Quasi-free states on algebras of multicomponent commutation relations

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Summary

Let $X = \mathbb{R}^2$ and let V be a finite-dimensional complex inner product space. Let $C : X^2 \to \mathcal{L}(V^{\otimes 2})$ be a continuous function such that, for each $(x, y) \in X^2$, $C(x, y)$ is a unitary operator in $V^{\otimes 2}$, $C^*(x, y) = C(y, x)$, and the functional Yang-Baxter equation is satisfied. The dissertation deals with the multicomponent commutation relations governed by the function C, see [A. Liguori, M. Mintchev, *Comm. Math. Phys.* **169** (1995) 635–652]. We introduce the $*$-algebra of the C-multicomponent commutation relations (C-MCR algebra). We propose definitions of a gauge-invariant quasi-free state and of a strongly quasi-free state on the C-MCR algebra, $\mathbb{A}$. Under restrictive assumptions on the function C, we construct a class of gauge-invariant quasi-free states on $\mathbb{A}$, which, for some functions C, are also strongly quasi-free. We show that, when $\dim V = 1$ (i.e., when we deal with the anyon commutation relations), among all gauge-invariant quasi-free states on $\mathbb{A}$, only the Fock state is strongly quasi-free. In the case $\dim V = 2$ (i.e., when we deal with two-component systems), we present a non-trivial class of examples of function C to which our theory is applicable, and hence, we can construct gauge-invariant quasi-free states, or even strongly quasi-free states on $\mathbb{A}$.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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STATEMENT 1

This dissertation is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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STATEMENT 2

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Contents

Chapter 1

Introduction

Multicomponent commutations relations (MCR) describe *plektons*, i.e., multicomponent quantum systems with a generalized statistics. In such systems, particle exchange is governed by a unitary matrix that depends on the position of particles. For such an exchange to be possible, the matrix must satisfy several conditions, including the functional Yang–Baxter equation.

The aim of the dissertation is to give an appropriate definition of a quasi-free state on an MCR algebra and to construct a class of such states on some MCR algebras.

Let us first recall the Araki-Woods [1] and Araki-Wyss [2] construction of gauge-invariant quasi-free states on the algebras of the canonical commutation relations (CCR) and the algebra of the canonical anticommutation relation (CAR), see also [8, Section 5.2].

Let $\mathcal{H}$ be a complex separable Hilbert space with an antilinear involution J (typically the complex conjugation in a complex L^2-space $\mathcal{H}$). Let $a^+(f)$, $a^-(f)$ ($f \in \mathcal{H}$) be linear operators in a complex separable Hilbert space $\mathfrak{F}$ defined on a dense subset $\mathfrak{D} \subset \mathfrak{F}$, and mapping $\mathfrak{D}$ into itself. (In the case of the CAR, the operators $a^+(f)$, $a^-(f)$ are actually bounded, hence $\mathfrak{D} = \mathfrak{F}$). Assume that the maps $\mathcal{H} \ni f \mapsto a^+(f)$, $\mathcal{H} \ni f \mapsto a^-(f)$ are linear and $a^-(f) = a^+(Jf)^* \restriction_{\mathfrak{D}}$, i.e., $a^-(f)$ is the restriction to $\mathfrak{D}$ of the adjoint operator of $a^+(Jf)$.

We assume that the operators $a^+(f)$, $a^-(f)$, called creation operators and annihilation operators, respectively, satisfy the commutation relations:

$$\begin{aligned} a^+(f)a^+(g) &= \pm a^+(g)a^+(f), \\ a^-(f)a^-(g) &= \pm a^-(g)a^-(f), \\ a^-(f)a^+(g) &= \pm a^+(g)a^-(f) + (g, Jf)_{\mathcal{H}}. \end{aligned} \tag{1.1}$$

The choice of the plus in (1.1) gives the CCR, describing bosons, and the choice of the minus gives the CAR, describing fermions.

Let $X = \mathbb{R}^p$ with $p \in \mathbb{N}$ (in the dissertation, we actually allow X to be a more general space) and let $\mathcal{H} = L^2(X, dx)$. One introduces creation and annihilation operators at point, $a^+(x)$ and $a^-(x)$ ($x \in X$), by

$$\begin{aligned} a^+(f) &= \int_X f(x)a^+(x)dx, \\ a^-(f) &= \int_X f(x)a^-(x)dx, \quad f \in \mathcal{H}. \end{aligned} \tag{1.2}$$

The $a^+(x)$ and $a^-(x)$ are a kind of 'operator-valued distributions'. In terms of these operators, the commutation relations (1.1) become

$$\begin{aligned} a^+(x)a^+(y) &= \pm a^+(y)a^+(x), \\ a^-(x)a^-(y) &= \pm a^-(y)a^-(x), \\ a^-(x)a^+(y) &= \pm a^+(y)a^-(x) + \delta(x-y). \end{aligned} \tag{1.3}$$

Here

$$\int_{X^2} \delta(x-y)f(x)g(y)dx\,dy = \int_X f(x)g(x)dx.$$

Let $\mathbb{A}$ be the complex $*$-algebra generated by the operators $a^+(f)$, $a^-(f)$ ($f \in \mathcal{H}$), satisfying either the CCR or CAR. Then $\mathbb{A}$ is called the CCR algebra or CAR algebra, respectively.

Define, for $f \in \mathcal{H}$, field (or Segal-type) operators

$$b(f) = a^+(f) + a^-(Jf). \tag{1.4}$$

Note that

$$a^+(f) = \frac{1}{2}\big(b(f) - ib(if)\big),$$

$$a^-(f)=\frac{1}{2}\big(b(Jf)+ib(iJf)\big).$$

Hence, the algebra $\mathbb{A}$ is generated by $b(f)$ ($f\in\mathcal{H}$).

Let τ be a state on $\mathbb{A}$. The state τ is called quasi-free if, for all $n\in\mathbb{N}$,

$$\tau\big(b(f_1)b(f_2)\cdots b(f_{2n-1})\big)=0, \tag{1.5}$$

and

$$\tau\big(b(f_1)b(f_2)\cdots b(f_{2n})\big)=\sum_{\xi}\operatorname{sgn}(\xi)\prod_{\substack{\{i,j\}\in\xi\\ i<j}}\tau\big(b(f_i)b(f_j)\big). \tag{1.6}$$

Here, the summation is over all partitions ξ of the set $\{1,2,\dots,2n\}$ into n two-point sets, $\operatorname{sgn}(\xi)=1$ for bosons and

$$\operatorname{sgn}(\xi)=\prod_{\substack{\{i,j\},\,\{k,l\}\in\xi\\ i<k<j<l}}(-1) \tag{1.7}$$

for fermions. Note that, in (1.7), $\operatorname{sgn}(\xi)$ is just one to the number of crossings in the partition ξ. We note also that, in the case of bosons, one may allow a slightly more general definition of a quasi-free state when the odd moments in (1.1) are not necessarily equal to zero.

A state τ is called gauge-invariant if, for any $q\in\mathbb{C}$, $|q|=1$, the state τ remains invariant under the transformation

$$a^+(f)\mapsto a^+(qf)=qa^+(f),$$
$$a^-(f)\mapsto a^-(\overline{q}f)=\overline{q}a^-(f).$$

This requirement can be written as follows:

$$\tau\big(b(f_1)\cdots b(f_n)\big)=\tau\big(b(qf_1)\cdots b(qf_n)\big)$$

for all n.

Note that, due to (1.1), any state τ is completely characterized by the so-called n-point functions,

$$\mathbf{S}^{(m,n)}(f_1,\dots,f_m,g_1,\dots,g_n)=\tau\big(a^+(f_1)\cdots a^+(f_m)a^-(g_1)\cdots a^-(g_n)\big). \tag{1.8}$$

As easily seen, a state τ is gauge-invariant if and only if $\mathbf{S}^{(m,n)} = 0$ if $m \neq n$. In fact, a state τ is gauge-invariant quasi-free if and only if $\mathbf{S}^{(m,n)} = 0$ if $m \neq n$ and

$$\begin{aligned} \mathbf{S}^{(n,n)}(f_n, \dots, f_1, g_1, \dots, g_n) &= \operatorname{per}\left[\mathbf{S}^{(1,1)}(f_i, g_j)\right]_{i,j=1,\dots,n} \\ &= \sum_{\pi \in S_n} \prod_{i=1}^{n} \mathbf{S}^{(1,1)}(f_i, g_{\pi(i)}) \end{aligned} \tag{1.9}$$

for bosons and

$$\begin{aligned} \mathbf{S}^{(n,n)}(f_n, \dots, f_1, g_1, \dots, g_n) &= \det\left[\mathbf{S}^{(1,1)}(f_i, g_j)\right]_{i,j=1,\dots,n} \\ &= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} \mathbf{S}^{(1,1)}(f_i, g_{\pi(i)}) \end{aligned} \tag{1.10}$$

for fermions. Here S_n is the symmetric group of order n, and for $\pi \in S_n$, $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π.

Gauge-invariant quasi-free states can be easily constructed by 'doubling' the underlying space. Let us consider this construction for fermions. Let $\mathfrak{F} = \mathcal{F}(\mathcal{H} \oplus \mathcal{H})$ be the antisymmetric Fock space over $\mathcal{H} \oplus \mathcal{H}$. For $(f, g) \in \mathcal{H} \oplus \mathcal{H}$, let $a^+(f, g)$ and $a^-(f, g)$ denote the standard creation and annihilation operators in $\mathcal{F}(\mathcal{H} \oplus \mathcal{H})$. Fix an arbitrary operator K in $\mathcal{H}$ satisfying $\mathbf{0} \leq K \leq \mathbf{1}$, and define $K_1 = \sqrt{K}$, $K_2 = \sqrt{1 - K}$. For each $f \in \mathcal{H}$, define operators

$$\begin{aligned} A^+(f) &= a^+(0, K_2 f) + a^-(K_1 f, 0), \\ A^-(f) &= a^-(0, K_2' f) + a^+(K_1' f, 0). \end{aligned} \tag{1.11}$$

Here $K_i' = J K_i J$, the transposed of K_i. The operators $A^+(f)$, $A^-(f)$ ($f \in \mathcal{H}$) satisfy the CAR. Let τ be the vacuum state on the corresponding CAR algebra $\mathbb{A}$. Then τ is a gauge-invariant quasi-free state with

$$\mathbf{S}^{(1,1)}(f, g) = \tau\big(A^+(f) A^-(g)\big) = (Kf, Jg)_{\mathcal{H}}.$$

Furthermore, for the corresponding field operators $B(f) = A^+(f) + A^-(Jf)$, we have

$$\tau\big(B(f)B(g)\big) = (g, f)_{\mathcal{H}} + 2 \operatorname{Re}(Kf, g)_{\mathcal{H}}\,.$$

Let us now briefly discuss generalized statistics. In physics, in the case $X = \mathbb{R}^2$, intermediate statistics have been discussed since Leinass and Myrheim (1977) [20] conjectured their existence. The first mathematically rigorous prediction of intermediate statistics was done by Goldin, Menikoff and Sharp (1980, 1981) [12, 13]. The name *anyon* was given to such statistics by Wilczek (1982) [29, 30].

We also refer to Bożejko, Speicher (1991) [6] for an alternative deformation of the CCR/CAR.

Liguori and Mintchev (1995) [21] and Goldin and Sharp (1996) [15] showed that anyon statistics can be described by certain generalized commutation relations. More precisely, let now $X = \mathbb{R}^2$ and consider a continuous function $Q : X^2 \to \mathbb{C}$ satisfying

$$Q(x, y) = \overline{Q(y, x)}, \quad |Q(x, y)| = 1.$$

Furthermore, we assume (with a slight abuse of of notation) that

$$Q(x, y) = Q(x_1, x_2, y_1, y_2) = Q(x_1, y_1),$$

i.e., the value of the function $Q(x, y)$ is completely determined by the first coordinates x_1 and y_1.

Heuristically, we are interested in creation operators $a^+(x)$ and annihilation operators $a^-(x)$ at points $x \in X$ such that $a^-(x)$ is the adjoint of $a^+(x)$ and these operators satisfy the following Q-anyon commutation relations (Q-ACR):

$$\begin{aligned} a^+(x)a^+(y) &= Q(y, x)a^+(y)a^+(x), \\ a^-(x)a^-(y) &= Q(y, x)a^-(y)a^-(x), \\ a^-(x)a^+(y) &= Q(x, y)a^+(y)a^-(x) + \delta(x - y), \end{aligned}$$

compare with (1.3). Liguori, Mintchev [21] (see also Goldin, Majid [11]) derived a rigorous representation of the Q-ACR in the Fock space of Q-symmetric functions.

More precisely, let as before $\mathcal{H} = L^2(X, dx)$. A function $f^{(n)} : X^n \to \mathbb{C}$ is called Q-symmetric if, for any $i \in \{1, \dots, n-1\}$ and $(x_1, \dots, x_n) \in X^n$,

$$f^{(n)}(x_1, \dots, x_n) = Q(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n).$$

We denote by $\mathcal{H}^{\circledast n}$ the subspace of $\mathcal{H}^{\otimes n}$ that consists of all Q-symmetric functions from $\mathcal{H}^{\otimes n}$. We call $\mathcal{H}^{\circledast n}$ the n-th Q-symmetric tensor power of $\mathcal{H}$. We define the Q-Fock space over $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledast n} n!\,.$$

(The factor $n!$ means that the square of the $\mathcal{F}(\mathcal{H})$-norm of $g^{(n)} \in \mathcal{H}^{\circledast n}$ is equal to the square of the $\mathcal{H}^{\circledast n}$-norm of $g^{(n)}$ times $n!$.) For $f \in \mathcal{H}$, we define a creation operator $a^+(f)$ in $\mathcal{F}(\mathcal{H})$ by

$$a^+(f)g^{(n)} = f \circledast g^{(n)} = P_{n+1}(f \otimes g^{(n)}), \quad g^{(n)} \in \mathcal{H}^{\circledast n}.$$

Here P_{n+1} is the Q-symmetrization in $\mathcal{H}^{\otimes(n+1)}$, i.e., the orthogonal projection of $\mathcal{H}^{\otimes(n+1)}$ onto $\mathcal{H}^{\circledast(n+1)}$. Respectively, we define an annihilation operator $a^-(f)$ in $\mathcal{F}(\mathcal{H})$ by

$$\left(a^-(f)g^{(n)}\right)(x_1, \dots, x_{n-1}) = n \int_X f(y) g^{(n)}(y, x_1, \dots, x_{n-1}) dy, \quad g^{(n)} \in \mathcal{H}^{\circledast n}.$$

Note that $a^+(f)$ and $a^-(f)$ are, generally speaking, unbounded operators and they are defined on the subspace $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ of all finite vectors from $\mathcal{F}(\mathcal{H})$. Then $a^-(f)$ is the adjoint operator of $a^+(Jf)$, restricted to $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$, and the corresponding operator-valued distributions $a^+(x)$, $a^-(x)$ ($x \in X$), defined as in (1.2), satisfy the Q-ACR.

Let $\mathbb{A}$ be the complex $*$-algebra generated by the operators $a^+(f)$, $a^-(f)$ ($f \in \mathcal{H}$), satisfying the Q-ACR. The $\mathbb{A}$ is then called the Q-ACR algebra[1].

Lytvynov [22] constructed a class of gauge-invariant quasi-free states on the Q-ACR algebra $\mathbb{A}$. The main idea of the construction was again a doubling of the space $\mathcal{H}$. Thus, one considers

$$\mathcal{H} \oplus \mathcal{H} = L^2\left(X_1 \sqcup X_2, dx\right),$$

where X_1 and X_2 are two copies of X. Let a function $\mathbf{Q} : (X_1 \sqcup X_2)^2 \to \mathbb{C}$ be defined by

$$\mathbf{Q}(x, y) := \begin{cases} Q(x, y), & \text{if } x, y \in X_1 \text{ or } x, y \in X_2, \\ Q(y, x), & \text{if } x \in X_1,\, y \in X_2 \text{ or } x \in X_2,\, y \in X_1. \end{cases} \tag{1.12}$$

Note that $\mathbf{Q}(x, y) = \overline{\mathbf{Q}(y, x)}$, $|\mathbf{Q}(x, y)| = 1$. One then considers the corresponding Fock space over $L^2\left(X_1 \sqcup X_2, dx\right)$, i.e., the space $\mathcal{F}\left(L^2\left(X_1 \sqcup X_2, dx\right)\right)$ constructed by using the function $\mathbf{Q}$.

[1]To be more precise, the Q-ACR algebra should contain not only products of the operators $a^+(f)$, $a^-(f)$ with $f \in \mathcal{H}$ but also special multiple integrals of $a^+(x)$ and $a^-(x)$ with $x \in X$. See [22] for detail.

Note that in [22], the function $Q(x,y)$ was not assumed to be continuous, in order to include the anyon statistics with

$$Q(x,y) = \begin{cases} q & \text{if } x_1 < y_1, \\ \overline{q} & \text{if } x_1 > y_1, \end{cases} \tag{1.13}$$

where $q \in \mathbb{C}$ is a fixed constant of modulus 1. Hence, one had to postulate a value of $Q(x,y)$ on the diagonal $\{(x,y) \in X^2 \mid x = y\}$.

Assume, for example, $Q(x,x) = -1$. Consider an operator K in $L^2(\mathbb{R}, dx)$ satisfying $\mathbf{0} \le K \le \mathbf{1}$. Let us preserve the notation K for the operator $\mathbf{1} \otimes K$ in

$$\mathcal{H} = L^2\left(\mathbb{R}^2, dx\right) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}).$$

Similarly to the case of fermions, define $K_1 = \sqrt{K}$, $K_2 = \sqrt{1-K}$, and for $f \in \mathcal{H}$ define operators $A^+(f)$, $A^-(f)$ just as in (1.11). These operators act in $\mathcal{F}\left(L^2\left(X_1 \sqcup X_2, dx\right)\right)$ and satisfy the Q-ACR. Furthermore, the vacuum state τ on the corresponding Q-ACR algebra is gauge-invariant quasi-free, in the sense that, for any $\varphi, \psi, f, g \in L^2(\mathbb{R})$, we have

$$\begin{aligned} \mathbf{S}^{(1,1)}(\varphi \otimes f, \psi \otimes g) &= \tau\big(A^+(\varphi \otimes f) A^-(\psi \otimes g)\big) \\ &= \int_{\mathbb{R}} \varphi(x)\psi(x)\, \gamma^{(2)}[f,g](dx), \end{aligned}$$

with

$$\gamma^{(2)}[f,g](dx) = (Kg, Jf)_{L^2(\mathbb{R})}\, dx$$

and

$$\begin{aligned} &\mathbf{S}^{(m,n)} = 0 \quad \text{if } m \neq n, \\ &\mathbf{S}^{(n,n)}(\varphi_n \otimes f_n, \dots, \varphi_1 \otimes f_1, \psi_1 \otimes g_1, \dots, \psi_n \otimes g_n) \\ &\quad = \tau\big(A^+(\varphi_n \otimes f_n) \cdots A^+(\varphi_1 \otimes f_1) A^-(\psi_1 \otimes g_1) \cdots A^-(\psi_n \otimes g_n)\big) \\ &\quad = \sum_{\pi \in S_n} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \varphi_i(x_i) \psi_{\pi(i)}(x_i) \right) Q_\pi(x_1, \dots, x_n) \bigotimes_{i=1}^n \gamma^{(2)}[f_i, g_{\pi(i)}](dx_i), \end{aligned} \tag{1.14}$$

with

$$Q_\pi(x_1, \dots, x_n) = \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} Q(x_i, x_j). \tag{1.15}$$

This dissertation deals with multicomponent quantum systems. Such systems are also called plektons, see e.g [11]. Plektons are generalized statistics (quasiparticles) over an underlying space of dimension 2, that are associated with higher-dimensional (non-abelian) unitary representation of the braid group. More recently, similar statistics have been actively studied in topological quantum computation under the name non-abelian anyons[2], see e.g. [24, 27].

The first paper pointing out the possibility of a multicomponent quantum system was the comment by Menikoff, Sharp and Goldin (1985) [14]. Such systems were rigorously derived and studied by Liguori, Mintchev (1995) [21], and later by Goldin, Majid (2004) [11].

The paper by Daletskii, Kalyuzhny, Lytvynov and Proskurin (2020) [9] gave an overview of multicomponent quantum systems with concrete examples when the number of components of a quantum system is two. That paper actually treated a more general case than the one considered in [21] and [11]. For other results that are related to multicomponent quantum systems, see e.g. Bożejko, Speicher (1994) [7] and Jørgensen, Schmitt, Werner (1995) [17].

Let us briefly explain in more detail what we mean under a multicomponent system. Let $X = \mathbb{R}^2$ and let V be a finite-dimensional complex inner product space, with a real orthonormal basis $\{e_1, \ldots, e_d\}$. We fix a map $C : X^2 \to \mathcal{L}(V^{\otimes 2})$ that satisfies the following conditions:

- $C(x, y) = C(x_1, y_1, x_2, y_2) = C(x_1, y_1)$;
- For each $(x, y) \in X^2$, $C(x, y)$ is a unitary operator in $V^{\otimes 2}$;
- For each $(x, y) \in X^2$, $C^*(x, y) = C(y, x)$;
- The functional Yang–Baxter equation is satisfied point-wise in $V^{\otimes 3}$,

$$C_1(x, y)C_2(x, z)C_1(y, z) = C_2(y, z)C_1(x, z)C_2(x, y).$$

[2]It is actually an interesting open problem to find a direct relation between non-abelian anyons as discussed in topological quantum computation and multicomponent quantum systems as discussed in this dissertation.

Here and below, $C_i(\cdot,\cdot)$ acts in the i-th and $(i+1)$-th components of $V^{\otimes 3}$.

Additionally, in this dissertation we will always assume that the map $C(\cdot,\cdot)$ is continuous.

We are interested in vectors of 'operator-valued distributions'

$$a^+(x) = (a_1^+(x), \dots, a_d^+(x)),$$
$$a^-(x) = (a_1^-(x), \dots, a_d^-(x))$$

that satisfy for, $f \in \mathcal{H} = L^2(X \to V, dx)$,

$$f(x) = \sum_{i=1}^{d} f_i(x) e_i \quad \text{i.e.,} \quad f_i(x) = (f(x), e_i)_V$$
$$a^+(f) = \sum_{i=1}^{d} \int_X f_i(x) a_i^+(x) dx,$$
$$a^-(f) = \sum_{i=1}^{d} \int_X f_i(x) a_i^-(x) dx,$$

with $a^-(f)$ being the (restriction) of the adjoint of $a^+(Jf)$. Here J is the complex conjugation in V, i.e., the antilinear operator in V satisfying $Je_i = e_i$ $(i = 1, \dots, d)$.

Denote, for $u, v \in V$, $\langle u, v\rangle_V = (u, Jv)_V$. Then we will use the heuristic notation

$$a^+(f) = \int_X \langle f(x), a^+(x)\rangle_V \, dx, \quad a^-(f) = \int_X \langle f(x), a^+(x)\rangle_V \, dx.$$

We will similarly write, for a product,

$$a^\pm(f) a^\pm(g) = \int_{X^2} \left\langle f(x) \otimes g(y), a^\pm(x) \otimes a^\pm(y)\right\rangle_{V^{\otimes 2}} dx\, dy.$$

We say that these operators satisfy the C-multicomponent commutation relations (C-MCR) if

$$\int_{X^2} \left\langle f(x) \otimes g(y), a^+(x) \otimes a^+(y)\right\rangle_{V^{\otimes 2}} dx\, dy$$
$$= \int_{X^2} \left\langle C(y,x) f(x) \otimes g(y), a^+(y) \otimes a^+(x)\right\rangle_{V^{\otimes 2}} dx\, dy,$$
$$\int_{X^2} \left\langle f(x) \otimes g(y), a^-(x) \otimes a^-(y)\right\rangle_{V^{\otimes 2}} dx\, dy$$
$$= \int_{X^2} \left\langle \widehat{C}(x,y) f(x) \otimes g(y), a^-(y) \otimes a^-(x)\right\rangle_{V^{\otimes 2}} dx\, dy,$$

$$\int_{X^2} \langle f(x)\otimes g(y), a^-(x)\otimes a^+(y)\rangle_{V^{\otimes 2}}\, dx\, dy = \int_X \langle f(x), g(x)\rangle_V\, dx$$
$$+ \int_{X^2} \big\langle \widetilde{C}(x,y) f(x)\otimes g(y), a^+(y)\otimes a^-(x)\big\rangle_{V^{\otimes 2}}\, dx\, dy. \tag{1.16}$$

The operator-valued integrals on the right hand side of formulas (1.16) are assumed to be well defined. Furthermore,

$$\widehat{C}(x,y) = \mathbb{S}\, C(x,y)\, \mathbb{S},$$

where the antilinear operator $\mathbb{S}$ in $V^{\otimes 2}$ is given by

$$\mathbb{S}(u\otimes v) = (Jv)\otimes(Ju),$$

and $\widetilde{C}(x,y)$ satisfies

$$\big\langle \widetilde{C}(x,y)e_i\otimes e_j, e_k\otimes e_l\big\rangle_{V^{\otimes 2}} = \big\langle C(x,y)e_k\otimes e_i, e_l\otimes e_j\big\rangle_{V^{\otimes 2}}. \tag{1.17}$$

Note that the C-MCR (1.16) can be written in the following shorthand form:

$$\begin{aligned} a^+(x)\otimes a^+(y) &= C(y,x)' a^+(y)\otimes a^+(x),\\ a^-(x)\otimes a^-(y) &= \widehat{C}(x,y)' a^-(y)\otimes a^-(x),\\ a^-(x)\otimes a^+(y) &= \delta(x-y)\,\mathrm{Tr}(\cdot) + \widetilde{C}(x,y)' a^+(y)\otimes a^-(x). \end{aligned} \tag{1.18}$$

Here, for $A\in\mathcal{L}(V^{\otimes 2})$, $A' := JAJ$ is the transposed of A; for $v^{(2)}\in V^{\otimes 2}$, $\mathrm{Tr}(v^{(2)}) := \sum_{k=1}^d (v^{(2)}, e_k\otimes e_k)_{V^{\otimes 2}}$ is the trace[3] of $v^{(2)}$.

