# Generalisations of Tropical Geometry over Hyperfields

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# Abstract

Hyperfields are structures that generalise the notion of a field by way of allowing the addition operation to be multivalued. The aim of this thesis is to examine generalisations of classical theory from algebraic geometry and its combinatorial shadow, tropical geometry. We present a thorough description of the hyperfield landscape, where the key concepts are introduced. Kapranov's theorem is a cornerstone result from tropical geometry, relating the tropicalisation function and solutions sets of polynomials. We generalise Kapranov's Theorem for a class of relatively algebraically closed hyperfield homomorphisms. Tropical ideals are reviewed and we propose the property of matroidal equivalence as a method of associating the geometric objects defined by tropical ideals. The definitions of conic and convex sets are appropriately adjusted allowing for convex geometry over ordered hyperfields to be studied.

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*Dedicated*
*to my parents*
*Tim and Susan,*
*and*
*my partner*
*Rachel.*

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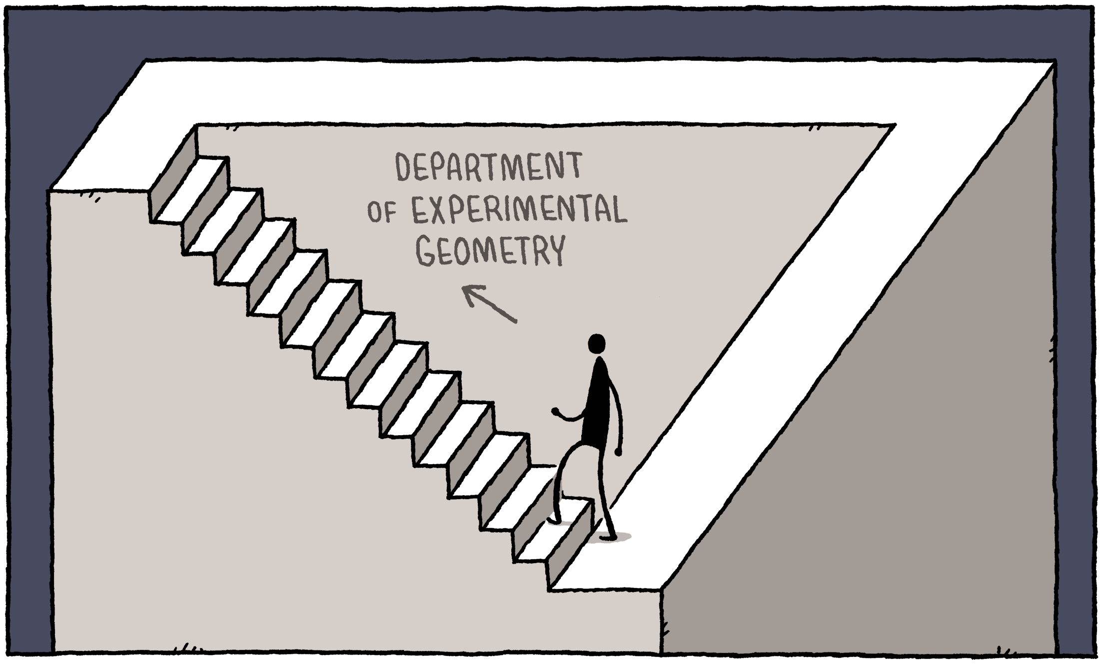


Figure 1: This image represents my PhD experience and journey; Credit the illustrator Tom Gauld [Gau21]

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# Chapter 1

# Introduction

Algebraic Geometry is the area of mathematics that explores the solution sets of polynomials and polynomial ideals. It studies the geometric properties of these solution sets using techniques from abstract and commutative algebra. The approach to algebraic geometry in this thesis is based on the work presented in [CLO13], [Har13] and [EH06]. Tropical geometry is the combinatorial shadow of algebraic geometry. It is used as a tool to take geometric problems and rephrase them as problems in combinatorics. Tropical geometry was brought to the attention of the wider mathematical community through being utilised in progressive work on enumerative geometry by Mikhalkin [Mik05], where there are calculations of the Gromov-Witten invariants of projective two space. In the main, research in tropical geometry has the objective to assess tropical analogues of classical theorems from algebraic geometry.

The re-framing of geometric problems in terms of combinatoics has enabled tropical geometry to be applied in several far reaching areas of mathematics. These include: game theory, machine learning and neural networks [ZNL18], mathematical biology and optimisation and linear programming [ABGJ21]. Although, a more publicly known application is to auction theory in [BK13], where the work by Klemperer in [Kle10] on product-mix auctions was understood by utilising tropical geometry. The work in [Kle10] was proposed to the Bank of England in the 2007 financial crash, but later in [BK13] it was demonstrated that the optimisation problem could be reformulated as a tropical problem and it was shown to have a solution.

More recently, tropical geometry has been connected with the area of hyperfields, structures where the addition is allowed to take on a multivalued output. The utility of this connection was proposed by Viro in both [Vir10] and [Vir11]. The tropical hyperfield is an analogue of the tropical semiring, for which this link provides motivation to attempt to understand theorems from tropical geometry over the multivalued setting of hyperfields.

Hyperfields will be introduced in detail in Chapter 2, but briefly, they were first employed by Krasner in [Kra83] for number theoretic problems and have more recently been used to unify the theory of matroids in [BB18]. They are a class of structures which generalises fields as the addition can be a mutlivalued operation.

The aim of the work presented in this thesis is to inspect generalisations of theory from algebraic and tropical geometry to the multivalued setting of hyperfields. Specific goals include; understanding and characterising roots and varieties of polynomials under hyperfields homomorphisms, connecting the combinatorial properties of tropical ideals to the geometric properties of the corresponding ideals and the studying convex geometry over hyperfields.

## 1.1 Tropical Geometry

In this section the tropical semiring will be introduced. This is one example of an idempotent semiring and the central example when working in the domain of tropical geometry. Denote the tropical semiring by $\overline{\mathbb{R}}$, which as a set is $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Endowing $\overline{\mathbb{R}}$ with the operations,

$$\oplus := \min, \quad \odot := +,$$

has the structure of an idempotent semiring. These operations are called tropical addition and multiplication respectively. The additive neutral element is $\infty$, as $x \oplus \infty = \infty \oplus x = \min\{x, \infty\} = x$. The multiplicative neutral element is 0, as $x \odot 0 = 0 \odot x =$

$x + 0 = x$. Using the tropical arithmetic, polynomials over $\overline{\mathbb{R}}$ are defined as,

$$f = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \odot \underline{X}^{\mathbf{u}} \in \overline{\mathbb{R}}[X_1, \ldots, X_n].$$

Due to the altered arithmetic, the notion of a solution, or root, to a polynomial over $\overline{\mathbb{R}}$ is redefined. An element $\mathbf{x} \in \overline{\mathbb{R}}^n$ is a root of $f \in \overline{\mathbb{R}}[X_1, \ldots, X_n]$, if $f(\mathbf{x})$ achieves its minimum with at least two of its monomials or is equal to $\infty$.

The collection of roots of a tropical polynomial is called its *tropical variety* or *tropical hypersurface*. It is shown in [MS15, Theorem 3.3.6] that tropical varieties are balanced rational polyhedral complexes. This polyhedral complex property makes tropical varieties easier to work with than their classical counterparts.

**Example 1.1.1.** Take the polynomial $f(X_0, X_1) = a \odot X_0 \oplus b \odot X_1 \oplus c \in \overline{\mathbb{R}}[X_0, X_1]$. Then, $f(X_0, X_1) = \min(a + X_0, b + X_1, c)$, and $f(1, 2) = \min(a + 1, b + 2, c)$. Setting $a = b = c = 0$, gives $f(X_0, X_1) = X_0 \oplus X_1 \oplus 0$, which has the geometric description of it's variety shown in Figure 1.1. Each top dimensional cell of $V(X_0 \oplus X_1 \oplus 0)$ is given a

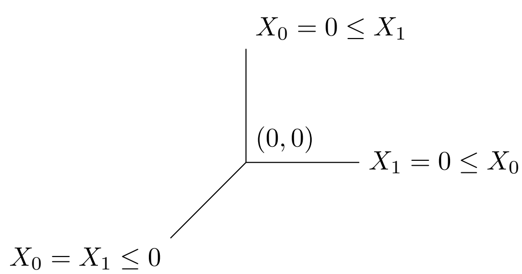


Figure 1.1: The variety of $f(X_0, X_1) = X_0 \oplus X_1 \oplus 0$

weight equal to one along with a direction vector corresponding to the primitive integer lattice point away from the co-dimension 1 point at (0,0). The variety is balanced as the sum of the weights directions is equal to zero.

$$\mathbf{1} \cdot (1, 0) + \mathbf{1} \cdot (0, 1) + \mathbf{1} \cdot (-1, -1) = 0$$

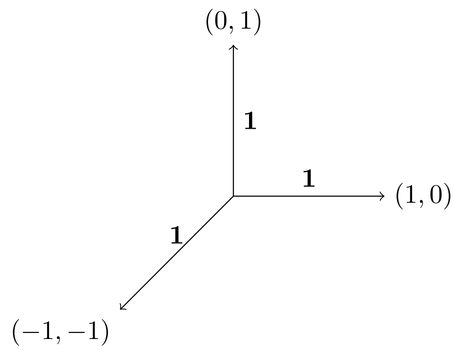


Figure 1.2: Balancing for $V(X_0 \oplus X_1 \oplus 0)$.

The description of the balancing condition in Example 1.1.1 has been streamlined in order to reduce the amount of detailed theory being presented at this introductory stage.

The tropical semiring can arise naturally under a valuation map from a valued field. A *valuation* on a field $K$ is defined as a map, $\mathrm{val} : K \to \overline{\mathbb{R}}$, such that;

- $\mathrm{val}(a) = \infty \Leftrightarrow a = 0$,
- $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,
- $\mathrm{val}(a + b) \geqslant \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.

The process of taking the coordinate-wise image under valuation maps in $\overline{\mathbb{R}}^n$ is referred to as *tropicalisation*, and denoted normally as *trop* : $K^n \to \overline{\mathbb{R}}^n$, where $K$ is a field with valuation.

The tropicalisation map can be extended from elements of a valued field $K$, to polynomials, thus resulting in tropical polynomials. Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \cdot \underline{X}^{\mathbf{u}} \in K[\underline{X}]$, then

$$trop(f) := \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} trop(a_{\mathbf{u}}) \odot \underline{X}^{\mathbf{u}} \in \overline{\mathbb{R}}[X_1, \ldots, X_n].$$

A cornerstone of tropical geometry is the Fundamental Theorem [MS15, Theorem 3.2.5], which builds on Kapranov's Theorem for tropical geometry ([MS15, Theorem 3.1.3], and recalled in Theorem 3.2.1). Let $K[X_1^{\pm}, \ldots X_n^{\pm}]$ be the ring of *Laurent* polynomials with coefficients in $K$, an algebraically closed field with surjective valuation $\mathrm{val} : K \to \overline{\mathbb{R}}$.

The variety of $I$ is the intersection of varieties of polynomials in $I$,

$$V(I) = \bigcap_{p \in I} V(p).$$

This can be pushed forward through the tropicalisation map, or in other words coordinate wise,

$$trop(V(I)) = \{trop(\mathbf{a}) \,:\, \mathbf{a} \in V(I)\}.$$

The image of the ideal $I$ under the tropicalisation map, or coefficient-wise valuation is,

$$trop(I) = \{trop(p) \,:\, p \in I\}.$$

In an analogous way to the definition of $V(I)$, the variety of $trop(I)$ is defined as the intersection of varieties of $trop(p)$, explicitly,

$$V(trop(I)) = \bigcap_{p \in I} V(trop(p)).$$

These notions are then combined to form the Fundamental Theorem of tropical geometry

**Theorem 1.1.2.** *[MS15, Theorem 3.2.5] Let $I$ be an ideal over $K[X_1^{\pm}, \ldots X_n^{\pm}]$, where $K$ is algebraicially closed with surjective valuation* $\mathrm{val} : K \to \overline{\mathbb{R}}$*, which extends to the a coordinate and coefficient wise tropicalisation map, then*

$$V(trop(I)) = trop(V(I)).$$

One recent development in tropical geometry has been the study of *tropical ideals*, chiefly in [MR18] and [MR20], then elsewhere in [FGG ], [AR22], [GG16] and [Zaj18]. Tropical ideals are polynomial ideals over $\overline{\mathbb{R}}[\underline{X}]$ that satisfy an underlying combinatorial property known as the *monomial elimination axiom.* This is based on the polynomial ideal being connected to valuated matroids on the support sets. These objects will be precisely defined and explored in Chapter 5.

## 1.2 Structure

There will now be a brief overview of the structure of the work and an outline of the main contributions from each section.

To begin, in Chapter 2 the definitions of hyper-structures, denoted $\mathbb{H}$, are presented, along with a thorough collection of detailed examples. Several properties of hyperfields are defined, including stringent and doubly distributive. The quotient construction of hypefields will be discussed. The main focus of this section is Table 2.1, where the properties of the examples are documented, as well as the structure of the hyperfields as a quotient.

Next, in Chapter 3 a property titled *Relatively Algebraically Closed* (RAC) is introduced for hyperfield homomorphisms. Intuitively, the RAC property states that when a univariate polynomial is pushed forward through a hyperfield homomorphism, the roots of the polynomial can be lifted back to roots of the original polynomial. The existence of a hyperfield homomorphism which has this property is demonstrated by the map $\eta = \log(|\cdot|) : \mathbb{TC} \to \mathbb{T}$, in Theorem 3.1.12, where $\mathbb{TC}$ is the tropical complex hyperfield and $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ is the tropical hyperfield. Furthermore, another class of examples of RAC hyperfield homomorphisms are those from algebraically closed fields to the Krasner hypefield. There is then an attempt to understand what conditions are needed for a hyperfield homomorphism to be RAC in Section 3.3. In addition, the multiplicities of roots over the hyperfield of signs and the signed tropical hyperfield are explored. Then to conclude, it is shown that the doubly distributive property for hyperfields implies a bound on the sum of multiplicities of roots for polynomials up to degree three. This is motivated by the goal to classify RAC maps, as this bound on the sum of root multiplicities is closely connected to the RAC property.

Chapter 4 discusses a range of quotient maps from the complex numbers. Firstly, the n-th roots of unity are taken as a subgroup and several properties are explored. Explicitly, it is shown that the map $\mathbb{C} \to \mathbb{C}/U_n$ is not a RAC map. This is then applied to Hahn series and the corresponding maps. In addition to this, varieties for univariate polynomials over the triangle hyperfield are surveyed. To conclude, there is a presentation of several results from the literature regarding amoebas and coamoebas.

In Chapter 5 there is a focus on objects over the tropical semiring rather than hyperfields. There is a description of tropical ideals, which satisfy a precise underlying combinatoric property, and examples are given. The notion of *Matroidal Equivalence* is introduced and is proposed as a method of controlling the behaviour of tropical ideals. It is suggested that this could be a method to view the geometric objects defined by two tropical ideals as equivalent. Several examples of tropical ideals that are matroidally equivalent are given and a collection of properties are stated. This chapter attempts to build a foundational theory over $\overline{\mathbb{R}}$ first and then in the future utilise this in a more general way for hyperfields. This is due to the fact that polynomial ideals over hyperfields are more challenging to work with. Whereas, a stronger understanding of ideals over $\overline{\mathbb{R}}$, specifically tropical ideals, has been developed in [MR18] and [MR20].

For Chapter 6 the motivation is due to [LV19], where a theory for convexity over the signed tropical semiring is developed. There is an attempt to extend this to a notion of convexity over hyperfields. Classical definitions are adjusted for the multivalued setting, and orderings over hyperfields are discussed. Halfspaces and varieties are explored over quotient and stringent hyperfields, in particular understanding the impact of pushing these objects forward through a hyperfield homomorphism. Then, building on the work presented in [BS20], there is a precise classification of ordered stringent hyperfields. Both the definitions of conic and convex sets over hyperfields are introduced and the properties of these types of sets are investigated. The section is concluded by stating versions of Radon's, Helly's and Caratheodory's Theorems for hyperfields admitting a order preserving homomorphism from an ordered field.

The final chapter outlines the open questions from all the topics and chapters in this work. These are questions which could be addressed in future research.

# Chapter 2

# Hyperfield Handbook

This chapter is an introduction to algebraic structures with multivalued addition, called hyper-structures. We discuss the history and background of hyper-structures with a description of the underlying motivation to why they are important and relevant. This will be done by presenting the foundational theory and laying out the landscape to demonstrate an understanding of the literature. The connection between the tropical semiring and the tropical hyperfield will be made, demonstrating why the aim is to understand generalisations of results from tropical geometry over the multivalued setting of hyperfields. There is a culmination of the relevant recent work in the area and new examples in order to produce a reference guide in Table 2.1 for future research.

## 2.1 Preliminaries

An algebraic structure with one multivalued operation is called a hyper-structure, where the multivalued operation is called a hyperoperation, but usually called hyper addition. Analogously to classical algebra, the hyper addition can be endowed with structured axioms allowing for the notions of a hypergroup, hyperring and hyperfield to be introduced. A hyperfield (resp. hypergroup and hyperring) is a generalisation of a field (resp. group and ring), where the addition operation is allowed to be a multivalued operation. Hyper-structures, in particular hypergroups, were initially introduced by Marty in the mid 1930's in [Mar34], with the development of hyperrings attributed to Krasner in 1956 in [Kra83]. Connes and Consani highlighted a relationship between hyper-structures and Number theory, based on Connes' *adele classes* [CC11], which progressed the theory.

Hyperfields are just one class of generalisations, others include Tracts, Fuzzy rings and Blueprints. There is a discussion of these algebraic structures in [BL18b], and a description of Blueprints can be found in [Lor12]. The connection between fuzzy rings and hyperrings is described in [GJL17], where a fully faithful functor from hyperfields to fuzzy rings with weak morphisms is shown to exist. One particular motivation for working with hyperfields is that the tropical semiring, where a foundational theory has been thoroughly developed, has an analogue in the multivalued setting called the tropical hyperfield. This connection and natural compatibility between tropical geometry and hyper-structures is explored in [Vir10]. Thus, indicating a link between the two areas, and hence motivates the aim to expand ideas from tropical geometry to the theory of hyperfields.

There has been substantial progress made in recent years in the development of the algebraic theory of hyperfields. Baker and Bowler developed a theory of matroids over hyperfields in [BB18], and there has been work done by Bowler and Lorscheid on roots and multiplicities, especially characterising multiplicities for the Krasner, sign and tropical hyperfields, in [BL18a]. The work completed by Jun, in [Jun18] and [Jun17], is a more recent study of algebraic geometry over hyperfields. In [Jun17], algebraic sets over hyperfields are introduced and connected to tropical varieties, along with a scheme-theoretic point of view. Leading to a demonstration that hyperrings without zero divisors can be realised as the hyperring of global regular functions in [Jun17, Theorem D]. One consequence of the multivalued addition is a necessary extension of the notion of a root. Over hyperfields roots are defined as elements at which the polynomial outputs a set which includes zero, rather than exactly equals zero. Topological aspects of hyperfields, in particular the Grassmannian, have been explored by Anderson and Davis in [AD19]. This section begins by outlining the necessary theory for hyper-structures, making the descriptive nature of these introductory paragraphs explicit.

Given a non-empty set $\mathbb{H}$, then a map $\boxplus : \mathbb{H} \times \mathbb{H} \to \boldsymbol{P}(\mathbb{H})^*$ will denote the hyperoperation of $\mathbb{H}$, where $\boldsymbol{P}(\mathbb{H})^*$ is the power set of $\mathbb{H}$. Explicitly, $\boldsymbol{P}(\mathbb{H})^*$ is the set of all nonempty subsets of $\mathbb{H}$. Then, for subsets $\boldsymbol{A}, \boldsymbol{B} \subseteq \mathbb{H}$,

$$A \boxplus B := \bigcup_{a \in A\,,\, b \in B} a \boxplus b.$$

This definition can be extended for a string of elements. Let $x_1, \ldots, x_k \in \mathbb{H}$ then we define this as follows;

$$x_1 \boxplus x_2 \boxplus \ldots \boxplus x_k = \bigcup_{x' \in x_2 \boxplus \ldots \boxplus x_k} x_1 \boxplus x'.$$

The hyperoperation $\boxplus$ is called commutative and associative if it satisfies

$$x \boxplus y = y \boxplus x \tag{2.1.1}$$

and

$$(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z) \tag{2.1.2}$$

respectively.

Analogously to classical algebra, a hyperoperation can be used to define structures on the set $\mathbb{H}$. In a standard way, the following definitions will generalise those of groups, rings and fields, using $\boxplus$ as the underlying operation.

**Definition 2.1.3.** A canonical hypergroup is a tuple $(\mathbb{H}, \boxplus, \mathbb{0})$, where $\boxplus$ is a commutative and associative hyper-operation (so (2.1.1) and (2.1.2) hold) on $\mathbb{H}$ such that:

(H0) $\mathbb{0} \boxplus x = \{x\}, \quad \forall\, x \in \mathbb{H}$.

(H1) For every $x \in \mathbb{H}$ there is a unique element of $\mathbb{H}$, denoted $-x$, such that $\mathbb{0} \in x \boxplus -x$.

(H2) $x \in y \boxplus z$ iff $z \in x \boxplus (-y)$. This is normally referred to as *reversibility*.

The reversibility condition is not required for non-canonical hypergroups, but throughout this work only canonical hypergroups will be used so the label canonical is dropped.

**Definition 2.1.4.** A hyperring is a tuple $(\mathbb{H}, \odot, \boxplus, \mathbb{1}, 0)$ such that:

- $(\mathbb{H}, \odot, \mathbb{1})$ is a commutative monoid.
- $(\mathbb{H}, \boxplus, \mathbb{0})$ is a commutative hypergroup.
- (Absorption rule) $\mathbb{0} \odot x = x \odot \mathbb{0} = \mathbb{0}$, for all $x \in \mathbb{H}$.
- (Distributive law) $a \odot (x \boxplus y) = (a \odot x) \boxplus (a \odot y)$, for all $a, x, y \in \mathbb{H}$.

**Definition 2.1.5.** A hyperring $\mathbb{H}$ is called a hyperfield if $\mathbb{0} \neq \mathbb{1}$ and every non-zero element of $\mathbb{H}$ has a multiplicative inverse.

*Remark* 2.1.6. The precise meaning of the neutral elements $\mathbb{0}$ and $\mathbb{1}$ will be made explicit in each specific example.

*Remark* 2.1.7. From this point onward, to clarify context, when discussing results over a field $K$ the following notation will be used; $+$, $\times$(or $\cdot$) and $\sum$. Whereas, when discussing results over a hyperfield $\mathbb{H}$ the following notation will be used; $\boxplus$, $\odot$ and $\boxed{+}$. This will enable the multivalued context to be easily identified.

There are a range of properties that hyper-structures can posses. The following definitions will outline the main properties that will be utilised in this work.

**Definition 2.1.8.** A hypergroup is called *stringent* if the addition for $x, y \in \mathbb{H}$, $x \boxplus y$ is a singleton whenever $x \neq -y$. A hyperring is called *stringent* if the underlying hypergroup is stringent.

**Definition 2.1.9.** A hyperring is said to be *doubly distributive* if for any $x, y, z, w \in \mathbb{H}$, it holds that

$$(x \boxplus y)(z \boxplus w) = xz \boxplus xw \boxplus yz \boxplus yw.$$

Note that in [Vir10] Theorem 4.B. demonstrates that the inclusion, $(x \boxplus y)(z \boxplus w) \subset xz \boxplus xw \boxplus yz \boxplus yw$, holds over all hyperrings.

*Remark* 2.1.10. The doubly distributive definition is extended to the following, in [BB18].

$$\Big(\underset{i\in I}{\boxed{+}} x_i\Big)\Big(\underset{j\in J}{\boxed{+}} y_j\Big) = \underset{i\in I, j\in J}{\boxed{+}} x_i y_j$$

There is a thorough classification of doubly distributive hyperfields in [BS20], where in addition a relationship between the doubly distributive property and the stringent property is presented.

**Lemma 2.1.11.** *[BS20, Proposition 1.3] Given a hyperfield $\mathbb{H}$ that is doubly distributive, then this implies that $\mathbb{H}$ is stringent.*

The maps between hyperfields are called hyperfield homomorphisms. In accordance with the multivalued nature of hyperoperations the definition of hyperfield homomorphisms encompass an inclusion rather than an equality.

**Definition 2.1.12.** Given hypergroups $\mathbb{H}_1$ and $\mathbb{H}_2$, with respective hyper-operations $\boxplus_1$ and $\boxplus_2$. Then a hypergroup homomorphism is a map $f : \mathbb{H}_1 \to \mathbb{H}_2$, such that $f(\mathbb{0}) = \mathbb{0}$ and $f(x \boxplus_1 y) \subseteq f(x) \boxplus_2 f(y)$ for all $x, y \in \mathbb{H}_1$.

This notion can immediatley be extended to hyperrings.

**Definition 2.1.13.** Given hyperrings $\mathbb{H}_1$ and $\mathbb{H}_2$, with respective hyper-operations $\boxplus_1$ and $\boxplus_2$ and multiplication $\odot_1$ and $\odot_2$. Then a homomorphism of hyperrings $f : \mathbb{H}_1 \to \mathbb{H}_2$ is defined by

1. $f(x \boxplus_1 y) \subseteq f(x) \boxplus_2 f(y) \quad \text{and} \quad f(\mathbb{0}) = \mathbb{0}$
2. $f(x \odot_1 y) = f(x) \odot_2 f(y) \quad \text{and} \quad f(\mathbb{1}) = \mathbb{1}$

Essentially, this is a homomorphism of additive hypergroups and a homomorphism of multiplicative monoids.

**Definition 2.1.14.** A hyperfield homomorphism is defined as a homomorphism of underling hyperrings.

**Definition 2.1.15.** A hyperfield homomorphism $f : \mathbb{H}_1 \to \mathbb{H}_2$ is called *strict* if,

$$f(x \boxplus_1 y) = f(x) \boxplus_2 f(y),$$

for all $x, y \in \mathbb{H}_1$.

**Definition 2.1.16** ([BS20])**.** A hypergroup (resp. hyperring and hyperfield) isomorphism is a bijection $f : \mathbb{H}_1 \to \mathbb{H}_2$, which is a hypergroup (resp. hyperring and hyperfield) homomorphism and whose inverse is a hypergroup (resp. hyperring and hyperfield) homomorphism.

## 2.2 Quotient Hyperfields

At first hyperfields may appear unfamiliar, due to the multivalued operations, but actually they can arise in a natural construction involving fields. This construction is called a quotient hyperfield. The quotient construction was developed in [CC11], aiming to build on the early hyperring theory established in [Kra83] and [Mas85]. This section will outline how quotient hyperfields are constructed. It will be shown that several of the most common hyperfields can be represented in the quotient form in Section 2.4. Although, not every hyperfield can be constructed from the quotient of a field, as will be seen in Example 2.2.4, first presented by [Mas85].

Given a field $K$, the set of units, denoted $K^\times$, is defined as $K^\times = \{x \in K \,|\, \exists\, y \in K \quad \text{s.t} \quad xy = 1\}$. Take a multiplicative subgroup $U \subseteq K^\times$, and take the quotient of $K$ by this subgroup $U$. The resulting object is;

$$K/U = \{xU|\, x \in K\}$$

where elements of $K/U$ are co-sets in the form,

$$[x] = xU = \{\, xu_i \,|\, u_i \in U,\, x \in K\}. \tag{2.2.1}$$

The set $K/U$ can be viewed as a hyperfield. The addition and multiplication for this quotient construction are defined in the following way:

$$[x] \odot [y] = [xy] = \{\, (xy)u_i \,|\, u_i \in U,\, x \in K\} \tag{2.2.2}$$

$$[x] \boxplus [y] = \{\, [z] \,|\, z = xu_i + yu_j \quad \text{s.t} \quad u_i, u_j \in U\}. \tag{2.2.3}$$

Or, in alternative notational convention but with equivalent meaning,

$$x + y = (xU + yU)/U.$$

From this point forward $K/U$ will be taken in the hyperfield sense, with the notation from above. This method of building hyperfields was formally updated into the language of hyperfields by Connes and Consani in [CC11], where a explicit proof that this quotient construction satisfies the hyperfield axioms can be see in the proof of the Theorem on page 310 in [CC11].

Quotient hyperfields are an interesting class of hyperfields to study, but in general not all hyperfields can be described in the quotient form. It has been shown by Massouros in [Mas85] that not all hyperfields are in this quotient form. The quotient construction allows us to build hyperfields, but does not encompass all of them.

**Example 2.2.4.** [Mas85, page 727] An *almost-group* is a semi-group which is the union of a group with a bilaterally absorbing neutral element. Consider a commutative multiplicative almost-group $(H, \cdot)$, which the hyperaddition can be defined as follows:

$$x \boxplus y = \begin{cases} \{x, y\}, & \text{if} \quad x \neq y \quad \text{and} \quad x, y \neq 0 \\ H\backslash\{0\}, & \text{if} \quad x = y \quad \text{and} \quad x \neq 0 \\ x, & \text{if} \quad y = 0 \end{cases} \tag{2.2.5}$$

Then the triple $(H, \boxplus, \cdot)$ is a hyperfield. It is shown in the proof that follows this example in [Mas85], that the class of hyperfields contains elements that are not in the quotient class of hyperfields. Thus, showing that not all hyperfields are in the quotient form.

Examples of hyperfields are defined in Section 2.4, along with a classification of their properties. To conclude this section there will be a discussion on how the quotient construction can be generalised by starting with a hyperfield rather a field.

Let $(\mathbb{H}, \boxplus, \odot)$ be a hyperfield and take $U \subseteq \mathbb{H}^\times$ a multiplicative subgroup of the non-zero elements of the hyperfield. Then, the quotient is defined as $\mathbb{H}/U := \mathbb{H}^\times/U \cup \{0\}$, which has a hyperfield structure due to the following operations. Elements of $\mathbb{H}/U$ are cosets, defined as $[x] := \{x \odot u \,:\, u \in U\}$. The multiplication is inherited from the hyperfield $\mathbb{H}$, $[x] \odot [y] = [x \odot y]$, and the multivalued addition is defined as;

$$[x] \boxplus [y] := \{[z] \,:\, [z] \in [x] \boxplus [y]\} = \{[z] \,:\, z \in x \odot u \boxplus y \odot v\,,\, u, v \in U\}.$$

It can be seen that this is a generalisation of the field case, as fields can trivially be viewed as hyperfields. There will now be a clarification of when elements in the quotient construction can be considered equal.

**Proposition 2.2.6.** *Given a hyperfield $\mathbb{H}$ and a multiplicative subgroup $U \subseteq \mathbb{H}^{\times}$, then for an element $y \in \mathbb{H}$, $y \odot U = U$ iff $y \in U$.*

*Proof.* Firstly, $y \in U \Rightarrow y^{-1} \in U \Rightarrow u \odot y^{-1} \in U$, $\forall u \in U$. Thus, $u = y \odot y^{-1} \odot u \in y \odot U$. Which gives that $U \subseteq y \odot U$ and as $U$ is a multiplicative subgroup $y \odot u \in U$, $\forall u \in U$, implying $y \odot U \subseteq U$. Which together shows that $y \odot U = U$.
Secondly, if $y \odot U = U$ then every $y \odot u \in U$, hence $y \in U$. This shows the other direction. □

**Proposition 2.2.7.** *Let $\mathbb{H}$ be a hyperfield, then for $a, b \in \mathbb{H}$, $[a] = [b]$ over $\mathbb{H}/U$ if and only if there exists and element $u \in U$ such that $a = b \odot u$.*

*Proof.* By Proposition 2.2.6, as $u \in U$, this implies that $[b \odot u] = b \odot u \odot U = b \odot U = [b]$, and by definition $a = b \odot u$, so $[a] = [b \odot u]$, hence $[a] = [b]$. Conversely, $[a] = [b]$ implies that every element of $[a]$ has a corresponding element of $[b]$. As $\mathbb{1} \in U$, then $a = a \odot \mathbb{1} \in [a]$ has a corresponding element of $[b]$ of the form $b \odot \hat{u}$, which is as required. □

## 2.3 Polynomials Over Hyperfields

This section will outline how polynomials and their roots are defined over hyperfields. These notions will be used throughout the following work. In particular, Chapter 3 explores the interaction between polynomials and their roots under hyperfield homomorphisms, and Chapter 6 uses linear polynomials to investigate convex geometry over hyper-structures.

**Definition 2.3.1.** The set of polynomials in $n$-variables over a hyperfield $\mathbb{H}$ will be denoted $\mathbb{H}[X_1, \ldots, X_n]$, where elements of this set are defined as,

$$p(X_1, \ldots, X_n) := \sum_I c_I X_1^{i_1} \cdots X_n^{i_n} = \sum_I c_I \underline{X}^I, \tag{2.3.2}$$

where the multi-index notation is used and $I = (i_1, \ldots, i_n) \subset \mathbb{Z}^n$ and $c_I \in \mathbb{H}$. Then, when the polynomial is evaluated at an element $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{H}^n$, the hyper addition and multiplication of the hyperfield $\mathbb{H}$ is used.

$$p(a_1, \ldots, a_n) := \bigboxplus_I c_I a_1^{i_1} \cdots a_n^{i_n} = \bigboxplus_I c_I \underline{a}^I \subseteq \mathbb{H} \tag{2.3.3}$$

The addition and product of the polynomials in (2.3.2) are both multivalued operations. The following example will demonstrate how to take the sum and product of univariate polynomials over a hyperfield.

**Example 2.3.4.** Let $p(X) = \sum_{i=0}^n c_i X^i$ and $q(X) = \sum_{j=0}^m d_j X^j$ be polynomials with coefficients in a hyperfield $\mathbb{H}$, where $n \leqslant m$.The addition is induced from the hyperfield $\mathbb{H}$;

$$p(X) \boxplus q(X) = \sum_{i=0}^{n} (c_i \boxplus d_i) X^i + \sum_{i=n+1}^{m} d_i X^i. \tag{2.3.5}$$

Furthermore, the multiplication is as follows;

$$p(X) \odot q(X) = \sum_{i=0}^{n+m} \Big( \bigboxplus_{j+k=i} c_j \odot d_k \Big) X^i. \tag{2.3.6}$$

*Remark* 2.3.7. Note that the notation $\mathbb{H}[X_1, \ldots, X_n]$ is used only to denote the *set* of polynomials over $\mathbb{H}$. In general, there is no ring or hyperring structure on $\mathbb{H}[X_1, \ldots, X_n]$, unlike the specialised case where $\mathbb{H}$ is a field. Although, it can be given additional algebraic structure as a $\mathbb{H}$-module. It can be illustrated in the univariate case that this object is not in general a hyperring; the multivalued nature of the addition in $\mathbb{H}$ combined with the distributivity leads to products of polynomials also being multivalued. For example, $(aX \boxplus b)(cX \boxplus d) \subseteq acX^2 \boxplus (ad \boxplus bc)X \boxplus bd$. The coefficient $(ad \boxplus bc)$ is not necessarily single valued, which shows that multiplication of polynomials is multivalued, hence $\mathbb{H}[X_1, \ldots, X_n]$ is not a hyperring. See Remark 4.4 and Example 4.13 in [Jun18] for an explicit example where the behaviour of $\mathbb{H}[X_1, \ldots, X_n]$ is more controllable.

Due to the output of a polynomial evaluated at an element producing a set of elements, as seen in (2.3.3), this leads to the following generalised definition of a root.

**Definition 2.3.8.** Let $p(X_1, \ldots, X_n) = \boxplus_I c_I \underline{X}^I$ be a polynomial defined over a hyperfield $\mathbb{H}$, then an element $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a *root* of the polynomial if $\mathbb{0} \in p(\boldsymbol{a}) = \boxplus_I c_I \boldsymbol{a}^I$.

**Definition 2.3.9.** This allows for a natural definition of the variety of $p(X_1, \ldots, X_n)$ as,

$$V(p) := \{\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{H}^n \mid \mathbb{0} \in p(\boldsymbol{a})\}.$$

The next definition recalls the notion of the multiplicity of a root for univariate polynomials defined over hyperfields.

**Definition 2.3.10** ([BL18a], Def. 1.5)**.** Let $p(X) \in \mathbb{H}[X]$, the multiplicity of an element $a \in \mathbb{H}$ is denoted $mult_a(p)$ and defined as follows. If $a$ is not a root of $p$ set $mult_a(p) = 0$. If $a$ is a root of $p$ define

$$mult_a(p) = 1 + \max\{mult_a(q) \,:\, p \in (X \boxplus -a) \odot q(X)\}. \tag{2.3.11}$$

The multivalued nature of the multiplication of polynomials with coefficients in a hyperfield implies that the polynomial $q(X)$ in (2.3.11) is not necessarily unique. This is the motivation behind the recursive definition of the multiplicity. See [BL18a] for details of the original definition and examples of the non-uniqueness.

*Remark* 2.3.12. There is a complete description of roots and corresponding multiplicities for univariate polynomials over $\mathbb{K}$, $\mathbb{S}$ and $\mathbb{T}$ in [BL18a], where the results are used to demonstrate proofs of Descartes' Rule of Signs and Newton's Polygon Rule. This work has been built on in [Gun19] where the multiplicities for roots over $\mathbb{TR}$ have been presented. The hyperfields $\mathbb{K}, \mathbb{S}, \mathbb{T}$ and $\mathbb{TR}$ are defined in the next section.

## 2.4 Hyperfield Zoo

This section focuses on describing a range of hyperfields and cataloguing their properties. The aim is to create a reference guide for hyperfields, in which the properties are documented collectively. The main product of this section is Table 2.1, which consists of a concise overview of the hyperfields and their properties. This will be followed by a description of each example hyperfield and a demonstration of the resources used to present the properties, along with calculations to highlight key points. The motivation to present the examples of hyperfields at this point is to allow all of the definitions to be first introduced, thus allowing for a more thorough discussion of each hyperfield to take place.

Table 2.1 has the purpose of acting as a reference guide for the hyperfield literature, when for future research information regarding these hypefields is required. The information in the table is a amalgamation of results from the literature, which will be explicitly cited in the detailed breakdown of each hyperfield, and some informative examples developed for this work. The overall aim is to create a database, or 'Zoo', of information on the commonly used hyperfields, bringing together work from the relevant areas of literature and presenting it collectively here.

The format for the description of each hyperfield will follow a similar pattern; define the operations, give the hyperfield as a quotient construction and then state and/or show what properties the hyperfield exhibits. Firstly, there will be additional definitions to clarify terms used in Table 2.1.

**Definition 2.4.1.** A hyperfield is called *algebraically closed* if for every univariate polynomial defined over the hyperfield there exists a root which belongs to this hyperfield.

**Definition 2.4.2.** A hyperfield is said to satisfy the *multiplicity bound* if for all univariate polynomials, $p(X) \in \mathbb{H}[X]$,

$$\sum_{a \in \mathbb{H}} \mathrm{mult}_a(p) \leqslant \deg(p).$$

Furthermore, a hyperfield is said to satisfy *multiplicity equality* if the above inequality is an equality for all univariate polynomials.

| Hyperfield | Quotient Form | Doubly Distributive | Stringent | Algebraically Closed | Multiplicity Bound |
|---|---|---|---|---|---|
| Field | - | ✓ | ✓ | $\overline{K}$ | ✓ |
| $\mathbb{K}$ | $K/K^\times$ | ✓ | ✓ | ✓ | ✓(equality) |
| $\mathbb{S}$ | $\mathbb{R}/\mathbb{R}_{>0}$ | ✓ | ✓ | ✗ | ✓ |
| $\mathbb{T}$ | $\overline{K}/v^{-1}(0)$ | ✓ | ✓ | ✓ | ✓(equality) |
| $\mathbb{P}$ | $\mathbb{C}/\mathbb{R}_{>0}$ | ✗ | ✗ | ✓ | ✗ |
| $\triangle$ | $\mathbb{C}/S^1$ | ✗ | ✗ | ✓ | ✗ |
| $\Phi$ | $\mathbb{TC}/\mathbb{R}_{>0}$ | ✗ | ✗ | ✓ | ✗ |
| $\mathbb{W}$ | $\mathbb{F}_p/(\mathbb{F}_p^\times)^2$ $p > 7$, $p \equiv 3 (\text{mod} 4)$ | ✗ | ✗ | ✗ | ✗ |
| $\mathbb{TR}$ | $\mathbb{R}[[t^\mathbb{R}]]/v_\mathbb{R}^{-1}(0)$ | ✓ | ✓ | ✗ | ✓ |
| $O_{\mathbb{TR}}$ | $O_\mathbb{R}[[t^\mathbb{R}]]/v_\mathbb{R}^{-1}(0)$ | ✓ | ✓ | ✗ | ✓ |
| $\mathbb{TC}$ | ? | ✗ | ✗ | ✓ | ✗ |

Table 2.1: A Hyperfield Reference Guide.

### 2.4.1 Fields

A field $K$ can be viewed as a hyperfield in a trivial manner, where the hyperaddition is defined as $x \boxplus y = \{x + y\}$. The quotient construction is also trivial as, $K = K/1$. It can be immediately seen that all fields are doubly distributive and stringent. Not all fields are algebraically closed, but there exists a unique, up to isomorphism, algebraic closure of any field which is not algebraically closed, normally denoted as $\overline{K}$. (Note that $\overline{\mathbb{R}}$ will not denote the algebraic closure of $\mathbb{R}$ in this work. It will be denoting the tropical semiring which is defined and used predominantly in Chapter 5.) Fields satisfy the multiplicity bound, and algebraically closed fields satisfy multiplicity equality due to the Fundamental Theorem of Algebra.

### 2.4.2 Krasner Hyperfield ( $\mathbb{K}$)

The Krasner hyperfield has the underlying set $\{0,1\}$, and denoted $\mathbb{K}$. It is defined with standard multiplication and hyperaddition is defined as;

$$0 \boxplus x = x \boxplus 0 = \{x\}, \quad \text{for} \quad x = 0, 1,$$

$$1 \boxplus 1 = \{0,1\}.$$

Where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. The Krasner hyperfield can be constructed as the quotient of a field $K$, by the set of invertible elements $K^{\times}$. Explicitly, $\mathbb{K} \cong K/K^{\times}$. It is stated in [BS20, Example 2.9] that $\mathbb{K}$ is doubly distributive, then by Lemma 2.1.11 $\mathbb{K}$ is also stringent. From [BL18a, Remark 1.11] it can be seen that the sum of multiplicities for any polynomial over $\mathbb{K}$ equals the degree of the polynomial. This gives that the multiplicity equality holds over $\mathbb{K}$. A direct consequence of this is that $\mathbb{K}$ is algebraically closed.

### 2.4.3 Hyperfield of Signs ( $\mathbb{S}$)

The hyperfield of signs has the underlying set $\{-1,0,1\}$, and denoted by $\mathbb{S}$. The multiplication is defined in the standard way and the hyperaddition defined as;

$$0 \boxplus 0 = 0, \quad 0 \boxplus 1 = 1, \quad 0 \boxplus -1 = -1,$$

$$1 \boxplus 1 = 1, \quad -1 \boxplus -1 = -1,$$

$$1 \boxplus -1 = \{0, 1, -1\}.$$

Where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. The quotient construction for $\mathbb{S} \cong \mathbb{R}/\mathbb{R}_{>0}$, is outlined in [BL18a]. It is stated in [BS20, Example 2.9] that $\mathbb{S}$ is doubly distributive, which again by Lemma 2.1.11 implies that $\mathbb{S}$ is stringent. Similarly to $\mathbb{K}$, a characterisation of the multiplicities for polynomials over $\mathbb{S}$ is explained in [BL18a]. It can be seen in [BL18a, Remark 1.12], that $\mathbb{S}$ satisfies the multiplicity bound, and in fact is it a bound rather than equality. The hyperfield of signs is not algebraically closed, as $V(p(X)) = V(X^2 \boxplus 1) = \varnothing$, over $\mathbb{S}$.

### 2.4.4 Tropical Hyperfield ( $\mathbb{T}$)

The tropical hyperfield has the underlying set $\mathbb{R} \cup \{-\infty\}$ and is denoted as $\mathbb{T}$. The multiplication on $\mathbb{T}$ is an extension of the addition on $\mathbb{R}$:

$$x \odot y := x + y.$$

$$x \odot -\infty = -\infty \odot x = -\infty + x = x + -\infty = -\infty,$$

$$x \odot 0 = 0 \odot x = x + 0 = 0.$$

The hyperaddition is defined as the following multivalued operation:

$$x \boxplus y = \begin{cases} \{\max(x, y)\}, & \text{if} \quad x \neq y \\ \{z \mid z \leqslant x\} \cup \{-\infty\}, & \text{if} \quad x = y \end{cases} \tag{2.4.3}$$

$$x \boxplus -\infty = -\infty \boxplus x = \max(x, -\infty) = x.$$

Where $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. The tropical hyperfield is a hyperfield analogue of the tropical semi-ring described in Section 1.1 and [MS15]. The tropical hyperfield can also be defined using *min* instead of *max*, and $\{\infty\}$ instead of $\{-\infty\}$, which yields an isomorphic structure. For further discussions of the tropical hyperfield and demonstrations of its usefulness, see both [Vir10] and [Vir11].

The tropical hyperfield is constructed in a quotient form by taking an algebraically closed field $\overline{K}$ with a surjective valuation $v : \overline{K}^{\times} \to \mathbb{R}$. Then, $\mathbb{T}$ is isomorphic to the quotient $\overline{K}/v^{-1}(0)$, as mentioned in [BL18a, Remark 1.1]. In exactly the same way as $\mathbb{K}$ and $\mathbb{S}$, the mulitplicities of roots have been characterised for $\mathbb{T}$ in [BL18a]. It is shown in [BL18a, Remark 1.17], that multiplicity equality holds for all polynomials over $\mathbb{T}$, which in turn implies that $\mathbb{T}$ is algebraically closed.

