Symmetry Breaking By Massive Fermions Under Sugimoto S-Duality

MSc Thesis by Henry Harper-Gardner
SID: [redacted]

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Abstract

In this project we investigate the behaviour of strongly coupled gauge theories with massive fermion fields under the S-duality proposed by Sugimoto [1]. In the massless case described by Sugimoto we see that the highly non-trivial dynamics of the fermion fields corresponds directly to the behaviour of a schematic scalar potential in the S-dual theory, which shows very clear global and gauge symmetry breaking as expected in the initial theory. We induce mass terms for the scalar and fermion fields by coupling the underlying string theory to a supergravity background and show that this preserves a subgroup of the global symmetry and lifts the vacuum degeneracy of the scalar potential in the S-dual side. Furthermore, we derive relations between the flux induced fermion masses to the masses induced in the Nambu-Goldstone modes of the S-dual scalar potential in an analogous way to the GMOR relation known from QCD. In this way we demonstrate a formal relation which is consistent with present QCD theory which derives from Sugimoto's duality, as well as providing a proof of concept for an insightful analytical tool for non-SUSY gauge theories.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed.....[redacted]...

Date........................31/12/2022...

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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I hereby give consent for my thesis, if accepted, to be available for electronic sharing

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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This thesis is dedicated to the memory of David Flynn.

Prologue

This thesis is written to present and discuss the body of work produced during the twelve month Master of Science project undertaken by the author, under the supervision of Professor Adi Armoni, at the Particle Physics and Cosmology Theory group of Swansea University. The core content and results of this project have been published by the author and Professor Armoni in a paper titled 'GMOR relation for a QCD-like theory from S-duality' [2], the objective of this thesis has then been to provide a more fleshed out and developed discussion of these findings. As per the title, the main characters of this story are symmetry breaking and strong-weak ('S-') duality, and while both of these concepts alone are not difficult to outline, the scope of their implications for the physical and mathematical contexts in which they arise in this work is extensive, and will require much discussion and exposition to appreciate. Our entry into the project in earnest will come as a further development of what is an already conceptually rich environment, involving quantum field theory, string theory, strongly-coupled gauge theory, confinement, supersymmetry and its breaking, and more.

These ideas will be introduced, to the extent that they are required, as we gradually set the stage to present the original work of this project. However, with the intention of keeping this thesis as self-contained as possible, this prologue will aim to introduce and recap the core ideas of the standard model, quantum chromodynamics, strongly-coupled theories, and symmetry breaking, as these are the perennial figures within modern physics to which our results are relevant. Extensions of these ideas such as supersymmetry and confinement will be discussed in the introduction proper, where they will aid to contextualise the background and motivation to this project.

The standard model of particle physics (often abbreviated to SM) is, to date, the most complete and vindicating success of modern theoretical physics. It is the achievement which ultimately established the position of quantum field theory as the central lens through which we can most accurately examine nature at its smallest accessible scales. The starting premise of the SM is that we can identify four 'fundamental' forces by which the observable universe evolves, these are electro-magnetism, the strong and weak nuclear forces, and gravity. The standard model is a theory which describes the dynamics of the electro-magnetic and nuclear forces, and their interactions with matter. At energy scales up to one tera-electron volt (1 TeV), currently accessible to collider experiments, predictions made by the SM have been incredibly accurate, culminating in the prediction of a previously unobserved theorised particle, the Higgs' boson, which was finally discovered in 2012.

In practice the standard model is described by a quantum field theory. That is to say; we describe the matter and interactions of the SM in terms of a system of fields, which are objects that exist continuously throughout space-time, the fields are then subject to the traditional rules and conditions of quantum mechanics, whereby their dynamical states are associated with vectors in a Hilbert space, whose moduli-squared have the interpretation of probability density. At the classical level, field theories are usually described by a Lagrangian function (there will be several examples in the main chapters), wherein the fields are expressed as continuously differentiable functions of space-time, however when we move to the quantum regime the fields are elevated to linear operators on the Hilbert space. The Lagrangian is then either transmuted into a Hamiltonian, from which we derive a unitary time evolution operator which is then applied to scattering-matrix methods; or it is used directly to generate a partition function, from which we extract correlation functions and expectation values to probe the quantum level physics.

As is the case in any physical theory, there are constraints in building the standard model, chiefest among which is symmetry. By symmetry we mean a transformation that we may impose on the system which will leave its physical behaviours and observables unchanged. A physical symmetry of the system must be reflected in the Lagrangian by a mathematical symmetry of its component terms with respect to the appropriate group. As the standard model is intended to describe short-range, short-lived phenomena which we can re-create with current accelerator technology, we treat the space-time environment of the theory as flat, meaning that we can disregard gravitational effects and assume maximal global symmetry for our theory. Physically this means our phenomena are seen as symmetric with respect to space and time translations, rotations, and Lorentz transformations (or boosts). Mathematically, symmetry is described in the language of group theory, and the appropriate group in this case is the Poincaré group, denoted $G_p := (\mathbb{R}^{3,1} \rtimes O(3,1))$, and this consequentially determines much of the mathematical structure of the theory.

Another layer of structure of the SM is how we couple the fields. By coupling, we are referring to terms in the Lagrangian which contain a products of fields. The reason this is constrained is because terms which include factors of different fields come to represent interactions between those fields, and at the quantum level this is responsible for describing particle interaction. A priori, we could include any term in the Lagrangian which respects the symmetries of the system, however, from observation we know that certain particles do and do not interact, and so to respect the phenomenology we are describing, certain interaction terms must be included in, and some must be omitted from, the Lagrangian. A familiar example is the neutrino, so called as it carries no electric charge (it is neutral, hence neutrino), as such it does not experience the electro-magnetic force, meaning it does not interact with photons. Neutrinos were first postulated to rectify an apparent anomaly in the existing description of neutron β-decay, therefore we know that it does interact with the weak nuclear force. The consequence of this is that the standard model includes a term which mixes the neutrino field with the gauge field describing the weak force, while it does not contain any term including both the electro-magnetic field and the neutrino field.

As we'll soon come to appreciate, the way that the fields are coupled in a theory has extensive consequences to the physics and the analysis of the system. Coupling is characterised by what are called 'coupling constants', which arise in Lagrangian field theory for two main reasons; the first is that multiplying constants are included in each term of the Lagrangian to introduce a sense of relative positivity or negativity between the terms. While an overall factor of -1 is irrelevant to the Lagrangian, as we are concerned ultimately with its differential behaviour, a relative factor of -1 between terms in the Lagrangian will often have very significant implications for the stationary points. The second reason is the requirement that the action of a dynamical evolution be mapped to a dimensionless, real number. The action of a field is the integral over space-time of the Lagrangian-density function. In the integral, the Lagrangian-density is naturally multiplied by the volume element, an inherently dimensionful quantity, therefore we know that each term of the Lagrangrian-density must contain appropriate inverse powers of the same dimension in order that the overall integral is dimensionless. In QFT we typically discuss the 'mass-dimension' of quantities within the Lagrangian, which gives us useful insight into how various processes scale with energy. From relativity we know that mass and energy are equivalent, as are space and time, and we know that at the quantum level there exists an inverse relationship between momentum-energy and space-time quantities. Hence, the mass-dimension of the volume element is $[d^d x] = -d$, where d is the dimension of the space-time our theory lives in, and therefore we know that the mass dimension of each term in the Lagrangian-density must be $[\mathcal{L}] = d$. With this information we can deduce the mass-dimension of our various fields, however, as we include terms in ascending powers of the fields the mass-dimension will increase, we will have to multiply the terms by a dimensionful constant in order to maintain $[\mathcal{L}] = d$. The coupling constant fulfills this role for us.

Much of the analyses undertaken in QFT, such as deriving scattering cross-sections, decay rates and expectation values, are performed using the techniques of perturbation theory, wherein the Lagrangian is split into a solvable free-field part, and an interacting part. The expectation values of the free-field theory are then corrected by a truncated power-series of the interacting part. This methodology is extremely convenient and surprisingly applicable to real world physics, but it has a strict criterion of applicability, in which the coupling controlling the interactions of a theory plays a critical role. Perturbation theory only works when a theory is weakly coupled. What this means in effect is that the interaction term power series, which contains ascending powers of the coupling constant, must be convergent, so that the interaction term constitutes a 'small' perturbation to the system. Naively, one might think that this can easily be achieved if the modulus coupling constant is significantly less than one, however, due to mass dimensionality and quantum effects it is often much less simple than this to be assured that the theory is weakly coupled. In the case of a dimensionless coupling, which we refer to as marginal, $|g_c| << 1$ (where g_c is the coupling constant) is sufficient to ensure weak coupling. However, when $[g_c] = 1$, which is called a relevant coupling, the interaction term is suppressed by a factor of the energy scale E and is larger at low energies and diminishes at high energies. Conversely, when $[g_c] = -n$, which is called an irrelevant coupling, the interaction is multiplied by a factor E^n, and so smaller at low energies but grows to become large at high energies. When couplings enter regions where the perturbation series becomes divergent, we say the theory is strongly coupled, and must proceed with different, often much more difficult means of analysis.

How does this information about couplings play into the standard model? Recall that we model three fundamental forces in the SM, the electromagnetic, the weak, and the strong nuclear force. Both the electromagnetic and the weak nuclear forces are weakly coupled, and perturbation theory is sufficient to tell us most of what we would like to know. However, the strong nuclear force (from here-on we'll refer to it simply as the strong force) is much less simple. The strong force is asymptotically free, an example of relevant coupling, and is strongly coupled at low energies, and becomes gradually weaker as the energy scale of a given process increases. This is convenient for analysing hard-processes, those which involve a large exchange of energy-momentum, however it makes it very difficult to study the lower energy phenomena, for example the ground-state. And we find in observation that this is where many of the more mysterious effects of the strong force appear, one of which is a central focus of this thesis; confinement. We will go into much further detail later on, but the essential characteristic of confinement in the strong force is that the massive, fundamental particles that interact with the force, the quarks, are never observed as free particles, but rather only in bound states as Hadrons, some familiar examples being protons and neutrons.

To make this all a bit more tangible, lets have a look at what we're talking about here. The theory which lives in the standard model that is responsible for describing the physics of the strong force is referred to as quantum chromodynamics (QCD). QCD is described by the Lagrangian

$$\mathcal{L}_{QCD} = \mathrm{Tr}[\overline{Q}_a(i\gamma^\mu\partial_\mu\delta^{ab} - g\gamma^\mu A_\mu\delta^{ab} - m^{ab})Q_b - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}] \quad (1)$$

where Q_a are the spinor fields representing the quarks, with flavour indices $a, b = 1, ..., N_f$, $G^{\mu\nu}$ is the gauge covariant field strength tensor, A_μ is the gauge field which takes values in the Lie algebra $su(3)$, and m^{ab} is the matrix whose entries define the mass of each quark field, and g is the QCD coupling constant, which controls the coupling of the quarks to the gauge field.

There is one final aspect of the standard model which is determined a priori, which is distinct from, but related to the symmetry and coupling. In fact we mentioned it obliquely towards the end of the last paragraph; gauge symmetry. When we wish to incorporate interactions between particles in QFT, for example between electrons, we do so by introducing a field to the theory that couples to the matter fields, which we interpret as mediating the appropriate force, (for electrons this is the electromagnetic force). These force mediating fields are typified by the name gauge fields, and the force carrying, integer spin particles that arise from their quantum treatment are referred to as gauge bosons, both so called because the fields in the Lagrangian exhibit the property of gauge symmetry.

Gauge symmetry is a property of a Lagrangian function, which is a symmetry in a more subtle sense than that of the global symmetry described previously. As we know, Lagrangian functions are used to map the evolution of physical systems to a real number, which we interpret as the action of that dynamical path, and then go on to use this information to deduce the dynamics of the system. However, in achieving this we have to account for the fact that many physical objects which we encounter and wish to investigate cannot be appropriately represented as simple real number valued functions, they may be tensor valued and complex, and still we must map them to real numbers to evaluate the action. A frequent consequence of this, particularly in the case of gauge fields, is that their resulting Euler-Lagrange equations of motion are not invertible. this means that there will exist various functions which can be input to them as representing the physical state of the system which will evolve identically, and which at a later point in time cannot be inverted and distinguished from each other. While this sounds principally the same as a true symmetry of the system, it is not. A true symmetry will act on the system by transforming it from one physical state to another, which will evolve in an identical way. To be explicit, a true symmetry represents a physical change in the conditions of the system, which will not affect how the state then evolves. A gauge symmetry arises when the non-invertibility of the equations of motion allows us a freedom of choice in representing a physical state, which will then regardless evolve to an identical physical state, because nothing has physically changed about the initial conditions. To put it another way, gauge symmetry allows us to re-name physical states. From this point of view, it is clear that re-naming the objects in a physical system can have no effect on its dynamics.

This physical equivalence of mathematically different terms is the essence of gauge symmetry. In the case of the electromagnetic theory we have a single, continuous mathematical degree of freedom, which we describe with a one dimensional Lie group, $U(1)$. This gauge theory forms part of the standard model, along with an $SU(2)$ and $SU(3)$ gauge theories, hence the full gauge group of the standard model is $G = SU(3) \times SU(2) \times U(1)$. However, the full $SU(3) \times SU(2) \times U(1)$ gauge symmetry of the SM does not survive dynamical evolution, due to the presence of the Higgs' field, which develops a non-zero vacuum expectation value that breaks the gauge symmetry from $SU(3) \times SU(2) \times U(1)$ to $SU(3) \times U(1)$, and this imparts masses to the W and Z bosons. This mass is responsible for the short range of the weak nuclear force. The Higgs' mechanism is an example of dynamical symmetry breaking, a phenomenon which is ubiquitous in physics but is especially central in systems consisting of large numbers of interacting elements or 'sites', such is the case with condensed matter theories, field theories, and by extension high-energy particle physics. The essence of dynamical, or sometimes called 'spontaneous', symmetry breaking is when a symmetry of the Lagrangian is not a symmetry of the ground-state, and this is often the case when an observable quantity takes a non-zero expectation value in the ground-state. This concept is easier to appreciate with the aid of an example, and it is known that QCD spontaneously breaks its global symmetry, and so we shall explore this.

