet boundary conditions. As $t \to \infty$, is there convergence in some sense to a travelling wave? Are there asymptotic front positions at $x_{N+1} = \pm \frac{L^*_{N+1}(t)}{2} + O(1)$ (where $L_{N+1}(t) = L^*_{N+1}(t)$ satisfies the 'critical' property for the linear equation on the box (8.1))?
- To conduct further research on time-periodic domains and gain a fuller understanding of the ω-dependence of the principal periodic eigenvalue $\mu(\omega)$. What happens in the $\omega \to \infty$ limit, in the cases where this remains unknown? Is $\mu(\omega)$ always monotonic increasing in $\omega > 0$, or are there cases where it is not monotonic in ω?

Additional topics to study on time-dependent domains could also include: (i) two-component (or multi-component) competition systems; (ii) equations involving a non-autonomous reaction term $f(u,t)$, with f having a certain structure or asymptotic behaviour with respect to t; (iii) bistable nonlinear terms; and (iv) exterior domains with time-dependent obstacles — among many other interesting possibilities.

Appendix A

Appendix of theorems

A.1 Elliptic and parabolic theorems

Theorem A.1. *(Interior/boundary/global elliptic estimates)*
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega$ sufficiently smooth (at least $C^{2+\gamma}$ for some $0 < \gamma < 1$). For $n \in \mathbb{N}$, let $c^{(n)}$ and $D^{(n)}$ be constant vectors in $\mathbb{R}^N$. Assume that there exist c_0, $\underline{D}$, $\overline{D}$, such that $|c^{(n)}| \leq c_0$ and $0 < \underline{D} \leq D_j^{(n)} \leq \overline{D}$ for $j = 1, \ldots, N$. Let f_n be Lipschitz continuous functions, and let u_n satisfy the elliptic equation

$$\sum_{j=1}^{N} D_j^{(n)} \frac{\partial^2 u_n}{\partial x_j^2} + c^{(n)} \cdot \nabla u_n + f_n(u_n) = 0 \qquad \text{in } \Omega. \tag{A.1}$$

Let $\Omega' \subset \Omega$ be such that either (i) $\Omega' \subset\subset \Omega$ or (ii) there exists $\Delta \subset \partial\Omega$ such that $u_n = 0$ on Δ, and Ω' is such that $(\partial\Omega' \cap \partial\Omega) \subset \Delta$ and (if $\Delta \neq \partial\Omega$) $dist(\partial\Omega', \partial\Omega \backslash \Delta) > 0$. Assume that $0 \leq u_n \leq K$ on Ω for some constant K, and that $|f_n| \leq M$ on $[0, K]$ and the Lipschitz constants of f_n are bounded by θ. Then the sequence u_n is bounded in $C^{2+\gamma}(\overline{\Omega'})$. Hence for each $0 < \gamma' < \gamma < 1$, there exists a subsequence u_{n_k} that is convergent in $C^{2+\gamma'}(\overline{\Omega'})$.

Proof. The existence of the convergent subsequence will follow from the compact embeddings of the Hölder spaces, so we just need to prove the boundedness.

Let $\Omega' \subset \Omega'' \subset \Omega$ be such that, in case (i) $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, or in case (ii)

$(\partial\Omega' \cap \partial\Omega'') \subset (\partial\Omega'' \cap \partial\Omega) \subset \Delta$ and (if $\Delta \neq \partial\Omega$) with $dist(\partial\Omega', \partial\Omega''\backslash\Delta) > 0$ and $dist(\partial\Omega'', \partial\Omega\backslash\Delta) > 0$. For $p > N$, the elliptic L^p estimates in [34, Theorem 9.13] imply that there is a constant C (depending on Ω, Ω'', p, N, $\underline{D}$, $\overline{D}$, and c_0) such that

$$||u_n||_{W_p^2(\Omega'')} \leq C \left(||u_n||_{L^p(\Omega)} + ||f_n(u_n)||_{L^p(\Omega)}\right) \tag{A.2}$$