Let $\mathbb{A}$ be the complex $*$-algebra generated by the operators $a^+(f)$, $a^-(f)$ ($f\in\mathcal{H}$), satisfying the C-MCR. Then $\mathbb{A}$ is called the C-MCR algebra

Note that, for $V=\mathbb{C}$ and $C(x,y)=Q(x,y)$, we get $\widehat{C}(x,y)=Q(y,x)$ and $\widetilde{C}(x,y) = Q(x,y)$. Hence, in this case, the C-MCR algebra becomes the Q-ACR algebra.

Let us now recall the Fock representation of the C-MCR [21]. We have $\mathcal{H}^{\otimes n} = L^2\big(X^n\to V^{\otimes n}, dx_1\dots dx_n\big)$. We say that a function $f^{(n)}\in\mathcal{H}^{\otimes n}$ is C-symmetric if

$$f^{(n)}(x_1,\dots,x_n) = C_i(x_i,x_{i+1}) f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n).$$

[3]Note that, in the standard way, $V^{\otimes 2}$ can be identified with $\mathcal{L}(V)$, and then $\mathrm{Tr}(v^{(2)})$ becomes the usual trace of the linear operator $v^{(2)}$.

We denote by $\mathcal{H}^{\circledast n}$ the subspace of $\mathcal{H}^{\otimes n}$ that consists of all C-symmetric functions from $\mathcal{H}^{\otimes n}$. We define the C-Fock space over $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledast n} n!\,.$$

For each $f \in \mathcal{H}$, we define a creation operator $a^+(f)$ by

$$a^+(f)g^{(n)} = f \circledast g^{(n)} = P_{n+1}\left(f \otimes g^{(n)}\right) \tag{1.19}$$

and an annihilation operator $a^-(f)$ by

$$\left(a^-(f)g^{(n)}\right)(x_1, \dots, x_{n-1}) = n \int_X \left\langle g^{(n)}(y, x_1, \dots, x_{n-1}), f(y)\right\rangle_V dy. \tag{1.20}$$

In (1.19), P_{n+1} is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}^{\circledast n}$, and in formula (1.20) we used the following notation: for $u_1, \dots, u_n, v \in V$,

$$\langle u_1 \otimes \cdots \otimes u_n, v\rangle_V = \langle u_1, v\rangle_V\, u_2 \otimes \cdots \otimes u_n \in V^{\otimes(n-1)}.$$

The operators $a^+(f)$ and $a^-(f)$ are defined on $\mathcal{F}_{\text{fin}}(\mathcal{H})$, the subspace of $\mathcal{F}(\mathcal{H})$ consisting of all finite sequences. Then $a^-(f)$ is the restriction to $\mathcal{F}_{\text{fin}}(\mathcal{H})$ of the adjoint of the operator $a^+(Jf)$. The operators $a^+(f)$, $a^-(f)$ satisfy the C-MCR.

The aim of the dissertation is to develop basics of a theory of quasi-free states for multicomponent quantum systems. The hope is that the construction of the gauge-invariant quasi-free states for the Q-ACR can be extended to the case of a multicomponent system. We show that this can be indeed achieved, however under very restrictive assumptions on the operator-valued function $C(x, y)$. It should be stressed that these assumptions are essentially necessary for the C-MCR to hold for our non-Fock representation.

So a natural question arises whether there exists any non-trivial class of examples of $C(x, y)$ to which our theory is applicable. We show that the answer to this question is positive: we present a non-trivial class of examples in the case where the dimension of the space V is two, i.e., when the quantum system has two components. More precisely, these C-MCR can be written as follows:

$$a_i^+(x)a_i^+(y) = Q_1(y, x)a_{\varphi(i)}^+(y)a_{\varphi(i)}^+(x),$$

$$
\begin{aligned}
a_i^+(x)a_{\varphi(i)}^+(y) &= Q_2(y,x)a_i^+(y)a_{\varphi(i)}^+(x),\\
a_i^-(x)a_i^-(y) &= Q_1(y,x)a_{\varphi(i)}^-(y)a_{\varphi(i)}^-(x),\\
a_i^-(x)a_{\varphi(i)}^-(y) &= Q_2(y,x)a_i^-(y)a_{\varphi(i)}^-(x),\\
a_i^-(x)a_i^+(y) &= \delta(x-y) + Q_2(x,y)a_{\varphi(i)}^+(y)a_{\varphi(i)}^-(x),\\
a_i^-(x)a_{\varphi(i)}^+(y) &= Q_1(x,y)a_i^+(y)a_{\varphi(i)}^-(x)
\end{aligned}
\tag{1.21}
$$

for $i = 1, 2$. Here, $Q_1(x, y)$ and $Q_2(x, y)$ are continuous functions on X^2 satisfying $Q_i(x, y) = \overline{Q_i(y, x)}$, $|Q_i(x, y)| = 1$, and the permutation $\varphi \in S_2$ is given by $\varphi(1) = 2$, $\varphi(2) = 1$. Note that, under the exchange (1.21), each operator $a_i^\pm(\cdot)$ changes its type to the opposite one, $a_{\varphi(i)}^\pm(\cdot)$.

Another problem that we discuss is related to the very definition of a quasi-free state. In the case of a state on the C-MCR algebra, we derive equations that could seemingly serve as a definition of a quasi-free state. These equations extend formulas (1.5)–(1.6) similarly to how formulas (1.14), (1.15) extend (1.8)–(1.10). It appears, however, that the gauge-invariant quasi-free states on the Q-ACR algebra constructed in [22] do not satisfy these equations. The reason for this is the definition (1.12) of the function $\mathbf{Q}(x, y)$. When one evaluates the n-point functions $\mathbf{S}^{(n,n)}$ for such a system, only the terms with all points from X_1 do not vanish. But for such points, $x, y \in X_1$, we have $\mathbf{Q}(x, y) = Q(x, y)$. Hence, one comes up with formula (1.14) in which the function $Q_\pi(x_1, \dots, x_n)$ is given by (1.15). However, when one evaluates $\tau\big(b(f_1)\cdots b(f_{2n})\big)$ (with $b(f)$ defined as in (1.4)), one has to deal with terms containing points from both parts, X_1 and X_2. Because of this, one is unable to come up with the function $Q_\pi(x_1, \dots, x_n)$, unless $Q(x, y) = Q(y, x)$. But the latter formula means that $Q(x, y)$ is either identically equal to 1 (bosons), or identically equal to -1 (fermions).

Thus, instead of using the term a 'quasi-free state on the C-MCR algebra,' we use the term a 'strongly quasi-free state on the C-MCR algebra' if a counterpart of formulas (1.5), (1.6) holds for a state τ on the C-MCR algebra. As a result, the gauge-invariant quasi-free states constructed on the Q-ACR algebra in [22] are not strongly quasi-free.

We prove that, for a C-MCR algebra, a gauge-invariant quasi-free state is strongly quasi-free if

$$\widetilde{C}(y,x) = C(x,y) \tag{1.22}$$

where $\widetilde{C}(y,x)$ is defined by (1.17).

It appears that the class of the C-MCR (1.21) contains a non-trivial subclass for which condition (1.22) is satisfied. More precisely, assume additionally that $Q_1(x,y) = Q_2(x,y) = Q(x,y)$, then the M-MCR (1.21) become

$$\begin{aligned} a_i^+(x)a_j^+(y) &= Q(y,x)a_{\varphi(i)}^+(y)a_{\varphi(j)}^+(x), \\ a_i^-(x)a_j^-(y) &= Q(y,x)a_{\varphi(i)}^-(y)a_{\varphi(j)}^-(x), \\ a_i^-(x)a_j^+(y) &= \delta(x-y)\delta_{i,j} + Q(x,y)a_{\varphi(i)}^+(y)a_{\varphi(j)}^-(x), \quad i,j \in \{1,2\}. \end{aligned} \tag{1.23}$$

Then (1.22) holds and so a gauge-invariant quasi-free state on the corresponding C-MCR algebra is strongly quasi-free. So, in a sense, such a state has better properties than a gauge-invariant quasi-free state on the Q-ACR algebra.

Let us briefly discuss the structure of the dissertation.

In Chapter 2, we discuss mostly known results related to abelian anyons (generalised statistics) and multicomponent quantum systems. For the reader's convenience, we present most of the key results with complete proofs. Unlike many other available sources, in this chapter we are dealing only with the situation where there is a unitary representation of the symmetric group, hence the corresponding 'symmetrization operator' is an orthogonal projection. This makes our proof sometimes different or easier than those available in the literature.

Chapter 3 deals with quasi-free states on the Q-ACR algebra. Recall that gauge-invariant quasi-free states on such an algebra were constructed in [22]. In order to include the case where Q is given by (1.13), the function Q was allowed in [22] to be discontinuous on the diagonal $\{x_1 = y_1\}$. In the case of gauge-invariant quasi-free representations of the Q-ACR, this made it necessary to *postulate* a (real) value of the function Q on the diagonal.

The aim of Chapter 3 is two-fold. First, we discuss a construction of gauge-invariant quasi-free states on the Q-ACR algebra in the case where the function Q is assumed to be continuous. This allows us to simplify the construction from [22], and make it more similar to the situation with bosons and fermions, see e.g. [8, Sections 5.2.1–5.2.3] and [10, Chapter 17].

Second, by analogy with bosons and fermions, we present a definition of a quasi-free state on the Q-ACR algebra. This definition mimics the behaviour of the Fock state applied to a product of field operators. In the classical setting, each gauge-invariant quasi-free state is automatically quasi-free. However, this is not true for the Q-ACR algebra. Hence, we call such states strongly quasi-free. We show that a gauge-invariant quasi-free state on the Q-ACR algebra is strongly quasi-free only if either the function $Q(x, y)$ is identically equal to 1 or -1 (i.e., for bosons and fermions) or the state is Fock.

The main results of the dissertation are in Chapter 4, which deals with quasi-free states on the C-MCR algebra.

In Section 4.1, we define the C-MCR algebra $\mathbb{A}$ as a complex $*$-algebra. Note that $\mathbb{A}$ contains not only product of operators $a^+(f)$, $a^-(f)$ satisfying the commutation relations (1.16) but also multiple integrals containing $a^+(x)$ and $a^-(x)$. We prove that the algebra $\mathbb{A}$ is of Wick type, i.e., it allows Wick ordering (Proposition 4.5). Also we show in Proposition 4.6 that, under an additional assumption on $C(x, y)$, we can also exchange the order in the product $a^+(x) \otimes a^-(y)$, which is important for construction of quasi-free representations of the C-MCR.

In Section 4.2, we briefly discuss generic states on the C-MCR algebra and introduce a certain continuity of such states.

In Section 4.3, we consider the Fock state on the C-MCR algebra, i.e., the vacuum state τ on the usual Fock representation of the C-MCR. The main result of this section, Corollary 4.16, is the formula for the moments of the field operators for the Fock state τ on the C-MCR algebra $\mathbb{A}$.

In Section 4.4, we present the definition of a strongly quasi-free state on the C-MCR algebra. This definition is inspired by Corollary 4.16. In particular, the Fock state is automatically strongly quasi-free. We also define a gauge-invariant quasi-free state on the C-MCR algebra.

Under very restrictive conditions (Assumtpions 4.21 and 4.22), we construct a class of gauge-invariant quasi-free states on the C-MRC algebra (Theorem 4.25). If additionally the condition (1.22) is satisfied, then such a state is strongly quasi-free (Theorem 4.26).

Finally, in Section 4.5, we consider examples of function $C(x, y)$ satisfying Assumptions 4.21, 4.22 and (1.22). More precisely, we consider two classes of examples. The first class of examples (Subsection 4.5.1) is obtained by a 'lifting' of gauge-invariant quasi-free states on the Q-ACR algebra. It should, however, be noted that such quasi-free states could have been constructed within the framework of the Q-ACR algebras by property modifying the underlying space X. Finally, the second class of examples (Subsection 4.5.2) concerns the C-MCR (1.21) for gauge-invariant quasi-free states and (1.23) for strongly quasi-free states.

Chapter 2

Symmetrization and commutation relations

The key references for this review chapter are [5, 7, 9, 11, 21]. For further reading on the topic, we refer to [12, 13, 14, 15, 16, 17, 18, 19, 28].

2.1 Symmetric group

Definition 2.1. Let $n \in \mathbb{N}$. A bijective mapping $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is called a *permutation*. The set of all such permutations is denoted by S_n.

Definition 2.2. The identity mapping $e : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ given by $e(i) = i$ is called the *identity permutation*.

Remark 2.3. Since a mapping $\pi \in S_n$ is bijective, the inverse mapping π^{-1} exists and $\pi^{-1} \in S_n$.

Remark 2.4. The product of two permutations $\pi, \nu \in S_n$ is defined as the composition of the mappings π and ν, i.e., $\pi\nu = \pi \circ \nu$. It is clear that $\pi\nu \in S_n$ and $\pi\nu \neq \nu\pi$ in general.

Definition 2.5. The set S_n with product of two permutations forms a group with identity element e and inverse of π being π^{-1}. This group is called a *symmetric group*.

Definition 2.6. For $i \in \{1, 2, \ldots, n-1\}$, define $\pi_i \in S_n$ by

$$\pi_i(j) := \begin{cases} j, & \text{if } j \neq i \text{ and } j \neq i+1, \\ i+1, & \text{if } j = i, \\ i, & \text{if } j = i+1. \end{cases}$$

The π_i is called an *adjacent transposition.*

Proposition 2.7. *The adjacent transpositions π_i satisfy the following equations:*

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \quad \text{if } i \in \{1, 2, \ldots, n-2\}, \tag{2.1}$$

$$\pi_i \pi_j = \pi_j \pi_i \quad \text{if } |i-j| \geq 2, \tag{2.2}$$

$$\pi_i^2 = e \quad \text{if } i \in \{1, 2, \ldots, n-1\}. \tag{2.3}$$

Proposition 2.8. *Each permutation $\pi \in S_n$ can be represented as a product of adjacent transpositions, i.e.,*

$$\pi = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} \tag{2.4}$$

for some $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n-1\}$.

Remark 2.9. The representation (2.4) of a permutation as a product of adjacent transpositions is not unique.

Remark 2.10. Equation (2.4) shows that the group S_n is generated by the adjacent transpositions $\pi_1, \pi_2, \ldots, \pi_{n-1}$.

Theorem 2.11 (The Coxeter representation of the symmetric group)**.** *The symmetric group S_n is isomorphic to an abstract group generated by elements $\pi_1, \pi_2, \ldots, \pi_{n-1}$ that satisfy equations* (2.1)*,* (2.2)*,* (2.3) *with e being the identity element of the group.*

2.2 Symmetric functions and symmetrization operators

2.2.1 The projection P_n

Let $n \in \mathbb{N}$, $n \geq 2$. Let H be a separable complex Hilbert space. Let $U_1, U_2, \dots, U_n$ be unitary operators acting on H that satisfy the following equations:

$$U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1} \quad \text{if } i \in \{1, 2, \dots, n-1\}, \tag{2.5}$$

$$U_i U_j = U_j U_i \quad \text{if } |i - j| \geq 2, \tag{2.6}$$

$$U_i^2 = \mathbf{1} \quad \text{if } i \in \{1, 2, \dots, n\}. \tag{2.7}$$

Here $\mathbf{1}$ denotes the identity operator. The equation (2.5) is called Yang–Baxter equation.

Remark 2.12. Since U_i is unitary operator, we get $U_i^{-1} = U_i^*$. On the other hand, formula (2.7) implies that $U_i^{-1} = U_i$. Therefore $U_i = U_i^*$, i.e., each U_i is also a self-adjoint operator. Since the spectrum of a self-adjoint operator is a subset of $\mathbb{R}$ and the spectrum of a unitary operator is a subset of the circle in $\mathbb{C}$ centered at 0 and of radius 1, the spectrum of each U_i is a subset of $\{-1, 1\}$. The latter implies that, for each $i = 1, 2, \dots, n$, the Hilbert space H can be represented as an orthogonal sum of closed subspaces $H_i^{(1)}$ and $H_i^{(2)}$, i.e., $H = H_i^{(1)} \oplus H_i^{(2)}$, and the operator U_i acts as the identity on $H_i^{(1)}$ and as the minus identity on $H_i^{(2)}$.

Definition 2.13. Let $\pi \in S_n$ be an arbitrary permutation. Represent π in the form (2.4) and define a unitary operator U_π by

$$U_\pi = U_{i_1} U_{i_2} \cdots U_{i_k}. \tag{2.8}$$

Proposition 2.14.

(i) For each $\pi \in S_n$ the definition (2.8) *of U_π does not depend on the choice of the representation* (2.4)*, i.e., two different representations of π in the form of a product of adjacent transpositions give the same operator U_π.*

(ii) The map

$$S_n \ni \pi \mapsto U_\pi \tag{2.9}$$

is a unitary representation of the group S_n.

Proof. Consider the set G of all finite products of operators $U_1, \dots, U_n$ in any order. By (2.7), we have $\mathbf{1} \in G$.

Again, by (2.7), $U_i^{-1} = U_i$ for all i. Hence, for any $i_1, \dots, i_k \in \{1, \dots, n\}$,

$$(U_{i_1} U_{i_2} \cdots U_{i_k})^{-1} = U_{i_k} \cdots U_{i_2} U_{i_1}.$$

Therefore, G is a group with respect to product of operators, and this group G is generated by $U_1, \dots, U_n$. By (2.5)–(2.7) and Theorem 2.11, the group G is isomorphic to S_n, where the isomorphism $I : G \to S_n$ is defined through the equality $IU_i = \pi_i$. From here both statements of the proposition follow. □

Remark 2.15. For the unitary representation (2.9) of S_n to be faithful, one should have $U_\pi \neq \mathbf{1}$ for each permutation $\pi \in S_n$ that is not the identity permutation. Obviously, this is not always the case. For example, by choosing all operators U_i to be $\mathbf{1}$, we get $U_\pi = \mathbf{1}$ for all $\pi \in S_n$. We advise the reader to compare this observation with the construction of the free Fock space in Section 2.7. Nevertheless, one can easily check that, in all the other examples of operators $U_1, \dots, U_n$ considered in this dissertation, the representation (2.9) is indeed faithful.

Definition 2.16. We define

$$P_n = \frac{1}{n!} \sum_{\pi \in S_n} U_\pi. \tag{2.10}$$

Proposition 2.17. *For each $n \geq 2$, P_n is an orthogonal projection, i.e., $P_n^* = P_n$ and $P_n^2 = P_n$.*

Proof. Let $\pi \in S_n$ be represented in the form (2.4). By Remark 2.12 and (2.8),

$$\begin{aligned} U_\pi^* &= (U_{i_1} U_{i_2} \cdots U_{i_k})^* \\ &= U_{i_k}^* U_{i_{k-1}}^* \cdots U_{i_1}^* \\ &= U_{i_k} U_{i_{k-1}} \cdots U_{i_1} \\ &= U_{\pi^{-1}}, \end{aligned}$$

since

$$\pi^{-1} = \pi_{i_k}\pi_{i_{k-1}}\cdots\pi_{i_1}.$$

Hence, by (2.10),

$$\begin{aligned}
P_n^* &= \frac{1}{n!}\sum_{\pi\in S_n} U_\pi^* \\
&= \frac{1}{n!}\sum_{\pi\in S_n} U_{\pi^{-1}} \\
&= \frac{1}{n!}\sum_{\pi\in S_n} U_\pi = P_n.
\end{aligned}$$

Next,

$$\begin{aligned}
P_n^2 &= \left(\frac{1}{n!}\right)^2 \sum_{\pi\in S_n}\sum_{\nu\in S_n} U_\pi U_\nu \\
&= \left(\frac{1}{n!}\right)^2 \sum_{\pi\in S_n}\sum_{\nu\in S_n} U_{\pi\nu} \\
&= \left(\frac{1}{n!}\right)^2 \sum_{\pi\in S_n}\sum_{\xi\in S_n} U_\xi \\
&= \frac{1}{n!}\sum_{\xi\in S_n} U_\xi = P_n.
\end{aligned}$$

Here we used the fact that, for a fixed $\pi \in S_n$, each $\xi \in S_n$ can be represented, in a unique way, as $\xi = \pi\nu$. □

Proposition 2.18. *For each* $n \geq 2$,

$$\operatorname{ran}(P_n) = \big\{ f \in H \mid U_i f = f \text{ for each } i = 1, 2, \ldots n-1 \big\}.$$

Here $\operatorname{ran}(P_n)$ *denotes the range of* P_n.

Proof. Let $f \in H$ be such that $U_i f = f$ for all $i = 1, 2, \ldots, n-1$. By (2.8), for each $\pi \in S_n$, we have $U_\pi f = f$, hence by (2.10),

$$P_n f = f. \tag{2.11}$$

Since P_n is an orthogonal projection, formula (2.11) means that $f \in \operatorname{ran}(P_n)$.

Now assume that $f \in \operatorname{ran}(P_n)$. We have to prove that, for each $i = 1, 2, \dots, n-1$, $U_i f = f$. Since $f \in \operatorname{ran}(P_n)$, formula (2.11) holds. Hence

$$\begin{aligned} U_i f &= U_i P_n f \\ &= U_i \frac{1}{n!} \sum_{\pi \in S_n} U_\pi f \\ &= \frac{1}{n!} \sum_{\pi \in S_n} U_i U_\pi f \\ &= \frac{1}{n!} \sum_{\xi \in S_n} U_\xi f \\ &= P_n f = f. \end{aligned}$$ □

2.2.2 Tensor product corresponding to a Yang–Baxter operator

Let $\mathcal{H}$ be a separable complex Hilbert space. Let $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be a self-adjoint, unitary operator satisfying the Yang–Baxter equation

$$U_1 U_2 U_1 = U_2 U_1 U_2, \tag{2.12}$$

where the operators $U_1, U_2 \in \mathcal{L}(\mathcal{H}^{\otimes 3})$ are defined by

$$U_1 := U \otimes \mathbf{1}_{\mathcal{H}}, \quad U_2 := \mathbf{1}_{\mathcal{H}} \otimes U.$$

Here and below, for a Hilbert space $\mathcal{G}$, we denote by $\mathbf{1}_{\mathcal{G}}$ the identity operator on $\mathcal{G}$.

Let us fix $n \in \mathbb{N}$, $n \geq 2$. Similarly to the above, we define for each $i = 1, 2, \dots, n-1$, operators $U_i \in \mathcal{L}(\mathcal{H}^{\otimes n})$ by

$$U_i := \mathbf{1}_{\mathcal{H}^{\otimes(i-1)}} \otimes U \otimes \mathbf{1}_{\mathcal{H}^{(n-i-1)}} .$$

The operators U_i are obviously unitary and self-adjoint on $\mathcal{H}^{\otimes n}$. The Yang–Baxter equation (2.12) implies (2.5). Obviously, if $|i-j| \geq 2$, the operators U_i and U_j commute, so that (2.6) holds. Finally, since the operators U_i are unitary and self-adjoint, they satisfy (2.7). Hence, the results of Subsection 2.2.1 are applicable to the operators U_i, and we get the corresponding projection operator P_n.

We denote $\mathcal{H}^{\circledast n}$ the range of P_n. By Proposition 2.18,

$$\mathcal{H}^{\circledast n} = \{f^{(n)} \in \mathcal{H}^{\otimes n} \mid U_i f^{(n)} = f^{(n)} \text{ for each } i = 1, 2, \dots n-1\}.$$

Also, for $f_1, f_2, \dots, f_n \in \mathcal{H}$, we denote

$$f_1 \circledast f_2 \circledast \cdots \circledast f_n = P_n \left[f_1 \otimes f_2 \otimes \cdots \otimes f_n\right].$$

Proposition 2.19. *Let $n \geq 2$, $m \in \mathbb{N}$. Then*

$$P_{m+n} = P_{m+n}\left[P_m \otimes \mathbf{1}_{\mathcal{H}^{\otimes n}}\right] \tag{2.13}$$

$$= P_{m+n}\left[\mathbf{1}_{\mathcal{H}^{\otimes n}} \otimes P_m\right] \tag{2.14}$$

$$= \left[\mathbf{1}_{\mathcal{H}^{\otimes n}} \otimes P_m\right] P_{m+n} \tag{2.15}$$

$$= \left[P_m \otimes \mathbf{1}_{\mathcal{H}^{\otimes n}}\right] P_{m+n}\,. \tag{2.16}$$

Proof. We have

$$P_{m+n}\left[P_m \otimes \mathbf{1}_{\mathcal{H}^{\otimes n}}\right] = \frac{1}{(m+n)!}\sum_{\pi \in S_{m+n}} U_\pi \frac{1}{n!}\sum_{\nu \in S_n} U_\nu \otimes \mathbf{1}_{\mathcal{H}^{\otimes n}}.$$

We have $S_n \subset S_{m+n}$ in the sense that we may identify $\nu \in S_n$ with the element of S_{m+n} that acts as ν for $i = 1, \dots, n$ and as the identity for $i = n+1, n+2, \dots, m+n$. Then,

$$\begin{aligned} P_{m+n}\left[P_m \otimes \mathbf{1}_{H^{\otimes n}}\right] &= \frac{1}{(m+n)!\,n!}\sum_{\pi \in S_{m+n}}\sum_{\nu \in S_n} U_\pi U_\nu \\ &= \frac{1}{(m+n)!\,n!}\sum_{\nu \in S_n}\sum_{\pi \in S_{m+n}} U_{\pi\nu} \\ &= \frac{1}{(m+n)!\,n!}\sum_{\nu \in S_n}\sum_{\pi \in S_{m+n}} U_\pi \\ &= \frac{1}{n!}\sum_{\nu \in S_n} P_{m+n} \\ &= P_{m+n}. \end{aligned}$$

□

Thus, (2.13) is proven. The proof of (2.14), (2.15) and (2.16) is similar.

