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Semilocal Milnor K-Theory

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In this paper, semilocal Milnor K-theory of fields is introduced and studied. A strongly convergent spectral sequence relating semilocal Milnor K-theory to semilocal motivic cohomology is constructed. In weight 2, the motivic cohomology groups $H^p_{\mathrm{Zar}}(k, \mathbb{Z}(2))$, $p \leqslant 1$, are computed as semilocal Milnor K-theory groups $\widehat{K}^M_{2,3-p}(k)$. The following applications are given: (i) several criteria for the Beilinson–Soulé Vanishing Conjecture; (ii) computation of K_4 of a field; (iii) the Beilinson conjecture for rational K-theory of fields of prime characteristic is shown to be equivalent to vanishing of rational semilocal Milnor K-theory.

1 Introduction

It is a classical fact of algebraic K-theory of fields that Milnor K-groups K^M_0, K^M_1, K^M_2 agree with Quillen's K_0, K_1, K_2. However, K^M_n is only a small piece of Quillen's K_n for $n \geqslant 3$. A key technical tool to make computations in algebraic K-theory is the motivic spectral sequence

$$E^2_{p,q} = H^{q-p}_{\mathrm{Zar}}(k, \mathbb{Z}(q)) \Longrightarrow K_{p+q}(k)$$

relating algebraic K-theory to motivic cohomology (see, e.g., [6]).

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By well-known theorems of Nesterenko–Suslin [23] and Totaro [28], the Milnor K-theory ring $K^M_*(k)$ is isomorphic to the ring $\bigoplus H^n_{\mathrm{Zar}}(k,\mathbb{Z}(n))$. We also know that $H^1_{\mathrm{Zar}}(k,\mathbb{Z}(2)) = K_3^{\mathrm{ind}}(k)$, where $K_3^{\mathrm{ind}}(k)$ is the indecomposable K-theory of k. The other motivic cohomology groups are a complete mystery. Their computation, and hence computation of algebraic K-theory, is one of the hardest problems in the field and several outstanding conjectures are related to this problem. For instance, the celebrated Beilinson–Soulé vanishing conjecture states that all motivic cohomology groups $H^p_{\mathrm{Zar}}(k,\mathbb{Z}(q))$ vanish for $p \leqslant 0$ and $q > 0$ [27, §3]. The finite coefficient motivic cohomology groups are much better understood due to the norm residue isomorphism theorem (formerly known as Milnor/Bloch–Kato conjectures) relating the motivic cohomology to étale cohomology [32]. The mysteries (such as the Beilinson–Soulé vanishing conjecture) mostly surround the K-theory and motivic cohomology with rational coefficients. In positive characteristic, the Beilinson conjecture states that Milnor K-theory and Quillen K-theory agree rationally:

$$K^M_n(k)_{\mathbb{Q}} \xrightarrow{\cong} K_n(k)_{\mathbb{Q}}.$$

As we have mentioned above, Milnor K-theory is isomorphic to the motivic cohomology diagonal $\bigoplus H^n_{\mathrm{Zar}}(k,\mathbb{Z}(n))$. The main purpose of this paper is to introduce and investigate "semilocal Milnor K-theory of fields". We show that it is precisely related to motivic cohomology outside the diagonal. An advantage of the theory is that it is defined in elementary terms whereas the motivic complexes are sophisticated and enormously hard for computations. All the definitions of the motivic complexes are strictly geometric, whereas the definition of semilocal Milnor K-theory is strictly algebraic (whenever the base field is infinite—see Remark 3.2).

By definition, semilocal Milnor K-theory of a field k consists of bigraded Abelian groups $\widehat{K}^M_{n,m}(k)$, $m,n \geqslant 0$ (see Definition 3.1). Precisely, let $\widehat{\Delta}^\bullet_k$ be the cosimplicial scheme, where each $\widehat{\Delta}^\ell_k$ is the semilocalization of the standard affine scheme Δ^ℓ_k at its vertices $v_0,\dots,v_\ell$ (see Definition 2.3). Let $\mathscr{K}^M_n$ be the Zariski sheaf of Milnor K-theory in degree $n \geqslant 0$. Semilocal Milnor K-theory complex is the chain complex $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)$ and

$$\widehat{K}^M_{n,m}(k) := H_m(\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)).$$

If k is infinite, the complex $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)$ is defined naively in terms of generators and relations (see Remark 3.2). So semilocal Milnor K-theory groups are defined as homology groups of complexes defined by generators and relations.

The main result of the paper, Theorem 3.5, says that there is a strongly convergent spectral sequence relating semilocal Milnor K-theory to semilocal motivic cohomology

$$E^2_{pq} = H_p(H^{n-1-q}_{\mathrm{Zar}}(\widehat{\Delta}^{\bullet}_k, \tau_{<n}\mathbb{Z}(n))) \Longrightarrow \widehat{K}^M_{n,p+q+2}(k).$$

Here $\tau_{<n}\mathbb{Z}(n)$ is the truncation complex of $\mathbb{Z}(n)$ for degrees smaller than n and $H^{n-1-q}_{\mathrm{Zar}}(\widehat{\Delta}^{\bullet}_k, \tau_{<n}\mathbb{Z}(n))$ is the chain complex obtained by the evaluation of the $(n-1-q)$-th Zariski cohomology sheaf of the complex $\tau_{<n}\mathbb{Z}(n)$ at the cosimplicial semilocal scheme $\widehat{\Delta}^{\bullet}_k$. Moreover, if $n=2$, then the spectral sequence above collapses, and hence for any $p \leqslant 1$ there is an isomorphism $H^p_{\mathrm{Zar}}(k, \mathbb{Z}(2)) = \widehat{K}^M_{2,3-p}(k)$. Thus, semilocal Milnor K-theory is related to motivic cohomology exactly outside the diagonal $\oplus H^n(k, \mathbb{Z}(n))$ in contrast with the classical Milnor K-theory. Another important and strong property of semilocal Milnor K-theory that distinguishes it, say, from motivic cohomology and Quillen's K-theory is that it is invariant under purely transcendental extensions (see Theorem 3.14). The spectral sequence of Theorem 3.5 also implies that plenty of information is removed from motivic cohomology groups to get semilocal Milnor K-theory groups (see Corollary 3.7 as well).

Various applications of semilocal Milnor K-theory are given in the paper. First, several criteria for the Beilinson–Soulé vanishing conjecture are established in Theorem 4.1. We next pass to computation of K_4 of a field. The group K_3 was actively investigated in the 80s—see Levine [17, 18], Merkurjev and Suslin [21, 24] (it is worth mentioning that semilocal PIDs play an important role in their analysis). Recall that $K_3(k)$ fits into an exact sequence

$$0 \to K^M_3(k) \to K_3(k) \to K^{\mathrm{ind}}_3(k) \to 0.$$

We compute the group $K^{\mathrm{ind}}_3(k)$ as $\widehat{K}^M_{2,2}(k)$ in Corollary 3.10, so that $K_3(k)$ is fully determined by Milnor K-theory and semilocal Milnor K-theory. The latter computation is of independent interest. Similarly to $K_3(k)$ we show in Theorem 5.4 that $K_4(k)$ is also fully determined by Milnor K-theory and semilocal Milnor K-theory. These are somewhat surprising results. As a whole, the relationship between the semilocal Milnor K-theory $\widehat{K}^M_{*,*}(k)$ and algebraic K-theory $K_*(k)$ is of great interest. Indeed, semilocal Milnor K-theory groups are homology groups of complexes defined naively by generators and relations (if k is infinite), whereas $K_*(k)$ is defined as homotopy groups of some infinite-dimensional space and thus seemingly very inaccessible to computations.

Another application is given for the Beilinson conjecture on the rational K-theory of fields of prime characteristic. Namely, it is shown in Theorem 6.1 that this conjecture is equivalent to vanishing of rational semilocal Milnor K-theory. Also, vanishing of rational semilocal Milnor K-theory is shown to be a necessary condition for Parshin's conjecture (see Theorem 6.3).

In the final Section 7, it is shown that in contrast with motivic cohomology groups with mod 2 coefficients, semilocal Milnor K-theory groups with $\mathbb{Z}/2$-coefficients $\widehat{K}^M_{n,*}(k,\mathbb{Z}/2)$ are zero for any $n > 1$. This is another property of semilocal Milnor K-theory distinguishing it with the classical Milnor K-theory/motivic cohomology of fields. We also raise a conjecture in this section on rational contractibility of the logarithmic de Rham–Witt sheaves $W_r\Omega^n_{\log}$.

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2 Preliminaries

Throughout the paper, we denote by Sm/k the category of smooth separated schemes of finite type over a field k. By a smooth semilocal scheme over k, we shall mean a k-scheme W for which there exists a smooth affine scheme $X \in Sm/k$ and a finite set $x_1, \dots, x_n$ of points of X such that W is the inverse limit of open neighborhoods of this set. We will deal with both complexes for which the differential has degree -1 (chain complexes) and those for which the differential has degree $+1$ (cochain complexes). By the homological shift of a chain complex A, we mean the chain complex $A[1]$ with $A[1]_n = A_{n+1}$ and differential $-d^A$. The author should stress that the machinery of motivic homotopy theory is not used here (except the proofs of Theorem 3.14 and Proposition 5.1) because we often work with non-$\mathbb{A}^1$-invariant (pre)-sheaves having no transfers.

Following [25, §2], for any presheaf $\mathscr{F} : Sm/k \to \mathrm{Ab}$, let $\widetilde{C}_1\mathscr{F}$ denote the following presheaf:

$$\widetilde{C}_1\mathscr{F}(X) = \varinjlim_{X\times\{0,1\}\subset U\subset X\times\mathbb{A}^1} \mathscr{F}(U).$$

There are two obvious presheaf homomorphisms (given by restrictions to $X\times 0$ and $X\times 1$, respectively) $i_0^*, i_1^* : \widetilde{C}_1\mathscr{F} \to \mathscr{F}$.

Definition 2.1 ([25, 27]). A presheaf $\mathscr{F}$ is said to be *rationally contractible* if there exists a presheaf homomorphism $s : \mathscr{F} \to \widetilde{C}_1\mathscr{F}$ such that $i_0^*s = 0$ and $i_1^*s = \mathrm{id}$.