*Remark* 2.4.4. These first three hyperfields are the most commonly used in the literature, and have been explored more thoroughly than the hyperfields that follow. The above properties are well known in the hyperfield research area. Below are some less extensively studied hyperfields. Some of the information presented below derives from the literature and has been cited appropriately. Although, some properties of the following hyperfields that are demonstrated explicitly below were not collated from the current literature.

Thus far, the algebraic closure of $\mathbb{K}$ and $\mathbb{T}$ has been implied from the multiplicity equality. The following result will be used to demonstrate algebraic closure in the absence of multiplicity equality.

**Lemma 2.4.5.** *Let $K = \bar{K}$, then the surjective quotient hyperfield homomorphism $f : K \to K/U$ implies that $K/U$ is algebraically closed.*

*Proof.* The proof for Lemma 2.4.5 will be consequence of Lemma 3.2.2, which is a more general statement regarding roots of polynomials pushing forward through hyperfield homomorphisms. Though explicitly, the algebraic closure of $K$ implies that every univariate polynomial has a root. Lemma 3.2.2 implies that these roots push-forward to roots over the quotient $K/U$, and as the map is surjective every polynomial over $K/U$ has a polynomial over $K$ which maps to it and hence a root that can be pushed forward. □

*Remark* 2.4.6. Lemma 3.2.2 is stated at a later stage rather than here as it is a more general result that encompasses one of the inclusions required for the equality in the hyperfield version of Kapranov's Theorem 3.2.5.

The next hyperfield will be used as the nominee to replace the valued field in a version of Kapranov's Theorem for tropical geometry for hyperfields in Chapter 3.

### 2.4.5 Tropical Complex Hyperfield ( $\mathbb{TC}$)

The tropical complex hyperfield is denoted $\mathbb{TC}$ and has the complex numbers $\mathbb{C}$ as its underlying set. The standard complex multiplication is given to $\mathbb{TC}$. The hyper-addition is defined in the following way for all $z, w \in \mathbb{C}$.

$$z \boxplus w = \begin{cases} \{c \in \mathbb{C} : |c| \leqslant z\}, & \text{if} \quad w = -z. \\ z, & \text{if} \quad |z| > |w|. \\ w, & \text{if} \quad |w| > |z|. \\ \text{Shortest arc connecting } z \text{ and } w, \text{ with radius } |z|, & \text{if } |z| = |w|,\ z \neq \pm w. \end{cases}$$

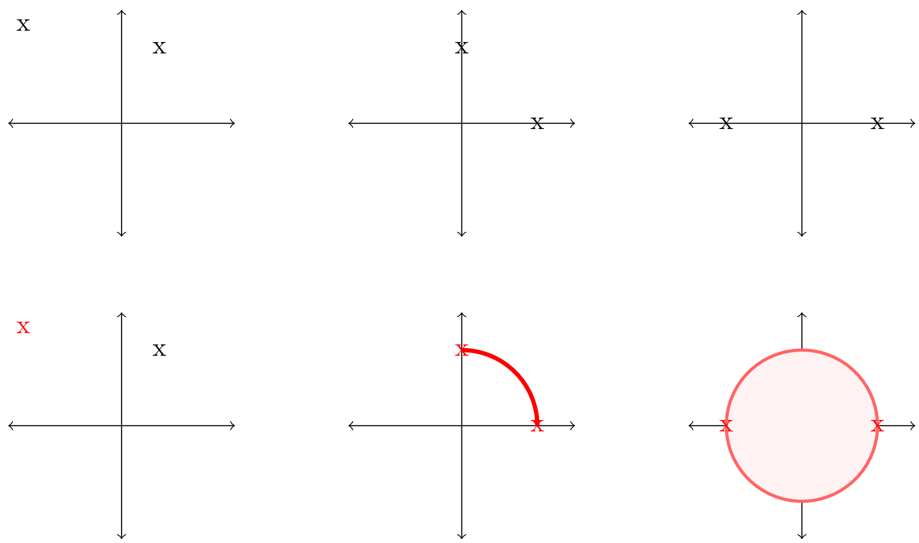

Figure 2.1: Hyper-addition over $\mathbb{TC}$ - Where the first row is two points in $\mathbb{TC}$ and the bottom row represents the outcome of the addition in red.

With this hyper-addition $\mathbb{TC}$ is a hyperfield, where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. This hyperfield is discussed in Example 9 in [AD12] and was introduced in [Vir10, Section 6] where it is described as the dequantization of the field of complex numbers. For further insight into the behaviour of the addition over $\mathbb{TC}$ see Figure 1 in Section 6 of [Vir10] and Figure 2.1.

It can be seen from the definition of the hyperoperation that $\mathbb{TC}$ is not stringent, and therefore by Lemma 2.1.11 not doubly distributive. The algebraic closure property will be discussed in Section 3, as a corollary to the main result of the section. The tropical complex hyperfield exceeds the multiplicity bound, which is outlined in the next example.

**Example 2.4.7.** Given the polynomial $p(X) = X^2 \boxplus X \boxplus 1 \in \mathbb{TC}[X]$, then

$$\begin{aligned} p(-1) &= (-1)^2 \boxplus -1 \boxplus 1 \\ &= 1 \boxplus -1 \boxplus 1 \\ &= \Big(\{a \in \mathbb{C} \,:\, |a| \leqslant 1\}\Big) \boxplus 1 \\ &\ni -1 \boxplus 1 \ni 0 \end{aligned}$$

$$p(i) = -1 \boxplus i \boxplus 1$$

$$= \Big(\{\text{shortest closed arc between } -1 \text{ and } i\}\Big) \boxplus 1$$
$$\ni -1 \boxplus 1 \ni 0$$

$$\begin{aligned} p(-i) &= -1 \boxplus -i \boxplus 1 \\ &= \Big(\{\text{shortest closed arc between } -1 \text{ and } -i\}\Big) \boxplus 1 \\ &\ni -1 \boxplus 1 \ni 0 \end{aligned}$$

These three calculations over $\mathbb{TC}$ imply that $\{-1, i, -i\} \subset V(p) \subset \mathbb{TC}$. This shows that a degree 2 polynomial can have three distinct root over $\mathbb{TC}$, thus the multiplicity bound does not hold.

The one question mark in Table 2.1 is the quotient form of $\mathbb{TC}$. Currently this can be summarised in the following open question.

**Question 2.4.8.** *Can $\mathbb{TC}$ be explicitly stated in the quotient form?*

### 2.4.6 Phase Hyperfield ( $\mathbb{P}$)

There are several different paradigms in which the arithmetic over the phase hyperfield can be viewed. The notation will stay consistent denoting the phase hyperfield by $\mathbb{P}$. The following two definitions are equivalent, but different ways of describing the hyperoperation of $\mathbb{P}$. It has the underlying set, $S^1 \cup \{0\}$, where $S^1 = \{e^{i\theta} \in \mathbb{C} \,|\, 0 \leqslant \theta < 2\pi\}$, which is the complex unit circle union with zero. In both instances the multiplication is inherited from $\mathbb{C}$.

a) The first is seen in [BL18a]. The hyperaddition is defined by the rule, where $S^1 = \{e^{i\theta} \in \mathbb{C} \,|\, 0 \leqslant \theta < 2\pi\}$.

$$\theta_1 = \theta_2 + \pi \quad \text{then} \quad e^{i\theta_1} \boxplus e^{i\theta_2} = \{0, e^{i\theta_1}, e^{i\theta_2}\}$$

$$\theta_1 < \theta_2 < \theta_1 + \pi \quad \text{then} \quad e^{i\theta_1} \boxplus e^{i\theta_2} = \{e^{i\theta} \,|\, \theta_1 < \theta < \theta_2\}$$

b) The second is seen in [BB18] and [BS20]. The hyperaddition is defined as,

$$0 \boxplus \{x\} = \{x\}, \quad x \boxplus -x = \{0, x, -x\},$$

$$x \boxplus y = \{\frac{ax+by}{|ax+by|} : a, b \in \mathbb{R}_{>0}\} \quad \text{for} \quad x, y \in S^1 \quad \text{such that} \quad x \neq -y.$$

Where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. The quotient form of $\mathbb{P}$ is given in [BL18a] as $\mathbb{C}/\mathbb{R}_{>0}$. The hyperaddition gives an arc around the unit circle which by definition shows the phase hyperfield is not stringent, and hence not doubly distributive by Lemma 2.1.11. The phase hyperfield is algebraically closed by Lemma 2.4.5, as there is a surjective hyperfield homomorphism, $ph(\cdot) : \mathbb{C} \to \mathbb{P}$, from the algebraically closed complex numbers. Drawing again from the work in [BL18a], $\mathbb{P}$ does not satisfy the multiplicity bound. This is demonstrated in the next example

**Example 2.4.9.** [BL18a, Remark 1.10] The polynomial $p(X) = X^2 \boxplus X \boxplus 1$ has infinitely many roots over $\mathbb{P}$, where $a \in V(X^2 \boxplus X \boxplus 1)$ are described by $a = e^{i\theta}$, for all $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

### 2.4.7 Triangle Hyperfield ($\triangle$)

The triangle hyperfield, denoted by $\triangle$, has the underlying set $\mathbb{R}_{\geqslant 0}$. The multiplication is defined in the standard way, and then the hyperaddition is defined as;

$$x \boxplus y = \{z : |x-y| \leqslant z \leqslant x+y\}.$$

Where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. It is stated in [BS20] that $\triangle$ is not doubly distributive and the triangle hyperfield is seen to be not stringent from the definition of the hyperoperation. The quotient construction of the triangle hyperfield is given by taking the quotient of the complex numbers by the unit circle in the complex plane. Explicitly, $\mathbb{C}/S^1$, which comes as a result of [AD19, Proposition 2.1]. Similarly to $\mathbb{P}$, the triangle hyperfield is algebraically closed, due to Lemma 2.4.5, as there is a surjective hyperfield homomorphism, $|\cdot| : \mathbb{C} \to \triangle$, from the algebraically closed complex numbers. It will be shown in the following example that the triangle hyperfield exceeds the multiplicity bound.

**Example 2.4.10.** Take the polynomial $p(X) = X^2 \boxplus X \boxplus 1$ over the triangle hyperfield.

$$\begin{aligned} 0 \in a \boxplus a &= \{c \in \mathbb{R}_{\geqslant 0} \mid |a - a| \leqslant c \leqslant a + a\} \\ &= [0, 2a] \subset \mathbb{R}_{\geqslant 0} \end{aligned}$$

By the reversibility property for hyperaddition,

$$0 \in X^2 \boxplus X \boxplus 1 \iff -1 = 1 \in X^2 \boxplus X.$$

By the definition of the hyperaddition over $\triangle$,

$$X^2 \boxplus X = \{c \in \mathbb{R}_{\geqslant 0} \mid |X^2 - X| \leqslant c \leqslant X^2 + X\}.$$

Therefore,

$$1 \in X^2 \boxplus X \iff |X^2 - X| \leqslant 1 \leqslant X^2 + X.$$

Then, splitting this statement into two separate inequalities and solving individually, they can be recombined to produce,

$$\frac{1}{2}(\sqrt{5} - 1) \leqslant X \leqslant \frac{1}{2}(1 + \sqrt{5}),$$

which defines an interval of roots to the polynomial $p(X) = X^2 \boxplus X \boxplus 1$ over $\triangle$, $V(p(X)) = \left[\frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(1 + \sqrt{5})\right]$. This demonstrates that the triangle hyperfield exceeds the multiplicity bound.

### 2.4.8 Tropical Phase Hyperfield ( $\Phi$)

The tropical phase hyperfield, as indicated by the name, is a close relation to both the phase hyperfield and the tropical hyperfield. It is denoted by $\mathbf{\Phi}$ with the ground set as $S^1 \cup \{0\}$, where the multiplication is defined as standard for the complex plane. The hyper-addition is defined as;

$$x \boxplus x = x, \quad x \boxplus -x = S^1 \cup \{0\}$$

Then, given $x \neq \pm y$,

$$x \boxplus y = \{\text{Shortest closed arc between the points}\, x \,\text{and}\, y \,\text{on the unit circle}\}.$$

The quotient form of the tropical phase hyperfield is another result of [AD19, Proposition 2.1], it is given as $\mathbb{TC}/\mathbb{R}_{>0}$. In an analogous way to $\mathbb{P}$ it can seen immediately from the definition of the hyper-addition that $\mathbf{\Phi}$ is no stringent. It can also be seen from the following example how the tropical phase hyperfield fails to be doubly distributive.

**Example 2.4.11.** Take the following four elements of $\mathbf{\Phi}$: $a = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $b = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, $c = 1$ and $d = i$. Then $0 \notin (a \boxplus b)$ and $0 \notin (c \boxplus d)$, giving $0 \notin (a \boxplus b)(c \boxplus d)$. Furthermore, $b \in (a \boxplus b) = c(a \boxplus b)$ and $-b \in i(a \boxplus b) = d(a \boxplus b)$. This leads to

$$\begin{aligned} ac \boxplus bc \boxplus ad \boxplus bd &= c(a \boxplus b) \boxplus d(a \boxplus b) \\ &\ni b \boxplus -b \\ &\ni 0 \end{aligned}$$

In conclusion,

$$0 \in ac \boxplus bc \boxplus ad \boxplus bd \neq (a \boxplus b)(c \boxplus d) \not\ni 0.$$

The map $\mathbb{P} \to \mathbf{\Phi}$ is a surjective hyperfield homomorphism, which implies that the tropical phase hyperfield is algebraically closed due to the phase hyperfield being algebraically closed by Lemma 2.4.5. This map also demonstrates that $\mathbf{\Phi}$ exceeds the multiplicity bound. In a similar manner to the proof of Lemma 2.4.5, the fact that the existence of this maps demonstrates that $\mathbf{\Phi}$ does not satisfy the multiplicity bound will be expanded on in Section 3.2.

### 2.4.9 Weak Hyperfield of Signs ( $\mathbb{W}$)

The weak hyperfield of signs denoted $\mathbb{W}$, is similar to $\mathbb{S}$, as it has the same underlying set $\{-1, 0, 1\}$. The multiplication is defined in the standard way, then the hyper-addition operation is defined as;

$$1 \boxplus 1 = -1 \boxplus -1 = \{1, -1\},$$

$$1 \boxplus -1 = \{0, -1, 1\}.$$

Where $\mathbb{0} = 0$ and $\mathbb{1} = 1$. It can be seen from the definition of the hyper-operation that $\mathbb{W}$ is not stringent. It can also be seen explicitly that $\mathbb{W}$ is not doubly distributive in the following way:

$$-1 \boxplus -1 \boxplus -1 \boxplus -1 = \mathbb{W} \neq \{-1, 1\} = (1 \boxplus 1)(-1 \boxplus -1).$$

The weak hyperfield of signs is not algebraically closed, as in an analogous way to $\mathbb{S}$, $V(p(X)) = V(X^2 \boxplus 1) = \varnothing$. Furthermore, it can be demonstrated that $\mathbb{W}$ does not satisfy the multiplicity bound either. In [BL18a, Remark 1.9], it is shown that $p(X) = X^2 \boxplus X \boxplus 1$ has two double roots at 1 and $-1$ over $\mathbb{W}$.

*Remark* 2.4.12. The above discussion puts $\mathbb{W}$ in an unusual position in the category of hyperfields. As it is not algebraically closed, this exhibits a suggestion that there are some 'roots' missing from the hyperfield, but the fact that it exceeds the multiplicity bound suggests that there are too many 'roots' in one sense. Taken together these points would form a contradiction in the classic setting. There is clearly more to be understood regarding roots, algebraic closure and the multiplicity bound in the category of hyperfields.

### 2.4.10 Signed Tropical Hyperfield ( $\mathbb{TR}$) and ($O_{\mathbb{TR}}$)

Here the signed tropical hyperfield, denoted $\mathbb{TR}$, and a sub-hyperring denoted $O_{\mathbb{TR}}$ are presented. The signed tropical hyperfield has the underlying set of $(\{\pm 1\} \times \mathbb{R}) \cup \{\infty\}$, with the underlying set of $O_{\mathbb{TR}}$ as the restriction to positive real numbers in the second component: $(\{\pm 1\} \times \mathbb{R}_{\geqslant 0}) \cup \{\infty\}$. Take $(a_1, b_1), (a_2, b_2) \in \mathbb{TR}$, then the multiplication is defined as:

$$(a_1, b_1) \odot (a_2, b_2) = (a_1 \cdot a_2, b_1 + b_2).$$

The hyper-addition is defined as:

$$(a_1, b_1) \boxplus (a_2, b_2) = \begin{cases} (a_1, b_1), & \text{if } b_1 < b_2, \\ (a_2, b_2), & \text{if } b_2 < b_1, \\ (a_1, b_1), & \text{if } a_1 = a_2, \text{ and } b_1 = b_2 \\ \{(\pm 1, c) \,:\, c \geqslant a_1\} \cup \{\infty\}, & \text{if } b_1 = b_2, \text{ and } a_1 = -a_2. \end{cases}$$

The sub-hyperring $O_{\mathbb{TR}}$ has the above operations but restricted to the subset $(\{\pm 1\} \times \mathbb{R}_{\geqslant 0}) \cup \{\infty\}$. Where $\mathbb{0} = \{\infty\}$ and $\mathbb{1} = (1, 0)$. It is stated in [BS20, Example 2.9] that $\mathbb{TR}$ is doubly distributive. This implies that $O_{\mathbb{TR}}$ is also doubly distributive and that both objects are stringent, by Lemma 2.1.11. The quotient construction of both of these objects uses Hahn series, which will now be defined formally. A detailed description of these can be found in [Gun19], including the Hahn series quotient construction.

**Definition 2.4.13.** Let $K$ be a real or algebraically closed field and let $\Omega \subset \mathbb{R}$ be an ordered subgroup. Then a *Hahn* series is defined as an element of

$$K[[t^{\Omega}]] = \left\{ \sum_{n \in N} c_n t^n \; : \; c_n \in K, \, N \subset \Omega \, \text{is well ordered.} \right\}$$

Fix $\Omega = \mathbb{R}$, then there is a valuation on $\mathbb{R}[[t^{\mathbb{R}}]]$ and $\mathbb{C}[[t^{\mathbb{R}}]]$ defined as

$$\mathrm{val}_{\mathbb{R}} : \mathbb{R}[[t^{\mathbb{R}}]] \to \mathbb{TR}, \quad \mathrm{val}_{\mathbb{R}}\Big( \sum_{n \in N} c_n t^n \Big) = (\mathrm{sgn}(c_{n_0}), n_0)$$

$$\mathrm{val}_{\mathbb{C}} : \mathbb{C}[[t^{\mathbb{R}}]] \to \mathbb{T}, \quad \mathrm{val}_{\mathbb{C}}\Big( \sum_{n \in N} c_n t^n \Big) = n_0.$$

Where $n_0 = \mathrm{Min}(N)$. The signed tropical hyperfield is isomorphic to the quotient $\mathbb{R}[[t^{\mathbb{R}}]]/\mathrm{val}_{\mathbb{R}}^{-1}(0)$, and $O_{\mathbb{TR}}$ is isomorphic to $O_{\mathbb{R}}[[t^{\mathbb{R}}]]/\mathrm{val}_{\mathbb{R}}^{-1}(0)$. Both of these objects are not algebraically closed as the polynomial $p(X) = X^2 \boxplus 1$ has no roots over $\mathbb{TR}$. This is a consequence of the definitions of the multiplicities of roots over $\mathbb{TR}$, which are characterised in [Gun19]. A corollary of the results in [Gun19] is that $\mathbb{TR}$ satisfies the multiplicity bound for all polynomials. The multiplicities are given as the sign changes along the edges of the Newton polytope defined by the push-forward polynomial over $\mathbb{T}$, see Section 4.1.4 for a detailed description. For all elements $X \in (\{\pm 1\} \times \mathbb{R}) \cup \{\infty\}$, $X^2 = (1, c)$ for some $c \in \mathbb{R}$ will have a positive first component. This can be used to see that there will be no sign changes on the corresponding Newton polytope, and hence has an empty variety.

*Remark* 2.4.14. The tropical hyperfield $\mathbb{T}$ can be constructed explicitly as a quotient of a Hahn series in the following way.

$$\mathbb{T} \cong \mathbb{C}[[t^{\mathbb{R}}]]/\mathrm{val}_{\mathbb{C}}^{-1}(0).$$

## 2.5 Hyperfield Homomorphisms

This subsection will present a range of examples of hyperfields homomorphisms between the hyperfields defined in Section 2.4. For a detailed representation of the majority of the examples refer to Diagram 1 in [AD19]. This presents them visually, enabling connections between the maps and hyperfields to be solidified. Many of these morphisms can be viewed as quotient maps, if this is the case, both forms will be given to be as comprehensive as possible.

**Example 2.5.1.** Given a hyperfield $\mathbb{H}$, then the trivial map to $\mathbb{K}$, sending every non-zero element to 1 is a hyperfield homomorphism.

$$f : \mathbb{H} \to \mathbb{K}, \quad f(x) = \begin{cases} 1, & \text{if} \quad x \neq 0. \\ 0, & \text{if} \quad x = 0. \end{cases}$$

This can be expressed as a quotient map but utilising the property that $\mathbb{K} \cong \mathbb{H}/\mathbb{H}^{\times}$. Thus,

$$f : \mathbb{H} \to \mathbb{H}/\mathbb{H}^{\times} \cong \mathbb{K}.$$

**Example 2.5.2.** Given the real numbers, then there exists a hyperfield homomorpshim to the hyperfields of signs, which takes each real number to its sign.

$$sgn : \mathbb{R} \to \mathbb{S}, \quad sgn(x) = \begin{cases} 1, & \text{if} \quad x \in \mathbb{R}_{>0}. \\ -1, & \text{if} \quad x \in \mathbb{R}_{<0}. \\ 0, & \text{if} \quad x = 0. \end{cases}$$

This can be expressed as a quotient map as follows,

$$\mathbb{R} \to \mathbb{R}/\mathbb{R}_{>0} \cong \mathbb{S}.$$

**Example 2.5.3.** There exists a hyperfield homomorphism from the signed tropical hyperfield to the hyperfield of signs. This maps elements of $\mathbb{TR}$ to their first component.

$$Sgn : \mathbb{TR} \to \mathbb{S}, \quad Sgn((a, b)) = \begin{cases} 1, & \text{if} \quad a = 1. \\ -1, & \text{if} \quad a = -1. \\ 0, & \text{if} \quad x = \infty. \end{cases}$$

The next several examples will be connecting the complex numbers to hyperfields by way of quotient maps.

**Example 2.5.4.** Given the complex numbers, $\mathbb{C}$, then there exists a map to the phase hyperfield, $\mathbb{P}$, which maps each complex number to its argument.

$$ph : \mathbb{C} \to \mathbb{P}, \quad ph(x) = \begin{cases} \frac{x}{|x|}, & \text{if} \quad x \in \mathbb{C}\backslash\{0\}. \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Similarly to the *sgn* map defined above, the *ph* map can be expressed as a quotient map, with $\mathbb{R}_{>0}$ as the quotient subgroup.

$$ph : \mathbb{C} \to \mathbb{C}/\mathbb{R}_{>0} \cong \mathbb{P}.$$

**Example 2.5.5.** Given the complex numbers, $\mathbb{C}$, then there exists a map to the triangle hyperfield, $\triangle$, which maps each complex number to its absolute value.

$$|\cdot| : \mathbb{C} \to \triangle, \quad |z| = |x + iy| = (x^2 + y^2)^{\frac{1}{2}}.$$

This map can be presented as a quotient map, with the unit circle $S^1$ as the quotient subgroup.

$$|\cdot| : \mathbb{C} \to \mathbb{C}/S^1 \cong \triangle.$$

**Example 2.5.6.** Given the real numbers, $\mathbb{R}$, then there exists a map to the real tropical hyperfield, $\mathbb{TR}$, which sends each real number to the pair representing the sign and the original number.

**Example 2.5.7.** Given the phase hyperfield, $\mathbb{P}$, then there is the identity map which sends elements in to the tropical phase hyperfield $\mathbf{\Phi}$.

**Example 2.5.8.** The standard tropicalisation map, from a valued field to the tropical semiring, $trop : K \to \overline{\mathbb{R}}$ is semi-field homomorphism. The tropical hyperfield $\mathbb{T}$ is a hyperfield analogue of the idempotent semiring structure that has been studied in [MS15] and [Vir11]. It is mentioned in the definition of the tropical hyperfield that it can be defined using the *min* convention. When taking this view, $\overline{\mathbb{R}}$ and $\mathbb{T}$ have the same underlying set. As one is a hyperfield and one is a semi-field they can be connected by the set-theoretic identity map. Given a valued field $K$ then the map $trop : K \to \mathbb{T}$, is a hyperfield homomorphism. This can be presented naturally as a quotient map, in the form

$$trop : K \to K/trop^{-1}(0) \cong \mathbb{T}$$

**Example 2.5.9.** Given the tropical complex hyperfield, then there exists a hyperfield homomorphism to the tropical phase hyperfield. This acts in an analogous way to the map $ph : \mathbb{C} \to \mathbb{P}$ as it sends the underlying complex numbers to their arguments, but preserves the hyperfield structure in the way a homomorphsim is defined.

$$ph : \mathbb{TC} \to \mathbf{\Phi}, \quad ph(x) = \begin{cases} \frac{x}{|x|}, & \text{if} \quad x \in \mathbb{C}\backslash\{0\}. \\ 0, & \text{if} \quad x = 0. \end{cases}$$

**Example 2.5.10.** There exists a hyperfield homomorphism between the tropical complex hyperfield and the tropical hyperfield. The map is denoted $\eta : \mathbb{TC} \to \mathbb{T}$, and is defined as:

$$\eta(z) := \log(|z|).$$

There are more details of how this map is constructed outlined in [AD19, Figure 1], and consequent discussion in part 9 of the hyperfield exmaples of [AD19] where the map is discussed in two parts.

*Remark* 2.5.11. The map $\eta : \mathbb{TC} \to \mathbb{T}$ will be explored in detail in Section 3. There will be a demonstration that $\eta$ satisfies a relative algebraic closure property in Theorem 3.1.12, where relative algebraic closure is defined in Definition 3.1.3. This property will be the base to proving a generalised version of Kapranov's theorem for tropical geometry over hyperfields in Theorem 3.2.5.

# Chapter 3

# Generalising Kapranov's Theorem

Kapranov's theorem is a foundational result in tropical geometry [MS15, Theorem 3.1.3]. It states that the set of tropicalisations of points on a hypersurface coincides precisely with the tropical variety of the tropicalisation of the defining polynomial. The aim of this section is to generalise Kapranov's theorem, replacing the role of a valuation, $trop : K \to \overline{\mathbb{R}}$, with a more general class of hyperfield homomorphisms, $\mathbb{H} \to \mathbb{T}$, which satisfy a *relatively algebraically closed* (RAC) condition.

The RAC property is precisely stated in Definition 3.1.3, but intuitively the RAC property states that when a univariate polynomial is pushed forward through a hyperfield homomorphism, the roots of the polynomial can be lifted back to roots of the original polynomial. The existence of a hyperfield homomorphism which has this property is demonstrated by the map $\eta = \log(|\cdot|) : \mathbb{TC} \to \mathbb{T}$, in Theorem 3.1.12. The main result of this section is a hyperfield version of Kapranov's theorem for RAC hyperfield homomorphism which map to $\mathbb{T}$.

A natural question regarding the RAC property of hyperfield homomorphisms is: can we give tractable sufficient conditions to guarantee that a map between hyperfields is RAC? This question is explored in Section 3.3 and partially answered, stating sufficient conditions for a specific class of hyperfield homomorphisms to be RAC.

The following results are presented in the author's recent paper [Max21].

## 3.1 Relative Algebraic Closure

The first step is to formalise how to combine the structure of polynomials defined over hyperfields with hyperfield homomorphisms. Below is the description of the induced map of polynomials over hyperfields.

**Definition 3.1.1.** Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be a hyperfield homomorphism. This induces a map from polynomials with coefficients in $\mathbb{H}_1$ to polynomials with coefficients in $\mathbb{H}_2$. This map is denoted $f_* : \mathbb{H}_1[X_1, \dots, X_n] \to \mathbb{H}_2[X_1, \dots, X_n]$, and is defined for $p(X_1, \dots, X_n) = \boxplus_I c_I \underline{X}^I \in \mathbb{H}_1[X_1, \dots, X_n]$ as;

$$f_*(p) = \boxplus_I f(c_I)\underline{X}^I \in \mathbb{H}_2[X_1, \dots, X_n].$$

(Note: the hyper-operations are now the operations over $\mathbb{H}_2$, and $f_*(p)$ will be called the push-forward of $p$.)

**Example 3.1.2.** Take the polynomial $p(X) = 4X^2 - 5X + 1 \in \mathbb{R}[X]$, the hyperfield homomorphism $sgn : \mathbb{R} \to \mathbb{S}$ induces the map $sgn_* : \mathbb{R}[X] \to \mathbb{S}[X]$, which gives $sgn_*(p)(X) = X^2 \boxplus -X \boxplus 1 \in \mathbb{S}[X]$.

Note that roots of $p \in \mathbb{H}_1[X]$ push-forward to roots of $f_*(p) \in \mathbb{H}_2[X]$. This is precisely described in Lemma 3.2.2. Next the definition of a *relatively algebraically closed* hyperfield homomorphism is introduced. This property identifies whether a root of polynomial can be pulled back through hyperfield homomorphism. This notion is presented for univariate polynomials below.

**Definition 3.1.3.** Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be a hyperfield homomorphism, with induced map $f_* : \mathbb{H}_1[X] \to \mathbb{H}_2[X]$. We say that $f$ is *relatively algebraically closed* (RAC) if for all univariate polynomials $f_*(p) \in \mathbb{H}_2[X]$ and every root $b \in V(f_*(p))$, there exists $a \in f^{-1}(b)$ such that $a \in V(p)$.

The map $f : \mathbb{H}_1 \to \mathbb{H}_2$ being RAC has the immediate consequence that $V(f_*(p)) \subseteq f(V(p))$ for all $p(X) \in \mathbb{H}_1[X]$. Then combining with Lemma 3.2.2, $V(f_*(p)) = f(V(p))$. The following example describes a RAC map which is the main motivating example for this chapter. It is the map that is studied in tropical geometry, and is the basis for the generalisation to hyperfield homomorpshisms.

**Example 3.1.4.** Let $K$ be an algebraically closed field with surjective valuation, $trop : K \to \mathbb{T}$, then this is a RAC hyperfield homomorphism. This is the underlying structure investigated when discussing valuations and tropicalisation maps in relation to tropical geometry. See Section 2.1 and Theorem 3.1.3 in [MS15] for a more detailed description.

The next collection of examples demonstrate that hyperfield homomorphisms are in general not *RAC*.

**Example 3.1.5.** Take the quotient map $sgn : \mathbb{R} \to \mathbb{S}$, and the polynomial $p(X) = X^2 - X + 1 \in \mathbb{R}[X]$. Then, $sgn_*(p) = X^2 \boxplus -X \boxplus 1 \in \mathbb{S}[X]$. The polynomial $p(X)$ has an empty variety, whereas $sgn_*(p)(1) = 1 \boxplus -1 \boxplus 1 = \mathbb{S}$, so $1 \in V(sgn_*(p))$. This demonstrates that the map $\mathbb{R} \to \mathbb{S}$ is not a RAC map.

**Example 3.1.6.** Take the map $ph : \mathbb{C} \to \mathbb{P}$, and the polynomial $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$. It is shown in [BL18a, Remark 1.10], that $ph_*(p)(X)$ has a root at each $a = e^{i\theta}$ for all $\pi/2 < \theta < 3\pi/2$. Not every element of this set can be lifted to roots of $p(X)$ as the variety of $p(X)$ is the set $\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, for instance there is not lift of $-1 \in \mathbb{P}$. Hence, $ph : \mathbb{C} \to \mathbb{P}$ is not a RAC map.

**Example 3.1.7.** Take the quotient map $|\cdot| : \mathbb{C} \to \triangle$, and again the polynomial $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$, then the push-forward is $|p|_* = X^2 \boxplus X \boxplus 1 \in \triangle[X]$. The variety of $p(X)$ is again the set $\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, whereas the variety of $|p|$ is the interval $[\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}]$, as shown in Example 2.4.10. Thus, $x \in [\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}] \Big\backslash \{|\frac{-1+i\sqrt{3}}{2}|, |\frac{-1-i\sqrt{3}}{2}|\}$ has no lift to a root of the original polynomial $p(X)$. Note there will be an exploration into a precise description of how $V(|p|_*)$ is calculated in Section 4.2.

**Proposition 3.1.8.** *The composition of two* RAC *hyperfields homomorphism is* RAC.

*Proof.* Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ and $g : \mathbb{H}_2 \to \mathbb{H}_3$ be *RAC* maps, then $h(x) = (f \circ g)(x) = f(g(x)) : \mathbb{H}_1 \to \mathbb{H}_3$. Let $p(X) \in \mathbb{H}_1[X]$, with the image under the induced polynomial composition map as $h_*(p) = f_*(g_*(p)) \in \mathbb{H}_3[X]$. As $f : \mathbb{H}_1 \to \mathbb{H}_2$ is *RAC*, this implies that $f(V(p)) = V(f_*(p))$, and similarly with $g : \mathbb{H}_2 \to \mathbb{H}_3$ being *RAC* this imples $g(V(p)) = V(g_*(p))$. Combining these two facts gives,

$$\begin{aligned} h(V(p)) &= f(g(V(p))) \\ &= f(V(g_*(p))) \\ &= V(f_*(g_*(p))) \\ &= V(h_*(p)). \end{aligned}$$

This is precisely equivalent to the map $h : \mathbb{H}_1 \to \mathbb{H}_3$ being *RAC*. □

The remainder of the section will be focussed on demonstrating that the map $\eta : \mathbb{TC} \to \mathbb{T}$ satisfies the RAC property.

**Proposition 3.1.9.** *Let $p(X) = \boxplus_{i=0}^{n} c_i \odot X^i \in \mathbb{TC}[X]$. An element $a \in \mathbb{TC}$ is a root of $p(X)$ if there exists $j_1, \dots, j_m \in \{1, \dots, n\}$ such that*

$$|c_{j_1} a^{j_1}| = \cdots = |c_{j_m} a^{j_m}| > |c_i a^i| \quad \forall i \notin \{j_1, \dots, j_m\} \quad \textit{and} \quad -c_0 \in \bigboxplus_{k=1}^{m} c_{j_k} \odot a^{j_k}.$$

*Proof.* By the definition of the hyperaddition over $\mathbb{TC}$ the monomial terms with the largest absolute value contribute to the hypersum when the polynomial $p$ is evaluated at $a$. Thus, by the hypothesis,

$$\begin{aligned} p(a) &= \bigboxplus_{i=0}^{n} c_i \odot a^i \\ &= \bigboxplus_{k=1}^{m} c_{j_k} \odot a^{j_k} \boxplus c_0, \end{aligned}$$

so, $\mathbb{0} \in p(a)$ if and only if $-c_0 \in \boxplus_{k=1}^{m} c_{j_k} \odot a^{j_k}$. □

**Proposition 3.1.10.** *Given $p(X) \in \mathbb{T}[X]$, an element $a \in \mathbb{T}$ is a root of $p$ if the maximum of $p(a)$ is achieved more than once.*

This is an analogue of the characterisation of roots over the tropical semi-ring. Focusing on the hyperfield homomorphism $\eta : \mathbb{TC} \to \mathbb{T}$, it can be seen that using the following lemma, $\eta$ is a RAC map.

**Lemma 3.1.11.** *Given $a, b \in \mathbb{T}$, if $a > b$, then for all $\alpha \in \eta^{-1}(a)$ and $\beta \in \eta^{-1}(b)$, it holds that $|\alpha| > |\beta|$.*

*Proof.* Taking $a, b \in \mathbb{T}$ such that $a > b$, if $\alpha \in \eta^{-1}(a)$, and $\beta \in \eta^{-1}(b)$ then $\log(|\alpha|) = a > b = \log(|\beta|)$. As both the logarithm and exponential functions preserve order, $\log(|\alpha|) > \log(|\beta|) \Rightarrow |\alpha| > |\beta|$, as required. □

**Theorem 3.1.12.** *The hyperfield homomorphism $\eta : \mathbb{TC} \to \mathbb{T}$ is RAC.*

*Proof.* A polynomial $p = \boxplus_{i=0}^{n} c_i \odot X^i \in \mathbb{TC}[X]$ has push-forward $q = \eta_*(p) = \boxplus_{i=0}^{n} \eta(c_i) \odot X^i \in \mathbb{T}[X]$. By Proposition 3.1.10, $a \in V(q)$ if and only if there are two or more monomials terms that are equal and great than or equal than the other monomial terms in $q(a)$. For this to occur there must exist $k_1, \ldots, k_m \in \{1, \ldots, n\}$ such that, $\eta(c_{k_1}) \odot a^{k_1} = \ldots = \eta(c_{k_m}) \odot a^{k_m} > \eta(c_i) \odot a^i$ for all $i \in \{1, \ldots, n\} \backslash \{k_1, \ldots, k_m\}$. Explicitly, the maximum is achieved more than once in the terms $\eta(c_1) \odot a^1, \ldots, \eta(c_n) \odot a^n$, and this maximum is greater than or equal to $\eta(c_0)$.
As $m \geqslant 2$, this allows for $t, t' \in \{k_1, \ldots, k_m\}$ to be chosen to construct a lift of $a$ as follows:

$$\widetilde{a} = \left(\frac{-c_{t'}}{c_t}\right)^{\frac{1}{t-t'}}.$$

To confirm that $\tilde{a}$ is a lift of $a$, apply the map $\eta$ to $\tilde{a}$ to give

$$\eta(\tilde{a}) = \left(\eta(-c_{t'}) \odot (\eta(c_t))^{-1}\right)^{\frac{1}{t-t'}}.$$

Furthermore, observe that $\eta(c_t) \odot a^t = \eta(c_{t'}) \odot a^{t'}$. Then,

$$\begin{aligned}\eta(c_t) \odot a^t = \eta(c_{t'}) \odot a^{t'} &\Rightarrow a^{t-t'} = \eta(c_{t'}) \odot (\eta(c_t))^{-1} \\ &\Rightarrow a = \left(\eta(c_{t'}) \odot (\eta(c_t))^{-1}\right)^{\frac{1}{t-t'}}.\end{aligned}$$

As $\eta(x) = \log(|x|) = \log(|-x|) = \eta(-x)$, this shows that $a = \eta(\tilde{a})$ as required.
It remains to shown that $\widetilde{a}$ is a root of the original polynomial $p$. Note that due to Lemma 3.1.11,

$$|c_{k_1}\widetilde{a}^{k_1}| = \ldots = |c_{k_m}\widetilde{a}^{k_m}| > |c_i\widetilde{a}^i|, \quad \text{for all} \quad i \notin \{k_1, \ldots, k_m\}. \tag{3.1.13}$$

Furthermore, observe the relationship between the monomial terms with $t, t'$ exponents,

$$\begin{aligned} c_t\widetilde{a}^t \boxplus c_{t'}\widetilde{a}^{t'} &= c_t\Big(\Big(\frac{-c_{t'}}{c_t}\Big)^{\frac{1}{t-t'}}\Big)^t \boxplus c_{t'}\Big(\Big(\frac{-c_{t'}}{c_t}\Big)^{\frac{1}{t-t'}}\Big)^{t'} \\ &= (-1)^{\frac{t}{t-t'}}(c_t)^{\frac{-t'}{t-t'}}(c_{t'})^{\frac{t}{t-t'}} \boxplus (-1)^{\frac{t'}{t-t'}}(c_t)^{\frac{-t'}{t-t'}}(c_{t'})^{\frac{t}{t-t'}} \\ &= (-1)(-1)^{\frac{t'}{t-t'}}(c_t)^{\frac{-t'}{t-t'}}(c_{t'})^{\frac{t}{t-t'}} \boxplus (-1)^{\frac{t'}{t-t'}}(c_t)^{\frac{-t'}{t-t'}}(c_{t'})^{\frac{t}{t-t'}} \\ &= \{z \in \mathbb{C} \,:\, |z| \leqslant R\}, \quad \text{where} \quad R = |c_t\widetilde{a}^t| \geqslant |c_0|. \end{aligned}$$

Then, $\boxplus_{j=1}^m c_{k_j} \odot \widetilde{a}^{k_j} \ni -c_0$, which combined with (3.1.13) and Proposition 3.1.9 gives that $\tilde{a} \in V(p)$. □

**Corollary 3.1.14.** *The tropical complex hyperfield* $\mathbb{TC}$ *is algebraically closed. (Recalling that here for a hyperfield, algebraically closed is explicitly defined as the hyperfield having a root for every univariate polynomial.)*

*Proof.* As a consequence of the results on roots and multiplicities in [BL18a], $\mathbb{T}$ is algebraically closed. As the map $\eta : \mathbb{TC} \to \mathbb{T}$ is RAC and surjective, every polynomial $p(X) \in \mathbb{TC}[X]$ has a corresponding polynomial $\eta_*(p)(X) \in \mathbb{T}[X]$, which has a root and can be lifted back to a root of $p(X)$. Hence, every polynomial in $\mathbb{TC}[X]$ has a root. □

**Example 3.1.15.** Take the polynomial $p(X) = iX^2 \boxplus \left(\frac{-1+i\sqrt{3}}{2}\right)X \boxplus -1 \in \mathbb{TC}[X]$, then the push-froward is $\eta_*(p)(X) = 0 \odot X^2 \boxplus 0 \odot X \boxplus 0 \in \mathbb{T}[X]$. It can be seen that $0 \in V(\eta_*(p))$. In accordance to the proof of Theorem 3.1.12, take $t = 2$ and $t' = 1$. This gives,

$$c_t = c_2 = i, \quad -c_{t'} = c_1 = \Big(\frac{1 - i\sqrt{3}}{2}\Big), \quad 1/(t - t') = 1$$

Taking the template for the lift,

$$\widetilde{a} = \Big(\frac{-c_{t'}}{c_t}\Big)^{\frac{1}{t-t'}} = \Big(\frac{1 - i\sqrt{3}}{2i}\Big).$$

Now to confirm that this is a pull back of 0 and is a root of $p(X)$.

$$\Big|\Big(\frac{1 - i\sqrt{3}}{2i}\Big)\Big| = \frac{1}{2}|1 - i\sqrt{3}| = 1 \Rightarrow f\Big(\frac{1 - i\sqrt{3}}{2i}\Big) = 0.$$

Finally,

$$p(\widetilde{a}) = p\Big(\frac{1 - i\sqrt{3}}{2i}\Big) = i\Big(\frac{1 - i\sqrt{3}}{2i}\Big)^2 \boxplus \Big(\frac{-1 + i\sqrt{3}}{2}\Big)\Big(\frac{1 - i\sqrt{3}}{2i}\Big) \boxplus -1$$

$$= \left(\frac{i-\sqrt{3}}{2}\right) \boxplus \left(\frac{1+i\sqrt{3}}{2i}\right) \boxplus -1$$
$$= \left(\frac{i-\sqrt{3}}{2}\right) \boxplus -\left(\frac{i-\sqrt{3}}{2}\right) \boxplus -1$$
$$= \{z \in \mathbb{C} \,:\, |z| \leqslant 1\} \boxplus -1 \ni \mathbb{0}.$$

This shows that $\widetilde{a}$ is a lifted root, which demonstrates the application of the structure of the proof of Theorem 3.1.12.

## 3.2 Kapranov's Theorem for Hyperfields

This section will outline a specific generalisation of Kapranov's Theorem over hyperfields. The original theorem by Kapranov (see Theorem 3.1.3 in [MS15]) is a key result in tropical geometry, which leads to the Fundamental Theorem of tropical geometry in [MS15]. A summary version of the original theorem in tropical geometry is stated here.

**Theorem 3.2.1.** *Given an algebraically closed field $K$ with surjective valuation, trop : $K \to \overline{\mathbb{R}}$, then for a Laurent polynomial $p(X) = \sum_{I \in \mathbb{Z}^n} c_I X^I \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$,*

$$V(trop(p)) = trop(V(p)).$$

*(For further details, see [MS15, Theorem 3.1.3]).*

An essential point to recognise is that by replacing the valuation with an arbitrary hyperfield homomorphism, one containment holds automatically.