Corollary 2.20. *The tensor product $\circledast$ is associative, i.e.,*

$$(f \circledast g) \circledast h = f \circledast (g \circledast h)$$

for all $f, g, h \in H$.

Proof. Immediate. □

Proposition 2.21. *We have*

$$P_{n+1} = \frac{1}{n+1} \left(\mathbf{1}_{\mathcal{H}} \otimes P_n\right) \left[\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2 \cdots U_n\right], \tag{2.17}$$

$$P_{n+1} = \frac{1}{n+1} \left[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_nU_{n-1} \cdots U_1\right] \left(\mathbf{1}_{\mathcal{H}} \otimes P_n\right). \tag{2.18}$$

Proof. From formula (2.10),

$$P_{n+1} = \frac{1}{(n+1)!} \sum_{\pi \in S_{n+1}} U_\pi.$$

For each $k \in \{1, 2, \ldots, n+1\}$, define $\nu_k \in S_{n+1}$ by

$$\nu_1 = e, \quad \nu_k = \pi_1\pi_2 \cdots \pi_{k-1} \quad \text{for } k = 2, \ldots, n+1. \tag{2.19}$$

Note that

$$\nu_k(1) = k,\ \nu_k(2) = 1, \ldots, \nu_k(k) = k-1, \text{ and } \nu_k(i) = i \text{ for } i = k+1, \ldots, n+1.$$

Each permutation $\nu \in S_{n+1}$ can be uniquely represented as the product $\nu = \pi\nu_k$ where $\pi \in S_{n+1}$ is such that $\pi(1) = 1$. Therefore,

$$S_{n+1} = \bigcup_{k=1}^{n+1} \left\{\pi\nu_k \mid \pi \in S_{n+1}, \pi(1) = 1\right\},$$

which implies

$$\begin{aligned}
P_{n+1} &= \frac{1}{(n+1)!} \sum_{\nu \in S_{n+1}} U_\nu \\
&= \frac{1}{(n+1)!} \sum_{k=1}^{n+1} \sum_{\pi \in S_{n+1}, \pi(1)=1} U_{\pi\nu_k} \\
&= \frac{1}{(n+1)!} \sum_{k=1}^{n+1} \sum_{\pi \in S_{n+1}, \pi(1)=1} U_\pi U_{\nu_k} \\
&= \frac{1}{n+1} \left(\frac{1}{n!} \sum_{\pi \in S_{n+1}, \pi(1)=1} U_\pi\right) \sum_{k=1}^{n+1} U_{\nu_k} \\
&= \frac{1}{n+1} \left(\mathbf{1}_{\mathcal{H}} \otimes P_n\right) \sum_{k=1}^{n+1} U_{\nu_k}
\end{aligned}$$

$$= \frac{1}{n+1}(\mathbf{1}_{\mathcal{H}} \otimes P_n)\left(\mathbf{1} + \sum_{k=2}^{n+1} U_{\nu_k}\right)$$

$$= \frac{1}{n+1}(\mathbf{1}_{\mathcal{H}} \otimes P_n)\left(\mathbf{1} + \sum_{k=1}^{n} U_{\nu_{k+1}}\right)$$

$$= \frac{1}{n+1}(\mathbf{1}_{\mathcal{H}} \otimes P_n)\left(\mathbf{1} + \sum_{k=1}^{n} U_1 U_2 \cdots U_k\right).$$

This implies (2.17). Formula (2.18) follows immediately from (2.17) by taking the adjoint operator. □

We will now consider several special cases of the tensor product $\circledast$.

2.3 Symmetric tensor product

Let $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be defined as follows

$$Uf \otimes g = g \otimes f. \tag{2.20}$$

Lemma 2.22. *U defined in* (2.20) *is self-adjoint and unitary operator.*

Proof. We have

$$\begin{aligned}(Uf \otimes g, u \otimes v) &= (g \otimes f, u \otimes v)\\ &= (g, u)(f, v)\\ &= (f \otimes g, v \otimes u)\\ &= (f \otimes g, Uu \otimes v).\end{aligned}$$

This proves that U in (2.20) is self-adjoint.

Our aim is to show that U is unitary. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, then $(e_i \otimes e_j)_{i,j\in\mathbb{N}}$ is orthonormal in $\mathcal{H}^{\otimes 2}$. Therefore every $f^{(2)}$ can be written as

$$f^{(2)} = \sum_{i,j\geq 1} f^{(2)}_{ij} e_i \otimes e_j \tag{2.21}$$

and $\left\|f^{(2)}\right\|^2 = \sum_{i,j\geq 1}\left|f^{(2)}_{ij}\right|^2$. Then

$$Uf^{(2)} = \sum_{i,j\geq 1} f^{(2)}_{ij} e_j \otimes e_i$$

$$= \sum_{i,j\geq 1} f_{ji}^{(2)} e_i \otimes e_j,$$

and so

$$\left\|Uf^{(2)}\right\|^2 = \sum_{i,j\geq 1} |f_{ji}^{(2)}|^2 = \left\|f^{(2)}\right\|^2.$$

It is clear that $\ker(U) = \{0\}$. Since U is self-adjoint and the range of U is closed, this implies $\operatorname{ran}(U) = \mathcal{H}^{\otimes 2}$, which proves that U is unitary. □

Lemma 2.23. *U defined in* (2.20) *satisfies the Yang–Baxter equation.*

Proof. We have, for any $f, g, h \in \mathcal{H}$,

$$\begin{aligned} U_1U_2U_1 f \otimes g \otimes h &= U_1U_2 g \otimes f \otimes h \\ &= U_1 g \otimes h \otimes f \\ &= h \otimes g \otimes f. \end{aligned}$$

Similarly,

$$\begin{aligned} U_2U_1U_2 f \otimes g \otimes h &= U_2U_1 f \otimes h \otimes g \\ &= U_2 h \otimes f \otimes g \\ &= h \otimes g \otimes f. \end{aligned}$$

This proves that $U_1U_2U_1 = U_2U_1U_2$. □

Definition 2.24. The corresponding tensor product $\circledast$ in this case is denoted by $\odot$ is called the *symmetric tensor product.* Thus,

$$\mathcal{H}^{\odot n} = \left\{ f^{(n)} \in \mathcal{H}^{\otimes n} \mid U_i f^{(n)} = f^{(n)} \ \forall i = 1, \ldots, n-1 \right\}.$$

Remark 2.25. If $\mathcal{H} = L^2(X, \sigma)$, then

$$\begin{aligned} \mathcal{H}^{\odot n} = \Big\{ f^{(n)} \in L^2\left(X^n, \sigma^{\otimes n}\right) \mid U f^{(n)} &\left(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n\right) \\ &= f^{(n)}\left(x_1, \ldots, x_n\right) \ \forall i = 1, \ldots, n-1 \Big\} \\ = \Big\{ f^{(n)} \in L^2\left(X^n, \sigma^{\otimes n}\right) \mid f^{(n)}&(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f^{(n)}(x_1, \ldots, x_n) \ \forall \pi \in S_n \Big\}. \end{aligned}$$

This implies that, for each $\pi \in S_n$, $f_{\pi(1)} \odot \cdots \odot f_{\pi(n)} = f_1 \odot \cdots \odot f_n$.

2.4 Antisymmetric tensor product

Let $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be defined as follows

$$Uf \otimes g = -g \otimes f. \tag{2.22}$$

The fact that U defined in (2.22) is self-adjoint, unitary and satisfies the Yang–Baxter equation follows immediately from Lemmas 2.22, 2.23.

Definition 2.26. The corresponding tensor product $\circledast$ in this case is denoted by $\wedge$ is called the *antisymmetric tensor product*. Thus,

$$\mathcal{H}^{\wedge n} = \left\{ f^{(n)} \in \mathcal{H}^{\otimes n} \mid U_i f^{(n)} = f^{(n)} \ \forall i = 1, \dots, n-1 \right\}.$$

Remark 2.27. If $\mathcal{H} = L^2(X, \sigma)$, then

$$\begin{aligned}\mathcal{H}^{\wedge n} = \Big\{ f^{(n)} \in L^2\left(X^n, \sigma^{\otimes n}\right) \mid U f^{(n)} &(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \\ &= -f^{(n)}(x_1, \dots, x_n) \ \ \forall i = 1, \dots, n-1 \Big\} \\ = \Big\{ f^{(n)} \in L^2\left(X^n, \sigma^{\otimes n}\right) \mid f^{(n)}&(x_{\pi(1)}, \dots, x_{\pi(n)}) = \operatorname{sgn}\pi\, f^{(n)}(x_1, \dots, x_n) \ \forall \pi \in S_n \Big\}.\end{aligned}$$

This implies that, for each $\pi \in S_n$, $f_{\pi(1)} \wedge \cdots \wedge f_{\pi(n)} = \operatorname{sgn}\pi\, f_1 \wedge \cdots \wedge f_n$.

2.5 Q-symmetric tensor product. Anyons

Let us recall some standard definitions.

Definition 2.28. A separable, completely mertizable topological space X is called a *Polish space*.

Definition 2.29. A topological space X is called *locally compact* if every point $x \in X$ has a compact neighbourhood.

Definition 2.30. Let X be a locally compact Polish space and let $\mathcal{B}(X)$ be the Borel σ-algebra on X. A measure σ on $(X, \mathcal{B}(X))$ is called a *Radon measure* if for each compact set $K \subset X$, $\sigma(K) < \infty$. Furthermore, if for any $x \in X$, $\sigma(\{x\}) = 0$, then σ is called a *non-atomic measure*.

Example 2.31. Let $X = \mathbb{R}^d$, which is a locally compact Polish space, and let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$. Let $\sigma(dx) = dx$ be the Lebesgue measure on $\mathbb{R}^d$. It is well known that the Lebesgue measure is non-atomic.

Let X be a locally compact Polish space and let σ be a non-atomic Radon measure on $(X, \mathcal{B}(X))$.

We define

$$O^{(2)} = \big\{(x, y) \in X^2 \mid x \neq y\big\},$$

i.e., $O^{(2)} = X^2 \setminus D$, where $D = \big\{(x, x) \in X^2 \mid x \in X\big\}$. ($D$ is the set of diagonal elements in X^2 and $O^{(2)}$ is the set of off-diagonal elements in X^2).

Lemma 2.32. *We have $\sigma^{\otimes 2}(D) = 0$, so that $\sigma^{\otimes 2}$ can be considered as a measure on $O^{(2)}$.*

Proof. By Fubini's theorem,

$$\begin{aligned}
\sigma^{\otimes 2}(D) &= \int_{X^2} \chi_D(x, y)\, \sigma(dx)\, \sigma(dy) \\
&= \int_X \int_X \chi_D(x, y)\, \sigma(dy)\, \sigma(dx) \\
&= \int_X \left[\int_X \chi_{\{x\}}(y) \sigma(dy)\right] \sigma(dx) \\
&= \int_X \sigma(\{x\})\, \sigma(dx) \\
&= \int_X 0\, \sigma(dx) = 0.
\end{aligned}$$

□

We similarly define, for $n \geq 3$,

$$O^{(n)} = \big\{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\big\}.$$

By Lemma 2.32, $\sigma^{\otimes n}(X^n \setminus O^{(n)}) = 0$, so that $\sigma^{\otimes n}$ can be considered as a measure on $O^{(n)}$.

For each $n \geq 2$, we fix a symmetric set $X^{(n)} \in \mathcal{B}(X^n)$ such that

$$X^{(n)} \subset O^{(n)} \quad \sigma^{\otimes n}(X^n \setminus X^{(n)}) = 0. \tag{2.23}$$

In particular, one can choose $X^{(n)} = O^{(n)}$.

Letting $\mathcal{H} = L^2(X, \sigma)$ (the L^2-space of complex-valued σ-square integrable functions on X), we have

$$\mathcal{H}^{\otimes n} = L^2(X^{(n)}, \sigma^{\otimes n}).$$

Consider a measurable function $Q : X^{(2)} \to \mathbb{C}$ that satisfies, for all $(x, y) \in X^{(2)}$,

$$Q(x,y) = \overline{Q(y,x)}, \tag{2.24}$$

$$|Q(x,y)| = 1. \tag{2.25}$$

Define $U \in \mathcal{L}\left(L^2\left(X^{(2)}, \sigma^{\otimes 2}\right)\right)$ by

$$\left(Uf^{(2)}\right)(x,y) = Q(x,y) f^{(2)}(y,x). \tag{2.26}$$

Lemma 2.33. *The linear operator* U *defined by* (2.26) *is self-adjoint and unitary.*

Proof. We have, by (2.24),

$$\begin{aligned}
\left(Uf^{(2)}, g^{(2)}\right)_{L^2\left(X^{(2)},\sigma^{\otimes 2}\right)} &= \int_{X^{(2)}} \left(Uf^{(2)}\right)(x,y)\overline{g^{(2)}(x,y)}\,\sigma(dx)\,\sigma(dy) \\
&= \int_{X^{(2)}} Q(x,y) f^{(2)}(y,x)\overline{g^{(2)}(x,y)}\,\sigma(dx)\,\sigma(dy) \\
&= \int_{X^{(2)}} Q(y,x) f^{(2)}(x,y)\overline{g^{(2)}(y,x)}\,\sigma(dy)\,\sigma(dx) \\
&= \int_{X^{(2)}} f^{(2)}(x,y)\overline{Q(x,y) g^{(2)}(y,x)}\,\sigma(dy)\,\sigma(dx) \\
&= \int_{X^{(2)}} f^{(2)}(x,y)\overline{Ug^{(2)}(x,y)}\,\sigma(dy)\,\sigma(dx) \\
&= \left(f^{(2)}, Ug^{(2)}\right)_{L^2\left(X^{(2)},\sigma^{\otimes 2}\right)}.
\end{aligned}$$

Hence, U is self-adjoint. By (2.25),

$$\begin{aligned}
\|Uf^{(2)}\|^2_{L^2\left(X^{(2)},\sigma^{\otimes 2}\right)} &= \int_{X^{(2)}} \left|Q(x,y) f^{(2)}(y,x)\right|^2 \sigma(dx)\,\sigma(dy) \\
&= \int_{X^{(2)}} \left|f^{(2)}(y,x)\right|^2 \sigma(dx)\,\sigma(dy) \\
&= \int_{X^{(2)}} \left|f^{(2)}(x,y)\right|^2 \sigma(dx)\,\sigma(dy) \\
&= \|f^{(2)}\|^2_{L^2\left(X^{(2)},\sigma^{\otimes 2}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\ker(U) &= \left\{f^{(2)} \in L^2\left(X^{(2)}, \sigma^{\otimes 2}\right) \mid Q(x,y) f^{(2)}(y,x) = 0 \ \sigma^{\otimes 2}\text{-a.e.}\right\} \\
&= \left\{f^{(2)} \in L^2\left(X^{(2)}, \sigma^{\otimes 2}\right) \mid f^{(2)}(y,x) = 0 \ \sigma^{\otimes 2}\text{-a.e.}\right\} \\
&= \left\{f^{(2)} \in L^2\left(X^{(2)}, \sigma^{\otimes 2}\right) \mid f^{(2)}(x,y) = 0 \ \sigma^{\otimes 2}\text{-a.e.}\right\} = \{0\}.
\end{aligned}$$

Hence, U is unitary. □

Lemma 2.34. *The operator U defined by* (2.26) *satisfies the Yang–Baxter equation* (2.12).

Proof. Let $f^{(3)} \in L^2\left(X^{(3)}, \sigma^{\otimes 3}\right)$, then

$$\left(U_1 f^{(3)}\right)(x,y,z) = Q(x,y) f^{(3)}(y,x,z).$$
$$\left(U_2 U_1 f^{(3)}\right)(x,y,z) = Q(y,z)Q(x,z) f^{(3)}(z,x,y).$$
$$\left(U_1 U_2 U_1 f^{(3)}\right)(x,y,z) = Q(x,y)Q(x,z)Q(y,z) f^{(3)}(z,y,x).$$

Similarly,

$$\left(U_2 f^{(3)}\right)(x,y,z) = Q(y,z) f^{(3)}(x,z,y).$$
$$\left(U_1 U_2 f^{(3)}\right)(x,y,z) = Q(x,y)Q(x,z) f^{(3)}(y,z,x).$$
$$\left(U_2 U_1 U_2 f^{(3)}\right)(x,y,z) = Q(y,z)Q(x,z)Q(x,y) f^{(3)}(z,y,x).$$

Therefore,

$$\left(U_1 U_2 U_1 f^{(3)}\right)(x,y,z) = \left(U_2 U_1 U_2 f^{(3)}\right)(x,y,z).$$

and this proves the lemma. □

Hence, using the operator U defined by (2.26), we can construct the corresponding spaces $\mathcal{H}^{\circledast n}$. We have

$$\mathcal{H}^{\circledast n} = \big\{ f^{(n)} \in \mathcal{H}^{\otimes n} \mid U f^{(n)}(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ = Q(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \ \forall i = 1, \dots, n-1 \big\}.$$

The following proposition is proved in [23].

Proposition 2.35 (Q-symmetrization formula)**.** *For each $f^{(n)} \in \mathcal{H}^{\otimes n}$, $n \geq 2$, we have*

$$P_n f^{(n)} = \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}), \tag{2.27}$$

where for $\pi \in S_n$

$$Q_\pi(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n,\, \pi(i) > \pi(j)} Q(x_i, x_j). \tag{2.28}$$

In particular, for any $f_1, \dots, f_n \in \mathcal{H}_{\mathbb{C}}$, we have:

$$(f_1 \circledast \cdots \circledast f_n)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi(x_1, \dots, x_n) \left(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}\right)(x_1, \dots, x_n).$$

Let now $X = \mathbb{R}^2$. Consider

$$X^{(2)} = \{(x, y) \in X^2 \mid x_1 \neq y_1\}, \tag{2.29}$$

where $x = (x_1, x_2)$. Obviously, $X^{(2)} \subset O^{(2)}$. Let $\sigma(dx) = dx$ be the Lebesgue measure on $\mathbb{R}^2$. Since $dx = dx_1\, dx_2$ and the Lebesgue measure dx_1 on $\mathbb{R}$ is non-atomic, we have $\int_{X^2 \setminus X^{(2)}} dx\, dy = 0$. Fix $q \in \mathbb{C}$ such that $|q| = 1$. Define

$$Q(x, y) := \begin{cases} q, & \text{if} \quad x_1 < y_1, \\ \overline{q}, & \text{if} \quad x_1 > y_1. \end{cases} \tag{2.30}$$

Such a choice of the function Q corresponds to the *(Abelian) anyons*.

2.6 Multicomponent quantum systems

Just as in Subsection 2.5, let X be a locally compact Polish space and let σ be a non-atomic Radon measure on $(X, \mathcal{B}(X))$. Let V be a separable complex Hilbert space, and let $\mathcal{B}(V)$ be the Borel σ-algebra on V. Let $\mathcal{H} = L^2(X \to V, \sigma)$ be the L^2-space of V-valued functions on X.

Note that this space is unitarily isomorphic to the tensor product $L^2(X, \sigma) \otimes V$. The corresponding unitary isomorphism $I : L^2(X \to V, \sigma) \to L^2(X, \sigma) \otimes V$ satisfies, for each $f \in L^2(X, \sigma)$ and $v \in V$, $I(fv) = f \otimes v$, where $(fv)(x) = f(x)v$. Below we will use both realizations of $\mathcal{H}$, depending on which one is more convenient for our current purpose.

For each $n \geq 2$, let $X^{(n)} \in \mathcal{B}(X^n)$ be as in (2.23). Then

$$\mathcal{H}^{\otimes n} = L^2(X^{(n)} \to V^{\otimes n}, \sigma^{\otimes n}) = L^2(X^{(n)}, \sigma^{\otimes n}) \otimes V^{\otimes n}.$$

We fix a map $C : X^{(2)} \to \mathcal{L}(V^{\otimes 2})$ that satisfies

(i) for each $(x, y) \in X^{(2)}$, $C(x, y)$ is a unitary operator on $V^{\otimes 2}$;

(ii) for each $(x, y) \in X^{(2)}$, $C^*(x, y) = C(y, x)$.

(iii) for each measurable function $f^{(2)} : X^{(2)} \to V^{\otimes 2}$, the map

$$X^{(2)} \ni (x, y) \mapsto C(x, y) f^{(2)}(x, y) \in V^{\otimes 2} \tag{2.31}$$

is measurable.

We define $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ by

$$(Uf^{(2)})(x,y) = C(x,y)f^{(2)}(y,x) \quad \text{for } f^{(2)} \in \mathcal{H}^{\otimes 2}. \tag{2.32}$$

Remark 2.36. If $V = \mathbb{C}$, then any linear operator on $\mathbb{C}^{\otimes 2} = \mathbb{C}$ is just an operator of multiplication by a complex number, i.e., $\mathcal{L}(V^{\otimes 2}) = \mathcal{L}(\mathbb{C})$. Hence in this case C is just a measurable complex-valued function on $X^{(2)}$ that satisfies $|C(x,y)| = 1$ and $\overline{C(x,y)} = C(y,x)$. Hence, U is the operator from Subsection 2.5.

Lemma 2.37. *The operator U defined by* (2.32) *is unitary and self-adjoint.*

Proof. We prove first that U is an isometry. We have

$$\begin{aligned}
\|Uf^{(2)}\|^2_{\mathcal{H}^{\otimes 2}} &= \int_{X^2} \left\|C(x,y)f^{(2)}(y,x)\right\|^2_{V^{\otimes 2}} \sigma^{\otimes 2}(dx\,dy) \\
&= \int_{X^2} \left\|f^{(2)}(y,x)\right\|^2_{V^{\otimes 2}} \sigma^{\otimes 2}(dx\,dy) \\
&= \int_{X^2} \left\|f^{(2)}(x,y)\right\|^2_{V^{\otimes 2}} \sigma^{\otimes 2}(dx\,dy) \\
&= \|f^{(2)}\|^2_{\mathcal{H}^{\otimes 2}}.
\end{aligned}$$

Furthermore, for each $g^{(2)} \in \mathcal{H}^{\otimes 2}$, we have $g^{(2)} = Uf^{(2)}$, where

$$f^{(2)}(x,y) = C(x,y)g^{(2)}(y,x) = (Ug^{(2)})(x,y). \tag{2.33}$$

Hence $\operatorname{ran}(U) = \mathcal{H}^{\otimes 2}$. Thus, U is a unitary operator.

Furthermore, by (2.33), we have $U^{-1} = U$. Since U is unitary, $U^* = U^{-1}$. Therefore, $U = U^*$, i.e., U is a self-adjoint operator. □

Lemma 2.38. *The operator U defined by* (2.32) *satisfies the Yang–Baxter equation* (2.12) *if and only if the following equation holds on $V^{\otimes 3}$ for a.a $(x,y,z) \in X^{(3)}$:*

$$C_1(x,y)C_2(x,z)C_1(y,z) = C_2(y,z)C_1(x,z)C_2(x,y). \tag{2.34}$$

Here $C_i(\cdot,\cdot)$ denotes the operator $C(\cdot,\cdot)$ acting on the i^{th} and $(i+1)^{th}$ components of the tensor product $V^{\otimes 3}$.

Remark 2.39. Formula (2.34) is called the *functional Yang–Baxter equation.*

Proof. For any $g^{(3)} \in L^2\big(X^{(3)}, \sigma^{\otimes 3}\big)$ and $v^{(3)} \in V^{\otimes 3}$, we have

$$
\begin{aligned}
&U_1\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(y,x,z) C_1(x,y)\, v^{(3)},\\
&U_2U_1\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(z,x,y) C_2(y,z) C_1(x,z) v^{(3)},\\
&U_1U_2U_1\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(z,y,x) C_1(x,y) C_2(x,z) C_1(y,z) v^{(3)},\\
&U_2\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(x,z,y) C_2(y,z)\, v^{(3)},\\
&U_1U_2\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(y,z,x) C_1(x,y) C_2(x,z)\, v^{(3)},\\
&U_2U_1U_2\left(g^{(3)} \otimes v^{(3)}\right)(x,y,z) = g^{(3)}(z,y,x) C_2(y,z) C_1(x,z) C_2(x,y)\, v^{(3)}.
\end{aligned}
$$

□

Now we will discuss two special cases of the operator U which is defined by (2.32).

2.6.1 Constant C

Let us assume that $C(x,y) = C$ is a constant operator. Hence, the operator C must satisfy $C = C^* = C^{-1}$, i.e., C is self-adjoint and unitary. Formula (2.34) now becomes

$$C_1C_2C_1 = C_2C_1C_2, \tag{2.35}$$

i.e., the operator C must satisfy the Yang–Baxter equation in $V^{\otimes 3}$.

2.6.2 Non-Abelian Anyon Quantum Systems

Just as in the end of Section 2.5, we set $X = \mathbb{R}^2$ and $X^{(2)}$ to be defined by (2.29). We will now discuss a choice of the operator-valued function C that determines a non-Abelian anyon quantum system.