Example 2.2. (1) Given $n, l > 0$, the Zariski sheaves with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) := Cor(-, \mathbb{G}_m^{\wedge n})$ and $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l$ defined in [27, Section 3] are rationally contractible by [27, 9.6].

(2) Let k be a perfect field of characteristic not 2. Then the presheaf with Milnor–Witt correspondences $\widetilde{\mathbb{Z}}(\mathbb{G}_m^{\times n})/\widetilde{\mathbb{Z}}((1,\dots,1)) := \widetilde{Cor}(-, \mathbb{G}_m^{\times n})/\widetilde{Cor}(-,(1,\dots,1))$ in the sense of [4] is rationally contractible by [2, 2.5], and hence its direct summand $\widetilde{\mathbb{Z}}(\mathbb{G}_m^{\wedge n}) := \widetilde{Cor}(-, \mathbb{G}_m^{\wedge n})$ is (see the proof of [27, 9.6] as well). $\widetilde{\mathbb{Z}}(\mathbb{G}_m^{\wedge n})$ is a Zariski sheaf by [4, 5.2.4].

Definition 2.3. Given a field k and $\ell \geqslant 0$, let $\mathscr{O}(\ell)_{k,v}$ denote the semilocal ring of the set $v = \{v_0 = (1, 0, \dots, 0), \dots, v_n = (0, \dots, 0, 1)\}$ of vertices of $\Delta_k^\ell = \mathrm{Spec}(k[t_0, \dots, t_\ell]/(t_0 + \cdots + t_\ell - 1))$ and set

$$\widehat{\Delta}_k^\ell := \mathrm{Spec}\mathscr{O}(\ell)_{k,v}.$$

Then $\ell \mapsto \widehat{\Delta}_k^\ell$ is a cosimplicial semilocal subscheme of $\Delta_k^\bullet$.

Proposition 2.4 (Suslin [25]). The following statements are true:

(1) Let $\mathscr{F} : Sm/k \to \mathrm{Ab}$ be a rationally contractible presheaf. Then the presheaf $C_n(\mathscr{F}) = \underline{\mathrm{Hom}}(\Delta_k^n, \mathscr{F})$ is also rationally contractible.

(2) Assume that the presheaf $\mathscr{F}$ is rationally contractible. Then the complex $\mathscr{F}(\widehat{\Delta}_k^\bullet)$ is contractible, and hence acyclic.

If $\mathscr{F}^\bullet$ is a cochain complex, then the canonical truncation $\tau_{<0}\mathscr{F}^\bullet$ of $\mathscr{F}^\bullet$ has the property that $H^i(\tau_{<0}\mathscr{F}^\bullet) = H^i(\mathscr{F}^\bullet)$ for $i < 0$ and $H^i(\tau_{<0}\mathscr{F}^\bullet) = 0$ if $i \geqslant 0$. If $\mathscr{F}^\bullet$ is a cochain complex of presheaves, we will write $\mathscr{H}^{-q}$ to denote the $-q^{th}$ cohomology presheaf of the complex $\mathscr{F}^\bullet$.

By a tower in a triangulated category $\mathscr{T}$, we mean a sequence of maps

$$\cdots \xrightarrow{f_{q+2}} X_{q+1} \xrightarrow{f_{q+1}} X_q \xrightarrow{f_q} \cdots \xrightarrow{f_1} X_0$$

in $\mathscr{T}$. The q-th layer of the tower is an object $B_q \in \mathscr{T}$ fitting in a cofiber sequence $X_{q+1} \xrightarrow{f_{q+1}} X_q \to B_q \to \Sigma X_{q+1}$ in $\mathscr{T}$.

The following result plays an important role in our analysis. It says that the zeroth cohomology of a non-positive cochain complex of rationally contractible presheaves evaluated at $\widehat{\Delta}^\bullet_k$ is recovered, up to homology, from negative cohomology evaluated at $\widehat{\Delta}^\bullet_k$.

Theorem 2.5. Suppose $\mathscr{F}^\bullet$

$$\cdots \xrightarrow{d^{-3}} \mathscr{F}^{-2} \xrightarrow{d^{-2}} \mathscr{F}^{-1} \xrightarrow{d^{-1}} \mathscr{F}^0 \to 0 \to \cdots \tag{1}$$

is a cochain complex of rationally contractible presheaves concentrated in non-positive degrees. Let $\mathscr{L}^{-n} := \operatorname{Ker} d^{-n}$, $n > 0$, and $\mathscr{L} := \operatorname{Coker} d^{-1}$. Then the chain complex of Abelian groups $\mathscr{L}(\widehat{\Delta}^\bullet_k)$ is zig-zag quasi-isomorphic to the chain complex $\mathscr{L}^{-1}(\widehat{\Delta}^\bullet_k)[-2]$. Moreover, there is a tower in the derived category $\mathscr{D}(\mathrm{Ab})$ of chain complexes of Abelian groups which are concentrated in non-positive degrees

$$\cdots \xrightarrow{\alpha^{-3}} \mathscr{L}^{-3}(\widehat{\Delta}^\bullet_k)[-2] \xrightarrow{\alpha^{-2}} \mathscr{L}^{-2}(\widehat{\Delta}^\bullet_k)[-1] \xrightarrow{\alpha^{-1}} \mathscr{L}^{-1}(\widehat{\Delta}^\bullet_k) \tag{2}$$

with q-th layer, $q \geqslant 0$, being the complex $\mathscr{H}^{-1-q}(\widehat{\Delta}^\bullet_k)[-q]$. In particular, the tower (2) gives rise to a strongly convergent spectral sequence

$$E^2_{pq} = H_p(\mathscr{H}^{-1-q}(\widehat{\Delta}^\bullet_k)) := H_p(\mathscr{H}^{-1-q}(\tau_{<0}\mathscr{F}^\bullet)(\widehat{\Delta}^\bullet_k)) \Rightarrow H_{p+q+2}(\mathscr{L}(\widehat{\Delta}^\bullet_k)).$$

Proof. Consider a short exact sequence of presheaves

$$0 \to \operatorname{Im} d^{-1} \to \mathscr{F}^0 \to \mathscr{L} \to 0.$$

It induces a short exact sequence of Abelian groups in each degree $n \geqslant 0$:

$$0 \to \varinjlim_{U \ni v_0,\dots,v_n} (\operatorname{Im} d^{-1})(U) \to \varinjlim_{U \ni v_0,\dots,v_n} (\mathscr{F}^0)(U) \to \varinjlim_{U \ni v_0,\dots,v_n} \mathscr{L}(U) \to 0,$$

where $v_0, \dots, v_n$ are the vertices of Δ^n_k and $U \subseteq \Delta^n_k$. We also use here the fact that the direct limit functor is exact. The latter is nothing but the short exact sequence

$$0 \to (\operatorname{Im} d^{-1})(\widehat{\Delta}^n_k) \to (\mathscr{F}^0)(\widehat{\Delta}^n_k) \to \mathscr{L}(\widehat{\Delta}^n_k) \to 0.$$

Since $\mathscr{F}^0$ is rationally contractible by assumption, it follows that the complex of Abelian groups $(\mathscr{F}^0)(\widehat{\Delta}^\bullet_k)$ is contractible by Proposition 2.4(2). Now the induced triangle in $\mathscr{D}(\mathrm{Ab})$

$$(\mathrm{Im}\ d^{-1})(\widehat{\Delta}^\bullet_k) \to (\mathscr{F}^0)(\widehat{\Delta}^\bullet_k) \to \mathscr{L}(\widehat{\Delta}^\bullet_k) \xrightarrow{\tau} (\mathrm{Im}\ d^{-1})(\widehat{\Delta}^\bullet_k)[-1]$$

yields a zig-zag quasi-isomorphism of chain complexes $\tau : \mathscr{L}(\widehat{\Delta}^\bullet_k) \xrightarrow{\sim} (\mathrm{Im} d^{-1})(\widehat{\Delta}^\bullet_k)[-1]$. For the same reasons, $(\mathrm{Im} d^{-1})(\widehat{\Delta}^\bullet_k)$ is zig-zag quasi-isomorphic to $\mathscr{L}^{-1}(\widehat{\Delta}^\bullet_k)[-1]$. For this, one uses the short exact sequence of presheaves $\mathscr{L}^{-1} \hookrightarrow \mathscr{F}^{-1} \twoheadrightarrow \mathrm{Im} d^{-1}$. So $\mathscr{L}(\widehat{\Delta}^\bullet_k)$ is zig-zag quasi-isomorphic to $\mathscr{L}^{-1}(\widehat{\Delta}^\bullet_k)[-2]$.

Similarly, each short exact sequence of presheaves

$$0 \to \mathscr{L}^{-n} \xrightarrow{i_n} \mathscr{F}^{-n} \xrightarrow{p_n} \mathrm{Im}\ d^{-n} \to 0$$

gives rise to a quasi-isomorphism of chain complexes $a_n : (\mathrm{Im} d^{-n})(\widehat{\Delta}^\bullet_k) \simeq \mathscr{L}^{-n}(\widehat{\Delta}^\bullet_k)[-1]$. It is induced by the map $b_n : \mathrm{Im} d^{-n} \to \mathscr{L}^{-n}[-1]$ in $\mathscr{D}(\mathrm{Ab})$ given by the zig-zag map of chain complexes

$$\begin{tikzcd}
 & \mathscr{L}^{-n} \arrow[r, "\mathrm{id}"] \arrow[d, "i_n"'] & \mathscr{L}^{-n} \\
\mathrm{Im}\ d^{-n} & \mathscr{F}^{-n} \arrow[l, "p_n"'] &
\end{tikzcd}$$

Here the chain complex in the middle is concentrated in degrees 1 and 0.