$$\leq C(K+M)|\Omega|^{\frac{1}{p}}. \tag{A.3}$$

So the sequence u_n is bounded in $W_p^2(\Omega'')$. Let $\alpha = 1 - \frac{N}{p}$. Then the space $W_p^2(\Omega'')$ is continuously embedded in $C^{1+\alpha}(\overline{\Omega''})$ (see [28, chapter 5, Theorem 6, page 270]) and so there is also some constant B_1 (depending on Ω, Ω'', p, N, K, M, $\underline{D}$, $\overline{D}$, and c_0) such that

$$||u_n||_{C^{1+\alpha}(\overline{\Omega''})} \leq B_1. \tag{A.4}$$

Then since each f_n is Lipschitz continuous with Lipschitz constant bounded by θ, the functions $f_n(u_n(x))$ are bounded in $C^\gamma(\overline{\Omega''})$, with the bound depending only on B_1, γ, and θ. The elliptic Schauder estimates in [34, Corollary 6.3 and Corollary 6.7] then imply that there is a constant C' (depending on Ω', Ω'', N, γ, $\underline{D}$, $\overline{D}$, and c_0) such that

$$||u_n||_{C^{2+\gamma}(\overline{\Omega'})} \leq C' \left(||u_n||_{C(\overline{\Omega''})} + ||f_n(u_n)||_{C^\gamma(\overline{\Omega''})}\right). \tag{A.5}$$

We therefore get the final result that

$$||u_n||_{C^{2+\gamma}(\overline{\Omega'})} \leq B_2, \tag{A.6}$$

where B_2 is a constant depending only on Ω, Ω', Ω'', p, N, K, M, γ, θ, $\underline{D}$, $\overline{D}$, and c_0. □

Remark A.2. *If $D^{(n)} \to D$, $c^{(n)} \to c$, and $||f_n - f||_{C^{0,1}([0,K])} \to 0$ as $n \to \infty$, then the limit of the subsequence u_{n_k} must satisfy the second-order elliptic equation*

$$\sum_{j=1}^{N} D_j \frac{\partial^2 u}{\partial x_j^2} + c \cdot \nabla u + f(u) = 0 \qquad in\ \Omega', \tag{A.7}$$

and (in case (ii)) $u = 0$ on $\partial\Omega' \cap \Delta$.

Theorem A.3. *(Interior/boundary/global parabolic estimates)*
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega$ sufficiently smooth (at least $C^{2+\gamma}$ for some $0 < \gamma < 1$), and let $T > 0$. For $n \in \mathbb{N}$, let $D^{(n)}(t) \in \mathbb{R}^N$ be uniformly continuous on $0 \le t \le T$, uniformly with respect to n, and assume that there exist $\underline{D}$, $\overline{D}$ such that $0 < \underline{D} \le D_j^{(n)}(t) \le \overline{D}$ for all $1 \le j \le N$, $0 \le t \le T$, and all n. Write $\Omega_T = \Omega \times (0, T]$, and let $c^{(n)}(x,t) \in \mathbb{R}^N$ be uniformly continuous on Ω_T, uniformly with respect to n. Let f be a Lipschitz continuous function, and let u_n satisfy the parabolic equation

$$\frac{\partial u_n}{\partial t} = \sum_{j=1}^{N} D_j^{(n)}(t) \frac{\partial^2 u_n}{\partial x_j^2} + c^{(n)}(x,t) \cdot \nabla u_n + f(u_n) \qquad \text{in } \Omega_T. \tag{A.8}$$

Assume that u_n is bounded (uniformly with respect to n) on Ω_T. Let $\Omega' \subset \Omega$ be such that either (i) $\Omega' \subset\subset \Omega$ or (ii) there exists $\Delta \subset \partial\Omega$ such that $u_n = 0$ on Δ, and Ω' is such that $(\partial\Omega' \cap \partial\Omega) \subset \Delta$ and (if $\Delta \neq \partial\Omega$) $dist(\partial\Omega', \partial\Omega \backslash \Delta) > 0$.