Let C be a unitary operator in $V^{\otimes 2}$ and define $C : X^{(2)} \to \mathcal{L}(V^{\otimes 2})$ by the formula

$$C(x,y) := \begin{cases} C, & \text{if } x_1 < y_1,\\ C^*, & \text{if } x_1 > y_1. \end{cases} \tag{2.36}$$

Obviously $C^*(x,y) = C(y,x)$ and for each measurable function $f^{(2)} : X^{(2)} \to V^{\otimes 2}$, the map (2.31) is measurable.

Lemma 2.40. *Let* $C : X^{(2)} \to \mathcal{L}(V^{\otimes 2})$ *be defined by* (2.36). *Then the functional Yang–Baxter equation* (2.34) *is satisfied if and only if the operator* C *satisfies the Yang–Baxter equation* (2.35) *on* $V^{\otimes 3}$.

Proof. We need to consider 6 cases, taking (2.36) into account.

Case 1: $x^1 < y^1 < z^1$. Equation (2.34) becomes (2.35).

Case 2: $y^1 < x^1 < z^1$. Equation (2.34) becomes

$$C_1^* C_2 C_1 = C_2 C_1 C_2^*. \tag{2.37}$$

Recall that, since C is unitary, $C^* = C^{-1}$. Multiplying equality (2.37) by C_1 from the left and by C_2 from the right, we get $C_2 C_1 C_2 = C_1 C_2 C_1$, which is (2.35).

Case 3: $x^1 < z^1 < y^1$. The equation (2.34) becomes

$$C_1 C_2 C_1^* = C_2^* C_1 C_2. \tag{2.38}$$

Since since C is unitary, $C^* = C^{-1}$. Multiplying equality (2.38) by C_1 from the right and by C_2 from the left, we get $C_2 C_1 C_2 = C_1 C_2 C_1$, which is (2.35).

Case 4: $y^1 < z^1 < x^1$. The equation (2.34) becomes

$$C_1^* C_2^* C_1 = C_2 C_1^* C_2^*. \tag{2.39}$$

Multiplying equality (2.39) by $C_2 C_1$ from the left and by $C_2 C_1$ from the right, we get $C_1 C_2 C_1 = C_2 C_1 C_2$, which is (2.35).

Case 5: $z^1 < y^1 < x^1$. The equation (2.34) becomes

$$C_1^* C_2^* C_1^* = C_2^* C_1^* C_2^*. \tag{2.40}$$

Take the adjoint of both sides of (2.40), we get $C_1 C_2 C_1 = C_2 C_1 C_2$, which is (2.35).

Case 6: $z^1 < x^1 < y^1$. The equation (2.34) becomes

$$C_1 C_2^* C_1^* = C_2^* C_1^* C_2. \tag{2.41}$$

Multiplying equality (2.41) by $C_1 C_2$ from the right and by $C_1 C_2$ from the left, we get $C_1 C_2 C_1 = C_2 C_1 C_2$, which is (2.35). □

Definition 2.41. For $n \in \mathbb{N}$, $n \geq 2$. A *braid group* B_n is defined as the group generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_n$ that satisfy the *braid relations*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{if } i \in \{1, 2, \ldots, n-1\}, \tag{2.42}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2. \tag{2.43}$$

Let $V = \mathbb{C}$ and recall Remark 2.36. As any operators of multiplication by a constant on $V^{\otimes(n+1)} = \mathbb{C}^{\otimes(n+1)} = \mathbb{C}$ commute, any constant $q \in \mathbb{C}$ with $|q| = 1$ determines an Abelian anyon statistics.

Next assume that the Hilbert space V is arbitrary, $q \in \mathbb{C}$ with $|q| = 1$, and let $C = q\,\mathrm{id}$, where id is the identity operator. It is clear that C is a unitary operator. Furthermore, we obviously have

$$\sigma_i \sigma_{i+1} = \sigma_{i+1} \sigma_i = q^2\,\mathrm{id}\,.$$

Therefore C determines an Abelian anyon statistics.

We will now consider several examples of C that determines a non-Abelian anyon statistics.

2.6.3 Examples

We first consider the simplest case where non-Abelian anyon statistics are possible: $V = \mathbb{C}^2$. We assume that (e_1, e_2) is an orthonormal basis of V. Then $V^{\otimes 2} = \mathbb{C}^4$ and

$$\big(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\big)$$

is an orthonormal basis of $V^{\otimes 2}$. In this basis, we will identify linear operators on $V^{\otimes 4}$ with 4×4 matrices acting on column vectors.

Example 2.42. Consider the operator C given by the matrix

$$C = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & q_4 \end{pmatrix}.$$

Here, $q_i \in \mathbb{C}$ and $|q_i| = 1$, $i = 1, 2, 3, 4$. It is equivalent to

$$
\begin{aligned}
Ce_1 \otimes e_1 &= q_1 e_1 \otimes e_1, \\
Ce_1 \otimes e_2 &= q_3 e_2 \otimes e_1, \\
Ce_2 \otimes e_1 &= q_2 e_1 \otimes e_2, \\
Ce_2 \otimes e_2 &= q_4 e_2 \otimes e_2.
\end{aligned}
$$

It is obvious that C is unitary.

Now we show that C satisfies the Yang–Baxter equation (2.35). We see that

$$\big(e_1\otimes e_1\otimes e_1,\ e_1\otimes e_1\otimes e_2,\ e_1\otimes e_2\otimes e_1,\ e_2\otimes e_1\otimes e_1,\ e_2\otimes e_2\otimes e_1,\ e_2\otimes e_1\otimes e_2,\ e_2\otimes e_2\otimes e_1,\ e_2\otimes e_2\otimes e_2\big)$$

is an orthonormal basis in $V^{\otimes 3} = \big(\mathbb{C}^2\big)^3$. We obviously have

$$
\begin{aligned}
C_1C_2C_1 e_1 \otimes e_1 \otimes e_1 &= C_2C_1C_2 e_1 \otimes e_1 \otimes e_1 = q_1^3 e_1 \otimes e_1 \otimes e_1, \\
C_1C_2C_1 e_2 \otimes e_2 \otimes e_2 &= C_2C_1C_2 e_2 \otimes e_1 \otimes e_1 = q_4^3 e_2 \otimes e_1 \otimes e_1.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
C_1C_2C_1 e_1 \otimes e_2 \otimes e_1 &= C_1C_2 q_3 e_2 \otimes e_1 \otimes e_1 \\
&= C_1 q_3 q_1 e_2 \otimes e_1 \otimes e_1 \\
&= q_1 q_2 q_3 e_1 \otimes e_2 \otimes e_1. \\
C_2C_1C_2 e_1 \otimes e_2 \otimes e_1 &= C_2C_1 q_2 e_1 \otimes e_1 \otimes e_2 \\
&= C_2 q_1 q_2 e_1 \otimes e_1 \otimes e_2 \\
&= q_1 q_2 q_3 e_1 \otimes e_2 \otimes e_1 = C_1C_2C_1 e_1 \otimes e_2 \otimes e_1. \\
C_1C_2C_1 e_2 \otimes e_1 \otimes e_2 &= C_1C_2 q_2 e_1 \otimes e_2 \otimes e_2 \\
&= C_1 q_4 q_2 e_1 \otimes e_2 \otimes e_2 \\
&= q_2 q_3 q_4 e_2 \otimes e_1 \otimes e_2. \\
C_2C_1C_2 e_2 \otimes e_1 \otimes e_2 &= C_2C_1 q_3 e_2 \otimes e_2 \otimes e_1 \\
&= C_2 q_4 q_3 e_2 \otimes e_2 \otimes e_1 \\
&= q_2 q_3 q_4 e_2 \otimes e_1 \otimes e_2 = C_1C_2C_1 e_2 \otimes e_1 \otimes e_2,
\end{aligned}
$$

and so forth.

The matrix C determines a non-Abelian anyon statistics, i.e., $C_1C_2 \neq C_2C_1$. Indeed,

$$C_1C_2e_1 \otimes e_2 \otimes e_1 = C_1q_2e_1 \otimes e_1 \otimes e_2 = q_1q_2e_1 \otimes e_1 \otimes e_2,$$

$$C_2C_1e_1 \otimes e_2 \otimes e_1 = C_2q_3e_2 \otimes e_1 \otimes e_1 = q_1q_3e_2 \otimes e_1 \otimes e_1.$$

Let us now consider the special case where $q_1 = k_1 \in \{-1, 1\}$, $q_4 = k_2 \in \{-1, 1\}$, $q_3 = q \in \mathbb{C}$, $|q| = 1$, and $q_2 = \bar{q}$, i.e.,

$$C = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & \bar{q} & 0 & 0 \\ 0 & 0 & 0 & k_2 \end{pmatrix}.$$

In this case, we additionally have $C = C^*$. Hence, the operator U is determined by the constant matrix C.

Example 2.43. Consider the operator C given by the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & q_2 \\ 0 & q_1 & 0 & 0 \\ 0 & 0 & q_1 & 0 \\ q_2 & 0 & 0 & 0 \end{pmatrix}.$$

Here, $q_i \in \mathbb{C}$ and $|q_i| = 1$, $i = 1, 2$. It is equivalent to

$$Ce_1 \otimes e_1 = q_2e_2 \otimes e_2,$$

$$Ce_1 \otimes e_2 = q_1e_1 \otimes e_2,$$

$$Ce_2 \otimes e_1 = q_1e_2 \otimes e_1,$$

$$Ce_2 \otimes e_2 = q_2e_1 \otimes e_1.$$

Similarly to Example 2.42, C is unitary and satisfies the Yang–Baxter equation (2.35). Furthermore, the matrix C determines a non-Abelian anyon statistics, i.e., $C_1C_2 \neq C_2C_1$.

Let us consider the special case where $q_2 = q \in \mathbb{C}$, $|q| = 1$ and $k = q_1 \in \{-1, 1\}$, i.e.,

$$C = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ q & 0 & 0 & 0 \end{pmatrix}.$$

In this case, we additionally have $C = C^*$. Hence, the operator U is determined by the constant matrix C.

Example 2.44. This example will be crucial for the dissertation. It generalizes Example 2.42 as follows. Let V be a separable complex Hilbert space and let $(e_i)_{i\geq 1}$ be an orthonormal basis in V. Then $(e_i \otimes e_j)_{i,j\geq 1}$ is an orthonormal basis in $V^{\otimes 2}$. Fix an arbitrary sequence of $q_{ij} \in \mathbb{C}$ such that $|q_{ij}| = 1$. Define $C \in \mathcal{L}(V^{\otimes 2})$ by

$$Ce_i \otimes e_j = q_{ij} e_j \otimes e_i. \tag{2.44}$$

It is clear that C is unitary. We state that C satisfies the Yang–Baxter equation (2.35). Indeed, for any $i, j, k \geq 1$, we have

$$\begin{aligned} C_1C_2C_1 e_i \otimes e_j \otimes e_k &= q_{ij} C_1C_2 e_j \otimes e_i \otimes e_k \\ &= q_{ij} q_{ik} C_1 e_j \otimes e_k \otimes e_i \\ &= q_{ij} q_{ik} q_{jk} e_k \otimes e_j \otimes e_i, \\ C_2C_1C_2 e_i \otimes e_j \otimes e_k &= q_{jk} C_2C_1 e_i \otimes e_k \otimes e_j \\ &= q_{jk} q_{ik} C_2 e_k \otimes e_i \otimes e_j \\ &= q_{jk} q_{ik} q_{ij} e_k \otimes e_j \otimes e_i. \end{aligned}$$

If $q_{ij} = \overline{q_{ji}}$, in particular $q_{ii} = k_i \in \{-1, 1\}$, then $C = C^*$.

Next,

$$\begin{aligned} C_1C_2 e_i \otimes e_j \otimes e_k &= q_{jk} q_{ik} e_k \otimes e_i \otimes e_j, \\ C_2C_1 e_i \otimes e_j \otimes e_k &= q_{ij} q_{ik} e_j \otimes e_k \otimes e_i. \end{aligned}$$

Hence, C is non-Abelian for any choice of q_{ij} if the dimension of V is ≥ 2.

Example 2.45. Let S be a locally compact Polish space and let ν be a Radon measure on S. Define $V = L^2(S, \nu)$. Consider a function $q : S^2 \to \mathbb{C}$ such that $|q(s,t)| = 1$ and $q(s,t) = \overline{q(t,s)}$ for all $(s,t) \in S^2$. Define $C : V^{\otimes 2} \to V^{\otimes 2}$ by

$$(C\varphi)(s,t) = q(s,t)\varphi(t,s). \tag{2.45}$$

By Lemma 2.34, the operator C satisfies the Yang–Baxter equation (2.35). Moreover

$$\begin{aligned} C_1C_2\varphi(s,t,u) &= C_1 q(t,u)\varphi(s,u,t) \\ &= q(s,t)q(s,u)\varphi(t,u,s), \\ C_2C_1\varphi(s,t,u) &= C_2 q(s,t)\varphi(t,s,u) \\ &= q(t,u)q(s,u)\varphi(u,s,t). \end{aligned}$$

Hence, if the measure ν is not concentrated at a single point (i.e., if V is not one-dimensional), we obtain $C_1C_2 \neq C_2C_1$, and so C is non-Abelian.

Consider the special case where the set S is discrete, i.e., $S = \{s_i\}_{i\geq 1}$ and ν is the counting measure, i.e., $\nu(\{s_i\})_{i\geq 1} = 1$ for all i. Define

$$e_i(t) := \begin{cases} 1, & \text{if } t = s_i, \\ 0, & \text{otherwise} . \end{cases}$$

Then $(e_i)_{i\geq 1}$ is an orthonormal basis of V. If we denote

$$q_{ij} = q(s_i, s_j),$$

then this example becomes Example 2.44.

2.7 Creation operators on the U-deformed Fock space

Let $\mathcal{H}$ be complex separable Hilbert space and let an operator $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be as in Subsection 2.2.2, i.e., U is unitary, self-adjoint and satisfies the Yang–Baxter equation (2.12).

Definition 2.46. We define the *U-deformed Fock space over* $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{H}). \tag{2.46}$$

Here $\mathcal{F}_0(\mathcal{H}) := \mathbb{C}$ and for $n \in \mathbb{N}$, $\mathcal{F}_n(\mathcal{H}) := \mathcal{H}^{\circledast n} n!$, i.e., $\mathcal{F}_n(\mathcal{H})$ coinsides with $\mathcal{H}^{\circledast n}$ as a set and $\|f^{(n)}\|^2_{\mathcal{F}_n(\mathcal{H})} = \|f^{(n)}\|^2_{\mathcal{H}^{\circledast n}}\, n!$ for each $f^{(n)} \in \mathcal{H}^{\circledast n}$.

The vector $\Omega = (1, 0, 0, \dots)$ is called the *vacuum*.

Definition 2.47. We denote by $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ the subspace of $\mathcal{F}(\mathcal{H})$ that consists of all finite sequences $f = \big(f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, 0, \dots\big)$ with $f^{(i)} \in \mathcal{F}_i(\mathcal{H})$ and $n \in \mathbb{N}$. We equip $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ with the topology of the topological direct sum of the $\mathcal{F}^{(n)}(\mathcal{H})$ spaces.

Thus, the convergence of a sequence in $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ means a uniform finiteness of the elements of the sequence and the coordinate-wise convergence of non-zero coordinates. We denote by $\mathcal{L}\left(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})\right)$ the space of all continuous linear operators on $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$.

Definition 2.48. Let $h \in \mathcal{H}$. We define a *creation operator* $a^+(h)$ as the linear operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given by

$$\begin{aligned} a^+(h)\Omega &= h, \\ a^+(h)f^{(n)} &= h \circledast f^{(n)}, \quad f^{(n)} \in \mathcal{F}_n(\mathcal{H}). \end{aligned} \tag{2.47}$$

Lemma 2.49. *For each* $h \in \mathcal{H}$, $a^+(h) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$.

Proof. Since $a^+(h)$ maps each $\mathcal{F}_n(\mathcal{H})$ into $\mathcal{F}_{n+1}(\mathcal{H})$ it is sufficient to prove that $a^+(h)$ is bounded as a linear operator from $\mathcal{F}_n(\mathcal{H})$ into $\mathcal{F}_{n+1}(\mathcal{H})$. For $n = 0$, we obviously have

$$\|a^+(h)f^{(0)}\|_{\mathcal{F}_1(\mathcal{H})} = \|h\|_{\mathcal{H}}\, |f^{(0)}|, \quad f^{(0)} \in \mathcal{F}_0(\mathcal{H}) = \mathbb{C},$$

and for each $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$, $n \in \mathbb{N}$,

$$\begin{aligned} \|a^+(h)f^{(n)}\|_{\mathcal{F}_{n+1}(\mathcal{H})} &= \|a^+(h)f^{(n)}\|_{\mathcal{H}^{\circledast(n+1)}} \sqrt{(n+1)!} \\ &= \|P_{n+1}(h \otimes f^{(n)})\|_{\mathcal{H}^{\otimes(n+1)}} \sqrt{(n+1)!} \\ &\le \|h \otimes f^{(n)}\|_{\mathcal{H}^{\otimes(n+1)}} \sqrt{(n+1)!} \\ &= \|h\|_{\mathcal{H}} \|f^{(n)}\|_{\mathcal{H}^{\otimes n}} \sqrt{(n+1)!} \end{aligned}$$

$$= \|h\|_{\mathcal{H}} \|f^{(n)}\|_{\mathcal{H}^{\circledast n}} \sqrt{(n+1)!}$$
$$= \|h\|_{\mathcal{H}} \|f^{(n)}\|_{\mathcal{F}_n(\mathcal{H})} \sqrt{n+1}\,.$$

Therefore,

$$\|a^+(h)\|_{\mathcal{L}(\mathcal{F}_n(\mathcal{H}),\mathcal{F}_{n+1}(\mathcal{H}))} \le \|h\|_{\mathcal{H}} \sqrt{n+1}\,. \quad \square$$

Definition 2.50. Let $g^{(2)} \in \mathcal{H}^{\otimes 2}$. We define a *double creation operator* $a^{++}(g^{(2)})$ as the linear operator on $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ given by

$$a^{++}\left(g^{(2)}\right)\Omega = P_2 g^{(2)}, \tag{2.48}$$
$$a^{++}\left(g^{(2)}\right) f^{(n)} = P_{n+2}\left(g^{(2)} \otimes f^{(n)}\right) = \left(P_2 g^{(2)}\right) \circledast f^{(n)}, \quad f^{(n)} \in \mathcal{F}_n(\mathcal{H}),\ n \in \mathbb{N}. \tag{2.49}$$

Lemma 2.51. *For each* $g^{(2)} \in \mathcal{H}^{\otimes 2}$, $a^{++}(g^{(2)}) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$.

Proof. Similar to the proof of Lemma 2.49. We only note that

$$\|a^{++}(g^{(2)})\|_{\mathcal{L}(\mathcal{F}_n(\mathcal{H}),\mathcal{F}_{n+2}(\mathcal{H}))} \le \|P_2 g^{(2)}\|_{\mathcal{H}^{\circledast 2}} \sqrt{(n+1)(n+2)}\,. \tag{2.50}$$

. $\square$

If $g, h \in \mathcal{H}$, we obviously have

$$a^+(g)a^+(h) = a^{++}(g \otimes h). \tag{2.51}$$

Furthermore, we have the following

Lemma 2.52. *Let* $(e_i)_{i\ge 1}$ *be an orthonormal basis in* $\mathcal{H}$. *Then, for any* $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$a^{++}\left(g^{(2)}\right) = \sum_{i,j\ge 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^+(e_i)a^+(e_j). \tag{2.52}$$

For each $n \ge 0$, *the series on the right-hand side of formula* (2.52) *converges in the norm of the space* $\mathcal{L}(\mathcal{F}_n(\mathcal{H}), \mathcal{F}_{n+2}(\mathcal{H}))$.

Proof. We first note that formula (2.50) implies

$$\|a^{++}(g^{(2)})\|_{\mathcal{L}(\mathcal{F}_n(\mathcal{H}),\mathcal{F}_{n+2}(\mathcal{H}))} \le \|g^{(2)}\|_{\mathcal{H}^{\otimes 2}} \sqrt{(n+1)(n+2)}\,.$$

From here and (2.51), the statement follows. $\square$

Proposition 2.53. *(i) Let $g^{(2)} \in \mathcal{H}^{\otimes 2}$. We have $a^{++}(g^{(2)}) = 0$ if and only if $Ug^{(2)} = -g^{(2)}$, i.e., the function $g^{(2)}$ is $-U$-symmetric.*

(ii) For each $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$a^{++}(g^{(2)}) = a^{++}(Ug^{(2)}). \tag{2.53}$$

In particular, for any $g, h \in \mathcal{H}$ we obtain

$$a^{+}(g)a^{+}(h) = a^{++}\left(Ug \otimes h\right), \tag{2.54}$$

which is called the commutation relation between creation operators.

Proof. (i) We first state that $a^{++}(g^{(2)}) = 0$ if and only if $P_2 g^{(2)} = 0$. Indeed, if $P_2 g^{(2)} = 0$, then, by (2.48) and (2.49), $a^{++}(g^{(2)}) = 0$. If $P_2 g^{(2)} \neq 0$, then, by (2.48), $a^{++}(g^{(2)})\Omega = P_2 g^{(2)} \neq 0$.

Since $P_2 = \frac{1}{2}(\mathbf{1} + U)$, we have $P_2 g^{(2)} = 0$ if and only if $Ug^{(2)} = -g^{(2)}$.

(ii) Consider $R_2 := \frac{1}{2}(\mathbf{1} - U)$, which is the orthogonal projection of $\mathcal{H}^{\otimes 2}$ onto the space of $-U$-symmetric functions. Then, by part (i), for each $g^{(2)} \in \mathcal{H}^{\otimes 2}$, $a^{++}(R_2 g^{(2)}) = 0$. By linearity, this implies (2.53). From here and (2.51), formula (2.54) follows. $\square$

Example 2.54. Let $U = \mathbf{1}$ be the identity operator. In this case, $\circledast$ is the usual tensor product $\otimes$, and the corresponding Fock space is the *full Fock space*. For future purposes, it will be convenient for us to denote this space by $\mathbb{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathbb{F}_n(\mathcal{H})$, where $\mathbb{F}_n(\mathcal{H}) := \mathcal{H}^{\otimes n} n!$. The subspace of $\mathbb{F}(\mathcal{H})$ consisting of 'finite vectors' will be denoted by $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$. The corresponding creation operators are called *free creation operators*. We will denote these operators by $a^{+}_{\mathrm{free}}(h)$. In particular, $a^{+}_{\mathrm{free}}(h)$ maps $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$ into itself. Equality (2.53) becomes now trivial, so there are no commutation relations between free creation operators.

Example 2.55. In the case of the symmetric tensor product, we get $Ug \otimes h = h \otimes g$. Hence, by (2.54),

$$a^{+}(g)a^{+}(h) = a^{+}(h)a^{+}(g).$$

Example 2.56. In the case of the antisymmetric tensor product, we get $Ug \otimes h = -h \otimes g$. Hence, by (2.54),

$$a^{+}(g)a^{+}(h) = -a^{+}(h)a^{+}(g).$$

Example 2.57. Recall the Q-symmetric tensor product discussed in Section 2.5. In particular, $\mathcal{H} = L^2(X, \sigma)$, $Q : X^{(2)} \to \mathbb{C}$ and

$$\left(Uf^{(2)}\right)(x, y) = Q(x, y)f^{(2)}(y, x).$$

We formally introduce creation operators $a^+(x)$ at points $x \in X$ that satisfy

$$a^+(h) = \int_X h(x)a^+(x)\sigma(dx), \quad h \in \mathcal{H}. \tag{2.55}$$

In view of Lemma 2.52, we then formally write

$$a^{++}(g^{(2)}) = \int_{X^2} g^{(2)}(x, y)a^+(x)a^+(y)\sigma(dx)\sigma(dy), \quad g^{(2)} \in \mathcal{H}^{\otimes 2}. \tag{2.56}$$

Our aim is to find the commutation relation between $a^+(x)$ and $a^+(y)$. We have

$$\begin{aligned} a^{++}\left(g^{(2)}\right) &= a^{++}\left(Ug^{(2)}\right), \\ &= \int_{X^2} \left(Ug^{(2)}\right)(x, y)a^+(x)a^+(y)\sigma(dx)\sigma(dy), \\ &= \int_{X^2} Q(x, y)g^{(2)}(y, x)a^+(x)a^+(y)\sigma(dx)\sigma(dy), \\ &= \int_{X^2} g^{(2)}(x, y)Q(y, x)a^+(y)a^+(x)\sigma(dx)\sigma(dy). \end{aligned} \tag{2.57}$$

Hence, by (2.56) and (2.57), we get the formal commutation relation

$$a^+(x)a^+(y) = Q(y, x)a^+(y)a^+(x). \tag{2.58}$$

Example 2.58. Let S be a locally compact Polish space and let ν be a Radon measure on S. Let $V = L^2(S, \nu)$ be the complex L^2-space on S with respect to the measure ν. In particular, V is a separable Hilbert space. With this space V let us consider the multicomponent quantum systems discussed in Subsection 2.6. In particular, X is a locally compact Polish space, σ is a non-atomic Radon measure on X and

$$\mathcal{H} = L^2(X \to V, \sigma) = L^2(X, \sigma) \otimes V = L^2(X, \sigma) \otimes L^2(S, \nu) = L^2(X \times S, \sigma \otimes \nu).$$

Hence,

$$\mathcal{H}^{\otimes 2} = L^2((X \times S)^2, (\sigma \otimes \nu)^{\otimes 2}) = L^2(X^{(2)} \times S^2, \sigma^{\otimes 2} \otimes \nu^{\otimes 2}).$$

Recall the map $C : X^{(2)} \to \mathcal{L}(V^{\otimes 2})$ from Subsection 2.6. In particular, for each $(x, y) \in X^{(2)}$, $C(x, y)$ is a unitary operator in $V^{\otimes 2} = L^2(S^2, \nu^{\otimes 2})$. We will assume that $C(x, y)$ is an integral operator with integral kernel $C(x, y, s, t, u, v)$,

$$\big(C(x,y)\varphi^{(2)}\big)(s,t) = \int_{S^2} C(x,y,s,t,u,v)\,\varphi^{(2)}(u,v)\,\nu(du)\,\nu(dv) \tag{2.59}$$

for $\varphi^{(2)} \in L^2(S^2, \nu^{\otimes 2})$.