An important point of fact about spontaneous symmetry breaking in QCD is that, while we have very reliable empirical indications that it does occur, the precise mechanism which causes it is not theoretically well understood, as of the time of writing. The reason for this comes back to the relevant couping of QCD, because it is strongly coupled at low energies, it is presently impossible to demonstrate the symmetry breaking explicitly. At present achieving this is an active project in theoretical physics, which this project aims to provide some resource to. Given that caveat, what are our indications that QCD spontaneously breaks its global symmetry? The primary piece of evidence comes from the spectrum of light mesons. If we look at the QCD Lagrangian in the limit that the masses of the light quarks (the up, down, and strange) vanish, known as the chiral limit, we see that the Lagrangian enjoys a $SU(3) \times SU(3) \times U(1)$ global symmetry, and if this were a symmetry of the groundstate, one would expect that the hadron spectrum of the theory would be organised according to the irreducible representations of this group. However, in real-world QCD we do not see this. One reason for this is that introducing mass to the quarks will break the global symmetry of the theory, however, the light mesons do exhibit a 'near-symmetry', that is, if one disregards mass differences which are small with respect to the total mass of the mesons, an $SU(3)$

symmetry emerges.

Why does this imply spontaneous symmetry breaking? The critical fact is that the light mesons are approximately organised into a representation of the $SU(3)$ group, and not $SU(3) \times SU(3)$, and this is rationalised by a very powerful result in particle theory, the Nambu-Goldstone theorem, the upshot of which is that symmetry breaking can tell us a lot about the low-energy physics of a system. The theorem runs as follows; if a system dynamically breaks a symmetry, represented by a Lie group G, to a Lie subgroup H, the implication is that the action of the full symmetry group contains the subgroup which preserves the ground-state, and a portion which generates transformations between ground-states. If we then consider a field in the groundstate, and subsequently introduce variation throughout space-time by continuously acting with the broken symmetry group, meaning that the variation corresponds to smooth transitions between a space of groundstates, then the only term in the Lagrangian which would raise the energy above the minimum is the derivative term, which we could minimise by making the configuration vary as a waveform with an infinitely long wavelength. When we are discussing quantum field theory, the quantisation of such configurations gives rise to particles which we call Nambu-Goldstone bosons. Goldstone's theorem states that the Nambu-Goldstone bosons associates with the symmetry breaking $G \to H$ are generated by the action of the coset G/H. By counting the generators of each Lie group, we can find the number of different Nambu-Goldstone bosons in the system $\#NG = \mathrm{dim}(G) - \mathrm{dim}(H)$. At low-energies Nambu-Goldstone bosons often dominate the dynamics of a system as they less suppressed in the partition function than typical excited states.

Using this knowledge, if we suppose that QCD in the chiral limit spontaneously breaks its global symmetry from $SU(3) \times SU(3) \times U(1) \to SU(3) \times U(1)$, which existing results independent of this investigation suggest that this is a symmetry of the chiral QCD groundstate, then this would explain why the light-mesons exhibit a $SU(3)$ near-symmetry, as this is the broken symmetry group which generates the spectrum of Nambu-Goldstone bosons. Indeed, this is not merely wishful thinking. We know empirically that mesons interactions dominate a lot of the low-energy processes in QCD, such as the exchanges which bind nucleons. Furthermore there exists a low-energy effective field theory of QCD, referred to as the Chiral Lagrangian, which models the light mesons as Nambu-Goldstone bosons of the $SU(3) \times SU(3) \times U(1) \to SU(3) \times U(1)$ exactly as we have discussed, and this theory has proven to be very accurate at modelling low-energy QCD phenomena.

Chapter 0. Introduction

The central focus of this project is to examine a conjectured electric-magnetic duality of a non-supersymmetric, confining gauge theory, proposed by Sugimoto in his paper of 2012 [1]. We aim to build upon and extend his results and in doing so we will explore the rich physics of symmetry breaking, we will see string theory as a fertile environment which allows us to construct interesting field theories and provides the structures which facilitate their analysis, and we shall encounter the mathematical language of representation theory as both an elusive obstacle and as a powerful analytic tool. Our procedure will be to analyse how the explicit symmetry breaking of massive fermions manifests under Sugimoto's duality, and we shall show that this leads to a class of relations between quark masses and light meson masses of the form of the GMOR relation, a result known from the chiral Lagrangian of QCD.

In his paper on confinement and dynamical symmetry breaking, Sugimoto prefaces his exposition of the S-duality concerned with a discussion of a mechanism of confinement known as the dual Meissner effect. We shall follow suit with our own terse recap of this topic here, as it provides a useful, if somewhat schematic, picture of the technical work that follows. It will also help frame the work and conclusions of this project amongst the background of existing literature on confining gauge theory and S-duality.

When we talk about confinement, we are referring to a phenomenon of certain strongly-coupled gauge theories in which particles, specifically those which are states belonging to non-trivial representations of the gauge group, are observed as existing within bound states, and never as individual, elementary particles. The present consensus of understanding of confinement is that the gauge flux associated with each non-singlet gauge particle forms a thin 'flux tube' of finite tension. The binding energy of the particles' bound state is proportional to the length of the flux tube, which is associated with the separation of the bound particles. Due to the strong coupling of confined gauge theories, perturbative techniques are not sufficient for useful analysis. As such, demonstrating the formation of gauge flux tubes is a highly non-trivial procedure. However, in cases where we have access to an electric-magnetic duality, there is a body of work which proposes a picture of the confinement mechanism which contains an exploit that allows us to circumvent the labour of analysing the strongly coupled phenomena directly. This picture is the dual Meissner mechanism.

The dual Meissner mechanism is a scenario of confinement based analagously upon the the electric-magnetic duality of the Meissner effect, observed in type II superconductors. In this class of superconductors, the $U(1)$ gauge symmetry of the electromagnetic theory is broken by the condensation of Cooper pairs (bound states of electrons), which causes the formation of magnetic flux tubes. The idea of the dual Meissner mechanism is then based upon the formation of gauge flux tubes on one side of an electric-magnetic duality when monopoles condense in the other side. Suppose we have a theory which is the magnetic dual to QCD, and in which the magnetic gauge symmetry was Higgsed by monopole condensation, the dual Meissner mechanism would cause the formation of colour flux tubes in the electric side, which would result in confinement in QCD.

Such a magnetic dual theory to QCD is not known to exist, and we discuss it here only hypothetically to illustrate how symmetry breaking and confinement would be related in a familiar environment by the dual Meissner mechanism. There are well understood examples of such electric-magnetic dualities in supersymmetric theories which undergo symmetry breaking on one side of the duality, which manifests as confinement on the other. Examples include $\mathcal{N} = 2$ super Yang-Mills (SYM), also referred to as Seiberg-Witten theory, and $\mathcal{N} = 1$ supersymmetric-QCD (SQCD), the latter being the subject of the well-know Seiberg's Duality. The existence of such dualities in supersymmetric theories has provided theoreticians with examples of strongly-coupled and confined systems which are accessible to established and well understood methods of analysis, significantly those of perturbation theory and holomorphicity, and this has expanded our understanding of strong-coupling regimes and the confined phase profoundly. However, this literature of results comes with certain scientific baggage.

This baggage is an unfortunate precipitate of both the fundamental implications of supersymmetry (SUSY) itself, and the present state of our ability to experimentally explore the parameter space of physical theories. Supersymmetry is a property of certain particle theories (there are examples in quantum field theory, string theory, and beyond), which describe both fermions and bosons. For any quantum theory which includes both fermions and bosons, the Hilbert space of quantum states can be decomposed into the fermionic and bosonic sectors. If there exists a linear operator that exchanges fermions and bosons within a given state, and which commutes with the Hamiltonian, the transformation this implies on the physical system is a symmetry of the theory. The operator defines a conserved quantity which we call a *supercharge*, while the transformation it generates is called a *supersymmetry*.

While the inclusion of supersymmetric structure to a theory has profound consequences for its analysis, there is also a significant phenomenological implication. The presence of supersymmetry in a theory implies an extension to the spectrum of the theory, in the form of new particles called *superpartners*. This is easy to see if one considers a solvable, non-supersymmetric QFT, with $|F_i\rangle$, $|B_i\rangle$, fermionic and bosonic eigenstates of the Hamiltonian respectively, which we know have the interpretation of species of particle. If we then include a supercharge, we can generate new eigenstates of the Hamiltonian $Q\,|F_i\rangle$, $Q\,|B_i\rangle$. We know that these are not states which existed in the spectrum before the SUSY was included, because we have that $Q^2 = 0$, and if we tried to express the original bosonic states in terms of the fermions, and vice-versa, with Q, i.e. $|B_i\rangle = Q\,|F_i\rangle$, then we would have $|F_i'\rangle = Q\,|B_i\rangle = Q^2\,|F_i\rangle = 0$, and thus the Hilbert space would be null.

Therefore, if we wished to extend standard model physics with supersymmetry, as has been suggested, we would then expect to find particles in collision experiements which are superpartners to already observed species. Thus far to the time of writing, however, none have been found. Whilst this is not fatal to SUSY, as it is possible for theories to predict significant energy differences between particles and their superpartners, which could place them out of range of our current level of observation, it does, for the time being, place our results which depend on SUSY, such as many well-understood examples of strong-weak duality, in a purgatorial realm of speculative and abstract study. Such abstract studies of structures are valid and, indeed, vital in theoretical physics in order for the field to make progress, but so long as at least part of the theoretician's job is concerned with distilling real world, phenomenological predictions, which may be observed and measured, there is compelling reason to study non-supersymmetric examples of S-duality.

The gauge theories examined in Sugimoto's paper, which we modify in our investigation, are realised as the low-energy effective theories of two string theories on an environment of N anti-D3 ($\overline{\text{D3}}$) branes suspended above an O3 orientifold. In [3], Uranga shows that such $\overline{\text{D3}}$-O3 systems completely break the supersymmetry of type IIB string theory, while the conjectured $SL(2, \mathbb{Z})$ symmetry of type IIB acts to generate strong-weak dualities by inverting the string coupling. The co-occurence of these mathematical exploits in this particular string theory set-up forms the basis for a procedure to construct a large class of non-SUSY S-dual gauge theories, proposed in [4], which aims to generalise the approach taken by Sugimoto in [1], which we shall outline in the following work. In this project we aim to build upon Sugimoto's proposed S-duality in a different way, by introducing fermions masses, in a way which may be extended to the more general class of non-SUSY S-dual gauge theories realised by $\overline{\text{D3}}$-O3 string theories. As we proceed we will observe significant, quantified similarities between Sugimoto's S-dual pair in the massive regime and standard model QCD, and our results will make contact with its known behaviours at low energy.

This thesis will be arranged as follows; chapters 1 and 2 will review the essential background concepts from which the original contribution of this project begins. Chapter 3 will detail our procedure for realising Sugimoto's duality in the case of massive fermions, chapter 4 will then follow how this fermion mass on the electric side of the duality is realised on the magnetic side. Finally, in chapter 5 we give the general relation between light meson masses in the magnetic theory and quark masses in the electric theory and show that this relationship is consistent with the GMOR relation of QCD, and in chapter 6 we will summarily discuss these results and potential future directions for further investigations.

Chapter 1. Review of 'Confinement & Symmetry Breaking' [1]

Before we begin to examine the dual gauge theories of interest, it is requisite to first fully establish where they come from and, most critically, the origin of their proposed duality.

To begin with, we consider a type IIB string theory, embedded in a 10-dimensional spacetime, that we parameterise with the usual co-ordinates $x^0, x^1, ..., x^9$. We then define a (3+1) dimensional hyperplane located at $(x^4, x^5, ..., x^9) = 0$, and we shall say that this plane is fixed with respect to the action of the operator; $I_6\Omega(-1)^{F_L}$, where I_6 generates a $\mathbb{Z}_2$ action that flips the sign of the spatial co-ordinates transverse to the fixed plane $(x^{4\sim 9})$, Ω is the world-sheet parity transformation operator and F_L is the left-moving spacetime fermion number. $I_6\Omega(-1)^{F_L}$ is called the orientifold action, and the hyperplane which is invariant under its action is called an orientifold plane, in this case it is an orientifold 3-plane, which we shall abbreviate to O3 [1].

As a non-perturbative object in a string theory, the O3 plane has various fields living on it, which when integrated over the O3 will define a kind of charge. For our purposes is it essential to introduce two of these charges explicitly;

$$\tau_{NS} = \exp(i\int_{\mathbb{RP}^2} B_2), \tau_{RR} = \exp(i\int_{\mathbb{RP}^2} C_2) \tag{2}$$

Where B_2 is the NSNS 2-form, C_2 is the RR 2-form, and $\mathbb{RP}^2$ is the 2-dimensional, real projective space defined by a 2-sphere surrounding the O3 plane in the transverse $x^{4\ 9}$ space, under the $\mathbb{Z}_2$ identification of antipodal points.

The charges τ_{NS} and τ_{RR}, called discrete torsions, can each take values ± 1, which together define 4 types of O3-plane; $(O3^-), (O3^+), (\widetilde{O3^-}), (\widetilde{O3^+})$ these are the O3 planes with $(\tau_{NS}, \tau_{RR}) = (+,+), (-,+), (+,-), (-,-)$ respectively. Type IIB string theory is believed to be invariant under the $SL(2,\mathbb{Z})$ action on the B_2 and C_2 fields [5];

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \tag{3}$$

Where $\Lambda \in SL(2,\mathbb{Z})$.