1. *Let $0 < t_0 < T$ and $0 < \alpha < 1$. Then the sequence u_n is bounded in $C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{\Omega'} \times [t_0, T])$. Hence there exists a subsequence u_{n_k} such that u_{n_k} and ∇u_{n_k} are uniformly convergent on $\overline{\Omega'} \times [t_0, T]$.*

2. *Suppose $D^{(n)}$ and $c^{(n)}$ are Hölder continuous, uniformly with respect to n. Then for $0 < t_0 < T$, the sequence u_n is bounded in $C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega'} \times [t_0, T])$. Hence there exists a subsequence u_{n_k} that converges in $C^{2,1}(\overline{\Omega'} \times [t_0, T])$, i.e. u_{n_k}, $\frac{\partial u_{n_k}}{\partial t}$, and the first and second spatial derivatives of u_{n_k}, are all uniformly convergent on $\overline{\Omega'} \times [t_0, T]$.*

Proof. The existence of the convergent subsequences will follow from the compact embeddings of the Hölder spaces, so we just need to prove the boundedness.

1. Given $0 < \alpha < 1$, let p be such that $\alpha = 1 - \frac{N+2}{p}$. The parabolic L^p estimates in [44, Theorem 7.30] together with [44, Lemma 7.20] imply that there is a constant C (depending on Ω, Ω', T, t_0, p, N, $\underline{D}$, the modulus of continuity of the $D^{(n)}$ and the uniform bound on the $c^{(n)}$), such that

$$||u_n||_{W_p^{2,1}(\Omega' \times (t_0, T))} \le C \left(||u_n||_{L^p(\Omega_T)} + ||f(u_n)||_{L^p(\Omega_T)} \right). \tag{A.9}$$

Therefore if $|u_n| \leq K$ and $|f| \leq M$, then for each n,

$$||u_n||_{W_p^{2,1}(\Omega' \times (t_0,T))} \leq C(K+M)|\Omega_T|^{\frac{1}{p}} \tag{A.10}$$

and so the sequence u_n is bounded in $W_p^{2,1}(\Omega' \times (t_0,T))$. Moreover, the space $W_p^{2,1}(\Omega' \times (t_0,T))$ is continuously embedded in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega'} \times [t_0,T])$ (see [38, Theorem 3.14(3)]). So there is also some constant B_1 (depending on Ω, Ω', T, t_0, p, N, K, M, $\underline{D}$, the modulus of continuity of the $D^{(n)}$ and the uniform bound on the $c^{(n)}$) such that

$$||u_n||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega'}\times[t_0,T])} \leq B_1. \tag{A.11}$$

2. Let $\Omega' \subset \Omega'' \subset \Omega$ be such that, in case (i) $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, or in case (ii) $(\partial\Omega' \cap \partial\Omega'') \subset (\partial\Omega'' \cap \partial\Omega) \subset \Delta$ and (if $\Delta \neq \partial\Omega$) with $dist(\partial\Omega', \partial\Omega''\backslash\Delta) > 0$ and $dist(\partial\Omega'', \partial\Omega\backslash\Delta) > 0$. By applying part 1 to $\Omega'' \subset \Omega$ and $(\frac{t_0}{2}, T] \subset (0,T]$ we deduce that the u_n are bounded in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega''} \times [\frac{t_0}{2}, T])$. Then since f is Lipschitz continuous, $f(u_n(x,t))$ is bounded in $C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega''} \times [\frac{t_0}{2}, T])$ independently of n, with the bound depending only on B_1, γ, and the Lipschitz constant of f. Now suppose we have Hölder bounds on the coefficients. Then the parabolic Schauder estimates in [43, chapter IV, Theorem 10.1] imply that there is a constant C' (depending on Ω', Ω'', T, t_0, N, γ, $\underline{D}$, the Hölder bound on $D^{(n)}$ and the Hölder bound on the $c^{(n)}$) such that

$$||u_n||_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega'}\times[t_0,T])} \leq C'\left(||u_n||_{C(\overline{\Omega''}\times[\frac{t_0}{2},T])} + ||f(u_n)||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega''}\times[\frac{t_0}{2},T])}\right). \tag{A.12}$$

We therefore get the final result that

$$||u_n||_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega'}\times[t_0,T])} \leq B_2, \tag{A.13}$$

where B_2 is a constant depending only on Ω, Ω', Ω'', T, t_0, p, N, γ, the Lipschitz constant of f, $\underline{D}$, the Hölder bound on $D^{(n)}$ and the Hölder bound on the $c^{(n)}$.