Remark 2.59. We note that the above assumptions on the structure of the vector space V and the form of the operators $C(x, y)$ do not essentially lead us to a loss of generality. Indeed, let V be a general separable Hilbert space. If V is finite-dimensional, choose $S = \{1, 2, \dots, n\}$, where n is the dimension of V, and if V is infinite dimensional, choose $V = \mathbb{N}$. Let ν be the counting measure on S. Fix an arbitrary orthonormal basis $(e_i)_{i \in S}$ in V. Construct the unitary operator $I : V \to L^2(S, \nu)$ that satisfies $Ie_i = \delta_i$, where

$$\delta_i(u) = \begin{cases} 1, & \text{if } u = i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C \in \mathcal{L}(V^{\otimes 2})$. Define $\mathfrak{C} \in \mathcal{L}(L^2(S, \nu))$ by $\mathfrak{C} = I^{\otimes 2} C (I^{\otimes 2})^{-1}$. Then $\mathfrak{S}$ is an integral operator with integral kernel $\mathfrak{C}(s, t, u, v)$, where $\mathfrak{C}(s, t, u, v)$ is the matrix of the operator C in the orthonormal basis $(e_i \otimes e_j)_{i,j \in S}$.

However, sometimes one can use $V = L^2(S, \nu)$ where S is a continuum, for example, $S = \mathbb{R}$ and $\nu(ds) = ds$. In that case, one typically can only think of (2.59) only as an informal equality. Still such an informal interpretation of the operators $C(x, y)$ would be useful.

By (2.32) and (2.59), for $g^{(2)} \in \mathcal{H}^{\otimes 2}$, we have

$$\begin{aligned}\big(Ug^{(2)}\big)(x,y,s,t) &= \big(C(x,y)g^{(2)}(y,x,\cdot,\cdot)\big)(s,t)\\ &= \int_{S^2} C(x,y,s,t,u,v)\,g^{(2)}(y,x,u,v)\,\nu(du)\,\nu(dv).\end{aligned} \tag{2.60}$$

Similarly to Example 2.57, formula (2.55), we define creation operators $a^+(x, s)$ at points $(x, s) \in X \times S$ so that, for each $h \in \mathcal{H}$,

$$a^+(h) = \int_{X \times S} h(x,s)\,a^+(x,s)\,\sigma(dx)\,\nu(ds).$$

Then, similarly to (2.56), for each $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$a^{++}\left(g^{(2)}\right) = \int_{X^2\times S^2} g^{(2)}(x,y,s,t)\, a^+(x,s)\, a^+(y,t)\, \sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt). \tag{2.61}$$

Now, we aim to find out the commutation relation between $a^+(x,s)$ and $a^+(y,t)$. Indeed, we have

$$\begin{aligned}
a^{++}\left(g^{(2)}\right) &= a^{++}\left(Ug^{(2)}\right)\\
&= \int_{X^2\times S^2} \left(Ug^{(2)}\right)(x,y,s,t)\, a^+(x,s)\, a^+(y,t)\, \sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt),\\
&= \int_{X^2\times S^4} C(x,y,s,t,u,v) g^{(2)}(y,x,u,v)\, a^+(x,s)\, a^+(y,t)\\
&\times \sigma(dx)\, \sigma(dy)\, \nu(du)\, \nu(dv)\, \nu(ds)\, \nu(dt).
\end{aligned}$$

Swapping the variables $x \leftrightarrow y$, $s \leftrightarrow u$, $t \leftrightarrow v$, we get

$$\begin{aligned}
a^{++}\left(g^{(2)}\right) &= \int_{X^2\times S^4} C(y,x,u,v,s,t) g^{(2)}(x,y,s,t)\, a^+(y,u)\, a^+(x,v)\\
&\times \sigma(dx)\, \sigma(dy)\, \nu(du)\, \nu(dv)\, \nu(ds)\, \nu(dt)\\
&= \int_{X^2\times S^2} g^{(2)}(x,y,s,t) \bigg(\int_{S^2} C(y,x,u,v,s,t) a^+(y,u)\, a^+(x,v)\, \nu(du)\, \nu(dv) \bigg)\\
&\times \sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt).
\end{aligned} \tag{2.62}$$

By (2.61) and (2.62),

$$a^+(x,s)\, a^+(y,t) = \int_{S^2} C(y,x,u,v,s,t)\, a^+(y,u) a^+(x,v)\, \nu(du)\, \nu(dv). \tag{2.63}$$

For a fixed $x \in X$, let us formally think of $a^+(x) = a^+(x,\cdot)$ as an operator-valued function on S. Furthermore, we formally denote, for fixed $x, y \in X$,

$$\left(a^+(x) \otimes a^+(y)\right)(s,t) := a^+(x,s) a^+(y,t).$$

Thus, $a^+(x) \otimes a^+(y)$ is an operator-valued function on S^2. By (2.63),

$$\left(a^+(x) \otimes a^+(y)\right)(s,t) = \int_{S^2} C(y,x,u,v,s,t)\, \left(a^+(y) \otimes a^+(x)\right)(u,v)\, \nu(du)\, \nu(dv). \tag{2.64}$$

If $K \in \mathcal{L}\left(V^{\otimes 2}\right)$, then the *transposed operator of* K, denoted by K^T, is defined by

$$\int_{S^2} \left(K\varphi^{(2)}\right)(s,t) \psi^{(2)}(s,t)\, \nu(ds)\, \nu(dt) = \int_{S^2} \varphi^{(2)}(s,t) \left(K^T \psi^{(2)}\right)(s,t)\, \nu(ds)\, \nu(dt).$$

for any $\varphi^{(2)}, \psi^{(2)} \in V^{\otimes 2}$. If K is an integral operator with integral kernel $K(s,t,u,v)$, then K^T is an integral operator with integral kernel $K(u,v,s,t)$.

Then, we formally have

$$\left(C^T(y,x)a^+(y) \otimes a^+(x)\right)(s,t) = \int_{S^2} C(y,x,u,v,s,t)\, \left(a^+(x) \otimes a^+(y)\right)(u,v)\, \nu(du)\, \nu(dv). \tag{2.65}$$

Now, by (2.64) and (2.65), we obtain the formal commutation relation

$$a^+(x) \otimes a^+(y) = C^T(y,x)a^+(y) \otimes a^+(x). \tag{2.66}$$

In the case of Abelian anyons, the space V is one dimensional, equivalently, $V = \{x\}$, $\nu(\{x\}) = 1$. Then $Q^T(y,x) = Q(y,x)$ and so the commutation relation (2.66) becomes the commutation relation (2.58).

In the case of non-Abelian anyons, the operators $C(x,y)$ are given by (2.36). We formally think of C as an integral operator with integral kernel $C(s,t,u,v)$. Then C^* has an integral kernel $\overline{C(u,v,s,t)}$. Therefore, if $x_1 > y_1$, we obtain

$$a^+(x) \otimes a^+(y) = (C^*)^T a^+(y) \otimes a^+(x).$$

The operator $(C^*)^T$ has integral kernel $\overline{C(s,t,u,v)}$. This implies

$$a^+(x,s)a^+(y,t) = \int_{S^2} \overline{C(s,t,u,v)}\, a^+(y,u)\, a^+(x,v)\, \nu(du)\, \nu(dv).$$

Similarly, if $x_1 < y_1$,

$$a^+(x,s)a^+(y,t) = \int_{S^2} C(u,v,s,t)\, a^+(y,u)\, a^+(x,v)\, \nu(du)\, \nu(dv).$$

2.8 Annihilation operators on the U-deformed Fock space

We make the same assumptions as in Section 2.7.

We now fix an operator $J : \mathcal{H} \to \mathcal{H}$ that satisfies the following assumptions:

- J is antilinear, i.e., for any $f, g \in \mathcal{H}$ and $a, b \in \mathbb{C}$, $J(af + bg) = \bar{a}Jf + \bar{b}Jg$;

- J is an involution, i.e., $J^2 = \mathbf{1}$;
- for any $f, g \in \mathcal{H}$,
$$(Jf, Jg)_{\mathcal{H}} = (g, f)_{\mathcal{H}}.$$

For each $h \in \mathcal{H}$, we define an *annihilation operator* $a^-(h) : \mathcal{F}_{\text{fin}}(\mathcal{H}) \to \mathcal{F}_{\text{fin}}(\mathcal{H})$ by

$$a^-(h) = a^+(Jh)^* \restriction_{\mathcal{F}_{\text{fin}}(\mathcal{H})} . \tag{2.67}$$

Here $a^+(Jh)^*$ is the adjoint of the operator $a^+(Jh)$ in $\mathcal{F}(\mathcal{H})$.

Let us now show that the above definition indeed makes sense and find the explicit form of the action of the annihilation operator $a^-(h)$. We start with the easy case where $U = \mathbf{1}$ and the corresponding Fock space is the full (also called free) Fock space, $\mathbb{F}(\mathcal{H})$, see Example 2.54. Analogously to free creation operators, we now call annihilation operators *free annihilation operators* and denote them by $a^-_{\text{free}}(h)$.

For any $f^{(n-1)}, g^{(n-1)} \in \mathcal{H}^{\otimes(n-1)}$ and $h, u \in \mathcal{H}$, we have

$$\begin{aligned}
(a^+_{\text{free}}(h)g^{(n-1)}, u \otimes f^{(n-1)})_{\mathbb{F}(\mathcal{H})} &= (h \otimes g^{(n-1)}, u \otimes f^{(n-1)})_{\mathcal{H}^{\otimes n}}\, n! \\
&= (h, u)_{\mathcal{H}} (g^{(n-1)}, f^{(n-1)})_{\mathcal{H}^{\otimes(n-1)}}\, n! \\
&= n(h, u)_{\mathcal{H}} (g^{(n-1)}, f^{(n-1)})_{\mathbb{F}(\mathcal{H})} \\
&= (g^{(n-1)}, n(u, Jh)_{\mathcal{H}}\, f^{(n-1)})_{\mathbb{F}(\mathcal{H})}.
\end{aligned}$$

Hence the operator $a^-_{\text{free}}(h) : \mathbb{F}_{\text{fin}}(\mathcal{H}) \to \mathbb{F}_{\text{fin}}(\mathcal{H})$ is well defined and

$$a^-_{\text{free}}(h)(u \otimes f^{(n-1)}) = n(u, Jh)_{\mathcal{H}}\, f^{(n-1)}. \tag{2.68}$$

We now consider the case of a general operator U and the corresponding U-deformed Fock space $\mathcal{F}(\mathcal{H})$. For any $h \in \mathcal{H}$, $g^{(n-1)} \in \mathcal{H}^{\circledast(n-1)}$, and $f^{(n)} \in \mathcal{H}^{\circledast n}$,

$$\begin{aligned}
(a^+(h)g^{(n-1)}, f^{(n)})_{\mathcal{F}(\mathcal{H})} &= (h \circledast g^{(n-1)}, f^{(n)})_{\mathcal{H}^{\circledast n}} n! \\
&= (P_n(h \otimes g^{(n-1)}), f^{(n)})_{\mathcal{H}^{\otimes n}} n! \\
&= (h \otimes g^{(n-1)}, f^{(n)})_{\mathcal{H}^{\otimes n}} n! \\
&= (a^+_{\text{free}}(h)g^{(n-1)}, f^{(n)})_{\mathcal{H}^{\otimes n}} n!
\end{aligned}$$

$$= (g^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\otimes(n-1)}}(n-1)!\,. \tag{2.69}$$

We state that $a^-_{\mathrm{free}}(Jh)f^{(n)} \in \mathcal{H}^{\circledast(n-1)}$. Indeed, let $u^{(n-1)} \in \mathcal{H}^{\otimes(n-1)}$ and $g^{(n-1)} = P_{n-1}u^{(n-1)}$. Then, similarly to (2.69), we calculate, by using Proposition 2.19,

$$\begin{aligned}(a^+(h)g^{(n-1)}, f^{(n)})_{\mathcal{F}(\mathcal{H})} &= (P_n(h\otimes g^{(n-1)}), f^{(n)})_{\mathcal{H}^{\otimes n}}n!\\ &= (P_n(\mathbf{1}_{\mathcal{H}}\otimes P_{n-1})(h\otimes u^{(n-1)}), f^{(n)})_{\mathcal{H}^{\otimes n}}n!\\ &= (P_n(h\otimes u^{(n-1)}), f^{(n)})_{\mathcal{H}^{\otimes n}}n!\\ &= (h\otimes u^{(n-1)}, f^{(n)})_{\mathcal{H}^{\otimes n}}n!\\ &= (u^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\otimes(n-1)}}(n-1)!\,. \end{aligned}\tag{2.70}$$

By (2.69) and (2.70)

$$\begin{aligned}(g^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\otimes(n-1)}} &= (P_{n-1}u^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\otimes(n-1)}}\\ &= (u^{(n-1)}, P_{n-1}a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\otimes(n-1)}}.\end{aligned}\tag{2.71}$$

Since (2.71) holds for all $u^{(n-1)} \in \mathcal{H}^{\otimes(n-1)}$, we get

$$a^-_{\mathrm{free}}(Jh)f^{(n)} = P_{n-1}a^-_{\mathrm{free}}(Jh)f^{(n)},$$

and so $a^-_{\mathrm{free}}(Jh)f^{(n)} \in \mathcal{H}^{\circledast(n-1)}$.

Now, by (2.69),

$$\begin{aligned}(a^+(h)g^{(n-1)}, f^{(n)})_{\mathcal{F}(\mathcal{H})} &= (g^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{H}^{\circledast(n-1)}}(n-1)!\\ &= (g^{(n-1)}, a^-_{\mathrm{free}}(Jh)f^{(n)})_{\mathcal{F}(\mathcal{H})}.\end{aligned}$$

Proposition 2.60. (i) *For each $h \in \mathcal{H}$, the annihilation operator $a^-(h) : \mathcal{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ is well-defined and acts as follows: for each $f^{(n)} \in \mathcal{H}^{\circledast n}$,*

$$a^-(h)f^{(n)} = a^-_{\mathrm{free}}(h)f^{(n)}, \tag{2.72}$$

where the free annihilation operator $a^-_{\mathrm{free}}(h) : \mathbb{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{F}_{\mathrm{fin}}(\mathcal{H})$ is given by (2.68).

(ii) *For each $u^{(n)} \in \mathcal{H}^{\otimes n}$,*

$$a^-(h)P_nu^{(n)} = \frac{1}{n}\,a^-_{\mathrm{free}}(h)\,(\mathbf{1}_{\mathcal{H}}\otimes P_{n-1})\,[\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}]\,u^{(n)} \tag{2.73}$$

$$= \frac{1}{n}\, P_{n-1}\, a^-_{\text{free}}(h)\, [\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}]\, u^{(n)}. \tag{2.74}$$

In particular, for any $f_1, \dots, f_n \in \mathcal{H}$,

$$\begin{aligned} &a^-(h)(f_1 \circledast \cdots \circledast f_n) \\ &\quad = \frac{1}{n}\, P_{n-1}\, a^-_{\text{free}}(h)\, [\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}]\, (f_1 \otimes \cdots \otimes f_n). \end{aligned} \tag{2.75}$$

Proof. Statement (i) is already proved. Formula (2.73) follows from (i) and Proposition 2.21. Formula (2.74) follows from (2.73) and the obvious formula

$$a^-_{\text{free}}(h)\, (\mathbf{1}_{\mathcal{H}} \otimes P_{n-1}) = P_{n-1} a^-_{\text{free}}(h) \quad \text{on } \mathcal{H}^{\otimes n}.$$

□

We show now examples.

Example 2.61. Recall that in the symmetric tensor product, U is given by

$$U f \otimes g = g \otimes f. \tag{2.76}$$

We have

$$U_1U_2\cdots U_{k-1}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = f_k \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_{k-1} \otimes f_{k+1} \otimes \cdots \otimes f_n.$$

Then

$$\begin{aligned} &(\mathbf{1} \otimes P_{n-1})\, [\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}]\, f_1 \otimes f_2 \otimes \cdots \otimes f_n \\ &\quad = f_1 \otimes (f_2 \circledast \cdots \circledast f_n) + f_2 \otimes (f_1 \circledast f_3 \circledast \cdots \circledast f_n) + \cdots + f_n \otimes (f_1 \circledast \cdots \circledast f_{n-1})\,. \end{aligned}$$

Thus, by formula (2.75),

$$\begin{aligned} &a^-(h) f_1 \circledast f_2 \circledast \cdots \circledast f_n \\ &\quad = (f_1, Jh)_{\mathcal{H}} f_2 \circledast f_3 \circledast \cdots \circledast f_n + (f_2, Jh)_{\mathcal{H}} f_1 \circledast f_3 \circledast \cdots \circledast f_n \\ &\qquad + \cdots + (f_n, Jh)_{\mathcal{H}} f_1 \circledast f_2 \circledast \cdots \circledast f_{n-1} \\ &\quad = \sum_{i=1}^n (f_i, Jh)_{\mathcal{H}}\, f_1 \circledast f_2 \circledast \cdots \circledast \check{f}_i \circledast \cdots \circledast f_n, \end{aligned}$$

where $\check{f}_i$ denotes the absence of f_i.

Example 2.62. Recall that, for the antisymmetric tensor product, U is given by

$$Uf \otimes g = -g \otimes f. \tag{2.77}$$

Similar to the Example 2.61, by using Proposition 2.60, we obtain

$$a^-(h) f_1 \circledast f_2 \circledast \cdots \circledast f_n = \sum_{i=1}^{n} (-1)^{i-1} (f_i, Jh)_{\mathcal{H}} \, f_1 \circledast f_2 \circledast \cdots \circledast \check{f}_i \circledast \cdots \circledast f_n$$

Example 2.63. Let $\mathcal{H} = L^2(X, \sigma)$ where X and σ are as in Section 2.5. Recall that, for the Q-symmetric tensor product, the operator U is given by

$$\left(Uf^{(2)}\right)(x, y) = Q(x, y) f^{(2)}(y, x).$$

Now, we need to find the formula for $a^-(h)$. For $k \in \{1, 2 \dots, n-1\}$, we have

$$\begin{aligned}
&(U_k f^{(n)})(x_1, \dots, x_n) = Q(x_k, x_{k+1}) f^{(n)}(x_1, \dots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \dots, x_n), \\
&(U_{k-1} U_k f^{(n)})(x_1, \dots, x_n) \\
&\qquad = Q(x_{k-1}, x_k) Q(x_{k-1}, x_{k+1}) f^{(n)}(x_1, \dots, x_{k-2}, x_k, x_{k+1}, x_{k-1}, x_{k+2}, \dots, x_n), \\
&(U_{k-2} U_{k-1} U_k f^{(n)}(x_1, \dots, x_n) = Q(x_{k-2}, x_{k-1}) Q(x_{k-2}, x_k) Q(x_{k-2}, x_{k+1}) \\
&\qquad \times f^{(n)}(x_1, \dots, x_{k-3}, x_{k-1}, x_k, x_{k+1}, x_{k-2}, x_{k+2}, \dots).
\end{aligned}$$

Continuing similarly, we obtain

$$\begin{aligned}
&(U_1 U_2 \cdots U_k f^{(n)})(x_1, \dots, x_n) = Q(x_1, x_2) Q(x_1, x_3) \cdots Q(x_1, x_{k+1}) \\
&\quad \times f^{(n)}(x_2, x_3, \dots, x_{k+1}, x_1, x_{k+2}, \dots, x_n),
\end{aligned}$$

which we may write in the form

$$\begin{aligned}
&(U_1 U_2 \cdots U_k f^{(n)})(y, x_1, \dots, x_{n-1}) = Q(y, x_1) Q(y, x_2) \cdots Q(y, x_k) \\
&\quad \times f^{(n)}(x_1, \dots, x_k, y, x_{k+1}, \dots, x_{n-1}),
\end{aligned} \tag{2.78}$$

Therefore, by formula (2.74), we have

$$\begin{aligned}
&\left(a^-(h) f^{(n)}\right)(x_1, x_2, \dots, x_{n-1}) \\
&\quad = P_{n-1}\Bigg(\int_X \sigma(dy)\, h(y) \Big[f^{(n)}(y, x_1, x_2, \dots, x_{n-1}) \\
&\qquad + \sum_{k=1}^{n-1} Q(y, x_1) Q(y, x_2) \cdots Q(y, x_k) f^{(n)}(x_1, \dots, x_k, y, x_{k+1}, \dots, x_{n-1}) \Big] \Bigg).
\end{aligned} \tag{2.79}$$

We next consider a multicomponent quantum system.

Theorem 2.64. *Consider a multicomponent quantum system as in Example 2.58, and (at least formally) we think think of $C(x,y)$ as an integral operator with integral kernel $C(x,y,s,t,u,v)$, see* (2.59)*. Then, for each $h\in\mathcal{H}$,*

$$\begin{aligned}
&a^-(h)f^{(n)}(x_1,\dots,x_{n-1},s_1,\dots,s_{n-1})\\
&\quad=P_{n-1}\Bigg(\int_X\sigma(dy)\int_S\nu(dt)h(y,t)\bigg[f^{(n)}(y,x_1,\dots,x_{n-1},t,s_1,\dots,s_{n-1})\\
&\qquad+\sum_{k=1}^{n-1}\int_{S^{2k}}\prod_{i=1}^{k+1}\nu(du_i)\prod_{j=2}^{k-1}\nu(du'_j)C(y,x_1,t,s_1,u_1,u'_2)\bigg(\prod_{l=2}^{k}C(y,x_l,u'_l,s_l,u_l,u'_{l+1})\bigg)\\
&\qquad\times f^{(n)}(x_1,\dots,x_k,y,x_{k+1},\dots,x_{n-1},t,u_1,u_2,\dots,u_{k+1},s_{k+1},s_{k+2},\dots,s_{n-1})\bigg]\Bigg).
\end{aligned}$$

Proof. We first consider the simplified setting. So we consider the Hilbert space $\mathcal{G}=L^2(S,\nu)$, and let $\mathcal{U}$ be an operator in $\mathcal{G}^{\otimes 2}=L^2\left(S^2,\nu^{\otimes 2}\right)$ defined by

$$\left(\mathcal{U}g^{(2)}\right)(s,t)=\int_{S^2}C(s,t,u,v)\,g^{(2)}(u,v)\,\nu(du)\,\nu(dv). \tag{2.80}$$

Lemma 2.65. *For any $g^{(n)}\in\mathcal{G}^{\otimes n}$ and $k\in\{1,2,\dots,n-1\}$, we have*

$$\begin{aligned}
&\left(\mathcal{U}_1\mathcal{U}_2\cdots\mathcal{U}_kg^{(n)}\right)(s_1,s_2,\dots,s_n)\\
&\quad=\int_{S^{2k}}\prod_{i=1}^{k+1}\nu(du_i)\prod_{j=2}^{k}\nu(du'_j)\\
&\qquad\times C(s_1,s_2,u_1,u'_2)\bigg(\prod_{l=2}^{k}C(u'_l,s_{l+1},u_l,u'_{l+1})\bigg)g^{(n)}(u_1,u_2,\dots,u_{k+1},s_{k+2},\dots,s_n).
\end{aligned} \tag{2.81}$$

Proof. We have

$$\begin{aligned}
&\left(\mathcal{U}_kg^{(n)}\right)(s_1,s_2,\dots,s_n)\\
&\quad=\int_{S^2}\nu(du_k)\nu(du_{k+1})C\left(s_k,s_{k+1},u_k,u_{k+1}\right)g^{(n)}\left(s_1,\dots,s_{k-1},u_k,u_{k+1},s_{k+2},\dots,s_n\right),\\
&\left(\mathcal{U}_{k-1}\mathcal{U}_kg^{(n)}\right)(s_1,s_2,\dots,s_n)\\
&\quad=\int_{S^4}\nu(du_{k-1})\nu(du_k)\nu(du_{k+1})\nu(du'_k)C(s_{k-1},s_k,u_{k-1},u'_k)C(u'_k,s_{k+1},u_k,u_{k+1})\\
&\qquad\times g^{(n)}\left(s_1,\dots,s_{k-2},u_{k-1},u_k,u_{k+1},s_{k+2},\dots,s_n\right),
\end{aligned}$$

$$\left(\mathcal{U}_{k-2}\mathcal{U}_{k-1}\mathcal{U}_k g^{(n)}\right)(s_1, s_2, \dots, s_n)$$
$$= \int_{S^6} \nu(du_{k-2})\nu(du_{k-1})\nu(du_k)\nu(du_{k+1})\nu(du'_{k-1})\nu(du'_k)$$
$$\times C(s_{k-2}, s_{k-1}, u_{k-2}, u'_{k-1})C(u'_{k-1}, s_k, u_{k-1}, u'_k)C(u'_k, s_{k+1}, u_k, u_{k+1})$$
$$\times g^{(n)}(s_1, \dots, s_{k-3}, u_{k-2}, u_{k-1}, u_k, u_{k+1}, s_{k+2}, \dots, s_n).$$

Continuing similarly, we get (2.82). □

We now rewrite formula (2.82) as follows:

$$\left(\mathcal{U}_1\mathcal{U}_2\cdots\mathcal{U}_k g^{(n)}\right)(t, s_1, \dots, s_{n-1}) = \int_{S^{2k}} \prod_{i=1}^{k+1}\nu(du_i)\prod_{j=2}^{k}\nu(du'_j)$$
$$\times C(t, s_1, u_1, u'_2)\left(\prod_{l=2}^{k} C(u'_l, s_l, u_l, u'_{l+1})\right)g^{(n)}(u_1, u_2, \dots, u_{k+1}, s_{k+1}, \dots, s_{n-1}). \quad (2.82)$$

Lemma 2.66. *For any $f^{(n)} \in \mathcal{H}^{\otimes(n)}$ and $k \in \{1, 2, \dots, n-1\}$, we have*

$$\left(U_1U_2\cdots U_k f^{(n)}\right)(y, x_1, \dots, x_{n-1}, t, s_1, \dots, s_{n-1})$$
$$= C(y, x_1, t, s_1, u_1, u'_2)\left(\prod_{l=2}^{k} C(y, x_l, u'_l, s_l, u_l, u'_{l+1})\right)$$
$$\times f^{(n)}(x_1, \dots, x_k, y, x_{k+1}, \dots, x_{n-1}, t, u_1, u_2, \dots, u_{k+1}, s_{k+1}, s_{k+2}, \dots, s_{n-1}). \quad (2.83)$$

Proof. Formula (2.83) is obtained by combining the arguments used to prove formula (2.79) in the case of the Q-symmetric tensor product and the arguments used to prove formula (2.82). □

Finally, the statement of the Theorem 2.64 follows from Proposition 2.60 and Lemma 2.66.