Next, each short exact sequence of presheaves

$$0 \to \mathrm{Im}\ d^{-n-1} \xrightarrow{j_n} \mathscr{L}^{-n} \to \mathscr{H}^{-n} \to 0$$

gives rise to a triangle in $\mathscr{D}(\mathrm{Ab})$

$$\mathscr{L}^{-n-1}(\widehat{\Delta}^\bullet_k)[-1] \to \mathscr{L}^{-n}(\widehat{\Delta}^\bullet_k) \to \mathscr{H}^{-n}(\widehat{\Delta}^\bullet_k) \to \mathscr{L}^{-n-1}(\widehat{\Delta}^\bullet_k).$$

In this way, we obtain the desired tower (2) with layers as stated. Up to shift the morphism α^{-n} equals the composite map $j_n(\widehat{\Delta}^\bullet_k) \circ a_n^{-1}$. Note that the nth complex of the tower $\mathscr{L}^{-n-1}(\widehat{\Delta}^\bullet_k)[-n]$ is $(n-1)$-connected, and hence the tower gives rise to a strongly convergent spectral sequence

$$E_2^{pq} = H_p(\mathscr{H}^{-1-q}(\widehat{\Delta}^\bullet_k)) \Rightarrow H_{p+q+2}(\mathscr{L}(\widehat{\Delta}^\bullet_k))$$

after applying [6, 6.1.1] to it. This completes the proof. ∎

Remark 2.6. Similarly to [6, 6.1], the spectral sequence of the preceding theorem occurs from an exact couple (with maps i, j, k of bidegrees $(1,-1),(0,0),(-2,1)$, respectively)—see [6, p. 798] for details.

Let $\mathscr{A}$ be a V-category of correspondences on Sm/k in the sense of [9] (V-categories are just a formal abstraction of basic properties for the category of finite correspondences Cor). We say that $\mathscr{A}$ is *nice* if for any smooth semilocal scheme W and any $\mathbb{A}^1$-invariant presheaf $\mathscr{F}$ with $\mathscr{A}$-correspondences the canonical morphism of presheaves $\mathscr{F} \to \mathscr{F}_{\mathrm{Zar}}$ induces an isomorphism of Abelian groups $\mathscr{F}(W) \xrightarrow{\cong} \mathscr{F}_{\mathrm{Zar}}(W)$. Here $\mathscr{F}_{\mathrm{Zar}}$ is the Zariski sheaf associated to the presheaf $\mathscr{F}$. For example, the category of finite correspondences Cor is nice by [29, 4.24]. If the base field k is infinite perfect of characteristic different from 2, then the category of finite MW-correspondences in the sense of [4] is nice by [2, 3.5].

Corollary 2.7. Under the conditions of Theorem 2.5 suppose that (1) is a cochain complex of Zariski sheaves with nice correspondences such that its presheaves $\mathscr{L}$ and $\mathscr{H}^{-q}$-s are $\mathbb{A}^1$-invariant. Then the chain complex of Abelian groups $\mathscr{L}_{\mathrm{Zar}}(\widehat{\Delta}_k^\bullet)$ is quasi-isomorphic to the chain complex of Abelian groups $\mathscr{L}^{-1}(\widehat{\Delta}_k^\bullet)[-2] = \mathscr{L}_{\mathrm{Zar}}^{-1}(\widehat{\Delta}_k^\bullet)[-2]$. Moreover, the q-th layer of the tower (2) equals the complex $\mathscr{H}_{\mathrm{Zar}}^{-1-q}(\widehat{\Delta}_k^\bullet)[-q]$. Here $\mathscr{H}_{\mathrm{Zar}}^{-q}$ stands for the $-q^{th}$ cohomology Zariski sheaf of the complex of Zariski sheaves (1). In particular, the tower (2) gives rise to a strongly convergent spectral sequence

$$E_{pq}^2 = H_p(\mathscr{H}_{\mathrm{Zar}}^{-1-q}(\widehat{\Delta}_k^\bullet)) := H_p(\mathscr{H}_{\mathrm{Zar}}^{-1-q}(\tau_{<0}\mathscr{F}^\bullet)(\widehat{\Delta}_k^\bullet)) \Rightarrow H_{p+q+2}(\mathscr{L}_{\mathrm{Zar}}(\widehat{\Delta}_k^\bullet)).$$

3 Semilocal Milnor K-Theory

Let $\mathbb{Z}(n)$ be Suslin–Voevodsky's motivic complex of Zariski sheaves of weight $n \geqslant 0$ on Sm/k (see [27, Definition 3.1]). By definition, it is concentrated in cohomological degrees $m \leqslant n$. More precisely, it equals the cochain complex with differential (of degree $+1$) equal to the alternating sum of face operations

$$\cdots \to Cor(\Delta_k^2 \times -, \mathbb{G}_m^{\wedge n}) \to Cor(\Delta_k^1 \times -, \mathbb{G}_m^{\wedge n}) \to Cor(-, \mathbb{G}_m^{\wedge n}) \to 0 \to \cdots \tag{3}$$

Here the Zariski sheaf $Cor(-, \mathbb{G}_m^{\wedge n})$ is in cohomological degree n. Basing on well-known theorems of Nesterenko–Suslin [23] and Totaro [28], define the *n-th Milnor K-theory sheaf* $\mathscr{K}_n^M$ as the Zariski sheaf $\mathscr{H}_{\mathrm{Zar}}^n(\mathbb{Z}(n))$.

Similarly, let $\widetilde{\mathbb{Z}}(n)$ be Calmès–Fasel's Milnor–Witt motivic complex of Zariski sheaves of weight $n \geqslant 0$ on Sm/k (see [4]). More precisely, it equals the cochain complex with differential (of degree $+1$) equal to the alternating sum of face operations

$$\cdots \to \widetilde{Cor}(\Delta^2_k \times -, \mathbb{G}_m^{\wedge n}) \to \widetilde{Cor}(\Delta^1_k \times -, \mathbb{G}_m^{\wedge n}) \to \widetilde{Cor}(-, \mathbb{G}_m^{\wedge n}) \to 0 \to \cdots \tag{4}$$

Denote by $\mathscr{K}_n^{MW}$ the Zariski sheaf $\mathscr{H}^n_{\mathrm{Zar}}(\widetilde{\mathbb{Z}}(n))$. We shall also refer to $\mathscr{K}_n^{MW}$ as the *n-th Milnor–Witt K-theory sheaf*.

Definition 3.1. Let k be any field and $n \geqslant 0$. The *n-th semilocal Milnor K-theory complex of the field k* is the chain complex of Abelian groups $\mathscr{K}_n^M(\widehat{\Delta}^\bullet_k)$.

The (n,q)*-th semilocal Milnor K-theory group* $\widehat{K}^M_{n,q}(k)$ *of k* is defined as the q-th homology group $H_q(\mathscr{K}_n^M(\widehat{\Delta}^\bullet_k))$ of the n-th semilocal Milnor K-theory complex of k. By definition, $\widehat{K}^M_{n,q}(k) = 0$ for all $q < 0$.

Let k be an infinite perfect field of characteristic not 2 and $n \geqslant 0$. The *n-th semilocal Milnor–Witt K-theory complex of the field k* is the chain complex of Abelian groups $\mathscr{K}_n^{MW}(\widehat{\Delta}^\bullet_k)$.

The (n,q)*-th semilocal Milnor–Witt K-theory group* $\widehat{K}^{MW}_{n,q}(k)$ *of k* is defined as the q-th homology group $H_q(\mathscr{K}_n^{MW}(\widehat{\Delta}^\bullet_k))$ of the n-th semilocal Milnor K-theory complex of k. By definition, $\widehat{K}^{MW}_{n,q}(k) = 0$ for all $q < 0$.

If A is an Abelian group, then the same definitions are given "with A-coefficients", in which case we just tensor the relevant complexes by A to get $\mathscr{K}_n^M(\widehat{\Delta}^\bullet_k) \otimes A$ and $\mathscr{K}_n^{MW}(\widehat{\Delta}^\bullet_k) \otimes A$ and then semilocal Milnor and Milnor–Witt K-theory groups with A-coefficients $\widehat{K}^M_{n,*}(k, A)$, $\widehat{K}^{MW}_{n,*}(k, A)$ are homology groups of these complexes. In what follows, we mostly deal with the case $A = \mathbb{Q}$, in which case we write the subscript $\mathbb{Q}$.

All statements that are proven below with integer coefficients will automatically be true with $\mathbb{Q}$-coefficients. The interested reader will always be able to repeat the relevant proofs rationally (we do not write them for brevity). Also, many statements are valid with any coefficients, say, when A is finite. Since motivic cohomology with finite coefficients is well studied, we do not discuss this case either, assuming that the interested reader will do this easily.

We also recall from [5, 16, 23] that the nth Milnor K-group $K_n^M(R)$ of a commutative ring R is the abelian group generated by symbols $\{a_1, \dots, a_n\}$, $a_i \in R^\times$, $i = 1, \dots, n$, subject to the following relations:

(1) for any i, $\{a_1,\dots,a_ia_i',\dots,a_n\}=\{a_1,\dots,a_i,\dots,a_n\}+\{a_1,\dots,a_i',\dots,a_n\}$;

(2) $\{a_1,\dots,a_n\}=0$ if there exist i,j, $i\neq j$, such that $a_i+a_j=0$ or 1.

Remark 3.2. If the field k is infinite, it follows from [5, 16] that $\mathscr{K}_n^M(\widehat{\Delta}_k^\ell)=K_n^M(\mathscr{O}(\ell)_{k,v})$. The n-th semilocal Milnor K-theory chain complex $\mathscr{K}_n^M(\widehat{\Delta}_k^\bullet)$ is therefore isomorphic to the chain complex $K_n^M(\mathscr{O}(\bullet)_{k,v})$. In particular, $\widehat{K}_{n,q}^M(k)=H_q(K_n^M(\mathscr{O}(\bullet)_{k,v}))$ for all $n,q\geqslant 0$. We see that $\mathscr{K}_n^M(\widehat{\Delta}_k^\ell)$ is defined naively in terms of generators and relations.

Lemma 3.3. Given any field k, the complex $\mathscr{K}_0^M(\widehat{\Delta}_k^\bullet)$ has only one non-zero homology group in degree zero, $\widehat{K}_{0,0}^M(k)$, which is isomorphic to $\mathbb{Z}$.