$\square$

Remark A.4. *If $D^{(n)}(t) \to D(t)$ and $c^{(n)}(x,t) \to c(x,t)$ as $n \to \infty$, then the limit in part 2 must satisfy the second-order parabolic equation*

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{N} D_j(t) \frac{\partial^2 u}{\partial x_j^2} + c(x,t) \cdot \nabla u + f(u) \qquad \text{in } \Omega', \tag{A.14}$$

and (in case (ii)) $u = 0$ on $\partial\Omega' \cap \Delta$.

Lemma A.5. *(Unique limits and convergence)*
Let X be a normed space and $u_n \in X$ a sequence. Suppose there exists $\hat{u} \in X$ such that, for every subsequence u_{n_m}, there exists a sub-subsequence $u_{n_{m_k}}$ that is convergent to $\hat{u}$. Then the whole sequence u_n converges to $\hat{u}$:

$$||u_n - \hat{u}||_X \to 0 \qquad \text{as } n \to \infty. \tag{A.15}$$

Proof. Suppose not. Then there exists $\varepsilon > 0$ and a subsequence $u^*_{n_m}$, such that $\left|\left|u^*_{n_m} - \hat{u}\right|\right|_X \geq \varepsilon$. But then applying the assumed property to $u^*_{n_m}$, there must be a sub-subsequence $u^*_{n_{m_k}}$ such that $\left|\left|u^*_{n_{m_k}} - \hat{u}\right|\right|_X \to 0$ which is a contradiction.

$\square$

Theorem A.6. *(Rayleigh-Ritz formula: minimisation of the Rayleigh quotient; see [53, Proposition 6.37 and Remark 6.38].)*
Let λ_1 be the principal eigenvalue of the regular Sturm-Liouville problem

$$(p(x)v')' + q(x)v + \lambda r(x)v = 0 \quad \text{on } a < x < b, \qquad v(a) = v(b) = 0 \tag{A.16}$$

where $p \in C^1([a,b])$, $q, r \in C([a,b])$, and $p(x) > 0$, $r(x) > 0$ on $[a,b]$. Then

$$\lambda_1 = \inf_u \frac{\int_a^b (p(u')^2 - qu^2)dx}{\int_a^b ru^2 dx} \tag{A.17}$$

where the infimum is taken over functions $u \in C^2([a,b])$ with $u(a) = u(b) = 0$, $u \not\equiv 0$. Moreover, the infimum is attained only by the principal eigenfunction.

Lemma A.7. *(Weak subsolutions)*
Consider a problem of the form

$$\mathcal{L}u := \frac{\partial u}{\partial t} - D(t)\frac{\partial^2 u}{\partial \xi^2} - V(\xi,t)u = 0 \quad \textit{for } 0 < \xi < L_0, \qquad u(0,t) = u(L_0,t) = 0 \tag{A.18}$$

where $D(t) > 0$ and $V(\xi,t)$ are continuous. Let $\hat{\xi}(t) \in (0, L_0)$, and let u_1, u_2 satisfy (in the classical sense) $\mathcal{L}u_1 \le 0$ on $(0, \hat{\xi}(t))$ and $\mathcal{L}u_2 \le 0$ on $(\hat{\xi}(t), L_0)$, with $u_1(0,t) = 0$, $u_2(L_0,t) = 0$, $u_1(\hat{\xi}(t),t) = u_2(\hat{\xi}(t),t)$ and $\frac{\partial u_1}{\partial \xi}(\hat{\xi}(t),t) \le \frac{\partial u_2}{\partial \xi}(\hat{\xi}(t),t)$. Then the function

$$u(\xi,t) := \begin{cases} u_1(\xi,t) & 0 \le \xi \le \hat{\xi}(t) \\ u_2(\xi,t) & \hat{\xi}(t) < \xi \le L_0 \end{cases} \tag{A.19}$$

is a weak subsolution for (A.18).