□

Similarly to Definition 2.50, we will now introduce a double annihilation operator. We start with the free case, where $U = \mathbf{1}$. Then we have the free Fock space $\mathbb{F}(\mathcal{H})$ and the subspace of $\mathbb{F}(\mathcal{H})$ consisting of finite vectors, $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$.

Let J be antilinear involution in $\mathcal{H}$ such that $(Jf, Jg)_{\mathcal{H}} = (g, f)$ for any $f, g \in \mathcal{H}$. Then J can be extended to an antilinear involution in $\mathcal{H}^{\otimes 2}$ such that

$$J(f \otimes g) = (Jf) \otimes (Jg).$$

Next, we define a continuous antilinear operator $\mathbb{S} : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ satisfying

$$\mathbb{S} f \otimes g = J(g \otimes f) = (Jg) \otimes (Jf). \tag{2.84}$$

Note that $\mathbb{S}$ is an involution on $\mathcal{H}^{\otimes 2}$, i.e., $\mathbb{S}^2 = \mathbf{1}$.

Definition 2.67. Let $g^{(2)} \in \mathcal{H}^{\otimes 2}$. The *free double annihilation operator* $a^{--}_{\mathrm{free}}(g^{(2)}) \in \mathcal{L}(\mathbb{F}_{\mathrm{fin}}(\mathcal{H}))$ *is defined by*

$$a^{--}_{\mathrm{free}}\left(g^{(2)}\right) = a^{++}_{\mathrm{free}}\left(\mathbb{S} g^{(2)}\right)^* \restriction_{\mathbb{F}_{\mathrm{fin}}(\mathcal{H})} . \tag{2.85}$$

Remark 2.68. The fact that formula (2.85) indeed defines an operator from $\mathcal{L}(\mathbb{F}_{\mathrm{fin}}(\mathcal{H}))$ follows immediately from Lemma 2.51.

Proposition 2.69. *For any $g, h \in \mathcal{H}$, we have*

$$a^{--}_{\mathrm{free}}(g \otimes h) = a^{-}_{\mathrm{free}}(g) a^{-}_{\mathrm{free}}(h). \tag{2.86}$$

Proof. By (2.51), (2.84), and (2.85), we get

$$\begin{aligned} a^{--}_{\mathrm{free}}(g \otimes h) &= a^{++}_{\mathrm{free}}\left(\mathbb{S}\left(g \otimes h\right)\right)^* \restriction_{\mathbb{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{++}_{\mathrm{free}}\left((Jh) \otimes (Jg)\right)^* \restriction_{\mathbb{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= \left(a^{+}_{\mathrm{free}}(Jh) a^{+}_{\mathrm{free}}(Jg)\right)^* \restriction_{\mathbb{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{+}_{\mathrm{free}}(Jg)^* a^{+}_{\mathrm{free}}(Jh)^* \restriction_{\mathbb{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{-}_{\mathrm{free}}(g) a^{-}_{\mathrm{free}}(h). \end{aligned}$$

□

Proposition 2.70. *For any $u^{(2)}, g^{(2)} \in \mathcal{H}^{\otimes 2}$ and $f^{(n)} \in \mathcal{H}^{\otimes n}$, we have*

$$a^{--}_{\mathrm{free}}\left(g^{(2)}\right)\left(u^{(2)} \otimes f^{(n)}\right) = (n+2)(n+1)\left(u^{(2)}, \mathbb{S}\, g^{(2)}\right)_{\mathcal{H}^{\otimes 2}} f^{(n)}. \tag{2.87}$$

Proof. Let $h^{(n)} \in \mathcal{H}^{\otimes n}$. Then

$$\begin{aligned} \left(a^{++}_{\mathrm{free}}\left(g^{(2)}\right) h^{(n)}, u^{(2)} \otimes f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} &= \left(g^{(2)} \otimes h^{(n)}, u^{(2)} \otimes f^{(n)}\right)_{\mathcal{H}^{\otimes(n+2)}} (n+2)! \\ &= \left(g^{(2)}, u^{(2)}\right)_{\mathcal{H}^{\otimes 2}} \left(h^{(n)}, f^{(n)}\right)_{\mathcal{H}^{\otimes n}} (n+2)(n+1)n! \\ &= \left(g^{(2)}, u^{(2)}\right)_{\mathcal{H}^{\otimes 2}} \left(h^{(n)}, f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} (n+2)(n+1) \\ &= \left(h^{(n)}, (n+2)(n+1)\left(u^{(2)}, g^{(2)}\right)_{\mathcal{H}^{\otimes 2}} f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} . \end{aligned}$$

This implies that

$$a^{++}_{\text{free}}\left(g^{(2)}\right)^{*}\left(u^{(2)}\otimes f^{(n)}\right)=(n+2)(n+1)\left(u^{(2)},g^{(2)}\right)_{\mathcal{H}^{\otimes 2}}f^{(n)}.$$

Thus,

$$a^{--}_{\text{free}}\left(g^{(2)}\right)\left(u^{(2)}\otimes f^{(n)}\right)=(n+2)(n+1)\left(u^{(2)},\mathbb{S}g^{(2)}\right)_{\mathcal{H}^{\otimes 2}}f^{(n)}.$$ □

Proposition 2.71. *Let $(e_i)_{i\geq 1}$ be an orthonormal basis in $\mathcal{H}$. Then, for any $g^{(2)}\in\mathcal{H}^{\otimes 2}$,*

$$a^{--}_{\text{free}}\left(g^{(2)}\right)=\sum_{i,j\geq 1}(g^{(2)},e_i\otimes e_j)_{\mathcal{H}^{\otimes 2}}\,a^{-}_{\text{free}}(e_i)a^{-}_{\text{free}}(e_j). \tag{2.88}$$

For each $n\geq 0$, the series on the right-hand side of formula (2.106) *converges in the norm of the space $\mathcal{L}\left(\mathbb{F}_{n+2}(\mathcal{H}),\mathbb{F}_n(\mathcal{H})\right)$.*

Proof. Let $u^{(2)}\in\mathcal{H}^{\otimes 2}$ and $f^{(n)}\in\mathcal{H}^{\otimes n}$. By Propositions 2.69 and 2.70, we have

$$\begin{aligned}
&\sum_{i,j\geq 1}(g^{(2)},e_i\otimes e_j)_{\mathcal{H}^{\otimes 2}}\,a^{-}_{\text{free}}(e_i)a^{-}_{\text{free}}(e_j)(u^{(2)}\otimes f^{(n)})\\
&\quad=\sum_{i,j\geq 1}(g^{(2)},e_i\otimes e_j)_{\mathcal{H}^{\otimes 2}}\,a^{--}_{\text{free}}(e_i\otimes e_j)(u^{(2)}\otimes f^{(n)})\\
&\quad=(n+2)(n+1)\sum_{i,j\geq 1}\left(g^{(2)},e_i\otimes e_j\right)_{\mathcal{H}^{\otimes 2}}\left(u^{(2)},\mathbb{S}\,e_i\otimes e_j\right)_{\mathcal{H}^{\otimes 2}}f^{(n)}\\
&\quad=(n+2)(n+1)\left(u^{(2)},\mathbb{S}\sum_{i,j\geq 1}\left(g^{(2)},e_i\otimes e_j\right)_{\mathcal{H}^{\otimes 2}}e_i\otimes e_j\right)_{\mathcal{H}^{\otimes 2}}f^{(n)}\\
&\quad=(n+2)(n+1)\left(u^{(2)},\mathbb{S}\,g^{(2)}\right)_{\mathcal{H}^{\otimes 2}}f^{(n)}\\
&\quad=a^{--}_{\text{free}}\left(g^{(2)}\right)\left(u^{(2)}\otimes f^{(n)}\right).
\end{aligned}$$

The statement of the proposition about the convergence follows immediately from the observation

$$\|a^{--}_{\text{free}}(g^{(2)})\|_{\mathcal{L}(\mathbb{F}_{n+2}(\mathcal{H}),\mathbb{F}_n(\mathcal{H}))}=\sqrt{(n+2)(n+1)}\,\|g^{(2)}\|_{\mathcal{H}^{\otimes 2}}.$$ □

We now proceed to consider the general case.

Definition 2.72. In the general case, we define, for each $g^{(2)}\in\mathcal{H}^{\otimes 2}$,

$$a^{--}(g^{(2)})=a^{++}\left(\mathbb{S}g^{(2)}\right)^{*}\restriction_{\mathcal{F}_{\text{fin}}(\mathcal{H})}.$$

Proposition 2.73. *For any* $f^{(n)} \in \mathcal{H}^{\circledast n}$ *and* $g^{(2)} \in H^{\otimes 2}$,

$$a^{--}\left(g^{(2)}\right) f^{(n)} = a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)}. \tag{2.89}$$

Proof. By (2.49) and (2.85), we have for $h^{(n-2)} \in \mathcal{H}^{\circledast(n-2)}$,

$$\begin{aligned}
\left(a^{++}\left(\mathbb{S}\, g^{(2)}\right) h^{(n-2)}, f^{(n)}\right)_{\mathcal{F}(\mathcal{H})} &= \left(P_n\left(\left(\mathbb{S}\, g^{(2)}\right) \otimes h^{(n-2)}\right), f^{(n)}\right)_{\mathcal{H}^{\circledast n}} n! \\
&= \left(\left(\mathbb{S}\, g^{(2)}\right) \otimes h^{(n-2)}, f^{(n)}\right)_{\mathcal{H}^{\otimes n}} n! \\
&= \left(a^{++}_{\text{free}}\left(\mathbb{S}\, g^{(2)}\right) h^{(n-2)}, f^{(n)}\right)_{\mathcal{H}^{\otimes n}} n! \\
&= \left(a^{++}_{\text{free}}\left(\mathbb{S}\, g^{(2)}\right) h^{(n-2)}, f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} \\
&= \left(h^{(n-2)}, a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} \\
&= \left(h^{(n-2)}, P_{n-2}\, a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)}\right)_{\mathbb{F}(\mathcal{H})} \\
&= \left(h^{(n-2)}, P_{n-2}\, a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)}\right)_{\mathcal{F}(\mathcal{H})}.
\end{aligned}$$

Thus

$$a^{--}\left(g^{(2)}\right) f^{(n)} = P_{n-2}\, a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)}. \tag{2.90}$$

Now, our aim is to get rid of P_{n-2}. By Proposition 2.60. (i) and by Proposition 2.71,

$$\begin{aligned}
a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)} &= \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^{-}_{\text{free}}(e_i) a^{-}_{\text{free}}(e_j) f^{(n)} \\
&= \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^{-}(e_i) a^{-}(e_j) f^{(n)}.
\end{aligned} \tag{2.91}$$

Since $a^{-}(e_i)a^{-}(e_j)f^{(n)} \in \mathcal{H}^{\circledast(n-2)}$, then the whole expression on the right hand-side of the equality (2.91) is in $\mathcal{H}^{\circledast(n-2)}$. Therefore,

$$a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)} = P_{n-2}\, a^{--}_{\text{free}}\left(g^{(2)}\right) f^{(n)},$$

and this proves (2.89). □

Below, for an operator $A \in \mathcal{L}(\mathcal{H}^{\otimes 2})$, we define an operator $\widehat{A} \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ by

$$\widehat{A} := \mathbb{S} A \mathbb{S}.$$

Proposition 2.74. *(i) Let $g^{(2)} \in \mathcal{H}^{\otimes 2}$. We have $a^{--}\left(g^{(2)}\right) = 0$ if and only if $-\widehat{U}g^{(2)} = g^{(2)}$, i.e., the function $g^{(2)}$ is $-\widehat{U}$ symmetric.*

(ii) For each $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$a^{--}\left(g^{(2)}\right) = a^{--}\left(\widehat{U}g^{(2)}\right). \tag{2.92}$$

Proof. (i) By Proposition 2.53 (i), we have

$$\begin{aligned} a^{--}\left(g^{(2)}\right) = a^{++}\left(\mathbb{S}g^{(2)}\right)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{H})} = 0 \;\Leftrightarrow\; a^{++}\left(\mathbb{S}g^{(2)}\right) = 0 \;\Leftrightarrow\; -U\mathbb{S}g^{(2)} = \mathbb{S}g^{(2)} \\ \Leftrightarrow\; -\mathbb{S}U\mathbb{S}g^{(2)} = g^{(2)} \;\Leftrightarrow\; -\widehat{U}g^{(2)} = g^{(2)}. \end{aligned}$$

(ii) By Proposition 2.53 (ii), we have

$$\begin{aligned} a^{--}\left(g^{(2)}\right) &= a^{++}\left(\mathbb{S}g^{(2)}\right)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{++}\left(U\mathbb{S}g^{(2)}\right)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{++}\left(\mathbb{S}^2U\mathbb{S}g^{(2)}\right)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{++}\left(\mathbb{S}\widehat{U}g^{(2)}\right)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{H})} \\ &= a^{--}\left(\widehat{U}g^{(2)}\right). \end{aligned}$$

□

Example 2.75. Consider the Q-symmetric tensor product discussed in Section 2.5. In particular, $\mathcal{H} = L^2(X, \sigma)$, $Q : X^{(2)} \to \mathbb{C}$ and

$$\left(Uf^{(2)}\right)(x, y) = Q(x, y)f^{(2)}(y, x).$$

In this case, the operator $\mathbb{S}$ acts as follows:

$$(\mathbb{S}f^{(2)})(x, y) = \overline{f^{(2)}(y, x)}, \quad f^{(2)} \in \mathcal{H}^{\otimes 2}. \tag{2.93}$$

Hence,

$$(U\mathbb{S}f^{(2)})(x, y) = Q(x, y)\,\overline{f^{(2)}(x, y)},$$

and so

$$\begin{aligned} (\widehat{U}f^{(2)})(x, y) = (\mathbb{S}U\mathbb{S}f^{(2)})(x, y) &= \overline{Q(y, x)}\, f^{(2)}(y, x) \\ &= Q(x, y)f^{(2)}(y, x) = (Uf^{(2)})(x, y). \end{aligned}$$

Therefore $\widehat{U} = U$.

Hence, similarly to (2.58), we get

$$a^-(x)a^-(y) = Q(y,x)a^-(y)a^-(x). \tag{2.94}$$

Example 2.76. Consider the multicomponent quantum system discussed in Section 2.6. In particular, the operator U is given by formula (2.32). Let us find the operator $\widehat{U} = \mathbb{S}U\mathbb{S}$.

For any $f, g \in \mathcal{H} = L^2(X \to V, \sigma)$, we have

$$(f \otimes g)(x,y) = f(x) \otimes g(y).$$

By (2.84),

$$\begin{aligned}(\mathbb{S}f \otimes g)(x,y) &= ((Jg) \otimes (Jf))(x,y) \\ &= (Jg(x)) \otimes (Jf(y)).\end{aligned} \tag{2.95}$$

Define an antilinear operator $\mathbb{S}_{V^{\otimes 2}} : V^{\otimes 2} \to V^{\otimes 2}$ by

$$\mathbb{S}_{V^{\otimes 2}}\, u \otimes v = (Jv) \otimes (Ju), \quad u, v \in V. \tag{2.96}$$

Further, we define a linear operator $\mathbb{E} : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ by

$$\left(\mathbb{E}f^{(2)}\right)(x,y) = f^{(2)}(y,x), \quad f^{(2)} \in \mathcal{H}^{\otimes 2}. \tag{2.97}$$

Then, by (2.95)–(2.97), we have

$$\mathbb{S} = \mathbb{E}\,\mathbb{S}_{V^{\otimes 2}} = \mathbb{S}_{V^{\otimes 2}}\,\mathbb{E}.$$

Therefore,

$$\widehat{U} = \mathbb{S}_{V^{\otimes 2}}\,\mathbb{E}\,U\mathbb{E}\,\mathbb{S}_{V^{\otimes 2}}.$$

Note that

$$\left(Uf^{(2)}\right)(x,y) = C(x,y)\,\mathbb{E}\,f^{(2)}(x,y). \tag{2.98}$$

Hence, for $f^{(2)} \in \mathcal{H}^{\otimes 2}$

$$\left(\widehat{U}f^{(2)}\right)(x,y) = \mathbb{S}_{V^{\otimes 2}}\,\mathbb{E}\,C(x,y)\,\mathbb{E}\,\mathbb{E}\,\mathbb{S}_{V^{\otimes 2}}\,f^{(2)}(x,y)$$

$$= \mathbb{S}_{V^{\otimes 2}}\, \mathbb{E}\, C(x,y)\, \mathbb{S}_{V^{\otimes 2}}\, f^{(2)}(x,y)$$
$$= \mathbb{S}_{V^{\otimes 2}}\, C(y,x)\, \mathbb{S}_{V^{\otimes 2}}\, f^{(2)}(y,x). \tag{2.99}$$

For a linear operator $A \in \mathcal{L}\left(V^{\otimes 2}\right)$, we define $\widehat{A} = \mathbb{S}_{V^{\otimes 2}}\, A\, \mathbb{S}_{V^{\otimes 2}}$. Then, by (2.99)

$$\big(\widehat{U} f^{(2)}\big)(x,y) = \widehat{C}(y,x) f^{(2)}(y,x). \tag{2.100}$$

Let now $V = \mathcal{L}^2(S,\nu)$ and the operator $C(x,y)$ be as in Example 2.58, in particular, (2.59) holds. Let us find the operator U in this case.

To this end, let us assume that $A \in \mathcal{L}\left(V^{\otimes 2}\right)$ is an integral operator, i.e.,

$$\left(A\, \varphi^{(2)}\right)(s,t) = \int_{S^2} A(s,t,u,v) \varphi^{(2)}(u,v) \nu(du)\nu(dv).$$

Since

$$\left(\mathbb{S}_{V^{\otimes 2}}\, \varphi^{(2)}\right)(s,t) = \overline{\varphi^{(2)}(t,s)},$$

we have

$$\left(A \mathbb{S}_{V^{\otimes 2}}\, \varphi^{(2)}\right)(s,t) = \int_{S^2} A(s,t,u,v) \overline{\varphi^{(2)}(v,u)} \nu(du)\nu(dv).$$

Hence,

$$\begin{aligned}
\big(\widehat{A} \varphi^{(2)}\big)(s,t) &= \overline{\int_{S^2} A(t,s,u,v) \overline{\varphi^{(2)}(v,u)} \nu(du)\nu(dv)} \\
&= \int_{S^2} \overline{A(t,s,u,v)} \varphi^{(2)}(v,u) \nu(du)\nu(dv) \\
&= \int_{S^2} \overline{A(t,s,v,u)} \varphi^{(2)}(u,v) \nu(du)\nu(dv).
\end{aligned}$$

Thus, the integral kernel of $\widehat{A}$ is

$$\widehat{A}(s,t,u,v) = \overline{A(t,s,v,u)}. \tag{2.101}$$

By (2.100) and (2.101),

$$\begin{aligned}
\big(\widehat{U} f^{(2)}\big)(x,y,s,t) &= \int_{S^2} \widehat{C}(y,x,s,t,u,v) f^{(2)}(y,x,u,v) \nu(du)\nu(dv) \\
&= \int_{S^2} \overline{C(y,x,t,s,v,u)} f^{(2)}(y,x,u,v) \nu(du)\nu(dv).
\end{aligned} \tag{2.102}$$

Remark 2.77. Since $C(y,x) = C^*(x,y)$, formula (2.102) implies that

$$\left(\widehat{U} f^{(2)}(y,x,s,t)\right) = \int_{S^2} C(x,y,v,u,t,s) f^{(2)}(y,x,u,v)\nu(du)\nu(dv). \tag{2.103}$$

Now, we will derive the commutation relation between $a^-(x,s)$ and $a^-(y,t)$ analogously to the commutation relation between $a^+(x,s)$ and $a^+(y,t)$. We have

$$a^{--}\left(f^{(2)}\right) = \int_{X^2\times S^2} f^{(2)}(x,y) a^-(x)\otimes a^-(y)\sigma(dx)\sigma(dy), \tag{2.104}$$

and by (2.100),

$$\begin{aligned}
&a^{--}\left(f^{(2)}\right) = a^{--}\big(\widehat{U} f^{(2)}\big)\\
&= \int_{X^2\times S^2} \big(\widehat{U} f^{(2)}\big)(x,y,s,t)\, a^-(x,s)\, a^-(y,t)\, \sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt),\\
&= \int_{X^2\times S^2} \left(\widehat{C}(y,x)\, f^{(2)}(y,x,\cdot,\cdot)\right)(s,t)\, a^-(x,s)\, a^-(y,t)\sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt)\\
&= \int_{X^2\times S^2} \left(\widehat{C}(x,y)\, f^{(2)}(x,y,\cdot,\cdot)\right)(s,t)\, a^-(y,s)\, a^-(x,t)\sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt)\\
&= \int_{X^2\times S^2} f^{(2)}(x,y,s,t)\, \widehat{C}^T(x,y)\big(a^-(y,\cdot)\, a^-(x,\cdot)\big)(s,t)\sigma(dx)\, \sigma(dy)\, \nu(ds)\, \nu(dt).
\end{aligned}$$

Hence,

$$a^-(x)\otimes a^-(y) = \widehat{C}^T(x,y) a^-(y)\otimes a^-(x). \tag{2.105}$$

2.9 Commutation between creation and annihilation operators

We will now find the commutation relations between $a^-(h)$ and $a^+(g)$.

Definition 2.78. Let $(e_i)_{i\geq 1}$ be an orthonormal basis in $\mathcal{H}$. Then, for any $g^{(2)}\in\mathcal{H}^{\otimes 2}$, we define $a^{+-}_{\mathrm{free}}\left(g^{(2)}\right)$ on $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$ by

$$a^{+-}_{\mathrm{free}}\left(g^{(2)}\right) = \sum_{i,j\geq 1} (g^{(2)}, e_i\otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^+_{\mathrm{free}}(e_i) a^-_{\mathrm{free}}(e_j). \tag{2.106}$$

Remark 2.79. As we will see from Proposition 2.80 below, the definition of $a^{+-}_{\mathrm{free}}\left(g^{(2)}\right)$ does not depend on the choice of an orthonormal basis $(e_i)_{i\geq 1}$.