This symmetry also has an action on the dilaton ϕ and RR 0-form C_0,

$$\text{if } \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \tag{4}$$
$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

where $\tau = C_0 + ie^{-\phi}$

If we consider specifically the action of;

$$\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z})$$

We find that this Λ will act in (3) to interchange $O3^+$ and $\widetilde{O3^-}$ planes, by flipping their respective values for τ_{NS} and τ_{RR}. Second, the string coupling $g_s = e^\phi$ transforms as $g_s \to 1/g_s$ under Λ, which is the defining action of a strong-weak duality. Therefore, in the type IIB scenario we can build string theories around $O3^+$ and $\widetilde{O3^-}$ planes with an intrinsic S-duality from the $SL(2,\mathbb{Z})$ symmetry at our disposal.

Two such theories are obtained by placing a stack of n $\overline{\text{D3}}$ (read: Anti-D3) branes in the transverse space to an $O3^+$ or $\widetilde{O3^-}$ plane. An $\overline{\text{D3}}$ brane is, like a D3 brane, a hypersurface defined by 3+1 Neumann boundary conditions on an open string, but which is oppositely charged, or 'rotated', under the NSNS and RR fields with respect to a standard D3 brane. For our discussion, their most important properties are that $\overline{\text{D3}}$, like D3 branes, are invariant under the S-duality described above, but preserve an opposite supersymmetry to O3 planes, meaning that a system of $\overline{\text{D3}}$ branes above an O3 plane of any type breaks supersymmetry completely. Therefore; a stack of n-$\overline{\text{D3}}$ branes suspended above an $O3^+$ or $\widetilde{O3^-}$ defines a pair of non-supersymmetric, S-dual string theories. At this point we shall implement some terminology, which will be useful for orienting our discussion moving forward. We shall refer to the theory of n $\overline{\text{D3}}$ branes above an $O3^+$ plane as the 'electric side' of our duality, while the theory of n $\overline{\text{D3}}$ branes above an $\widetilde{O3^-}$ will be refered to as the 'magnetic side'.

The S-Dual Gauge theories

Recall that prior to the above exposition we had begun by discussing S-dual gauge theories, and had framed these as the central subject of this study. This being so, our attention to the S-dual $\overline{\text{D3}}$-O3 string theories may appear conceptually parallel. However, the $SL(2,\mathbb{Z})$ symmetry as a generator of S-duality in the type IIB set-up is precisely how Sugimoto realises a pair of QCD-like, non-SUSY S-dual gauge theories, which in tandem provide an illuminating and novel scenario for exploring the relationship between symmetry-breaking and confinement. The mathematical relationships between string theory and gauge theory are an area of active and on-going research, and while the particular connection we employ to transport our string S-duality to a gauge theoretic picture is not a new one, it does make contact with some non-trivial complications, and so requires some detail before we discuss how this project attempts to build on its results.

One of the early results of string theory is that the quantization of any string theory gives rise to infinitely many species of elementary particles. Furthermore, the quantization of open strings ending on D-branes means we have infinite species of particles living on the brane, and in the case of type IIB string theory these species include fermions and gauge bosons, as well as scalar particles. As with any quantum theory of particles, we're free to interpret these as excited states of fields. In his paper, Sugimoto, examines the tree-level massless field contents of the electric and magnetic string theories outlined above and posits that these effective field theories should also be s-dual to eachother. This is sensible as they each describe the low-energy physics of two string theories which are related by an exact internal symmetry, and should therefore be 'physically' equivalent. We will see more evidence to suggest this is correct, beyond the plausibility argument here, as we go on.

We can now get properly acquainted with our S-dual gauge theories. We shall from here on refer to the low-energy effective theory of the 'electric side' string theory, as simply; the electric theory, and we list its tree level massless fields here;

$$\mathcal{L}^{tree}_{electric} \sim \mathrm{Tr}\, F^2_{\mu\nu} + \mathrm{Tr}\,(\bar{Q}_i \sigma^\mu (\delta_\mu Q^i + [A_\mu, Q^i])) + \mathrm{Tr}\,((\delta_\mu \Phi^I + [A_\mu, \Phi^I])^2) \\ + \mathrm{Tr}\,(Q^i \Sigma^I_{ij} [\Phi^I, Q^j]) + \mathrm{Tr}\,([\Phi^I, \Phi^J]^2) + h.c. \tag{5}$$

Where A is the gauge field, Q is the left-handed fermion field, Φ^I are the six transverse scalar fields, and Σ^I are matrices which form a dirac-like algebra. The electric theory has the gauge symmetry $USp(2n)$ and global symmetry $SO(6)$

In line with this naming convention, we call the low-energy effective theory of the 'magnetic side' string theory, the magnetic theory, and again list its tree level massless contents;

$$\mathcal{L}^{tree}_{magnetic(I)} \sim \mathrm{Tr}\, f^2_{\mu\nu} + \mathrm{Tr}\,(\bar{q}_i \sigma^\mu (\delta_\mu q^i + [a_\mu, q^i])) + \mathrm{Tr}\,((\delta_\mu \phi^I + [a_\mu, \phi^I])^2) \\ + ((\delta_\mu + a_\mu) t)^2 + V(t) + \bar{\psi}^i \sigma^\mu (\delta_\mu \phi_i + a_\mu \psi_i) + \mathrm{Tr}\,(q^i \Sigma^I_{ij} [\phi^I, q^j]) \\ + \mathrm{Tr}\,([\phi^I, \phi^J]^2) + t^T \phi^I \phi^I t + \bar{\psi}^{iT} \Sigma^I_{ij} \phi^I \bar{\psi}^j + t^T q^i \psi_i + h.c. \tag{6}$$

Where a is the gauge field, q and ψ are left-handed weyl fermions, t is a tachyonic scalar field and again ϕ^I are six tranverse scalar fields. The magnetic theory has gauge symmetry $SO(2n)$ and global symmetry $SO(6)$.

The electric and magnetic theory Lagrangians are in schematic form, and the couplings have been omitted. There are some issues to clarify before any productive analyses of these theories can proceed. Starting with the most obvious point of concern, we have a tachyon in our magnetic theory, making it unstable. After tachyon condensation the gauge symmetry will be broken to $SO(2n) \to SO(2n-1)$, and through the terms $t^T \phi^I \phi^I t$ and $t^T q^i \psi_i$ we see that this condensation will give mass to some components of ϕ^I and q^i. The massless field Lagrangian after tachyon condensation is given here;

$$\mathcal{L}^{tree}_{magnetic(II)} \sim \mathrm{Tr}\, f^2_{\mu\nu} + \mathrm{Tr}\,(\bar{q}_i \sigma^\mu (\delta_\mu q^i + [a_\mu, q^i])) + \mathrm{Tr}\,((\delta_\mu \phi^I + [a_\mu, \phi^I])^2) \\ + \bar{\psi}^i \sigma^\mu (\delta_\mu \phi_i + a_\mu \psi_i) + \mathrm{Tr}\,(q^i \Sigma^I_{ij} [\phi^I, q^j]) + \mathrm{Tr}\,([\phi^I, \phi^J]^2) + h.c. \tag{7}$$

Going forward, this is the theory we will be referring to as the magnetic theory.

There have been analyses of the electric theory which tell us about the dynamics of this theory when we take into account the 1-loop beta function. Firstly, we have that this theory is asymptotically free, in other words, it is best described at high energies. The fields in (5) are all massless at tree level, however as the supersymmetry is completely broken, the scalar fields take on cut-off scale masses from the quantum corrections. Significantly, it is conjectured in the literature that the $USp(2n)$ gauge theory with four Weyl fermions in the anti-symmetry representation lies outside the conformal window and is in the confined phase, also, for the case that $n > 1$ the global $SO(6)$ symmetry is believed to be dynamically broken to $SO(4)$ by the condensation of a fermion bilinear; $\epsilon^{\alpha\beta} \langle \mathrm{Tr}\,(Q^i_\alpha Q^j_\beta) \rangle \propto \delta^{ij}$.

The magnetic theory is more opaque. It is not asymptotically free, which raises obstacles when taking a decoupling limit in a controlled way, however, some useful insight is extracted from a comparison of 1-loop corrections to the scalar masses between the electric and magnetic theories. The 1-loop calculation shows that the mass-squared for the Φ^I in the electric theory is positive, while the mass-squared for the ϕ^I in the magnetic theory is negative;

$$m_\Phi^2 = +Cg_s l_s^{-2},\ m_\phi^2 = -C' g_s l_s^{-2} \tag{8}$$

Where g_s is the string coupling and C and C' are positive constants.

The scalar fields on the $\overline{\text{D3}}$ brane have the interpretation of the position of the brane in the transverse space to the central O3 plane, as such, the mass-squared values in (8) suggest that the $\overline{\text{D3}}$ in the electric theory are attracted to the O3 plane, while in the magnetic theory they are repulsed. Initially this might seem strange, as we stated before that both the magnetic and electric theories describe low-energy dynamics of two 'physically' equivalent string theories, and yet it appears that we are seeing two contradictory behaviours. However, this is in fact not the case, and to understand why is what makes knowledge of the 1-loop corrections to each theory necessary to making progress.

The mass-squared results in (8) are only reliable when the coupling in each theory is small. Being that the electric theory is asymptotically free, and the magnetic theory is asymptotically non-free, the couplings in each theory are suppressed at different energy scales, and so we find that the $\overline{\text{D3}}$ branes are not in fact attracted and repelled at the same time, but instead are attracted at high energies (when the electric theory is best described), corresponding to a large displacement from the O3, and are repulsed at whichever energy (and corresponding distance from O3) minimises the coupling of the magnetic theory. Since the magnetic theory is non-free and we lack a smooth decoupling limit we cannot be certain where the coupling is small, however, as the S-duality of the parent string theories inverts the string coupling, it is reasonable to proceed with the assumption that it will be at a lower energy (and therefore closer to the O3) than the electric theory. Though, it should be noted that this does not mean that the magnetic theory will necessarily be weakly coupled at any energy.

With this interpretation, (8) gives us a picture of a scalar theory which is unstable at the origin, but becomes attractive towards the origin at large distances. This Higgs potential-like behaviour suggests that our scalar fields will develop a non-trivial expectation value (vev) and will thereby spontaneously break the global symmetry of the system. This qualitative picture is of course best described mathematically on the magnetic side of our duality, in spite of the difficulties concerning the energy scales at which it is strongly or weakly coupled, because the scalar fields are completely decoupled from the electric theory. Rather than going through the hardship of attempting to calculate the exact potential of the magnetic theory, Sugimoto makes fruitful progress by what he refers to as a 'toy' potential, or model, for the magnetic theory scalars. Note that we are now disregarding the dynamics of the fermions on the magnetic side as they are not relevant to the global symmetry breaking that we want to realise.

Sugimoto's model potential is as follows;

$$V(\phi^I) = -\frac{\mu^2}{2}\,\mathrm{tr}(\phi^I\phi^I) - \frac{g}{4}\,\mathrm{tr}([\phi^I,\phi^J]^2) + \frac{\lambda}{2}\,\mathrm{tr}((\phi^I\phi^I)^2) \tag{9}$$

Where the first term is the tachyonic mass term, the second is imported from the potential portion of the Lagrangian (7), and the final, quartic term is included to stabilise the potential at long-distance, to reflect the behaviour we know to expect. It should be said that the only term in the model potential not given an explicit origin is this quartic term, however, Sugimoto is not pulling this out of the air for convenience, rather this term is certain to exist in the full potential of the magnetic theory, which we are not privy to, and we are simply disregarding any further corrections as being unnecessary to capturing the symmetry breaking behaviour of the theory and needlessly burdensome to try to extract more precisely.

Differentiating (9) yields the following equation of motion for the magnetic theory;

$$-\mu^2\phi^I - g[\phi^J,[\phi^I,\phi^J]] + \lambda(\phi^I(\phi^J\phi^J) + (\phi^J\phi^J)\phi^I) = 0 \tag{10}$$

Which admits several vacua, depending on the choice of (in)equality between the positive coefficients λ and g.

The scalar field ϕ^I takes values in the Lie algebra of the gauge group, for the case n=2; $SO(3)$. Hence we define

$$\phi^I = A^I_i J^i \tag{11}$$

Where J^i are basis elements of the Lie algebra $so(3)$ (the spin-1 representation of $su(2)$). For the choice $\lambda > g$, it is straight-forward to show that the following value for ϕ^I is a solution to (10) and there-by a vacuum of the theory

$$\phi^1 = aJ^1, \phi^2 = aJ^2, \phi^3 = aJ^3, \phi^{4\sim 6} = 0 \tag{12}$$

This vacuum for (9) is clearly invariant under the group $SO(3) \times SO(3)$, where one SO(3) acts on the non-zero components of $\langle\phi^I\rangle$ and is freely undone by a gauge rotation, and the other is the SO(3) which acts on the null-components of $\langle\phi^I\rangle$ which are trivially invariant under its action. The isomorphism $SO(3) \times SO(3) \simeq SO(4)$ is a known result; therefore, Sugimoto's duality allows us to realise the dynamical $SO(6) \to SO(4)$ symmetry breaking expected of the strongly-coupled electric theory very simply and elegantly in terms of the condensation of a non-zero vev for scalar fields in its magnetic-dual description. Furthermore, had we been without prior indication of the phase of the electric theory, by showing that its dual magnetic description breaks global symmetry dynamically, it would be consistent with the dual-Meissner mechanism of confinement to conjecture that the electric theory was confining.

Chapter 2. The Chiral Lagrangian

We are now in a position to evaluate our mass term for the pions (Nambu-Goldstone bosons) in more explicit terms. To make proper sense of our term, it is very helpful to quickly review the pion mass term in the chiral Lagrangian of QCD.

In a $USp(2N)$ theory with four antisymmetric quarks there exists a $U(4) = U(1) \otimes SU(4)$ global symmetry. The $U(1)$ part is anomalous and hence the theory admits a massive η' pseudo-scalar meson. According to Witten-Veneziano formula [6, 7] we expect it to have a mass $M_{\eta'}^2 \sim \frac{2N-2}{2N}\Lambda_{\rm QCD}^2$. Unlike ordinary QCD where the η' becomes light in the 't Hooft large-N limit, in the present case the η' is always heavy.