Proof. This follows from the fact that the complex $\mathbb{Z}(0)$ is canonically quasi-isomorphic to the constant sheaf $\mathbb{Z}$, positioned in degree 0 (see [27, 3.2]). ■

Recall from [15, 2.3.1] that a presheaf with transfers $\mathscr{F}$ is *birationally invariant* if $\mathscr{F}(X)\xrightarrow{\simeq}\mathscr{F}(U)$ for any dense open immersion $j:U\to X$. Birationally invariant homotopy invariant presheaves with transfers are called *birational sheaves*.

Lemma 3.4. Given a birational sheaf $\mathscr{F}$ and an irreducible $X\in Sm/k$, the natural morphism of complexes of Abelian groups

$$\mathscr{F}(k(X))\to\mathscr{F}(\widehat{\Delta}_{k(X)}^\bullet)$$

is a quasi-isomorphism.

Proof. By [14, 4.1.3], $\mathscr{F}$ is a birational motivic sheaf. By [14, p. 513], $\mathscr{F}(k(X))\to\mathscr{F}(\widehat{\Delta}_{k(X)}^\bullet)$ is a quasi-isomorphism. ■

We are now in a position to prove the main result of the paper.

Theorem 3.5. Suppose k is any field. The following statements are true:

(1) For any $n\geqslant 1$, $\widehat{K}_{n,0}^M(k)=\widehat{K}_{n,1}^M(k)=0$.

(2) For any $n\geqslant 1$, there is a strongly convergent spectral sequence

$$E_{pq}^2=H_p(\mathscr{H}_{\mathrm{Zar}}^{n-1-q}(\tau_{<n}\mathbb{Z}(n))(\widehat{\Delta}_k^\bullet))\Longrightarrow\widehat{K}_{n,p+q+2}^M(k),$$

where $\tau_{<n}\mathbb{Z}(n)$ is the truncation complex of $\mathbb{Z}(n)$ for degrees smaller than n.

(3) If $n = 2$ and $p \leqslant 1$, there is an isomorphism of Abelian groups $H^p_{\mathrm{Zar}}(k, \mathbb{Z}(2)) = \widehat{K}^M_{2,3-p}(k)$.

(4) If the field k is infinite perfect of characteristic different from 2 and $n \geqslant 1$, then the natural morphism of chain complexes of Abelian groups $\mathscr{K}^{MW}_n(\widehat{\Delta}^\bullet_k) \to \mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)$ is a quasi-isomorphism. In particular, it induces isomorphisms of Abelian groups $\widehat{K}^{MW}_{n,q}(k) \xrightarrow{\cong} \widehat{K}^M_{n,q}(k)$ for all $q \in \mathbb{Z}$.

Proof. (1)-(2). By Example 2.2 the Zariski sheaf with transfers $Cor(-, \mathbb{G}_m^{\wedge n})$, $n \geqslant 1$, is rationally contractible. It follows from Proposition 2.4 that the Zariski sheaf with transfers $Cor(\Delta^\ell_k \times -, \mathbb{G}_m^{\wedge n})$ is rationally contractible for every $\ell \geqslant 0$. Now the desired spectral sequence of the second assertion follows from Corollary 2.7 if we apply it to the cochain complex of rationally contractible Zariski sheaves with transfers (3). Another argument proving (2) is that there is an exact triangle

$$0 \to Tot(\tau_{<n}\mathbb{Z}(n)(\widehat{\Delta}^\bullet_k)) \to Tot(\mathbb{Z}(n)(\widehat{\Delta}^\bullet_k)) \to K^M_n(\widehat{\Delta}^\bullet_k)[-n] \to 0$$

using the fact that the complex in the middle is acyclic.

Next, by Theorem 2.5 and Corollary 2.7, there is a quasi-isomorphism $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k) \simeq \mathscr{L}^{-1}_{\mathrm{Zar}}(\widehat{\Delta}^\bullet_k)[-2]$ of chain complexes for some presheaf with transfers $\mathscr{L}^{-1}$ (the shift is homological). Assertion (1) now follows.

(3). Suppose k is perfect. Since the motivic complex of weight one $\mathbb{Z}(1)$ is acyclic in non-positive cohomological degrees by [27, 3.2], the proof of [29, 4.34] and Voevodsky's cancellation theorem [33] imply that

$$\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))) \cong (\mathscr{H}^p(\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}(2))))_{\mathrm{Zar}} = 0$$

for all $p \leqslant 1$. It follows from [15, 2.5.2] that each $\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))$, $p \leqslant 1$, is a birational (Nisnevich) sheaf.

Lemma 3.4 implies that the natural map of chain complexes

$$\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(k) \to \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_k), \quad p \leqslant 1,$$

is a quasi-isomorphism. Thus, $H_i(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_k)) = 0$ for all $i \neq 0$ and $H_0(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_k)) = \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(k)$. Using Corollary 2.7 applied to the cochain complex of rationally contractible Zariski sheaves with transfers (3), the qth layer $\mathscr{H}^{-1-q}_{\mathrm{Zar}}(\widehat{\Delta}^\bullet_k)[-q]$ of the tower (2) is a complex having only one homology group in degree q. Using induction in q

and applying homology functor to each triangle $\mathscr{L}^{-q-2}_{\mathrm{Zar}}(\widehat{\Delta}^\bullet_k)[-q-1] \to \mathscr{L}^{-q-1}_{\mathrm{Zar}}(\widehat{\Delta}^\bullet_k)[-q] \to \mathscr{H}^{-1-q}_{\mathrm{Zar}}(\widehat{\Delta}^\bullet_k)[-q] \xrightarrow{+}$ coming from the tower (2), we get an isomorphism of Abelian groups $H^p_{\mathrm{Zar}}(k, \mathbb{Z}(2)) = \widehat{K}^M_{2,3-p}(k)$ for any $p \leqslant 1$.

Next, suppose K/k is a finitely generated field extension of the perfect field k. Then $K = k(U)$ for some $U \in Sm/k$. Each scheme $\widehat{\Delta}^\ell_K$ is the semilocalization of $\Delta^\ell_k \times U$ at the points $(v_0, \eta), \dots, (v_n, \eta)$, where η is the generic point of U. Lemma 3.4 implies that the natural map of chain complexes

$$\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(K) \to \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_K), \quad p \leqslant 1,$$

is a quasi-isomorphism. Thus, $H_i(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_K)) = 0$ for all $i \neq 0$ and $H_0(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(\widehat{\Delta}^\bullet_K)) = \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(2))(K)$. As above, we get isomorphisms of Abelian groups $H^p_{\mathrm{Zar}}(K, \mathbb{Z}(2)) = \widehat{K}^M_{2,3-p}(K)$, $p \leqslant 1$, for any finitely generated field extension K/k.

Finally, for any field K of characteristic p we follow the same argument as in [25, p. 244]. We use the fact that it can be written as a directed limit $K = \varinjlim_i K_i$ of fields finitely generated over $\mathbb{Z}/p$, and the fact that the above homology/cohomology groups commute with directed limits (we use [20, Lemma 3.9]). We also use here the fact that the cohomology groups $H^*_{\mathrm{Zar}}(K, \mathbb{Z}(n)) = \varinjlim H^*_{\mathrm{Zar}}(K_i, \mathbb{Z}(n))$ are defined intrinsically in terms of the field K and are independent of the choice of the base field.

(4). The proof is based on Bachmann's results on the MW-motivic cohomology [1]. By Example 2.2, the Zariski sheaf $\widetilde{Cor}(-, \mathbb{G}_m^{\wedge n})$ is rationally contractible for every $n > 0$.

If we consider the cochain complex of Zariski sheaves (4) and repeat the arguments for the proof of the second assertion, we shall get a strongly convergent spectral sequence

$$\widetilde{E}^2_{pq} := H_p(\mathscr{H}^{n-1-q}_{\mathrm{Zar}}(\widetilde{\mathbb{Z}}(n))(\widehat{\Delta}^\bullet_k)) \Longrightarrow \widehat{K}^{MW}_{n,p+q+2}(k).$$

The natural functor of additive categories of correspondences $\widetilde{Cor} \to Cor$ induces a map of spectral sequences $\widetilde{E}^2_{pq} \to E^2_{pq}$, where E^2_{pq} is the spectral sequence of the second assertion.

It follows from [1, Theorem 17] that the morphism of complexes $\tau_{<n}\widetilde{\mathbb{Z}}(n) \to \tau_{<n}\mathbb{Z}(n)$ is a quasi-isomorphism, locally in the Nisnevich topology. This means that each morphism of Nisnevich sheaves $\mathscr{H}^p_{\mathrm{Nis}}(\widetilde{\mathbb{Z}}(n)_{\mathrm{Nis}}) \to \mathscr{H}^p_{\mathrm{Nis}}(\mathbb{Z}(n))$, $p \neq n$, is an isomorphism. The relation between notations of [1] and this paper is as follows: $\pi_i(\widetilde{H}\mathbb{Z})_j = \mathscr{H}^{j-i}_{\mathrm{Nis}}(\widetilde{\mathbb{Z}}(j)_{\mathrm{Nis}})$, $\pi_i(H_\mu\mathbb{Z})_j = \mathscr{H}^{j-i}_{\mathrm{Nis}}(\mathbb{Z}(j)_{\mathrm{Nis}})$.

It follows from [2, 3.5] that

$$\mathscr{H}^p_{\mathrm{Nis}}(\widetilde{\mathbb{Z}}(n)_{\mathrm{Nis}})(\widehat{\Delta}^\ell_k) = \mathscr{H}^p_{\mathrm{Zar}}(\widetilde{\mathbb{Z}}(n))(\widehat{\Delta}^\ell_k), \quad \ell \geqslant 0.$$

Using [29, 5.5], one has

$$\mathscr{H}^p_{\mathrm{Nis}}(\mathbb{Z}(n))(\widehat{\Delta}^\ell_k) = \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(\widehat{\Delta}^\ell_k), \quad \ell \geqslant 0.$$

Therefore, the morphism of strongly convergent spectral sequences $\widetilde{E}^2_{pq} \to E^2_{pq}$ is an isomorphism. This isomorphism implies that the map of chain complexes $\mathscr{K}^{MW}_n(\widehat{\Delta}^\bullet_k) \to \mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)$ is a quasi-isomorphism, as was to be proved. ∎

Corollary 3.6. Given any field k, semilocal Milnor K-theory complex $\mathscr{K}^M_1(\widehat{\Delta}^\bullet_k)$ is acyclic. In particular, $\widehat{K}^M_{1,q}(k) = 0$ for all $q \in \mathbb{Z}$.