Proof. We need to show that, for every $T > 0$ and $\Omega_T = (0, L_0) \times (0, T)$, we have

$$\int_0^T \int_0^{L_0} \left(-u\frac{\partial \varphi}{\partial t} + D(t)\frac{\partial u}{\partial \xi}\frac{\partial \varphi}{\partial \xi} - Vu\varphi \right) d\xi dt \le 0 \tag{A.20}$$

for all C^∞, non-negative test functions φ compactly supported within Ω_T. By splitting the ξ integral at $\hat{\xi}(t)$, we find that the left hand side of (A.20) is:

$$\int_0^T \int_0^{\hat{\xi}(t)} \mathcal{L}u_1 d\xi dt + \int_0^T \int_{\hat{\xi}(t)}^{L_0} \mathcal{L}u_2 d\xi dt + \int_0^T D(t) \left(\frac{\partial u_1}{\partial \xi}(\hat{\xi}(t),t) - \frac{\partial u_2}{\partial \xi}(\hat{\xi}(t),t) \right) \varphi(\hat{\xi}(t),t) dt \tag{A.21}$$

which is ≤ 0 as required. □

Lemma A.8. *For $c \in \mathbb{R}^N$ and initial conditions $u_c(\xi,0)$ that depend continuously on c, let $u_c(\xi,t)$ satisfy the parabolic problem* (6.1), (6.2)*. Then $u_c(\xi,t)$ is continuous with respect to c for each $t > 0$ and uniformly in Ω_0.*

Proof. Fix $T > 0$ and $c \in \mathbb{R}^N$. Let $c^{(n)} \to c$ and $u_n(\xi,t) = u_{c^{(n)}}(\xi,t)$. We need to show that $u_n(\xi,T) \to u_c(\xi,T)$ uniformly in Ω_0 as $n \to \infty$. Let $v_n(\xi,t)$ satisfy

$$\frac{\partial v_n}{\partial t} = D\nabla^2 v_n + c \cdot \nabla v_n + f(u_n) \qquad \text{for } \xi \in \Omega_0 \tag{A.22}$$

with $v_n(\xi,t) = 0$ on $\partial\Omega_0$ and with initial conditions $v_n(\xi,0) = u_n(\xi,0)$. Similar arguments to those in Theorem A.3 show that for every $0 < t_0 < T$ there is a subsequence n_k and a limit v such that v_{n_k} converges to v in $C^{2,1}(\overline{\Omega_0} \times [t_0,T])$. Since this holds for every $0 < t_0 < T$, we can take a sequence $t_m \to 0$ and take v_{n_k} to be a diagonal subsequence that converges to v in $C^{2,1}(\overline{\Omega_0} \times [t_m,T])$ for all $t_m \in (0,T)$. Since the initial conditions vary continuously with c, we also have $v_{n_k}(\xi,0) = u_{c(n_k)}(\xi,0) \to u_c(\xi,0)$.

Let G_c be the Green's function of the parabolic operator $\frac{\partial}{\partial t} - D\nabla^2 - c\cdot\nabla$ on Ω_0. Then $u_n - v_n$ satisfies the integral equation

$$\begin{aligned} u_n(\xi,t) - v_n(\xi,t) = &\int_{\Omega_0} (G_{c_n}(\xi,y,t,0) - G_c(\xi,y,t,0))u_n(y,0)dy \\ &+ \int_0^t \int_{\Omega_0} (G_{c_n}(\xi,y,t,\tau) - G_c(\xi,y,t,\tau))f(u_n(y,\tau))dyd\tau \end{aligned} \tag{A.23}$$

for each $0 < t \le T$. Let $n \to \infty$ in this equation. Using the continuity of G_c with respect to c, the boundedness of $u_n(\cdot,0)$ and f, and the dominated convergence theorem, we find that the right hand side tends to zero uniformly in ξ. Therefore $u_n(\xi,t) - v_n(\xi,t) \to 0$ uniformly in ξ, and so $u_{n_k}(\xi,t) \to v(\xi,t)$ as $k \to \infty$. Thus, for each $0 < t \le T$,

$$\begin{aligned} v(\xi,t) &= \lim_{k\to\infty} v_{n_k}(\xi,t) \\ &= \lim_{k\to\infty}\left(\int_{\Omega_0} G_c(\xi,y,t,0)u_{n_k}(y,0)dy \right. \\ &\qquad \left. + \int_0^t\int_{\Omega_0} G_c(\xi,y,t,\tau)f(u_{n_k}(y,\tau))dyd\tau\right) \qquad \text{(A.24)} \\ &= \int_{\Omega_0} G_c(\xi,y,t,0)u_c(y,0)dy + \int_0^t\int_{\Omega_0} G_c(\xi,y,t,\tau)f(v(y,\tau))dyd\tau \qquad \text{(A.25)} \end{aligned}$$

and so v is a solution to (6.1), (6.2) with initial conditions $u_c(\cdot,0)$. By uniqueness, it follows that $v \equiv u_c$. Finally by Lemma A.5, the whole sequence u_n must converge to u_c and not just the subsequence u_{n_k}. □