The global $SU(4)$ is expected to break dynamically, according to the pattern

$$SU(4) \to SO(4). \tag{13}$$

The order parameter for the breaking is the quark condensate

$$\langle Q^a Q^b \rangle = c\delta^{ab} \ \ a, b = 1...4\,, \tag{14}$$

where $c \neq 0$ is the value of the condensate.

The breaking of the global symmetry results in a multiplet of nine massless Nambu-Goldstone (NG) bosons. The NG bosons belong to the coset $U \equiv G/H = SU(4)/SO(4)$. The fifteen generators of the $SU(4)$ are either symmetric (and real) or antisymmetric (and imaginary) Hermitian matrices. The six antisymmetric generators form the generators of the $SO(4)$ group. The remaning nine symmetric generators of the $SU(4)$ group transform in the two-index traceless symmetric representation of $SO(4)$.

The chiral Lagrangian of the present theory is written in terms of U, with

$$U = \exp i\pi \tag{15}$$

where π is a matrix that transforms in the two-index traceless symmetric representation of the $SO(4)$ algebra.

The relevant terms that will be at the centre of our interest are the kinetic term and the mass term for the NG bosons (the 'pions')

$$S \sim \int d^4x \, \mathrm{tr}\left((U^{-1}\partial_\mu U)(U^{-1}\partial^\mu U) + c(MU + \mathrm{h.c.})\right)\,, \tag{16}$$

where M is the quarks' mass matrix, namely the same 4×4 symmetric matrix that gives mass to the quarks

$$M_{ab}Q^aQ^b + \mathrm{h.c.} \tag{17}$$

We will choose M to be real. Note that we set $f_\pi = 1$.

We will mostly be interested in the kinetic term and the mass term of the pions

$$S = \int d^4x \, \mathrm{tr}\left(\partial_\mu \pi \partial^\mu \pi - cM\pi^2\right) + \dots\,. \tag{18}$$

In the simplest case, where all four quarks have the same mass $M = m\mathbb{1}$, we recover the celebrated GMOR relation

$$M_\pi^2 \sim cm\,. \tag{19}$$

Another interesting case that we will discuss later is the special case when the four pions consist of two pairs of pions of equal mass. In this case the global $SO(4)$ symmetry is further explicitly broken to $SO(2) \times SO(2)$ and the resulting mass spectrum of the pions is five degenerate pions with mass $M^2 \sim c\frac{1}{2}(m_1 + m_2)$, two pions with mass $M^2 \sim cm_1$ and two pions with mass $M^2 \sim cm_2$.

In the most general case where the four quark masses have arbitrary values we can proceed as follows. We parametrize the symmetric 4×4 pion matrix using ten entries, such that $\pi_{ij} = \pi_{ji}$. Note that the diagonal is not traceless, namely we have ten Nambu-Goldstone bosons instead of nine. We thus add the constraint

$$\sum_i \pi_{ii} = 0\,. \tag{20}$$

The mass terms in (18) together with the constraint (20) take the form

$$\mathcal{L} = -\frac{c}{2}\sum_{ij}(m_i + m_j)\pi_{ij}^2 - \Lambda^2(\sum_i \pi_{ii})^2\,, \tag{21}$$

with $\Lambda \to \infty$. We may think about $\sum_i \pi_{ii}$ as an infinitely heavy η'. If, instead, we consider a hypothetical theory where $\Lambda = 0$, namely we ignore the constraint (20), we obtain at low eneregy ten light particles whose masses are given by $M_{ij}^2 = c(m_i + m_j)$, where four of them contain a quark anti-quark pair of same flavour ($m_i = m_j$) and the other six contain a quark anti-quark of different flavours ($m_i \neq m_j$). Imagine that we continuously vary the value of Λ from 0 to ∞. As we increase Λ the mass of the η' increases, the masses of six NG bosons do not change and the mass of the three remaining NG bosons become a mixture of the four quark masses. The precise eigenvalues are determined by diagonalising a 3×3 matrix. We will discuss it in more detail in the next section.

At the Lie Algebra level $so(4)$ is isomorphic to $so(3) \times so(3)$. For comparison with the results of S-duality, it will be more convenient to write the chiral Lagrangian in the language of $SO(3) \times SO(3)$. The nine pions which transform in the traceless symmetric representation of $SO(4)$ transform in the bi-fundamental of $SO(3)\times SO(3)$. The ten entries of the symmetric mass matrix M can be decomposed into a singlet m and nine bifundamentals $m_i^{\tilde{i}}$ of $SO(3) \times SO(3)$ $(i, \tilde{i} = 1..3)$, as listed in table (1) below

	$SO(3)$	$SO(3)$
m	$\bullet$	$\bullet$
$m_i^{\tilde{i}}$	$\square$	$\square$
$\pi_i^{\tilde{i}}$	$\square$	$\square$

Table 1: *Content of the chiral Lagrangian.*

The relation between the ten parameters of M_{ab} and $m, m_i^{\tilde{i}}$ is given in Chapter 5.

The explicit form of the action (18) is

$$S = \int d^4x \left(\partial_\mu \pi_i^{\tilde{i}} \partial^\mu \pi_{\tilde{i}}^i - cm\pi_i^{\tilde{i}} \pi_{\tilde{i}}^i + c\epsilon_{\tilde{i}\tilde{j}\tilde{k}} \epsilon^{ijk} m_i^{\tilde{i}} \pi_j^{\tilde{j}} \pi_k^{\tilde{k}} \right). \tag{22}$$

Chapter 3. Massive Quarks Under Sugimoto's Duality

We have the scalar potential (9) as a model of the magnetic-dual theory to our confined electric theory

$$V(\phi^I) = -\frac{\mu^2}{2}\operatorname{tr}(\phi^I\phi^I) - \frac{g}{4}\operatorname{tr}([\phi^I,\phi^J]^2) + \frac{\lambda}{2}\operatorname{tr}((\phi^I\phi^I)^2)$$

Which, as we know from chapter one, admits several vacua, depending on the choice of (in)equality between the coefficients λ and g. For the choice $\lambda > g$ the e.o.m admits a vacuum

$$\phi^1 = aJ^1, \phi^2 = aJ^2, \phi^3 = aJ^3, \phi^{4\ \ 6} = 0$$

We focus on this solution as it dynamically breaks the global $SO(6)$ symmetry to $SO(3) \times SO(3) \sim SO(4)$. We have stated from the outset that our intention is to examine the case for massive fermions, and to relate these masses to the spectrum of pions, which are Nambu-Goldstone bosons of this symmetry breaking. Therefore we treat Sugimoto's fuzzy sphere vacuum to the potential (9), as the limiting case where the mass of the fermions vanish.

Introducing a general perturbation to the scalar fields

$$\phi^I = \langle \phi^I \rangle + \delta\phi^I \tag{23}$$
$$\text{Where } \delta\phi^I = A_a^I J^a$$

Substituting into potential (9) and evaluating to order $\mathcal{O}(\delta\phi^2)$ yields the following

$$V(\phi^I) = V(\langle\phi^I\rangle) + \frac{\mu^2}{2(g+2\lambda)}((\lambda - g)(A_a^b A_a^b + A_a^b A_b^a) + 2(\lambda+g)(A_a^a)^2) \tag{24}$$

Where $(a, b = 1, 2, 3)$.

Note that we ignore terms in the expansion which are linear in the perturbation. In the language of QFT, such terms are called tadpoles, and do not contribute to the dynaimcs of the scalar fields. There are also terms which are cubic and quartic in the perturbation, however these are not relevant to the mass of the Nambu-Goldstone bosons, hence we truncate at order $\mathcal{O}(\delta\phi^2)$. Note that the terms which appear in (24) involve the perturbation components A_a^b, which are the perturbations around the non-zero components of the vacuum, which is invariant under the action of $SO(3)$ by virtue of the gauge symmetry. As such, these perturbations are absorbed by the gauge field and are referred to as 'would-be' Nambu-Goldstone bosons.

The true Nambu-Goldstone bosons of the dynamical symmetry breaking, which are our pions, are associated with the 9 perturbation components A^m_a, where $(m = 4, 5, 6, a = 1, 2, 3)$, which do not have mass terms in (24). Therefore we have that the case of massless quarks in the electric theory corresponds to massless pions in the magnetic theory. To explore the relation between non-trivial quark and pion masses we look to introduce an additional physics to the string theory, which will confer masses to the fermions, and when we perturb around the vacuum (12), yield massive pions.

Such a modification is presented in the paper by Uranga et al. [8], which details the coupling of a 3-form flux background (as encountered in supergravity) to a D3-brane action. The term added to the Lagrangian of the string theories which is relevant to our discussion is as follows

$$\mathcal{L}_{soft} = \mathrm{Tr}\Big(\frac{ig_s}{6}(*_6G_3 - iG_3)_{IJK}\phi^I\phi^J\phi^K + h.c. + \frac{g_s}{96}(*_6G_3 - iG_3)_{IJK}\psi\gamma^{IJK}\psi + h.c.\Big) \quad (25)$$

Note that this term is added to the Lagrangians of both the electric and magnetic string theories, however, we expect only the fermionic term to survive in our electric gauge theory, as the scalars decouple from the low-energy physics by acquiring cut-off scale masses. Likewise, we expect the cubic scalar term to remain in the magnetic theory while the fermionic term becomes irrelevant to the physics.

Dimensional analysis tells us that the term $\frac{g_s}{96}(*_6G_3 - iG_3)_{IJK}\psi\gamma^{IJK}\psi$ is a mass term for the quarks. The only a priori constraint is on the three-form $(*_6G_3 - iG_3)_{IJK}$ in that it must be anti-self-dual, though G_3 itself is an arbitrary 3-form. Therefore we may engineer the components of the three-form to produce a controllable quark mass term. The same three-form couples to the a cubic scalar term in the magnetic theory, therefore, the coupling introduces a term to our magnetic theory potential which will produce a pion mass when we perturb around the fuzzy sphere solution (12). This pion mass term should be directly relatable to the quark mass term as they are both linear in the three-form. i.e.

$$\begin{aligned} \text{Defining: } (*_6G_{IJK} - iG_{IJK}) &= C_{IJK} \\ \text{Under the Perturbation: } \phi^I &= \langle\phi^I\rangle + \delta\phi^I \\ C_{IJK}\phi^I\phi^J\phi^K &\to ... + C_{IJK}\langle\phi^I\rangle\delta\phi^J\delta\phi^K + ... \end{aligned} \quad (26)$$

Three-Form Flux Components

We have that both the quark masses of the electric theory, and the pion masses of the magnetic theory are linear in the three-form background C. This common factor of the three-form flux, and a naive dimensional analysis, conspire to reveal that the quark and pion masses are related, schematically, as $m^2_\pi \sim M_q$. The full form of the relation is $m^2_\pi f^2 = -2M_q\langle\overline{q}q\rangle$ [9], where f is the pion decay constant and $\langle\overline{q}q\rangle$ is the quark condensate. This result is known from the chiral Lagrangian of QCD and is referred to as the Gell-Mann, Oakes, Renner (GMOR) relation [10]. We wish to go further than a schematic comparison with known results however, rather, we aim to extract the full, general relationship between the quark and pion masses under Sugimoto's duality.

The full expression will allow us to compare not only the form of the quark mass/pion mass relation with chiral QCD, but will also facilitate comparison between specific cases of quark mass degeneracies and how they affect the distribution of pion masses. In order that we should be able to tune the electric-theory fermion masses at will, and to extract the exact relationship between the quark and pion masses under the S-duality, we first must have an explicit expression of the 3-form C.

To derive an expression for C, we first look at a general term which couples a 3-form, which we will call G, to the fermions in our electric theory;

$$G_{IJK}(\gamma^{IJK})_{ij} = (m_q)_{ij} \tag{27}$$

Where m_q is a symmetric, 4×4 matrix with real eigenvalues. I,J,K are indices of the $SO(6)$ space, taking values $(1, ..., 6)$. i,j are indices over the $SO(4)$ space and take values $(1, ..., 4)$. We progress with the following procedure.

$$\gamma^{IJK} = \gamma^{[I}\gamma^{J}\gamma^{K]} \tag{28}$$

$$G_{IJK}\gamma^{IJK}\gamma^{[I'}\gamma^{J'}\gamma^{K']} = m_q\gamma^{[I'}\gamma^{J'}\gamma^{K']} \tag{29}$$

Given that γ^I satisfy the Dirac algebra $[\gamma^I, \gamma^J] = 2(\gamma^I\gamma^J - \delta^{IJ}\mathbb{1})$ we can show that,

$$\mathrm{Tr}(\gamma^{[I}\gamma^{J}\gamma^{K]}\gamma^{[I'}\gamma^{J'}\gamma^{K']}) = \tag{30}$$
$$4\delta^{KI'}\delta^{JJ'}\delta^{IK'} - 4\delta^{KI'}\delta^{IJ'}\delta^{JK'} + 4\delta^{II'}\delta^{KJ'}\delta^{JK'} - 4\delta^{II'}\delta^{JJ'}\delta^{KK'} + 4\delta^{JI'}\delta^{IJ'}\delta^{KK'} - 4\delta^{JI'}\delta^{KJ'}\delta^{IK'}$$

Which we substitute into (29) and evaluate.

$$\mathrm{Tr}(G_{IJK}\gamma^{IJK}\gamma^{[I'}\gamma^{J'}\gamma^{K']}) = \mathrm{Tr}(m_q\gamma^{[I'}\gamma^{J'}\gamma^{K']}) \tag{31}$$
$$= G_{IJK}(4\delta^{KI'}\delta^{JJ'}\delta^{IK'} - 4\delta^{KI'}\delta^{IJ'}\delta^{JK'} + 4\delta^{II'}\delta^{KJ'}\delta^{JK'} - 4\delta^{II'}\delta^{JJ'}\delta^{KK'} + 4\delta^{JI'}\delta^{IJ'}\delta^{KK'} - 4\delta^{JI'}\delta^{KJ'}\delta^{IK'})$$

$$= 4G^{K'J'I'} - 4G^{J'K'I'} + 4G^{I'K'J'} - 4G^{I'J'K'} + 4G^{J'I'K'} - 4G^{K'I'J'}$$
$$= -24G^{[I'J'K']} = \mathrm{Tr}(m_q\gamma^{[I'}\gamma^{J'}\gamma^{K']})$$

The reader will notice that we have derived an expression for the components of a vector field, with raised indices, whereas we started this procedure with the aim of finding the components of a three-form, which would have lowered indices. To lower the indices we need the metric on the 6-dimensional space transverse to the $\overline{\mathrm{D3}}$ branes. In [8] we learn that the dynamical metric of the full string theory is perturbed about the 10-dimensional flat Minkowski metric. It follows then that the portion of the metric which lives on the transverse space is, to leading order, the 6-dimensional flat Euclidean metric. Therefore, in order to lower the indices of the derived expression, to give find the components of the three-form G, we need only contract with a Kronecker delta. Since this does not introduce any multiplying factors or sign changes on $G^{[IJK]}$, we can forego an explicit contraction and simply raise or lower indices at our convenience, i.e. $G_{[IJK]} = G^{[IJK]}$.