Proposition 2.80. *For each $n \geq 1$, the series on the right-hand side of formula* (2.106) *converges in the norm of the space $\mathcal{L}(\mathbb{F}_n(\mathcal{H}))$. Furthermore, for any $g, h \in \mathcal{H}$,*

$$a^{+-}_{\text{free}}(g \otimes h) = a^{+}_{\text{free}}(g) a^{-}_{\text{free}}(h). \tag{2.107}$$

Proof. We initially prove the first statement of the proposition for $n = 1$. Let $M, N \in \mathbb{N}$ and $u \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{i=1}^{M} \sum_{j=1}^{N} \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} a^{+}_{\text{free}}(e_i)\, a^{-}_{\text{free}}(e_j) u \\
&\quad = \sum_{i=1}^{M} \left(\sum_{j=1}^{N} \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (u, Je_j)_{\mathcal{H}} \right) e_i \,.
\end{aligned}$$

By Cauchy's inequality,

$$\begin{aligned}
&\left\| \sum_{i=1}^{M} \left(\sum_{j=1}^{N} \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (u, Je_j)_{\mathcal{H}} \right) e_i \right\|_{\mathcal{H}}^2 \\
&= \sum_{i=1}^{M} \left| \sum_{j=1}^{N} \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (u, Je_j)_{\mathcal{H}} \right|^2 \\
&\leqslant \sum_{i=1}^{M} \sum_{j=1}^{N} \left| \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} \right|^2 \sum_{l=1}^{N} \left|(u, Je_l)_{\mathcal{H}}\right|^2 \\
&\leqslant \left(\sum_{i=1}^{M} \sum_{j=1}^{N} \left| \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} \right|^2 \right) \sum_{l \geqslant 1} \left|(u, Je_l)_{\mathcal{H}}\right|^2 \\
&= \|g^{(2)}_{M,N}\|^2_{\mathcal{H}^{\otimes 2}} \, \|u\|^2,
\end{aligned}$$

where

$$g^{(2)}_{M,N} = \sum_{i=1}^{M} \sum_{j=1}^{N} \left(g^{(2)}, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} e_i \otimes e_j.$$

From here the statement follows, and furthermore we obtain

$$\|a^{+-}_{\text{free}}\left(g^{(2)}\right) u\|_{\mathcal{H}} \leqslant \|g^{(2)}\|_{\mathcal{H}^{\otimes 2}} \, \|u\|_{\mathcal{H}}.$$

Therefore,

$$\|a^{+-}_{\text{free}}\left(g^{(2)}\right)\|_{\mathcal{L}(\mathbb{F}_1(\mathcal{H}))} \leqslant \|g^{(2)}\|_{\mathcal{H}^{\otimes 2}}.$$

For $n \geqslant 2$, similar calculations give, for $u^{(n)} \in \mathcal{H}^{\otimes n}$,

$$\|a^{+-}_{\text{free}}\left(g^{(2)}\right)\|_{(\mathcal{L}(\mathbb{F}_n(\mathcal{H}))} \leqslant n\|g^{(2)}\|_{\mathcal{H}^{\otimes 2}}.$$

Next, for $g, h \in \mathcal{H}$,

$$\begin{aligned}
a^{+-}_{\text{free}}(g \otimes h) &= \sum_{i,j\geq 1} (g \otimes h, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\; a^{+}_{\text{free}}(e_i)\, a^{-}_{\text{free}}(e_j) \\
&= \sum_{i\geq 1} (g, e_i)_{\mathcal{H}}\, a^{+}_{\text{free}}(e_i) \sum_{j\geq 1} (h, e_j)_{\mathcal{H}}\, a^{-}_{\text{free}}(e_j) \\
&= a^{+}_{\text{free}}(g) a^{-}_{\text{free}}(h).
\end{aligned}$$

□

Definition 2.81. For any $g^{(2)} \in \mathcal{H}^{\otimes 2}$, we define $a^{+-}\left(g^{(2)}\right)$ on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ by

$$a^{+-}(g^{(2)})f^{(n)} = P_n\, a^{+-}_{\text{free}}\left(g^{(2)}\right) f^{(n)}, \quad f^{(n)} \in \mathcal{F}_n(\mathcal{H}). \tag{2.108}$$

Proposition 2.82. *If $(e_i)_{i\geq 1}$ is an orthonormal basis in $\mathcal{H}$, then*

$$a^{+-}\left(g^{(2)}\right) = \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^{+}(e_i) a^{-}(e_j). \tag{2.109}$$

Furthermore, for each $n \in \mathbb{N}$, the series on the right-hand side of (2.109) *is strongly convergent in $\mathcal{L}\left(\mathcal{F}_n(\mathcal{H})\right)$.*

Proof. By Proposition 2.60 (i), for any $g^{(2)} \in \mathcal{H}^{\otimes 2}$ and $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$,

$$\begin{aligned}
a^{+-}\left(g^{(2)}\right) f^{(n)} &= P_n\, a^{+-}_{\text{free}}\left(g^{(2)}\right) f^{(n)} \\
&= P_n \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^{+}_{\text{free}}(e_i) a^{-}_{\text{free}}(e_j) f^{(n)} \\
&= \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, P_n a^{+}_{\text{free}}(e_i) a^{-}_{\text{free}}(e_j) f^{(n)} \\
&= \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, \left(P_n a^{+}_{\text{free}}(e_i)\right)\, a^{-}_{\text{free}}(e_j) f^{(n)} \\
&= \sum_{i,j\geq 1} (g^{(2)}, e_i \otimes e_j)_{\mathcal{H}^{\otimes 2}}\, a^{+}(e_i)\, a^{-}(e_j) f^{(n)},
\end{aligned}$$

where the series converges in $\mathcal{F}_n(\mathcal{H})$. □.

In order to determine a commutation relation between the creation and annihilation operators, we first calculate $a^-(h)a^+(g)$ applied to a vector $u \in \mathcal{F}_1(\mathcal{H})$. We have

$$\begin{aligned} a^-(g)a^+(h)u &= a^-(g)h \circledast u \\ &= a^-(g)P_2(h \otimes u) \\ &= a^-_{\text{free}}(g)P_2(h \otimes u) \\ &= a^-_{\text{free}}(g)\frac{1}{2}(\mathbf{1} + U)h \otimes u \\ &= \frac{1}{2}\left[a^-_{\text{free}}(g)h \otimes u + a^-_{\text{free}}(g)Uh \otimes u\right] \\ &= (h, Jg)_{\mathcal{H}}\, u + \frac{1}{2}a^-_{\text{free}}(g)Uh \otimes u. \end{aligned} \tag{2.110}$$

Let $\widetilde{U} \in \mathcal{L}\left(\mathcal{H}^{\otimes 2}\right)$. Then

$$\begin{aligned} a^{+-}\left(\widetilde{U}g \otimes h\right) u &= \sum_{i,j\geq 1} \left(\widetilde{U}g \otimes h, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} a^+(e_i)a^-(e_j)u \\ &= \sum_{i,j\geq 1} \left(\widetilde{U}g \otimes h, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (u, Je_j)_{\mathcal{H}}\, e_i. \end{aligned} \tag{2.111}$$

Let $f \in \mathcal{H}$. Then

$$\begin{aligned} \left(\frac{1}{2}a^-_{\text{free}}(g)Uh \otimes u, f\right)_{\mathcal{H}} &= \frac{1}{2}\left(Uh \otimes u, a^+_{\text{free}}(Jg)f\right)_{\mathbb{F}_2(\mathcal{H})} \\ &= (Uh \otimes u, (Jg) \otimes f)_{H^{\otimes 2}}, \end{aligned} \tag{2.112}$$

and, by (2.111),

$$\begin{aligned} \left(a^{+-}(\widetilde{U}g \otimes h)u, f\right)_{\mathcal{H}} &= \sum_{i,j\geq 1} \left(\widetilde{U}g \otimes h, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (u, Je_j)_{\mathcal{H}}(e_i, f)_{\mathcal{H}} \\ &= \sum_{i,j\geq 1} \left(\widetilde{U}g \otimes h, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (e_j, Ju)_{\mathcal{H}}(e_i, f)_{\mathcal{H}} \\ &= \sum_{i,j\geq 1} \left(\widetilde{U}g \otimes h, e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} (e_i \otimes e_j, f \otimes (Ju)) \\ &= \left(\widetilde{U}g \otimes h, \sum_{i,j\geq 1} (f \otimes (Ju), e_i \otimes e_j)e_i \otimes e_j\right)_{\mathcal{H}^{\otimes 2}} \\ &= \left(\widetilde{U}g \otimes h, f \otimes (Ju)\right)_{\mathcal{H}^{\otimes 2}}. \end{aligned} \tag{2.113}$$

Assume that

$$(U(h \otimes u), (Jg) \otimes f)_{\mathcal{H}^{\otimes 2}} = (\widetilde{U} g \otimes h, f \otimes (Ju))_{\mathcal{H}^{\otimes 2}}. \tag{2.114}$$

Then by (2.110) and (2.112)–(2.114), we obtain

$$a^-(g)a^+(h)u = (h, Jg)_{\mathcal{H}} u + a^{+-}(\widetilde{U} g \otimes h)u \tag{2.115}$$

We will now see that the above calculations can be generalised to the case where $u \in \mathcal{F}_1(\mathcal{H})$ can be replaced with $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$ for any n.

Theorem 2.83. *Assume that there exists an operator $\widetilde{U} \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ that satisfies, for all $u_1, u_2, u_3, u_4 \in \mathcal{H}$ such that $Ju_i = u_i$ $(i = 1, 2, 3, 4)$,*

$$(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} = (\widetilde{U} u_3 \otimes u_1, u_4 \otimes u_2)_{\mathcal{H}^{\otimes 2}} . \tag{2.116}$$

Then, for any $g, h \in \mathcal{H}$, we have

$$a^-(g)a^+(h) = (h, Jg)_{\mathcal{H}} + a^{+-}(\widetilde{U} g \otimes h). \tag{2.117}$$

Remark 2.84. If $\widetilde{U}$ exists, then it is unique.

Remark 2.85. Formula (2.116) is equivalent to requiring that, for any $u_1, u_2, u_3, u_4 \in \mathcal{H}$,

$$(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} = (\widetilde{U}(Ju_3) \otimes u_1, u_4 \otimes (Ju_2)). \tag{2.118}$$

Remark 2.86. The operator $\widetilde{U}$ satisfying (2.116) or (2.118) always exists if the Hilbert space $\mathcal{H}$ is finite-dimensional. In the case of an infinite dimensional Hilbert space $\mathcal{H}$, we shall see that this formula is satisfied in all special cases we are interested in.

Proof of Theorem 2.83. We divide the proof into several steps.

Step 1. Let $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$. By Proposition 2.60, (i), we have

$$a^-(g)a^+(h)f^{(n)} = a^-(g)P_{n+1}\left[h \otimes f^{(n)}\right] = a^-_{\text{free}}(g)P_{n+1}\left[h \otimes f^{(n)}\right].$$

By Proposition 2.21, formula (2.18),

$$a^-(g)a^+(h)f^{(n)}$$

$$= a^-_{\text{free}}(g)\frac{1}{n+1}\left[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_nU_{n-1}\cdots U_1\right](\mathbf{1}_{\mathcal{H}}\otimes P_n)\, h\otimes f^{(n)}$$
$$= a^-_{\text{free}}(g)\frac{1}{n+1}\left[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_nU_{n-1}\cdots U_1\right] h\otimes f^{(n)}. \tag{2.119}$$

Step 2. We note that

$$a^-_{\text{free}}(g)\frac{1}{n+1}h\otimes f^{(n)} = (h, Jg)_{\mathcal{H}}\, f^{(n)}. \tag{2.120}$$

Step 3. We will now prove the following lemma.

Lemma 2.87. *Let $n \geq 3$ and $i = 2, 3, \ldots, n-1$. Then*

$$a^-_{\text{free}}(g)U_i = U_{i-1}a^-_{\text{free}}(g) \tag{2.121}$$

on $\mathcal{H}^{\otimes n}$.

Proof. Due to the continuity, we only need to check the equality

$$a^-_{\text{free}}(g)U_i f_1\otimes f_2\otimes f_3\otimes\cdots\otimes f_n = U_{i-1}a^-_{\text{free}}(g) f_1\otimes f_2\otimes f_3\otimes\cdots\otimes f_n \tag{2.122}$$

for any $f_1, \ldots, f_n \in \mathcal{H}$. We have

$$\begin{aligned}
&a^-_{\text{free}}(g)U_i f_1\otimes\cdots\otimes f_i\otimes f_{i+1}\otimes\cdots\otimes f_n\\
&\quad= a^-_{\text{free}}(g) f_1\otimes\cdots\otimes(Uf_i\otimes f_{i+1})\otimes\cdots\otimes f_n\\
&\quad= n\,(f_1, Jg)_{\mathcal{H}}\, f_2\otimes\cdots\otimes(Uf_i\otimes f_{i+1})\otimes\cdots\otimes f_n\\
&\quad= n\,(f_1, Jg)_{\mathcal{H}}\, U_{i-1}f_2\otimes\cdots\otimes f_i\otimes f_{i+1}\otimes\cdots\otimes f_n\\
&\quad= U_{i-1}\, n\,(f_1, Jg)_{\mathcal{H}}\, f_2\otimes\cdots\otimes f_i\otimes f_{i+1}\otimes\cdots\otimes f_n\\
&\quad= U_{i-1}a^-_{\text{free}}(g) f_1\otimes\cdots\otimes f_i\otimes f_{i+1}\otimes\cdots\otimes f_n.
\end{aligned}$$

This proves (2.122). □

Step 4. By Lemma 2.87,

$$\begin{aligned}
&a^-_{\text{free}}(g)\frac{1}{n+1}\left[U_1 + U_2U_1 + \cdots + U_nU_{n-1}\cdots U_1\right] h\otimes f^{(n)}\\
&\quad= \frac{1}{n+1}\left[a^-_{\text{free}}(g)U_1 + U_1a^-_{\text{free}}(g)U_1 + \cdots + U_{n-1}\cdots U_1a^-_{\text{free}}(g)U_1\right] h\otimes f^{(n)}\\
&\quad= \frac{1}{n+1}\left[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_{n-1}U_{n-2}\cdots U_1\right] a^-_{\text{free}}(g)U_1 h\otimes f^{(n)}.
\end{aligned}$$

Step 5. Without loss the generality, we next look at the case when

$$f^{(n)} = f_1 \otimes f_2 \otimes \cdots \otimes f_n.$$

In this case,

$$\begin{aligned} a^-_{\text{free}}(g)U_1 h \otimes f^{(n)} &= a^-_{\text{free}}(g)U_1\, h \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_n \\ &= a^-_{\text{free}}(g)\,(Uh \otimes f_1) \otimes f_2 \otimes \cdots \otimes f_n \\ &= \frac{n+1}{2}\big(a^-_{\text{free}}(g)Uh \otimes f_1\big) \otimes f_2 \otimes \cdots \otimes f_n. \end{aligned}$$

By formulas (2.112)–(2.113)–(2.114)

$$\frac{1}{2}\, a^-_{\text{free}}(g)Uh \otimes f_1 = a^{+-}(\widetilde{U} g \otimes h) f_1 = a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f_1.$$

Therefore,

$$\begin{aligned} a^-_{\text{free}}(g)U_1 h \otimes f^{(n)} &= (n+1)\big(a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f_1\big) \otimes f_2 \otimes \cdots \otimes f_n \\ &= \frac{n+1}{n}\, a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f_1 \otimes f_2 \otimes \cdots \otimes f_n. \end{aligned}$$

Step 6. By Steps 4 and 5, Proposition 2.21, formula (2.18), and formula (2.109),

$$\begin{aligned} & a^-_{\text{free}}(g)\frac{1}{n+1}\,[U_1 + U_2U_1 + \cdots + U_nU_{n-1}\cdots U_1]\, h \otimes f^{(n)} \\ &= \frac{1}{n}\,[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_{n-1}U_{n-2}\cdots U_1]\, a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f^{(n)} \\ &= \frac{1}{n}\,[\mathbf{1} + U_1 + U_2U_1 + \cdots + U_{n-1}U_{n-2}\cdots U_1]\,(\mathbf{1}_{\mathcal{H}} \otimes P_{n-1})\, a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f^{(n)} \\ &= P_n a^{+-}_{\text{free}}(\widetilde{U} g \otimes h) f^{(n)} \\ &= a^{+-}(\widetilde{U} g \otimes h) f^{(n)}. \end{aligned} \tag{2.123}$$

By (2.119), (2.120), and (2.123), the theorem follows. □

Example 2.88. Recall that, for the symmetric tensor product, U is given by $Uf \otimes g = g \otimes f$. Therefore, for any $u_1, u_2, u_3, u_4 \in \mathcal{H}$ such that $Ju_i = u_i$,

$$\begin{aligned} (Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} &= (u_2 \otimes u_1, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} \\ &= (u_2, u_3)_{\mathcal{H}}\,(u_1, u_4)_{\mathcal{H}}, \end{aligned}$$

$$(Uu_3 \otimes u_1, u_4 \otimes u_2)_{\mathcal{H}^{\otimes 2}} = (u_1 \otimes u_3, u_4 \otimes u_2)_{\mathcal{H}^{\otimes 2}}$$
$$= (u_1, u_4)_{\mathcal{H}} (u_3, u_2)_{\mathcal{H}}.$$

This implies that $U = \widetilde{U}$.

By Proposition 2.53 (ii), Proposition 2.74 (ii), and Theorem 2.83.

Example 2.89. Recall that, for the antisymmetric tensor product, U is given by $Uf \otimes g = -g \otimes f$. Similarly to Example 2.88, we conclude that $U = \widetilde{U}$.

Example 2.90. Recall the Q-symmetric tensor product discussed in Section 2.5. In particular, $\mathcal{H} = L^2(X, \sigma)$, $Q : X^{(2)} \to \mathbb{C}$ and

$$\left(Uf^{(2)}\right)(x, y) = Q(x, y) f^{(2)}(y, x).$$

We shall find the formula of $\widetilde{U}$. Let $f^{(2)} \in \mathcal{H}^{\otimes 2}$. Then, for any $u_1, u_2, u_3, u_4 \in \mathcal{H}$ such that $Ju_i = u_i$,

$$\begin{aligned}
(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} &= \int_{X^2} (Uu_1 \otimes u_2)(x, y) (u_3 \otimes u_4)(x, y) \sigma(dx)\sigma(dy) \\
&= \int_{X^2} Q(x, y)(u_1 \otimes u_2)(y, x) u_3(x) u_4(y) \sigma(dx)\sigma(dy) \\
&= \int_{X^2} Q(x, y) u_1(y) u_2(x) u_3(x) u_4(y) \sigma(dx)\sigma(dy).
\end{aligned}$$

Swapping the variables $x \leftrightarrow y$,

$$\begin{aligned}
(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} &= \int_{X^2} Q(y, x) u_1(x) u_2(y) u_3(y) u_4(x) \sigma(dx)\sigma(dy) \\
&= \int_{X^2} \big(Q(y, x) u_1(x) u_3(y)\big) u_2(y) u_4(x) \sigma(dx)\sigma(dy) \\
&= \int_{X^2} (Q(y, x)(u_3 \otimes u_1)(y, x)) (u_4 \otimes u_2)(x, y) \sigma(dx)\sigma(dy) \\
&= \int_{X^2} \left(\widetilde{U}(u_3 \otimes u_1)(y, x)\right) (u_4 \otimes u_2)(x, y) \sigma(dx)\sigma(dy) \\
&= \left(\widetilde{U} u_3 \otimes u_1, u_4 \otimes u_2\right)_{\mathcal{H}^{\otimes 2}},
\end{aligned}$$

where

$$\widetilde{U} f^{(2)}(x, y) = Q(y, x) f^{(2)}(y, x). \tag{2.124}$$

Example 2.91. We consider the case of a multicomponent quantum system as discussed in Section 2.6. In particular, $\mathcal{H} = L^2(X \to V, \sigma)$ and the operator U is given by (2.32). Then, for any $u_1, u_2, u_3, u_4 \in \mathcal{H}$ such that $Ju_i = u_i$,

$$\begin{aligned}(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} &= \int_{X^{(2)}} \big((Uu_1 \otimes u_2)(x,y), (u_3 \otimes u_4)(x,y)\big)_{V^{\otimes 2}}\, \sigma(dx)\sigma(dy)\\ &= \int_{X^{(2)}} \big(C(x,y)u_1(y) \otimes u_2(x), u_3(x) \otimes u_4(y)\big)_{V^{\otimes 2}}\, \sigma(dx)\sigma(dy).\end{aligned}$$

Swapping the variables $x \leftrightarrow y$, we obtain

$$(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} = \int_{X^{(2)}} \big(C(y,x)u_1(x) \otimes u_2(y), u_3(y) \otimes u_4(x)\big)_{V^{\otimes 2}}\, \sigma(dx)\sigma(dy). \tag{2.125}$$

For an operator $C \in \mathcal{L}(V^{\otimes 2})$, we denote by $\widetilde{C}$ the operator from $\mathcal{L}(V^{\otimes 2})$ satisfying

$$(Cv_1 \otimes v_2, v_3 \otimes v_4)_{V^{\otimes 2}} = (\widetilde{C}v_3 \otimes v_1, v_4 \otimes v_2)_{V^{\otimes 2}}$$

for all $v_1, v_2, v_3, v_4 \in V$ such that $Jv_i = v_i$ $(i = 1, 2, 3, 4)$. (We have assumed that $\widetilde{C}$ exists.)

Assuming that $\widetilde{C}(x,y)$ exists for all $(x,y) \in X^{(2)}$, we obtain from (2.125)

$$\begin{aligned}(Uu_1 \otimes u_2, u_3 \otimes u_4)_{\mathcal{H}^{\otimes 2}} &= \int_{X^{(2)}} \big(\widetilde{C}(y,x)u_3(y) \otimes u_1(x), u_4(x) \otimes u_2(y)\big)_{V^{\otimes 2}}\, \sigma(dx)\sigma(dy)\\ &= (\widetilde{U}u_3 \otimes u_1, u_4 \otimes u_2)_{\mathcal{H}^{\otimes 2}},\end{aligned}$$

where

$$(\widetilde{U}f^{(2)})(x,y) = \widetilde{C}(y,x)f^{(2)}(y,x), \quad f^{(2)} \in \mathcal{H}^{\otimes 2}, \tag{2.126}$$

compare with (2.124).

Next, similarly to Example 2.58, let us assume that $V = L^2(S, \nu)$ and let us think (at least informally) of an operator $C \in \mathcal{L}(V^{\otimes 2})$ as an integral operator with integral kernel $C(s,t,u,v)$. Let us find the operator $\widetilde{C}$. Let $g_1, g_2, g_3, g_4 \in V$ be real-valued. Then

$$(Cg_1 \otimes g_2, g_3 \otimes g_4)_{V^{\otimes 2}} = \int_{S^4} C(s,t,u,v)g_1(u)g_2(v)g_3(s)g_4(t)\nu(du)\nu(dv)\nu(ds)\nu(dt).$$

Changing the variables

$$t \to s,\ s \to u,\ u \to v,\ v \to t,$$

we get

$$
\begin{aligned}
&(Cg_1 \otimes g_2, g_3 \otimes g_4)_{V^{\otimes 2}} \\
&\quad = \int_{S^4} C(u,s,v,t) g_1(v) g_2(t) g_3(u) g_4(s) \nu(du)\nu(dv)\nu(ds)\nu(dt) \\
&\quad = \int_{S^2} \left(\int_{S^2} C(u,s,v,t) g_3(u) g_1(v) \nu(du)\nu(dv) \right) g_4(s) g_2(t) \nu(ds)\nu(dt) \\
&\quad = \int_{S^2} \left(\int_{S^2} C(u,s,v,t) (g_3 \otimes g_1)(u,v) \nu(du)\nu(dv) \right) (g_4 \otimes g_2)(s,t) \nu(ds)\nu(dt) \\
&\quad = \left(\widetilde{C} g_3 \otimes g_1, g_4 \otimes g_2 \right)_{V^{\otimes 2}}.
\end{aligned}
$$

Here, for $g^{(2)} \in V^{\otimes 2}$,

$$\left(\widetilde{C} g^{(2)} \right)(s,t) = \int_{S^2} \widetilde{C}(s,t,u,v) g^{(2)}(u,v) \nu(du)\nu(dv), \tag{2.127}$$

where

$$\widetilde{C}(s,t,u,v) = C(u,s,v,t). \tag{2.128}$$

Let us now think (at least informally) of $C(x,y)$ as an integral operator with integral kernel $C(x,y,s,t,u,v)$. Then, by (2.126)–(2.128), for any $f^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$(\widetilde{U} f^{(2)})(x,y,s,t) = \int_{S^2} C(y,x,u,s,v,t) f^{(2)}(y,x,u,v) \nu(du)\nu(dv). \tag{2.129}$$

2.10 Conclusions

We aim in this section to briefly sum up the main results we discussed regarding the commutation of creation and annihilation operators.

Corollary 2.92 (The general case)**.** *Let $\mathcal{H}$ be a separable complex Hilbert space, and let $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be a self-adjoint, unitary operator satisfying the Yang–Baxter equation* (2.12)*. Furthermore, assume that there exists an operator $\widetilde{U} \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ that satisfies* (2.118) *for any $u_1, u_2, u_3, u_4 \in \mathcal{H}$. Then the creation and annihilation operators in the U-deformed Fock space $\mathcal{F}(\mathcal{H})$ satisfy the following commutation relations.*

(i) For any $g^{(2)} \in \mathcal{H}^{\otimes 2}$, we have

$$a^{++}\left(g^{(2)}\right) = a^{++}\left(U g^{(2)}\right).$$

In particular, for any $g, h \in \mathcal{H}$,

$$a^+(g)a^+(h) = a^{++}(Ug \otimes h).$$

(ii) For any $g^{(2)} \in \mathcal{H}^{\otimes 2}$, *we have*

$$a^{--}\left(g^{(2)}\right) = a^{--}\left(\widehat{U} g^{(2)}\right).$$

In particular, for any $g, h \in \mathcal{H}$,

$$a^-(g)a^-(h) = a^{--}\left(\widehat{U} g \otimes h\right).$$

Here $\widehat{U} = \mathbb{S}U\mathbb{S}$, *where* $\mathbb{S}f \otimes g = (Jg) \otimes (Jf)$.