Proof. By [27, 3.2], the complex $\tau_{<1}\mathbb{Z}(1)$ is acyclic. Therefore, the E^2-page of the strongly convergent spectral sequence of Theorem 3.5(2) for $n = 1$ is trivial. Our statement now follows. ∎

Corollary 3.7. $\widehat{K}^M_{n,2}(k) = \mathrm{Coker}(H^{n-1}_{\mathrm{Zar}}(\widehat{\Delta}^1_k, \mathbb{Z}(n)) \xrightarrow{\partial_1 - \partial_0} H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}(n)))$ for any field k and $n > 1$.

If k is a field of positive characteristic p, then Milnor K-theory groups of k are p-torsionfree by a theorem of Izhboldin [13]. The following statement says that semilocal Milnor K-theory groups are p-uniquely divisible.

Corollary 3.8. Given any field k of positive characteristic $p > 0$, semilocal Milnor K-theory groups $\widehat{K}^M_{n,m}(k)$ are p-uniquely divisible for all $n > 0$ and $m \in \mathbb{Z}$. In particular, $\widehat{K}^M_{n,m}(k) = \widehat{K}^M_{n,m}(k) \otimes \mathbb{Z}[1/p]$.

Proof. We claim that $\mathscr{H}^s_{\mathrm{Zar}}(\mathbb{Z}(n))$ is a sheaf of $\mathbb{Z}[1/p]$-modules for all $n > 0$ and $s < n$. The Geisser–Levine theorem [11, 1.1] implies that $\mathscr{H}^s_{\mathrm{Zar}}(\mathbb{Z}(n))(K)$ is p-uniquely divisible for any field extension K/k. It follows that the morphism of sheaves $\mathscr{H}^s_{\mathrm{Zar}}(\mathbb{Z}(n)) \to \mathscr{H}^s_{\mathrm{Zar}}(\mathbb{Z}(n)) \otimes \mathbb{Z}[1/p]$ is an isomorphism on field extensions K/k, and hence it is an isomorphism of sheaves by [29, 4.20].

We see that the E^2-term of the strongly convergent spectral sequence of Theorem 3.5(2) consists of p-uniquely divisible Abelian groups, and hence the semilocal Milnor K-theory groups of the statement are p-uniquely divisible. ∎

Remark 3.9. The preceding theorem implies that the evaluation of the Milnor–Witt sheaf $\mathscr{K}_n^{MW}$, $n \geqslant 1$, at $\widehat{\Delta}_k^\bullet$ "deletes" the information about quadratic forms.

By Remark 3.2 if the base field k is infinite, Milnor K-theory of semilocal schemes like $\widehat{\Delta}_k^\ell$ has an explicit, naive description, whereas motivic cohomology involves sophisticated constructions. Thus Theorem 3.5 computes some motivic cohomology groups as homology groups of certain naive complexes. In particular, we can apply "symbolic" computations to cycles in motivic complexes. It is worth pointing out that the term "symbolic" here refers to the symbols in the definition of Milnor K-theory.

Furthermore, a theorem of Kerz [16, 1.2] implies that the norm residue homomorphism induces an isomorphism of complexes

$$\mathscr{K}_n^M(\widehat{\Delta}_k^\bullet) \otimes \mathbb{Z}/\ell \xrightarrow{\cong} H_{et}^n(\widehat{\Delta}_k^\bullet, \mu_\ell^{\otimes n}), \quad n > 0,$$

if the field k is infinite of characteristic not dividing ℓ. Since the second and the third statement of Theorem 3.5 are true with finite coefficients, it follows that semilocal Milnor K-theory groups with finite coefficients can be computed as homology groups of complexes $H_{et}^n(\widehat{\Delta}_k^\bullet, \mu_\ell^{\otimes n})$.

Recall from [24] that the indecomposable K_3-group of a field k, denoted by $K_3^{\mathrm{ind}}(k)$, is defined as the cokernel of the canonical homomorphism $K_3^M(k) \to K_3(k)$.

Corollary 3.10. For any field k, there is an isomorphism $K_3^{\mathrm{ind}}(k) = \widehat{K}_{2,2}^M(k)$. In particular, $K_3(k)_{\mathbb{Q}}^{(2)} = \widehat{K}_{2,2}^M(k)_{\mathbb{Q}}$.

Proof. The motivic spectral sequence gives an isomorphism $K_3^{\mathrm{ind}}(k) = H_{\mathrm{Zar}}^1(k, \mathbb{Z}(2))$. Now Theorem 3.5 implies the claim. ■

Corollary 3.11. For any perfect field k, any connected $X \in Sm/k$ and any $p \leqslant 1$, there is an isomorphism

$$H_{\mathrm{Zar}}^p(X, \mathbb{Z}(2)) = \widehat{K}_{2,3-p}^M(k(X)),$$

where $k(X)$ is the function field of X.

Proof. This follows from Lemma 3.4, the proof of Theorem 3.5(3) and the fact that for any $p \leqslant 1$ there is an isomorphism of groups $H_{\mathrm{Zar}}^p(X, \mathbb{Z}(2)) = H_{\mathrm{Zar}}^p(k(X), \mathbb{Z}(2))$. ■

Since $H^p_{\mathrm{Zar}}(k, \mathbb{Z}(q))$ is uniquely divisible for $p \leqslant 0$ and any field k (see, e.g., [34, Exercise VI.4.6]), Theorem 3.5(3) implies the following

Corollary 3.12. For any field k and any $n \geqslant 3$, the group $\widehat{K}^M_{2,n}(k)$ is uniquely divisible.

Corollary 3.13. For any field k, there are isomorphisms of rational vector spaces

$$K_{4+p}(k)^{(2)}_{\mathbb{Q}} = \widehat{K}^M_{2,3+p}(k), \quad p \geqslant 0.$$

Moreover, $K_3(k)_{\mathbb{Q}} = K^M_3(k)_{\mathbb{Q}} \oplus \widehat{K}^M_{2,2}(k)_{\mathbb{Q}}$.

Proof. This follows from Theorem 3.5, Corollary 3.12, and the fact that for any $p \geqslant -1$ there is an isomorphism $H^{-p}_{\mathrm{Zar}}(k, \mathbb{Q}(2)) = K_{4+p}(k)^{(2)}_{\mathbb{Q}}$. ■

Theorem 3.14. Semilocal Milnor K-theory is invariant under purely transcendental extensions. Namely, $\widehat{K}^M_{n,q}(k) = \widehat{K}^M_{n,q}(k(x))$ for any field k and $n, q \geqslant 0$.

Proof. Suppose the base field k is perfect. Then the Zariski sheaf $\mathscr{K}^M_n$ on Sm/k is strictly homotopy invariant. Its Eilenberg–Mac Lane motivic S^1-spectrum $EM(\mathscr{K}^M_n)$ is $\mathbb{A}^1$-local by [22, 6.2.2], hence $H^n_{\mathrm{Nis}}(X, \mathscr{K}^M_n) = SH_{S^1}(k)(X_+, EM(\mathscr{K}^M_n)[n])$, where $SH_{S^1}(k)$ is the homotopy category of motivic S^1-spectra. In particular, the homology groups of the complex $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_{k(X)/k})$ are isomorphic to the homotopy groups of the spectrum $EM(\mathscr{K}^M_n)(\widehat{\Delta}^\bullet_{k(X)/k})$ for any irreducible $X \in Sm/k$. By [14, 2.2.6], the latter spectrum is stably weakly equivalent to $s_0(EM(\mathscr{K}^M_n))(X)$, where $s_0(EM(\mathscr{K}^M_n))$ is the zeroth slice of $EM(\mathscr{K}^M_n)$ in the motivic Postnikov tower (see [14, §1.2]). Applying $s_0(EM(\mathscr{K}^M_n))$ to the morphism $\mathbb{A}^1 \to pt$, one gets a stable weak equivalence of spectra $s_0(EM(\mathscr{K}^M_n))(pt) \to s_0(EM(\mathscr{K}^M_n))(\mathbb{A}^1)$. We see that $\widehat{K}^M_{n,q}(k) = \widehat{K}^M_{n,q}(k(x))$ for any $n, q \geqslant 0$.

Suppose now k is any field of characteristic $p > 0$. Denote by k_∞, the perfect closure of k. Following Suslin [26], the theory of motivic cohomology and associated categories of motives with p^{-1}-coefficients over k is essentially the same with the theory over k_∞. We shall write $\mathscr{K}^M_{n,\infty}$ for the nth Milnor K-theory sheaf on Sm/k_∞. By [26, 1.1 and 1.3] the canonical functor $\varphi^\sharp : Shv_{\mathrm{Nis}}(k) \to Shv_{\mathrm{Nis}}(k_\infty)$ between the categories of Nisnevich sheaves takes $\mathscr{K}^M_n$ to $\mathscr{K}^M_{n,\infty}$ (they are Nisnevich sheaves as well). It follows from [26, 2.13] that the natural morphism of chain complexes $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k)[p^{-1}] \to \mathscr{K}^M_{n,\infty}(\widehat{\Delta}^\bullet_{k_\infty})[p^{-1}]$ is an isomorphism. It remains to apply Corollary 3.8 and invariance under purely transcendental extensions for perfect fields proven above. ■

We say that two smooth k-varieties X and Y are *stably* $\mathbb{A}^1$*-equivalent* if their suspension motivic S^1-spectra $\Sigma^\infty_{S^1} X_+$ and $\Sigma^\infty_{S^1} Y_+$ are isomorphic in the homotopy category of motivic S^1-spectra $SH_{S^1}(k)$. If $Y = pt$ we call X *stably* $\mathbb{A}^1$*-contractible*. For example, any motivic equivalence between X and Y in the category of motivic spaces induces an isomorphism of $\Sigma^\infty_{S^1} X_+$ and $\Sigma^\infty_{S^1} Y_+$.