As G_{IJK} are the components of a 3-form, we drop the brackets on the lower indices, which indicate an antisymmetrisation that from here on we will assume tacitly. Therefore, we have;

$$G_{IJK} = \frac{-1}{24} \operatorname{Tr}(M\gamma_{[I}\gamma_J\gamma_{K]}) \tag{32}$$

Anti-Self-Duality of C

So far we have an expression for a three-form, that we've called G, which contracts with the anti-symmetric product of three Dirac matrices to give the 4×4 symmetric, real matrix m_q. This criterion being satisfied is sufficient to support the interpretation of the coupling of G to the fermions as a sensible mass term. However, the three-form which contracts with the γ triple index in Uranga's coupling term in [8] was anti-self-dual, and so far G is not. We must therefore go further to assimilate this property into a new three-form, C, which is derived from G.

Let us see what this anti-self-dual property requires: First, recall that on a Riemannian manifold, the square of the Hodge dual upon a 3-form evaluates to -1, i.e.

$$(*_6)^2\omega = -\omega$$

Where ω is a three-form.

Uranga gives us the expression for the anti-self-dual three-form C in terms of an arbitrary three-form, which he calls G, the components of which we have derived explicitly such that its coupling to the fermions is a reasonable mass term. From [8] we have:

$$C = (*_6G - iG) \tag{33}$$

$$\therefore *_6C = (-i *_6 G - G) = -iC$$

The Hodge dual convention we follow here is as follows

$$(*\omega)_{IJK} = i\epsilon_{IJK}^{I'J'K'}\omega_{\,I'J'K'} \tag{34}$$

Therefore we substitute our expression for the components of G into (33) to derive the anti-self-dual components of C

$$C_{IJK} = \frac{-1}{48} \operatorname{Tr}(m_q(\epsilon^{IJKI'J'K'}\gamma_{[I'}\gamma_{J'}\gamma_{K']} - i\gamma_{[I}\gamma_J\gamma_{K]})) \tag{35}$$

Where we have included an additional factor of 1/2 so as to avoid unwanted scaling of the matrix M when C couples to the fermions, as opposed to G. Also, the factor of i in the Hodge dual is absorbed into Sugimoto's employed representation of the Dirac algebra in [1].

Chapter 4. Representation Theory & The Pion Mass Term

Let us introduce mass to the quarks of the electric theory and examine how it affects the mass of the pions in the magnetic side of the duality. To this end we will introduce a three-form flux $G_3 = F_3 - \tau H_3$ in the type IIB background following [11, 12]. F_3 and H_3 are RR and NSNS three-form fluxes. As we shall see in a moment the flux encodes the quark mass matrix $(m_q)_{ij}$.

The action of a D3 brane in a background that includes a three-form flux is given in [11] contains the following terms.

$$\mathcal{L}_{\text{soft}} = ... + i\frac{g_s}{6}(\star_6 G_3 - iG_3)_{IJK}\phi^I\phi^J\phi^K + i\frac{g_s}{96}(\star_6 G_3 - iG_3)_{IJK}Q\gamma^{[I}\gamma^J\gamma^{K]}Q + \text{h.c.} \quad (36)$$

Substituting the following components for the three-form term.

$$(\star_6 G_3 - iG_3)_{IJK} = C_{IJK} = \frac{-1}{48}\operatorname{Tr}(m_q(\epsilon_{IJK}^{I'J'K'}\gamma_{[I'}\gamma_{J'}\gamma_{K']} - i\gamma_{[I}\gamma_J\gamma_{K]})) \quad (37)$$

Note that this trace is carried out over the $SO(4)$ indices.

We find that the three-form coupling confers a fermion mass term to the electric theory of the form $Q^i(m_q)_{ij}Q^j$ where we have full control over the entries of the matrix m_q.

The reader will recall, from chapter one, that in [1] the scalar fields of the electric theory acquire cut-off scale masses and decouple. Consequently, only the flux-induced quark mass term of (36) carries into the electric theory. However, in the magnetic theory, the scalar fields are where the critical behaviours of the physics are realised, and as such only the scalar coupling term of (36) is of interest to the magnetic theory. Our specific aim is to relate the quark masses and pion masses due to this three-form coupling, therefore we introduce the scalar coupling in (36) as a perturbation around the fuzzy sphere vacuum (12) of the original magnetic theory potential.

$$V' = V_0 + \operatorname{Tr}(C_{IJK}\phi^I\phi^J\phi^K) \quad (38)$$

Where V_0 is the potential (9). The trace is over the gauge group of the magentic theory $SO(2N-1)$. For simplicity we will consider the case with $N = 2$, namely $SO(3)$. The generalisation to arbitrary N is straightforward.

We introduce perturbations to the scalar fields of the following form

$$\phi^I = \langle\phi^I\rangle + \delta\phi^I \quad (39)$$

where $\delta\phi^{1\sim3} = 0, \delta\phi^{I=4\sim6} = \delta A_a^I J^a$.

Substituting this into the three-form coupling term in (38) we obtain several terms of varying order in the vev and perturbations, but most important for our purposes is the following

$$\begin{aligned}\operatorname{Tr} C_{IJK}(\langle\phi^I\rangle + \delta\phi^I)(\langle\phi^J\rangle + \delta\phi^J)(\langle\phi^K\rangle + \delta\phi^K) = \\ ... + 3\operatorname{Tr} C_{IJK}(\langle\phi^I\rangle\delta\phi^J\delta\phi^K) + ...\end{aligned} \quad (40)$$

Which, as discussed in chapter 3, is a mass term for the pions.

To proceed to the full relation between the quark and pion masses we must first expand (40). We find immediately that there are some important mathematical details in (40) which must be properly understood in order that we can make progress correctly. Firstly, note that the trace over the scalar fields in the magnetic theory potential is a trace over $SO(3)$, namely the gauge group of the magnetic theory. While the three-form C contains a trace over $SO(4)$. So in (40) we are dealing with a term which involves a nested trace over two different groups.

$$3\,\mathrm{Tr}_{SO(3)}(C_{IJK}\langle\phi^I\rangle\delta\phi^J\delta\phi^K) = \frac{-1}{16}\,\mathrm{Tr}_{SO(3)}(\mathrm{Tr}_{SO(4)}(m_q(\epsilon^{I'J'K'}_{IJK}\gamma_{[I'}\gamma_{J'}\gamma_{K']} - i\gamma_{[I}\gamma_J\gamma_{K]}))\langle\phi^I\rangle\delta\phi^J\delta\phi^K) \tag{41}$$

Where our expression for the components of the background three-form C_{IJK} enters into the pion mass term, we are free to use our non-chiral expression which is derived in chapter 3, as the first portion which contracts with the epsilon is not relevant to calculations concerning the pion masses. With this in mind, we can simplify our term to a more compact form.

$$3\,\mathrm{Tr}_{SO(3)}(C_{IJK}\langle\phi^I\rangle\delta\phi^J\delta\phi^K) = \frac{-1}{8}\,\mathrm{Tr}_{SO(3)}(\mathrm{Tr}_{SO(4)}(m_q\gamma_{[I}\gamma_J\gamma_{K]})\langle\phi^I\rangle\delta\phi^J\delta\phi^K) \tag{42}$$

The nested trace of (42) is a bar to progress, as it mixes the $SO(4)$ language of the electric theory with the $SO(3)$ language of the gauge symmetry of the magnetic theory. However, we can resolve this by reconsidering the space our pions live in. Our magnetic theory pions are Nambu-Goldstone bosons, which take values as the generators of the coset $(SO(6)/(SO(3)\times SO(3)))$, associated with the dynamical symmetry breaking. It is easiest to realise these generators if we represent the coset as $SU(4)/SO(4)$. The isomorphism $SU(4)\simeq SO(6)$ (modulo $\mathbb{Z}_2$) is well known. Less well known is the isomorphism $SO(4)\simeq SO(3)\times SO(3)$, so we briefly present its origin. In [13], Pegoraro demonstrates by means of defining a particular basis, the existence of an isomorphism between $SO(4)$ and $SO(3)\times SO(3)$ at the level of the Lie algebra. A basis for the Lie algebra $so(4)$ in the fundamental representation is given as

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that we have, by our notation, defined the basis as consisting of two 'blocks'. This is made natural by the Lie bracket structure on the basis

$$[L_a, L_b] = \epsilon_{abc}L_c,\ [K_a, K_b] = \epsilon_{abc}L_c,\ [L_a, K_b] = \epsilon_{abc}K_c \tag{43}$$

Pegoraro then defines a new basis, also consisting of two blocks, $P = \frac{1}{2}(L+K)$, $Q = \frac{1}{2}(L-K)$, which obey the following Lie bracket structure

$$[P_a, P_b] = \epsilon_{abc}P_c,\ [Q_a, Q_b] = \epsilon_{abc}Q_c,\ [P_a, Q_b] = 0 \tag{44}$$

From this we see that each block in this basis of $so(4)$ independently realises the Lie algebra $so(3)$. Therefore we can map one block of $so(4)$ to one copy of $so(3)$ and map the other block to the other copy, the resulting total map over the basis of $so(4)$ is linear, injective and preserves the Lie bracket structure, and therefore defines an isomorphism. This verifies that we can indeed represent the coset defined the symmetry breaking, $SO(6)/SO(3) \times SO(3)$ as $SU(4)/SO(4)$. This helps us in realising the generators of the coset explicitly as $SO(4)$ is a Lie subgorup of $SU(4)$, and therefore we can define its action on $SU(4)$ very simply.

The coset space of $SU(4)/SO(4)$ is defined as $SU(4)$ modulo the equivalence relation defined by the action of $SO(4)$, where $g \sim g'$, where $g, g' \in SU(4)$, iff $\exists h \in SO(4) : g' = hg$. $SU(4)/SO(4)$ then comprises the elements of $SU(4)$ which are not related under this equivalence class. In general we express the generators of a Lie group as a basis of the tangent space to curves on the Lie group which pass through the identity. On $SU(4)$, we can generate curves through the identity by the action of $SU(4)$ on itself, and by the action of $SO(4)$. $SU(4)/SO(4)$ is a non-empty set which implies that there exist elements of $SU(4)$ which are not connected to the identity by the action of $SO(4)$. However, $SU(4)$ is a connected Lie group, therefore we can define a curve which connects every element to the identity by the action of $SU(4)$ on itself. These two facts together imply that there exist curves which are generated by the action of $SU(4)$ which are not generated by the action of $SO(4)$, these curves are defined by the action of the coset space. This implies that the tangent space at the identity of $SU(4)$ can be decomposed as the direct sum of tangent vectors to curves generated by $SU(4)/SO(4)$ and tangent vectors to curves generated by $SO(4)$. To put it explicitly

$$su(4) = \mathcal{L}\{SU(4)/SO(4)\} \oplus so(4) \tag{45}$$

We have, therefore, that the generators of the coset space of our symmetry breaking is defined by the orthogonal complement of the Lie algebra $su(4)$ with respect to the Lie algebra $so(4)$.

The Lie algebra $su(4)$ in the fundamental representation is realised by the vector space of traceless, anti-Hermitian four-by-four matrices, which we can decompose as the direct sum of the space of traceless, symmetric imaginary 4×4 matrices, and the space of anti-symmetric real 4×4 matrices. The latter portion being exactly the Lie algebra $so(4)$. Therefore, the orthogonal complement of $su(4)$ w.r.t. $so(4)$, which defines the generators of $SU(4)/SO(4)$, is the space of symmetric, traceless imaginary 4×4 matrices. The space of symmetric, traceless imaginary 4×4 matrices is clearly isomorphic to the space of symmetric, traceless real 4×4 matrices. In [13], Pegoraro tells us that this space transforms under the adjoint action of $SO(4)$. The isomorphism we detailed earlier tells that we can then realise the space of $SU(4)/SO(4)$ generators as a representation space of $SO(3) \times SO(3)$, and in fact Pegoraro describes the appropriate space, and it is simply the space of 3×3 real matrices under the action

$$SO(3) \times SO(3) : Mat(3, \mathbb{R}) \to Mat(3, \mathbb{R}) = n \to O_1 n O_2^T \tag{46}$$

Where $n \in Mat(3, \mathbb{R})$, O_1, $O_2 \in SO(3)$.