Theorem A.9. *(A result for periodic-parabolic operators; see [46, Lemma 2.1].)* *Let* Ω_0 *be a smooth bounded domain,* $V \in C^{1,1}(\overline{\Omega_0} \times [0,1])$*, and* $A \in C^1([0,1])$ *with* $A(s) > 0$*. Assume both* V *and* A *are periodic in* $s \in [0,1]$ *with period* 1*. For* $\omega > 0$ *consider the operator*

$$\mathcal{P}_\omega u = \frac{\omega}{2\pi}\frac{\partial u}{\partial s} - A(s)\nabla^2 u + V(\xi, s)u. \tag{A.26}$$

Let $u_\omega(\xi, s) > 0$ *be the principal periodic eigenfunction of the operator* $\mathcal{P}_\omega$*, and* $v_\omega(\xi, s) > 0$ *the principal periodic eigenfunction of its adjoint operator* $\mathcal{P}^*_\omega = -\frac{\omega}{2\pi}\frac{\partial}{\partial s} - A(s)\nabla^2 + V(\xi, s)$*, with zero Dirichlet boundary conditions on* $\partial\Omega_0$*, and with* u_ω *and* v_ω *normalised so that* $\int_0^1 \int_{\Omega_0} u_\omega^2 d\xi ds = \int_0^1 \int_{\Omega_0} u_\omega v_\omega d\xi ds = 1$*.*

Let $\mathcal{S}$ *denote the set of functions* $\zeta \in C^{2,1}(\Omega_0 \times (0,1)) \cap C^{1,1}(\overline{\Omega_0} \times [0,1])$ *that are positive on* $\Omega_0 \times [0,1]$*, satisfy the zero Dirichlet boundary conditions and have non-zero normal derivative on* $\partial\Omega_0$*, and are* 1*-periodic in* s*. For* $\zeta \in \mathcal{S}$ *define the functional* J_ω *by*

$$J_\omega(\zeta) = \int_0^1 \int_{\Omega_0} u_\omega v_\omega \left(\frac{\mathcal{P}_\omega \zeta}{\zeta}\right) d\xi ds. \tag{A.27}$$

Then, for all $\zeta \in \mathcal{S}$*,*

$$J_\omega(u_\omega) - J_\omega(\zeta) = \int_0^1 \int_{\Omega_0} A(s) u_\omega v_\omega \left|\nabla \log\left(\frac{\zeta}{u_\omega}\right)\right|^2 d\xi ds \geq 0. \tag{A.28}$$

In particular, for every $\omega > 0$*,*

$$\int_0^1 \int_{\Omega_0} (v_\omega \mathcal{P}_\omega u_\omega - u_\omega \mathcal{P}_\omega v_\omega) d\xi ds = J_\omega(u_\omega) - J_\omega(v_\omega) \geq 0. \tag{A.29}$$

Theorem A.10. *(A Liouville-type property for monostable reaction-diffusion equations on* $\mathbb{R}^N$*; see [8, Theorem 3.7].)*
Let $\mathcal{L}v = \sum_{i,j} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i q_i(x)\frac{\partial v}{\partial x_i}$ *where* a_{ij}*,* $q_i \in L^\infty(\mathbb{R}^N)$ *and are locally Hölder continuous, and* a_{ij} *is a uniformly elliptic matrix field with minimum eigenvalue* $\alpha(x) > 0$*. Suppose that* $v \in C^2(\mathbb{R}^N)$ *satisfies* $0 \leq v \leq 1$ *and*

$$-\mathcal{L}v \geq f(v(x)) \qquad \text{in } \mathbb{R}^N \tag{A.30}$$

where f *is a Lipschitz continuous function,* C^1 *on a neighbourhood of* 0*, and*

$$f(0) = f(1) = 0, \qquad f > 0 \text{ on } (0,1), \qquad 4\alpha(x)f'(0) - |q(x)|^2 > 0. \tag{A.31}$$

Then either $v \equiv 0$ or $v \equiv 1$ in $\mathbb{R}^N$.