The following result says that semilocal Milnor K-theory is an invariant for stably $\mathbb{A}^1$-equivalent varieties.

Corollary 3.15. Suppose k is a perfect field. If $X, Y \in Sm/k$ are irreducible and stably $\mathbb{A}^1$-equivalent, then $\widehat{K}^M_{n,q}(k(X))$ is isomorphic to $\widehat{K}^M_{n,q}(k(Y))$ for any $n, q \geqslant 0$. In particular, if X is irreducible and stably $\mathbb{A}^1$-contractible, then $\widehat{K}^M_{n,q}(k(X))$ is isomorphic to $\widehat{K}^M_{n,q}(k)$ for any $n, q \geqslant 0$.

Proof. We use the proof of Theorem 3.14 showing that the $\widehat{K}^M_{n,q}(k(X))$ (respectively, $\widehat{K}^M_{n,q}(k(Y))$) are isomorphic to homotopy groups $\pi_*(s_0(EM(\mathscr{K}^M_n))(X)) = SH_{S^1}(k)(\Sigma^\infty_{S^1} X_+[*], s_0(EM(\mathscr{K}^M_n)))$ (respectively, $\pi_*(s_0(EM(\mathscr{K}^M_n))(X)) = SH_{S^1}(k)(\Sigma^\infty_{S^1} Y_+[*], s_0(EM(\mathscr{K}^M_n))))$. ■

4 Some Criteria for the Beilinson–Soulé Vanishing Conjecture

In this section, an application of the technique developed in the previous sections is given. Recall that the Beilinson–Soulé vanishing conjecture states that each complex $\mathbb{Z}(n)$, $n > 0$, on Sm/k is acyclic outside the interval of cohomological degrees $[1, n]$. It follows from [31, p. 352] that it suffices to verify acyclicity of the complex $\mathbb{Z}(n)$ on Sm/k outside the interval $[1, n]$ whenever k is perfect. Therefore, the base field k is assumed to be perfect throughout this section.

The main result of this section, Theorem 4.1, gives equivalent conditions for the Beilinson–Soulé vanishing conjecture. In particular, it says that instead of verifying acyclicity of the sophisticated complexes $\mathbb{Z}(n)$ in non-positive cohomological degrees, it is enough to verify acyclicity of the chain complexes of Abelian groups $\mathscr{Z}^0(n)(\widehat{\Delta}^\bullet_{K/k})$, where $\mathscr{Z}^0(n) := \operatorname{Ker}\partial^0$ with $\partial^0 : Cor(\Delta^n_k \times -, \mathbb{G}_m^{\wedge n}) \to Cor(\Delta^{n-1}_k \times -, \mathbb{G}_m^{\wedge n})$ being the zeroth differential of the complex $\mathbb{Z}(n)$, and K/k is a finitely generated field extension.

Theorem 4.1. The following conditions are equivalent:

(1) the Beilinson–Soulé vanishing conjecture is true for complexes $\mathbb{Z}(n)$, $n \geqslant 1$, on Sm/k;

(2) for every $n \geqslant 1$ and every finitely generated field extension K/k, $\mathscr{Z}^0(n)(\widehat{\Delta}^\bullet_{K/k})$ is an acyclic chain complex;

(3) for every $n \geqslant 1$, $\mathscr{Z}^0(n)$ has a resolution in the category of Zariski sheaves

$$\dots \xrightarrow{d^{-2}} \mathscr{R}^{-1} \xrightarrow{d^{-1}} \mathscr{R}^0 \to \mathscr{Z}^0(n)$$

such that each $\mathscr{R}^i$ is rationally contractible and cohomology presheaves $\mathscr{H}^{i<0}$ are trivial on the semilocal schemes of the form $\widehat{\Delta}^{\ell}_{K/k}$, where K/k is a finitely generated field extension;

(4) for every $n \geqslant 1$ and every finitely generated field extension K/k, the total complex of the bicomplex $\tau_{\leqslant 0}(\mathbb{Z}(n))(\widehat{\Delta}^{\bullet}_{K/k})$ is acyclic, where $\tau_{\leqslant 0}$ is the truncation in corresponding cohomological degrees;

(5) for every $n \geqslant 1$ and every finitely generated field extension K/k, the total complex of the bicomplex $\tau_{[1,n]}(\mathbb{Z}(n))(\widehat{\Delta}^{\bullet}_{K/k})$ is acyclic.

Proof. (1) $\Rightarrow$ (2). Given $\ell \geqslant 0$ and $n > 0$, set

$$\mathscr{W}^{-1-\ell}(n) := \mathrm{Ker}(\partial^{-\ell} : Cor(\Delta^{n+\ell}_k \times -, \mathbb{G}_m^{\wedge n}) \to Cor(\Delta^{n+\ell-1}_k \times -, \mathbb{G}_m^{\wedge n})).$$

The proof of Theorem 2.5 and Corollary 2.7 shows that there is a tower in the derived category $\mathscr{D}(\mathrm{Ab})$ of chain complexes of Abelian groups

$$\dots \xrightarrow{\alpha^{-3}} \mathscr{W}^{-3}(n)(\widehat{\Delta}^{\bullet}_{K/k})[-2] \xrightarrow{\alpha^{-2}} \mathscr{W}^{-2}(n)(\widehat{\Delta}^{\bullet}_{K/k})[-1] \xrightarrow{\alpha^{-1}} \mathscr{W}^{-1}(n)(\widehat{\Delta}^{\bullet}_{K/k}) \tag{5}$$

with q-th layer, $q \geqslant 0$, being the complex $\mathscr{H}^{-q}_{\mathrm{Zar}}(\widehat{\Delta}^{\bullet}_{K/k})[-q]$. Here $\mathscr{H}^{-q}_{\mathrm{Zar}}$ stands for the $-q^{\mathrm{th}}$ cohomology sheaf of the complex $\mathbb{Z}(n)$. By definition, $\mathscr{W}^{-1}(n) = \mathscr{Z}^0(n)$. We use here the fact that $Cor(\Delta^{n+\ell}_k \times -, \mathbb{G}_m^{\wedge n})$ is a rationally contractible sheaf by Example 2.2 and Proposition 2.4. Similarly to Theorem 3.5, the tower (5) yields a strongly convergent spectral sequence

$$E^2_{pq} = H_{p+q}(\mathscr{H}^{-q}_{\mathrm{Zar}}(\widehat{\Delta}^{\bullet}_{K/k})) \Rightarrow H_{p+q}(\mathscr{Z}^0(n)(\widehat{\Delta}^{\bullet}_{K/k})) \tag{6}$$

By assumption, $\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n)) = 0$ for $p \leqslant 0$. Therefore, the strongly convergent spectral sequence (6) is trivial, and hence $\mathscr{Z}^0(n)(\widehat{\Delta}^{\bullet}_{K/k})$ is acyclic.

(2) $\Rightarrow$ (1). We use induction in n. By [27, 3.2] $\mathbb{Z}(1)$ is acyclic in non-positive degrees, hence the base case $n = 1$. Suppose $\mathbb{Z}(n-1)$ is acyclic outside the interval $[1, n-1]$. We want to show that $\mathbb{Z}(n)$ is acyclic outside the interval $[1, n]$. Voevodsky's

cancellation theorem [33] together with [29, 4.34] implies that

$$\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))) = \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathscr{H}^p_{\mathrm{Nis}}(\mathbb{Z}(n))) = \mathscr{H}^{p-1}_{\mathrm{Nis}}(\mathbb{Z}(n-1)) = 0$$

for all $p \leqslant 0$. We use here the fact that $\mathscr{F}_{\mathrm{Zar}} = \mathscr{F}_{\mathrm{Nis}}$ for any homotopy invariant presheaf with transfers (see [29, 5.5]). It follows from [15, 2.5.2] that each $\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))$, $p \leqslant 0$, is a birational (Nisnevich) sheaf.

Let K/k be a finitely generated field extension. Then $K = k(X)$ for some $X \in Sm/k$. Lemma 3.4 implies that the natural map of chain complexes

$$\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(K) \to \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(\widehat{\Delta}^\bullet_K), \quad p \leqslant 0,$$

is a quasi-isomorphism. Thus, $H_i(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(\widehat{\Delta}^\bullet_K)) = 0$ for all $i \neq 0$ and $H_0(\mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(\widehat{\Delta}^\bullet_K)) = \mathscr{H}^p_{\mathrm{Zar}}(\mathbb{Z}(n))(K)$. Therefore, the strongly convergent spectral sequence (6) collapses, and hence

$$0 = H_i(\mathscr{Z}^0(n)(\widehat{\Delta}^\bullet_{K/k})) = \mathscr{H}^{-i}_{\mathrm{Zar}}(\mathbb{Z}(n))(K), \quad i \geqslant 0.$$

Each sheaf $\mathscr{H}^{-i}_{\mathrm{Zar}}(\mathbb{Z}(n))$ is homotopy invariant by [29] and trivial on finitely generated field extensions. It follows from [29, 4.20] that $\mathscr{H}^{-i}_{\mathrm{Zar}}(\mathbb{Z}(n)) = 0$, hence $\mathbb{Z}(n)$ is acyclic outside the interval $[1, n]$.

(1) $\Rightarrow$ (3). This is straightforward: set $\mathscr{R}^\ell := Cor(\Delta_k^{n+\ell+1} \times -, \mathbb{G}_m^{\wedge n})$ with differentials being those of $\mathbb{Z}(n)$. We use here the facts that $Cor(\Delta_k^{n+\ell} \times -, \mathbb{G}_m^{\wedge n})$ is rationally contractible (see Example 2.2 and Proposition 2.4) and that for any smooth semilocal scheme W, any $\mathbb{A}^1$-invariant presheaf $\mathscr{F}$ with transfers the canonical morphism $\mathscr{F}(W) \xrightarrow{\cong} \mathscr{F}_{\mathrm{Zar}}(W)$ is an isomorphism [29, 4.24].

(3) $\Rightarrow$ (2). This follows from the spectral sequence of Theorem 2.5.

(1) $\Rightarrow$ (4). This is obvious.