**Lemma 3.2.2.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be a hyperfield homomorphism. For $p(X_1, \ldots, X_n) = \boxplus_I c_I \odot \underline{X}^I \in \mathbb{H}_1[X_1, \ldots, X_n]$,*

$$f(V(p)) \subseteq V(f_*(p)).$$

*Proof.* By definition $f_*(p)(X_1, \ldots, X_n) = \boxplus_I f(c_I) \odot \underline{X}^I \in \mathbb{H}_2[X_1, \ldots, X_n]$. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{H}_1^n$ be a root of $p(X_1, \ldots, X_n)$, meaning $\boldsymbol{a} \in V(p)$ and

$$\mathbb{0} \in p(\boldsymbol{a}) = p(a_1, \ldots, a_n) = \boxplus_I c_I \odot \boldsymbol{a}^I. \tag{3.2.3}$$

The aim is to demonstrate that $f(\boldsymbol{a}) \in V(f_*(p))$ holds. Firstly,

$$\begin{aligned} f_*(p)(f(\boldsymbol{a})) &= \boxplus_I f(c_I) \odot f(\boldsymbol{a})^I \\ &= \boxplus_I f(c_I \odot \boldsymbol{a}^I) \\ &\supseteq f\Big(\boxplus_I c_I \odot \boldsymbol{a}^I\Big) \\ &\ni f(\mathbb{0}) = \mathbb{0} \end{aligned}$$

The above steps use the properties of a hyperfield homomorphism to give that $\mathbb{0} \in f(p)(f(\boldsymbol{a}))$, yielding $f(\boldsymbol{a}) \in V(f_*(p))$. Hence, proving that $f(V(p)) \subseteq V(f_*(p))$. ☐

For a field $K$ and the classical notion of an ideal $I \subseteq K[X_1, \ldots, X_n]$, Lemma 3.2.2 can be extended. To be explicit, before stating the next result, see the following definitions

$$f_*(I) := \{f_*(p) \,:\, p \in I\},$$

$$V(f_*(I)) := \bigcap_{p \in f_*(I)} V(p).$$

**Lemma 3.2.4.** *Given a polynomial ideal $I \subseteq K[X_1, \ldots, X_n]$, where $K$ is field and $\mathbb{H}$ is a hyperfield such that $f : K \to \mathbb{H}$, is a hyperfield homomorphism. Then, under the induced polynomial map, $f_* : K[X_1, \ldots, X_n] \to \mathbb{H}[X_1, \ldots, X_n]$.*

$$f(V(I)) \subseteq V(f_*(I)).$$

*Proof.* Take $y \in V(I)$, which is equivalent to $y \in V(p)$ for all $p \in I$. For every $p \in I \subseteq K[X_1, \ldots, X_n]$, Lemma 3.2.2 implies that $f(V(p)) \subseteq V(f_*(p))$, hence $f(y) \in V(f_*(p))$ for all $p \in I$. This is exactly stating that $f(y) \in V(f_*(I))$, due to the definition, as required. ☐

The main purpose of this chapter is to specifically generalise Kapranov's Theorem to hyperfield homomorphisms $f : \mathbb{H} \to \mathbb{T}$, which satisfy the RAC property. The following theorem demonstrates that the RAC property for a hyperfield homomorphism $\mathbb{H} \to \mathbb{T}$ can be used to deduce the existence of a pull-back of roots for polynomials in $n$-variables.

**Theorem 3.2.5** (Generalised Kapranov's Theorem)**.** *Given a polynomial* $p \in \mathbb{H}[X_1, \ldots, X_n]$*, then for a RAC hyperfield homomorphism* $f : \mathbb{H} \to \mathbb{T}$*,*

$$V(f_*(p)) = f(V(p)).$$

*Proof.* The inclusion $f(V(p)) \subseteq V(f_*(p))$ is a direct consequence of Lemma 3.2.2.

The inclusion in the reverse direction, $V(f_*(p)) \subseteq f(V(p))$, is more interesting and requires an argument. Take a point $\underline{a} \in V(f_*(p))$, so $\mathbb{0} \in f_*(p)(\underline{a})$. The aim is to demonstrate that there exists an element in $V(p)$ that pushes forward to $\underline{a}$. This will be done by restricting to univariate polynomials and using the property that $f : \mathbb{H} \to \mathbb{T}$ is a RAC map, to find an appropriate lift of $\underline{a}$.

Firstly, choose lifts $\lambda_i \in f^{-1}(a_i)$, where $\underline{a} = (a_1, \ldots, a_n) \in V(f_*(p))$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{H}^n$. The map $f : \mathbb{H} \to \mathbb{T}$ can be used to define the coordinate wise map $F : \mathbb{H}^n \to \mathbb{T}^n$, $F(x_1, \ldots, x_n) := (f(x_1), \ldots, f(x_n))$. For any non-zero $D = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ the map $\varphi(x) := (\lambda_1 \odot x^{d_1}, \ldots, \lambda_n \odot x^{d_n})$ defines an inclusion $\varphi : \mathbb{H}^* \to (\mathbb{H}^*)^n$, where $\mathbb{H}^*$ denotes $\mathbb{H}/\{\mathbb{0}\}$. Then, if the map $\psi : \mathbb{T}^* \to (\mathbb{T}^*)^n$ is defined as

$$\begin{aligned} \psi(x) :&= (f(\lambda_1) \odot x^{d_1}, \ldots, f(\lambda_n) \odot x^{d_n}) \\ &= (f(\lambda_1) + d_1 x, \ldots, f(\lambda_n) + d_n x), \end{aligned}$$

the diagram below is commutative:

$$\begin{tikzcd} \mathbb{H}^* \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{T}^* \arrow[d, "\psi"] \\ (\mathbb{H}^*)^n \arrow[r, "F"'] & (\mathbb{T}^*)^n \end{tikzcd}$$

Note that $\psi(0) = F(\underline{\lambda}) = \underline{a}$. The polynomial $p$ will be pulled back through $\varphi$ to a univariate polynomial, then pushed forward through $f_*$ and it will be shown that this polynomial has a root at 0. The RAC property will be used to lift this root back. The pullback of $p$, denoted $\varphi^* p$, is the univariate polynomial defined by the expression for $p$ where $X_i$ is replaced with $\lambda_i \odot X^{d_i}$. Explicitly,

$$\begin{aligned}\varphi^* p &= \boxplus_I c_I \odot (\lambda_1 \odot X^{d_1})^{i_1} \odot \ldots \odot (\lambda_n \odot X^{d_n})^{i_n}, \\ &= \boxplus_I c_I \odot \underline{\lambda}^I \odot X^{D \cdot I} \in \mathbb{H}[X].\end{aligned}$$

The pullback polynomial $\varphi^* p$ is pushed forward to $f_*(\varphi^* p) \in \mathbb{T}[X]$. The image of $f_*(\varphi^* p)$ is equal to the image of $f_*(p)$ when restricted to $\psi$.

$$\begin{aligned}f_*(p)(\psi(X)) &= f_*(p)\Big(f(\lambda_1) \odot X^{d_1}, \ldots, f(\lambda_n) \odot X^{d_n}\Big) \\ &= \boxplus_I f(c_I) \odot f(\underline{\lambda})^I \odot X^{D \cdot I} \\ &= f_*(\varphi^* p)(X).\end{aligned}$$

The next step is to show that $f_*(\varphi^* p)$ has a root at $0 \in \mathbb{T}$. This can be seen as,

$$f_*(\varphi^* p)(0) = \boxplus_I f(c_I) \odot f(\underline{\lambda})^I \odot 0^{D \cdot I}.$$

Then, due to the arithmetic over $\mathbb{T}$,

$$\begin{aligned}\boxplus_I f(c_I) \odot f(\underline{\lambda})^I \odot 0^{D \cdot I} &= \boxplus_I f(c_I) \odot f(\underline{\lambda})^I \\ &= \boxplus_I f(c_I) \odot \underline{a}^I \\ &= f_*(p)(\underline{a}) \ni \mathbb{0}. \end{aligned} \tag{3.2.6}$$

This shows that $0 \in V(f_*(\varphi^* p))$. Then, the property that the map $f : \mathbb{H} \to \mathbb{T}$ is a RAC homomorphism gives that there exists an element $\tilde{a} \in \mathbb{H}$ such that $f(\tilde{a}) = 0$ and $\tilde{a} \in V(\varphi^* p)$.

Furthermore, to ensure that $\tilde{a}$ can be pushed forward to a root of $p$ the tuple $D \in \mathbb{Z}^n$ must be chosen with the following property. Choose $D \in \mathbb{Z}^n$ such that the dot products of $D$ taken with exponent vectors of monomial terms of $p = \boxplus_I c_I \odot \underline{X}^I$, are all distinct. Explicitly,

$$D \cdot J = d_1 \cdot j_1 + \cdots + d_n \cdot j_n \neq d_1 \cdot j_1' + \cdots + d_n \cdot j_n' = D \cdot J',$$

for all pairs $J, J' \in I$. The condition $D \cdot J \neq D \cdot J'$ can be interpreted as $D$ not lying on the hyperplane defined by $\underline{X} \cdot (J - J')$. Therefore, it is possible to pick such a $D \in \mathbb{Z}^n$, as the number of possible pairs $J, J'$ is finite and $\mathbb{Z}^n$ can not be covered by a finite union of hyperplanes.

The pullback $\varphi^* p$ utilises the requirement imposed on the tuple $D = (d_1, \ldots, d_n)$. If the requirement was not imposed, this would allow multiple monomials in the restricted polynomial to have equal exponents. This would lead to the corresponding coefficient being a hypersum, thus the potentially detrimental possibility of the restriction becoming a set of polynomials rather than a single polynomial, which is what is needed here.

This condition on $D$ implies that the element $\widetilde{a}$ can then be pushed forward through $\varphi$ to give an element $\varphi(\widetilde{a}) \in V(p)$. As the diagram commutes, this shows that the $F(\varphi(\widetilde{a})) = \psi(f(\widetilde{a})) = \psi(0) = \underline{a}$, which is sufficient to show that for every element of $V(f_*(p))$ there is a lift to an element of $V(p)$. This demonstrates that $V(f_*(p)) \subseteq f(V(p))$, giving the desired result.

Hence, it has been shown that both $f(V(p)) \subseteq V(f_*(p))$ and $V(f_*(p)) \subseteq f(V(p))$ hold. These taken together demonstrate the required equality, $V(f_*(p)) = f(V(p))$.
(The structure of the proof, restricting to univariate polynomials to use the RAC property is based on the argument presented in a proof of the original theorem in tropical geometry, as seen in [Bog15]. Therefore, this adapted proof along with Example 3.1.4 gives a proof which encompasses that of the original Kapranov's Theorem ). □

In particular, since $\eta : \mathbb{TC} \to \mathbb{T}$ is RAC, the above theorem applies to it.

## 3.3 Characterising RAC Maps.

This section aims to present sufficient conditions for a hyperfield homomorphism to be a RAC map. This will not include a complete description of the necessary conditions for a hyperfield homomorphism to be a RAC; this remains an open topic. To begin the exploration into RAC hyperfield homomorphisms it is necessary to recall the definition linking multiplicities of roots to the degree of the polynomial.

Definition 2.4.2 states that a hyperfield is said to satisfy the *multiplicity bound* if for all univariate polynomials, $p(X) \in \mathbb{H}[X]$,

$$\sum_{a \in \mathbb{H}} \mathrm{mult}_a(p) \leqslant \deg(p).$$

Furthermore, a hyperfield is said to satisfy *multiplicity equality* if the above inequality is an equality for all univariate polynomials.

Proposition B from [BL18a] describes a relationship between the multiplicities of roots over a field and the multiplicities of the push-forwards of these roots over a hyperfield. It is stated below, in notation consistent with this work.

**Proposition 3.3.1** ([BL18a], Prop. B)**.** *Let $K$ be a field and $\mathbb{H}$ a hyperfield with hyperfield homomorphism $f : K \to \mathbb{H}$. Let $p(X) \in K[X]$, with push-forward $f_*(p)(X) \in \mathbb{H}[X]$. Then,*

$$\mathrm{mult}_b(f_*(p)) \geqslant \sum_{a \in f^{-1}(b)} \mathrm{mult}_a(p)$$

*for all $b \in \mathbb{H}$. Moreover, if $\mathbb{H}$ satisfies the multiplicity bound for the polynomial $f_*(p)$ and $p \in K[X]$ splits into linear factors then,*

$$\mathrm{mult}_b(f_*(p)) = \sum_{a \in f^{-1}(b)} \mathrm{mult}_a(p).$$

**Lemma 3.3.2.** *Let $f : K \to \mathbb{H}$ be a hyperfield homomorphism, with $\mathbb{H}$ satisfying the multiplicity bound. If the polynomial $p \in K[X]$ splits into linear factors, then $V(f_*(p)) = f(V(p))$.*

*Proof.* Take the polynomial $p \in K[X]$, $f_*(p) \in \mathbb{H}[X]$. Take $b \in V(f_*(p))$, so $\mathbb{0} \in f_*(p)(b)$ and hence $\mathrm{mult}_b(f_*(p)) > 0$. As $p$ splits into linear factors over $K$ and $\mathbb{H}$ satisfies the multiplicity bound, by Proposition 3.3.1 we have,

$$0 < \mathrm{mult}_b(f_*(p)) = \sum_{a \in f^{-1}(b)} \mathrm{mult}_a(p).$$

Therefore, there exists $\hat{a} \in f^{-1}(b)$ such that $\mathrm{mult}_{\hat{a}}(p) > 0$. Thus, $\hat{a}$ is a root of $p \in K[X]$. This demonstrates that given an element $b \in V(f_*(p))$ there exists an element $\hat{a}$, such that $f(\hat{a}) = b$ and $p(\hat{a}) = 0$. This shows that $V(f_*(p)) \subseteq f(V(p))$. Then the reverse inslcusion is demonstrated by Lemma 3.2.2. □

The above lemma provides a sufficient condition for a hyperfield homomorphism from a field to a hyperfield to be a RAC map. The next stage is to investigate whether this view can be extended to maps $f : \mathbb{H}_1 \to \mathbb{H}_2$. Proposition 3.3.1 is the key tool used in the proof of Lemma 3.3.2. The following discussion aims to explore a generalisation of this result for maps $f : \mathbb{H}_1 \to \mathbb{H}_2$. Firstly, it is important to recognise that this is not a simple generalisation and the property from Prop. B in [BL18a] does not always hold in this less restrictive setting.

**Example 3.3.3.** Take the hyperfield homomorphism $f : \mathbb{P} \to \mathbb{K}$. Note that, over the Krasner hyperfield $\mathbb{K}$, the multiplicity bound achieves equality for all polynomials: $\sum_{b \in \mathbb{K}} \mathrm{mult}_b(p) = \deg(p)$ for all $p \in \mathbb{K}[X]$, (see Remark 1.11 in [BL18a] for details). Take the polynomial $p(X) = X^2 \boxplus X \boxplus 1 \in \mathbb{P}[X]$, then due to the argument presented in [BL18a] in Remark 1.10, $\sum_{a \in \mathbb{P}} \mathrm{mult}_a(p) = \infty$, in particular $a = e^{i\theta}$ is a root of $p$ for all $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Now the push-forward coefficients of the polynomial are unchanged, $f_*(p)(X) = X^2 \boxplus X \boxplus 1$, with $\sum_{b \in \mathbb{K}} \mathrm{mult}_b(f(p)) = 2$. This then leads to,

$$\infty = \sum_{a \in \mathbb{P} \setminus \{0\}} \mathrm{mult}_a(p) = \sum_{a \in f^{-1}(1)} \mathrm{mult}_a(p) \nleqslant \mathrm{mult}_1(f(p)) = 2.$$

This demonstrates that the property does not hold in total generality over all hyperfield homomorphisms.

There are several key properties of fields that underpin the result in Proposition 3.3.1. These properties do not automatically hold for hyperfields. The first key property is that all fields satisfy the multiplicity bound. The second is regarding the factorisation process of polynomials. Restricting to hyperfields with the multiplicity bound is a solution to half of the problem, whereas the factorisation property needs to be discussed in further detail.

If $K$ is a field, then $K[X]$ is a unique factorisation domain, and if $K$ is algebraically closed then and polynomial $p$ factors into a product of linear factors. It is not obvious that these properties extend to hyperfields. This is due to the non-uniqueness of the choice of factorisation, even for more well-behaved hyperfields, such as those with the doubly distributive property (see [BS20] for a description of doubly distributive hyperfields). It could occur that for two distinct roots, the maximum multiplicity is achieved with different factorisations. In an attempt to overcome this the next definition is introduced.

**Definition 3.3.4.** A polynomial $p \in \mathbb{H}[X]$ is said to have the *inheritance* property if given the list of its roots $\{a_1 \ldots a_k\}$, inclusive of repetitions corresponding to multiplicities, then for a subset $\{a_{j_1} \ldots a_{j_m}\} \subseteq \{a_1 \ldots a_k\}$, such that $m \leqslant \deg(p)$ there exists $q \in \mathbb{H}[X]$ such that

$$p \in (X - a_{j_1}) \odot (X - a_{j_2}) \odot \cdots \odot (X - a_{j_m}) \odot q.$$

A hyperfield $\mathbb{H}$ is said to have the *inheritance* property if every polynomial $p \in \mathbb{H}[X]$ satisfies the inheritance property.

The results in this work do not extend to fully characterising the multiplicity bound or inheritance properties, but rather opens this area up for exploration. The following conjectures are based on the current knowledge of doubly distributive hyperfields, specifically including $\mathbb{K}, \mathbb{S}$ and $\mathbb{T}$.

**Conjecture 3.3.5.** *All hyperfields with the doubly distributive property satisfies the multiplicity bound. (This is shown to hold for polynomials of degree up to three in Section 3.5.)*

**Conjecture 3.3.6.** *All hyperfields with the doubly distributive property satisfies the inheritance property.*

There will now be an demonstration of the implications of the multiplicity bound and the inheritance property.

**Lemma 3.3.7.** *Given the hyperfield homomorphism $f : \mathbb{H}_1 \to \mathbb{H}_2$, then for all $p(X) = \boxplus_{i=0}^{n} c_i \odot X^i \in \mathbb{H}_1[X]$,*

$$\operatorname{mult}_b(f(p)) \geqslant \sum_{a \in f^{-1}(b)} \operatorname{mult}_a(p) \tag{3.3.8}$$

*holds if $\mathbb{H}_1$ satisfies the multiplicity bound and inheritance property.*

*Proof.* As $\mathbb{H}_1$ satisfies the multiplicity bound the list of roots, inclusive of multiplicities, $\{a_1 \ldots a_k\}$ is a finite set with $k \leqslant \deg(p)$. Take the subset $\{a_{j_1} \ldots a_{j_m}\} \subseteq \{a_1 \ldots a_k\}$, such that $f(a_{j_1}) = \cdots = f(a_{j_m}) = b$. These are the only elements of $\mathbb{H}_1$ that are roots of $p$ and push-forward to $b$. By the inheritance property, there exists a $q \in \mathbb{H}_1[X]$, such that,

$$p \in (X - a_{j_1}) \odot \cdots \odot (X - a_{j_m}) \odot q.$$

By the hyperfield homomorphism properties it can be seen that under $f_*$,

$$\begin{aligned} f_*(p) &\in (X - f(a_{j_1})) \odot \cdots \odot (X - f(a_{j_m})) \odot f_*(q) \\ &\in (X - b) \odot \cdots \odot (X - b) \odot f_*(q). \end{aligned}$$

This gives, $\operatorname{mult}_b(f_*(p)) \geqslant m$, implying that,

$$\operatorname{mult}_b(f_*(p)) \geqslant \sum_{a \in f^{-1}(b)} \operatorname{mult}_a(p).$$

□

**Theorem 3.3.9.** *Given $\mathbb{H}_1$ which statisfies the multiplicity equality and has the inheritance property, and $\mathbb{H}_2$ that satisifes the multiplicity bound, then a surjective homomorphism $f : \mathbb{H}_1 \to \mathbb{H}_2$ is a RAC map.*

*Proof.* The hyperfield $\mathbb{H}_1$ satisfying the multiplicity equality and the inheritance property implies by Lemma 3.3.7 that, $\operatorname{mult}_b(f_*(p)) \geqslant \sum_{a \in f^{-1}(b)} \operatorname{mult}_a(p)$ holds. These together then imply that $\operatorname{mult}_b(f_*(p)) = \sum_{a \in f^{-1}(b)} \operatorname{mult}_a(p)$, by the following logic;

$$\deg(p) = \sum_{a \in \mathbb{H}_1} \operatorname{mult}_a(p) \leqslant \sum_{b \in \mathbb{H}_2} \operatorname{mult}_b(f(p)) \leqslant \deg(f_*(p)) = \deg(p).$$

Then finally, $\operatorname{mult}_b(f_*(p)) = \sum_{a \in f^{-1}(b)} \operatorname{mult}_a(p)$ gives that the map $f : \mathbb{H}_1 \to \mathbb{H}_2$ is RAC, using analogous logic to the proof of Lemma 3.3.2. □

*Remark* 3.3.10. Theorem 3.3.9 demonstrates that there are sufficient conditions that can be given to both hyperfields to give the corresponding homomorphism as RAC, although this does not classify all RAC maps. It does incorporate the motivating example for the the paper, $trop : K \to \overline{\mathbb{R}}$. Although, it can be seen that $\mathbb{TC}$ does not satisfy the multiplicity bound, and hence does not fulfil the conditions of Theorem 3.3.9. This demonstrates the theoretical complexity in attempting to outline the conditions for a hyperfield homomorphism to be RAC.

## 3.4 Multiplicity Bound

When dealing with polynomials over a field, the number of roots, counting multiplicity, is bounded by the degree of the polynomial. This is not the case when discussing roots and multiplicities for polynomials defined over a hyperfield. There are two "pathological" examples in [BL18a], (see Remarks 1.9 and 1.10), which demonstrate the possibly unbounded nature of the sum of multiplicities for polynomials over hyperfields. The purpose of this section is to characterise the sufficient properties of hyperfields for which the roots and multiplicities are bounded by the degree of the polynomial. This classification is in part motivated by Theorem 3.3.9. If there is a classification of hyperfields with the multiplicity bound, then this contributes to the understanding of RAC hyperfield homomorphisms.

From Definition 2.4.2 it can be recalled that a hyperfield $\mathbb{H}$ is said to satisfies the *multiplicity bound* if for every polynomial $p \in \mathbb{H}[X]$, the following bound on the roots of the polynomial holds,

$$\sum_{b \in \mathbb{H}} \mathrm{mult}_b(p) \leqslant \deg(p).$$

For this section it will be presumed that the polynomials are monic. This does not change the variety of the polynomial. Recalling the addition and multiplication of polynomials over hyperfields in Example 2.3.4, the restriction to monic polynomials can be made due to the Lemma 2.1 in [Liu19], which states that for every polynomial $p \in \mathbb{H}[X]$ there exists a monic polynomial $q \in \mathbb{H}[X]$ such that $p = a \odot q(x)$ where $a \in \mathbb{H}$.

The varieties of $p(x)$ and $q(x)$ are equal, by the following inclusions:

$$\mathbb{0} \in p(b) = a \odot q(b)$$

$$\mathbb{0} \in q(b) \Rightarrow 0 \in a \odot q(b) = p(b).$$

**Proposition 3.4.1.** *A degree one polynomial over $\mathbb{H}$ has $\sum_{b \in \mathbb{H}} \operatorname{mult}_b(p) = 1$.*

*Proof.* A monic degree one polynomial over $\mathbb{H}$ is in the form $x \boxplus a$. By the hyperfield axioms there exists a unique element $-a \in \mathbb{H}$ such that $-a \boxplus a \ni \mathbb{0}$. Thus, the polynomial has one root and the multiplicity is automatically one due to the recursive nature of the definition. □

**Lemma 3.4.2.** *Let $\mathbb{H}$ be a doubly distributive hyperfield, with polynomials $p_1, p_2, p_3 \in \mathbb{H}[X]$. If $p_1 \in p_2 \odot p_3$, then for every element of the hyperfield $\alpha \in \mathbb{H}$, the following inclusion occurs,*

$$p_1(\alpha) \subseteq p_2(\alpha) \odot p_3(\alpha).$$

*Proof.* The multiplication of polynomials over hyperfields is actually multi-valued, so observing the structure of $p_2 \odot p_3$ first is important.

$$p_2(X) \odot p_3(X) = \left( \boxplus_{j=0}^{n} b_j X^j \right) \left( \boxplus_{k=0}^{m} c_k X^k \right)$$

Then due to the doubly distributive property of $\mathbb{H}$,

$$\left( \boxplus_{j=0}^{n} b_j X^j \right) \left( \boxplus_{k=0}^{m} c_k x^k \right) = \boxplus_{i,j} b_j X^j c_k X^k = \boxplus_{i,j} b_j c_k X^{j+k}.$$

Define the polynomial as $p(x) = \boxplus_{i=0}^{r} a_i X^i$. The statement $p \in p_2 \odot p_3$ is taken to mean, $a_i \in \boxplus_{i=j+k} b_j c_k$. Now take $\alpha \in \mathbb{H}$ and evaluate $p_2 \odot p_3$ at $\alpha$.

$$\begin{aligned} p_2(\alpha) \cdot p_3(\alpha) &= \left( \boxplus_{j=0}^{n} b_j \alpha^j \right) \left( \boxplus_{k=0}^{m} c_k \alpha^k \right) \\ &= \boxplus_{j,k} b_j \alpha^j c_k \alpha^k \\ &= \boxplus_{j,k} b_j c_k \alpha^j \alpha^k \end{aligned}$$

$$
\begin{aligned}
&= \boxplus_{i} \Big( \boxplus_{i=j+k} b_j c_k \Big) \alpha^i \\
&\ni \boxplus_{i} a_i \alpha^i \\
&= p(\alpha)
\end{aligned}
$$

This demonstrates that $p(\alpha) \subseteq p_2(\alpha) \odot p_3(\alpha)$. □

The next result is a specific case of Lemma 3.4.2, with an analogous proof but is shown here as it will be used more frequently than Lemma 3.4.2. The proof also connects with the definition of roots and multiplicity from [BL18a].

**Lemma 3.4.3.** *Given a polynomial $p(X) \in \mathbb{H}[X]$, where $\mathbb{H}$ is doubly distributive, for which $p(X) \in (X \boxplus -a) \odot q(X)$, then*

$$p(\alpha) \subseteq (\alpha \boxplus -a) \odot q(\alpha).$$

*Proof.* Set $p(X) = \boxplus_{i=0}^{n} c_i X^n$, then $q(X) = \boxplus_{i=0}^{n-1} d_i X^i$ is defined, as in Lemma A [BL18a], such that $c_0 = -ad_0$, $c_i \in (-ad_i) \boxplus d_{i-1}$ and $c_n = d_{n-1}$. Then when $(X \boxplus -a) \odot q(X)$ is evaluated,

$$
\begin{aligned}
(\alpha \boxplus -a) q(\alpha) =& (\alpha \boxplus -a)(d_0 \boxplus d_1 \alpha \boxplus \dots \boxplus d_{n-1} \alpha^{n-1}) \\
=& \alpha d_0 \boxplus \alpha d_1 \alpha \boxplus \dots \boxplus \alpha d_{n-1} \alpha^{n-1} \\
& \quad \boxplus -a d_0 \boxplus -a d_1 \alpha \boxplus \dots \boxplus -a d_{n-1} \alpha^{n-1} \\
=& -a d_0 \boxplus (d_0 \boxplus -a d_1) \alpha \boxplus (d_1 \boxplus -a d_2) \alpha^2 \boxplus \dots \\
& \quad \boxplus (d_{n-2} \boxplus -a d_{n-1}) \alpha^{n-1} \boxplus d_{n-1} \alpha^n \\
\ni& c_0 \boxplus c_1 \alpha \boxplus \dots \boxplus c_{n-1} \alpha^{n-1} \boxplus c_n \alpha^n \\
=& p(\alpha).
\end{aligned}
$$

□

**Proposition 3.4.4.** *Let $p \in \mathbb{H}[X]$, with $\deg(p) = n$. Then, if $b \in \mathbb{H}$ is a root of the polynomial $p(X)$, the multiplicity of $b$ is at most $n$. Explicitly,*

$$\mathrm{mult}_b(p) \leqslant n.$$

*Proof.* The multiplicity of a root is defined in [BL18a] and recalled in Definition 2.3.10 as,

$$\mathrm{mult}_b(p) = 1 + \max\{\mathrm{mult}_b(q) \mid p \in (X \boxplus -b) \odot q(X)\},$$

if $b$ is a root of $p(X)$, and zero if $b$ is not a root. The degree of the factor $q(X)$ will decrease by one from the degree of $p(X)$. Giving, $\deg(q) = \deg(p) - 1$. The degree of the remaining factor will continue to decrease by 1 as $(X \boxplus -b)$ is continued to being factored out if possible. Therefore the maximum amount of times a root can continued to be factor out is the degree of the polynomial. Hence, the multiplicity of a single root can not exceed the degree of the polynomial. □

**Proposition 3.4.5.** *For a finite hyperfield $\mathbb{H}$, with $|\mathbb{H}| = m$, then for all polynomials $p(X) \in \mathbb{H}[X]$, with $\deg(p) = n$,*

$$\sum_{b \in \mathbb{H}} \mathrm{mult}_b(p) \leqslant n \cdot m.$$

*Proof.* From Proposition 3.4.4, each element of the hyperfield can have maximum multiplicity of $n$. This gives that for all $m$ elements of the finite $\mathbb{H}$, the bound on the sum of the multiplicities is $n \cdot m$. □

**Lemma 3.4.6.** *Let $\mathbb{H}$ be a doubly distributive hyperfield, then for every polynomial $p \in \mathbb{H}[X]$ there are at most $n$ distinct roots of $p(X)$, where $n = \deg(p)$.*

*Proof.* Given a list of $n+1$ distinct elements of $\mathbb{H}$, $a_1, \ldots, a_n$, $b$, which will be assumed to be roots of the polynomial $p(X)$, with $\deg(p) = n$. Then the aim of the proof is to demonstrate a contradiction to this assumption of $n+1$ distinct roots.

Take $a_1$, by the definition of multiplicity of a root of $p(X)$, it can be seen that, $p \in (X \boxplus -a_1) \odot q(X)$, for some choice of $q(X)$. Then using the result from Lemma 3.4.3,

$$p(\alpha) \subseteq (\alpha \boxplus -a_1) \odot q(\alpha) \quad \forall \alpha \in \mathbb{H},$$

for doubly distributive hyperfields. This property can be used, along with the fact hyperfields do not have zero divisors, to show that the remaining roots $a_2, \ldots, a_n$ are roots of the factor $q(X)$. This gives the following factorisation:

$$p(X) \in (X \boxplus -a_1) \odot (X \boxplus -a_2) \odot \cdots \odot (X \boxplus -a_n).$$

Now there is a single distinct root remaining, $b \in \mathbb{H}$, which means $0 \in p(b)$. Compare this to the factorisation above.

$$\mathbb{0} \in p(b) \subseteq (b \boxplus -a_1) \odot (b \boxplus -a_2) \odot \cdots \odot (b \boxplus -a_n)$$

$$\Rightarrow \mathbb{0} \in (b \boxplus -a_1) \odot (b \boxplus -a_2) \odot \cdots \odot (b \boxplus -a_n).$$

There are no zero divisors over $\mathbb{H}$, so for the above to hold $0 \in (b \boxplus -a_i)$ for some $i \in \{1, 2, \ldots, n\}$, but this only happens when $b = a_i$, which is contradiction as $b$ was defined to be a distinct root. Therefore, either $b = a_i$ for some $i$, or $b$ is not a root of $p(X)$. Both of these scenarios gives that there is a maximum $n = \deg(p)$ distinct roots. □

*Remark* 3.4.7. An immediate consequence of Lemma 3.4.6 is that there is now a slightly more strict bound on the sum of the multiplicities of polynomials over doubly distributive hyperfields:

$$\sum_{b \in \mathbb{H}} \mathrm{mult}_b(p) \leqslant n^2.$$

This updated bound is not overly restrictive, but does mean that all polynomials over doubly distributive hyperfields with finite degree have a finite sum of multiplicities.

**Proposition 3.4.8.** *If a polynomial $p_1 \in p_2 \odot p_3$, where $p_2 \odot p_3$ is the set of polynomials produced by the multiplication over $\mathbb{H}$, and in the same sense, $p_3 \in p_4 \odot p_5$. Then, if the hyperfield $\mathbb{H}$ is doubly distributive,*

$$p_1(\alpha) \subset p_2(\alpha) \odot p_4(\alpha) \odot p_5(\alpha).$$

*Proof.* This follows directly from the inclusion property shown in Lemma 3.4.2, applied twice. Both to $p_1 \in p_2 \odot p_3 \Rightarrow p_1(\alpha) \subseteq p_2(\alpha) \odot p_3(\alpha)$, and $p_3 \in p_4 \odot p_5 \Rightarrow p_3(\alpha) \in p_4(\alpha) \odot p_5(\alpha)$. □

### 3.4.1 Multiplicities over $\mathbb{S}$

The multiplicites of roots for polynomials defined over the hyperfield of signs are characterised in full in [BL18a, Theorem C]. The results in this section will build on this characterisation with the goal of understanding the multiplicity bound in general.

**Proposition 3.4.9.** *Given a polynomial $p(X) = a_nX^n + \ldots + a_rX^r$, where $n > r \geqslant 0$. Then the sign changes of $p(X)$ along with the sign changes of $p(-X)$ are bounded by $n-r$. Explictly, $\sigma(p)+\sigma(p(-X)) \leqslant n-r$, where $\sigma(p), \sigma(p(-X))$ are the sign changes of $p(X)$ and $p(-X)$, respectively.*

*Proof.* The proof relies on an inductive argument, for which the base case is a degree one polynomial. It can be seen for any degree one polynomial the sign will only change in the $X$ term, meaning there is at most one sign change in either $p(X)$ or $p(-X)$, demonstrating the bound.

Now assume that the bound holds for all polynomials of deg $= n-1$. Let $a_d$ be the first coefficient which has a sign change from the leading coefficient $a_n$. Set $h(X) = a_dX^d + \ldots + a_rX^r$ and $g(X) = a_nX^n + \ldots + a_{d+1}X^{d+1}$. Then,

$$\sigma(p) + \sigma(p(-X)) = \sigma(h(X)) + \sigma(h(-X)) + \sigma(g(X)) + \sigma(g(-X)) + 1.$$

Then $(-1)^{d+1}a_{d+1}$ and $(-1)^d a_d$ have the same signs. There are no sign changes in $g(X)$, implying $\sigma(g(X)) = 0$, and $\sigma(g(-X)) \leqslant n-d-1$ by induction as $d < n$. Also,

$$\sigma(h(X)) + \sigma(h(-X)) \leqslant d - r$$

Leading to,

$$\begin{aligned}\sigma(p) + \sigma(p(-X)) &\leqslant d - r + (n-d-1) + 1 \\ &= n - r.\end{aligned}$$

□

*Remark* 3.4.10. If the lowest power of the polynomial $p(X)$ is zero, $r = 0$, then this means that the signs changes are bound by the degree of the polynomial. Giving that,

$$\sigma(p) + \sigma(p(-X)) \leqslant d - r + (n-d-1) + 1 = n - r = n = \deg(p).$$

The next result is stated as an exercise in [BL18a], but it is outlined in full in this section to clarify the multiplicity bound on the hyperfield of signs. It also then enables the multiplicity bound to be extended to the signed tropical hyperfield.

**Lemma 3.4.11** ([BL18a], Remark 1.12)**.** *The hyperfield of signs, $\mathbb{S}$, satisfies the multiplicity bound.*

*Proof.* Given the polynomial $p(X) = c_n X^n \boxplus \ldots \boxplus c_r X^r$, the multiplicities are defined as,

$$\mathrm{mult}_1(p) = \sigma(p) \quad \mathrm{mult}_{-1}(p) = \sigma(p(-X)) \quad \mathrm{mult}_0(p) = r.$$

These definitions of the multiplicities give an explicit expression for the sum over all elements of $\mathbb{S}$.

$$\begin{aligned}\sum_{b \in \mathbb{S}} \mathrm{mult}_b(p) &= \mathrm{mult}_1(p) + \mathrm{mult}_{-1}(p) + \mathrm{mult}_0(p) \\ &= \sigma(p) + \sigma(p(-X)) + r \\ &\leqslant n - r + r \\ &= n\end{aligned}$$

This point is emphasised, because the number of sign changes of a polynomial added to the number of sign changes of the polynomial evaluated at minus $X$, is dependant on the highest and lower power terms. This is a bound rather than equality, which is why roots over $\mathbb{S}$ maybe not always add up to the degree but will never exceed it. This links to the fact that the hyperfield of signs is not algebraically closed. □

The above result shows that $\mathbb{S}$ satisfies the multiplicity bound. In [BL18a] it is shown that $\mathbb{T}$ and $\mathbb{K}$ also satisfy the multiplicity bound, and actually have equality rather than simply bounded. The next section will demonstrate the multiplicity bound for the signed tropical hyperfield.

### 3.4.2 Multiplicities over $\mathbb{TR}$

There has been work done in [Gun19] regarding the multiplicities of roots over the signed tropical hyperfield, $\mathbb{TR}$. The author has shown there is an explicit description of the multiplicity of a root over $\mathbb{TR}$. The multiplicity of the root is a combination of the multiplicity description for roots over both the tropical hyperfield and the hyperfield of signs described above.

**Theorem 3.4.12** ([Gun19] , Theorem A). *Given a positive root, $a = (1, r) \in \mathbb{TR}$, then for a polynomial $p(X) \in \mathbb{TR}[X]$,*

$$\mathrm{mult}_a(p) = \Delta(p_\sigma) = \sigma(p_\sigma).$$

*Where $p_\sigma$ is the initial form of $p$ corresponding to the edge of the Newton Polygon with slope $-r$. Furthermore, $\mathrm{mult}_0(p) = k$ if and only if $p = c_k X^k + \ldots + c_n X^n$, with $c_k \nplus o$. Then for a negative root, $a = (-1, m) \in \mathbb{TR}$, $\mathrm{mult}_a(p(X)) = \mathrm{mult}_{-a}(p(X))$.*

The descriptions of the multiplicities for roots over the signed tropical hyperfield can be used to state the multiplicity bound for $\mathbb{TR}$, which is not explicitly stated in [Gun19].

**Theorem 3.4.13.** *The signed tropical hyperfield, $\mathbb{TR}$, satisfies the multiplicity bound for all polynomials $p(X) \in \mathbb{TR}[X]$.*

*Proof.* The expression for the multiplicities outlined in (3.4.12) can be substituted into the formal sum over all elements.

$$\begin{aligned}\sum_{a \in \mathbb{TR}} \mathrm{mult}_a(p) &= \sum_{\substack{a \in \mathbb{TR} \\ a>0}} \mathrm{mult}_a(p) + \sum_{\substack{a \in \mathbb{TR} \\ a<0}} \mathrm{mult}_a(p) + \mathrm{mult}_0(p) \\ &= \sum_{\substack{a \in \mathbb{TR} \\ a>0}} \mathrm{mult}_a(p(x)) + \sum_{\substack{b \in \mathbb{TR} \\ b>0}} \mathrm{mult}_b(p(-x)) + \mathrm{mult}_0(p) \\ &= \sum_{\substack{a \in \mathbb{TR} \\ a>0}} \Delta(p_\sigma) + \sum_{\substack{b \in \mathbb{TR} \\ b>0}} \Delta(p_\sigma(-x)) + \mathrm{mult}_0(p)\end{aligned} \tag{3.4.14}$$

As defined in (3.4.12), $\Delta(p_\sigma) = \sigma(p_\sigma)$, is the sign changes of the initial form corresponding to the part of the Newton polygon which has slope $-r$. The section of the Newton polygon corresponding to this initial form will have at most the number of sign changes that there are monomials included in the initial form. This gives implies the number of sign changes of $p(X)$ and $p(-X)$, when added over all the initial forms for each different gradient $-r$, will be bounded by the number of monomials terms in the polynomial expression, as it was over the hyperfield of signs. This leads to,

$$\sum_{a \in \mathbb{TR}} \mathrm{mult}_a(p) \leqslant n - k + k = n = \deg(p).$$

Which demonstrates the multiplicity bound for $\mathbb{TR}$. □

Both the hyperfield of signs and the signed tropical hyperfield have been shown to satisfy the multiplicity bound, but unlike $\mathbb{T}$ and $\mathbb{K}$ they do not satisfy multiplicity equality in totally generality.

## 3.5 Double Distributivity and the Multiplicity Bound

The aim of the final section of this chapter is to demonstrate a precise condition under which a hyperfield satisfies the multiplicity bound. The main result of this section states that the multiplicity bound holds in doubly distributive hyperfields, up to polynomials of degree three. There is motivation to extend this to higher degrees and see if the doubly distributive property is sufficient for polynomials of any degree. Several of the techniques in the proof are applicable to certain cases for higher degrees, especially degree four but are not exhaustive. Therefore, there needs to be an improvement, refinement or extension of the techniques in order to deal with higher degrees.

**Theorem 3.5.1.** *Let $\mathbb{H}$ be a doubly distributive hyperfield. Then all polynomials $p(X) \in \mathbb{H}[X]$, where $\deg(p) \leqslant 3$, have the sum of there multiplicities bounded by the degree of the polynomial.*

*Proof.* Explicitly, the following bound:

$$\sum_{b \in \mathbb{H}} \operatorname{mult}_b(p) \leqslant \deg(p) \tag{3.5.2}$$

needs to be shown for polynomials with degree up to three. The proof will be discussed in three parts, one for each of the degrees from one to three.

1. Let $\deg(p) = 1$, by Proposition 3.4.1 the result holds.

2. Let $\deg(p) = 2$ and assume that

$$\sum_{b \in \mathbb{H}} \operatorname{mult}_b(p) \geqslant 3.$$

The possible lists of roots and multiplicities correspond to integer partitions of $n \geqslant 3$. This property is used to split up this case even further.

(a) Say $\sum_{b\in\mathbb{H}} \mathrm{mult}_b(p) = 3$, then due to Lemma 3.4.6, there can not be more than 2 distinct roots for the polynomial $p(X)$ and by Proposition 3.4.4, there can not be a single root with multiplicity greater than 2. The only partition that remains is $(2,1)$. Take two distinct roots $a_1, a_2$ with multiplicities,

$$\mathrm{mult}_{a_1}(p) = 2 \quad \mathrm{mult}_{a_2}(p) = 1.$$

Then to proceed, factor out a linear polynomial corresponding to the root $a_1$. This gives,

$$p(X) \in (X \boxplus -a_1) \odot q(X).$$

Choose the polynomial $q(X)$ such that it maximises the multiplicity of $a_1$. The degree of $q(X)$ is one less than the degree of $p(X)$, $\deg(q) = \deg(p) - 1 = 1$. Therefore the choice of $q(X)$ gives that $a_1$ must be the only root of $q(X)$. Due to the doubly distributive property, Lemma 3.4.3, it has been shown that,

$$p(\alpha) \subseteq (\alpha \boxplus -a_1) \odot q(\alpha), \quad \forall \alpha \in \mathbb{H}.$$

As $a_2$ is a distinct root of $p(X)$, zero belongs to the set of elements when $p(X)$ is evaluated at $a_2$. This leads to,

$$\mathbb{0} \in p(a_2) \subseteq (a_2 \boxplus -a_1) \odot q(a_2).$$

This causes a contradiction as there are no zero divisors in $\mathbb{H}$, and $a_1 \neq a_2$, so $\mathbb{0} \notin (a_2 \boxplus -a_1)$. Therefore, $\mathbb{0} \in q(a_2)$ has to be the case, but $a_1$ is the unique root of $q(X)$, so this is a contradiction.

(b) Say $\sum_{b\in\mathbb{H}} \mathrm{mult}_b(p) = 4$, then again due to Lemma 3.4.6 and Proposition 3.4.4 the only partition of 4 that remains is $(2,2)$. Although, this is an extension of the $(2,1)$ partition from the previous part, hence nothing more to show here.

(c) Say $\sum_{b\in\mathbb{H}} \mathrm{mult}_b(p) \geqslant 5$, as every partition of an integer $n \geqslant 5$ either includes a number greater that 2 or has size greater than 2 the results in Lemma 3.4.6 and Proposition 3.4.4 show a contradiction.

Therefore, by combining this logic the multiplicity bound has been shown for quadratic polynomials over doubly distributive hyperfields.

3. Let $\deg(p) = 3$ and assume that

$$\sum_{b \in \mathbb{H}} \text{mult}_b(p) \geqslant 4.$$

The correspondence to integer partitions is utilised once more here.

(a) Say $\sum_{b \in \mathbb{H}} \text{mult}_b(p) = 4$, then by Lemma 3.4.6 and Proposition 3.4.4 this reduces to three possible cases for the sum of the multiplicities. Either $(3,1), (2,2)$ or $(2,1,1)$, these cases will be dealt with separately.

(i) For $(3,1)$ the multiplicities are $\text{mult}_{a_1}(p) = 3$ and $\text{mult}_{a_2}(p) = 1$. By selecting factors which maximise the multiplicity of $a_1$ this gives

$$p(X) \in (X \boxplus -a_1) \odot (X \boxplus -a_1) \odot (X \boxplus -a_1).$$

Then by Lemma 3.4.3, as the hyperfield is doubly distributive gives,

$$\mathbb{0} \in p(a_2) \subseteq (a_2 \boxplus -a_1) \odot (a_2 \boxplus -a_1) \odot (a_2 \boxplus -a_1).$$

As before in the quadratic case, the fact there are no zero divisors and $a_1$ and $a_2$ were defined to be distinct implies,

$$\mathbb{0} \notin (a_2 \boxplus -a_1) \odot (a_2 \boxplus -a_1) \odot (a_2 \boxplus -a_1).$$

This shows a contradiction.

(ii) For $(2,1,1)$ the multiplicities are $\text{mult}_{a_1}(p) = 2$, $\text{mult}_{a_2}(p) = 1$ and $\text{mult}_{a_3}(p) = 1$. Take the factor corresponding to $a_1$ out, choosing $q(X)$ to maximise the multiplicity.

$$p(X) \in (X \boxplus -a_1) \odot (X \boxplus -a_1) \odot t(X)$$

Then, by Lemma 3.4.3, and $\mathbb{0} \in p(a_2)$,

$$\mathbb{0} \in p(a_2) \subseteq (a_2 \boxplus -a_1) \odot (a_2 \boxplus -a_1) \odot t(a_2).$$

This implies that $t(X) = X \boxplus -a_2$. The same step can be applied again with the root $a_3$.

$$\mathbb{0} \in p(a_3) \subseteq (a_3 \boxplus -a_1) \odot (a_3 \boxplus -a_1) \odot (a_3 \boxplus -a_2).$$

Again, the roots are defined to be distinct and there are no zero divisors in $\mathbb{H}$, which gives,

$$\mathbb{0} \notin (a_3 \boxplus -a_1) \odot (a_3 \boxplus -a_1) \odot (a_3 \boxplus -a_2).$$

This causes a contradiction, either $a_3$ is not a root or it is equal to another one of the roots, whereas it was defined to be a distinct root.