Theorem A.11. *(A strong maximum principle for strict super-solutions in unbounded domains; see [8, Lemma 2.1(iii)].)*
Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain, with a boundary $\partial\Omega$, and let

$$\mathcal{L}u = \sum_{i,j} b_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i p_i(x)\frac{\partial u}{\partial x_i} + c(x)u(x) \tag{A.32}$$

where b_{ij}, p_i, $c \in L^\infty(\Omega)$ and b_{ij} is a uniformly elliptic matrix field. Suppose u, $v \in C^2(\Omega) \cap C(\overline{\Omega})$ are such that for some positive number $\varepsilon > 0$

$$-\mathcal{L}u \le 0 \quad \text{in } \Omega, \qquad \sup_\Omega u < \infty, \tag{A.33}$$

$$-\mathcal{L}v \ge \varepsilon > 0 \quad \text{in } \Omega, \qquad \inf_{\partial\Omega} v > 0, \tag{A.34}$$

$$\text{and} \quad u \le v \quad \text{on } \partial\Omega. \tag{A.35}$$

Then $u \le v$ on Ω.

A.2 Laplace transforms and singularities

For functions $h(t) \ge 0$ denote the Laplace transform as $\hat{h}(s)$ for $s > 0$:

$$\hat{h}(s) = \int_0^\infty h(t)e^{-st}dt. \tag{A.36}$$

Theorem A.12. *Let $0 \le h_1(t) \le h_2(t)$. Then as $s \to 0$ the first singular term in $\hat{h}_2(s)$ is at least as bad as that in $\hat{h}_1(s)$, in the sense that $0 \le \hat{h}_1(s) \le \hat{h}_2(s)$, and that if for some $N \ge 0$ we can write*

$$\hat{h}_j(s) = \sum_{n=0}^{N} a_{jn}s^n + \theta_{jN}(s), \qquad \theta_{jN}(s) = o(s^N) \text{ as } s \to 0 \qquad (\text{for } j = 1, 2), \tag{A.37}$$

then $|\theta_{1N}(s)| \le |\theta_{2N}(s)|$.

Proof. It is clear from the definition that $0 \le \hat{h}_1(s) \le \hat{h}_2(s)$. For the statement involving θ_{1N} and θ_{2N}, write

$$E_N(s,t) = e^{-st} - \sum_{n=0}^{N} \frac{(-st)^n}{n!}. \tag{A.38}$$

For N even, $E_N \leq 0$ and so

$$0 \geq \int_0^\infty h_1(t) E_N(s,t)dt \geq \int_0^\infty h_2(t) E_N(s,t)dt. \tag{A.39}$$

This is equivalent to

$$0 \geq \hat{h}_1(s) - \sum_{n=0}^{N} \frac{\hat{h}_1^{(n)}(0)}{n!} s^n \geq \hat{h}_2(s) - \sum_{n=0}^{N} \frac{\hat{h}_2^{(n)}(0)}{n!} s^n, \tag{A.40}$$

which is precisely $0 \geq \theta_{1N}(s) \geq \theta_{2N}(s)$.

For N odd, $E_N \geq 0$ and so (A.39) holds with all the inequalities reversed, and this becomes $0 \leq \theta_{1N}(s) \leq \theta_{2N}(s)$. □

Theorem A.13. *(See [12, equation (47)].)*
For $s > 0$ and $\nu \in \mathbb{R}$,

$$\int_1^\infty t^\nu e^{-st} dt = [\textit{analytic function of } s] + \begin{cases} \Gamma(\nu+1)s^{-(\nu+1)} & \textit{for } -\nu \notin \mathbb{N} \\ \frac{(-1)^\nu s^{-(\nu+1)} \log(s)}{(-(\nu+1))!} & \textit{for } -\nu \in \mathbb{N}. \end{cases} \tag{A.41}$$

Proof. Let $I_\nu(s) = \int_1^\infty t^\nu e^{-st} dt$. For $\nu > -1$, the definition of the Γ function gives

$$I_\nu(s) = -\int_0^1 t^\nu e^{-st} dt + \Gamma(\nu+1)s^{-(\nu+1)}, \tag{A.42}$$

and the first term is an analytic function of s.