We see now that we have come full circle, and shown that our Nambu-Goldstone bosons transform under the symmetry group of the vacuum, and as such the trace taken over (42) should be a trace over the bifundamental representation space of $SO(3) \times SO(3)$. This is achieved by making a 'colour-flavour' identification in the trace of (42), that is, we identify the $SO(3)$ gauge symmetry of the magnetic theory with one of the copies of $SO(3)$ which lives in the $SO(4)$ symmetry of the electric theory. i.e.

$$SO(3)_{\text{col.}} \sim SO(3)_{\text{flav.}}$$

$$\implies \operatorname{Tr}_{SO(3)} \operatorname{Tr}_{SO(4)} \to \operatorname{Tr}_{SO(3)\times SO(3)}$$

Of course in order to perform an $SO(3) \times SO(3)$ trace over the pion mass term, all the factors in the operand of the trace must be in the bi-fundamental representation of $SO(3) \times SO(3)$. However, from the outset, m_q and the Dirac matrices γ^I belong to a representation space of $SO(4)$. m_q is a real, symmetric, 4×4 matrix, which we may view as having two components, a traceful, and a traceless. Both of these parts may be treated as representation spaces of $SO(4)$. The traceless part of m_q is a nine dimensional representation with the $SO(4)$ action $m \to OmO^T$, where $m \in$ traceless, symm. $\text{Mat}(4, \mathbb{R})$. The traceful component of m_q is the one dimensional 'singlet' representation of $SO(4)$. Furthermore, in [1] representation of the Dirac algebra, the elements of which we have labelled γ^I are generators of $SO(4)$, which can in turn be viewed as a double copy of the Lie algebra of $SO(3)$.

To arrive at a mass term for the pions which can be fully evaluated in the $so(3) \times so(3)$ language which is the natural to the magnetic theory moduli space, we must map the traceful and traceless components of m_q from their respective $SO(4)$ to the appropriate representations of $SO(3) \times SO(3)$. To be explicit, we wish to map,

$$(SO(4)) : m\mathbb{1}_{4\times 4} + m^{\mu}T_{\mu} \to (SO(3) \times SO(3)) : m' 1 \otimes 1 + m'^{\,a}_{\tilde{a}} J_a \otimes J^{\tilde{a}} \tag{47}$$

Where T_{μ} (with $\mu = 1, ..., 9$), are a basis of traceless, symmetric $\text{Mat}(4, \mathbb{R})$. $J^a, J_{\tilde{a}}$ (with $a, \tilde{a} = 1, 2, 3$), are generators of the Lie algebra $so(3)$.

We are only concerned with mapping m_q of the form $m_q = \text{diag}(m_1, m_2, m_3, m_4)$, as this corresponds to the most general quark mass term in our electric theory. Therefore we may decompose m_q in the form given on the left-hand side of (47) as follows

$$\begin{aligned} m_q = {} & \frac{1}{2\sqrt{2}}(m_1 + m_2 + m_3 + m_4)(\frac{1}{\sqrt{2}}\mathbb{1}) \\ & + \frac{1}{2\sqrt{2}}(m_1 + m_2 - m_3 - m_4)(\frac{1}{\sqrt{2}} \text{diag}(1, 1, -1, -1)) \\ & + \frac{1}{2\sqrt{2}}(m_1 + m_4 - m_2 - m_3)(\frac{1}{\sqrt{2}} \text{diag}(1, -1, -1, 1)) \\ & + \frac{1}{2\sqrt{2}}(m_2 + m_4 - m_1 - m_3)(\frac{1}{\sqrt{2}} \text{diag}(-1, 1, -1, 1)) \end{aligned} \tag{48}$$

Note that the generators are normalised such that $\mathrm{Tr}(T_\mu T_\nu) = 2\delta_{\mu\nu}$. The pion mass term (42) may then be expressed in a form which makes the realisation of the map (47) straightforward.

$$\mathrm{Tr}(M_\pi^2 \delta\phi\delta\phi) = \frac{-1}{8}\,\mathrm{Tr}((\frac{m}{\sqrt{2}}\mathbb{1}_{4\times 4} + m^\mu T_\mu)\langle\phi^I\rangle\gamma_I\delta\phi^J\gamma_J\delta\phi^K\gamma_K) \tag{49}$$

Note that $m = (\frac{m_1+m_2+m_3+m_4}{2\sqrt{2}})$ and m^μ has non-zero components $(\frac{m_1+m_2-m_3-m_4}{2\sqrt{2}})$, $(\frac{m_1+m_4-m_2-m_3}{2\sqrt{2}})$, $(\frac{m_2+m_4-m_1-m_3}{2\sqrt{2}})$.

We now implement the isomorphism $SO(3)\times SO(3) \simeq SO(4)$. Naturally, the tracefull part of $SO(4)$ is mapped to the singlet of $SO(3)\times SO(3)$. The traceless, diagonal matrices of (48) T_μ, are mapped to elements of the bifundamental algebra of $SO(3)\times SO(3)$. The Dirac matrices (γ_a) are mapped to basis elements (J_a) of the Lie algebra $so(3)$. Finally, the factor of $\langle\phi^I\rangle\gamma_I$, which is the singlet of the vacuum symmetry, is mapped to the singlet of $SO(3)\otimes SO(3)$, multiplied by the constant (a) which we associate with the fermion bilinear condensate. To summarise the map, we tabulate the transformation of each factor in (49) below

$SO(4)$	$SO(3)\times SO(3)$
$\frac{m}{\sqrt{2}}\mathbb{1}$	$2m'1\otimes 1$
T^μ	$J_a \otimes J^{\tilde{a}}$
m^μ	$m'^{\,a}_{\,\tilde{a}}$
$\langle\phi^I\rangle\gamma_I$	$a1\otimes 1$

Note that $\mathrm{Tr}(J_a J_b) = 2\delta_{ab}$, and therefore $\mathrm{Tr}(J_a\otimes J^{\tilde{a}} J_b \otimes J^{\tilde{b}}) = 4\delta_{ab}\delta^{a'b'}$. Due to this difference in normalisation between our bases of $so(4)$ and $so(3)\times so(3)$, the coefficients m' and $m'^a_{\tilde{a}}$ of the $so(3)\times so(3)$ expression of m_q receive an additional factor of $\frac{1}{\sqrt{2}}$ relative to the $so(4)$ components, in order to preserve $\mathrm{Tr}(m_q^2)$, which must be invariant under changes of basis.

Recall that the isomorphism $SO(3) \times SO(3) \simeq SO(4)$ is actually an isomorphism at the level of the Lie algebra. As such, we are at liberty to map $T_\mu \to J_a \otimes J^{\tilde{a}}$ in whatever way is convenient. For reasons of neatness further along in the process, we have chosen $(T_1 \to J_1 \otimes J^{\tilde{1}}, T_2 \to J_2 \otimes J^{\tilde{2}}, T_3 \to J_3 \otimes J^{\tilde{3}})$. This gives $(m'{}^{a}_{\tilde{a}})$ as follows

$$m'{}^{a}_{\tilde{a}} = \begin{pmatrix} \frac{m_1+m_2-m_3-m_4}{4} & 0 & 0 \\ 0 & \frac{m_1+m_4-m_2-m_3}{4} & 0 \\ 0 & 0 & \frac{m_2+m_4-m_1-m_3}{4} \end{pmatrix} \tag{50}$$

and in addition $m' = m = \frac{1}{4}(m_1 + m_2 + m_3 + m_4)$.

We can now express the pion mass term fully in terms of the $SO(3) \times SO(3)$ language.

$$\mathrm{Tr}(M_\pi^2 \delta\phi\delta\phi) = \frac{-a}{8} \mathrm{Tr}((2m' 1 \otimes 1 + m'{}^{a}_{\tilde{a}} J_a \otimes J^{\tilde{a}})(\delta A^{b}_{\tilde{b}} J^{\tilde{b}} \otimes J_b)(\delta A^{c}_{\tilde{c}} J^{\tilde{c}} \otimes J_c)) \tag{51}$$

There is subtlety we need to address: the matrix m_q acts on Dirac fermions. For this reason the group we need to consider is actually the group $SU(2) \times SU(2)$ and we therefore choose $J_i \equiv \sigma_i$, the Pauli matrices.

We can now begin to explicitly evaluate our pion mass term.

$$\begin{aligned} \mathrm{Tr}(m'(\delta A^{b}_{\tilde{b}} J^{\tilde{b}} \otimes J_b)(\delta A^{c}_{\tilde{c}} J^{\tilde{c}} \otimes J_c)) &= m' \delta A^{b}_{\tilde{b}} \delta A^{c}_{\tilde{c}} \mathrm{Tr}(J^{\tilde{b}} \otimes J_b J^{\tilde{c}} \otimes J_c) \\ &= 4m' \delta A^{b}_{\tilde{b}} \delta A^{c}_{\tilde{c}} \delta^{\tilde{b}\tilde{c}} \delta_{bc} \end{aligned} \tag{52}$$

$$\begin{aligned} \mathrm{Tr}((m'{}^{a}_{\tilde{a}} J_a \otimes J^{\tilde{a}})(\delta A^{b}_{\tilde{b}} J^{\tilde{b}} \otimes J_b)(\delta A^{c}_{\tilde{c}} J^{\tilde{c}} \otimes J_c)) &= m'{}^{a}_{\tilde{a}} \delta A^{b}_{\tilde{b}} \delta A^{c}_{\tilde{c}} \mathrm{Tr}(J_a \otimes J^{\tilde{a}})(J^{\tilde{b}} \otimes J_b)(J^{\tilde{c}} \otimes J_c) \\ &= -4m'{}^{a}_{\tilde{a}} \delta A^{b}_{\tilde{b}} \delta A^{c}_{\tilde{c}} \epsilon^{\tilde{a}\tilde{b}\tilde{c}} \epsilon_{abc} \end{aligned} \tag{53}$$

Our full mass term for the magnetic theory pions is then,

$$M_\pi^2 \delta A^{a}_{\tilde{a}} \delta A^{a}_{\tilde{a}} = \frac{-a}{8}(8m' \delta A^{a}_{\tilde{a}} \delta A^{a}_{\tilde{a}} - 4m'{}^{a}_{\tilde{a}} \delta A^{b}_{\tilde{b}} \delta A^{c}_{\tilde{c}} \epsilon^{\tilde{a}\tilde{b}\tilde{c}} \epsilon_{abc}) \tag{54}$$

We see that this is equivalent to the pion mass term derived from the chiral Lagrangian in (22) if the radius of the fuzzy sphere, namely the constant a, is identified with the value of the quark condensate c in field theory. We can justify why this should be the case with a brief, semi-qualitative outline. We saw in the review of the chiral Lagrangian how the bi-linear quark condensate, which is responsible for the global symmetry breaking in QCD, percolates into the mass term of the pions as an overall factor that we label c. While we see that in the pion mass term for our magnetic theory we have an overall factor of a. Given that we know our electric theory is QCD-like, the formal similarity of how the two factors c and a arise in the their respective theories is good motivation to identify them. It is important to stress however, that this identification does not constitute a dictionary between the electric and magnetic theories. We cannot show the relationship between the electric theory quark condensate and the magnetic theory fuzzy sphere radius in the same, explicit way as we do with the corresponding quantities in QCD, because the relationship between the electric and magnetic theories is fundamentally different to that between the full theory of QCD and the chiral Lagrangian. Where the chiral theory is a low-energy effective theory of QCD, the magnetic theory is dual to the electric theory, so we are not in a position to directly import the electric bi-linear quark condensate into the magnetic scalar equations because the same quark fields do not live in the magnetic theory.

Chapter 5. Pion Spectra & GMOR-like Relations

However, (54) is not yet an entirely sensible mass-squared term for our pions. This becomes apparent when we contract the indices on the right-hand side of the equation. The first portion is well-behaved as we get nine terms of the form $m'(\delta A^{\tilde{a}}_{a})^2$, but a problem occurs in the second portion.

$$\begin{aligned} m'^{a}_{\tilde{a}}\delta A^{b}_{\tilde{b}}\delta A^{c}_{\tilde{c}}\epsilon^{\tilde{a}\tilde{b}\tilde{c}}\epsilon_{abc} = {} & m'^{1}_{\tilde{1}}(\delta A^{2}_{\tilde{2}}\delta A^{3}_{\tilde{3}} + \delta A^{3}_{\tilde{3}}\delta A^{2}_{\tilde{2}} - \delta A^{2}_{\tilde{3}}\delta A^{3}_{\tilde{2}} - \delta A^{3}_{\tilde{2}}\delta A^{2}_{\tilde{3}}) + \\ & m'^{2}_{\tilde{2}}(\delta A^{3}_{\tilde{3}}\delta A^{1}_{\tilde{1}} + \delta A^{1}_{\tilde{1}}\delta A^{3}_{\tilde{3}} - \delta A^{1}_{\tilde{3}}\delta A^{3}_{\tilde{1}} - \delta A^{3}_{\tilde{1}}\delta A^{1}_{\tilde{3}}) + \\ & m'^{3}_{\tilde{3}}(\delta A^{1}_{\tilde{1}}\delta A^{2}_{\tilde{2}} + \delta A^{2}_{\tilde{2}}\delta A^{1}_{\tilde{1}} - \delta A^{2}_{\tilde{1}}\delta A^{1}_{\tilde{2}} - \delta A^{1}_{\tilde{2}}\delta A^{2}_{\tilde{1}}) \end{aligned} \tag{55}$$

The complication that emerges here is that we have terms in this sum which have coefficients which are dimensionally mass-squared, but the factors of the fields are mixed. That is, rather than terms of the form $M^2_\pi(\delta A^1_{\tilde{1}})^2$ as is usual for a scalar mass-term, we instead have terms like $M^2_\pi(\delta A^1_{\tilde{1}}\delta A^2_{\tilde{2}})$. We can un-mix the fields in these terms by a change of basis, however as we are only interested in extracting the mass-squared of each pion field, it is more direct to perform a series of diagonalisations on (55), the eigenvalues of which will sum with the overall value of m' from the first portion, to yield the mass-squared values for the pions.