(iii) For $(2, 2)$ the multiplicities are $\text{mult}_{a_1}(p) = 2$ and $\text{mult}_{a_2}(p) = 2$. This requires a different approach than the previous combinations. Choosing the factor $q(X)$ which maximises the multiplicity individually, for both of these roots when factoring, then using the Lemma 3.4.3 leads to,

$$p(X) \in (X \boxplus -a_1) \odot (X \boxplus -a_1) \odot (X \boxplus -a_2)$$

$$p(X) \in (X \boxplus -a_2) \odot (X \boxplus -a_2) \odot (X \boxplus -a_1).$$

There must be a non-empty intersection of the sets produced from both polynomial multiplications. By looking at the constant term for both multiplications, it can be seen that for a non-empty intersection,

$$a_1^2 a_2 = a_2^2 a_1.$$

This then implies by cancellation that $a_1 = a_2$, which is a contradiction to original assumption, where they were defined as distinct elements.

(b) Say $5 \leqslant \sum_{b \in \mathbb{H}} \text{mult}_b(p) \leqslant 6$, then $(3, 3), (3, 2, 1), (2, 2, 2), (3, 2), (3, 1, 1)$ and $(2, 2, 1)$ are the remaining partitions after invoking the results from Lemma 3.4.6 and Proposition 3.4.4. The partitions $(3, 3), (3, 2, 1), (3, 2)$ and $(3, 1, 1)$ are an extension of the partition $(3, 1)$ of 4. Furthermore, both $(2, 2, 2)$ and $(2, 2, 1)$ are extensions of the partition $(2, 2)$ of 4. These are all accounted for by the previous case.

(c) Say $7 \leqslant \sum_{b \in \mathbb{H}} \mathrm{mult}_b(p) \leqslant 9$, then the only partitions not eliminated by Lemma 3.4.6 and Proposition 3.4.4 are $(3,3,3),(3,3,2)$ and $(3,3,1)$ but these are extensions of the partition $(3,3)$ of 6. Hence, they are covered by the previous arguments.

(d) Say $\sum_{b \in \mathbb{H}} \mathrm{mult}_b(p) \geqslant 10$, as every partition of an integer $n \geqslant 10$ either includes a number greater that 3 or has size greater than 3 the results in Lemma 3.4.6 and Proposition 3.4.4 show a contradiction.

Thus, the multiplicity bound has been shown for cubic polynomials over doubly distributive hyperfields.

This demonstrates that all polynomials of degree three over doubly distributive hyperfields have the sum of the multiplicities bounded by the degree of the polynomial, which concludes the proof. □

The construction of the proof is very clearly difficult to generalise to higher degree polynomials. It is shown by checking case by case, which is not the most efficient way to look for a proof to hold for general degrees. This is a suggestion that there might be more to be understood, which leads to the following questions:

**Question 3.5.3.** *Is there a proof that demonstrates that all doubly distributive hyperfields satisfy the multiplicity bound?*

**Question 3.5.4.** *Can a counter example by constructed to demonstrate that in fact this does not hold in general?*

**Question 3.5.5.** *If the multiplicity bound does not hold for arbitrary degree, then is there a power (possibly higher than 3) such that this is the highest power the multiplicity bound holds for?*

These questions lend themselves naturally to a possible computer search for a counter example. Although, another potential option is to use the characterisation of stringent hyperfields in [BS20] and the results for the signed tropical hyperfield in [Gun19] to explore the multiplicity bound and inheritance property for stringent hyperfields.

There is clearly a link here to the way polynomials can be factored and hence the inheritance property in Definition 3.3.4. This would be an interesting area to explore and hopefully understand if doubly distributivity controls the sum of the multiplicities.

# Chapter 4

# Polynomials Over Quotient Hyperfields

This chapter explores the details of polynomials and varieties over quotient hyperfields, and the corresponding quotient homomorphisms. In Chapter 6 the behaviour of varieties under quotient homomorphisms will be investigated in its most general form. Here quotients of $\mathbb{C}$ are specifically explored, in particular looking at the case of $\mathbb{C}/U_n$, where $U_n$ is the group of n-th roots of unity, where both properties of the hyperfields and the corresponding maps between them are discussed. These ideas are then developed for the more familiar triangle hyperfield, which can be described as $\triangle \cong \mathbb{C}/S^1$, which connects to the quotients by $U_n$.

It can be seen by the results in Chapter 3 that quotient maps are not RAC maps in general. For instance, this can be viewed in Example 3.1.6. Over the triangle and phase hyperfield this occurs because polynomials over these hyperfields can exceed the multiplicity bound. One possible method of encoding the information of the variety more precisely would be to look at the principal ideal generated by a polynomial, rather than just the polynomial individually. This idea is best understood by utilising amoebas and coamoebas for the triangle and phase hyperfields respectively. The existing literature is surveyed, with the addition of several more detailed results.

## 4.1 Quotients by Roots of Unity

When describing hyperfields in terms of the quotient of a field, there is key information which is encoded in the multiplicative subgroup of the original field. Properties of the hyperfield can be characterised by properties of the subgroup. The most prominent link between a subgroup $U \subseteq K^{\times}$ and the hyperfield $K/U$, is the ability to determine when the hyperfield is a stringent hyperfield. This relationship is shown in [BS20], where it is proved that all stringent hyperfields are in the quotient form with a specific type of subgroup. Before the result is stated here, a property of multiplicative subgroups is introduced.

**Definition 4.1.1.** A subgroup $U \subset K$ is called *Hüllenbildend* if for $x, y \in K$,

$$x + y - xy \in U \Longrightarrow x \in U \quad \text{or} \quad y \in U$$

*Remark* 4.1.2. The word Hüllenbildend means hull producing, which is based in the work on convexity completed by [DG73] using these subgroups.

**Theorem 4.1.3.** *[BS20, Thm. 7.4 and Cor 7.5] Every stringent skew hyperfield is the quotient of a skew field and hence every doubly distributive skew hyperfield is the quotient of a skew field.*

*Remark* 4.1.4. The subgroups forming the quotients in Theorem 4.1.3 are shown to be Hüllenbildend in [BS20].

These results lead to a precise property of the subgroup which can be checked to demonstrate whether the resulting quotient hyperfield is stringent. Theorem 4.1.3 along with Remark 4.1.4 states that every stringent hyperfield is the quotient of a field by a Hüllenbildend subgroup. Therefore, if a quotient hyperfield is not the quotient of a field by a Hüllenbildend subgroup then this implies that the hyperfield is not stringent. This section explores several hyperfields constructed in the quotient form and describes the classification of the stringent property for these quotients. There will now be a collection of examples outlining the Hüllenbildend property for several of the hyperfields discussed in Table 2.1.

**Example 4.1.5.** Given $\mathbb{P} \cong \mathbb{C}/\mathbb{R}_{>0}$, this example will show that the subgroup $\mathbb{R}_{>0} \subseteq \mathbb{C}$ is not Hüllenbildend.

Take, $x = a + ib$ and $y = c + id$, with $b, d \neq 0$, then

$$\begin{aligned} x + y - xy &= a + ib + c + id - (a + ib)(c + id) \\ &= (a + c) + (b + d)i - (ac + ibc + iad - bd) \\ &= (a + c) + (b + d)i - ac - (bc + ad)i + bd \\ &= (a + c - ac + bd) + i(b + d - bc - ad) \end{aligned}$$

Thus, the requirement for $x + y - xy \in U$ would be,

$$a + c - ac + bd > 0 \Longleftrightarrow a + c + bd > ac$$

$$b + d - bc - ad = 0 \Longleftrightarrow b + d = bc + ad$$

This could occur when $c = a = 1$, then the first line would be $1 + 1 + bd > 1$, or $2 + bd > 1$, or as long as $bd > -1$. Thus, for exmaple take $a = c = 1$, $b = 2$, $d = 2$, giving $x = 1 + 2i$, $y = 1 + 2i$.

$$\begin{aligned} x + y - xy &= 1 + 2i + 1 + 2i - (1 + 2i)(1 + 2i) \\ &= 2 + 4i - (1 + 2i)(1 + 2i) \\ &= 2 + 4i - (1 + 2i + 2i - 4) \\ &= 2 + 4i - 1 - 4i + 4 \\ &= 5 \in \mathbb{R}_{>0} \end{aligned}$$

Therefore, this gives a contradiction as $x = y = 1 + 2i \notin \mathbb{R}_{>0}$. This shows that for the field $K = \mathbb{C}$, the subgroup $U = \mathbb{R}_{>0}$ is not Hüllenbildend.

Note that the ambient group the subgroup belongs to is key in the context of Hüllenbildend subgroups. The same set, treated as a subgroup of two different ambient groups can be seen as both Hüllenbildend and not Hüllenbildend depending on the setting.

**Example 4.1.6.** It is seen in Table 2.1 that $\mathbb{S} \cong \mathbb{R}/\mathbb{R}_{>0}$. This example will show that $U = \mathbb{R}_{>0}$ is Hüllenbildend when it is a subgroup of $\mathbb{R}$.

Firstly, it is proposed that the Hüllenbildend property does not hold. Then there must exist $x, y \in \mathbb{R}\backslash\mathbb{R}_{>0}$, such that $x + y - xy \in U = \mathbb{R}_{>0}$. $\mathbb{R}\backslash\mathbb{R}_{>0} = \mathbb{R}_{\leqslant 0}$, so there are negative numbers or zero such that $x + y - xy \in U$. This gives three possibilities;

1. $x = y = 0 \Longrightarrow 0 + 0 - 0 = 0 \notin U$
2. Wlog $x = 0\,,\ y \in \mathbb{R}_{<0}$.
   $\Longrightarrow x + y - xy = 0 + y - 0 = y < 0 \notin U$
3. $x, y \in \mathbb{R}_{<0} \Longrightarrow xy \in \mathbb{R}_{>0} \Longrightarrow -xy \in \mathbb{R}_{<0}$.
   $x + y - xy \notin U$, as all three parts are negative.

Hence, there are not possible choices of $x$ and $y$ not belonging to the subgroup, where $x + y - xy$ does belong to it. Thus, giving a contradiction, showing that here the subgroup $U = \mathbb{R}_{>0}$ is Hüllenbildend.

The previous two examples show the subtlety in the relationship between the ambient group or field and a subgroup being Hüllenbildend or not. In both examples the subgroup is $\mathbb{R}_{>0}$, but in the first, as a subgroup of the complex numbers, it is not Hüllenbildend and in the second, as a subgroup of the real numbers, it is.

### 4.1.1 Properties of $\mathbb{C}/U_n$

There will now be a focus on a class on quotient hyperfields defined by taking the quotient of the complex numbers by the roots of unity.

**Definition 4.1.7.** For a positive integer $n$, the $n$-th roots of unity are defined as the complex numbers which when raised to the $n$-th power equal 1. They are denoted $U_n$.

$$U_n := \{z \in \mathbb{C} \,|\, z^n = 1\}.$$

**Lemma 4.1.8.** *Take $U_n$ as a multiplicative subgroup over $\mathbb{C}$. Then $U_n$ is not a Hüllenbildend subgroup over $\mathbb{C}$, for any $n > 1$.*

*Proof.* The argument will be designed to construct $x, y \in \mathbb{C}\backslash U_n$, such that $x+y-xy \in U_n$, causing a contradiction. Firstly, let $y \in \mathbb{R}_{>0}\backslash\{1\} \not\subset U_n$, then fix $u = a + ib \in U$, such that $u \neq 1$. Define,

$$x = \frac{a + ib - y}{1 - y},$$

then with some algebraic manipulation it can be seen that $x + y - xy = a + ib = u \in U_n$. The aim for the rest of the proof is to demonstrate that $x \notin U_n$. (Note that as $y \notin U_n$, this means that $y \neq 1$, which removes the unfavourable possibility in the denominator). To do this observe the absolute value of $x$ constructed as above.

$$|x| = \left|\frac{a + ib - y}{1 - y}\right| = \frac{|a + ib - y|}{|1 - y|}.$$

Where,

$$|a + ib - y| = |a - y + ib| = (a^2 + y^2 - 2ay + b^2)^{\frac{1}{2}},$$

and $|1 - y| = 1 - y$, both due to the fact $y \in \mathbb{R}_{>0}$. This leads to an expression for $|x|$.

$$|x| = \frac{1}{1 - y}(a^2 + y^2 - 2ay + b^2)^{\frac{1}{2}} = \left(\frac{a^2 + y^2 - 2ay + b^2}{y^2 - 2y + 1}\right)^{\frac{1}{2}}.$$

All roots of unity have an absolute value equal to one. To show $x \notin U_n$, it is sufficient to show that $|x| \neq 1$. If $|x| = 1$, then

$$\begin{aligned} & a^2 + y^2 - 2ay + b^2 = y^2 - 2y + 1 \\ \iff & a^2 - 2ay + b^2 = -2y + 1 \\ \iff & a^2 - 2y(a - 1) + b^2 = 1 \end{aligned}$$

As $u = a + ib \in U_n$, then $|u| = 1$, which is equivalent to $a^2 + b^2 = 1$, which gives $2y(a - 1) = 0$. This then implies that $a = 1$, as $y \in \mathbb{R}_{>0} \backslash \{1\}$ and therefore $b = 0$, yielding $u = a + ib = 1$. This is a contradiction as the construction fixed $u \neq 1$. This implies that $|x| \neq 1$, so $x \notin U_n$. In conclusion, this demonstrates a construction of $x$ and $y$ such that neither are elements of $U_n$, but $x + y - xy = u \in U$. This shows that $U_n$ does not satisfy the Hüllenbildend definition. □

**Example 4.1.9.** Take $U_2 = \{\pm 1\}$, then as in Lemma (4.1.8), take $u = -1 \neq 1$.Then,

$$x = \frac{u - y}{1 - y} = \frac{-1 - y}{1 - y}.$$

Choose $y = 7$, giving $x = \frac{-1-7}{1-7} = \frac{4}{3}$. Thus,$7, \frac{4}{3} \notin S^1 \Rightarrow 7, \frac{4}{3} \notin U_n$. Although,

$$x + y - xy = \frac{4}{3} + 7 - \frac{4}{3} \cdot 7 = \frac{4}{3} + 7 - \frac{28}{3} = \frac{25 - 28}{3} = -1 \in U_2.$$

*Remark* 4.1.10. Example 4.1.9 can be used as an explicit example for all $U_n$ where $n$ is even. Setting $x = \frac{4}{3}$ and $y = 7$, then $x, y \notin S^1 \Rightarrow x, y \notin U_n$, actually for all $n$. It can be seen that $-1 \in U_n$ when $n$ is even, or equivalently $n = 2m$. Then,

$$\begin{aligned} x + y - xy \in U_n &\iff (x + y - xy)^n = 1 \\ &\iff (x + y - xy)^{2m} = 1 \\ &\iff \left((x + y - xy)^2\right)^m = 1 \end{aligned}$$

It has been shown in Example (4.1.9) that when $x = \frac{4}{3}$ and $y = 7$, $x + y - xy = -1 \Rightarrow (x + y - xy)^2 = 1$. This then gives the required construction for all $n = 2m$.

**Example 4.1.11.** Take $U_3 = \{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, and in an analogous way to Example (4.1.9) set $u = \frac{-1+i\sqrt{3}}{2} \neq 1$. Then a choice of $y = 2$ which gives $x = \frac{5-i\sqrt{3}}{2}$, so $x, y \notin U_3$, and

$$x + y - xy = \frac{5 - i\sqrt{3}}{2} + 2 - \frac{5 - i\sqrt{3}}{2} \cdot 2 = \frac{5 - i\sqrt{3} + 4 - 10 + 2i\sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2} \in U_3.$$

**Theorem 4.1.12.** *The hyperfield $\mathbb{C}/U_n$ is not a stringent hyperfield, for any $n > 1$*

*Proof.* Section 7 of [BS20] is designed to outlining that all stringent hyperfields are in the quotient form, with the subgroup having the Hüllenbildend property, as stated here in Theorem 4.1.3. As shown in Lemma 4.1.8 the $n$-th roots of unity are not Hüllenbildend subgroups of the complex number for all $n > 1$. Hence, the hyperfields $\mathbb{C}/U_n$ are not stringent. □

**Corollary 4.1.13.** *The hyperfield $\mathbb{C}/U_n$ is not a doubly distributive hyperfield, for any $n > 1$.*

*Proof.* By Lemma (2.1.11), all doubly distributive hyperfields are stringent hyperfields. As it is shown in the previous Theorem that the hyperfields $\mathbb{C}/U_n$ are not stringent for all $n > 1$, they therefore can not be doubly distributive either. □

### 4.1.2 Relating $U_n$ to Hahn Series

The aim of this section is to use the properties and ideas developed for $U_n$ in the previous section and apply them to subgroups of Hahn series, in turn demonstrating that certain subgroups of $\mathbb{C}[[t^{\mathbb{R}}]]$ are not Hüllenbildend. Refer to Definition 2.4.13 for the definition of Hahn series, $K[[t^{\Omega}]]$.

Define a subgroup $V_n$ of the Hahn series $\mathbb{C}[[t^{\mathbb{R}}]]$ as:

$$V_n := \{\sum_{\gamma \in I} a_\gamma t^\gamma \mid \min(I) = 0\,,\ a_0 \in U_n\}.$$

This gives that $V_n \subset v_{\mathbb{C}}^{-1}(0)$. The valuation map is defined as

$$v_{\mathbb{C}} : \mathbb{C}[[t^{\mathbb{R}}]] \to \mathbb{R} \cup \{\infty\},$$

$$v_{\mathbb{C}}\Big(\sum_{\gamma \in I} a_\gamma t^\gamma\Big) = \gamma_0,$$

where $\gamma_0 = \min(I)$ and $I \subseteq \mathbb{R}$ is a well ordered group. One way to view $v_{\mathbb{C}}^{-1}(0)$ as a subset of $\mathbb{C}[[t^{\mathbb{R}}]]$, is the set of Hahn series for which the lowest power if zero, so the leading term is a constant term. The set $V_n \subset v_{\mathbb{C}}^{-1}(0)$ imposes the extra condition that this constant term has to belong to $U_n$. The properties of $V_n$ will be explored where $V_n$ is viewed as a subgroup of $v_{\mathbb{C}}^{-1}(0)$, which can be extended to discuss $V_n$ as a subgroup of $\mathbb{C}[[t^{\mathbb{R}}]]$.

**Lemma 4.1.14.** *The subgroup $V_n$ is not a Hüllenbildend subgroup of $v_{\mathbb{C}}^{-1}(0)$.*

*Proof.* Let $x$ and $y$ be Hahn series such that $x, y \in v_{\mathbb{C}}^{-1}(0) \backslash V_n$, so

$$v_{\mathbb{C}}(x) = 0 \quad v_{\mathbb{C}}(y) = 0.$$

Explicitly,

$$x = \sum_{\gamma \in I} a_\gamma t^\gamma\,, \quad \min(I) = 0\,, \quad a_0 \notin U_n.$$

$$y = \sum_{\alpha \in J} b_\alpha t^\alpha\,, \quad \min(J) = 0\,, \quad b_0 \notin U_n.$$

Then, observing the form of the multiplication of these two Hahn series,

$$xy = \Big(\sum_{\gamma \in I} a_\gamma t^\gamma\Big)\Big(\sum_{\alpha \in J} b_\alpha t^\alpha\Big)\,, \quad v_{\mathbb{C}}(xy) = 0.$$

The valuation of the multiplication is zero due to the subgroup properties of $v_{\mathbb{C}}^{-1}(0)$. Furthermore, this gives that the constant term of $xy$ is equal to $a_0 \cdot b_0$. As these are the only possible coefficients from $x$ and $y$ respectively that when multiplied together give a constant term. To continue the behaviour of $x + y - xy$ is explored. All three terms $x$, $y$ and $xy$ have valuation zero. Due to the definition of addition of Hahn series this gives $v_{\mathbb{C}}(x + y - xy) = 0$. The constant term of $x + y - xy$ is exactly the term $a_0 + b_0 - a_0 \cdot b_0 \in \mathbb{C}$. This reduces the question of whether $V_n$ is a Hüllenbildend subgroup of $v_{\mathbb{C}}^{-1}(0)$, to whether $U_n$ is a Hüllenbildend subgroup of $\mathbb{C}$. Explicitly, does there exist $a_0, b_0 \in \mathbb{C}\backslash U_n$, such that $a_0 + b_0 - a_0 \cdot b_0 \in U_n$. It has been shown in Lemma 4.1.8 that the answer to this question is that $U_n$ is not a U-hüll subgroup of $\mathbb{C}$ for any $n > 1$. Therefore, $V_n$ is not a Hüllenbildend subgroup of $v_{\mathbb{C}}^{-1}(0)$. Lemma 4.1.8 shows that it is possible to find $a_0, b_0 \in \mathbb{C}\backslash U_n$, which can be used to define $x, y \in v_{\mathbb{C}}^{-1}(0)\backslash V_n$ such that $x + y - xy \in V_n$. □

**Corollary 4.1.15.** *The subgroup $V_n$ is not a Hüllenbildend subgroup of $\mathbb{C}[[t^{\mathbb{R}}]]$. Thus, the hyperfield built as the quotient $\mathbb{C}[[t^{\mathbb{R}}]]/V_n$ is neither stringent or doubly distributive.*

*Remark* 4.1.16. The result in Lemma 4.1.8 allows for an observation to be made about the properties of subgroups of a Hüllenbildend subgroup. Given a Hüllenbildend subgroup, then not all subgroups of this given subgroup are themselves Hüllenbildend.

**Example 4.1.17.** Take $U_4 = \{\pm 1\,,\, \pm i\} \subset \mathbb{C}$, then define the Hahn series,

$$x = \sum_{\gamma \in I} a_\gamma t^\gamma = a_0 + \ \dots\ , \quad \min(I) = 0,$$
$$y = \sum_{\alpha \in J} b_\alpha t^\alpha = b_0 + \ \dots\ , \quad \min(J) = 0.$$

Define $a_0 = 7$ and $b_0 = \frac{4}{3}$, which together with above gives, $v_{\mathbb{C}}(x) = v_{\mathbb{C}}(y) = 0$, and $x, y \in v_{\mathbb{C}}^{-1}(0)\backslash V_4$. Then,

$$x + y - xy = (a_0 + b_0 - a_0 \cdot b_0) + \ \dots$$

and it has been shown in Example (4.1.9) and Remark (4.1.10) that,

$$a_0 + b_0 - a_0 \cdot b_0 = \frac{4}{3} + 7 - \frac{4}{3} \cdot 7 \in U_4.$$

This implies that $x + y - xy \in V_4$.

### 4.1.3 Applications to RAC Maps

The aim of this section is explore the properties of hyperfield homomorphisms between $\mathbb{C}$ and the quotient hyperfields $\mathbb{C}/U_n$ discussed in the previous sections. The main aim is to explore whether the maps $\mathbb{C} \to \mathbb{C}/U_n$ are RAC. The properties of roots and multiplicities of these roots over $\mathbb{C}/U_n$ will come to play an important role, due to the next result.

**Proposition 4.1.18.** *The hyperfield homomorphism $K \to K/U$ is not RAC if $K/U$ exceeds the multiplicity bound, by way of having too many distinct roots.*

*Proof.* If $K/U$ does not satisfy the multiplicity bound because it has too many distinct roots for a particular polynomial, then not all of these roots can be lifted back because the polynomials in the pre-image will have at most $n$ distinct roots over the field $K$. As not all the roots can be lifted back, the map is not RAC. □

**Example 4.1.19.** Take the polynomial $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$, and $f_*(p(X)) = X^2 \boxplus X \boxplus 1 \in \mathbb{C}/U_2[X]$. The variety of the push-forward polynomial is

$$V(f_*(p(X))) = \left\{\left[\frac{-1-\sqrt{5}}{2}\right], \left[\frac{\sqrt{5}-1}{2}\right], \left[\frac{-1+i\sqrt{3}}{2}\right], \left[\frac{-1-i\sqrt{i}}{2}\right]\right\}.$$

The map $\mathbb{C} \to \mathbb{C}/U_2$ is a surjective map, and the pre-images of distinct elements are distinct. Therefore, to lift back all the elements of $V(f_*(p(X)))$, this would require 4 distinct lifts and all these needs to be roots of $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$ for the map to be RAC. This can not happen as degree two polynomials in $\mathbb{C}[X]$ can not have four distinct roots. Thus, this map is not RAC, by means of the push-forward polynomial having too many distinct roots.

This example motivates the remainder of the section. The goal is to generalise this argument for all maps $\mathbb{C} \to \mathbb{C}/U_n$. This would then demonstrate that they are all not RAC maps. The argument above can not be generalised in a striaght-forward way, as when $n$ increases, $U_n$ causes more elements to be equal in the quotient. The focus will though remain on the polynomial $p(X) = X^2 \boxplus X \boxplus 1$.

**Proposition 4.1.20.** *The polynomial* $p(X) = X^2 \boxplus X \boxplus 1 \in \mathbb{C}/U_n[X]$ *has a root at* $X$ *when* $[-1] \in X^2 \boxplus X$.

*Proof.*

$$\begin{aligned} \mathbb{0} \in p(X) &\iff \mathbb{0} \in X^2 \boxplus X \boxplus 1 \\ &\iff -[1] \in X^2 \boxplus X \\ &\iff [-1] \in X^2 \boxplus X \end{aligned}$$

□

The behaviour of $X^2 \boxplus X$ is characterised by the structure of the hyper-addition defined for hyperfields built from a quotient (See 2.2.3). By definition

$$X^2 \boxplus X = \{[c] \mid c = x^2u + xv\,,\ u, v \in U_n\},$$

where $X = [x] = \{xU\} \in \mathbb{C}/U_n$. For $X$ to be a root there is requirement that $[c] = [-1]$. This occurs when $c = -1 \cdot w$, where $w \in U_n$. Inside the hyper-sum there can be some rearrangement,

$$c = x^2u + xv \iff 0 = x^2u + xv - c.$$

Then if it is fixed that $u = 1$, as $1 \in U_n$, then this gives a set of monic quadratics, $0 = x^2 + xv - c$, where $v \in U_n$ and $c \in \{-u \mid u \in U_n\}$. The aim is to find values for $x \in \mathbb{C}$ such that the equations above give zero when $c \in [-1]$. These values of $x$ will then give roots, $[x]$, of the polynomial $p(X) = X^2 \boxplus X \boxplus 1$. Another way of viewing this would be a solution to $0 = x^2 + xv - c$ where $c \in [-1]$, gives a solution to $c = x^2 + xv$, and thus the elements $X = [x]$ will gives $[-1] \in X^2 \boxplus X$. Which is the exact condition required for there to be a root of $p(X) = X^2 \boxplus X \boxplus 1$. If there can be a sufficient amount of distinct elements $x$ found for these equations then this will be able to show that there are a certain amount of distinct roots for $p(X) = X^2 \boxplus X \boxplus 1$ over $\mathbb{C}/U_n$.

**Proposition 4.1.21.** *Let* $\{a_1, \ldots, a_m\}$, *be distinct elements of a field* $K$. *Then over* $K/U$,

$$\left|\left\{[a_1], \ldots, [a_m]\right\}\right| \geqslant \left\lceil \frac{m}{n} \right\rceil$$

*where $n = |U|$. Concretely, the minimum amount of distinct elements in the quotient coming from $m$ distinct elements in $K$ is given by the ceiling function of $\frac{m}{n}$.*

**Proposition 4.1.22.** *Given a field $K$, the minimum number of distinct elements over $K$ needed to produce at least $t$ distinct elements over $K/U$ is $n(t-1)+1 = nt - t + 1$, where $n = |U|$.*

**Proposition 4.1.23.** *The equations $0 = x^2 + x - c$ each have two distinct roots for every choice of $c \in [-1]$.*

*Proof.* By the quadratic formula, the discriminant determines whether there are repeating roots, and here

$$b^2 - 4ac = 1 + 4c = 0 \Rightarrow -4c = 1 \Rightarrow c = -\frac{1}{4} \notin [-1] = \{-U_n\}.$$

Which is a contradiction to the assumption that $c \in [-1]$. Thus, $b^2 - 4ac \neq 0$, and this implies no double roots. □

Given the choice $v = 1$, then this restricts to the equations $0 = x^2 + x - c$. For each of these $n$ equations, there are no double/repeating roots for individual equations.

**Proposition 4.1.24.** *The equations $0 = x^2 + x - c$ share no common roots for every choice of $c \in [-1]$.*

*Proof.* Say, $f_1 = x^2 + x - c_1$ and $f_2 = x^2 + x - c_2$, with $c_1, c_2 \in [-1]$ and $c_1 \neq c_2$. Then as these polynomials are defined over $\mathbb{C}$,

$$f_1 = (x - a_1)(x - b_1) \quad \text{and} \quad f_2 = (x - a_2)(x - b_2)$$

$$\Rightarrow a_1 \cdot b_1 = -c_1, \quad a_2 \cdot b_2 = -c_2,$$

and

$$-a_1 + -b_1 = 1, \quad -a_2 + -b_2 = .$$

If $f_1$ and $f_2$ had a common root then $\{a_, b_1\} \cap \{a_2, b_2\} \neq \varnothing$. Any of the four choices available to give a non-empty intersection, along with the observation that,

$$-a_1 + -b_1 = 1 = -a_2 + -b_2,$$

gives then the other pair have to be equal. Which gives $\{a_,b_1\} = \{a_2, b_2\}$, implying in turn that,

$$-c_1 = a_1 \cdot b_1 = a_2 \cdot b_2 = -c_2.$$

This is a contradiction, hence all choices of $c \in [-1]$ give distinct varieties for $x^2 + x - c$. □

**Proposition 4.1.25.** *For the equations $0 = x^2 + x - c$, where $c \in [-1]$, $\bigcup_c V(x^2 + x - c)$, contains $2n$ distinct elements.*

*Proof.* This is a direct consequence of Propositions 4.1.23 and 4.1.24. □

**Proposition 4.1.26.** *Take the equation $0 = x^2u + x - c$, for some $u \in U_n$ and $u \neq 1$. Then, this equation has at least one root which is distinct from the collection of roots for all the equations $0 = x^2 + x - c_i$, where $c_i$ varies over all non-zero elements of $\mathbb{C}$.*

*Proof.* The equation $0 = x^2u + x - c$ has the same roots as $0 = x^2 + xu^{-1} - cu^{-1}$. For convenience set $cu^{-1} = \hat{c}$. Then, $0 = x^2 + xu^{-1} - \hat{c} = (x - d_1)(x - d_2)$, which gives $d_1 \cdot d_2 = \hat{c}$, and $-d_1 + -d_2 = u^{-1} \neq 1$. For an equation $f_i = x^2 + x + c_i = (x - a_i)(x - b_i)$, $a_i \cdot b_i = -c_i$ and $-a_i + -b_i = 1$ in a similar manner as before. Presume that $\{a_i, b_i\} = \{d_1, d_2\}$, then $u^{-1} = -d_1 + -d_2 = -a_i + -b_i = 1$. This is a contradiction as $u \neq 1$ hence $u^{-1} \neq 1$. Therefore, at least one of $d_1$ and $d_2$ is not equal to either $a_i$ and $b_i$, giving the desired result. □

This then leads to the following result regarding the multiplicity bound for $\mathbb{C}/U_n$.

**Lemma 4.1.27.** *The hyperfield $\mathbb{C}/U_n$ exceeds the multiplicity bound, by way of having too many distinct roots.*

*Proof.* Take the polynomial $p(X) = X^2 \boxplus X \boxplus 1$ over $\mathbb{C}/U_n$, then by Proposition (4.1.20), $\mathbb{0} \in p(X) \iff [-1] \in X^2 \boxplus X$. The behaviour of $X^2 \boxplus X$ is characterised by the structure of the hyper-addition defined for hyperfields built from a quotient. By definition

$$X^2 \boxplus X = \{[c] \mid c = x^2u + xv\,,\ u, v \in U_n\},$$

where $X = [x] = \{xU\} \in \mathbb{C}/U_n$. For $X$ to be a root there is requirement that $[c] = [-1]$. This occurs when $c = -1 \cdot w$, where $w \in U_n$. Inside the hyper-sum there can be some rearrangement,

$$c = x^2 u + xv \iff 0 = x^2 u + xv - c.$$

Then,

$$\left( \bigcup_{c \in [-1]} V(x^2 + x - c) \right) \cup V(x^2 \tilde{u} + x - c) \subseteq \bigcup V(x^2 u + xv - c),$$

where $\tilde{u} \in U_n$ is a fixed. By Propositions (4.1.25) and (4.1.26), the set on the left has at least $2n+1$ elements, therefore the set on the right has at least $2n+1$. Then by Proposition (4.1.22), this gives at least 3 elements $[x] = X$ over $\mathbb{C}/U_n$, which are by construction roots of $p(X) = X^2 \boxplus X \boxplus 1$. Hence, $p(X)$ has at least three distinct roots over $\mathbb{C}/U_n$, which implies that $\mathbb{C}/U_n$ exceeds the multiplicity bound by way of having too many distinct roots. □

**Corollary 4.1.28.** *The map $\mathbb{C} \to \mathbb{C}/U_n$ is not a RAC map.*

*Proof.* By Lemma 4.1.27, $\mathbb{C}/U_n$ exceeds the multiplicity bound, by way of having too many distinct roots, then by Proposition 4.1.18, the map $\mathbb{C} \to \mathbb{C}/U_n$ is not a RAC map. □

### 4.1.4 The Signed Tropical Hyperfield

As outlined in Definition 2.4.13 the signed tropical hyperfield can be constructed as a quotient of the Hahn series, $\mathbb{R}[[t^{\mathbb{R}}]]/v_{\mathbb{R}}^{-1}(0)$. It will now be shown that when this construction is viewed as a quotient map $v_{\mathbb{R}} : \mathbb{R}[[t^{\mathbb{R}}]] \to \mathbb{TR}$ this is not a RAC hyperfield homomorphism.

In [Gun19], there is an explicit description of multiplicities for roots over $\mathbb{TR}$. This can also been seen in Theorem 3.4.12, but to recall they are characterised by sign changes on edges of the Newton polytope. The Newton Polytope for polynomials over $\mathbb{TR}$ is defined as $\mathrm{Newt}(p) = \mathrm{Newt}(|p|)$, where $|p|$ is the image of the polynomial under the induced map $|\cdot| : \mathbb{TR} \to \mathbb{T}$ defined by restricting to the second component. The Newton polytope over $\mathbb{T}$ for a polynomial $q(X) = \sum_{i=0}^{n} c_i X^i$ is defined as,

$$\mathrm{Newt}(q) := \mathrm{Convhull}\{(i, c_i) \, : \, i = 1, \ldots, n\}.$$

Using these ideas the next example will demonstrate that the map $v_{\mathbb{R}} : \mathbb{R}[[t^{\mathbb{R}}]] \to \mathbb{TR}$ is not *RAC*, by taking a polynomial with coefficients in $\mathbb{R}[[t^{\mathbb{R}}]]$ but roots in $\mathbb{C}[[t^{\mathbb{R}}]]$, and showing these roots push-forward to roots over $\mathbb{T}$ and then correspond, due to the sign changes, to roots over $\mathbb{TR}$. Hence showing there exists a polynomial over $\mathbb{TR}$ with non-empty variety with no lifts of these roots back to roots of the original polynomial over $\mathbb{R}[[t^{\mathbb{R}}]]$.

**Example 4.1.29.** Let $p(X) = X^2 - 2X + (1 + t^2) \in \mathbb{R}[[t^{\mathbb{R}}]]$, this has a precise and complete factorisation over the field extension $\mathbb{C}[[t^{\mathbb{R}}]]$,

$$p(X) = (X - (1 - it))(X - (1 + it)) = X^2 - (1 - it + 1 + it)X + (1 - it)(1 + it).$$

Taking the image under the maps $v_{\mathbb{R}}$ and $v_{\mathbb{C}}$ gives,

$$v_{\mathbb{R}}(p) = (1, 0)X^2 \boxplus (-1, 0)X \boxplus (1, 0) \in \mathbb{TR}[X],$$

$$v_{\mathbb{C}}(p) = 0X^2 \boxplus 0X \boxplus 0 \in \mathbb{T}[X].$$

The resulting Newton polytope of $v_{\mathbb{R}}(p)$ is equal to the Newton polytope of $v_{\mathbb{C}}(p)$ with inherited signs.

$$\mathrm{Newt}(v_{\mathbb{R}}(p)) = \mathrm{Newt}(|v_{\mathbb{R}}(p)|) = \mathrm{Newt}(v_{\mathbb{C}}(p))$$

$\mathrm{Newt}(v_{\mathbb{R}}(p))$ is a horizontal line of length 2, with two sign changes, as seen in Figure 4.1. Hence, it is an edge of slope zero with two sign changes. This implies that there exists a root of multiplicity two for $v_{\mathbb{R}}(p)$ of multiplicity two over $\mathbb{TR}$ which does not have a lift to the any of the roots of the original polynomial $p(X) = X^2 - 2X + (1 + t^2) \in \mathbb{R}[[t^{\mathbb{R}}]]$, as this was constructed with roots from $\mathbb{C}[[t^{\mathbb{R}}]]$. Explicitly, this root is $(1, 0)$, and lifts of this root must have a Hahn series starting with a constant term. By construction the roots of the original polynomial $p(X)$ do have Hahn series starting with constant terms but the other terms contain complex components. This implies that there does not exist any lift for the root $(1, 0)$, and thus demonstrating that the map $v_{\mathbb{R}} : \mathbb{R}[[t^{\mathbb{R}}]] \to \mathbb{TR}$ is not RAC.

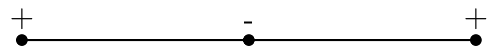


Figure 4.1: The Newton polytope for $v_{\mathbb{R}}(p) = (1,0)X^2 \boxplus (-1,0)X \boxplus (1,0) \in \mathbb{TR}[X]$.

*Remark* 4.1.30. This demonstrates that there are instances where roots over $\mathbb{T}$ can be lifted to roots over $\mathbb{TR}$, which would suggest that the map $|\cdot| : \mathbb{TR} \to \mathbb{T}$ is $RAC$, but to clarify, this is not the case. The tropical hyperfield is algebraically closed and the tropical signed hyperfield is not algebraically closed, as seen in Table 2.1. Therefore, the polynomial $X^2 \boxplus 1$ has an empty variety over $\mathbb{TR}$ but there exists roots over $\mathbb{T}$.

## 4.2 Polynomials over the Triangle Hyperfield

Quotients of the the complex numbers by roots of unity have been explored in Section 4.1, the attention is focused on a more general quotient which has broader applications. This section will explore the properties of the triangle hyperfield. As stated in Table 2.4 the triangle hyperfield is isomorphic to the quotient of the complex numbers by the unit circle $S^1$. The quotient map from the complex numbers to the triangle hyperfield links closely to amoebas from classical algebra.

This section will investigate varieties over the triangle hyperfield and other related areas. In particular, there is a presentation of a result regarding the variety of push-forward ideals over $\triangle$.

### 4.2.1 Varieties over $\triangle$

The aim of this section is to study the properties of varieties specifically over the triangle hyperfield. This will include results characterising the structure of varieties and examples to complement these descriptions. The first result demonstrates that varieties of univariate polynomials over the triangle hyperfield are closed intervals and gives a bound with respect to the degree of the polynomial, on the number of intervals.

**Theorem 4.2.1.** *Let $p \in \triangle[X]$, with* $\deg(p) = d$*, then $V(p) \subseteq \triangle$ is the union of at most $d$-distinct closed intervals.*

*Proof.* From Corollary 6.3.4, $V(p) \subseteq \triangle$ is equal to the union of push-forward roots of all the polynomials in the pre-image of $p$ under the quotient map from the complex numbers. Every polynomial in the pre-image of $p$ has degree equal to $\deg(p) = d$, and as $\mathbb{C}$ is algebraically closed, has at most $d$-distinct roots. Each polynomial in the pre-image of $p$ can be related to every other polynomial by a continuous change in coefficients by multiplication of scalars from $S^1$. The roots of the pre-image polynomials can be viewed as a continuous map from the coefficients of the polynomial. Thus, as the coefficient continuously change by scalar multiplication from $S^1$, the roots continuously change in the complex plane. This gives $d$ paths in the complex plane, one for each root, when viewing all the roots of all the pre-image polynomials. When these are mapped back through the quotient map to the triangle hyperfield this is will give at most $d$-intervals. $\square$

It will be shown later in this section that this bound is sharp for quadratic and cubic polynomials.

In general, Corollary 6.3.4 demonstrates that $V(f_*(p)) \neq f(V(p))$ for quotient hyperfields. This phenomenon occurs as $V(f_*(p))$ may be larger than $f(V(p))$. By replacing $p$ with the principal ideal $\langle p\rangle$, $f(V(\langle p\rangle)) = f(V(p))$ is unchanged, but $V(f_*(\langle p\rangle))$ can only reduce in size as;

$$V(f_*(\langle p\rangle)) = \bigcap_{q\in\langle p\rangle} V(f_*(q)).$$

This leads to the natural question of is there equality between $f(V(\langle p\rangle))$ and $V(f_*(\langle p\rangle))$ in general? The next result demonstrates that for a specific class of polynomials, using the principal ideal $\langle p\rangle$ gives the desired equality.

**Theorem 4.2.2.** *Let $p \in \mathbb{C}[X]$, where each root has rational argument the same absolute value. Then,*

$$V(f_*\langle p\rangle) = f(V(p))$$

*Proof.* From the statement of the theorem it can be seen that $V(p) = \{a_1, \dots, a_m\}$, where $m \leqslant \deg(p)$, and $|a_1| = \cdots = |a_m| = r$. This gives that $f(V(p)) = \{r\}$. For each

$a_i \in V(p)$ a point $\tilde{a}_i \in S^1$ can be identified by, $\tilde{a}_i = \frac{a_i}{|a_i|}$. Due to the property that each $a_i$ has rational argument, there exists an $n_i \in \mathbb{N}$ for each $\tilde{a}_i$ such that $\tilde{a}_i{}^{n_i} = 1$. Hence, as $a_i = r \cdot \tilde{a}_i$, then $a_i^{n_i} = (r \cdot \tilde{a}_i)^{n_i} = r^{n_i}$. Denote $R = n_1 \cdot n_2 \cdot \cdots \cdot n_m$, and define a polynomial $q(X) := X^R - r^R$. Then,

$$\begin{aligned} q(a_i) &= (a_i)^R - r^R \\ &= (a_i^{n_i})^{n_1 \cdot \ldots \cdot \hat{n}_i \cdot \ldots \cdot n_m} - r^R \\ &= r^R - r^R = 0 \end{aligned}$$

Thus, this implies that $V(p) \subseteq V(q)$. The polynomial $q$ has $R$ roots equally distributed around the circle of radius $r$ in the complex plane. Explicitly, $V(q) = \{r, r\omega, r\omega^2, \ldots, r\omega^{R-1}\}$, where $\omega, \omega^2, \ldots, \omega^{R-1}$ are the $R$th roots of unity.

The next step is to show that $q(X) \in \langle p \rangle$. This is a consequence of the properties of the variety of $q(X)$. Define a polynomial,

$$t(X) = \prod_{s \in \{r, \ldots, r\omega^{R-1}\} \setminus \{a_1, \ldots, a_m\}} (X - s).$$

Then, $q(X) = t(X) \cdot p(X)$, and hence $q(X) \in \langle p \rangle$. Furthermore, $f_*(q) = X^R + r^R \in f_*(\langle p \rangle)$, with $V(f_*(q)) = \{r\}$. When computing $V(f_*(\langle p \rangle))$ this gives,

$$\begin{aligned} \{r\} = f(V(p)) &\subseteq V(f_*(\langle p \rangle)) \\ &= \bigcap_{\tilde{p} \in \langle p \rangle} V(f_*(\tilde{p})) \\ &\subseteq V(f_*(q)) = \{r\} \end{aligned}$$

Which allows the conclusion of $V(f_*(\langle p \rangle)) = f(V(p))$ to be made. □

**Example 4.2.3.** Let $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$, then the push-forward though the map $|\cdot| : \mathbb{C} \to \triangle$ is $|p(X)| = X^2 \boxplus X \boxplus 1 \in \triangle[X]$. It is shown in Example 3.1.7 that the variety of $|p|$ is the interval $[\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}]$. Take $X - 1 \in \mathbb{C}[X]$, then $(X-1)p(X) = X^3 - 1 \in \langle p \rangle \subset \mathbb{C}[X]$, the principal ideal generated by $p(X)$. The push-forward is $X^3 \boxplus 1 \in |\langle p \rangle| \subset \triangle[X]$, which has a single root, $V(X^3 \boxplus 1) = \{1\} \subset \triangle$. Moreover, this gives $V(|\langle p \rangle|) = \bigcap_{q \in \langle p \rangle} V(|q|) = \{1\}$, which is equal to $|V(p)| = \{1\}$ as the roots of $p(X)$ are $\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\} \subset U_3$, where $U_n$ are the n-th roots of unity and thus have absolute value equal to one.

The results that have been outlined over the triangle hyperfield raise some questions about how precisely varieties can be calculated. When can the number of intervals be precisely stated? What are the end points of the intervals? Can the end points be calculated in terms of the coefficients? Given an interval, can there be a polynomial constructed such that this interval is the polynomials variety? These questions will be answered for quadratics polynomials over the triangle hyperfield next.

Define a general quadratic polynomial over $\triangle[X]$ as, $p(X) = X^2 \boxplus aX \boxplus b$, where $a, b \in \mathbb{R}_{\geqslant 0}$. Then, the variety is

$$V(p) = \{y \in \triangle : \mathbb{0} \in p(y)\}.$$

Noting that the reversibility axiom allows this to be stated as $b = -b \in y^2 \boxplus ay$. Then, using the definition of the hyper-addition over $\triangle$ it can be seen that this is characterised our by two inequalities,

$$V(p) = \{y \in \triangle : |y^2 - ay| \leqslant c \leqslant y^2 + ay\}. \tag{4.2.4}$$

These two inequalities can be viewed graphically, which allows for the intervals to be read off the graph. This can be seen in the next example.