For $\nu = -1$,

$$I_{-1}(s) = \int_1^\infty t^{-1} e^{-st} dt = \int_s^\infty y^{-1} e^{-y} dy \tag{A.43}$$

$$= \int_s^1 \sum_{n=0}^{\infty} \frac{(-1)^n y^{n-1}}{n!} dy + \int_1^\infty y^{-1} e^{-y} dy \tag{A.44}$$

$$= -\log(s) + \sum_{n=1}^{\infty} \frac{(-1)^n (1-s^n)}{n!n} + \int_1^\infty y^{-1} e^{-y} dy. \tag{A.45}$$

The second terms is a convergent Taylor series and so an analytic function of s, and the third term is a constant.

Finally, for $\nu < -1$, integrating by parts gives:

$$I_\nu(s) = \frac{-e^{-s}}{\nu+1} + \frac{s}{\nu+1}\int_1^\infty t^{\nu+1}e^{-st}dt = \frac{-e^{-s}}{\nu+1} + \frac{s}{\nu+1}I_{\nu+1}(s). \qquad \text{(A.46)}$$

It follows that (A.41) holds for every $\nu \in \mathbb{R}$. □

Remark A.14. *By Theorem A.13, the singular term in*

$$\int_1^\infty t^\nu e^{-\varepsilon^2 t}dt \qquad \text{(A.47)}$$

is proportional to $|\varepsilon|^{-2(\nu+1)}$ *if* $-\nu \notin \mathbb{N}$, *or* $\varepsilon^{-2(\nu+1)}\log|\varepsilon|$ *if* $-\nu \in \mathbb{N}$. *Therefore the integral* (A.47) *will be an analytic function of* $|\varepsilon|$ *if and only if*

$$-\nu \notin \mathbb{N} \qquad \textit{and} \qquad -2(\nu+1) \in \mathbb{N}, \qquad \text{(A.48)}$$

which is if and only if $-(\nu+\frac{1}{2}) \in \mathbb{N}$, *i.e.* $\nu \in \{-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots\}$. *In these cases* (A.47) *has one-sided derivatives of all orders with respect to* ε *at* $\varepsilon = 0$, *but it is not analytic on any neighbourhood* $(-\varepsilon_0, \varepsilon_0)$ *of* 0.

Theorem A.15. *[12, Equations (51), (52), (53), (54), and (55).]*
For $s > 0$ *and* $\nu \in \mathbb{R}$*:*

$$\begin{aligned}\int_1^\infty \log(z)z^\nu e^{-sz}dz &= [\textit{analytic function of } s] \\ &\quad + s^{-(\nu+1)}\left(-\Gamma(\nu+1)\log(s) + \Gamma'(\nu+1)\right) \quad \textit{for } -\nu \notin \mathbb{N}.\end{aligned} \qquad \text{(A.49)}$$

$$\int_1^\infty \log(z)z^{-\frac{3}{2}}e^{-sz}dz = [\textit{analytic function of } s] + s^{\frac{1}{2}}\left(2\sqrt{\pi}\log(s) + \textit{const}\right). \qquad \text{(A.50)}$$

$$\int_1^\infty \log(z)z^{-\frac{5}{2}}e^{-sz}dz = [\textit{analytic function of } s] + s^{\frac{3}{2}}\left(-\frac{4}{3}\sqrt{\pi}\log(s) + \textit{const}\right). \qquad \text{(A.51)}$$

$$\begin{aligned}\int_1^\infty \log^2(z)z^{-\frac{3}{2}}e^{-sz}dz =& [\textit{analytic function of } s] \\ & - 2\pi^{\frac{1}{2}}s^{\frac{1}{2}}\left(\log^2(s) - (4 - 4\log 2 - 2\gamma_E)\log(s) + \textit{const}\right).\end{aligned} \qquad \text{(A.52)}$$

$$\int_1^\infty \log(z) z^{-2} e^{-sz} dz = 1 - s\left(\frac{\log^2(s)}{2} - (1-\gamma_E)\log(s) + const\right) + O(s^2)$$

$$\text{as } s \to 0. \qquad \text{(A.53)}$$

Here γ_E is Euler's constant.

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