In equation (55), we have mixed terms which involve all nine pion fields, however, they do not mix homogeneously, instead they mix as three pairs and one triple. To see this more clearly we shall shuffle the series

$$\begin{aligned} m'^{a}_{\tilde{a}}\delta A^{b}_{\tilde{b}}\delta A^{c}_{\tilde{c}}\epsilon^{\tilde{a}\tilde{b}\tilde{c}}\epsilon_{abc} = {} & m'^{1}_{\tilde{1}}(-\delta A^{2}_{\tilde{3}}\delta A^{3}_{\tilde{2}} - \delta A^{3}_{\tilde{2}}\delta A^{2}_{\tilde{3}}) + \\ & m'^{2}_{\tilde{2}}(-\delta A^{1}_{\tilde{3}}\delta A^{3}_{\tilde{1}} - \delta A^{3}_{\tilde{1}}\delta A^{1}_{\tilde{3}}) + \\ & m'^{3}_{\tilde{3}}(-\delta A^{2}_{\tilde{1}}\delta A^{1}_{\tilde{2}} - \delta A^{1}_{\tilde{2}}\delta A^{2}_{\tilde{1}}) + \\ & m'^{1}_{\tilde{1}}(\delta A^{2}_{\tilde{2}}\delta A^{3}_{\tilde{3}} + \delta A^{3}_{\tilde{3}}\delta A^{2}_{\tilde{2}}) + \\ & m'^{2}_{\tilde{2}}(\delta A^{3}_{\tilde{3}}\delta A^{1}_{\tilde{1}} + \delta A^{1}_{\tilde{1}}\delta A^{3}_{\tilde{3}}) + \\ & m'^{3}_{\tilde{3}}(\delta A^{1}_{\tilde{1}}\delta A^{2}_{\tilde{2}} + \delta A^{2}_{\tilde{2}}\delta A^{1}_{\tilde{1}}) \end{aligned} \tag{56}$$

We see that fields are not mixed between the first three terms in (56). We can treat each of these terms as actions of metrics on three, two-dimensional, vector sub-spaces, which we can represent as matrix equations, which makes their diagonalisation very easy.

If we define a vector A_1, with components $(\delta A^2_{\tilde{3}}, \delta A^3_{\tilde{2}})$, with a metric μ_1 with entries $\begin{pmatrix} 0 & -m'^1_{\tilde{1}} \\ -m'^1_{\tilde{1}} & 0 \end{pmatrix}$, we can represent the first term in (56) as follows

$$m'^1_{\tilde{1}}(-\delta A^2_{\tilde{3}}\delta A^3_{\tilde{2}} - \delta A^3_{\tilde{2}}\delta A^2_{\tilde{3}}) = \mu_1(A_1, A_1) \tag{57}$$

The diagonalisation of μ_1 is trivial, and yields eigenvalues $(+m'^1_{\tilde{1}}, -m'^1_{\tilde{1}})$. We can repeat this procedure for the second and third terms of (56). Doing so we get eigenvalues $(+m'^2_{\tilde{2}}, -m'^2_{\tilde{2}})$ and $(+m'^3_{\tilde{3}}, -m'^3_{\tilde{3}})$, respectively. As stated, these eigenvalues combine with the contribution of the first term in (55), which is m'. We can therefore express the masses of six of the pions in the general case where $m_q = \mathrm{diag}(m_1, m_2, m_3, m_4)$.

M_π^2	Degeneracy
$a(m_1 + m_2)$	1
$a(m_1 + m_3)$	1
$a(m_1 + m_4)$	1
$a(m_2 + m_3)$	1
$a(m_2 + m_4)$	1
$a(m_3 + m_4)$	1

We can more succinctly write, for this case, that we have six pions with mass-squared values given by $M_\pi^2 = a(m_i + m_j)$ where $i \neq j$.

An aside on mixed quadratic scalar terms and factors of two.

The author is conscious that if one follows the preceding calculations closely, its is quite easy to lose track of the factors of two, and think that an error has been made in cancelling terms. In this aside we will briefly run over where the appropriate factors of two come from, and how the above and further results are recovered. The above result is not naively apparent from equation (54), as the factor of two difference between the two terms on the right-hand side is likely to mislead the reader to think that the appropriate factors of the quark masses will not cancel in the sum. However. In fact this factor of two is only apparent, and enters into our mass term due to the normalisation of the singlet. It is easiest to demonstrate where this additional factor of 2 comes from in the case of QCD with two quark flavours, which breaks symmetry from $U(2) \times U(2)$ to $U(2)$.

The generators of the coset $(U(2) \times U(2))/U(2)$ form a basis for the Lie algebra of $U(2)$. If we use a Hermitian basis for the generators, such as

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{58}$$

We can express the pion fields as a vector in the index notation, or as a matrix

$$\Pi = \pi^i T_i = \begin{pmatrix} \pi^1 + \pi^2 & \pi^3 + i\pi^4 \\ \pi^3 - i\pi^4 & \pi^1 - \pi^2 \end{pmatrix} \tag{59}$$

The pion mass-squared term which appears in the chiral Lagrangian is $\mathrm{Tr}(m_q \Pi^2)$, where in this case $m_q = \mathrm{diag}(m_1, m_2)$. We can evaluate $\mathrm{Tr}(m_q \Pi^2)$ as a matrix expression or as a vector expression in terms of the generators T_i. The matrix calculation proceeds as follows

$$\mathrm{Tr}(m_q \Pi^2) = \tag{60}$$

$$\mathrm{Tr} \begin{pmatrix} m_1((\pi^1 + \pi^2)^2 + (\pi^3 + i\pi^4)(\pi^3 - i\pi^4)) & m_1((\pi^1 + \pi^2)(\pi^3 + i\pi^4) + (\pi^3 + i\pi^4)(\pi^1 - \pi^2) \\ m_2((\pi^3 - i\pi^4)(\pi^1 + \pi^2) = (\pi^1 - \pi^2)(\pi^3 - i\pi^4)) & m_2((\pi^3 - i\pi^4)(\pi^3 + i\pi^4) + (\pi^1 - \pi^2)^2)) \end{pmatrix}$$

$$= m_1((\pi^1)^2 + (\pi^2)^2 + 2\pi^1\pi^2 + (\pi^3)^2 + (\pi^4)^2) + m_2((\pi^1)^2 + (\pi^2)^2 - 2\pi^1\pi^2 + (\pi^3)^2 + (\pi^4)^2)$$

By inspection we have the mass-squared values for two pions, $M^2_{\pi^3} = M^2_{\pi^4} = (m_1 + m_2)$. while the remaining mass-squared term for the other two pions contain terms which mix fields, like the pions of our magnetic theory. We can un-mix these terms by the same diagonalisation procedure as before.

Isolating the terms only in π^1 and π^2 we have

$$(m_1 + m_2)(\pi^1)^2 + (m_1 + m_2)(\pi^2)^2 + 2(m_1 - m_2)\pi^1\pi^2 \tag{61}$$

We can re-write this as a matrix expression

$$M^2_{\alpha\beta}\pi^\alpha\pi^\beta \tag{62}$$

Where $\alpha, \beta = 1, 2$. And $M^2_{\alpha\beta}$ has the entries

$$M^2_{\alpha\beta} = \begin{pmatrix} m_1 + m_2 & m_1 - m_2 \\ m_1 - m_2 & m_1 + m_2 \end{pmatrix} \tag{63}$$

It is clear that we can derive the masses of the pions π^1 and π^2 by diagonalising the matrix $M^2_{\alpha\beta}$, which we find has eigenvalues $M^2_{\pi^1,\pi^2} = (m_1 + m_2) \pm (m_1 - m_2)$. Therefore our pion spectrum is given by

M^2_π	Degeneracy
$(m_1 + m_2)$	2
$2(m_1)$	1
$2(m_2)$	1

Now let's evaluate the mass term (60) in terms of the generators of $U(2)$ contracted with factors of the pion fields as the components of vectors. First note that we decompose m_q in terms of a singlet and a $U(2)$ generator.

$$m_q = \sqrt{2}(\frac{m_1 + m_2}{2})(\mathbf{1}) + (\frac{m_1 - m_2}{2})\,\mathrm{diag}(1, -1) \tag{64}$$

There are two important points to clarify in explicit detail here. First is that **1** does not denote the 2×2 unit matrix, but is simply the positive unit integer 1. We use this for the singlet as opposed to the unit matrix, as the singlet necessarily means a 1-dimensional representation, and while unit matrix multiplied by a scalar variable is a 1-dimensional term, it acts on a representation space that has dimension greater than one, which is morally unsatisfactory. Secondly, the factor of $\sqrt{2}$ multiplies the singlet term so that this decomposition of m_q preserves $\mathrm{Tr}(m_q^2) = (m_1^2 + m_2^2)$, which is required to be basis independent. These formal implications of decomposing m_q into a direct sum of a singlet and multiplet representations is crucial to understanding the mysterious factor of 2 in (54).

Evaluating (60)

$$\mathrm{Tr}(m_q \Pi^2) = \sqrt{2}(m_1 + m_2)\pi^i \pi^j \delta_{ij} + i(m_1 - m_2)\pi^a \pi^b \epsilon_{3ab} \tag{65}$$

When we diagonalise the second portion of this sum by the established procedure, we again get the eigenvalues $\Delta M^2 = \pm(m_1 - m_2)$. However, as in (54), we have a factor multiplying the first portion which would naively appear to prevent the necessary cancellations between $(m_1 + m_2)$ and $(m_1 - m_2)$ to yield the mass spectrum we derived from the matrix calculation. We can be sure the spectrum of the matrix calculation is correct, as it is basis invariant, while the factor difference between the terms in (65) is not. The matrix calculated spectrum is also supported by the phenomenological sanity check, if we recall that the pions are bound-states of the quarks, and should therefore have masses which are combinatorically related.

End of aside.

The other three eigenvalue are obtained by diagonalising the matrix

$$\begin{pmatrix} m' & m'^1_{\tilde{1}} & m'^2_{\tilde{2}} \\ m'^1_{\tilde{1}} & m' & m'^3_{\tilde{3}} \\ m'^2_{\tilde{2}} & m'^3_{\tilde{3}} & m' \end{pmatrix} \tag{66}$$

The eigenvalues of the remaining three NG bosons are therefore $M_\pi^2 = 2a(m' + \Delta_{1,2,3})$, where $\Delta_{1,2,3}$ are the three roots of the cubic equation

$$\Delta^3 - \Delta((m'^1_{\tilde{1}})^2 + (m'^2_{\tilde{2}})^2 + (m'^3_{\tilde{3}})^2) - 2m'^1_{\tilde{1}}\tilde{m}'^2_{\tilde{2}}\tilde{m}'^3_{\tilde{3}} = 0\,. \tag{67}$$

We now review in detail the solution of this cubic equation and the resulting masses of the pions in various special cases.

The nine massess of the NG bosons obtained by S-duality using the $SO(3) \times SO(3)$ language match exactly the massess obtained by the chiral lagrangian using the $SO(4)$ language, upon the identification $a = c$. Thus the radius of the fuzzy sphere is identified with the value of the quark condensate.

In section 4 we derived the pion mass term of the magnetic theory under Sugimoto's S-duality. We stated that six of the pions have masses $M_{ij}^2 = a(m_i + m_j)$ where $i \neq j$, and that the remaining three pions have masses $2a(m' + \Delta_{1,2,3})$, where $\Delta_{1,2,3}$ are the roots of the following polynomial.

$$\Delta^3 - \Delta((m'^1_{\tilde{1}})^2 + (m'^2_{\tilde{2}})^2 + (m'^3_{\tilde{3}})^2) - 2m'^1_{\tilde{1}}m'^2_{\tilde{2}}m'^3_{\tilde{3}} = 0 \tag{68}$$

This is a depressed cubic equation, which has a set of known solution methods. The method we employ here is Vieta's substitution, which proceeds as follows.

For the general depressed cubic

$$t^3 + pt + q = 0 \tag{69}$$

We make the substitution $t = w - \frac{p}{3w}$, which transforms (69) to the form

$$(w^3)^2 + q(w^3) - \frac{p^3}{27} = 0 \tag{70}$$

We can solve this quadratic by the standard formula. For W, any non-zero root of the quadratic (70), let w_1, w_2, w_3 be the cube-roots. The roots of the initial cubic (69) are then $t_{1,2,3} = w_{1,2,3} - \frac{p}{3w_{1,2,3}}$.

Applying Vieta's substitution to (68) yields, firstly, the following quadratic

$$\Omega^2 - \Omega(2m'^1_{\tilde{1}}m'^2_{\tilde{2}}m'^3_{\tilde{3}}) + \frac{((m'^1_{\tilde{1}})^2 + (m'^2_{\tilde{2}})^2 + (m'^3_{\tilde{3}})^2)^3}{27} = 0 \tag{71}$$

Which has a root

$$\Omega = m'^1_{\tilde{1}}m'^2_{\tilde{2}}m'^3_{\tilde{3}} + \sqrt{(m'^1_{\tilde{1}}m'^2_{\tilde{2}}m'^3_{\tilde{3}})^2 - \frac{((m'^1_{\tilde{1}})^2 + (m'^2_{\tilde{2}})^2 + (m'^3_{\tilde{3}})^2)^3}{27}} \tag{72}$$

Our aim for this paper is, of course, the extraction of GMOR-like relations, which requires that we express our pion masses in terms of the electric-theory quark masses. In section 4 we stated $m'^1_{\tilde{1}}, m'^2_{\tilde{2}}, m'^3_{\tilde{3}}$ in terms of the quark masses, and repeat here for convenience.

$$\begin{aligned} m'^1_{\tilde{1}} &= \frac{m_1 + m_2 - m_3 - m_4}{4} \\ m'^2_{\tilde{2}} &= \frac{m_1 + m_4 - m_2 - m_3}{4} \\ m'^3_{\tilde{3}} &= \frac{m_1 + m_3 - m_2 - m_4}{4} \end{aligned} \tag{73}$$

When we expand the factors of $m'^1_{\tilde{1}}, m'^2_{\tilde{2}}, m'^3_{\tilde{3}}$ in (72) in terms of m_1, m_2, m_3, m_4, we derive a pair of very large polynomials, which we shall label $Q(m_q), P(m_q)$. For convenience we express the quadratic root Ω in terms of these polynomials.

$$\Omega = \frac{Q(m_q)}{64} + \frac{1}{48\sqrt{3}}\sqrt{P(m_q)} \tag{74}$$

$P(m_q)$ is related to the discriminant of the depressed cubic by a real, negative factor. It is known for cubic polynomials that a positive discriminant implies that the equation has three real, distinct roots. It can be shown that the polynomial $P(m_q)$ is non-positive for any choice of the quark masses, and therefore the discriminant is non-negative. There are specific cases of quark mass degeneracy which yield a discriminant of zero, and the effect of this in the magnetic theory pion masses will be explored in example calculations. Excluding these special cases however, we are assured that we will always have real pion masses.