**Example 4.2.5.** Let $p(X) = X^2 \boxplus 8X \boxplus 15 \in \triangle[X]$, then the variety can be calculated graphically by looking at the intersection of the following three lines:

$$y = |X^2 - 8X|, \quad y = X^2 + 8X, \quad y = 15.$$

This is shown in Figure 4.2, where the intersection points gives the intervals $[1.568, 3] \cup [5, 9.568]$ as the variety of $p(X)$ to three decimal places.

This viewpoint can be developed further and used to set up formulas for the endpoints of the intervals in terms of the coefficients $a, b \in \triangle$. The critical points, or equality, of

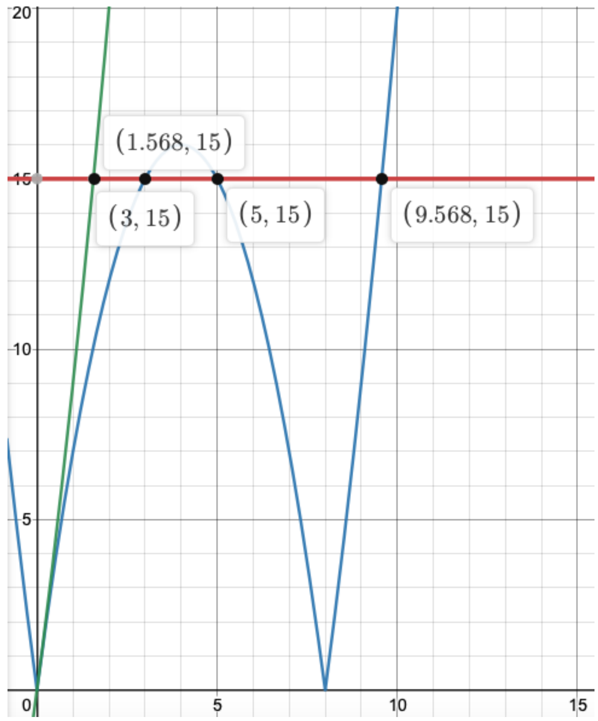


Figure 4.2: Representation of the graphs of the equations in Example 4.2.5, this was created using the online tool https://www.desmos.com/calculator

the two sided inequality will characterise the boundary of the variety. The following discussion will construct this method explicitly and demonstrate that the intervals of the variety can be calculated by purely using the coefficients. This process closely mirrors the use of the quadratic formula for polynomials over $\mathbb{R}$ and $\mathbb{C}$.

Firstly, set up the equations for which equality is achieved in (4.2.4),

$$|y^2 - ay| = b \quad \text{and} \quad y^2 + ay = b.$$

The first equation can have two variants due to the absolute value. Taking this into consideration three quadratic equations can be constructed.

$$y^2 - ay - b = 0, \quad y^2 - ay + b = 0 \quad \text{and} \quad y^2 + ay - b = 0. \tag{4.2.6}$$

The solutions to these three equations will give the intersection points of the lines $f(y) = |y^2 - ay|$, $f(y) = y^2 + ay$ and $f(y) = b$, which in turn define the candidates for the end points of the intervals of the variety of $p(X) = X^2 \boxplus aX \boxplus b$. It will be shown the order in which these points lie and confirmed that these are indeed the endpoints that are required. The negative solutions to (4.2.6) can be immediately removed, as the triangle hyperfield only deals with the positive real numbers. Applying the quadratic formula, along with the consideration about negative solutions, it can be seen that the solutions that are of interest are;

$$\begin{aligned} y^2 - ay - b = 0 &\rightarrow \frac{a + \sqrt{a^2+4b}}{2}, \\ y^2 - ay + b = 0 &\rightarrow \frac{a \pm \sqrt{a^2-4b}}{2}, \\ y^2 + ay - b = 0 &\rightarrow \frac{-a + \sqrt{a^2+4b}}{2}. \end{aligned}$$

From these solutions it can be seen that the second equation does not always achieve solutions over the positive real numbers, due to the possibility that $a^2 - 4b$ can be negative value and hence over $\mathbb{R}_{\geqslant 0}$, it has no solutions. Once the ordering of the solutions above has been given, it will be stated that this condition on the discriminant determines when the variety to a quadratic is one or two intervals.

**Proposition 4.2.7.** *Let $p(X) = X^2 \boxplus aX \boxplus b \in \triangle[X]$, then $V(p)$ is a single closed interval when $b > \frac{a^2}{4}$, and $V(p)$ is the union of two closed intervals when $b \leqslant \frac{a^2}{4}$.*

*Proof.* It has been shown that $\frac{a+\sqrt{a^2+4b}}{2}$, $\frac{a\pm\sqrt{a^2-4b}}{2}$ and $\frac{-a+\sqrt{a^2+4b}}{2}$ are the end points of the intervals and thus when $b > \frac{a^2}{4}$ there are no solutions to $\frac{a\pm\sqrt{a^2-4b}}{2}$. This shows there is only one interval. Furthermore, when $b \leqslant \frac{a^2}{4}$ there exists two solutions to $\frac{a\pm\sqrt{a^2-4b}}{2}$ and hence gives two distinct closed intervals as the variety of $p(X)$. □

**Proposition 4.2.8.** *Let $p(X) = X^2 \boxplus aX \boxplus b \in \triangle[X]$, then the cumulative length of $V(p)$ is bounded by $a$.*

*Proof.* If the variety is one interval then the endpoints are $\frac{-a+\sqrt{a^2+4b}}{2}$ and $\frac{a+\sqrt{a^2+4b}}{2}$. This gives the length of the interval as;

$$\frac{a+\sqrt{a^2+4b}}{2} - \frac{-a+\sqrt{a^2+4b}}{2} = a.$$

Alternatively, if the variety is the union of two closed varieties, the intervals are $\left[\frac{-a+\sqrt{a^2+4b}}{2}, \frac{a-\sqrt{a^2-4b}}{2}\right]$ and $\left[\frac{a+\sqrt{a^2-4b}}{2}, \frac{a+\sqrt{a^2+4b}}{2}\right]$. This gives the combined length as,

$$\Big(\frac{a-\sqrt{a^2-4b}}{2}-\frac{-a+\sqrt{a^2+4b}}{2}\Big)+\Big(\frac{a+\sqrt{a^2+4b}}{2}-\frac{a+\sqrt{a^2-4b}}{2}\Big)=a-\sqrt{a^2-4b}<a.$$

Hence, for either one or two intervals the cumulative length is bound by $a$. □

### 4.2.2 Amoebas over Hyperfields

When observing the variety of a univariate polynomial over the triangle hyperfield, it can be seen that the number of roots is generally much larger that expected in classical algebraic geometry, as can be demonstrated in Theorem 6.3.1. This gives an indication that a single polynomial doesn't encode the precise information about the roots of a specific lift. In terms of Theorem 6.3.1, it contains the information about the push-forward of the roots of every polynomial lift. This gives rise to a question regarding how the information of the roots over the triangle hyperfield can be more precise. One object to consider as a candidate for encoding more precise information is the principal ideal generated by the original polynomial over the complex numbers. Instead of only taking the starting polynomial over $\mathbb{C}$, and pushing this forward to look at the roots. Use the starting polynomials to generate an ideal and push this forward and look at the roots of this object over the triangle hyperfield.

Classically a complex amoeba is the image of the zero locus or variety of a polynomial or collection of polynomials under the map coordinate wise map $\log|\cdot| : \mathbb{C} \to \mathbb{R}$. They were introduced in [GKZD94] and are closely connected to tropical geometry, as taking the limit of the logarithmic base to infinity gives that amoebas converge to tropical varieties. Here the term amoebas refers to the image of the variety of a complex polynomial or ideal under the maps $|\cdot| : \mathbb{C} \to \triangle$ and $ph : \mathbb{C} \to \mathbb{P}$.

The next Theorem from [Pur08] outlines a relationship between the amoeba of an ideal and the roots of the push-forward of the ideal. In the paper the author discusses the result, although not in the hyperfield language that has been used throughout this work.

Therefore, the results terminology has been adjusted to align with the notation used thus far in this work. The result is stronger than the principal ideal univariate case, which initially motivated this section.

**Theorem 4.2.9.** *[Pur08, Thm.2] Given an ideal, $I \subseteq \mathbb{C}[X_1, \dots, X_n]$ and the quotient map $|\cdot| : \mathbb{C} \to \triangle$, then*

$$V(|(I)|_*) = |(V(I))|.$$

*Where $|\cdot|_*$ is the induced map of polynomials, $V(I) = \bigcap_{g\in I} V(g)$, and $V(|(I)|_*) = \bigcap_{g\in I} V(|(g)|_*)$.*

*Remark* 4.2.10. In the univariate case, where the ideal is a principle ideal, so generated by a single polynomial, the result becomes more explicit in demonstrating that individual push forward polynomials do not posses the precise information regarding the original variety. Given $p(X) \in \mathbb{C}[X]$, then Theorem 4.2.9 states that,

$$|(V(p))| = |((\langle p\rangle))| = V(|(\langle p\rangle)|).$$

there is clearly a requirement to use the full ideal generated by the polynomial to achieve the required restriction on the varieties over the triangle hyperfield.

*Remark* 4.2.11. Theorem 4.2.9 is a strong broad result and generalises Theorem 4.2.2. An important point to draw to attention is that the techniques in the proofs of the two results are independent. Theorem 4.2.2 views the construction differently to the proof outlined in [Pur08]. There is evidently more to the picture to be explored when phrasing the results in the hyperfield setting.

A similar result has been shown for the coameoba case in [For15], it is stated below in the hyperfield language.

**Theorem 4.2.12.** *[For15] Given the principal ideal $\langle p\rangle \subseteq \mathbb{C}[X]$ then under the phase hyperfield homomorphism, $ph : \mathbb{C} \to \mathbb{P}$,*

$$V(ph_*(\langle p\rangle)) = ph(V(p)).$$

It would be interesting to explore the underlying structure of Theorem 4.2.9 and Theorem 4.2.12, with a hope to using specific hyperfield techniques to unify the arguments There will now be several examples to demonstrate the results regarding taking the variety of a push-forward ideal rather than just the polynomial. The first example will specifically be a case of Theorem 4.2.2, but as stated in Remark 4.2.11 this is one case of the general result presented in Theorem 4.2.9.

**Example 4.2.13.** The polynomial, $p(X) = X^2 + X + 1 \in \mathbb{C}[X]$, and the principal ideal it generates have been discussed in Example 4.2.3, where it is shown that $V(|\langle p \rangle|) = |V(p)|$. This is also an example demonstrating the result and method from Theorem 4.2.9.

Theorem 4.2.9, precisely gives a fundamental theorem for the quotient map $|\cdot| : \mathbb{C} \to \mathbb{C}/S^1$, although as already explored, this is not the only quotient map from the complex numbers which results in a hyperfield. There is a class of quotient hyperfields constructed with the roots of unity, $U_n$, as discussed in Section 4.1. This opens up a question: does the fundamental theorem, stated in Theorem 4.2.9, hold for these other quotient maps? This question and related notions will be discussed in the remainder of the section.

The hyperfields $\mathbb{C}/U_n$, along with $\mathbb{C}$ and $\mathbb{C}/S^1$, form a commutative diagram as presented in Figure 4.3. The hyperfields homomorphisms are defined as;

$$[\cdot]_n : \mathbb{C} \to \mathbb{C}/U_n\,, \quad [z]_n = z \cdot U_n$$

$$\varphi : \mathbb{C} \to \mathbb{C}/S^1\,, \quad \varphi(w) = w \cdot S^1$$

These maps are compatible in the sense of quotients, as every $U_n \subset S^1$. With relation to Figure 4.3, Theorem 4.2.9 demonstrates a fundamental theorem for ideals along the vertical map, $|\cdot|$. The next result will use this fact to explore the relation along the diagonal map $\varphi$.

**Theorem 4.2.14.** *Let $I \subset \mathbb{C}[X_1 \, \dots \, X_n]$ be an ideal then,*

$$\varphi([V(I))]_n) = V(\varphi([I]_{n*})).$$

*Where, to clarify notation, both $\varphi_*$ and $[\cdot]_{n*}$ are the induced polynomial maps.*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, Rightarrow, "{[\cdot]_n}"] \arrow[d, Rightarrow, "|\cdot|"'] & \mathbb{C}/U_n \arrow[dl, Rightarrow, "\varphi"] \\
\triangle \cong \mathbb{C}/S^1 &
\end{tikzcd}$$

Figure 4.3: Commutative maps from $\mathbb{C}$ to $\triangle$ and $\mathbb{C}/U_n$.

*Proof.* The diagram commutes so $|V(I)| = \varphi([V(I)]_n)$. Lemma 3.2.4 implies both $[V(I)]_n \subseteq V[I]_n$ and $\varphi(V([I]_n)) \subseteq V(\varphi_*([I]_{n*}))$. For which the former then gives $\varphi([V(I)]_n) \subseteq \varphi(V[I]_n)$. Thus, combining these shows,

$$\begin{aligned}
|V(I)| &\subset \varphi([V(I)]_n) \\
&\subset \varphi(V([I]_n) \\
&\subset V(\varphi([I]_n)) \\
&= V(|I|) = |V(I)|
\end{aligned} \tag{4.2.15}$$

Hence, the inclusions become equalities, which yields $\varphi([V(I)]_n) = V(\varphi([I]_{n*}))$. □

*Remark* 4.2.16. This Theorem gives a version of the Fundamental Theorem for the map $\varphi : \mathbb{C}/U_n \to \triangle$, as the ideal is defined over the complex numbers rather that the hyperfields $\mathbb{C}/U_n$. Although, from the proof of the Theorem it can be seen that $\varphi([V(I)]_n) = \varphi(V([I]_n)$, or the images of $[V(I)]_n$ and $V([I]_{n*})$ are equal under the map $\varphi$ to $\triangle$. This is a step towards a Fundamental Theorem for the horizontal map in Figure 4.3, $[\cdot]_n : \mathbb{C} \to \mathbb{C}/U_n$.

### 4.2.3 Further Questions

There are many questions that would be worth investigating with regards to the content of this section. They will be outlined here. Firstly, note that this section builds on the notion of a RAC map, although it still remains a question to find a way to construct RAC maps in general.

It is shown that for the maps $ph : \mathbb{C} \to \mathbb{P}$ and $|\cdot| : \mathbb{C} \to \triangle$ satisfy a version of the fundamental theorem. Is this due to some underlying feature of $\mathbb{C}$ or the subgroups in the quotient construction of these maps? One aim in this area would be to characterise quotient maps from $\mathbb{C}$ which satisfy the fundamental theorem. How does this compare to quotients of other fields? Can a generic construction $\mathbb{C} \to \mathbb{C}/U$ be given such that it satisfies the fundamental theorem? For instance, do the maps $[\cdot]_n : \mathbb{C} \to \mathbb{C}/U_n$ satisfy the fundamental theorem? It is clear that the route forward is to utilise ideals rather than individual polynomials; as very few maps are RAC but it is seen here that they can still satisfy the fundamental theorem.

# Chapter 5

# Equivalence of Tropical Ideals

In this chapter, tropical ideals will be discussed. Specifically, these are polynomial ideals over $\overline{\mathbb{R}}$ that satisfy additional combinatorial properties. These combinatorial properties are based on valuated matroids which will also be introduced here. The matroid properties of tropical ideals are utilised to define a notion of equivalence between tropical ideals, called *matroidal equivalence*. Examples of tropical ideals will be presented and then connected by the matroidal equivalence. Properties of the matroidal equivalence are demonstrated, in particular it is seen that matroidal equivalence is preserved under tropicalisation.

To clarify notation, for this chapter the results are presented over the tropical semiring $\overline{\mathbb{R}}$. Therefore, the operations $\oplus$ and $\odot$ will be denoting tropical operations as follows for this chapter; $x \oplus y = \min\{x, y\}$ and $x \odot y = x + y$.

## 5.1 Motivation and Background

When observing the zero locus of a tropical polynomial, the set in $\overline{\mathbb{R}}^n$ is a balanced rational polyhedral complex, see [MS15, Theorem 3.3.6]. This notion of a tropical variety can be extracted and studied independently. In algebraic geometry the endeavour to study solutions sets of polynomials extends to varieties of polynomial ideals. Difficulty arises in the tropical setting when considering the set of all polynomial ideals over $\overline{\mathbb{R}}$. There exists polynomial ideals such that the variety is not a polyhedral complex, see

[MR18, Example 5.14].

The proposed solution to this is to restrict to a class of ideals called tropical ideals. Tropical ideals have been chiefly outlined and investigated by Maclagan and Rincon in [MR18] and [MR20]. Tropical ideals satisfy extra combinatorial properties, by way of a monomial elimination axiom. Furthermore, other contributions progressing the understanding of tropical ideals can be seen in [GG16], [GG18], [DR21], [AR22], [FGG ], [Zaj18] and [War20].

The class of tropical ideals has been shown to produce varieties that are balanced polyhedral complexes, [MR18] and [MR20]. They also include the set of tropicalisations of classical varieties, [GG16]. Although, not all tropical ideals are realisable, [MR18, Example 2.8], and tropical ideals do not realise all possible tropical varieties, [DR21]. It is shown that when a Hilbert function is defined for a tropical ideal, it is eventually polynomial, [MR18, Proposition 3.8]. Tropical ideals with Hilbert polynomial equal to one have been shown to be realisable in [Zaj18], precisely in for the form $trop(\langle X_1 - a_1, \ldots, X_n - a_n \rangle)$. Zero dimensional tropical ideals with Boolean coefficients are studied in [AR22].

The motivating question for the work presented here is; when should the geometric objects defined by two tropical ideals be considered equivalent? This is due to the fact that tropically there are many schemes structures that can be endowed to such tropical varieties. There is a requirement to make this choice of schemes more precise. Can the association of the varieties of tropical ideals be captured combinatorially by the matroidal structure? The aim is to define the notion matroidal equivalence, Definition 5.3.2, to enable an attempt in understanding when the geometric objects should be considered equivalent.

### 5.1.1 Matroids

This section will introduce matroids which combinatorially capture the notion of linear dependence. Matroids will be key when studying tropical ideals in the following sections, as they are defined as ideals which extra matroidal structure. They were first introduced by Whitney in [Whi92]. Matroids can be thought of as analogous to linear subspaces. They can arise in several different ways, including from matrices and graphs. There is a large number of equivalent methods to defining them. For a comprehensive overview of matroid theory see [Oxl03], [Oxl06] and [W$^+$95].

**Definition 5.1.1.** A matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$ consisting of the finite ground set $E$ and a collection $\mathcal{I}$ of subsets of $E$, called independent sets, satisfying the following axioms:

**(I1)** $\varnothing \in \mathcal{I}$.

**(I2)** If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

**(I3)** If $X, Y \in \mathcal{I}$, with $|X| = |Y| + 1$, then there exists an element $e$ of $X \backslash Y$ such that $Y \cup \{e\} \in \mathcal{I}$.

A subset $Z$ is called dependent if $Z \notin \mathcal{I}$

The next set of definitions give alternative, but equivalent, methods of defining matroids in terms of circuits and bases. It will be noted how they are linked to the notion of independence.

**Definition 5.1.2.** A matroid $\mathcal{M}$ is a pair $(E, \mathcal{C})$ consisting of the finite ground set $E$ and a collection $\mathcal{C}$ of subsets of $E$, called circuits, satisfying the following axioms:

**(C1)** $\varnothing \notin \mathcal{C}$.

**(C2)** No members of $\mathcal{C}$ are proper subsets of each other. This explicitly means, given $X, Y \in \mathcal{C}$, then if $X \subseteq Y$, then $X = Y$.

**(C3)** (*Circuit Elimination*) Given distinct members $X_1$ and $X_2$ of $\mathcal{C}$. Then if $e \in X_1 \cap X_2$, there is a member $X_3$ of $\mathcal{C}$ such that $X_3 \subseteq (X_1 \cup X_2) \backslash e$.

With regard to dependence, the circuits of a matroid are the minimal dependent sets.

**Definition 5.1.3.** A matroid $\mathcal{M}$ is a pair $(E, \mathcal{B})$ consisting of the finite ground set $E$ and a collection $\mathcal{B}$ of subsets of $E$, called bases, satisfying the following axioms:

**(B1)** $\mathcal{B}$ is non empty.

**(B2)** (*Basis Exchange*) Given $B_1, B_2 \in \mathcal{B}$, and $e \in B_1 \backslash B_2$. Then there is an element $g \in B_2 \backslash B_1$ such that, $(B_1 \backslash \{e\}) \cup \{g\} \in \mathcal{B}$.

With regard to dependence, the bases are the maximal independent sets. The size of all basis elements is equal and this defines the *rank* of a matroid. It has been stated that the set of circuits are the minimal dependent sets of the matroid, but the set of all dependent sets has not be described yet. In classical matroid theory a *cycle* is the union of circuits. This notion will be generalised when discussing valuated matroids.

**Example 5.1.4.** The uniform matroid, denoted by $U_{n,r}$, is defined over a ground set $E$, such that $|E| = n$. The independent subsets of $E$ are defined as subsets of $E$ with at most $r$ elements, $\mathcal{I} = \{I \subseteq E : |I| \leqslant r\}$. The bases are the subsets with cardinality $r$, or $\mathcal{B} = \{B \subseteq E : |B| = r\}$. The circuits are hence defined as $\mathcal{C} = \{C \subseteq E : C \backslash x \in \mathcal{I}, \forall x \in C\}$.
One explicit example of an uniform matroid is $U_{4,2}$, with the following defining sets.

$$\mathcal{B} = \Big\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\Big\},$$
$$\mathcal{C} = \Big\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\Big\}.$$

**Example 5.1.5.** As previously mentioned matroids can arise in many different manners. Take the graph $G$ shown in Figure 5.1, a matroid can be constructed from $G$. The ground set is $E = \{1, 2, \dots, 9\}$, which is the edge set of $G$. The independent sets $\mathcal{I}$ of the graphic matroid $G$, are the collections of edges which do not contain a cycle of $G$. A cycle is any simple closed path of $G$. Here is the list of the cycles, rather than the independent sets. The cycles are:

$$\{2\}, \{1,3,4\}.\{1,3,5,6\}, \{4,5,6\}, \{8,9\}, \{6,7,8\}, \{6,7,9\},$$

$$\{1,3,5,8,7\},\{1,3,5,9,7\},\{4,5,8,7\},\{4,5,9,7\}.$$

There is an explicit method to construct a matrix from a graphic matroid. The columns

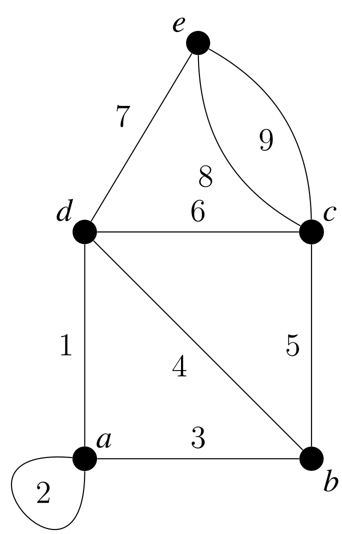


Figure 5.1: The graphic matroid G.

of the matrix are indexed by the ground set $E$, which is the edge set of the graphic matroid. Then the rows are indexed by the vertices of the graphic matroid. The entries of the columns are either 1 or 0. For every column, place a 1 in every row where this edge corresponding to the column meets the vertex corresponding to the row. Place a 0 if the edge does not meet that vertex. If the edge is a loop, then place a zero in every row for that column. The matrix that represents the graphic matroid $G$ in Figure 5.1 is stated now.

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} & \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} & \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} \end{array}$$

To reverse this construction, the independent sets correspond to the sets of columns of the matrix which are linearly independent.

**Definition 5.1.6.** A *spanning set* of a matroid is a set which contains a basis set of the matroid.

### 5.1.2 Valuated Matroids

Matroids can be endowed with extra structure, examples include the class of oriented matroids and the class of valuated matroids. Basis elements of oriented matroids are assigned signs. A detailed description of oriented matroids can be found in [BLV78] and [BBLV+99]. For valuated matroids, each element is assigned a value, determined by a valuation function. A more thorough introduction of valuated matroids can be in found in [DW92]. The class of valuated matroids is used to define tropical ideals.

**Definition 5.1.7.** [MR18, Section 2.1] Given a finite ground set $E$, then $\binom{E}{r}$, where $r \in \mathbb{N}$, denotes subsets of the ground set of size $r$. A valuated matroid on the ground set $E$, with values in $\overline{\mathbb{R}}$, is a pair $\mathcal{M} = (E, \rho)$, where $\rho : \binom{E}{r} \to \overline{\mathbb{R}}$ satisfies the following:

**(V1)** There exists $B \in \binom{E}{r}$, such that $\rho(B) \neq \infty$.

**(V2)** (*Valuated Basis Exchange Axiom*) For every $X, Y \in \binom{E}{r}$, and every $x \in X \backslash Y$ there is a $y \in Y \backslash X$, such that

$$\rho(X) \cdot \rho(Y) \geqslant \rho((X \cup y) \backslash \{x\}) \cdot \rho((Y \cup x) \backslash \{y\})$$

Then, $\rho$ is called the basis valuation function.

Valuated matroids, like standard matroids, have several cryptomorphic axiomatic descriptions. To align with the literature on tropical ideals, such as [MR18], the definitions of the set of circuits and the set of vectors for valuated matroids are now presented. The definitions are based on the descriptions appearing in [MR18, Section 2.1] and formerly in Theorem 3.1 and Theorem 3.4 in [MT01].

**Definition 5.1.8.** [MT01, Theorem 3.1] A collection of elements is the circuits of a valuated matroid, denoted $\mathcal{C}(\mathcal{M}) \subseteq \overline{\mathbb{R}}^E$, if the following properties hold:

(VC0) $(\infty, \ldots, \infty) \notin \mathcal{C}(\mathcal{M})$.

(VC1) If $X \in \mathcal{C}(\mathcal{M})$ and $\lambda \in \mathbb{R}$ then, $\lambda \odot X \in \mathcal{C}(\mathcal{M})$.

(VC2) If $X, Y \in \mathcal{C}(\mathcal{M})$ with $X \neq Y$, then $X \nsubseteq Y$.

(VC3) (*Circuit Elimination*) For any $X, Y \in \mathcal{C}(\mathcal{M})$ and $e, e' \in E$ such that $X_e = Y_e \neq \infty$ and $X_{e'} < Y_{e'}$, then there exists $Z \in \mathcal{C}(\mathcal{M})$ satisfying $Z_e = \infty$, $Z_{e'} = X_{e'}$ and $Z \geqslant X \oplus Y$.

This phrasing of the circuit elimination axiom has been mirror from [MR18], where $X \geqslant X'$ if $X_e \geqslant X'_e$ and $(X \oplus Y)_e = X_e \oplus Y_e$.

The set of circuits in the above definition generates a subsemimodule of $\overline{\mathbb{R}}^E$, for which elements are called vectors of the valuated matroid. The set is denoted $\mathcal{V}(\mathcal{M})$ and defined as

$$\mathcal{V}(\mathcal{M}) := \{ \bigoplus_{C \in \mathcal{C}(\mathcal{M})} \lambda_C \odot C \ : \ \lambda_C \in \overline{\mathbb{R}} \}.$$

In an analogous way to circuits, the set of vectors can be intrinsically defined by axioms described in the following definition.

**Definition 5.1.9.** [MT01, Theorem 3.4] A subset $\mathcal{V} \subseteq \overline{\mathbb{R}}^E$ is the set of vectors of a valuated matroid if and only if it is a subsemimodule of $\overline{\mathbb{R}}^E$ that satisfies the following elimination property:

(*Vector Elimination*) For any $X, Y \in \mathcal{V}$ and any $e \in E$ such that $X_e = Y_e \neq \infty$, there exists $Z \in \mathcal{V}$ with $Z_e = \infty$, $Z \geqslant X \oplus Y$, and $Z_{e'} = X_{e'} \oplus Y_{e'}$ for all $e' \in E$ such that $X_{e'} \neq Y_{e'}$.

Valuated matroids, as above, are just one way of characterising tropical linear spaces. An alternative method is presented in [Ham15], where they are connected to tropical convexity theory.

## 5.2 Tropical Ideals

Denote $\mathrm{Mon}_d$ as the monomials of degree equal to $d$ in the variables $X_0, \ldots, X_n$. When studying matroidal equivalence these sets of monomials may need clarification regarding the variables. If this is the case $\mathrm{Mon}_d\{X_0, \ldots, X_n\}$ will be used to clarify the variables. There is an identification of $\overline{\mathbb{R}}^{\mathrm{Mon}_d}$ with $\overline{\mathbb{R}}[X_0, \ldots, X_n]_d$. This gives that elements of a valuated matroid on the ground set $\mathrm{Mon}_d$ can be viwed as homogenous polynomials with degree equal to $d$ in $\overline{\mathbb{R}}[X_0, \ldots, X_n]$. The definition of a tropical ideal, [MR18, Definition 2.1], is now presented.

**Definition 5.2.1.** (*Tropical Ideals.*) A *homogeneous tropical ideal* is a homogeneous ideal $I \subseteq \overline{\mathbb{R}}[X_0, \ldots, X_n]$ such that for each $d \geqslant 0$, $I_d$ is the collection of vectors of a valuated matroid $\mathcal{M}_d(I)$ on the ground set $\mathrm{Mon}_d$.

In [MR18, Definition 1.1] the definition of a tropical ideal is extended to non-homogeneous ideals over $\overline{\mathbb{R}}$. The aim is to be able to understand matroidal equivalence on both homogeneous and non-homogeneous tropical ideals so we recall the next definition.

**Definition 5.2.2.** Let $I \subseteq \overline{\mathbb{R}}[X_1, \ldots, X_n]$ be a non-homogeneous ideal. $I$ is a tropical ideal if for each $d \geqslant 0$, the set $I_{\leqslant d}$ of polynomials in $I$ with degree at most $d$ is the set of vectors of a valuated matroid.

More explicitly, $I$ satisfies the following *monomial elimination axiom*: for any $f, g \in I_{\leqslant d}$ and any monomial $\underline{X}^{\mathbf{u}} \in \overline{\mathbb{R}}[X_1, \ldots, X_n]_{\leqslant d}$ where the coefficient of $f$ and $g$ are equal and not $\infty$ at $\underline{X}^{\mathbf{u}}$, then there exists a polynomial $h \in I_{\leqslant d}$ such that the coefficient of $h$ at $\underline{X}^{\mathbf{u}}$ is equal to $\infty$ and the coefficient of $h$ at all other monomials is greater than or equal to the minimum of the corresponding coefficients in $f$ and $g$.

Homogeneous tropical ideals can be viewed as the "tower" of compatible valuated matroids, which determine its homogeneous part at each degree. This viewpoint is discussed in detail in [MR18] in Definition 2.5 and Proposition 2.6.

The Hilbert function of a homogeneous ideal $I \subseteq K[X_0, \ldots, X_n]$, is a function which enumerates the dimension of the degree $d$ piece of $K[X_0, \ldots, X_n]/I$ and encodes several invariants of the ideal, for a detailed description of Hilbert functions see [EH06] and

[Har13]. This notion is extended and generalised in [MR18] to tropical ideals in a combinatorial manner involving the underlying matroid of the ideal.

**Definition 5.2.3.** (*Hilbert Function for Tropical ideals.*) Let $I \subseteq \overline{\mathbb{R}}[X_0, \ldots, X_n]$ be a homogeneous tropical ideal. The *Hilbert function* of $I$ is the function $H_I : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ defined by

$$H_I(d) := \operatorname{rank}(\mathcal{M}_d(I))$$

This notion can be generalised to non-homogeneous tropical ideals. This can be done by taking the *homogenisation* of an ideal. Take a tropical polynomial $f = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \odot \underline{X}^{\mathbf{u}} \in \overline{\mathbb{R}}[X_1, \ldots, X_n]$, then the homogenisation of $f$ is defined as

$$f^h := \bigoplus X_0^{d-|\mathbf{u}|} \odot c_{\mathbf{u}} \odot \underline{X}^{\mathbf{u}} \in \overline{\mathbb{R}}[X_0, \ldots, X_n],$$

where $|\mathbf{u}| = \sum_{i=1}^n u_i$ and $d = \max(\,|\mathbf{u}| \,:\, c_{\mathbf{u}} \neq \infty)$. This leads to the homogenisation of an ideal $I$ to be defined as,

$$I^h := \langle f^h \mid f \in I \rangle \subset \overline{\mathbb{R}}[X_0, \ldots, X_n].$$

It is shown in [MR20, Lemma 2.1] that if $I$ is a tropical ideal then so is $I^h$. In the case that $I$ is a non-homogeneous tropical ideal, define its Hilbert function as that of its homogenisation, i.e.

$$H_I = H_{I^h}.$$

It is demonstrated in [MR18, Prop. 3.8] that the Hilbert function of a tropical ideal is eventually polynomial, this then leads to the following definition of the Hilbert polynomial of a homogeneous tropical ideal.

**Definition 5.2.4.** (*Hilbert Polynomial.*) Let $I$ be a homogeneous tropical ideal. The *Hilbert polynomial* of $I$ is the polynomial $P_I$ that agrees with the Hilbert function $H_I$ for $d \gg 0$.

In an analogous manner to the classical Hilbert polynomial, several invariants are encoded in the Hilbert polynomial of a tropical ideal. The *dimension* of a tropical ideal is one example, denoted $\dim(I)$, it is the degree of the associated Hilbert polynomial.

As discussed, tropical ideals have been studied chiefly in [MR18] and [MR20], where deep properties and results have been presented. The set of tropical ideals is strictly contained in the set of all polynomial ideals over $\overline{\mathbb{R}}$. Take the ideal $\langle X \oplus Y \rangle \subseteq \overline{\mathbb{R}}[X, Y]$ as an example on a non-tropical ideal from [MR18]. A tropical ideal is non-realisable if it is not the tropicalisation of any classical ideal over any valued field. An example of a non-realisable tropical is presented in [MR18, Example 2.8] and recalled in the next section. Two more non-trivial results are that tropical ideals satisfy the *Nullstullensatz*, [MR18, Theorem 5.16], and that the varieties of tropical ideals are balanced polyhedral complexes, [MR18, Theorem 5.11] and [MR20, Theorem 1.2].

### 5.2.1 Examples of Tropical Ideals

The following are a selection of examples of tropical ideals from the literature which will be used throughout this work.

**Example 5.2.5.** Take $I \subset K[X_1, \ldots, X_n]$, then under $trop : K \to \overline{\mathbb{R}}$,

$$trop(I) := \langle trop(f) \mid f \in I \rangle.$$

It is discussed in [MR18] that $trop(I) \subset \overline{\mathbb{R}}[X_1, \ldots, X_n]$ is a tropical ideal. Tropical ideals over $\overline{\mathbb{R}}$ of this form are called *realisable* over $K$.

**Example 5.2.6.** An example of a construction which gives non-realisable tropical ideals is presented in [MR18, Ex.2.8], which builds on the result [AB07, Thm.4.1]. It is recalled here. In [MR18] there are descriptions in terms of the basis valuation function and geometrically. Here the geometric interpretation will be discussed as this will be used more prominently in the examples in Section 5.3.3.

Over $\overline{\mathbb{R}}[X_0, \ldots, X_n]$, where $n \geqslant 2$, $I_{\mathrm{NR}}$ will denote the non-realisable, homogeneous tropical ideal defined in the following way. View $\mathrm{Mon}_d$ as the lattice points of a simplex of side length $d+1$. An independent set $\mathcal{I}$ is a collection of $d+1$ or more lattice points,

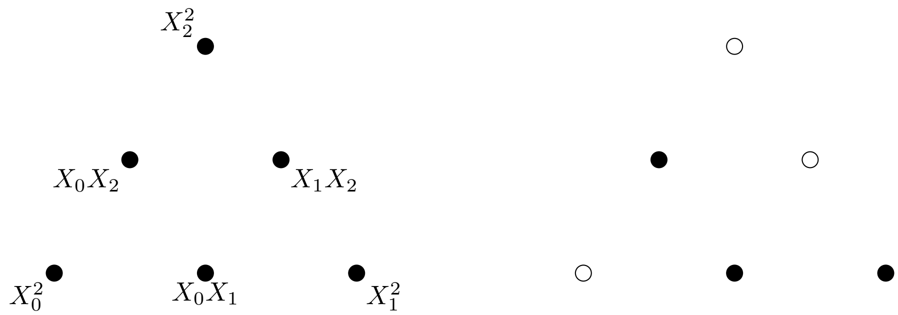


Figure 5.2: $\mathrm{Mon}_2$ of $X_0, X_1, X_2$ as lattice points and a basis set $\{X_0X_2, X_0X_1, X_1^2\}$.

such that for any $k \leqslant d$, $\mathcal{I}$ contains at most $d+1-k$ lattice points of any subsimplex of size $d+1-k$. This condition will be referred at as the *density bound* from here onward. See Figure 5.2 for a visual representation. The basis valuation in [MR18, Ex.2.8] gives a set $\mathcal{B}$ as a basis of $I_{\mathrm{NR}}$, if $|\mathcal{B}| = d+1$ and it satisfies the density bound.

The following example will use the notation $[f]$ where $f$ is a homogeneous tropical polynomial. This is separate to the co-set notation used when discussing quotient hyperfields. In particular, $[f]$ will refer to the *Macaulay tropical ideal* constructed in Example 5.2.7, whereas $[x] = x \cdot U$ refers to a co-set in the quotient construction of hyperfields and $[p]_*$ is the push-forward of a polynomial under the quotient map.

**Example 5.2.7.** In section 4 of [FGG ] there is a description of the method used to construct *tropically principal* ideals (see [FGG , Def.3.1.1]), called the *Macaulay tropical ideals*. The definition and construction is recalled here.

Take a non-zero homogeneous polynomial $f \in \overline{\mathbb{R}}[X_0, \dots, X_n]$, for $d \geqslant \deg(f)$ construct its *Macaulay matrix* $D_d(f)$ in the following way. The rows of $D_d(f)$ are indexed by monomials in the variables $\{X_0, \dots, X_n\}$ with degree equal to $d - \deg(f)$. The columns are indexed by monomials in the variables $\{X_0, \dots, X_n\}$ with degree equal to $d$. The entry $(\underline{X}, \underline{Y})$ of $D_d(f)$ is the coefficient of $\underline{Y}$ in the polynomial $\underline{X}f$. Unless otherwise stated, the monomials are ordered with the lexicographic order in both the rows and columns. Let $[f]_d$ denote the stable sums of the rows of $D_d(f)$, and for $d < \deg(f)$ let

$[f]_d := 0$. Then denote,

$$[f] := \bigoplus_{d \geqslant 0} [f]_d \subset \overline{\mathbb{R}}[X_0, \dots, X_n].$$

$[f]$ is a graded $\overline{\mathbb{R}}$-submodule and referred to as the *Macaulay tropical ideal*. A more detail description of this construction, along with a formal proof as to why $[f]$ is a tropical ideal can be found in [FGG , Section 4], explicitly *Lemma 4.1.2* and *Definition 4.1.3*.

**Example 5.2.8.** A matroid $\mathcal{M}$ is a *paving* matroid if all the circuits are of size $\operatorname{rank}(\mathcal{M})$ or $\operatorname{rank}(\mathcal{M}) + 1$. In [AR22] the authors study zero dimensional tropical ideals, whose underlying matroid is a paving matroid.

## 5.3 Matroidal Equivalence

In this section the notion of matroidal equivalence is introduced. It will be defined for (non)homogeneous ideals over both a field $K$ and the tropical semiring $\overline{\mathbb{R}}$ separately. When defined over $\overline{\mathbb{R}}$ it is a property which relies of the underlying combinatorial structure of tropical ideals. The aim is to use this property to understand when the geometric objects defined by tropical ideals should be consider as equivalent. A number of examples of ideals that are matroidally equivalent are presented both over a field and the tropical semiring.

### 5.3.1 Matroidal Equivalence Over Fields

**Definition 5.3.1.** (*Matroidal Equivalence*) Let $J \subseteq K[X_0, \dots, X_m]$ and $I \subseteq K[X_0, \dots, X_n]$, be homogeneous ideals, with $m < n$. Then, $J$ is called *matroidally equivalent* to $I$ if the following three conditions hold:

(ME1) The inclusion map, $K[X_0, \dots, X_m] \hookrightarrow K[X_0, \dots, X_n]$, sends $J$ into $I$.

(ME2) The set $S_d = \{\text{monomials of degree } d, \textit{ not } \text{divisble by } X_{m+1}, \dots, X_n\}$ is a spanning set over $\mathcal{M}_d(I)$ for all $d > 0$.

(ME3) The Hilbert polynomials of $J$ and $I$ are equal, i.e.

$$P_J = P_I.$$

To be as complete as possible the appropriate adjustments are made for non-homogeneous ideals below.

**Definition 5.3.2.** (*Affine Matroidal Equivalence*) Let $J \subseteq K[X_1, \ldots, X_m]$ and $I \subseteq K[X_1, \ldots, X_n]$ be non-homogenous ideals. Then, $J$ is called *matroidally equivalent* to $I$ if the following three conditions holds:

(ME1) The inclusion map, $K[X_0, \ldots, X_m] \hookrightarrow K[X_0, \ldots, X_n]$, sends $J$ into $I$.

(ME2) The set $S_{\leqslant d} = \{\text{monomials of degree } \leqslant d, \textit{ not} \text{ divisble by } X_{m+1}, \ldots, X_n\}$ is a spanning set over $\mathcal{M}_{\leqslant d}(I)$ for all $d > 0$.

(ME3)

$$\dim\Big(K[X_1, \ldots, X_m]_{\leqslant d}/J_{\leqslant d}\Big) = \dim\Big(K[X_1, \ldots, X_n]_{\leqslant d}/I_{\leqslant d}\Big).$$

### 5.3.2 Matroidal Equivalence Over The Tropical Semiring

The definitions in Definition 5.3.1 and Definition 5.3.2 are stated here in a style suitable for tropical ideals.

**Definition 5.3.3.** (*Matroidal Equivalence*) Let $J \subseteq \overline{\mathbb{R}}[X_0, \ldots, X_m]$ and $I \subseteq \overline{\mathbb{R}}[X_0, \ldots, X_n]$, be homogeneous tropical ideals, with $m < n$. Then, $J$ is called *matroidally equivalent* to $I$ if the following three conditions hold:

(ME1) The circuits of $J$ are included in the vectors of $I$ for each degree $d > 0$,

$$\mathcal{C}(\mathcal{M}_d(J)) \subseteq \mathcal{V}(\mathcal{M}_d(I)).$$

(ME2) The set $S_d = \{\text{monomials of degree } d, \textit{ not} \text{ divisble by } X_{m+1}, \ldots, X_n\}$ is a spanning set over $\mathcal{M}_d(I)$ for all $d > 0$.

(ME3) The Hilbert polynomials of $J$ and $I$ are equal, i.e.

$$P_J = P_I.$$

Equivalently to working over $K$, the definition of matroidal equivalence is appropriately adjusted for non-homogeneous tropical ideals over $\overline{\mathbb{R}}$ below.

**Definition 5.3.4.** (*Affine Matroidal Equivalence*) Let $J \subseteq \overline{\mathbb{R}}[X_1, \ldots, X_m]$ and $I \subseteq \overline{\mathbb{R}}[X_1, \ldots, X_n]$ be non-homogenous tropical ideals. Then $J$ is called *matroidally equivalent* to $I$ if the following three conditions holds:

(ME1) The circuits of $J$ are included in the vectors of $I$ for each degree $d > 0$,

$$\mathcal{C}(\mathcal{M}_{\leqslant d}(J)) \subseteq \mathcal{V}(\mathcal{M}_{\leqslant d}(I)).$$

(ME2) The set $S_{\leqslant d} = \{\text{monomials of degree } \leqslant d, \textit{not} \text{ divisble by } X_{m+1}, \ldots, X_n\}$ is a spanning set over $\mathcal{M}_{\leqslant d}(I)$ for all $d > 0$.

(ME3) $\operatorname{rank}(\mathcal{M}(J)_{\leqslant d}) = \operatorname{rank}(\mathcal{M}(I)_{\leqslant d})$.

Note that, it is most logical to discuss matroidal equivalence between either two homogeneous or two non-homogeneous tropical ideals, and not between a homogeneous and non-homogeneous directly. The matroidal equivalence conditions are denoted by the same notation for both homogeneous and non-homogeneous in Definition 5.3.1 and Definition 5.3.2 respectively. As the context will be clear when discussing either the projective or affine case the notation is unaltered.

*Remark* 5.3.5. The property of matroidal equivalence is slightly misleading in the nature of its name. The property is not an equivalence relation as the property is in general not symmetric.

From this point onward when discussing two ideals that are connected directly by the matroidal equivalance definition this will be called an *elementary matroidal equivalence*. Whereas in more generality, two tropical ideals are said to be *matroidally equivalent* if there exists a chain of matroidal equivalences connecting them. This is a broader notion of matroidal equivalence, enabling ideals to be related in more extensive manner. For instance, if $J_1, J_2 \subset \overline{\mathbb{R}}[X_0, \ldots, X_m]$ both satisfy elementary matroidal equivalences with $I \subset \overline{\mathbb{R}}[X_0, \ldots, X_n]$, then $J_1$ is matroidally equivalent to $J_2$ by the chain $J_1 \dashrightarrow I \dashleftarrow J_2$.

### 5.3.3 Examples of Matroidal Equivalence

There will now be a description of examples of elementary matroidal equivalence between tropical ideals. The first is an example is over a field $K$ for homogeneous ideals so utilises Definition 5.3.1, then the remaining examples are set over $\overline{\mathbb{R}}$, hence will use corresponding defintions for tropical ideals.