(4) $\Leftrightarrow$ (5). It is enough to observe that the total complex of the bicomplex $(\mathbb{Z}(n))(\widehat{\Delta}^\bullet_{K/k})$ is acyclic for $n > 0$. The latter easily follows from [25, 2.2; 2.4] (see [2, 2.3] as well).

(4) $\Rightarrow$ (2). The complex $\tau_{\leqslant 0}(\mathbb{Z}(n))$ equals

$$\cdots \to Cor(\Delta_k^{n+2} \times -, \mathbb{G}_m^{\wedge n}) \to Cor(\Delta_k^{n+1} \times -, \mathbb{G}_m^{\wedge n}) \to \mathscr{Z}^0(n) \to 0 \to \cdots$$

Example 2.2(1), Proposition 2.4 and [26, 4.7] imply that the complex $Cor(\Delta_k^{n+\ell} \times -, \mathbb{G}_m^{\wedge n})(\widehat{\Delta}^\bullet_{K/k})$ is acyclic for all $n > 0$. The spectral sequence for a double complex implies that the homology groups of the complex $\tau_{\leqslant 0}(\mathbb{Z}(n))(\widehat{\Delta}^{\bullet}_{K/k})$ are those of the complex $\mathscr{Z}^0(n)(\widehat{\Delta}^{\bullet}_{K/k})$, and hence the remaining implication follows. ■

5 K_4 of a Field

In this section, another application of semilocal Milnor K-theory is given. We show that the group $K_4(k)$ is completely determined by extensions involving the classical Milnor K-theory and semilocal Milnor K-theory. If k is algebraically closed, then $K_4(k)$ is a direct sum of relevant Milnor K-theory and semilocal Milnor K-theory groups of k.

Recall that the motivic spectral sequence relates algebraic K-theory to motivic cohomology [6]

$$E^2_{p,q} = H^{q-p}_{\mathrm{Zar}}(k, \mathbb{Z}(q)) \Rightarrow K_{p+q}(k). \tag{7}$$

It is a strongly convergent spectral sequence concentrated in the first quadrant. It is obtained from a tower of connected S^1-spectra

$$\cdots \to \mathbb{K}^3 \to \mathbb{K}^2 \to \mathbb{K}^1 \to \mathbb{K}^0 := K(k), \tag{8}$$

where $K(k)$ is Quillen's K-theory spectrum of k. Rationally, the motivic spectral sequence collapses at $E^2 = E^\infty$ and

$$K_n(k)_{\mathbb{Q}} = \bigoplus_q H^{2q-n}_{\mathrm{Zar}}(k, \mathbb{Q}(q)) \tag{9}$$

(see [7] for details).

Proposition 5.1. Let k be a perfect field (respectively, any field with $\mathrm{char}(k) = p > 0$) and let $\mathscr{F}$ be a homotopy invariant Nisnevich sheaf with transfers of Abelian groups (respectively, $\mathbb{Z}[1/p]$-modules). Let $\widehat{\mathbb{A}}^1_k$ be the semilocalization of the affine line at $0, 1$. Then

$$\mathscr{F}(\widehat{\mathbb{A}}^1_k) = \mathscr{F}(k) \bigoplus \left(\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} \mathscr{F}_{-1}(k(x)) \right),$$

where each x in the direct sum is a closed point of $\mathbb{A}^1_k$.

Proof. Suppose k is perfect and U is an open subset of $\mathbb{A}^1_k$ with $Z = \mathbb{A}^1_k \setminus U = \{x_1, \dots, x_n\}$. The Gysin triangle for motives [30] gives a triangle in $DM^{eff}(k)$

$$\bigoplus_1^n M(\mathbb{G}^{\wedge 1}_{m,k(x_i)}) \to M(U) \to M(pt) \xrightarrow{+}$$

If $x_0 \in U$ is a rational point, then this triangle splits. If we apply $DM^{eff}(k)(-, \mathscr{F})$ to this triangle, one gets a canonical isomorphism $\mathscr{F}(U) = \mathscr{F}(k) \oplus (\oplus_{i=1}^n \mathscr{F}_{-1}(k(x_i)))$. It follows that

$$\mathscr{F}(\widehat{\mathbb{A}}^1_k) = \operatorname{colim}_{U \ni \{0,1\}} \mathscr{F}(U) = \mathscr{F}(k) \bigoplus \left(\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} \mathscr{F}_{-1}(k(x)) \right),$$

as required. Here the splitting onto the first summand is given by $x_0 := 0 \in \mathbb{A}^1_k$.

The statement for fields of positive characteristic is the same if we use Suslin's results [26] saying that Voevodsky's theory for motivic complexes works for non-perfect fields as well provided that we deal with sheaves with transfers of $\mathbb{Z}[1/p]$-modules. ■

The following result says that the motivic cohomology groups $H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}(n))$ fit into a finite tower of homomorphisms of groups with subsequent quotients being semilocal Milnor K-theory groups.

Theorem 5.2. There are exact sequences of Abelian groups

$$\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} H^{n-2}_{\mathrm{Zar}}(k(x), \mathbb{Z}(n-1)) \xrightarrow{u} H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}(n)) \to \widehat{K}^M_{n,2}(k) \to 0$$

and

$$\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} \widehat{K}^M_{2,2}(k(x)) \xrightarrow{u} H^2_{\mathrm{Zar}}(k, \mathbb{Z}(3)) \to \widehat{K}^M_{3,2}(k) \to 0.$$

Here $n > 1$ and each x of the left direct sums is a closed point. The homomorphism u is the restriction of

$$\partial_1 - \partial_0 : H^{n-1}_{\mathrm{Zar}}(\widehat{\mathbb{A}}^1_k, \mathbb{Z}(n)) \to H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}(n))$$

to $\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} H^{n-2}_{\mathrm{Zar}}(k(x), \mathbb{Z}(n-1)) \subset H^{n-1}_{\mathrm{Zar}}(\widehat{\mathbb{A}}^1_k, \mathbb{Z}(n))$.

Proof. The first exact sequence follows from Proposition 5.1 and Corollary 3.7. The second exact sequence is a particular case of the first one if we apply Theorem 3.5(3). ■

Corollary 5.3. Let $n > 1$ and $\mathscr{X} = \{\widehat{K}^M_{\ell,2}(k(x)) \mid x \text{ is a closed point in } \mathbb{A}^1_k \text{ and } 2 \leqslant \ell \leqslant n\}$. Then $H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}(n))$ belongs to the smallest localizing Serre subcategory of Ab containing $\mathscr{X}$.

Proof. This is a consequence of Theorem 5.2 and [8, Proposition 2]. ■

The motivic spectral sequence (7) gives a long exact sequence of Abelian groups

$$H^{-1}_{\mathrm{Zar}}(k, \mathbb{Z}(2)) \to \pi_1(\mathbb{K}^3) \to K_4(k) \to H^0_{\mathrm{Zar}}(k, \mathbb{Z}(2)) \xrightarrow{d} K^M_3(k) \to K_3(k) \to H^1_{\mathrm{Zar}}(k, \mathbb{Z}(2)) \to 0.$$

Here $\mathbb{K}^3$ is the fourth entry of the tower (8). It follows from [34, VI.4.3.2] that $d = 0$. By Corollary 3.11 the latter long exact sequence can be rewritten as

$$\widehat{K}^M_{2,4}(k) \to \pi_1(\mathbb{K}^3) \to K_4(k) \to \widehat{K}^M_{2,3}(k) \xrightarrow{0} K^M_3(k) \to K_3(k) \to \widehat{K}^M_{2,2}(k) \to 0.$$

Next, by using the motivic spectral sequence, we find that $\pi_1(\mathbb{K}^3)$ fits into an exact sequence

$$K^M_4(k) \to \pi_1(\mathbb{K}^3) \to H^2_{\mathrm{Zar}}(k, \mathbb{Z}(3)) \to 0.$$

By Theorem 5.2, $H^2_{\mathrm{Zar}}(k, \mathbb{Z}(3))$ fits into an exact sequence

$$\bigoplus_{x \in \mathbb{A}^1_k \setminus \{0,1\}} \widehat{K}^M_{2,2}(k(x)) \xrightarrow{u} H^2_{\mathrm{Zar}}(k, \mathbb{Z}(3)) \to \widehat{K}^M_{3,2}(k) \to 0,$$

where each x in the direct sum is a closed point of $\mathbb{A}^1_k$.

We see that $\pi_1(\mathbb{K}^3)$ is expressed in terms of Milnor K-theory and semilocal Milnor K-theory groups, and hence so is $K_4(k)$.

We are now in a position to prove the main result of the section.

Theorem 5.4. Let k be any field. The following statements are true:

(1) $K_4(k)$ is entirely expressed in terms of Milnor K-theory and semilocal Milnor K-theory groups. Precisely, $K_4(k)$ fits into an exact sequence

$$\widehat{K}^M_{2,4}(k) \to A \to K_4(k) \to \widehat{K}^M_{2,3}(k) \to 0,$$

where A is an Abelian group fitted into an exact sequence

$$K^M_4(k) \to A \to B \to 0$$

with B fitted into an exact sequence

$$\bigoplus_{x\in\mathbb{A}^1_k\setminus\{0,1\}} \widehat{K}^M_{2,2}(k(x)) \to B \to \widehat{K}^M_{3,2}(k) \to 0.$$

(2) There is an isomorphism of Abelian groups

$$K_4(k)_{\mathbb{Q}} \cong K^M_4(k)_{\mathbb{Q}} \oplus \widehat{K}^M_{3,2}(k)_{\mathbb{Q}} \oplus \widehat{K}^M_{2,3}(k) \oplus F,$$

where F is a direct summand of $\bigoplus_{x\in\mathbb{A}^1_k\setminus\{0,1\}} \widehat{K}^M_{2,2}(k(x))_{\mathbb{Q}}$.

(3) If k is algebraically closed, then there is an isomorphism of Abelian groups

$$K_4(k) \cong K^M_4(k) \oplus \widehat{K}^M_{3,2}(k)_{\mathbb{Q}} \oplus \widehat{K}^M_{2,3}(k) \oplus F,$$

where F is a direct summand of $\bigoplus_{k^\times\setminus\{1\}} \widehat{K}^M_{2,2}(k)_{\mathbb{Q}}$.