To be complete, we provide the full expressions of $Q(m_q)$ and $P(m_q)$.

$$
\begin{gathered}
Q(m_q) = (m_1)^3 + (m_2)^3 + (m_3)^3 + (m_4)^3 \\
-m_1(m_2)^2 - m_1(m_3)^2 - m_1(m_4)^2 \\
-m_2(m_1)^2 - m_2(m_3)^2 - m_2(m_4)^2 \\
-m_3(m_1)^2 - m_3(m_2)^2 - m_3(m_4)^2 \\
-m_4(m_1)^2 - m_4(m_2)^2 - m_4(m_3)^2 \\
+2m_1m_2m_3 + 2m_1m_2m_4 + 2m_1m_3m_4 + 2m_2m_3m_4
\end{gathered}
$$

$$
\begin{gathered}
P(m_q) = \\
-(9(m_1)^4(m_2)^2 - 9(m_1)^4(m_2)(m_3) - 9(m_1)^4(m_2)(m_4) + 9(m_1)^4(m_3)^2 - 9(m_1)^4(m_3)(m_4) \\
+9(m_1)^4(m_4)^2 - 14(m_1)^3(m_2)^3 + 3(m_1)^3(m_2)^2(m_3) + 3(m_1)^3(m_2)^2(m_4) + 3(m_1)^3(m_2)(m_3)^2 \\
+24(m_1)^3(m_2)(m_3)(m_4) + 3(m_1)^3(m_2)(m_4)^2 - 14(m_1)^3(m_3)^3 + 3(m_1)^3(m_3)^2(m_4) \\
+3(m_1)^3(m_3)(m_4)^2 - 14(m_1)^3(m_4)^3 + 9(m_1)^2(m_4)^4 + 3(m_1)^2(m_2)^3(m_3) + 3(m_1)^2(m_2)^3(m_4) \\
-3(m_1)^2(m_2)^2(m_3)^2 - 12(m_1)^2(m_2)^2(m_3)(m_4) - 3(m_1)^2(m_2)^2(m_4)^2 + 3(m_1)^2(m_2)(m_3)^3 \\
-12(m_1)^2(m_2)(m_3)^2(m_4) - 12(m_1)^2(m_2)(m_3)(m_4)^2 + 3(m_1)^2(m_2)(m_4)^4 + 9(m_1)^2(m_3)^4 \\
+3(m_1)^2(m_3)^3(m_4) - 3(m_1)^2(m_3)^2(m_4)^2 + 3(m_1)^2(m_3)(m_4)^3 + 9(m_1)^1(m_4)^4 - 9(m_1)(m_2)^4(m_3) \\
-9(m_1)(m_2)^4(m_4) + 3(m_1)(m_2)^3(m_3)^2 + 24(m_1)(m_2)^3(m_3)(m_4) + 3(m_1)(m_2)^3(m_4)^2 \\
+3(m_1)(m_2)^2(m_3)^3 - 12(m_1)(m_2)^2(m_3)^2(m_4) - 12(m_1)(m_2)^2(m_3)(m_4)^2 + 3(m_1)(m_2)^2(m_4)^3 \\
-9(m_1)(m_2)(m_3)^4 + 24(m_1)(m_2)(m_3)^3(m_4) - 12(m_1)(m_2)(m_3)^2(m_4)^2 + 24(m_1)(m_2)(m_3)(m_4)^3 \\
-9(m_1)(m_2)(m_4)^4 - 9(m_1)(m_3)^4(m_4) + 3(m_1)(m_3)^3(m_4)^2 + 3(m_1)(m_3)^2(m_4)^3 \\
-9(m_1)(m_3)(m_4)^4 + 9(m_2)^4(m_3)^2 + 9(m_2)^4(m_3)(m_4) + 9(m_2)^4(m_4)^2 - 14(m_2)^3(m_3)^3 \\
+3(m_2)^3(m_3)^2(m_4) + 3(m_2)^3(m_3)(m_4)^2 - 14(m_2)^3(m_4)^3 + 9(m_2)^2(m_3)^4 + 3(m_2)^2(m_3)^3(m_4) \\
-3(m_2)^2(m_3)^2(m_4)^2 + 3(m_2)^2(m_3)(m_4)^3 + 9(m_2)^2(m_4)^4 - 9(m_2)(m_3)^4(m_4) + 3(m_2)(m_3)^3(m_4)^2 \\
+3(m_2)(m_3)^2(m_4)^3 - 9(m_2)(m_3)(m_4)^4 + 9(m_3)^4(m_4)^2 - 14(m_3)^3(m_4)^3 + 9(m_3)^2(m_4)^4)
\end{gathered}
$$

Note that in the most general case (where $m_1 \neq m_2 \neq m_3 \neq m_4$), $Q(m_q)$ and $P(m_q)$ cannot be factorized such that the cube-roots of Ω and the roots of (68) can be expressed generally and explicitly in linear terms of the quark masses. Therefore, in order to calculate the resulting pion masses explicitly it is necessary to fix the degeneracy of the quark masses a priori.

In terms of the polynomials $Q(m_q), P(m_q)$, the roots of (68) are given as follows.

$$
\begin{gathered}
\Delta_{1,2,3} = \sqrt[3]{\frac{Q(m_q)}{64} + \frac{1}{48\sqrt{3}}\sqrt{P(m_q)}} \\
+\frac{3(m_1)^2+3(m_2)^2+3(m_3)^2+3(m_4)^2-2m_1m_2-2m_1m_3-2m_1m_4-2m_2m_3-2m_2m_4-2m_3m_4}{48\sqrt[3]{\frac{Q(m_q)}{64}+\frac{1}{48\sqrt{3}}\sqrt{P(m_q)}}}
\end{gathered}
$$

Special Degeneracy Cases

($m_1 = m_2 = m_3 = m_4$):

For the case of full quark mass degeneracy, we see immediately that m'^1_1, m'^2_2, m'^3_3 all vanish. Therefore $Q(m_q)$ and $P(m_q)$ (equivalent to the discriminant of (68)) also vanish, giving trivial roots for (68), which means the three non-trivial pions receive no shift from m'. To state it explicitly

$$\Delta_{1,2,3} = 0$$

M_π^2	Degeneracy
$2am_1$	9

($m_1 \neq m_2$), ($m_2 = m_3 = m_4$):

$$m'^{1}_{\tilde{1}} = \frac{1}{4}(m_1 - m_2)$$
$$m'^{2}_{\tilde{2}} = \frac{1}{4}(m_1 - m_2)$$
$$m'^{3}_{\tilde{3}} = \frac{1}{4}(m_1 - m_2)$$

This reduces (72) to

$$\Omega = \frac{1}{64}(m_1 - m_2)^3 \tag{75}$$

We see again that $P(m_q)$ has vanished, hence the discriminant of (68) is also zero in this case.

(75) has cube-roots

$$\sqrt[3]{\Omega} = \omega_{1,2,3} = \frac{1}{4}(m_1 - m_2),\ \frac{\sqrt{3}+i}{8}(m_1 - m_2),\ \frac{\sqrt{3}-i}{8}(m_1 - m_2) \tag{76}$$

As stated previously, the cube-roots $\omega_{1,2,3}$ relate to $\Delta_{1,2,3}$ as follows

$$\Delta_{1,2,3} = \omega_{1,2,3} + \frac{(m'^{1}_{\tilde{1}})^2 + (m'^{2}_{\tilde{2}})^2 + (m'^{3}_{\tilde{3}})^2}{3\omega_{1,2,3}} \tag{77}$$

Evaluating this with the roots (76), we find the following values for the mass shifts

$$\begin{aligned}
\Delta_1 &= \tfrac{1}{4}(m_1 - m_2) + \tfrac{3(\frac{1}{4}(m_1-m_2))^2}{3(\frac{1}{4}(m_1-m_2))} = \tfrac{1}{2}(m_1 - m_2) \\
\Delta_2 &= \tfrac{\sqrt{3}+i}{8}(m_1 - m_2) + \tfrac{3(\frac{1}{4}(m_1-m_2))^2}{3(\frac{1}{4}(m_1-m_2))(\frac{\sqrt{3}+i}{2})} \\
&= \tfrac{\sqrt{3}+i}{8}(m_1 - m_2) + \tfrac{\sqrt{3}-i}{8}(m_1 - m_2) \\
&= \tfrac{\sqrt{3}}{4}(m_1 - m_2) \\
\Delta_3 &= \tfrac{\sqrt{3}-i}{8}(m_1 - m_2) + \tfrac{3(\frac{1}{4}(m_1-m_2))^2}{3(\frac{1}{4}(m_1-m_2))(\frac{\sqrt{3}-i}{2})} \\
&= \tfrac{\sqrt{3}-i}{8}(m_1 - m_2) + \tfrac{\sqrt{3}+i}{8}(m_1 - m_2) \\
&= \tfrac{\sqrt{3}}{4}(m_1 - m_2)
\end{aligned}$$

Summarily

$$\Delta_1 = \frac{1}{2}(m_1 - m_2),\ \Delta_2 = \Delta_3 = \frac{\sqrt{3}}{4}(m_1 - m_2) \tag{78}$$

Note that while these masses at first look dubious, they are consistent with the result of the previous degeneracy case. If one takes ($m_1 = m2$), above pion masses reduce appropriately to $2am_1$ with a degeneracy of 9.

M_π^2	Degeneracy
$a(m_1+m_2)$	3
$2am_2$	3
$a(\frac{3m_1+m_2}{2})$	1
$a\frac{(1+\sqrt{3})m_1+(3-\sqrt{3})m_2}{2}$	2

$(m_1 = m_2) \neq (m_3 = m_4)$**:**

This degeneracy yields the following

$$m'^{1}_{\tilde{1}} = \frac{1}{2}(m_1 - m_4),\ m'^{2}_{\tilde{2}} = m'^{3}_{\tilde{3}} = 0$$

Substituting into (68) we get

$$\Delta^3 - \frac{(m_1 - m_4)^2}{4}\Delta = 0 \tag{79}$$

Which by inspection has the solution

$$\Delta_1 = 0,\ \Delta_2 = \frac{1}{2}(m_1 - m_4),\ \Delta_3 = \frac{1}{2}(m_4 - m_1) \tag{80}$$

This yields the pion spectrum

M_π^2	Degeneracy
$a(m_1+m_4)$	5
$2am_4$	2
$2am_1$	2

Chapter 6. Summary

Now that our procedure and results have been presented fully, it is appropriate to briefly summarise our project from a point of retrospect. We began by introducing the primary theme of this work, strong-weak duality, and the phenomenological and methodological motivations for developing a more holistic picture of S-duality, giving particular development to cases in which we are not strictly dependent on the presence of supersymmetry to obtain S-dual theories. We then went on to briefly detail the S-duality for a tree-level massless, non-SUSY QCD-like theory, proposed by Sugimoto in [1]. In the introduction we discussed the dual-Meissner scenario of confinement, which relates the dynamical symmetry breaking of an S-dual theory to the confinement of its S-dual partner, and Sugimoto notes in the conclusion to his paper that his proposed S-duality is consistent with this picture of confinement, in that the electric theory has been previously conjectured to be confining, and he strongly motivates the proposition that the magnetic dual theory spontaneously breaks global symmetry.

Our aim in this project has been to realise Sugimoto's duality in a massive case, which we achieve by coupling the parent string theories to a three-form flux background, thereby adding a fermionic mass-term to the electric theory, while also adding a scalar interaction term to the magnetic theory. From this procedure we derived our main result, a spectrum of GMOR-like relations between light meson masses in the magnetic theory, and the quark masses of the electric theory. While this result does not make direct contact with questions about the confinement of the electric theory, or the exactness of the $SL(2,\mathbb{Z})$ symmetry of type IIB string theory (which generates Sugimoto's duality), it does quantitatively show that Sugimoto's magnetic dual to the QCD-like magnetic theory allows us to capture the low-energy physics of the electric theory in a way that is demonstrably consistent with known results about the low-energy physics of QCD. This does not prove, but does meaningfully support Sugimoto's proposal, and by extension, the dual-Meissner mechanism of confinement.

Beyond our works' relevance as evidence to support Sugimoto's proposed duality, it also has potential to be significant to the broader literature concerning non-SUSY S-duality. In [4], Hook and Torroba look to extend the procedure proposed by Uranga, and applied by Sugimoto, to construct a large class of non-SUSY, S-dual pairs of gauge theories. The three-form flux coupling procedure we employ to realise Sugimoto's duality for a massive theory could be plausibly be applied more generally to extend this class of S-dual theories by including controllable fermionic mass deformations and related scalar interaction terms.

In addition to our results there open question that we purposefully omitted from our consideration, as it would constitute a significant addition to the scope of this project's investigation, and that is to identify the η' meson within the magnetic theory. The η' transforms, as with the nine NG bosons, in the coset $U(4)/SO(4)$, which is the full symmetry breaking which occurs, rather than the $SU(4)/SO(4)$ breaking we have examined. In terms of the chiral Lagrangian presented in chapter 3, it is the 'missing component' of the two-index symmetric representation of $SO(4)$, namely the 4×4 unit matrix. The presence of massive W bosons in the magnetic theory suggests a 'hidden local symmetry'. It is tempting to identify the W boson with the ρ-meson. Similar to the discussion in ref.[14] we expect a rich phenomenolgy, in particular the relation $M_W = gv$ automatically translates into $M_\rho^2 = 2g_{\rho\pi\pi}^2 f_\pi^2$.

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