**Example 5.3.6.** Take $J = \langle 0 \rangle \subseteq K[X_0, X_1]$ and $I = \langle X_0 + X_1 + X_2 \rangle \subseteq K[X_0, X_1, X_2]$, then $J$ is matroidally equivalent to $I$.

(ME1) $\mathcal{C}(\mathcal{M}_d(J))$ is the empty set so trivially includes into $\mathcal{V}(\mathcal{M}_d(I))$.

(ME2) Set $S_d = \{$monomials of degree $d$, *not* divisble by $X_2\}$. Firstly, $|S_d| = d + 1$ which is the rank of $\mathcal{M}(I)$ as $P_I = m + 1$, (see third part for explicit details). So if $S_d$ is a spanning set, it is precisely a basis element. Assume that $S_d$ is an element of $I$ and hence a dependent set in $\mathcal{M}_d(I)$. As $K$ is a field, for $S_d$ to be an element of $I$ is has to be a multiple of $X_0 + X_1 + X_2$, or $S_d = \text{supp}\{(X_0 + X_1 + X_2) \cdot q\}$. For this to hold, the polynomial $q$ has to have some term $X_0^{d-1}$ for $X_0^d \in S_d$. This gives an element $X_0^{d-1}X_2 \in (X_0 + X_1 + X_2) \cdot q$, although $X_0^{d-1}X_2 \notin S_d$. To rectify this $q$ must have another term which results in a cancellation. This can only happen when $X_0^{d-2}X_2 \in \text{supp}(q)$. This gives $X_0^{d-2}X_2 \in (X_0 + X_1 + X_2) \cdot q$ , but again $X_0^{d-2}X_2 \notin S_d$. This logic can be iterated to give a conclusion that $X_2^d \in S_d$. This is clearly a contradiction by the definition of $S_d$. Hence, $S_d \neq \text{supp}\{(X_0 + X_1 + X_2) \cdot q\}$, for any $q$. Therefore, it is an independent set of size equal to the rank of the matroid of $I$, which implies it is a basis set and furthermore a spanning set.

(ME3) The Hilbert function of $J$ is;

$$\begin{aligned} H_J(d) &= \dim_K \Big( K[X_0, X_1]/\langle 0 \rangle \Big)_d, \\ &= \dim_K(K[X_0, X_1]), \\ &= \dim_K(\text{span}\{X_0^d, X_0^{d-1}X_1, \ldots, X_0X_1^{d-1}, X_1^d\}), \\ &= d + 1. \end{aligned}$$

The Hilbert function of $I$ can be calculated in a similar way;

$$\begin{aligned}
H_I(d) &= \dim_K\Big(K[X_0, X_1, X_2]/\langle X_0 + X_1 + X_2\rangle\Big)_d, \\
&= \dim_K(K[X_0, X_1, X_2]_d) - \dim_K(K[X_0, X_1, X_2]_{d-1}), \\
&= \binom{d+2}{2} - \binom{d+1}{2}, \\
&= d + 1.
\end{aligned}$$

This demonstrates that $P_I = d + 1 = P_J$.

Together, these three parts show that all $J$ is matroidally equivalent to $I$. Matroidal equivalence over a field $K$, as in this example, is given in relation to the varieties of the ideals in Theorem 5.3.12.

**Example 5.3.7.** The tropical ideal $J = \langle\infty\rangle \subseteq \overline{\mathbb{R}}[X_0, \ldots X_n]$ is matroidally equivalent to $I = [f] \subseteq \overline{\mathbb{R}}[X_0, \ldots X_{n+1}]$, where $f \in \overline{\mathbb{R}}[X_0, \ldots X_{n+1}]$ has full support and $\deg(f) = 1$.

(ME1) The set $\mathcal{C}(\mathcal{M}_d(J))$ is empty so includes trivially into $\mathcal{V}(\mathcal{M}_d(I))$.

(ME2) Set $S_d = \{$monomials of degree $d$, *not* divisble by $X_{n+1}\}$. The Stiefel construction in [FGG , Section 4.1] is phrased in terms of the dual matroid, whereas thus far the convention has been aligned with the standard matroid picture, as described in [MR18] and [MR20]. Therefore, the aim will be to demonstrate that the monomials *divisible* by $X_{n+1}$ are independent in the [FGG ] convention. This is equivalent to the monomials *divisible* by $X_{n+1}$ being contained in a basis.

The bases of $[f]$ are the sets of columns which have non-zero permanents of the Macaulay matrix, which is invariant under column and row reordering. The *reverse-lexicographic* monomial term ordering is explained in [MR20, Ex.2.4]. Reorder both the rows and columns with the reverse-lexicographic ordering, starting with the least most element. This gives that the right most square submatrix, denoted $A_{X_{n+1}}$, is column indexed by the monomials divisible by $X_{n+1}$. The diagonal elements of $A_{X_{n+1}}$ are of the form $(\underline{X}, \underline{X}')$, where $\underline{X}$ is a monomial of degree

$d-1$ and $\underline{X}' = X_{n+1} \cdot \underline{X}$. The coefficient of $\underline{X}'$ in $\underline{X} \cdot f$ is then the corresponding coefficient of $X_{n+1}$ in $f$. By the reordering, the entries to the right of the diagonal entries of $A_{X_{n+1}}$, on all rows are zero. As the maximum power of $X_{n+1}$ in $\underline{X} \cdot f$ for any $\underline{X}$ has to come from the monomial multiplied with $X_{n+1}$, but by construction this is the diagonal entry. This gives the description of $A_{X_{n+1}}$ as a lower triangular matrix, with diagonal entries as the coefficient of $X_{n+1}$ in $f$. As $f$ has full support, the coefficient of $X_{n+1}$ in $f$ in non-zero. This gives that the permanent of $A_{X_{n+1}}$ is the product of the non-zero diagonal elements and hence, non-zero. Therefore, the monomials of degree equal to $d$, divisible by $X_{n+1}$ is a basis over $[f]$.

To make this logic more explicit, take $f = aX_0 + aX_1 + cX_2 \in \overline{\mathbb{R}}[X_0, X_1, X_2]$. Then the corresponding reordered Macaulay matrices up to degree three are,

$$D_1(f) = \begin{pmatrix} a & b & c \end{pmatrix},$$

$$D_2(f) = \begin{array}{c} \\ X_0 \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cccccc} X_0^2 & X_0X1 & X_1^2 & X_0X_2 & X_1X_2 & X_2^2 \end{array} \\ \left(\begin{array}{ccc|ccc} a & b & 0 & c & 0 & 0 \\ 0 & a & b & 0 & c & 0 \\ 0 & 0 & 0 & a & b & c \end{array}\right) \end{array}$$

$$D_3(f) = \begin{array}{c} \\ X_0^2 \\ X_0X1 \\ X_1^2 \\ X_0X_2 \\ X_1X_2 \\ X_2^2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} X_0^3 & X_0^2X_1 & X_0X_1^2 & X_1^3 & X_0^2X_2 & X_0X_1X_2 & X_1^2X_2 & X_0X_2^2 & X_1X_2^2 & X_2^3 \end{array} \\ \left(\begin{array}{cccc|cccccc} a & b & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & \mathbf{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & 0 & \mathbf{c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 & \mathbf{c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & \mathbf{c} \end{array}\right) \end{array}$$

(ME3) The Hilbert function of $J$ is,

$$H_J(d) = \binom{n+d}{d}.$$

This is counting the number of monomials of degree equal to $d$ in the variables $X_0, \ldots, X_n$. The Hilbert function of $I$, as described in [FGG , Prop.4.2.2], is,

$$\begin{aligned} H_I(d) &= \binom{n+1+d}{d} - \binom{n+1+d-\deg(f)}{d-\deg(f)}, \\ &= \binom{n+1+d}{d} - \binom{n+d}{d-1}. \end{aligned}$$

The first term is counting the number of monomials with degree equal to $d$ in the variables $X_0, \ldots, X_{n+1}$. The second term is counting the number of monomials of degree equal to $d-1$ in the variables $X_0, \ldots, X_{n+1}$, which is equal to the number of monomials with degree equal to $d$ and divisible by $X_{n+1}$ in the variables $X_0, \ldots, X_{n+1}$. Taking the second term from the first results in the counting of the number of monomials with degree equal to $d$, *not* divisible by $X_{n+1}$ in the variables $X_0, \ldots, X_{n+1}$. In other words the Hilbert function of $I$ counts the number of monomials with degree equal to $d$ in the variables $X_0, \ldots, X_n$. Hence, both $H_J(d)$ and $H_I(d)$ are counting monomials of degree equal to $d$ in the variables $X_0, \ldots, X_n$, so are equal. Yielding, $P_J = P_I$.

**Example 5.3.8.** The tropical ideal $J = \langle \infty \rangle \subseteq \overline{\mathbb{R}}[X_0, X_1]$ is matroidally equivalent to $I_{\text{NR}} \subseteq \overline{\mathbb{R}}[X_0, \ldots, X_n]$ for $n \geqslant 2$.

(ME1) The set $C(\mathcal{M}_d(J))$ is empty so includes trivially into $\mathcal{V}(\mathcal{M}_d(I))$.

(ME2) It can be seen that,

$$\begin{aligned} S_d &= \{\text{monomials of degree}\, d,\ \textit{not}\ \text{divisble by}\, X_2, \ldots X_n\}, \\ &= \{\text{monomials of degree}\, d,\ \text{in the variables}\, X_0, X_1\}. \end{aligned}$$

In $\text{Mon}_d$, the lattice points corresponding to $S_d$ will form a one dimensional edge of the whole simplex of lattice points. By the Hilbert polynomial calculation, $|S_d| = d+1$, so if $S_d$ is a spanning set it will be a basis. For every $k \leqslant d$, a subsimplex of $\text{Mon}_d$ of length $d-k+1$, will at most have $S_d$ as a full one dimensional edge. Hence, the number of elements of $S_d$ in the subsimplex is at most $d-k+1$. Therefore, $S_d$ is a basis and thus spanning set over the matroid of $I_{\text{NR}}$. See Figure 5.3 for visual details.

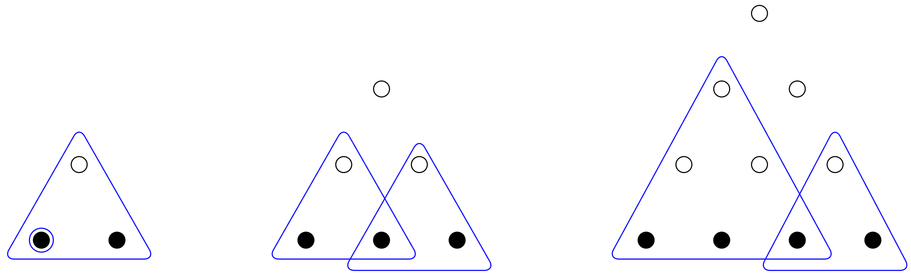

Figure 5.3: The lattices $\mathrm{Mon}_1, \mathrm{Mon}_2$ and $\mathrm{Mon}_3$ of $\{X_0, X_1, X_2\}$, with the shaded sets $S_1 = \{X_0, X_1\}, S_2 = \{X_0^2, X_0X_1\}$ and $S_3 = \{X_0^3, X_0^2X_1, X_0X_1^2, X_1^3\}$ respectively and a sample of blue subsimplices.

(ME3) The NR construction from [MR18, Ex.2.8] and [AB07] gives a towering set of matroids of rank $d+1$ over $\mathrm{Mon}_d$. Hence, $P_{I_{\mathrm{NR}}} = d+1$. Using the calculation of $H_J(d)$, previously shown in Example 5.3.7, with $n = 1$ as $J \subseteq \overline{\mathbb{R}}[X_0, X_1]$,

$$H_J(d) = \binom{n+d}{d} = \binom{1+d}{d} = d+1.$$

This gives, $P_{I_{\mathrm{NR}}} = d+1 = P_J$.

**Example 5.3.9.** Furthermore, $J_{\mathrm{NR}} \subseteq \overline{\mathbb{R}}[X_0, \dots, X_n]$ is matroidally equivalent to $I_{\mathrm{NR}} \subseteq \overline{\mathbb{R}}[X_0, \dots, X_{n+r}]$, where $r > 0$.

(ME1) A circuit of $J_{\mathrm{NR}}$ is a set of lattice points in $\mathrm{Mon}_d\{X_0, \dots, X_n\}$,which exceeds the density bound, where the removal of a singular lattice point would result in the density bound holding globally. Embedding this set of lattice points corresponding to the circuit, into the simplex of lattice points of $\mathrm{Mon}_d\{X_0, \dots, X_{n+r}\}$, will still cause the density bound to be exceeded. With the condition that if the correct point is removed, the density bound is satisfied. This logic demonstrates that $\mathcal{C}(\mathcal{M}_d(J_{\mathrm{NR}})) \subseteq \mathcal{C}(\mathcal{M}_d(I_{\mathrm{NR}}))$, hence $\mathcal{C}(\mathcal{M}_d(J_{\mathrm{NR}})) \subseteq \mathcal{V}(\mathcal{M}_d(I_{\mathrm{NR}}))$.

(ME2) It can be seen that,

$$\begin{aligned} S_d &= \{\text{monomials of degree } d,\ \textit{not} \text{ divisble by } X_{n+1}, \dots, X_{n+r}\}, \\ &\supseteq \{\text{monomials of degree } d,\ \textit{not} \text{ divisble by } X_2, \dots X_{n+r}\}, \\ &= \{\text{monomials of degree } d, \text{ in the variables } X_0, X_1\}. \end{aligned}$$

Then, by Example 5.3.8, this $S_d$ contains a basis, and hence spanning set.

(ME3) The construction, in both cases, gives towering matroids of rank $d+1$, as stated in [MR18, Ex.2.8]. Thus, $P_{J_{\mathrm{NR}}} = d+1 = P_{I_{\mathrm{NR}}}$.

### 5.3.4 Properties of Matroidal Equivalence

This section will focus on the properties of matroidal equivalence. In particular, it will be shown that matroidal equivalence is preserved under tropicalisation and that there is a close relationship between the varieties of matroidally equivalent ideals over $K$.

When discussing the axioms of the matroidal equivalence definition, this implicitly has an underlying inclusion map of the variables at its heart. For instance over $K$

$$i : K[X_1, \dots, X_m] \hookrightarrow K[X_1, \dots, X_n],$$

where $n > m$. In particular, this is the only natural way to interpret the axiom (ME1). The map here is a monomial homomorphism of ambient algebras and corresponds to a coordinate projection of the variety over $K^n$ onto the variety over $K^m$. This connection will be made clearer in the next results.

**Proposition 5.3.10.** *Let $L \subseteq K^E$ be a linear space, where $E$ is the ground set of the associated matroid of $L$. For $W \subset E$, then*

$$K^W \to K^E \to K^E/L \tag{5.3.11}$$

*is surjective if and only if $W$ is a spanning set of the matroid of $L$*

*Proof.* In the matroid convention of Maclagan and Rincon in both [MR18] and [MR20], a set $W \subset E$ is independent in the matroid of $L$ if and only if (5.3.11) is injective. In other words, the basis vectors corresponding to $W$ remain linearly independent after quotienting by $L$. As a basis is a maximal independent set, this gives that a basis $W \subset E$ yields (5.3.11) is an isomorphism. By definition, a spanning set contains a basis, which is equivalent to (5.3.11) being surjective. $\square$

**Theorem 5.3.12.** *The ideals $J \subseteq K[X_1, \dots, X_m]$ and $I \subseteq K[X_1, \dots, X_n]$, with $m < n$, are matroidally equivalent if and only if $V(I)$ maps isomorphically onto $V(J)$.*

*Proof.* Considering the inclusion $i : K[X_1, \dots, X_m] \hookrightarrow K[X_1, \dots, X_n]$, where $m < n$. The condition (ME1) corresponds to the inclusion map $i$ mapping $J$ into $I$. This is equivalent to the composition

$$K[X_1, \dots, X_m] \hookrightarrow K[X_1, \dots, X_n] \to K[X_1, \dots, X_n]/I, \tag{5.3.13}$$

descending to the quotient by $J$. Hence, there exists a map

$$K[X_1, \dots, X_m]/J \to K[X_1, \dots, X_n]/I. \tag{5.3.14}$$

For (5.3.14) to be an isomorphism, as required, both sides must have the same dimension. This is precisely capture by (ME3) which requires the Hilbert polynomials $P_J$ and $P_I$ to be equal.

As the dimensions are equal, (5.3.14) is an isomorphism if and only if it is surjective, which is equivalent to (5.3.13) being surjective. By Proposition 5.3.10 this is equivalent to stating that

$$S_{\leqslant d} = \{\text{monomials of degree at most } d, \textit{ not} \text{ divisble by some } X_{m+1}, \dots, X_n\},$$

is a spanning set in $\mathcal{M}_{\leqslant d}(I)$ for all $d > 0$ which is exactly (ME2).

Therefore, (5.3.14) is an isomorphism of algebras which corresponds to an isomorphism of varieties $V(I) \to V(J)$. □

As the definitions of matroidal equivalence differ for non-homogeneous and homogeneous ideals the next result is stated to thoroughly adjust for those differences.

**Theorem 5.3.15.** *Let $J \subseteq K[X_0, \dots, X_m]$ and $I \subseteq K[X_0, \dots, X_n]$ be homogeneous ideals, with $m < n$. The morphism of ambient space polynomial algebras induces an isomorphism on the quotients,*

$$K[X_0, \dots, X_m]/J, \quad \textit{and} \quad K[X_0, \dots, X_n]/I,$$

*if and only if $J$ is matroidally equivalent to $I$.*

*Proof.* This proof will follow a similar pattern to the proof of Theorem 5.3.12 with the appropriate adjustments for the homogeneous structure.

Considering the inclusion $i : K[X_1, \ldots, X_m] \hookrightarrow K[X_1, \ldots, X_n]$, where $m < n$. The condition (ME1) corresponds to the inclusion map $i$ mapping $J$ into $I$. This is equivalent to the composition

$$K[X_1, \ldots, X_m] \hookrightarrow K[X_1, \ldots, X_n] \to K[X_1, \ldots, X_n]/I, \tag{5.3.16}$$

descending to the quotient by $J$. Hence, there exists a map

$$K[X_1, \ldots, X_m]/J \to K[X_1, \ldots, X_n]/I. \tag{5.3.17}$$

For (5.3.17) to be an isomorphism, as required, both sides must have the same dimension. this is precisely capture by (ME3) which requires

$$\dim\Big(K[X_1, \ldots, X_m]_{\leqslant d}/J_{\leqslant d}\Big) = \dim\Big(K[X_1, \ldots, X_n]_{\leqslant d}/I_{\leqslant d}\Big).$$

As the dimensions are equal, (5.3.17) is an isomorphism if and only if it is surjective, which is equivalent to (5.3.16) being surjective. By Proposition 5.3.10 this is equivalent to stating that

$$S_d = \{\text{monomials of degree}\, d,\ \textit{not}\ \text{divisible by some}\, X_{m+1}, \ldots, X_n\},$$

is a spanning set in $\mathcal{M}_d(I)$ for all $d > 0$ which is exactly (ME2).

Therefore, (5.3.17) is an isomorphism of the algebras $K[X_0, \ldots, X_m]/J$ and $K[X_0, \ldots, X_n]/I$.

□

The result in Theorem 5.3.12 is stated geometrically in terms of the varieties of $J$ and $I$, whereas the result in Theorem 5.3.15 is stated in terms of algebras and quotients of the polynomial ring. This is due to the fact that not all morphisms of projective schemes come from morphisms of algebras. Furthermore, not all morphisms of algebras induce morphisms of projective schemes. For example, take the morphism $K[X_0, X_1] \hookrightarrow K[X_0, X_1, X_2]$, then $K\mathbb{P}^2 \to K\mathbb{P}^1$, is not defined at the point corresponding to the $X_2$-axis.

As discussed one key aspect of tropical geometry is the tropicalisation map, *trop* : $K \to \overline{\mathbb{R}}$, where $K$ is a valued field. Matroidal equivalence has been discussed over both $K$

(Example 5.3.6) and $\overline{\mathbb{R}}$. As $I$ and $trop(I)$ are both tropical ideals the natural question would be to ask whether tropicalisation preserves matroidal equivalence? The next result will show that matroidal equivalence is preserved under $trop : K \to \overline{\mathbb{R}}$.

**Lemma 5.3.18.** *Let $J \subseteq K[X_0, \ldots, X_m]$ and $I \subseteq K[X_0, \ldots, X_n]$ with $m < n$, be matroidally equivalent. Then, under a tropicalisation map $trop : K \to \overline{\mathbb{R}}$, $trop(J)$ is matroidally equivalent to $trop(I)$.*

*Proof.*

(ME1) The inclusion, over $K$ is equivalent to the linear space defined by $J$ being included in the linear space defined by $I$. The tropicalisation map preserves inclusions of linear spaces. This implies that the tropical linear space defined by $trop(J)$ includes into the tropical linear space defined by $trop(I)$, which is equivalent to, $\mathcal{C}(\mathcal{M}_d(trop(J))) \subseteq \mathcal{V}(\mathcal{M}_d(trop(I)))$, for all $d$.

(ME2) It is shown in [GG16, Thm.7.1.6] that the the Hilbert function is preserved under tropicalisation, thus the Hilbert polynomial is consequently preserved. Hence, as $P_J = P_I$, it can be seen that

$$P_{trop(J)} = P_J = P_I = P_{trop(I)}.$$

(ME3) The set $S_d = \{\text{monomials of degree } d, \textit{ not} \text{ divisble by } X_{n+1}, \ldots, X_m\}$ is a spanning set over $\mathcal{M}_d(I)$ for all $d > 0$. Then, by definition $S_d$ contains a basis of $\mathcal{M}_d(I)$. The tropicalisation map sends basis to basis, therefore $S_d$ contains a basis of $\mathcal{M}_d(trop(I))$, and thus a spanning set.

□

**Lemma 5.3.19.** *Take $J \subseteq K[X_0, \ldots, X_m], I \subseteq K[X_0, \ldots, X_n]$ and $W \subseteq K[X_0, \ldots, X_t]$, where $m < n < t$, such that $J$ is matroidally equivalent to $I$ and $I$ is matroidally equivalent to $W$. Then, $J$ is matroidally equivalent to $W$.*

*Proof.*

(ME1) Over $K$ the vectors of a matroid are the linear combinations of the circuits, and the set of vectors is closed under linear combinations. Therefore, $\mathcal{C}(\mathcal{M}_d(I)) \subseteq \mathcal{V}(\mathcal{M}_d(W)) \Rightarrow \mathcal{V}(\mathcal{M}_d(I)) \subseteq \mathcal{V}(\mathcal{M}_d(W))$. This implies, $\mathcal{C}(\mathcal{M}_d(J)) \subseteq \mathcal{V}(\mathcal{M}_d(I)) \subseteq \mathcal{V}(\mathcal{M}_d(W))$.

(ME2) By the matroidal equivalences,

$$P_J = P_I = P_W.$$

(ME3) The monomials divisible by $X_0, \dots, X_m$ are a spanning set in each degree for $I$. This implies that each variable $X_i \in \{X_{m+1}, \dots, X_n\}$ has a polynomial $X_i + p_i(X_0, \dots, X_m) \in I$, so $X_i \sim_{\mathrm{Mod}_I} -p_i(X_0, \dots, X_m)$, noting that $p_i(X_0, \dots, X_m)$ may not be unique. Similarly, the monomials divisible by $X_0, \dots, X_n$ are a spanning set for $W$ in each degree. Therefore, each $X_w \in \{X_{n+1}, \dots, X_t\}$ has a polynomial $X_w + Q_w(X_0, \dots, X_n) \in W$. As (ME1) holds between $J$ and $W$, the relations $X_w + Q_w(X_0, \dots, X_n)$ can be converted to $X_w + \hat{Q}_w(X_0, \dots, X_m)$, by the property that each variable $X_i \in \{X_{m+1}, \dots, X_n\}$ has a polynomial $X_i + p_i(X_0, \dots, X_m) \in I$. Thus, each $X \in \{X_{m+1}, \dots, X_t\}$ can be expressed in terms of just the variables $X_0, \dots, X_m$. Hence, the monomials not divisible by $X_{m+1}, \dots, X_t$ are a spanning set for $W$.

□

## 5.4 Questions and Aims

There are many open questions remaining in relation to the property of matroidal equivalence. For instance, how is matroidal equivalence affected after taking the (de)homogenisation, saturation and initial form of a tropical ideal? Exploring this would enable matroidal equivalence to be discuss between non-homogeneous and homogeneous tropical ideals.

In a deeper sense, it would be prudent to determine the coarseness of matroidal equivalence when comparing with each axiom independently over $\overline{\mathbb{R}}$. Say for example, does it encode more information that simply associating tropical ideals that have equal Hilbert polynomials.

The motivation for the introduction of matroidal equivalence was to understand when the geometric objects defined by tropical ideals should be associate with each other. It has been demonstrated that for ideals over $K$ this relation connects to induced isomophisms of varieties. It can be seen that this precise relationship does not hold in the less restrictive setting of the tropical semi-ring. A substantial goal is to understand how the geometric objects defined by matroidally equivalent tropical ideals are connected. Specically, using the techniques of tropical modification, such as in [CM16], [Kal15] and [Sha15].

More precisely, we conjecture that if there is an elementary matroidal equivalence between two tropical ideals, then the corresponding varieties are related by a tropical modification. This is a key question and the next aim of the project is to understand this. Additionally, we conjecture that the class of tropical ideals whose variety is a tropical linear space and whose Hilbert polynomial is that of a tropical linear space are matroidally equivalent. Although, tropical ideals with varieties of different degrees should not be connected by matroidal equivalence. This is because the Hilbert polynomial encodes the degree of the corresponding variety and (ME2) specifies that the Hilbert polynomials are equal.

A long term aspiration is to employ the definition of matroidal equivalence to present a viable notion of a polynomial ideal defined over a hyperfield. It has been seen that the set $\mathbb{H}[X_1, \ldots, X_n]$ does not have the structure of a hyperring. This is due to the additional multivalued nature of the mutliplication. This subtle difference immediately challenges the classical notion of a polynomial ideal. The recent establishment of a unifying theory of matroids over hyperfields presented in [BB18] highlights a possible solution. Can the matroid theory be leveraged, in an analogous way to the defintion of tropical ideals, as an underlying tool to construct polynomial ideals over hyperfields?

If this is the case, it would be important to investigate the implications of matroidal equivalence over hyperfields.

# Chapter 6

# Convex Geometry over Hyperfields

In convex geometry, the properties of convex sets are explored. The standard setting for convex geometry is $\mathbb{R}^n$ and can be referred to as real convex geometry. Convex sets are intuitively sets that for any two elements the straight line connecting them is entirely contained within the set. Examples of classical convex sets are linear halfspaces. Convex sets occur naturally in many distinct areas of mathematics, including game theory, probability and functional analysis.

This chapter focuses on generalising the notions of real convex geometry to the multi-valued setting of hyper-structures. This simultaneously builds on the results developed on signed tropical convexity in [LV19]. Real convex geometry has applications to optimisation and linear programming, this connection has been further developed in [LV19] for the signed tropical setting. This relationship is one motivating factor to the results presented in this chapter.

One of the areas in which there is scope for applications of convex geometry over hyperfields is the problem of complexity of linear programming. There is a connection between linear programming over $\mathbb{R}$ and tropical linear programming, and hence to signed tropical programming. Linear programming in these senses is interconnected to the respective convex optimisation. Therefore, by understanding how useful the framework of hyperfields is for studying convex geometry, there is potential for impact on the questions of complexity in linear programming over $\mathbb{R}$.

This chapter is part of a collaborative endeavour with B.Smith, for which there are plans to continue the research in the future.

## 6.1 Background & Motivation

Orderings on hyper-structures are recalled from [AD19], [LS20] and [KLS21]. These are then used to present several properties of ordered hyperfields. Further work on real multirings is presented in [Mar06], although this chapter will focus on ordered hyperfields. The main results of this chapter will be established for ordered stringent hyperfields. The work outlined in [BS20] presents a classification of stringent hyperfields as semi-direct products and quotient hyperfields. This is specialised here to demonstrate a precise classification of ordered stringent hyperfields.

The concepts of orderings and polynomials are connected by presenting definitions of open and closed halfspaces. The behaviour of open and closed halfspaces, as well as varieties, under hyperfields homomorphims is examined. It is asserted that a version of Kapranov's theorem holds for linear polynomials under quotient maps. There is more thoroughness needed when investigating open and closed halfspaces, as can be seen in Example 6.3.14.

The definitions of conic and convex sets are introduced. These are an algebraic generalisation of the corresponding notions in classic convex geometry. It is explored how they interact with hyperfield homomorphisms, with particular focus on quotient homomorphisms. Several classical results from real convex geometry are recalled and proved for ordered fields in more generality, which are then in turn utilised to develop Radon's and Helly's Theorems for ordered hyperfields that admit a order preserving homomorphism from an ordered field. Then finally, Caratheodory's Theorem is discussed for ordered quotient hyperfields. The chapter is concluded with further questions that the candidate and B.Smith would be interested in working on in the future. For instance, the existence of separation theorems for hyperfields.

## 6.2 Orderings On Hyper-structures

The main results of this section will be stated for stringent and quotient hyperfields. They will be recalled here. (Note that in the literature on ordered hyerpfields, such as [KLS21], the notion of quotient hyperfields is referred to as *Factor* hyperfields.)

**Definition 6.2.1.** A hypergroup is called *stringent* if the addition for $x, y \in \mathbb{H}$, $x \boxplus y$ is a singleton whenever $x \neq -y$. A hyperring is called *stringent* if the underlying hypergroup is stringent.

**Example 6.2.2.** Let $(\mathbb{H}, \boxplus, \odot)$ be a hyperfield and take $U \subseteq \mathbb{H}^\times$ a subgroup of the non-zero elements of the hyperfield. Then the quotient is defined as $\mathbb{H}/U := \mathbb{H}^\times/U \cup \{0\}$, which has a hyperfield structure due to the following operations. Elements of $\mathbb{H}/U$ are cosets, defined as $[x] := \{x \odot u \ : \ u \in U\}$. The multiplication is inherited from the hyperfield $\mathbb{H}$, $[x] \odot [y] = [x \odot y]$, and the multivalued addition is defined as;

$$[x] \boxplus [y] := \{[z] \ : \ [z] \in [x] \boxplus [y]\} = \{[z] \ : \ z \in x \odot u \boxplus y \odot v \, , \, u, v \in U\}.$$

The notion of an ordering over a hyperstructure is now introduced.

**Definition 6.2.3.** [AD19, Def.2.2] An *ordering* on a hyperfield $(\mathbb{H}, \boxplus, \odot, \mathbb{0}, \mathbb{1})$ is a subset $\mathbb{H}^+ \subset \mathbb{H}$ satisfying:

- $\mathbb{H}^+ \boxplus \mathbb{H}^+ \subseteq \mathbb{H}^+$,
- $\mathbb{H}^+ \odot \mathbb{H}^+ \subseteq \mathbb{H}^+$,
- $\mathbb{H} = \mathbb{H}^+ \sqcup \{\mathbb{0}\} \sqcup \mathbb{H}^-$ where $\mathbb{H}^- = -\mathbb{H}^+ = \{-x \mid x \in \mathbb{H}^+\}$.

A hyperfield is called *real* if its set of orderings is non-empty. (See [Mar06] for further results on real hyperfields)

There are discussions of equivalent notions of orderings for hyperfields in [LS20] and [KLS21] and for multirings in [Mar06]. The intuition one should have is that the subset $\mathbb{H}^+$ distinguishes the positive elements of $\mathbb{H}$. The key example of this is the sign hyperfield $\mathbb{S}$ where $\mathbb{S}^+ = \{\mathbb{1}\}$ is an ordering. In fact, one can check that $\mathbb{H}^+ \subset \mathbb{H}$ is an ordering of $\mathbb{H}$ if and only if there exists a hyperfield homomorphism $f \colon \mathbb{H} \to \mathbb{S}$ such that $\mathbb{H}^+ = f^{-1}(\mathbb{1}_{\mathbb{S}})$, see [AD19] Section 2.3.1 for more details. By the properties of a hyperring homomorphism and $\mathbb{H}^+ = f^{-1}(\mathbb{1}_{\mathbb{S}})$, it can be seen that $\mathbb{1} \in \mathbb{H}^+$ for any ordering. There are no self inverses over ordered hyperfields. Suppose that $x \in \mathbb{H}$ is self inverse, then $0 \in x \boxplus x$. When taking the image under the morphism to $\mathbb{S}$, this gives,

$$\mathbb{0} \in sgn(x \boxplus x) \subseteq sgn(x) \boxplus sgn(x) = sgn(x) \not\ni \mathbb{0}.$$

Demonstrating a contradiction.

**Example 6.2.4.** Consider the signed tropical hyperfield $\mathbb{TR}$, a viable ordering is

$$\mathbb{TR}^+ = \{(1, x) \mid x \in \mathbb{R}\} .$$

With,

$$\mathbb{TR}^- = \{(-1, x) \mid x \in \mathbb{R}\} .$$

**Example 6.2.5.** For an ordered field $K$, one can define a strict total order compatible with addition given by $x < y$ if and only if $y - x \in K^+$. The same does not precisely hold for hyperfields. Given an ordering $\mathbb{H}^+$, define the associated relation $<_{\mathbb{H}^+}$ by

$$x <_{\mathbb{H}^+} y \;\Leftrightarrow\; y \boxplus -x \subseteq \mathbb{H}^+ .$$

This is a strict partial order on $\mathbb{H}$. It also has a corresponding non-strict partial order $\leqslant$ defined by

$$x \leqslant_{\mathbb{H}^+} y \Leftrightarrow x = y \text{ or } x <_{\mathbb{H}^+} y\,,$$

Unlike the analogous construction over fields, compatibility is particularly troublesome, as $a \boxplus c$ and $b \boxplus c$ may be sets and there is no canonical way to extend $\leqslant$ to sets. There is a discussion in Section 2 of [LS20] and Section 3 of [KLS21].

There is a study of the number of orderings of quotient hyperfields presented in [KLS21, Theorem 3.4]. Explicitly, the set of orderings on $\mathbb{H}$ is denoted $\chi(\mathbb{H})$ and the set of orderings on $\mathbb{H}$ containing a subset $U$ is denoted $\chi(\mathbb{H} \mid U)$. Then, [KLS21, Theorem 3.4] shows that $|\chi(K/U)| = |\chi(K \mid U)|$. This relationship is used to demonstrate that hyperfields can have an uncountable number of orderings.

**Example 6.2.6.** Take $\mathbb{Q}[X]$, then its field of fractions, denoted $\mathbb{Q}(X)$, is defined as,

$$\mathbb{Q}(X) := \left\{\frac{f(X)}{g(X)} \mid f(X), g(X) \in \mathbb{Q}[X], g(X) \neq 0\right\} \Big/ \frac{f}{g} \sim \frac{f'}{g'} \iff f'g = f'g.$$

For any transcendental $\alpha \in \mathbb{R}$, one can define an ordering: $f <_\alpha g \iff f(\alpha) < g(\alpha)$, with $P_\alpha = \{f \mid f(\alpha) > 0\}$. Furthermore, $P_\alpha = P_\beta$ if and only if $\alpha = \beta$. This implies, $|\chi(\mathbb{Q}(X))| > \infty$, as $\{P_\alpha \mid \alpha \in \mathbb{R}, \text{transcendental}\} \subset \mathbb{Q}(X)$. As $\mathbb{Q}(X)$ is a field the quotient construction can be used to generate non-trivial hyperfields. For $f \in \mathbb{Q}(X)^\times$, let $\langle f \rangle = \{f^k \mid k \in \mathbb{Z}\}$ be the multiplicative subgroup generated by $f$. If $f(\alpha) > 0$, then $f^k(\alpha) > 0 \Rightarrow \langle f \rangle \subset P_\alpha$. Then $\mathbb{H} = \mathbb{Q}(X)/\langle f \rangle$ is a quotient hyperfield. Generically, $\{P_\alpha \mid \alpha \in \mathbb{R}, \text{transcendental}, f(\alpha) > 0\} \subset \chi(\mathbb{Q}(X) \mid \langle f \rangle)$ is uncountable. Then by [KLS21, Theorem 3.4], $|\chi(\mathbb{H})| = |\chi(\mathbb{Q}(X) \mid \langle f \rangle)| > \infty$, is uncountable.

**Definition 6.2.7.** Let $\mathbb{H}_1, \mathbb{H}_2$ be ordered hyperfields. The map $f\colon \mathbb{H}_1 \to \mathbb{H}_2$ is a homomorphism of ordered hyperfields if it is a hyperfield homomorphism, and $f(\mathbb{H}_1^+) \subseteq \mathbb{H}_2^+$. This will be referred to as *order preserving*. Hyperfield homomorphsisms satifsfy $f(-x) = -f(x)$, so this can be defined equivalently as $f(\mathbb{H}_1^-) \subseteq \mathbb{H}_2^-$.

**Example 6.2.8.** The following hyperfield homomorphsims are order preserving; $sgn : \mathbb{R} \to \mathbb{S}$, $Sgn : \mathbb{TR} \to \mathbb{S}$ and $\mathrm{val}_\mathbb{R} : \mathbb{R}[[t^\mathbb{R}]] \to \mathbb{TR}$.

**Proposition 6.2.9.** *Let $f : \mathbb{H}^1 \to \mathbb{H}^2$ be a surjective homomorphism of ordered hyperfields, then $f(\mathbb{H}_1^+) = \mathbb{H}_2^+$.*

*Proof.* If $x \in \mathbb{H}_2^+$ such that there exists $y \in \mathbb{H}_1^-$, where $f(y) = x$, then $f(\mathbb{H}_1^-) \nsubseteq \mathbb{H}_2^-$, which is a contradiction. ◻

To conclude this section, a compatible total order is introduced that can be defined on any real hyperfield, and aligns with the total order in Example 6.2.5 over stringent hyperfields. Given an ordering $\mathbb{H}^+$, define $\overline{\mathbb{H}^+} = \mathbb{H}^+ \cup \{\mathbb{0}\}$; this is sometimes called a *positive cone*, see [LS20, Def. 2.14]. It can also be checked that the properties of $\mathbb{H}^+$ imply

$$\overline{\mathbb{H}^+} \boxplus \overline{\mathbb{H}^+} = \overline{\mathbb{H}^+}\,, \qquad \overline{\mathbb{H}^+} \cap -\overline{\mathbb{H}^+} = \{\mathbb{0}\}\,,$$
$$\overline{\mathbb{H}^+} \odot \overline{\mathbb{H}^+} = \overline{\mathbb{H}^+}\,, \qquad \overline{\mathbb{H}^+} \cup -\overline{\mathbb{H}^+} = \mathbb{H}\,.$$

It can moreover be seen that a homomorphism of ordered hyperfields $f\colon \mathbb{H}_1 \to \mathbb{H}_2$ satisfies $f(\overline{\mathbb{H}_1^+}) \subseteq \overline{\mathbb{H}_2^+}$. Then, define a binary relation $\preceq$ on $\mathbb{H}$ given by

$$x \preceq y \;\Leftrightarrow (y \boxplus -x) \cap \overline{\mathbb{H}^+} \neq \varnothing\,.$$

**Proposition 6.2.10.** *Let $\mathbb{H}^+$ be an ordering on a stringent $\mathbb{H}$. Then $\preceq$ is a compatible total order.*

*Proof.*

- (Reflexive) For any $a \in \mathbb{H}$, it holds that $\mathbb{0} \in (a \boxplus -a) \cap \overline{\mathbb{H}^+}$.

- (Antisymmetric) If both $a \preceq b$ and $b \preceq a$, then the second by definition gives $(a \boxplus -b) \cap \overline{\mathbb{H}^+} \neq \varnothing$. This implies $(b \boxplus -a) = (-a \boxplus b) \cap -\overline{\mathbb{H}^+} \neq \varnothing$. Along with $(b \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing$, this results in $(b \boxplus -a)$ being a set not a singleton, as it has both positive and negative elements, or zero belonging to it. Therefore, this gives $a = b$.

- (Transitivity) Let $a \preceq b$ and $b \preceq c$, by definition $(b \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing$ and $(c \boxplus -b) \cap \overline{\mathbb{H}^+} \neq \varnothing$. As $\mathbb{H}$ is stringent, there are two options for each $(b \boxplus -a)$ and $(c \boxplus -b)$ such that they intersect with $\overline{\mathbb{H}^+}$. They either contain zero or are a singleton belonging to $\mathbb{H}^+$. If both $(b \boxplus -a)$ and $(c \boxplus -b)$ contain zero then $a = b = c$, yielding $\mathbb{0} \in (c \boxplus -a)$, and hence $(c \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing \Leftrightarrow a \preceq c$. If both $(b \boxplus -a)$ and $(c \boxplus -b)$ are singletons contained in $\mathbb{H}^+$, then by the properties of $\mathbb{H}^+$ and $\mathbb{H}$ being stringent, $(c \boxplus -a) \subseteq (c \boxplus -b) \boxplus (b \boxplus -a) = \{\text{singleton}\} \in \mathbb{H}^+$. This implies $(c \boxplus -a) \in \mathbb{H}^+ \Rightarrow (c \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing \Leftrightarrow a \preceq c$. If $(b \boxplus -a)$ contains zero and $(c \boxplus -b)$ is a singleton contained in $\mathbb{H}^+$, then $b = a$. Thus, $(c \boxplus -a) = (c \boxplus -b)$, which is a singleton contained in $\mathbb{H}^+$, implying $(c \boxplus -a) \in$

$\mathbb{H}^+ \Rightarrow (c \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing \Leftrightarrow a \preceq c$. The argument can be mirrored for $(c \boxplus -b)$ contains zero and $(b \boxplus -a)$ is a singleton contained in $\mathbb{H}^+$.

- (Total) If $(b \boxplus -a) \cap \overline{\mathbb{H}^+} = \varnothing$ then it must be a subset of $\mathbb{H}^-$. This implies that $-b \boxplus a \subseteq \mathbb{H}^+$ and therefore $b \preceq a$.

- (Compatible) Let $a \preceq b$, by definition $(b \boxplus -a) \cap \overline{\mathbb{H}^+} \neq \varnothing$. For any $c \in \mathbb{H}$

$$b \boxplus -a \subseteq (b \boxplus c) \boxplus (-c \boxplus -a) \Rightarrow ((b \boxplus c) \boxplus -(c \boxplus a)) \cap \overline{\mathbb{H}^+} \neq \varnothing ,$$

  implying $a \boxplus c \preceq b \boxplus c$.

□

**Proposition 6.2.11.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be an order preserving hyperfield homomorphism. If $a \preceq b$ then $f(a) \preceq f(b)$.*

*Proof.* By definition $a \preceq b \Leftrightarrow (b \boxplus -a) \cap \overline{\mathbb{H}_1^+} \neq \varnothing$, as $f$ is order preserving $f(b \boxplus -a) \cap \overline{\mathbb{H}_2^+} \neq \varnothing$. By the homomorphism properties of $f$, $f(b \boxplus -a) \subseteq f(b) \boxplus -f(a)$. Thus, $f(b) \boxplus -f(a) \cap \overline{\mathbb{H}_2^+} \neq \varnothing$, which yields $f(a) \preceq f(b)$. □

**Proposition 6.2.12.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be a hyperfield homomorphism such that $a \preceq b \Leftrightarrow f(a) \preceq f(b)$. Then, $f$ is order preserving.*

*Proof.* Note that $a = a \boxplus \mathbb{0}_1 \in \overline{\mathbb{H}_1^+}$, *iff* $\mathbb{0}_1 \preceq a$. Thus, $\mathbb{0}_2 = f(\mathbb{0}_1) \preceq f(a)$, so $f(a) \boxplus \mathbb{0}_2 \in \overline{\mathbb{H}_2^+}$. This implies that $f(\overline{\mathbb{H}_1^+}) \subseteq \overline{\mathbb{H}_2^+}$. □

The reverse implication of Proposition 6.2.11 does not hold in general, even for a surjective homomorphism between stringent hyerfields. Explicitly, if $f$ is order preserving then, $f(a) \preceq f(b) \not\Rightarrow a \preceq b$.

**Example 6.2.13.** Take $sgn : \mathbb{R} \to \mathbb{S}$, then $a = 8, b = 4$, gives $a \npreceq b$ as $4 + -8 \in \mathbb{R}^-$. Then, the images under $sgn$ result in,

$$sgn(b) \boxplus -sgn(a) = \mathbb{1}_\mathbb{S} \boxplus -\mathbb{1}_\mathbb{S} = \mathbb{S}$$

This implies that $sgn(b) \boxplus -sgn(a) \cap \mathbb{S}^+ \neq \varnothing$, and hence, $sgn(a) \preceq sgn(b)$. Therefore, $a \npreceq b \not\Rightarrow sgn(a) \npreceq sgn(b)$, which is equivalent to $sgn(a) \preceq sgn(b) \not\Rightarrow a \preceq b$.

**Lemma 6.2.14.** *Let $\mathbb{H}$ be a stringent hyperfield with some ordering $\mathbb{H}^+$. Then $\leqslant_{\mathbb{H}^+}$, from Example 6.2.5, and $\preceq$ are the same total orders.*

*Proof.* Suppose $a \leqslant b$. If $a < b$ then $b \boxplus -a \subseteq \mathbb{H}^+ \subset \overline{\mathbb{H}^+}$ and so clearly $a \preceq b$. Otherwise $a = b$, then $\mathbb{0} \in (b \boxplus (-a)) \cap \overline{\mathbb{H}^+} \neq \varnothing$.