The isomorphisms from (2) and (3) are not canonical.

Proof. The first statement follows from the arguments above the theorem. The second statement is a consequence of the first statement and isomorphism (9). It also uses a rational splitting of the exact sequence for $H^2_{\mathrm{Zar}}(k,\mathbb{Z}(3))$ from Theorem 5.2. We also use here the fact that the group $\widehat{K}^M_{2,3}(k)$ is uniquely divisible by Corollary 3.12. Finally, the third statement follows from the second one and the fact that $K^M_4(k)$ and $K_4(k)$ are uniquely divisible Abelian groups if k is algebraically closed (see, e.g., [34, pp. 267, 511, 514]). ■

It is worth mentioning that since $H^{-2}_{\mathrm{Zar}}(k,\mathbb{Z}(1)) = 0$ by [27, 3.2], the Beilinson–Soulé vanishing conjecture for K_4 requires only $\widehat{K}^M_{2,3}(k) = 0$.

6 On Conjectures of Beilinson and Parshin

Let k be a field of characteristic $p > 0$. A conjecture of Beilinson [3, 2.4.2.2] says that Milnor K-theory and Quillen K-theory agree rationally:

$$K_n^M(k)_{\mathbb{Q}} \xrightarrow{\cong} K_n(k)_{\mathbb{Q}}.$$

The purpose of this section is to show that the Beilinson conjecture is equivalent to vanishing of the rational semilocal Milnor K-theory. Since semilocal Milnor K-theory is defined in elementary terms, its vanishing with rational coefficients should be much easier for verification than the original Beilinson conjecture. We shall also show in this section that vanishing of the rational semilocal Milnor K-theory is a necessary condition for Parshin's conjecture.

Theorem 6.1. The Beilinson conjecture for rational algebraic K-theory of fields of positive characteristic is true if and only if rational semilocal Milnor K-theory groups $\widehat{K}^M_{n,m}(k)_{\mathbb{Q}}$ of such fields vanish for all $n > 0, m \geqslant 0$.

Proof. Assume the Beilinson conjecture. Then the isomorphism (9) implies $H^i_{\mathrm{Zar}}(k, \mathbb{Q}(n)) = 0$ for $i \neq n$ and all fields of prime characteristic. It follows that Zariski cohomology sheaves except the nth cohomology are zero (we use here [29, 4.20]). Now the spectral sequence of Theorem 3.5 and Corollary 3.6 imply $\widehat{K}^M_{n,m}(k)_{\mathbb{Q}}$ vanish for all $n > 0, m \geqslant 0$.

Conversely, suppose that $\widehat{K}^M_{n,m}(k)_{\mathbb{Q}}$ vanish for all $n > 0, m \geqslant 0$ and all fields of prime characteristic. We claim that each complex $\mathbb{Q}(n)$, $n \geqslant 1$, has only one non-zero cohomology sheaf in degree n. We use induction in n. By [27, 3.2], $\mathbb{Q}(1)$ is acyclic in non-positive degrees, hence the base case $n = 1$.

Assume that the complex $\mathbb{Q}(n)$, $n \geqslant 1$, has only one non-zero cohomology sheaf in degree n. Repeating the proof of Theorem 3.5(3) word for word (see the proof of (2) $\Rightarrow$ (1) in Theorem 4.1 as well), we obtain that

$$H^m_{\mathrm{Zar}}(k, \mathbb{Q}(n+1)) = \widehat{K}^M_{n+1,n-m+2}(k)_{\mathbb{Q}} = 0, \quad m \leqslant n.$$

Then Zariski cohomology sheaves except the $(n+1)$th cohomology of $\mathbb{Q}(n+1)$ are zero (we use here [29, 4.20]), and our claim follows.

The isomorphism (9) now implies that the natural homomorphism $K_n^M(k)_{\mathbb{Q}} \to K_n(k)_{\mathbb{Q}}$ is an isomorphism, as was to be shown. ■

Remark 6.2. (1) We should stress that in the proof of Theorem 6.1 we work with all fields of prime characteristic (not with individual ones) in order to annihilate the relevant Zariski cohomology sheaves.

(2) By Corollary 3.8, $\widehat{K}^M_{n,m}(k) = \widehat{K}^M_{n,m}(k) \otimes \mathbb{Z}[1/p]$ for all $n > 0, m \geqslant 0$. It follows that $\widehat{K}^M_{n,m}(k)_{\mathbb{Q}} = \widehat{K}^M_{n,m}(k) \otimes \mathbb{Z}_{(p)}$. Therefore, $\widehat{K}^M_{n,m}(k)_{\mathbb{Q}} = 0$ if and only if $\widehat{K}^M_{n,m}(k) \otimes \mathbb{Z}_{(p)} = 0$.

Recall that Parshin's conjecture [3, 2.4.2.3] states that for any smooth projective variety X defined over a finite field, the higher algebraic K-groups vanish rationally:

$$K_i(X)_{\mathbb{Q}} = 0, \quad i > 0.$$

We finish the section by the following

Theorem 6.3. Let k be a field of characteristic $p > 0$ and assume Parshin's conjecture. Then rational semilocal Milnor K-theory groups $\widehat{K}_{n,m}(k)_{\mathbb{Q}}$ vanish for all $n > 0, m \geqslant 0$.

Proof. It follows from [10, p. 203] that $H^i_{\mathrm{Zar}}(k, \mathbb{Q}(n)) = 0$ for $i \neq n$. The proof of Theorem 6.1 shows that rational semilocal Milnor K-theory groups $\widehat{K}_{n,m}(k)_{\mathbb{Q}}$ vanish for all $n > 0, m \geqslant 0$, as required. ■

7 Concluding Remarks

In contrast with motivic cohomology with mod 2 coefficients, we show in this section that semilocal Milnor K-theory groups with $\mathbb{Z}/2$-coefficients $\widehat{K}^M_{n,*}(k, \mathbb{Z}/2)$ are zero for any $n > 1$ (see Definition 3.1). This is another property of semilocal Milnor K-theory distinguishing it with the classical Milnor K-theory/motivic cohomology of fields. This also distinguishes semilocal Milnor K-theory with relative Milnor K-theory in the sense of Levine [19]. More precisely, we have the following:

Theorem 7.1. For any infinite perfect field k and any $n > 1$ the n-th semilocal Milnor K-theory complex with $\mathbb{Z}/2$-coefficients $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k) \otimes \mathbb{Z}/2$ is acyclic or, equivalently, the semilocal Milnor K-theory groups with $\mathbb{Z}/2$-coefficients $\widehat{K}^M_{n,*}(k, \mathbb{Z}/2)$ are zero.

Proof. We separate two cases in the proof: when $\mathrm{char}(k) = 2$ and $\mathrm{char}(k) \neq 2$. If $\mathrm{char}(k) = 2$, then Geisser–Levine's theorem [11] implies that the sheaf $\mathscr{K}^M_n/2$ is quasi-isomorphic to a shift of $\mathbb{Z}/2(n)$. It follows from Example 2.2 and Proposition 2.4 that the total complex of the bicomplex $\mathbb{Z}/2(n)(\widehat{\Delta}^\bullet_k)$ is acyclic, hence so is the complex $\mathscr{K}^M_n(\widehat{\Delta}^\bullet_k) \otimes \mathbb{Z}/2$.

Suppose now that $\mathrm{char}(k) \neq 2$. Set,

$$I^{n+1} := \ker(\mathscr{K}_n^{MW} \to \mathscr{K}_n^{M}), \quad n \geqslant 0,$$

where $\mathscr{K}_n^{MW}, \mathscr{K}_n^{M}$ are Nisnevich sheaves of Milnor–Witt and Milnor K-theory, respectively. By [16, 7.10] and [12, 6.3] there is an isomorphism of sheaves

$$I^n/I^{n+1} = \mathscr{K}_n^{M}/2. \tag{10}$$

Consider a short exact sequence of sheaves

$$0 \to I^{n+1} \to \mathscr{K}_n^{MW} \to \mathscr{K}_n^{M} \to 0, \quad n \geqslant 1.$$

It follows from Theorem 3.5(4) and [2, 3.6] that the complex $I^{n+1}(\widehat{\Delta}_k^\bullet)$ is acyclic for all $n \geqslant 1$. The isomorphism (10) and [2, 3.6] imply that the complex $\mathscr{K}_n^{M}(\widehat{\Delta}_k^\bullet) \otimes \mathbb{Z}/2$ is acyclic if $n > 1$. ∎

In characteristic p, it follows from the Geisser–Levine theorem [11] that the logarithmic de Rham–Witt sheaf $W_r\Omega^n_{\log}$ is quasi-isomorphic to a shift of $\mathbb{Z}/p^r(n)$. Using this quasi-isomorphism, the proof of the preceding theorem shows that the complex $W_r\Omega^n_{\log}(\widehat{\Delta}_k^\bullet)$ is acyclic. The converse is also true. More precisely, using the technique developed in this paper for semilocal Milnor K-theory, one can show that if the complex $W_1\Omega^n_{\log}(\widehat{\Delta}_k^\bullet)$ is acyclic for any $n > 0$ then the complex $\mathbb{Z}/p(n)$ has only one cohomology sheaf isomorphic to $W_1\Omega^n_{\log}$. The latter implies (using induction in r) that the only non-trivial cohomology sheaf of $\mathbb{Z}/p^r(n)$ is $W_r\Omega^n_{\log}$.

The above arguments justify to raise the following

Conjecture. Each logarithmic de Rham–Witt sheaf $W_r\Omega^n_{\log}$ is rationally contractible.

This conjecture will shed new light not only on the fundamental theorem of Geisser–Levine [11] but also on further properties of the logarithmic de Rham–Witt sheaf $W_r\Omega^n_{\log}$ which is of fundamental importance. In particular, if the conjecture were solved in the affirmative then the complex $W_r\Omega^n_{\log}(\widehat{\Delta}_k^\bullet)$ would be contractible by Proposition 2.4(2). A good starting point for the conjecture would be rational contractibility of $W_1\Omega^n_{\log}$.

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