Conversely, suppose $a \preceq b$. As $\mathbb{H}$ is stringent, $b \boxplus -a$ is either a singleton or contains $\mathbb{0}$. If the latter, then by uniqueness of inverses $a = b$. Otherwise, $b \boxplus -a = \{c\} \subseteq \overline{\mathbb{H}^+}$ is a singleton $c$: if $c$ contained in $\mathbb{H}^+$ then $a < b$, otherwise $c = \mathbb{0}$ implying $a = b$. □

### 6.2.1 Classifying Ordered Stringent Hyperfields

Bowler and Su [BS20] classified stringent (skew) hyperfields via the following family of hyperfields. Their construction for hyperfields is recalled. Let $(\mathbb{H}, \boxplus_{\mathbb{H}}, \odot_{\mathbb{H}}, \mathbb{0}_{\mathbb{H}}, \mathbb{1}_{\mathbb{H}})$ be a hyperfield and $(G, +, 0)$ a totally ordered abelian group. Then,define $\mathbb{H} \rtimes G$ to be the hyperfield with ground set $(\mathbb{H}^\times \times G) \cup \mathbb{0}$, where $\mathbb{0}$ is some new element acting as the additive identity. The multiplication is defined by

$$(x, g) \odot (y, h) = (x \odot_{\mathbb{H}} y, g + h) \quad , \quad (x, g) \odot \mathbb{0} = \mathbb{0} \odot (x, g) = \mathbb{0} \,,$$

where $\mathbb{1} := (\mathbb{1}_{\mathbb{H}}, 0)$ is the multiplicative identity. The addition is defined in a more involved way as follows:

$$(x, g) \boxplus (y, h) = \begin{cases} \{(x, g)\}, & g > h \\ \{(y, h)\}, & g < h \\ \{(z, g) \mid z \in (x \boxplus_{\mathbb{H}} y) \cap \mathbb{H}^\times\}, & g = h \,, \\ & \mathbb{0}_{\mathbb{H}} \notin x \boxplus_{\mathbb{H}} y \\ \{(z, g) \mid z \in (x \boxplus_{\mathbb{H}} y) \cap \mathbb{H}^\times\} & g = h \,, \\ \qquad \cup \{(x, g') \mid x \in \mathbb{H}^\times \,,\, g' < g\} \cup \{\mathbb{0}\}, & \mathbb{0}_{\mathbb{H}} \in x \boxplus_{\mathbb{H}} y \end{cases}$$

It follows therefore that $\mathbb{H} \rtimes G$ is a hyperfield, and that it is stringent if $\mathbb{H}$ is stringent also. Furthermore, the following classification theorem shows all stringent hyperfields can be written this way.

**Theorem 6.2.15.** *[BS20, Theorem 4.10] Every stringent hyperfield has the form $\mathbb{H} \rtimes G$, where $\mathbb{H}$ is either the Krasner hyperfield $\mathbb{K}$, the sign hyperfield $\mathbb{S}$ or a field $K$.*

This classification of stringent hyperfields is used to completely classify ordered stringent hyperfields.

**Proposition 6.2.16.** *Every ordered stringent hyperfield has the form* $\mathbb{H} \rtimes G$*, where* $\mathbb{H}$ *is either the sign hyperfield* $\mathbb{S}$ *or an ordered field* $K$*.*

*Proof.* It will be shown that $\mathbb{H} \rtimes G$ has an ordering if and only if $\mathbb{H}$ has an ordering. As the sign hyperfield and ordered fields are the only possibilities for $\mathbb{H}$ that are ordered, this completes the proof.

It is mathematically possible to check that if $\mathbb{H}^+$ is an ordering of $\mathbb{H}$, then $\{(x, g) \mid x \in \mathbb{H}^+, g \in G\}$ satisfies the properties of being an ordering on $\mathbb{H} \rtimes G$. Conversely, suppose $\mathbb{H} \rtimes G$ has an ordering i.e. a hyperfield homomorphism $\phi \colon \mathbb{H} \rtimes G \to \mathbb{S}$. There is an injective homomorphism $i \colon \mathbb{H} \to \mathbb{H} \rtimes G$ that sends $\mathbb{0}_{\mathbb{H}} \mapsto \mathbb{0}$ and all nonzero elements $x \mapsto (x, 0)$. Therefore the composition $\phi \circ i$ is a hyperfield homomorphism from $\mathbb{H}$ to $\mathbb{S}$ and so $\mathbb{H}$ is also ordered. □

It was also shown in [BS20] that every (skew) stringent hyperfield is the quotient of a field. Their results are recalled, specialising to ordered stringent hyperfields; note that removing the skew condition simplifies the construction considerably.

Let $K[[t^G]]$ be the field of *Hahn series* with value group $G$ and coefficients in some arbitrary field $K$. This is the field of formal power series

$$\gamma = \sum_{g \in G} c_g t^g \,,\; c_g \in K$$

whose support $\operatorname{supp}(\gamma) = \{g \in G \mid c_g \neq 0\}$ is well-ordered. The *leading coefficient* $\operatorname{lc}(\gamma)$ of $\gamma$ is the coefficient of the term with smallest exponent,

$$\operatorname{lc}(\gamma) = \left\{c_g \mid g \leqslant g' \, \forall g' \in \operatorname{supp}(\gamma)\right\} \,.$$

Addition and multiplication are the usual operations on power series:

$$\left(\sum_{g \in G} c_g t^g\right) + \left(\sum_{g \in G} d_g t^g\right) = \sum_{g \in G} (c_g + d_g) t^g$$

$$\left(\sum_{g\in G} c_g t^g\right)\cdot\left(\sum_{g\in G} d_g t^g\right) = \sum_{g\in G}\left(\sum_{\substack{h,h'\in G\\ h+_G h'=g}} c_h\cdot d_{h'}\right)t^g$$

If $K$ is an ordered field, then an ordering can be defined on $K[[t^G]]$ given by power series whose leading coefficient is positive, i.e.

$$K[[t^G]]^+ := \{\gamma \mid \mathrm{lc}(\gamma)\in K^+\}\,.$$

**Theorem 6.2.17.** *[BS20, Theorem 7.5] Every ordered stringent hyperfield can be realised as a quotient of an ordered field. Explicitly,*

$$\mathbb{S}\rtimes G \simeq K[[t^G]]/U \qquad U = \left\{\gamma \mid \mathrm{lc}(\gamma) = c_0\in K^+\right\}\,,$$
$$K\rtimes G \simeq K[[t^G]]/V \qquad V = \{\gamma \mid \mathrm{lc}(\gamma) = c_0 = 1\}\,.$$

## 6.3 Hyperplanes and Halfspaces

Let $\mathbb{H}$ be a real hyperfield with some fixed ordering $\mathbb{H}^+$. Consider the "vector space" $\mathbb{H}^d$ obtained by extending the hyperaddition $\boxplus$ to tuples, and multiplication $\odot$ to scalars acting on tuples as follows:

$$\boxplus\colon \mathbb{H}^d\times\mathbb{H}^d\to P(\mathbb{H}^d)^* \qquad \mathbf{u}\boxplus\mathbf{v} = \bigcup_{w_i\in u_i\boxplus v_i}(w_1,\dots,w_d)\,,$$
$$\odot\colon \mathbb{H}\times\mathbb{H}^d\to\mathbb{H}^d \qquad a\odot\mathbf{v} = (a\odot v_1,\dots,a\odot v_d)\,.$$

Formally, $\mathbb{H}^d$ has the structure of a vector space over a hyperfield, as defined by [TNSL17]. Also note that for any hyperfield homomorphism $f$, it can be extended to $\mathbb{H}^d$ by applying entrywise.

Let $K$ be a field with ordering $K^+$. Take a polynomial $p(X_1\dots X_n)\in K[X_1\dots X_n]$. Then there are three sets regarding this polynomial and the ordering $K^+$ on $K$ that can be defined;

$$V(p) := \{(X_1\dots X_n)\in K^n \,:\, p(X_1\dots X_n) = 0\}$$

$$\mathcal{H}^+(p) := \{(X_1 \dots X_n) \in K^n \,:\, p(X_1 \dots X_n) \in K^+\}$$

$$\overline{\mathcal{H}^+}(p) := \{(X_1 \dots X_n) \in K^n \,:\, p(X_1 \dots X_n) \in \overline{K^+}\}.$$

The first is the variety of the polynomial $p$, the second and third are the open and closed positive halfspaces respectively, defined by $p$. These notions can be generalised to polynomials over hyperfields. This follows from the notions discussed in Section 3.3 of [LV19] and Section 5.2 in [JSY22]. Given $\mathbb{H}$ a hyperfield with ordering $\mathbb{H}^+$. Take a polynomial $p(X_1 \dots X_n) \in \mathbb{H}[X_1 \dots X_n]$. The addition becomes the hyper-addition when evaluating at elements of $\mathbb{H}^n$.

$$V(p) := \{(X_1 \dots X_n) \in \mathbb{H}^n \,:\, p(X_1 \dots X_n) \ni 0\}$$

$$\mathcal{H}^+(p) := \{(X_1 \dots X_n) \in \mathbb{H}^n \,:\, p(X_1 \dots X_n) \subseteq \mathbb{H}^+\}$$

$$\overline{\mathcal{H}^+}(p) := \{(X_1 \dots X_n) \in \mathbb{H}^n \,:\, p(X_1 \dots X_n) \cap \overline{\mathbb{H}^+} \neq \varnothing\}.$$

In an analogous manner to working over a field, these objects are called the variety of $p$, the positive open halfspace of $p$ and the closed positive halfspace of $p$ respectively.

Note that the closed halfspace is defined with intersection rather than inclusion as a subset. This is due to the fact that this generalised notion of a closed halfspace is attempting to capture both elements of the open halfspace and roots of the polynomial. When an element is a root of the polynomial this can cause both positive and negative elements along with $\mathbb{0}$ to belong to the output set. This can occur even for stringent hyperfields. Therefore, non-empty is proposed as the most appropriate definition.

### 6.3.1 Varieties of Polynomials

This section will explore the first structure outlined above, the variety of a polynomial over a hyperfield.

**Theorem 6.3.1.** *Given a polynomial $p(X_1, \dots, X_n) \in \mathbb{H}[X_1, \dots, X_n]$, with $[p]_* \in \mathbb{H}/U[X_1, \dots, X_n]$. Let $[\boldsymbol{a}] = ([a_1], \dots, [a_n]) \in V([p]_*) \subseteq \left(\mathbb{H}/U\right)^n$, then for any lift $\tilde{\boldsymbol{a}}$ such that $[\tilde{\boldsymbol{a}}] = [\boldsymbol{a}]$, there exists $q(X_1, \dots, X_n) \in \mathbb{H}[X_1, \dots, X_n]$ such that $[q]_* = [p]_*$ and $\tilde{\boldsymbol{a}} \in V(q)$.*

*Proof.* The polynomial $p(X_1, \dots, X_n) = \boxplus_I c_I X_1^{i_1} \dots X_n^{i_n}$, where $I = (i_1, \dots, i_n)$, then, $[p]_* = \boxplus_I [c_I] \odot X_1^{i_1} \dots X_n^{i_n}$, with appropriately adjusted operations. Take $[\boldsymbol{a}] = ([a_1], \dots, [a_n]) \in V([p]_*)$, which by definition gives:

$$\begin{aligned} \mathbb{0} \in [p]_*([\boldsymbol{a}]) &\iff \mathbb{0} \in \boxplus_I [c_I] \odot [a_1]^{i_1} \odot \dots \odot [a_n^{i_n}] \\ &\iff \mathbb{0} \in \boxplus_I [c_I \odot a_1^{i_1} \odot \dots \odot a_n^{i_n}]. \end{aligned} \tag{6.3.2}$$

Selecting an arbitrary lift $\tilde{\boldsymbol{a}}$ of $[\boldsymbol{a}]$, precisely $\tilde{\boldsymbol{a}} = (\tilde{a_1}, \dots, \tilde{a_n})$ such that each $[\tilde{a_i}] = [a_i]$. This implies the existence of $\tilde{u_i}$ for each $\tilde{a_i}$ such that $\tilde{a_i} \odot \tilde{u_i} = a_i$.

The way the addition is construction for the quotient hyperfield $\mathbb{H}/U$ gives that when (6.3.2) holds there is some collection $\{u_I\}_I \subset U$ such that,

$$\mathbb{0} \in \boxplus_I c_I \odot a_1^{i_1} \odot \dots \odot a_n^{i_n} \odot u_I,$$

over $\mathbb{H}$. Substituting in the expression for each $a_i$ in terms of $\tilde{a_i}$ gives,

$$\begin{aligned} &\mathbb{0} \in \boxplus_I c_I \odot (\tilde{a_1} \odot \tilde{u_1})^{i_1} \odot \dots \odot (\tilde{a_n} \odot \tilde{u_n})^{i_n} \odot u_I \\ \iff &\mathbb{0} \in \boxplus_I c_I \odot (\tilde{u_1})^{i_1} \odot \dots \odot (\tilde{u_n})^{i_n} \odot u_I \odot (\tilde{a_1})^{i_1} \odot \dots \odot (\tilde{a_n})^{i_n}. \end{aligned} \tag{6.3.3}$$

Denote $(\tilde{u_1})^{i_1} \odot \dots \odot (\tilde{u_n})^{i_n} \odot u_I = \tilde{u_I}$, then due to $U$ being a multiplicative subgroup of $\mathbb{H}^\times$ it can be seen that $\tilde{u_I} \in U$. This is sufficient to be able to define the lifted polynomial. Set,

$$q(X_1, \dots, X_n) = \boxplus_I c_I \odot \tilde{u_I} \odot X_1^{i_1} \dots X_n^{i_n} \in \mathbb{H}[X_1, \dots, X_n].$$

It follows from (6.3.3) that $q(\tilde{a_1}, \dots, \tilde{a_n}) \ni \mathbb{0}$ and $[c_I] = [c_I \odot \tilde{u_I}]$ hence $[q]_* = [p]_*$. This then demonstrates that there exists a $q \in \mathbb{H}[X_1, \dots, X_n]$ for any lift of the root $[\boldsymbol{a}]$ such that the $\tilde{\boldsymbol{a}}$ is a root of $q$. As the original root $[\boldsymbol{a}]$ was chosen arbitrarily this holds for all roots and any corresponding lift. ◻

The result in Theorem 6.3.1 is a strongly applicable result. It shows that for any root regardless of the lift that is chosen, there will exists a suitable lifted polynomial for which it is a root.

**Corollary 6.3.4.** *Given a polynomial $p(X_1, \dots, X_n) \in \mathbb{H}[X_1, \dots, X_n]$ with $P = [p]_* \in \mathbb{H}/U[X_1, \dots, X_n]$. Then,*

$$V(P) = \bigcup_{\substack{q \in \mathbb{H}[X_1, \dots, X_n] \\ s.t\ [q]_* = P}} [V(q)]. \tag{6.3.5}$$

*Proof.* This is a combination of two intermediate results. Firstly, Lemma 3.2.2 gives that each $[V(q)] \subseteq V(P)$, as $[q]_* = P$. Therefore, the union of all push-forward varieties also belongs to $V(P)$, yielding

$$\bigcup_{\substack{q \in \mathbb{H}[X_1, \dots, X_n] \\ s.t\ [q]_* = P}} [V(q)] \subseteq V(P).$$

Secondly, Theorem 6.3.1 demonstrates that for every root of $P$ there exists a lift such that this lift is a root of a polynomial $q \in \mathbb{H}[X_1, \dots, X_n]$, where $[q]_* = P$. Hence, this shows the converse inclusion;

$$V(P) \subseteq \bigcup_{\substack{q \in \mathbb{H}[X_1, \dots, X_n] \\ s.t\ [q]_* = P}} [V(q)].$$

These opposite inclusions give the desired equality. ◻

The following example will make the constructive proof of Theorem 6.3.1 explicit when choosing specific lifts of roots.

**Example 6.3.6.** Take the quotient map $sgn : \mathbb{R} \to \mathbb{S}$, the polynomial $p(X) = X^2 - X + 1 \in \mathbb{R}[X]$ has push-forward $sgn_*(p) = X^2 \boxplus -X \boxplus 1 \in \mathbb{S}[X]$. $1 \in V(sgn_*(p))$, but $V(p) = \varnothing$. Choosing a lift of $1 \in \mathbb{S}[X]$ as $4 \in sgn^{-1}(1)$, gives us the following representation in terms of cosets, $[1] = [4]$. Hence, $[4]^2 \boxplus [-4] \boxplus [1] \ni \mathbb{0}$. The first hyper addition gives the set, $\{[z] \mid z \in 16u - 4v\}$ with the requirement that there exists some $[z] = [-1]$. This is equivalent to $16u - 4v < 0$ for some choice of $u, v \in \mathbb{R}_{>0}$. This can be reduced to the inequality $4u < v$. Therefore, one choice is $u = 1$ and $v = 8$. This leaves the constant of the polynomial to be lifted to 16 as $16 - 32 = -16$ and the whole equation should sum to 0. This finally gives the lifted polynomial as, $q(X) = X^2 - 8X + 16 = (X - 4)(X - 4) \in \mathbb{R}[X]$, with $sgn_*(q) = sgn_*(p)$, with the lifted root $4 \in sgn^{-1}(1)$ as a root of $q(X)$.

The next result will demonstrate a stronger connection between $f(V(p))$ and $V(f_*(p))$ over quotient hyperfields for the specific case of linear polynomials.

**Theorem 6.3.7.** *Given a linear polynomial $p(X_1, \dots, X_n) \in \mathbb{H}[X_1, \dots, X_n]$, then under the quotient map $[\cdot] : \mathbb{H} \to \mathbb{H}/U$,*

$$V([p]_*) = [V(p)].$$

*Proof.* Given the linear polynomial $p(X_1, \dots, X_n) = c_0 \boxplus c_1 X_1 \boxplus \dots \boxplus c_n X_n$, then the push-forward is $[p]_* = [c_0] \boxplus [c_1] X_1 \boxplus \dots \boxplus [c_n] X_n$. If $[\underline{y}] = ([y_1], \dots [y_n]) \in \left(\mathbb{H}/U\right)^n$ is a root of $[p]_*$, then due to the quotient construction, there are collections of $\{u_i\}, \{v_i\} \subset U$ such that,

$$c_0 u_0 \boxplus c_1 u_1 y_1 v_1 \boxplus \dots c_n u_n y_n v_n \ni \mathbb{0}.$$

With some rearranging this can be stated as,

$$c_0 \boxplus c_1 \left(\frac{u_1 v_1}{u_0}\right) y_1 \boxplus \dots \boxplus c_n \left(\frac{u_n v_n}{u_0}\right) y_n \ni \mathbb{0}. \tag{6.3.8}$$

For each $i \in \{1, \dots, n\}$, $\left[\left(\frac{u_i v_i}{u_0}\right) y_i\right] = [y_i]$, which gives,

$$\left(\left[\left(\frac{u_1 v_1}{u_0}\right) y_1\right], \dots, \left[\left(\frac{u_n v_n}{u_0}\right) y_n\right]\right) = ([y_1], \dots [y_n]). \tag{6.3.9}$$

By (6.3.8), the lift defined in (6.3.9) belongs to $V(p)$. Hence, for every element in $V([p]_*)$ there exists a lift back to an element of $V(p)$, yielding the inclusion, $V([p]_*) \subseteq [V(p)]$. The reverse inclusion is an immediate consequence of Lemma 3.2.2. The two inclusions taken together produce the desired equality $V([p]_*) = [V(p)]$. □

**Example 6.3.10.** Let $p(X, Y, Z) = 3X - Y + 2Z \in \mathbb{R}[X, Y, Z]$, then under the map $sgn : \mathbb{R} \to \mathbb{S}$, the push-forward is $sgn_*(p) = X - Y + Z \in \mathbb{S}[X, Y, Z]$. Then the element $(1, 1, 1) \in \mathbb{S}^3$ is a root of $sgn_*(p)$, as

$$sgn_*(p)(1, 1, 1) = 1 \boxplus -1 \boxplus 1 = \mathbb{S} \ni \mathbb{0}.$$

Then, $(1, 5, 1) \in sgn^{-1}(1, 1, 1)$, is a lift to a root of $p(X, Y, Z)$,

$$p(1, 5, 1) = 3 - 5 + 2 = 0.$$

This example demonstrates the freedom of choice in the lifts for each component, due to the linearity of the original polynomial. The variables are not interacting with each other in the polynomial, hence each component can be lifted separately whilst maintaining the fact it is a root. This example mirrors the lifting techniques for non-linear polynomials, but as shown in Theorem 6.3.7, there exists a lift back to a root of the original polynomial rather than the more broad result in Theorem 6.3.1, which demonstrates there may only exist a lift to another polynomial in the pre-image, whilst maintaining the root property.

### 6.3.2 Halfspaces and Quotient Maps

There will now be a discussion of the properties of halfspaces, in particular under quotient hyperfield homomorphisms. This is due to a strong connection between orders on hyperfields and orders on a quotient with a restiction on the multiplicative subgroup. The next result generalises that of Lemma 3.3 in [KLS21], with analogous proof outlined here.

**Lemma 6.3.11.** *Given a hyperfield $\mathbb{H}$ with ordering $\mathbb{H}^+$. Assume a multiplicative subgroup $U \subseteq \mathbb{H}^\times$ is contained in the ordering $\mathbb{H}^+$, thus consider the quotient hyperfield $\mathbb{H}/U$, then*

1. *If $x \in \mathbb{H}^+$, then $[x] \subseteq \mathbb{H}^+$.*
2. *The set $\mathbb{H}_U^+ := \{[x] \,:\, x \in \mathbb{H}^+\}$ is an ordering of $\mathbb{H}/U$.*

*Proof.* 1. Take $x \in \mathbb{H}^+$, as $U \subseteq \mathbb{H}^+$ then by $\mathbb{H}^+ \odot \mathbb{H}^+ \subseteq \mathbb{H}^+$, $[x] = \{au \,:\, u \in U\} \subseteq \mathbb{H}^+$.

2. Take $[x], [y] \in \mathbb{H}_U^+$, then by the first part $x, y \in \mathbb{H}^+$ for any representative. As $U \subseteq \mathbb{H}^+$, the combination $xu \boxplus yv \subseteq \mathbb{H}^+$ for all $u, v \in U$. Therefore, $[x] \boxplus [y] = \{[xu \boxplus yv] \,:\, u, v \in U\} \subseteq \mathbb{H}_U^+$. This implies $\mathbb{H}_U^+ \boxplus \mathbb{H}_U^+ \subseteq \mathbb{H}_U^+$. Next, for $[x], [y] \in \mathbb{H}_U^+$, $[x] \odot [y] = [x \odot y]$ by definition. Then by the first part $x, y \in \mathbb{H}^+$, hence $x \odot y \in \mathbb{H}^+$, which yields $[x \odot y] \in \mathbb{H}^+$. This implies $\mathbb{H}_U^+ \odot \mathbb{H}_U^+ \subseteq \mathbb{H}_U^+$. As $\mathbb{H}^+ \cup \mathbb{H}^+ = \mathbb{H}^\times$, $\mathbb{H}_U^+ \cup \mathbb{H}_U^+ \subseteq \mathbb{H}_U^\times$ holds. Finally, assume $[x], -[x] = [-x] \in \mathbb{H}_U^+$. Then, $x, -x\mathbb{H}^+$, which implies $0 \in x \boxplus -x \subseteq \mathbb{H}^+$, which is contradiction. This implies $\mathbb{H}_U^+ \cap \mathbb{H}_U^+ = \varnothing$.

□

The construction in Lemma 6.3.11 can be viewed as a hyperfield homomorphism, $[\cdot] : \mathbb{H} \to \mathbb{H}/U$. This can be extended to polynomials coefficient wise, and denoted $[\cdot]_* : \mathbb{H}[X_1 \dots X_n] \to \mathbb{H}/U[X_1 \dots X_n]$. This section will present results focused on understanding how the map $[\cdot]$ interacts with halfspaces. For the next results let $\mathbb{H}$ be a hyperfield with fixed order $\mathbb{H}^+$, and multiplicative subgroup of the units such that $U \subseteq \mathbb{H}^+$.

**Lemma 6.3.12.** *Given a polynomial $p \in \mathbb{H}[X_1 \dots X_n]$, then*

$$[\mathcal{H}^+(p)] \subseteq \overline{\mathcal{H}^+}([p]_*).$$

*Proof.* Let $(y_1, \dots, y_n) \in \mathcal{H}^+(p)$, then $([y_1], \dots, [y_n]) \in (\mathbb{H}/U)^n$. The push-forward of the polynomial under the induced map is $[p]_* = \boxplus_I [c_I]\underline{X}^I$. Evaluating $[p]_*$ at $([y_1], \dots, [y_n])$ yields,

$$\begin{aligned} [p]_*([y_1], \dots, [y_n]) &= \boxplus_I [c_I] \odot [y_1]^{i_1} \odot \dots \odot [y_n]^{i_n} \\ &= \boxplus_I [c_I \odot y_1^{i_1} \odot \dots \odot y_n]^{i_n} \\ &\supseteq \Big[ \boxplus_I c_I \odot y_1^{i_1} \odot \dots \odot y_n^{i_n} \Big] \end{aligned}$$

As $(y_1, \dots, y_n) \in \mathcal{H}^+(p)$, this gives $\boxplus_I c_I \odot y_1^{i_1} \odot \dots \odot y_n^{i_n} \subseteq \mathbb{H}^+$. This implies that $\big[\boxplus_I c_I \odot y_1^{i_1} \odot \dots \odot y_n^{i_n}\big] \subseteq \mathbb{H}_U^+$. Therefore, $[p]_*([y_1], \dots, [y_n]) \cap \overline{\mathbb{H}_U^+} \neq \varnothing$, demonstrating $[\mathcal{H}^+(p)] \subseteq \overline{\mathcal{H}^+}([p]_*)$. □

*Remark* 6.3.13. The inclusion $[\mathcal{H}^+(p)] \subseteq \mathcal{H}^+([p]_*)$ does not hold in general, as elements of $\mathcal{H}^+(p)$ may push-forward to roots of $[p]_*$ and hence the output set is not contained in the ordering $\mathbb{H}_U^+$. This can be seen explicitly in the following example.

**Example 6.3.14.** Let $sgn : \mathbb{R} \to \mathbb{S}$ and take the polynomial $p(X, Y) = aX + bY \in \mathbb{R}[X, Y]$, where $a, b \in \mathbb{R}_{>0}$. The set $\mathcal{H}^+(p)$ is a open halfspace above a negatively sloped line through the origin in $\mathbb{R}^2$. The push-forward of $p$ is $sgn_*(p) = X \boxplus Y \in \mathbb{S}[X, Y]$. This gives the set $\mathcal{H}^+(sgn_*(p)) = \{(0, 1), (1, 0), (1, 1)\}$, which does not include every element $sgn(\mathcal{H}^+(p))$, such as $(-1, 1)$. See Figure 6.1 for a visualisation of how this occurrs.

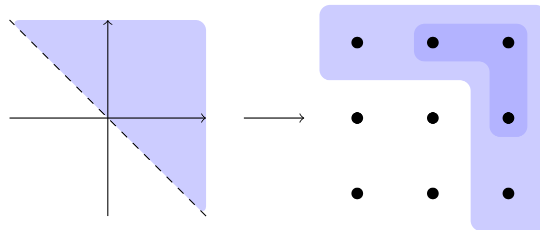

Figure 6.1: $\mathcal{H}^+(X+Y) \in \mathbb{R}^2$, with $sgn(\mathcal{H}^+(X+Y)) \notin \mathcal{H}^+(X \boxplus Y) \in \mathbb{S}^2$.

**Proposition 6.3.15.** *Let $p$ be a linear polynomial over $\mathbb{H}$, then $\mathcal{H}^+([p]_*) \subseteq [\mathcal{H}^+(p)]$.*

*Proof.* This method will follow and analogous pattern to the first part of Theorem 6.3.7. Take $[\mathbf{x}] \in \mathcal{H}^+([p]_*)$, then $[p]_*([\mathbf{x}]) \subseteq (\mathbb{H}/U)^+$. This is equivalent to $\boxplus_{i=0}^n [c_i \odot x_i] \subseteq (\mathbb{H}/U)^+$, which by the definition of $\boxplus$ for quotient hyperfields implies that there exists $\{u_i\}_i \subseteq U$ such that $\boxplus_{i=0}^n c_i \odot x_i \odot u_i \subseteq \mathbb{H}^+$. Define $\tilde{\mathbf{x}} = (x_1 \odot u_1, \ldots, x_n \odot u_n)$, thus $[\tilde{\mathbf{x}}] = [\mathbf{x}]$ and $p(\tilde{\mathbf{x}}) \subseteq \mathbb{H}^+$, which gives the inclusion. □

**Lemma 6.3.16.** *Let $\mathbb{H}$ be an ordered stringent hyperfield, then*

$$\overline{\mathcal{H}^+}(p) = \mathcal{H}^+(p) \sqcup V(p).$$

*Proof.* If $\mathbf{x} \in \overline{\mathcal{H}^+}(p)$, then $p(\mathbf{x}) \cap \overline{\mathbb{H}^+} \neq \varnothing$. By Lemma 39 in [BP19], $p(\mathbf{x})$ is a singleton unless it contains contains zero. If $p(\mathbf{x})$ is a singleton then $p(\mathbf{x}) \in \mathbb{H}^+$ and hence an element of $\mathcal{H}^+(p)$. Whereas, if $p(\mathbf{x})$ contains zero, then $\mathbf{x}$ is a root and thus an element of $V(p)$. By this description $\mathcal{H}^+(p)$ and $V(p)$ are disjoint sets. □

*Remark* 6.3.17. It could be the case over non-stringent ordered hyperfields that the polynomial produces an output set which does not contain zero but has both positive and negative elements. This would be an element of $\overline{\mathcal{H}^+}(p)$, but would not be an element of either $\mathcal{H}^+(p)$ or $V(p)$. See the next example for a demonstration of this.

**Example 6.3.18.** In [Liu20] there is a description of finite hyperfields of order less than five. The first case of Prop. 2.12 in [Liu20] is an example of a non-stringent ordered hyperfield in the following way:

$$\mathbb{H} := \{0, 1, 2, -1, -2\}, \quad \mathbb{H}^+ := \{1, 2\}.$$

| $\boxplus$ | $0$ | $1$ | $2$ | $-2$ | $-1$ |
|---|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{1\}$ | $\{2\}$ | $\{-2\}$ | $\{-1\}$ |
| $1$ | $\{1\}$ | $\{1\}$ | $\{1,2\}$ | $\{1,-2\}$ | $\mathbb{H}$ |
| $2$ | $\{2\}$ | $\{1,2\}$ | $\{2\}$ | $\mathbb{H}$ | $\{2,-1\}$ |
| $-2$ | $\{-2\}$ | $\{1,-2\}$ | $\mathbb{H}$ | $\{-2\}$ | $\{-2,-1\}$ |
| $-1$ | $\{-1\}$ | $\mathbb{H}$ | $\{2,-1\}$ | $\{-2,-1\}$ | $\{-1\}$ |

(Note that to return to the notation in [Liu20] take $ab = -1, a = 2, b = -2$.) Then, take $p(X,Y) = X \boxplus Y \in \mathbb{H}[X,Y]$ and evaluate at $(2,-1) \in \mathbb{H}^2$. This gives $p(2,-1) = 2\boxplus -1 = \{2,-1\}$. This set does not contain zero hence, $(2,-1) \notin V(p)$. The set contains both positive and negative elements, thus does not belong to $\mathcal{H}^+(p)$, but does belong to $\overline{\mathcal{H}^+}(p)$.

**Theorem 6.3.19.** *Let $p$ be a linear polynomial in $\mathbb{H}[X_1,\dots,X_n]$, with the induced map of polynomials $[\cdot]_* : \mathbb{H}[X_1,\dots,X_n] \to \mathbb{H}/U[X_1,\dots,X_n]$. Then,*

$$\left[\overline{\mathcal{H}^+}(p)\right] = \overline{\mathcal{H}^+}([p]_*).$$

*Proof.* For the inclusion $\left[\overline{\mathcal{H}^+}(p)\right] \subseteq \overline{\mathcal{H}^+}([p]_*)$, take $(y_1,\dots,y_n) \in \overline{\mathcal{H}^+}(p)$. Then using the same logic as the proof to Lemma 3.2.2, it can obtained that,

$$[p]_*([y_1],\dots,[y_n]) \supseteq [c_1 \odot y_1 \boxplus \cdots \boxplus c_n \odot y_n].$$

As $(y_1,\dots,y_n) \in \overline{\mathcal{H}^+}(p)$, this implies $[c_1 \odot y_1 \boxplus \cdots \boxplus c_n \odot y_n] \cap \overline{\mathbb{H}_U^+} \neq \emptyset$. In turn, implying that $[p]_*([y_1],\dots,[y_n]) \cap \overline{\mathbb{H}_U^+} \neq \emptyset$, hence $([y_1],\dots,[y_n]) \in \overline{\mathcal{H}^+}([p]_*)$.
For the second inclusion, $\overline{\mathcal{H}^+}([p]_*) \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$, take $([w_1],\dots,[w_n]) \in \overline{\mathcal{H}^+}([p]_*)$, then $[c_1][w_1] \boxplus \cdots \boxplus [c_n][w_n] \cap \overline{\mathbb{H}_U^+} \neq \emptyset$. By the definition of the addition over quotient hyperfields,

$$\begin{aligned}[c_1][w_1] \boxplus \cdots \boxplus [c_n][w_n] &= [c_1 w_1] \boxplus \cdots \boxplus [c_n w_n] \\ &= \{[\lambda] \,:\, \lambda \in c_1 v_1 \boxplus \cdots \boxplus c_n v_n,\ v_i = w_i \cdot u,\ u \in U\}\end{aligned}$$

This implies that there exists a $\lambda$ such that $[\lambda] \in \overline{\mathbb{H}_U^+}$. This gives $\lambda \in \overline{\mathbb{H}^+}$, hence the corresponding combination of $(v_1,\dots,v_n)$ is a lift of the $([w_1],\dots,[w_n])$ such that, $\lambda \in a_1 v_1 \boxplus \cdots \boxplus a_n v_n \cap \overline{\mathbb{H}^+} \neq \emptyset$. Therefore, $(v_1, \ldots, v_n) \in \overline{\mathbb{H}^+(p)}$, giving $([w_1], \ldots, [w_n]) \in \left[\overline{\mathcal{H}^+}(p)\right]$. $\square$

*Remark* 6.3.20. If the hyperfield $\mathbb{H}/U$ is stringent the inclusion $\overline{\mathcal{H}^+}([p]_*) \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$ in Theorem 6.3.19 can be shown with a combination of alternative techniques from previous results. As $\mathbb{H}/U$ is stringent, Lemma 6.3.16 gives $\overline{\mathcal{H}^+}([p]_*) = \mathcal{H}^+([p]_*) \sqcup V([p]_*)$. Then, Theorem 6.3.7 implies $V([p]_*) = [V(p)] \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$, and Proposition 6.3.15 implies $\mathcal{H}^+([p]_*) \subseteq [\mathcal{H}^+(p)]$. Then, as $\mathcal{H}^+(p) \subseteq \overline{\mathcal{H}^+}(p)$, $[\mathcal{H}^+(p)] \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$. This yields $\mathcal{H}^+([p]_*) \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$. Taking these together demonstrates the inclusion,

$$\overline{\mathcal{H}^+}([p]_*) = \mathcal{H}^+([p]_*) \sqcup V([p]_*) \subseteq \left[\overline{\mathcal{H}^+}(p)\right]$$

**Question 6.3.21.** *Can the result in Theorem 6.3.19 be understood for non-quotient hyperfield homomorphisms?*

*Remark* 6.3.22. The result in Theorem 6.3.19 could be asked of open halfspaces if the definition of $\mathcal{H}^+(p)$ implemented non-empty intersection with $\overline{\mathbb{H}^+}$ rather than inclusion as a subset in $\mathbb{H}^+$. Actually, if $\mathbb{H}$ is stringent then an open halfspace defined using non-empty intersection would then be exactly the definition of a closed halfspace.

**Theorem 6.3.23.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$, be an order preserving surjective hyperfield homomorphism between stringent hyperfields. Then, for a linear polynomial $p \in \mathbb{H}_1[X_1, \ldots, X_n]$,*

$$\mathcal{H}^+(f_*(p)) = f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big).$$

*Proof.* To be explicit with notation $p = \boxplus_{i=0}^d a_i \odot X_i$, a linear polynomial.

1. Take $\mathbf{Y} \in \mathcal{H}^+(f_*(p))$, then $\boxplus_{i=0}^d f(a_i) \odot Y_i = A \in \mathbb{H}_2^+$, as $\mathbb{H}_2$ is stringent. Then, for all $\mathbf{y} \in f^{-1}(\mathbf{Y})$ and $q \in \mathbb{H}_1[\underline{X}]$ such that $f_*(q) = f_*(p)$,

$$\begin{aligned} f(q(\mathbf{y})) = f\Big(\boxplus_{i=0}^{d} q_i \odot y_i\Big) &\subseteq \boxplus_{i=0}^{d} f(q_i) \odot f(y_i) \\ &= \boxplus_{i=0}^{d} f(a_i) \odot Y_i \\ &= A, \end{aligned}$$

by order preserving $q(\mathbf{y}) \in \mathbb{H}_1^+$. Thus, $\mathbf{y} \in \bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)$, implying that $\mathbf{Y} \in f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big)$. Demonstrating the inclusion,

$$\mathcal{H}^+(f_*(p)) \subseteq f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big).$$

2. Take $\mathbf{Y} \notin \mathcal{H}^+(f_*(p))$, then this occurs either by $f_*(p)(\mathbf{Y}) < \mathbb{0}_2$ or $f_*(p)(\mathbf{Y}) \ni \mathbb{0}_2$. If $f_*(p)(\mathbf{Y}) < \mathbb{0}_2$, then in an analogous way to part (1), the order preserving property of $f$ implies that $q(\mathbf{y}) < \mathbb{0}_1$, for any $\mathbf{y} \in f^{-1}(\mathbf{Y})$ and $q$ such that $f_*(q) = f_*(p)$. Explicitly,

$$f(q(\mathbf{y})) \subseteq f_*(q)(\mathbf{Y}) = f_*(p)(\mathbf{Y}) = B \in \mathbb{H}_2^-.$$

Therefore, $\mathbf{y} \notin \Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big)$, for all $\mathbf{y} \in f^{-1}(\mathbf{Y})$. This gives that $\mathbf{Y} \notin f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big)$.

Finally, take $\mathbf{Y}$ such that $f_*(p)(\mathbf{Y}) \ni \mathbb{0}_2$. Then, by Theorem 6.3.1, for any lift $\mathbf{y} \in f^{-1}(\mathbf{Y})$, there exists $q$, such that $\mathbb{0}_1 \in q(\mathbf{y})$ and $f_*(q) = f_*(p)$. Thus, for all $\mathbf{y} \in f^{-1}(\mathbf{Y})$, $\mathbf{y} \notin \mathcal{H}^+(q)$, for some $q$. Implying that for all $\mathbf{y} \in f^{-1}(\mathbf{Y})$, $\mathbf{y} \notin \Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big)$, which shows that $\mathbf{Y} \notin f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big)$. Demonstrating the inclusion,

$$f\Big(\bigcap_{f_*(p)=f_*(q)} \mathcal{H}^+(q)\Big) \subseteq \mathcal{H}^+(f_*(p)).$$

□

**Example 6.3.24.** It has been shown in Example 6.3.14 that $\mathcal{H}^+(sgn_*(p)) = \{(0,1),(1,0),(1,1)\}$, where $sgn_*(p) = X \boxplus Y$. Using the description of open halfspaces given in Theorem 6.3.23, this gives,

$$\mathcal{H}^+(X \boxplus Y) = sgn\Big(\bigcap_{q \in Q} \mathcal{H}^+(q)\Big),$$

where $Q = \{q = aX + bY \,:\, a, b \in \mathbb{R}_{>0}\}$. The visual representation of this is presented in Figure 6.2.

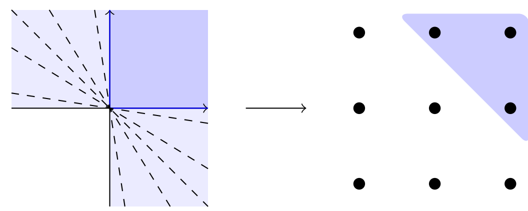

Figure 6.2: Demonstrating the intersection of open halfspaces.

## 6.4 Conic and Convex Sets over Hyperfields

This section will introduce the definitions of conic and convex sets over hyperfields. These will generalise the classical definitions in an algebraic sense. It will be explored how these sets interact with hyperfield homomorphisms and it will be discussed whether objects such as open/closed halfspaces and varieties are conic and convex in the hyperfield setting. There will be a focus on understanding conic and convex sets over quotient hyperfields and the corresponding quotient maps. One way that convex geometry over hyperfields differs from the classical theory is that hyperplanes of linear polynomials are not in general convex, see Example 6.4.18.

### 6.4.1 Conic Sets

**Definition 6.4.1.** A subset $S \subseteq \mathbb{H}^d$ is *conic* if for any two (not necessarily distinct) elements $\mathbf{u}, \mathbf{v} \in S$

$$\left\{ \lambda \odot \mathbf{u} \boxplus \mu \odot \mathbf{v} \;\middle|\; \lambda, \mu \in \overline{\mathbb{H}^+} \right\} \subseteq S\,,$$

where at least one of $\lambda, \mu$ is non-zero. The *conic hull* $\operatorname{cone}(S)$ of a set $S \subseteq \mathbb{H}^d$ is the smallest conic set that contains it.

**Proposition 6.4.2.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be a homomorphism of ordered hyperfields, and $S \subset \mathbb{H}_1^d$ a finite set. Then $f(\operatorname{cone}(S)) \subseteq \operatorname{cone}(f(S))$.*

*Proof.* Let $f(\mathbf{x}) \in f(\operatorname{cone}(S))$ where $\mathbf{x} \in \operatorname{cone}(S)$ and so is contained in a conic linear combination:

$$\begin{aligned} \mathbf{x} &\in \boxplus_{\mathbf{s}\in S} \lambda_{\mathbf{s}} \odot \mathbf{s} \quad , \quad \lambda_{\mathbf{s}} \in \overline{\mathbb{H}_1^+} \\ \Rightarrow f(\mathbf{x}) &\in f\big(\boxplus_{\mathbf{s}\in S} \lambda_{\mathbf{s}} \odot \mathbf{s}\big) \end{aligned}$$

$$\subseteq \boxplus_{f(\mathbf{s})\in f(S)} \lambda_{f(\mathbf{s})} \odot f(\mathbf{s}) \quad , \quad \lambda_{f(\mathbf{s})} = f(\lambda_{\mathbf{s}}) \in \overline{\mathbb{H}_2^+}$$
$$\subseteq \operatorname{cone}(f(S))$$

□

The inclusion $\operatorname{cone}(f(S)) \subseteq f(\operatorname{cone}(S))$ does not hold in general. Even under a surjective hyperfield homomorphism between stringent hyperfields, as can be seen in Example 6.4.3.

**Example 6.4.3.** The following calculation will show that $\operatorname{cone}(f(S)) \neq f(\operatorname{cone}(S))$ for the surjective homomorphism $Sgn\colon \mathbb{TR} \to \mathbb{S}$ from the signed tropical hyperfield $\mathbb{TR}$ to the hyperfield of signs $\mathbb{S}$.

Let $S = \{((1,0),(1,0)),((-1,0),(-1,1))\} \subset \mathbb{TR}^2$. The image of $S$ is $Sgn(S) = \{(\mathbb{1}_{\mathbb{S}},\mathbb{1}_{\mathbb{S}}),(-\mathbb{1}_{\mathbb{S}},-\mathbb{1}_{\mathbb{S}})\}$, the cone of which is $\operatorname{cone}(Sgn(S)) = \mathbb{S}^2$. However, the point $(\mathbb{1}_{\mathbb{S}},-\mathbb{1}_{\mathbb{S}})$ is not contained in the image $Sgn(\operatorname{cone}(S))$. To see this, consider a point $((1,x),(-1,y))$ in the preimage of $(-\mathbb{1}_{\mathbb{S}},\mathbb{1}_{\mathbb{S}})$. If $((1,x),(-1,y)) \in \operatorname{cone}(S)$, there exists scalars $(1,\lambda),(1,\mu) \in \mathbb{TR}^+$ such that

$$\begin{aligned}((1,x),(-1,y)) &\in (1,\lambda)\odot((1,0),(1,0)) \boxplus (1,\mu)\odot((-1,0),(-1,1)) \\ &= ((1,\lambda),(1,\lambda)) \boxplus ((1,\mu),(1,\mu+1)) .\end{aligned}$$

Comparing tropical signs, this implies $\mu > \lambda$ and $\mu + 1 < \lambda$, a contradiction.

**Lemma 6.4.4.** *Take the quotient map, $[\cdot] : \mathbb{H} \to \mathbb{H}/U$ and $S \subseteq \mathbb{H}^d$, then*

$$\operatorname{cone}([S]) = \bigcup_{f(S)=f(T)} [\operatorname{cone}(T)].$$

*Proof.* By Proposition 6.4.2,

$$\bigcup_{f(S)=f(T)} [\operatorname{cone}(T)] \subseteq \operatorname{cone}([S]).$$

Then, take $[\mathbf{x}] = ([x_1],\ldots,[x_d]) \in \operatorname{cone}([S])$. This admits a description as a conic linear combination,

$$[\mathbf{x}] \in \boxplus_{[\mathbf{s}]\in[S]} [\lambda_{\mathbf{s}}] \odot [\mathbf{s}] = \boxplus_{[\mathbf{s}]\in[S]} [\lambda_{\mathbf{s}} \odot \mathbf{s}], \quad [\lambda_{\mathbf{s}}] \in \overline{\left(\mathbb{H}/U\right)}^{+}.$$

This gives component wise,

$$[x_j] \in \boxplus_{[\mathbf{s}]\in[S]} [\lambda_{\mathbf{s}} \odot \mathbf{s}_j].$$

Thus, by the definition of the quotient addition, there exists $\tilde{x_j} \in \boxplus \lambda_{\mathbf{s}} \odot \mathbf{s}_j \odot u_{\mathbf{s}_j}$. Construct, for each $[\mathbf{s}] \in [S]$, $\tilde{\mathbf{s}} = (\mathbf{s}_1 \odot u_{\mathbf{s}_1}, \ldots, \mathbf{s}_d \odot u_{\mathbf{s}_d}) \in \mathbb{H}^d$, giving $[\tilde{\mathbf{s}}] = [\mathbf{s}]$. This implies that there exists $\tilde{\mathbf{x}} = (\tilde{x_1}, \ldots, \tilde{x_d}) \in \mathbb{H}^d$ such that $\tilde{\mathbf{x}} \in \boxplus \lambda_{\mathbf{s}} \odot \tilde{\mathbf{s}}$, and $[\tilde{\mathbf{x}}] = [\mathbf{x}]$. Then, $T = \{\tilde{\mathbf{s}} \mid [\mathbf{s}] \in [S]\}$, has $\tilde{\mathbf{x}} \in \mathrm{cone}(T)$. This shows the reverse inclusion, as by construction $[T] = \{[\tilde{\mathbf{s}}] \mid \tilde{\mathbf{s}} \in T\} = \{[\mathbf{s}] \mid \mathbf{s} \in S\}$. □

*Remark* 6.4.5. Apart the exceptional case of the field $\mathbb{F}_2$, the Krasner hyperfield is isomorphic to a quotient of every hyperfield as $\mathbb{K} \cong \mathbb{H}/\mathbb{H}^*$. Therefore, the morphism $\mathbb{H} \to \mathbb{K}$ can be represented as the quotient morphism $\mathbb{H} \to \mathbb{H}/\mathbb{H}^*$, thus Lemma 6.4.4 holds for all $\mathbb{H} \to \mathbb{K}$.

**Lemma 6.4.6.** *For a linear polynomial $p \in \mathbb{H}[X_1, \ldots, X_n]$, and $\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{H}^n$, then $p(\boldsymbol{x} \boxplus \boldsymbol{y}) = p(\boldsymbol{x}) \boxplus p(\boldsymbol{y})$.*

*Proof.* Firstly, $\mathbf{x} \boxplus \mathbf{y}$ is defined component wise as $\mathbf{x} \boxplus \mathbf{y} = (x_1 \boxplus y_1, \ldots, x_n \boxplus y_n)$. Then, evaluating at this combined element

$$\begin{aligned}
p(x_1 \boxplus y_1, \ldots, x_n \boxplus y_n) &= \boxplus_{i=1}^{n} a_i(x_i \boxplus y_i) \\
&= \boxplus_{i=1}^{n} (a_i x_i \boxplus a_i y_i) \\
&= (a_1 x_1 \boxplus a_1 y_1) \boxplus \cdots \boxplus (a_n x_n \boxplus a_n y_n) \\
&= (a_1 x_1 \boxplus \cdots \boxplus a_n x_n) \boxplus (a_1 y_1 \boxplus \cdots \boxplus a_n y_n) \\
&= p(\mathbf{x}) \boxplus p(\mathbf{y})
\end{aligned}$$

□

**Theorem 6.4.7.** *Let $p$ be a linear polynomial over an ordered $\mathbb{H}$, then $\mathcal{H}^+(p)$ is conic.*

*Proof.* Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{H}^+(p)$, which is equivalent to $p(\mathbf{x}), p(\mathbf{y}) \in \mathbb{H}^+$. Then with elements $\lambda, \mu \in \mathbb{H}^+$ construct the vectors,

$$\lambda\mathbf{x} = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n),$$

$$\mu\mathbf{y} = \mu(y_1, \ldots, y_n) = (\mu y_1, \ldots, \mu y_n).$$

By Lemma 6.4.6, $p(\lambda\mathbf{x} \boxplus \mu\mathbf{y}) = p(\lambda\mathbf{x}) \boxplus p(\mu\mathbf{y})$. Now, the positive scalars can be factorised out of the expression, due to the polynomial being linear. Explicitly,

$$\begin{aligned} p(\lambda\mathbf{x}) &= \boxplus_{i=1}^{n} a_i(\lambda x_i) \\ &= \boxplus_{i=1}^{n} a_i \lambda x_i \\ &= \lambda(\boxplus_{i=1}^{n} a_i x_i) \\ &= \lambda p(\mathbf{x}) \end{aligned}$$

Similarly, $p(\mu\mathbf{y}) = \mu p(\mathbf{y})$. Formally this yields,

$$p(\lambda\mathbf{x} \boxplus \mu\mathbf{y}) = \lambda p(\mathbf{x}) \boxplus \mu p(\mathbf{y}).$$

Recalling that $\lambda, \mu \in \mathbb{H}^+$ and $p(\mathbf{x}), p(\mathbf{y}) \subseteq \mathbb{H}^+$, by the multiplication closure of the order $\mathbb{H}^+$, $\lambda p(\mathbf{x}), \mu p(\mathbf{y}) \subseteq \mathbb{H}^+$. Furthermore, by the additive closure of the order $\mathbb{H}^+$, $\lambda p(\mathbf{x}) \boxplus \mu p(\mathbf{y}) \subseteq \mathbb{H}^+$. This gives $p(\lambda\mathbf{x} \boxplus \mu\mathbf{y}) \subseteq \mathbb{H}^+$, which is equivalent to $\lambda\mathbf{x} \boxplus \mu\mathbf{y} \in \mathcal{H}^+(p)$. Therefore, $\mathcal{H}^+(p)$ is conic. □

It can be seen that $\overline{\mathcal{H}^+}(p)$ is *not* conic in general, even for stringent hyperfields. This is due, more precisely, to varieties over stringent hyperfields not being convex in general. This can be seen in Example 6.4.18.

### 6.4.2 Convex Sets

Convex sets require more precision in the definition due to addition constraints requiring hyperaddition. Over $\mathbb{R}$, convex sets can be defined geometrically. Concretely, any two points of the set be joined by a straight line segment that completely remains within the set. This approach will be set aside and the algebraic approach, with regard to positive scalars, will be used. The reason for this being that for finite hyperfields, such as $\mathbb{S}$ the notion of a line segment is not yet established.

**Definition 6.4.8.** A subset $S \subseteq \mathbb{H}^d$ is *convex* if for any two elements $\mathbf{u}, \mathbf{v} \in S$

$$\left\{ \lambda \odot \mathbf{u} \boxplus \mu \odot \mathbf{v} \;\middle|\; \lambda, \mu \in \overline{\mathbb{H}^+},\ \mathbb{1} \in \lambda \boxplus \mu \right\} \subseteq S\,.$$

Given a set $V \subseteq \mathbb{H}^d$, the *convex hull* $\operatorname{conv}(V)$ of $V$ is (unique) minimal convex set containing $V$.

*Remark* 6.4.9. Note that unlike the classical definition, it is not demanded that $\lambda \boxplus \mu = \mathbb{1}$, thus making convex sets "larger" than one might expect. However, when the underlying hyperfield $\mathbb{H}$ is stringent, the conditions $\mathbb{1} \in \lambda \boxplus \mu$ and $\mathbb{1} = \lambda \boxplus \mu$ are equivalent. This is because $\lambda, \mu \in \overline{\mathbb{H}^+}$, then they cannot be additive inverses of one another, implying $\lambda \boxplus \mu$ is a singleton.

Let $X$ be a finite (multi)set of (not necessarily distinct) points $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq \mathbb{H}^d$. A *convex combination* of $X$ is an expression of the form

$$\boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}} \odot \mathbf{v} \subseteq \mathbb{H}^d \quad , \quad \lambda_{\mathbf{v}} \in \overline{\mathbb{H}^+} \, , \; \mathbb{1} \in \boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}} \, .$$

This occasionally denotes the set of all convex combinations of $X$ as

$$\Delta(X) = \bigcup_{\lambda_{\mathbf{v}} \in \overline{\mathbb{H}^+} \, , \, \mathbb{1} \in \boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}}} \left( \boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}} \odot \mathbf{v} \right) .$$

Note that unlike over fields, repetitions must be allowed in the set of points that a combination is being taken of, as $\lambda_1 \odot \mathbf{v} \boxplus \lambda_2 \odot \mathbf{v} = (\lambda_1 \boxplus \lambda_2) \odot \mathbf{v}$, but $\lambda_1 \boxplus \lambda_2$ may be a subset of $\mathbb{H}^+$ rather than an element. If $\mathbb{H}$ is stringent, then this is not an issue as $\lambda_1 \boxplus \lambda_2$ is always a singleton.

The following lemma shows a useful way to compute the convex hull of a set.

**Lemma 6.4.10.** *The convex hull of $V$ is equal to the set of all finite convex combinations of elements of $V$, i.e.*

$$\operatorname{conv}(V) = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \Delta(X) \, , \; X \subseteq V \; \textit{finite multiset} \, \}$$

*Proof.* To begin it is shown that $\operatorname{conv}(V)$ must contain all finite convex combinations of elements of $V$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$ be a finite set of points in $V$, possibly with repetition. It is claimed via induction on $k$ that $\boxplus_{j=1}^k \lambda_j \mathbf{v}_j \subseteq \operatorname{conv}(V)$, where $\lambda_j \in \mathbb{H}^+$

and $\mathbb{1} \in \boxplus_{j=1}^{k} \lambda_j$. For the base case $k = 2$, this holds by the definition of convex sets. Assume that the claim holds for $k-1$. As

$$\mathbb{1} \in \lambda_1 \boxplus \cdots \boxplus \lambda_{k-1} \boxplus \lambda_k = \bigcup_{\lambda \in \lambda_1 \boxplus \cdots \boxplus \lambda_{k-1}} \lambda \boxplus \lambda_k ,$$

there exists some $\gamma \in \lambda_1 \boxplus \cdots \boxplus \lambda_{k-1}$ such that $\mathbb{1} \in \gamma \boxplus \lambda_k$. Then

$$\boxplus_{j=1}^{k} \lambda_j \mathbf{v}_j = \gamma \odot \left( \boxplus_{j=1}^{k-1} \lambda_j \gamma^{-1} \mathbf{v}_j \right) \boxplus \lambda_k \mathbf{v}_k . \tag{6.4.11}$$

As $\mathbb{H}^+$ is closed under hyperaddition and multiplication, note that $\gamma \in \mathbb{H}^+$ and so $\lambda_j \gamma^{-1} \in \mathbb{H}^+$ also for all $j$. Furthermore

$$\mathbb{1} = \gamma \odot \gamma^{-1} \in (\boxplus_{j=1}^{k-1} \lambda_j) \odot \gamma^{-1} = \boxplus_{j=1}^{k-1} \lambda_j \gamma^{-1} ,$$

and so by the induction hypothesis, it holds that $\boxplus_{j=1}^{k-1} \lambda_j \gamma^{-1} \mathbf{v}_j \subseteq \operatorname{conv}(V)$. By definition of convex sets, equation (6.4.11) must also be in $\operatorname{conv}(V)$.

Conversely, it is shown that the set of finite convex combinations of $V$ forms a convex set. By the minimality of $\operatorname{conv}(V)$, this completes the proof. Let $\mathbf{x} = \boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}} \odot \mathbf{v}$ and $\mathbf{y} = \boxplus_{\mathbf{v} \in Y} \gamma_{\mathbf{v}} \odot \mathbf{v}$ be finite convex combinations of points in $V$, where $X, Y$ are finite subsets of points of $V$. By letting $\lambda_{\mathbf{v}} = \mathbb{0}$ for $\mathbf{v} \in Y \setminus X$ and $\gamma_{\mathbf{v}} = \mathbb{0}$ for $\mathbf{v} \in X \setminus Y$, it can be assumed that $X = Y$. Then for $\alpha, \beta \in \overline{\mathbb{H}^+}$ such that $\mathbb{1} \in \alpha \boxplus \beta$, the following holds;

$$\begin{aligned} \alpha \mathbf{x} \boxplus \beta \mathbf{y} &\in \alpha \left( \boxplus_{\mathbf{v} \in X} \lambda_{\mathbf{v}} \odot \mathbf{v} \right) \boxplus \beta \left( \boxplus_{\mathbf{v} \in Y} \gamma_{\mathbf{v}} \odot \mathbf{v} \right) \\ &= \left( \boxplus_{\mathbf{v}} \alpha \lambda_{\mathbf{v}} \odot \mathbf{v} \right) \boxplus \left( \boxplus_{\mathbf{v}} \beta \gamma_{\mathbf{v}} \odot \mathbf{v} \right) \\ &= \boxplus_{\mathbf{v}} (\alpha \lambda_{\mathbf{v}} \boxplus \beta \gamma_{\mathbf{v}}) \odot \mathbf{v} . \end{aligned}$$

Note that as $\overline{\mathbb{H}^+}$ is closed under multiplication, each $\alpha \lambda_{\mathbf{v}}, \beta \gamma_{\mathbf{v}} \in \overline{\mathbb{H}^+}$. Furthermore,

$$\begin{aligned} \mathbb{1} \in \alpha \boxplus \beta &\subseteq \alpha \left( \boxplus \lambda_{\mathbf{v}} \right) \boxplus \beta \left( \boxplus \lambda_{\mathbf{v}} \right) \\ &= \left( \boxplus \alpha \lambda_{\mathbf{v}} \right) \boxplus \left( \boxplus \beta \lambda_{\mathbf{v}} \right) \\ &= \boxplus \left( \alpha \lambda_{\mathbf{v}} \boxplus \beta \lambda_{\mathbf{v}} \right) . \end{aligned}$$

Therefore, the set of finite convex combinations of elements of $V$ is itself a convex set. □

**Lemma 6.4.12.** *Let $\mathbb{H} = K/U$ where $K$ is an ordered field, and $V \subseteq \mathbb{H}^d$. If $[\boldsymbol{x}] \in \operatorname{conv}(V) \subseteq \mathbb{H}^d$, then there exists $\tilde{\boldsymbol{x}} \in [\boldsymbol{x}]$ and $\tilde{V} \subseteq V \cdot U$ such that $\tilde{x} \in \operatorname{conv}(\tilde{V}) \subseteq K^d$.*

*Proof.* As $\mathbf{x} \in \operatorname{conv}(V)$, by Lemma 6.4.10 there exists a finite set of elements $\{[\mathbf{v}_1], \dots [\mathbf{v}_k]\} \subseteq V$ such that

$$[\mathbf{x}] \in \boxplus_{i=1}^{k} [\lambda_i] \odot [\mathbf{v}_i] \,, \; \mathbb{1} \in \boxplus [\lambda_i] \,.$$

By the quotient addition construction, this means that there exists $\tilde{\lambda}_i = \lambda_i \cdot r_i \in [\lambda_i]$ for some $r_i \in U$ such that $\sum_{i=1}^k \tilde{\lambda}_i = 1$ over $K$. In particular, the representative can be switched to $\tilde{\lambda}_i$.

Considering the $j$-th coordinate of $[\mathbf{x}]$, it can be seen that

$$[x_j] \in \boxplus [\tilde{\lambda}_i] \odot [v_{ij}] = \boxplus [\tilde{\lambda}_i \cdot v_{ij}] \,.$$

Again, by the quotient addition construction there must exist $u_{ij} \in U$ such that

$$\sum_{i=1}^{d} \tilde{\lambda}_i \cdot v_{ij} \cdot u_{ij} = x_j \,.$$

Therefore, picking $\tilde{\mathbf{v}}_i = (v_{i1} \cdot u_{i1}, \dots, v_{id} \cdot u_{id}) \in [\mathbf{v}_i]$ suffices. □

The next result will be an analogous version of Proposition 6.4.2 for convex sets. The proof will follow a similar pattern but presented here to emphasis the conditions on the positive scalars.

**Proposition 6.4.13.** *Let $f \colon \mathbb{H}_1 \to \mathbb{H}_2$ be a homomorphism of ordered hyperfields, and $S \subset \mathbb{H}_1^d$ a finite set. Then $f(\operatorname{conv}(S)) \subseteq \operatorname{conv}(f(S))$.*

*Proof.* Let $\mathbf{x} \in \operatorname{conv}(S)$, then by Lemma 6.4.10, $\mathbf{x} \in \boxplus_{\mathbf{s} \in S} \lambda_{\mathbf{s}} \odot \mathbf{s}$, where $\lambda_{\mathbf{s}} \in \overline{\mathbb{H}_1^+}$ and $\boxplus \lambda_{\mathbf{s}} \ni \mathbb{1}$. Then,

$$\begin{aligned} \Rightarrow f(\mathbf{x}) &\in f(\boxplus_{\mathbf{s} \in S} \lambda_{\mathbf{s}} \odot \mathbf{s}) \\ &\subseteq \boxplus_{f(\mathbf{s}) \in f(S)} \lambda_{f(\mathbf{s})} \odot f(\mathbf{s}) \quad , \quad \lambda_{f(\mathbf{s})} = f(\lambda_{\mathbf{s}}) \in \overline{\mathbb{H}_2^+} . \end{aligned}$$

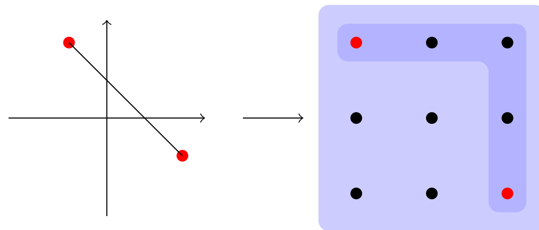

Figure 6.3: Visual representation of Example 6.4.14 .

This is a convex combination over $\mathbb{H}_2$ as,

$$\mathbb{1} = f(\mathbb{1}) \in f\Big(\boxplus_{\mathbf{s}\in S} \lambda_{\mathbf{s}}\Big) \subseteq \boxplus_{\mathbf{s}\in S} f(\lambda_{\mathbf{s}}) = \boxplus_{\mathbf{s}\in S} \lambda_{f(\mathbf{s})}.$$

Therefore, $f(\mathbf{x}) \in \mathrm{conv}(f(S))$. □

In a similar manner to conic sets over hyperfields the inclusion $\mathrm{conv}(f(S)) \subseteq f(\mathrm{conv}(S))$ does not hold in general.

**Example 6.4.14.** Take $S = \{(8,-2),(-2,8)\} \subseteq \mathbb{R}^2$. Then, $\mathrm{conv}(S)$ is the line connecting $(8,-2)$ and $(-2,8)$ passing through the upper right quadrant. This pushes forward as $sgn(\mathrm{conv}(S)) = \{(-1,1),(0,1),(1,1),(1,0),(1,-1)\}$. Furthermore, $sgn(S) = \{(1,-1),(-1,1)\}$, giving $\mathrm{conv}(sgn(S)) = \mathbb{S}^2$ as $(1,-1) \boxplus (-1,1) = \mathbb{S}^2$. This is depicted in Figure 6.3.

Even though the equality $\mathrm{conv}(f(S)) = f(\mathrm{conv}(S))$ is not true in general, presented next is a result for convex sets which is analogous to Corollary 6.3.4.

**Lemma 6.4.15.** *Take the quotient map, $[\cdot] : \mathbb{H} \to \mathbb{H}/U$ and $S \subseteq \mathbb{H}^d$, then*

$$\mathrm{conv}([S]) = \bigcup_{f(S)=f(T)} [\mathrm{conv}(T)].$$

*Proof.* By Proposition 6.4.13,

$$\bigcup_{f(S)=f(T)} [\mathrm{conv}(T)] \subseteq \mathrm{conv}([S]).$$

Then, by Lemma 6.4.12,

$$\mathrm{conv}([S]) \subseteq \bigcup_{f(S)=f(T)} [\mathrm{conv}(T)].$$

□

**Proposition 6.4.16.** *The intersection of convex sets over $\mathbb{H}^d$ is convex.*

*Proof.* Let $S_1, \dots, S_n \subseteq \mathbb{H}^d$ be convex sets. If $\bigcap_{i=1}^n S_i = \varnothing$ or is a singleton, then trivially convex. Say, $\left|\bigcap_{i=1}^n S_i\right| > 1$, take $\mathbf{x}, \mathbf{y} \in \bigcap_{i=1}^n S_i$. Then, $\lambda\mathbf{x} \boxplus \mu\mathbf{y} \subseteq S_i$, for all $\lambda, \mu \geqslant \mathbb{0}$ such that $\lambda \boxplus \mu \ni \mathbb{1}$. This implies that $\lambda\mathbf{x} \boxplus \mu\mathbf{y} \subseteq \bigcap_{i=1}^n S_i$, hence convex. □

**Proposition 6.4.17.** *For a linear polynomial $p$, $\mathcal{H}^+(p)$ is convex.*

*Proof.* This is a direct consequence of Theorem 6.4.7. □

Unlike open halfspaces, closed halfspaces are not convex. It is useful to note that the definitions of *conic* and *convex* align over $\mathbb{S}$. As the only possible positive scalar is 1, and $1 \boxplus 1 = 1$.

**Example 6.4.18.** Let $p = -X \boxplus Y \in \mathbb{S}[X,Y]$, the closed halfspace and variety defined by $p$ are;

$$\overline{\mathcal{H}^+}(p) = \{(-1,-1),(0,0),(1,1),(-1,1),(0,1),(-1,0)\}.$$

$$V(p) = \{(-1,-1),(0,0),(1,1)\} \subseteq \overline{\mathcal{H}^+}(p)$$

Then, $(1,1) \boxplus (-1,-1) = \mathbb{S}^2 \nsubseteq V(p)$ and hence, $(1,1) \boxplus (-1,-1) = \mathbb{S}^2 \nsubseteq \overline{\mathcal{H}^+}(p)$. Demonstrating that neither $V(p)$ or $\overline{\mathcal{H}^+}(p)$ are convex for $p = -X \boxplus Y$, and hence not in general either.

*Remark* 6.4.19. In particular, this example demonstrates that in contrast to the classical setting, linear hyperplanes are not convex. This emphasises the nuances of convex geometry over hyperfields.

The next step is to use the property that open halfspaces for linear polynomials are convex to show that open halfspaces for affine polynomials are also convex.

**Lemma 6.4.20.** *Let $p = a_0 \boxplus a_1 \odot x_1 \boxplus \cdots \boxplus a_n \odot x_n$ be a affine polynomial over stringent $\mathbb{H}$. Then, $\mathcal{H}^+(p)$ is convex.*

*Proof.* The open halfspace is defined as;

$$\mathcal{H}^+(p) := \{\mathbf{x} \in \mathbb{H}^d \;:\; a_0 \boxplus a_1 \odot x_1 \boxplus \cdots \boxplus a_d \odot x_d > \mathbb{0}\}$$

$$= \{\tilde{\mathbf{x}} = (\mathbb{1}, \mathbf{x}) \in \mathbb{H}^{d+1} \,:\, \boxplus_{i=0}^{d} a_i \odot \tilde{x}_i > \mathbb{0}\}$$

$$= \left\{\mathbf{y} \in \mathbb{H}^{d+1} \,:\, \boxplus_{i=0}^{d} a_i \odot y_i > \mathbb{0}\right\} \bigcap \left\{\mathbf{y} \,:\, y_1 = \mathbb{1}\right\}$$

The set on the left is a open halfspace for a linear polynomial in $\mathbb{H}^{d+1}$ and by Proposition 6.4.17 this is convex. The set on the right is equal to the variety, $V(y_1 \boxplus -\mathbb{1})$. In general in has been shown that varieties are not convex, but it will be shown that $V(y_1 \boxplus -\mathbb{1})$ is convex. Take $\mathbf{s}, \mathbf{w} \in V(y_1 \boxplus -\mathbb{1})$, giving $s_1 = w_1 = \mathbb{1}$. As $\mathbb{H}$ is stringent the positive scalars must be equal to $\mathbb{1}$, explicitly $\lambda \boxplus \mu = \mathbb{1}$. This gives,

$$\lambda s_1 \boxplus \mu w_1 = (\lambda \boxplus \mu)\mathbb{1} = \mathbb{1} \odot \mathbb{1} = \mathbb{1}.$$

Therefore, $(\lambda \mathbf{s} \boxplus \mu \mathbf{w})_1 = \mathbb{1}$, independent of whether the other coordinates produce sets. Thus, $(\lambda \mathbf{s} \boxplus \mu \mathbf{w}) \in V(y_1 \boxplus -\mathbb{1})$ and hence convex. Then, by Proposition 6.4.16, $\mathcal{H}^+(p)$ is convex, as it is the intersection is convex sets. □

**Proposition 6.4.21.** *Let $S \subseteq \mathbb{H}^d$, where $\mathbb{H}$ is stringent, then*

$$\mathrm{conv}(S) \subseteq \bigcap_{S \subseteq \mathcal{H}^+(p)} \mathcal{H}^+(p).$$

*Proof.* By Proposition 6.4.17 and Lemma 6.4.20 each $\mathcal{H}^+(p)$, whether defined by a linear or affine polynomial, is a convex set containing $S$. By Proposition 6.4.16, the intersection of these hafspaces is convex. By definition $\mathrm{conv}(S)$ is the smallest convex set containing $S$, hence giving the inclusion. □

## 6.5 Radon's, Helly's and Carathéodory's Theorems

This section will investigate generalisations of three theorems of classical convex geometry: Radon's, Helly's and Carathéodory's. The main technique, as prominent throughout the work so far, will be to lift back to an ordered field where the manipulation is more concrete, then push-forward back to the hyperfield. Firstly, the theorems are recalled over ordered field in general and then these are used to relate to hyperfields admitting a order preserving morphism from the ordered field.

### 6.5.1 Theorems over Ordered Fields

Radon's, Helly's and Carathéodory's theorems over ordered fields are recalled (and proven). Note that the proofs are practically identical over an arbitrary ordered field to the proofs over $\mathbb{R}$, however this has not been documented to the authors knowledge. Also note that this is unusual for theorems in convex geometry, as it makes no assumption on the existence of a metric. The proofs are included for completeness. Throughout the following, let $K$ be an arbitrary ordered field.

**Theorem 6.5.1** (Radon's Theorem)**.** *Let $K$ be an ordered field and $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+2}\} \subset K^d$. Then there exists a point $\boldsymbol{p} \in K^d$ and a nonempty subset $I \subseteq [d+2]$ such that*

$$\boldsymbol{p} \in \operatorname{conv}(\boldsymbol{x}_i \mid i \in I) \cap \operatorname{conv}(\boldsymbol{x}_j \mid j \notin I)\,.$$

*Proof.* Let $\lambda_1, \ldots, \lambda_{d+2} \in K$ be a non-zero solution to the following system of $d+1$ linear equations in $d+2$ unknowns:

$$\sum_{i=1}^{d+2} \lambda_i = 0 \quad , \quad \sum_{i=1}^{d+2} \lambda_i \mathbf{x}_{ij} = 0\,,\; 1 \leqslant j \leqslant d\,.$$

Let $I = \{i \mid \lambda_i > 0\}$ and set

$$\mathbf{p} = \sum_{i \in I} \frac{\lambda_i}{\gamma} \mathbf{x}_i = \sum_{j \notin I} \frac{-\lambda_j}{\gamma} \mathbf{x}_j\,,$$

where $\gamma = \sum_{i \in I} \lambda_i = -\sum_{j \notin I} \lambda_j$. Note that $\gamma$ is positive as the sum of positive elements, and so the coefficients $\lambda_i/\gamma$ are all positive for $i \in I$ and sum to one. Similarly, $\lambda_j$ are all negative for $j \notin I$ and so $-\lambda_j/\gamma$ are all positive and sum to one. Therefore, $\mathbf{p}$ suffices as the point in the statement. □

**Theorem 6.5.2** (Helly's Theorem)**.** *Let $X_1, \ldots, X_n$ be a finite collection of convex sets in $K^d$, with $n \geqslant d+1$. The intersection of each $d+1$ collection of sets is non-empty if and only if the intersection of all the sets is non-empty.*

*Proof.* Note that as one direction is trivially true, it suffices to show each $d+1$ intersection is non-empty implies the whole intersection is non-empty. Proceeding by induction on $n$; note that $n = d+1$ is trivially true, so take $n = d+2$ as the base case.

For $1 \leqslant j \leqslant d+2$, there exists some $\mathbf{x}_j \in \bigcap_{i \neq j} X_i$. By Radon's theorem, an $I \subset [d+2]$ can be found such that

$$\mathbf{p} \in \operatorname{conv}(\mathbf{x}_i \mid i \in I) \cap \operatorname{conv}(\mathbf{x}_j \mid j \notin I) .$$

If $i \in I$, then $x_j \in X_i$ for all $j \notin I$, and so $\operatorname{conv}(\mathbf{x}_j \mid j \notin I) \subseteq X_i$ by convexity. The same argument shows if $j \notin I$ then $\operatorname{conv}(\mathbf{x}_i \mid i \in I) \subseteq X_j$. This implies $\mathbf{p} \in X_i$ for all $1 \leqslant i \leqslant d+2$, and so $\mathbf{p} \in \bigcap_{i=1}^{d+2} X_i$.

For the induction step, $X_{n-1}$ and $X_n$ can be replaced with $X_{n-1} \cap X_n$: this set is non-empty and convex and so the induction hypothesis completes the proof. □

*Remark* 6.5.3. Over $\mathbb{R}$, Helly's theorem can be strengthened to an infinite version, provided that the convex sets are also compact. Topological concerns are more intricate over arbitrary ordered fields: in general, a metric cannot be placed on $K$ as done in $\mathbb{R}$, see [Dob00]. Therefore, only the finite version is discussed, which requires no compactness condition.

**Theorem 6.5.4** (Carathéodory's Theorem). *Let $V \subseteq K^d$. If $\boldsymbol{x} \in \operatorname{conv}(V)$, then $\boldsymbol{x}$ can be written as a convex combination of at most $d+1$ points in $V$.*

*Proof.* As $\mathbf{x} \in \operatorname{conv}(V)$, it can be written as a finite convex combination. Suppose that it can expressed as the convex combination $\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{v}_j$ where $k > d+1$ is minimal, i.e. $\lambda_j > 0$ for all $j$. The vectors $\mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_k - \mathbf{v}_1$ must be linearly independent, therefore there exist $\gamma_2, \ldots, \gamma_k \in K$ not all zero such that

$$\sum_{j=2}^{k} \gamma_j (\mathbf{v}_j - \mathbf{v}_1) = 0 .$$

Setting $\gamma_1 = -\sum_{j=2}^{k} \gamma_j$ gives us a linear dependence $\sum_{i=1}^{k} \gamma_i \mathbf{v}_i = 0$ such that the sum of the scalars is zero. As $\gamma_i$ are not all zero, there exists at least one such that $\gamma_i > 0$. Define

$$\alpha = \min_{1 \leqslant i \leqslant k} \left\{ \frac{\lambda_i}{\gamma_i} \mid \gamma_i > 0 \right\} = \frac{\lambda_\ell}{\gamma_\ell} > 0 .$$

Note that

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i - \alpha \left( \sum_{i=1}^{k} \gamma_i \mathbf{v}_i \right) = \sum_{i=1}^{k} (\lambda_i - \alpha \gamma_i) \mathbf{v}_i , \tag{6.5.5}$$

where the sum of the coefficients $\lambda_i - \alpha\gamma_i$ is one. Furthermore

$$\lambda_i - \alpha\gamma_i \geqslant \lambda_i - \frac{\lambda_i}{\gamma_i}\gamma_i = 0 \quad , \quad 1 \leqslant i \leqslant k$$

with $\lambda_\ell - \alpha\gamma_\ell = 0$. Therefore, the $\ell$-th term from (6.5.5) can be removed, giving $\mathbf{x}$ as a convex combination of $k-1$ points, a contradiction. □

### 6.5.2 Pushing Forward to Ordered Hyperfields

In this section, the previous theorems over ordered fields are used to prove the corresponding results for ordered quotient hyperfields.

**Theorem 6.5.6.** *Let $\mathbb{H}$ be an ordered hyperfield, $K$ an ordered field and $f\colon K \to \mathbb{H}$ a surjective order-preseriving homomorphism between them. Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+2}\} \subseteq \mathbb{H}^d$, then there exists a point $\boldsymbol{p} \in \mathbb{H}^d$ and a nonempty subset $I \subseteq [d+2]$ such that*

$$\boldsymbol{p} \in \operatorname{conv}(\boldsymbol{x}_i \mid i \in I) \cap \operatorname{conv}(\boldsymbol{x}_j \mid j \notin I)\,.$$

*Proof.* For each $\mathbf{x}_i$, take some lift $\tilde{\mathbf{x}}_i \in f^{-1}(\mathbf{x}_i)$. By Radon's theorem for ordered fields, there exists some non-empty $I \subset [d+2]$ such that there exists some $\tilde{\mathbf{p}}$ that can be written as two different convex combinations:

$$\tilde{\mathbf{p}} = \sum_{i \in I} \lambda_i \tilde{\mathbf{x}}_i = \sum_{j \notin I} \gamma_j \tilde{\mathbf{x}}_j\,.$$

Then,

$$\tilde{\mathbf{p}} \in \operatorname{conv}(\tilde{\mathbf{x}}_i \mid i \in I) \cap \operatorname{conv}(\tilde{\mathbf{x}}_j \mid j \notin I) \Rightarrow f(\tilde{\mathbf{p}}) \in f(\operatorname{conv}(\tilde{\mathbf{x}}_i \mid i \in I)) \cap f(\operatorname{conv}(\tilde{\mathbf{x}}_j \mid j \notin I)),$$

which by Proposition 6.4.13 gives,

$$\begin{aligned} \mathbf{p} = f(\tilde{\mathbf{p}}) &\in \operatorname{conv}(f(\tilde{\mathbf{x}}_i) \mid i \in I) \cap \operatorname{conv}(f(\tilde{\mathbf{x}}_j) \mid j \notin I) \\ &= \operatorname{conv}(\mathbf{x}_i \mid i \in I) \cap \operatorname{conv}(\mathbf{x}_j \mid j \notin I). \end{aligned}$$

□

**Lemma 6.5.7.** *Let $f : \mathbb{H}_1 \to \mathbb{H}_2$ be an order preserving hyperfield homomorphism. Take $S \subseteq \mathbb{H}_2^d$, then $f^{-1}(\operatorname{conv}(S))$ is convex.*

*Proof.* Let $C$ denote $f^{-1}(\mathrm{conv}(S))$, then for $\mathbf{x}, \mathbf{y} \in C$,

$$f(\lambda \mathbf{x} \boxplus \mu \mathbf{y}) \subseteq f(\lambda) \odot f(\mathbf{x}) \boxplus f(\mu) \odot f(\mathbf{y}).$$

As $f(\lambda) > \mathbb{0}_2$ and $f(\mu) > \mathbb{0}_2$ and $f(\mathbf{x}), f(\mathbf{y}) \in \mathrm{conv}(S)$, along with $\mathbb{1}_1 \in \lambda \boxplus \mu$, then $\mathbb{1}_2 = f(\mathbb{1}_1) \in f(\lambda \boxplus \mu) \subseteq f(\lambda) \boxplus f(\mu)$. This implies $f(\lambda \mathbf{x} \boxplus \mu \mathbf{y}) \subseteq \mathrm{conv}(S)$. Therefore, $C$ is convex. □

**Theorem 6.5.8.** *Let $\mathbb{H}$ be an ordered hyperfield, $K$ an ordered field and $f : K \to \mathbb{H}$ a surjective order-preseriving homomorphism between them. Let $X_1, \ldots, X_n$ be a finite collection of convex sets in $\mathbb{H}^d$, with $n \geqslant d+1$. The intersection of each $d+1$ collection of sets is non-empty if and only if the intersection of all the sets is non-empty.*

*Proof.* The pre-image of a convex set is convex by Lemma 6.5.7, thus $Y_1, \ldots, Y_n$, where $Y_i = f^{-1}(X_i)$, are convex sets over $K^d$. If the intersection of each $d+1$ collection of $X_1, \ldots, X_n$ is non-empty over $\mathbb{H}^d$, then this is also the case for $Y_1, \ldots, Y_n$. Then, by Theorem 6.5.2, the intersection of the complete collection is non-empty, i.e.

$$\varnothing \neq A = \bigcap_{i=1}^{n} Y_i.$$

Yielding,

$$\varnothing \neq f(A) = \bigcap_{i=1}^{n} f(Y_i) = \bigcap_{i=1}^{n} X_i.$$

□

*Remark* 6.5.9. The existence of an order preserving hyperfield homomorphism from an ordered field to a ordered stringent hyperfield can be proven by the classification in Section 6.2.1. It is stated that every ordered stringent hyperfield is the quotient of a ordered field, thus a quotient map exists from the ordered field to the stringent ordered hyperfield. This map is order preserving when multiplicative subgroup $U$ is contained in the order. As [KLS21, Theorem 3.4] shows that $|\chi(K/U)| = |\chi(K \mid U)|$, there is an order that contains $U$. Hence, by Lemma 6.3.11 this compatible map is order preserving.

**Theorem 6.5.10.** *Let $\mathbb{H} = K/U$ be a hyperfield, where $K$ is an ordered field. Let $V \subseteq \mathbb{H}^d$. If $\boldsymbol{x} \in \operatorname{conv}(S)$, then $\boldsymbol{x}$ can be written as a convex combination of at most $d+1$ points in $S$.*

*Proof.* By Lemma 6.4.12, lift $\mathbf{x} \in \operatorname{conv}(S)$ to $\tilde{\mathbf{x}} \in \operatorname{conv}(\tilde{S})$. By Theorem 6.5.4, $\tilde{\mathbf{x}}$ can be written as,

$$\tilde{\mathbf{x}} = \sum_{j=1}^{d+1} \tilde{\lambda_j} \cdot \tilde{s_{i_j}}.$$

where $\tilde{s_{i_j}} \in \tilde{S}$. This implies,

$$\mathbf{x} = [\tilde{\mathbf{x}}] = \Big[ \sum_{j=1}^{d+1} \tilde{\lambda_j} \cdot \tilde{s_{i_j}} \Big] \subseteq \boxplus_{j=1}^{d+1} [\tilde{\lambda_j}] \odot [\tilde{s_{i_j}}].$$

Where $[\tilde{\lambda_j}] \in S$, and $1 = \sum_{j=1}^{d+1} \tilde{\lambda_j}$ gives $\mathbb{1} \in \boxplus_{j=1}^{d+1} [\tilde{\lambda_j}]$. Thus, $\mathbf{x}$ can be written as a convex combination of at most $d+1$ points in $S$ □

### 6.5.3 Applications of Convex Geometry over Hyperfields

Convex geometry over hyperfields in it's present state has applications as a general algebraic framework and meaningful future aims as a tool to be applied in linear programming.

Firstly, the contributions presented in this chapter can be viewed as a general algebraic framework which encompasses several recent areas of interest. The tropical semi-ring has limitations for studying convexity, the lack of subtraction being a key one. Therefore, recently the study of convexity has required an extended foundation to progress. This includes tow distinct directions: signed tropical convexity in [LV19] and [LS22], and higher rank tropical geometry, for example in [JS18]. It can be seen that convex geometry over hyperfields is a generalisation of both.

More concretely, under the classification of stringent ordered hyperfields in section 6.2.1 both signed and higher rank versions of the tropical setting can be constructed. The signed tropical hyperfield can be seen as $\mathbb{TR} = \mathbb{S} \rtimes \mathbb{R}$ and higher rank tropical geometry can be seen as $\mathbb{K} \rtimes \mathbb{R}^n$. Moreover, it also encompasses tropical and classical convexity in an analogous way. Thus, the progress in this section can be seen to build on, encompass and generalise these works. This demonstrates where convex geometry can be seen in action.

Secondly, in long term aspirations there is hope to apply these tools and techniques to contribute to complexity problems in linear programming and optimisation. This will require more substantial work and significant effort to fully connect these theories, but it is hoped that the work will continue to strive forward in this direction.

## 6.6 Further Questions

To conclude, there is a discussion of the open questions and paths for future research regarding convex geometry over ordered hyperfields. The work outlined in this chapter is an ongoing project with B.Smith and the following are the possible suggested areas for future investigation.

In classical group theory the following theorem classifies linearly ordered groups.

**Theorem 6.6.1** (Hahn's Embedding Theorem)**.** *Every linearly ordered abelian group $G$ can be embedded as an ordered subgroup of the additive group $\mathbb{R}^k$ with the lexicographical ordering.*

Can one give a hypergroup analogue of Hahn's embedding theorem? This would build on the classification of ordered hyperfields in Section 6.2.1.

There is now a workable definition of vector spaces over $\mathbb{H}$. Is it possible to develop a theory of matrices over hyperfields and connect this, as done classically, to the vector space definition. This is motivated by the link between linear programming and optimisation and then classical matrix theory.

Furthermore, there are many results for matrices over tropical-like semirings, see [AGG09] and [AGG14]. Using the correspondence between hyperfields and ($\mathbb{T}$-)semirings, can one recover these results for hyperfields? There are three distinct notions of rank of a tropical matrix, see [MS15, Section 5.3]. Do these align over hyperfields?

Finally, the main aim of future research on convex geometry over hyperfields is to study separation theorems. for example, does it hold that there exists a hyperplane that separates disjoint convex sets over hyperfields? There are a range of theorems in this area which are yet to be studied over hyperfields.

# Chapter 7

# Open Questions and Future Aims

This section highlights the key open questions from each section of this work and presents potential areas for future research.

***Chapter 2: Hyperfield Zoo***

There is a single gap, and hence one open question, in Table 2.1.

**Question 7.0.1.** *Is there a way to express* $\mathbb{TC}$ *as the quotient of a field/hyperfield?*

***Chapter 3: Generalising Kapranov's Theorem***

To give a more detailed description of RAC maps it would be essential to understand how to fully characterise them. This could be by further examples, or linking the inheritance property to a well studied property, such as the doubly distributive property.

**Question 7.0.2.** *Are there any other non-trivial examples of RAC hyperfield homomorpshims?*

**Question 7.0.3.** *Do doubly distributive hyperfields satisfy the inheritance property?*

One way to further progress the knowledge of RAC maps and hyperfields would be to deconstruct the confusion regarding the classification of hyperfields satisfying the multiplicity bound.

**Question 7.0.4.** *Does the multiplicity bound hold in general for doubly distributive hypefields?*

***Chapter 4: Polynomials Over Quotient Hyperfields***

There is a broad area of possible research stemming from this chapter. The work presented here suggests that instead of computing varieties of single polynomials and pushing forward through hyperfield homomorphisms, it makes mathematical sense to use the information stored in the ideal generated by the polynomial to precisely cut out the corresponding points over the hyperfield. Equally, rather than exploring Kapranov's theorem, there should be a view to investigate a generalise fundamental theorem of tropical geometry.

**Question 7.0.5.** *Can a version of the fundamental theorem of tropical geometry be stated in general for a class of hyperfield homomorphism?*

***Chapter 5: Equivalence of Tropical Ideals***

The matroidal structure of tropical ideals is preserved after acting on them with certain actions, as seen in [MR18] and [MR20]. One way to develop the theory introduced here would be to see how matroidal equivalence is affected by these actions.

**Question 7.0.6.** *How does matroidal equivalence interact with taking the initial form, homogenisation and saturation of the ideals?*

Currently, matroidal equivalence is composed of three axioms. It would be natural to investigate if they are all necessary, and whether matroidal equivalence is coarser than relations based on a single axiom.

**Question 7.0.7.** *How does matroidal equivalence compare, as a relation, to each individual axiom taken independently?*

The motivation for introducing matroidal equivalence was to understand where the geometric objects defined by tropical ideals should be associated with each other. The natural question is then to understand how the geometric objects are connected to one another when they are matroidally equivalent. One possible tool that could be utilised to understand this is the idea of tropical modifications, see [Kal15] and [CM16].

**Question 7.0.8.** *Can matroidal equivalence be connected with tropical modification?*

In more long term aspirations, once this theory has been sufficiently developed for tropical ideals, can this then be extended to hyperfields. More explicitly, defining ideals over hyperfields by the matroidal structure and generalising the equivalence to study them.

**Question 7.0.9.** *Can matroidal equivalence be utilised to study a coherent notion of polynomial ideals over hyperfields?*

***Chapter 6: Convex Geometry over Hyperfields***

The initial motivation to studying convex geometry over hyperfields came from the progression of tropical and signed tropical convex geometry, [DS03] and [LV19] respectively. Along with the applications convex geometry has linear programming and optimisation. It would be productive to make a concrete link between the theory presented here and the literature and its applications.

The work done in [LV19] comments on separation for signed tropical convexity. Although, thus far there has been no presentation of a generalised version of any separation theorems over hyperfields analogous to the classical or signed tropical theory. This would be an interesting direction to spend time investigating further.

**Question 7.0.10.** *Can there be a formulation of separation theorems over ordered stringent hyperfields?*

In classical convex geometry there are colourful version of Radon's, Helly's and Caratheodory's theorems, see [BO95] and [BO96]. As the standard versions have been generalised to hyperfields in this work it would be natural to explored the colourful versions.

**Question 7.0.11.** *Do the colourful versions of Radon's, Helly's and Caratheodory's theorems hold over hyperfields admitting an ordered preserving map from an ordered field?*

In section 6.2.1, there is a classification of ordered stringent hyperfields using results given in [BS20]. In classical group theory Hahn's theorem gives a precise description of ordered groups in relation to an embedding into $\mathbb{R}^k$, see Theorem 6.6.1 for details. To further classify ordered hyper-structures it would be of interest to investigate whether a results analogous to this holds for hypergroups.

**Question 7.0.12.** *Can a hypergroup analogue of Hahn's embedding theorem be stated?*

Classical convexity theory is connected to linear programming and optimisation problems. These are regularly phrased in terms of matrices. One direction for future study would be to develop a theory of matrices over hyper-structures.

**Question 7.0.13.** *Can a theory of matrices be developed over hyperfields that unifies the ideas of classical and tropical matrices?*

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