(iii) For any $g, h \in \mathcal{H}$,

$$a^-(g)a^+(h) = (h, Jg)_{\mathcal{H}} + a^{+-}\left(\widetilde{U} g \otimes h\right).$$

Corollary 2.93 (Q-symmetric tensor product, Abelian anyons)**.** *Let* X *be a locally compact Polish space, and let* σ *be a non-atomic Radon measure on* X. *Let* $\mathcal{H} = L^2(X, \sigma)$. *Let* $X^{(2)}$ *be a symmetric subset of* X^2 *such that, for any* $(x, y) \in X^2$ $x \neq y$, *and* $\sigma^{\otimes 2}(X^2 \setminus X^{(2)}) = 0$. *Let* $Q : X^{(2)} \to \mathbb{C}$ *satisfy* (2.24), (2.25), *and let* $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ *be given by*

$$\left(Uf^{(2)}\right)(x, y) = Q(x, y) f^{(2)}(y, x).$$

Then the creation and annihilation operators in the U*-deformed Fock space* $\mathcal{F}(\mathcal{H})$ *satisfy the following commutation relations.*

(i) For $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$\int_{X^{(2)}} g^{(2)}(x, y) a^+(x) a^+(y) \sigma(dx)\sigma(dy) = \int_{X^{(2)}} g^{(2)}(x, y) Q(y, x) a^+(y) a^+(x) \sigma(dx)\sigma(dy),$$
$$\int_{X^{(2)}} g^{(2)}(x, y) a^-(x) a^-(y) \sigma(dx)\sigma(dy) = \int_{X^{(2)}} g^{(2)}(x, y) Q(y, x) a^-(y) a^-(x) \sigma(dx)\sigma(dy),$$

and for any $g, h \in \mathcal{H}$

$$\int_{X^{(2)}} g(x)h(y) a^-(x) a^+(y) \sigma(dx)\sigma(dy) = \int_X g(x)h(x)\sigma(dx) + \int_{X^{(2)}} Q(x, y) g(x) h(y) a^+(y) a^-(x) \sigma(dx)\sigma(dy).$$

These relations can be formally written as

$$
\begin{aligned}
a^+(x)a^+(y) &= Q(y,x)a^+(y)a^+(x),\\
a^-(x)a^-(y) &= Q(y,x)a^-(y)a^-(x),\\
a^-(x)a^+(y) &= \delta(x,y) + Q(x,y)a^+(y)a^-(x).
\end{aligned}
\tag{2.130}
$$

Here, $\delta(x,y)$ satisfies, for $g, h \in \mathcal{H}$,

$$\int_{X^{(2)}} g(x)h(y)\delta(x,y)\sigma(dx)\sigma(dy) = \int_X g(x)h(x)\sigma(dx). \tag{2.131}$$

(ii) In particular, in the case of Abelian anyons, $X = \mathbb{R}^2$, $\sigma(dx) = dx = dx_1\,dx_2$, $\mathcal{H} = L^2(\mathbb{R}^2, dx)$, $X^{(2)}$ is given by (2.29)*, $Q(x,y)$ is given by* (2.30) *with $q \in \mathbb{C}$, $|q| = 1$, for any $x, y \in \mathbb{R}^2$ with $x_1 < y_1$*

$$
\begin{aligned}
a^+(x)a^+(y) &= \bar{q}a^+(y)a^+(x),\\
a^-(x)a^-(y) &= \bar{q}a^-(y)a^-(x),\\
a^-(x)a^+(y) &= \delta(x,y) + qa^+(y)a^-(x),
\end{aligned}
$$

and for any $x, y \in \mathbb{R}^2$ with $x_1 > y_1$

$$
\begin{aligned}
a^+(x)a^+(y) &= qa^+(y)a^+(x),\\
a^-(x)a^-(y) &= qa^-(y)a^-(x),\\
a^-(x)a^+(y) &= \delta(x,y) + \bar{q}a^+(y)a^-(x).
\end{aligned}
$$

Corollary 2.94 (Multicomponent system, non-Abelian anyons)**.** *Let X, σ, and $X^{(2)}$ be as in Corollary 2.93. Let S be a locally compact Polish space and let ν be a Radon measure on S. Let $V = L^2(S, \nu)$ and denote $\langle v, w\rangle_V = \int_S v(s)w(s)\nu(ds)$ for $v, w \in V$, and similarly $\langle v^{(2)}, w^{(2)}\rangle_{V^{\otimes 2}} = \int_{S^2} v^{(2)}(s,t)w^{(2)}(s,t)\nu(ds)\nu(dt)$ for $v^{(2)}, w^{(2)} \in V^{\otimes 2} = L^2(S^2, \nu^{\otimes 2})$.*

Let $\mathcal{H} = L^2(X \to V, \sigma) = L^2(X \times S, \sigma \otimes \nu)$, hence $\mathcal{H}^{\otimes 2} = L^2(X^{(2)} \to V^{\otimes 2}, \sigma^{\otimes 2}) = L^2(X^{(2)} \times S^2, \sigma^{\otimes 2} \otimes \nu^{\otimes 2})$. Let $C : X^{(2)} \to \mathcal{L}(V^{\otimes 2})$ be such that, for each $(x,y) \in X^{(2)}$, $C(x,y)$ is unitary, $C^*(x,y) = C(y,x)$, and (2.31) *holds. We also assume that each $C(x,y)$ is an integral operator in $V^{\otimes 2}$ with integral kernel $C(x,y,s,t,u,v)$.*

Let $U \in \mathcal{L}(\mathcal{H}^{\otimes 2})$ be given by

$$(Uf^{(2)})(x,y) = C(x,y)f^{(2)}(y,x).$$

Consider the creation and annihilation operators in the corresponding U-deformed Fock space $\mathcal{F}(\mathcal{H})$. We will formally write

$$a^+(g) = \int_X \langle g(x), a^+(x)\rangle_V \, \sigma(dx),$$
$$a^-(g) = \int_X \langle g(x), a^-(x)\rangle_V \, \sigma(dx)$$

for $g \in \mathcal{H}$. Similarly, for $g^{(2)} \in \mathcal{H}^{\otimes 2}$, we will formally write

$$a^{++}(g^{(2)}) = \int_{X^{(2)}} \langle g^{(2)}(x,y), a^+(x) \otimes a^+(y)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy),$$

and similarly for $a^{--}(g^{(2)})$ and $a^{+-}(g^{(2)})$.

(i) For $g^{(2)} \in \mathcal{H}^{\otimes 2}$,

$$\int_{X^{(2)}} \langle g^{(2)}(x,y), a^+(x) \otimes a^+(y)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy)$$
$$= \int_{X^{(2)}} \langle C(y,x) g^{(2)}(x,y), a^+(y) \otimes a^+(x)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy),$$
$$\int_{X^{(2)}} \langle g^{(2)}(x,y), a^-(x) \otimes a^-(y)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy)$$
$$= \int_{X^{(2)}} \langle \widehat{C}(x,y) g^{(2)}(x,y), a^-(y) \otimes a^-(x)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy).$$

Here for $v^{(2)} \in V^{\otimes 2}$,

$$\big(\widehat{C}(x,y)v^{(2)}\big)(s,t) = \int_{S^2} \overline{C(x,y,t,s,v,u)}\, v^{(2)}(u,v)\nu(du)\nu(dv)$$
$$= \int_{S^2} C(y,x,v,u,t,s) v^{(2)}(u,v)\nu(du)\nu(dv).$$

Next, for any $g, h \in \mathcal{H}$,

$$\int_{X^{(2)}} \langle g(x) \otimes h(y), a^-(x) \otimes a^+(y)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy) = \int_X \langle g(x), h(x)\rangle_V \, \sigma(dx)$$
$$+ \int_{X^{(2)}} \langle \widetilde{C}(x,y) g(x) \otimes h(y), a^+(y) \otimes a^-(x)\rangle_{V^{\otimes 2}} \, \sigma(dx)\sigma(dy).$$

Here, for $v^{(2)} \in V^{\otimes 2}$,

$$\big(\widetilde{C}(x,y)v^{(2)}\big)(s,t) = \int_{S^2} \overline{C(x,y,t,s,v,u)}\, v^{(2)}(u,v)\nu(du)\nu(dv)$$
$$= \int_{S^2} C(y,x,u,s,v,t) v^{(2)}(u,v)\nu(du)\nu(dv).$$

These relations can be formally written as

$$
\begin{aligned}
a^+(x)\otimes a^+(y) &= C^T(y,x)a^+(y)\otimes a^+(x),\\
a^-(x)\otimes a^-(y) &= \widehat{C}^T(x,y)a^-(y)\otimes a^-(x),\\
a^-(x)\otimes a^+(y) &= \Delta(x,y)+\widetilde{C}^T(x,y)a^+(y)\otimes a^-(x).
\end{aligned}
\tag{2.132}
$$

Here, $\Delta(x,y)$ *satisfies, for* $g,h\in\mathcal{H}$,

$$
\int_{X^{(2)}}\langle g(x)\otimes h(y),\Delta(x,y)\rangle_{V^{\otimes 2}}\,\sigma(dx)\sigma(dy)=\int_X\langle g(x),h(x)\rangle_V\,\sigma(dx),
\tag{2.133}
$$

and for $v^{(2)}\in V^{\otimes 2}$, *we have*

$$
\begin{aligned}
(C^T(x,y)v^{(2)})(s,t) &= \int_{S^2} C(y,x,u,v,s,t)v^{(2)}(u,v)\nu(du)\nu(dv),\\
(\widehat{C}^T(x,y)v^{(2)})(s,t) &= \int_{S^2}\overline{C(x,y,v,u,t,s)}v^{(2)}(u,v)\nu(du)\nu(dv)\\
&= \int_{S^2} C(y,x,t,s,v,u)v^{(2)}(u,v)\nu(du)\nu(dv)\\
(\widetilde{C}^T(x,y)u^{(2)})(s,t) &= \int_{S^2} C(x,y,s,u,t,v)v^{(2)}(u,v)\nu(du)\nu(dv).
\end{aligned}
$$

(ii) In particular, in the case of non-Abelian anyons, $X=\mathbb{R}^2$, $\sigma(dx)=dx=dx_1\,dx_2$, $X^{(2)}$ *is given by* (2.29), $C(x,y)$ *is given by* (2.36) *in which* C *is a unitary operator in* V *satisfying the Yang–Baxter equation* (2.35) *in* $V^{\otimes 3}$. *Let also* C *be an integral operator in* V *with integral kernel* $C(s,t,u,v)$. *Then, for any* $x,y\in\mathbb{R}^2$ *with* $x_1<y_1$,

$$
\begin{aligned}
\big(a^+(x)\otimes a^+(y)\big)(s,t) &= \int_{S^2}\overline{C(s,t,u,v)}\,\big(a^+(y)\otimes a^+(x)\big)(u,v)\nu(du)\nu(dv),\\
\big(a^-(x)\otimes a^-(y)\big)(s,t) &= \int_{S^2}\overline{C(v,u,t,s)}\,\big(a^-(y)\otimes a^-(x)\big)(u,v)\nu(du)\nu(dv),\\
\big(a^-(x)\otimes a^+(y)\big)(s,t) &= \Delta(x,y)\Delta(s,t)+\int_{S^2}C(s,u,t,v)\big(a^+(y)\otimes a^-(x)\big)(u,v)\nu(du)\nu(dv),
\end{aligned}
$$

and for $x,y\in\mathbb{R}^2$ *with* $y_1>x_1$

$$
\begin{aligned}
\big(a^+(x)\otimes a^+(y)\big)(s,t) &= \int_{S^2}C(u,v,s,t)\big(a^+(y)\otimes a^+(x)\big)(u,v)\nu(du)\nu(dv),\\
\big(a^-(x)\otimes a^-(y)\big)(s,t) &= \int_{S^2}C(t,s,v,u)\,\big(a^-(y)\otimes a^-(x)\big)(u,v)\nu(du)\nu(dv),\\
\big(a^-(x)\otimes a^+(y)\big)(s,t) &= \Delta(x,y)\Delta(s,t)+\int_{S^2}\overline{C(t,v,s,u)}\,\big(a^+(y)\otimes a^-(x)\big)(u,v)\nu(du)\nu(dv).
\end{aligned}
$$

Chapter 3

Quasi-free states on the Wick algebra of the Q-anyon commutation relations

3.1 The Q-ACR algebra

Assume that Y and Z are locally compact Polish spaces and consider $X = Y \times Z$. Let ν and μ be Radon measures on Y and Z, respectively. We define a Radon measure σ on X by $\sigma = \nu \otimes \mu$. We fix a function $Q : Y^2 \to \mathbb{C}$ which is continuous and satisfies

$$|Q(y_1, y_2)| = 1, \quad Q(y_1, y_2) = \overline{Q(y_2, y_1)}, \quad \text{for all } (y_1, y_2) \in Y^2.$$

With an abuse of notation, we will also consider Q as a (continuous) function on X^2 defined by

$$Q(x_1, x_2) = Q(y_1, y_2), \quad x_i = (y_i, z_i), \; i = 1, 2. \tag{3.1}$$

Thus, we also have

$$|Q(x_1, x_2)| = 1, \quad Q(x_1, x_2) = \overline{Q(x_2, x_1)}, \quad \text{for all } (x_1, x_2) \in X^2.$$

We obviously have $Q(y, y) = 1$ or -1 for all $y \in Y$. We will assume that $Q(y, y)$ is constant for all $y \in Y$. This assumption is always satisfied if Y is a connected space (like $\mathbb{R}$).

Example 3.1. Let $Y = \mathbb{R}$, $Z = \mathbb{R}^{d-1}$ $(d \geq 2)$, $\nu(dy) = dy$, $\mu(dz) = dz$. Then $X = \mathbb{R}^d$ and $\sigma(dx) = dx$. Fix an arbitrary $\alpha \in \mathbb{R}$, and define $Q : Y^2 \to \mathbb{C}$ by

$$Q(y_1, y_2) = e^{i\alpha(y_2 - y_1)}.$$

Such a function Q satisfies the above assumptions and $Q(y, y) = 1$ for all $y \in Y$. Alternatively, we may choose

$$Q(y_1, y_2) = -e^{i\alpha(y_2 - y_1)},$$

in which case $Q(y, y) = -1$ for all $y \in Y$.

Example 3.2. Let X, Y, Z, σ, ν, μ, and α be as in Example 3.1. Fix an arbitrary $\varepsilon > 0$. Define $Q : Y^2 \to \mathbb{C}$ by

$$Q(y_1, y_2) = \begin{cases} e^{i\alpha(y_2 - y_1)}, & \text{if } |y_2 - y_1| < \varepsilon, \\ q, & \text{if } y_1 \leq y_2 - \varepsilon, \\ \bar{q}, & \text{if } y_1 \geq y_2 + \varepsilon, \end{cases}$$

where $q = e^{i\alpha\varepsilon}$. Thus, this function is equal to the Q-function in the case of the abelian anyons if the distance between y_1 and y_2 is $\geq \varepsilon$. Clearly, $Q(y, y) = 1$ for all $y \in Y$. Alternatively, we may chose

$$Q(y_1, y_2) = \begin{cases} -e^{i\alpha(y_2 - y_1)}, & \text{if } |y_2 - y_1| < \varepsilon, \\ q, & \text{if } y_1 \leq y_2 - \varepsilon, \\ \bar{q}, & \text{if } y_1 \geq y_2 + \varepsilon, \end{cases}$$

where $q = -e^{i\alpha\varepsilon}$, in which case $Q(y, y) = -1$ for all $y \in Y$.

We denote by $C_0(Y^n)$ the space of all continuous functions $\varphi^{(n)} : Y^n \to \mathbb{C}$ with compact support.

Denote $\mathcal{G} = L^2(Z, \mu)$ and define

$$\mathcal{G}^{\otimes_{\mathrm{alg}} n} = \mathrm{l.\,s.}\{g_1 \otimes g_2 \otimes \cdots \otimes g_n \mid g_1, \dots, g_n \in \mathcal{G}\}.$$

Here l. s. denotes the linear span.

Now, we define

$$\begin{aligned}\mathfrak{F}(X^n) &= C_0(Y^n) \otimes_{\text{alg}} \mathcal{G}^{\otimes \text{alg}\, n} \\ &= \text{l. s.}\{f^{(n)}(x_1, \dots, x_n) = \varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n) \mid \\ &\qquad \varphi^{(n)} \in C_0(Y^n),\ g_1, g_2, \dots, g_n \in \mathcal{G}\}. \end{aligned} \tag{3.2}$$

Remark 3.3. If $f^{(n)} \in \mathfrak{F}(X^n)$, then $\overline{f^{(n)}} \in \mathfrak{F}(X^n)$. Furthermore, if $f^{(n)} \in \mathfrak{F}(X^n)$, then $f^{(n)}\left(x_{\xi(1)}, \dots, x_{\xi(n)}\right) \in \mathfrak{F}(X^n)$ for each permutation $\xi \in S_n$.

Remark 3.4. If $f^{(n)} \in \mathfrak{F}(X^n)$, then

$$Q(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_n),\ Q(x_{i+1}, x_i) f^{(n)}(x_1, \dots, x_n) \in \mathfrak{F}(X^n).$$

Remark 3.5. Let $f^{(n)}(x_1, \dots, x_n) = \varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n) \in \mathfrak{F}(X^n)$. Then

$$\begin{aligned}&\int_X \sigma(dx) f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \\ &\quad= \int_Y \nu(dy) \varphi^{(n)}(y_1, \dots, y_{i-1}, y, y, y_i, \dots, y_{n-2})\, g_1(z_1) \dots g_{i-1}(z_{i-1}) \\ &\quad\times \left(\int_Z \mu(dz) g_i(z) g_{i+1}(z) \right) g_{i+2}(z_i) \cdots g_n(z_{n-2}).\end{aligned}$$

Therefore, for each $f^{(n)} \in \mathfrak{F}(X^n)$,

$$\int_X \sigma(dx) f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \in \mathfrak{F}\left(X^{n-2}\right).$$

Intuitively, elements of the algebra of the Q-anyon commutation relations (Q-ACR algebra) can be represented by operator-valued integrals

$$\int_{X^n} f^{(n)}(x_1, \dots, x_n) a^{\sharp_1}(x_1) \cdots a^{\sharp_n}(x_n) \sigma(dx_1) \cdots \sigma(dx_n), \tag{3.3}$$

where $f^{(n)}(x_1, \dots, x_n) \in \mathfrak{F}(X^n)$ and $\sharp_1, \dots, \sharp_n \in \{+, -\}$. These operator-valued integrals must be subject to the commutation relations (2.130) which hold 'pointwise.' More precisely, if $i \in \{1, \dots, n-1\}$ and $\sharp_i = \sharp_{i+1}$, then

$$\begin{aligned}&\int_{X^n} f^{(n)}(x_1, \dots, x_n) a^{\sharp_1}(x_1) \cdots a^{\sharp_n}(x_n) \sigma(dx_1) \cdots \sigma(dx_n) \\ &\quad= \int_{X^n} f^{(n)}(x_1, \dots, x_n) Q(x_{i+1}, x_i) a^{\sharp_1}(x_1) \cdots a^{\sharp_{i+1}}(x_{i+1}) a^{\sharp_i}(x_i) \cdots a^{\sharp_n}(x_n)\end{aligned}$$

$$\times\, \sigma(dx_1)\cdots\sigma(dx_n)$$

$$= \int_{X^n} Q(x_i,x_{i+1}) f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,\dots,x_n)) a^{\sharp_1}(x_1)\cdots a^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n).$$

If $\sharp_i = -$, $\sharp_{i+1} = +$, then

$$\int_{X^n} f^{(n)}(x_1,\dots,x_n) a^{\sharp_1}(x_1)\cdots a^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n)$$
$$= \int_{X^n} f^{(n)}(x_1,\dots,x_n) Q(x_i,x_{i+1}) a^{\sharp_1}(x_1)\cdots a^{\sharp_{i+1}}(x_{i+1}) a^{\sharp_i}(x_i)\cdots a^{\sharp_n}(x_n)$$
$$\times\, \sigma(dx_1)\cdots\sigma(dx_n) + \int_{X^{n-2}} \left(\int_X f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx)\right)$$
$$\times\, a^{\sharp_1}(x_1)\cdots a^{\sharp_{i-1}}(x_{i-1}) a^{\sharp_{i+2}}(x_i)\cdots a^{\sharp_n}(x_{n-2})\sigma(dx_1)\cdots\sigma(dx_{n-2})$$
$$= \int_{X^n} Q(x_{i+1},x_i) f^{(n)}(x_1,\dots,x_{i+1},x_i,\dots,x_n) a^{\sharp_1}(x_1)\cdots a^{\sharp_{i+1}}(x_i) a^{\sharp_i}(x_{i+1})\cdots a^{\sharp_n}(x_n)$$
$$\times\, \sigma(dx_1)\dots\sigma(dx_n) + \int_{X^{n-2}} \left(\int_X f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx)\right)$$
$$\times\, a^{\sharp_1}(x_1)\cdots a^{\sharp_{i-1}}(x_{i-1}) a^{\sharp_{i+2}}(x_i)\cdots a^{\sharp_n}(x_{n-2})\sigma(dx_1)\cdots\sigma(dx_{n-2}).$$

We note that the adjoint of

$$\int_{X^n} f^{(n)}(x_1,\dots,x_n) a^{\sharp_1}(x_1)\cdots a^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n)$$

should be (at least formally)

$$\int_{X^n} \overline{f^{(n)}(x_1,\dots,x_n)}\, a^{-\sharp_n}(x_n)\cdots a^{-\sharp_1}(x_1)\sigma(dx_1)\cdots\sigma(dx_n)$$
$$= \int_{X^n} \overline{f^{(n)}(x_n,\dots,x_1)}\, a^{-\sharp_n}(x_1)\cdots a^{-\sharp_1}(x_n)\sigma(dx_1)\cdots\sigma(dx_n),$$

where

$$-\sharp = \begin{cases} -, & \text{if } \sharp = +,\\ +, & \text{if } \sharp = -. \end{cases}$$

Next, we have, for any $f^{(m)} \in \mathfrak{F}(X^m)$ and $g^{(n)} \in \mathfrak{F}(X^n)$,

$$\int_{X^m} f^{(m)}(x_1,\dots,x_m) a^{\sharp_1}(x_1)\cdots a^{\sharp_m}(x_m)\sigma(dx_1)\cdots\sigma(dx_m)$$
$$\times \int_{X^n} g^{(n)}(x_{m+1},\dots,x_{m+n}) a^{\sharp_{m+1}}(x_{m+1})\cdots a^{\sharp_{m+n}}(x_{m+n})\sigma(dx_{m+1})\cdots\sigma(dx_{m+n})$$
$$= \int_{X^{m+n}} \left(f^{(m)}\otimes g^{(n)}\right)(x_1,\dots,x_{m+n}) a^{\sharp_1}(x_1)\cdots a^{\sharp_{m+n}}(x_{m+n})\sigma(dx_1)\cdots\sigma(dx_{m+n}),$$

where $f^{(m)} \otimes g^{(n)} \in \mathfrak{F}(X^{m+n})$ is defined by

$$f^{(m)} \otimes g^{(n)}(x_1, \ldots, x_{m+n}) = f^{(m)}(x_1, \ldots, x_m) g^{(n)}(x_{m+1}, \ldots, x_{m+n}).$$

To define the Q-ACR algebra, we will write $\Phi^{(n)}\left(f^{(n)}; \sharp_1, \ldots, \sharp_n\right)$ for

$$\int_{X^n} f^{(n)}(x_1, \ldots, x_n) a^{\sharp_1}(x_1) \cdots a^{\sharp_n}(x_n) \sigma(dx_1) \cdots \sigma(dx_n).$$

We will sometimes also write $\Phi^{(n)}\left(f^{(n)}(x_1, \ldots, x_n); \sharp_1, \ldots, \sharp_n\right)$.

Recall that a unital $*$-algebra $\mathbb{A}$ is an algebra with identity element for multiplication and a $*$-operation $* : \mathbb{A} \to \mathbb{A}$ satisfying the standard axioms.

Definition 3.6. The *Q-ACR algebra* is defined as the unital $*$-algebra that is generated by elements of the form $\Phi^{(n)}\left(f^{(n)}; \sharp_1, \ldots, \sharp_n\right)$ for $f^{(n)} \in \mathfrak{F}(X^n)$ $(n \in \mathbb{N})$ and $\sharp_1, \ldots, \sharp_n \in \{+, -\}$. These elements satisfy the following relations:

- If $i \in \{1, \ldots, n-1\}$ and $\sharp_i = \sharp_{i+1}$, then

$$\Phi^{(n)}\left(f^{(n)}; \sharp_1, \ldots, \sharp_n\right) = \Phi^{(n)}\left(Q(x_i, x_{i+1}) f^{(n)}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n); \sharp_1, \ldots, \sharp_n\right). \tag{3.4}$$

- If $i \in \{1, \ldots, n-1\}$ and $\sharp_i = -$, $\sharp_{i+1} = +$, then

$$\begin{aligned}&\Phi^{(n)}\left(f^{(n)}; \sharp_1, \ldots, \sharp_n\right)\\ &= \Phi^{(n)}\left(Q(x_{i+1}, x_i) f^{(n)}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n);\ \sharp_1, \ldots, \sharp_{i+1}, \sharp_i, \ldots, \sharp_n\right)\\ &\quad + \Phi^{(n-2)}\left(\int_X f^{(n)}(x_1, \ldots, x_{i-1}, x, x, x_i, \ldots, x_{n-2}) \sigma(dx); \sharp_1, \ldots, \sharp_{i-1}, \sharp_{i+2}, \ldots, \sharp_n\right).\end{aligned} \tag{3.5}$$

The multiplication in the Q-ACR algebra is given by

$$\Phi^{(m)}\left(f^{(m)}; \sharp_1, \ldots, \sharp_m\right) \Phi^{(n)}\left(g^{(n)}; \sharp_{m+1}, \ldots, \sharp_{m+n}\right) = \Phi^{(m+n)}\left(f^{(m)} \otimes g^{(n)}; \sharp_1, \ldots, \sharp_{m+n}\right).$$

The addition in the Q-ACR algebra satisfies, for any $\lambda, \mu \in \mathbb{C}$ and $f^{(n)}, g^{(n)} \in \mathfrak{F}(X^n)$,

$$\lambda \Phi^{(n)}(f^{(n)}; \sharp_1, \ldots, \sharp_n) + \mu \Phi^{(n)}(g^{(n)}; \sharp_1, \ldots, \sharp_n) = \Phi^{(n)}(\lambda f^{(n)} + \mu g^{(n)}; \sharp_1, \ldots, \sharp_n).$$

The $*$-operation in the Q-ACR algebra is defined by

$$\Phi^{(n)}(f^{(n)}; \sharp_1, \ldots, \sharp_n)^* = \Phi^{(n)}\left(\overline{f^{(n)}(x_n, \ldots, x_1)}; -\sharp_n, \ldots, -\sharp_1\right). \tag{3.6}$$

We define $P_n : \mathfrak{F}(X^n) \to \mathfrak{F}(X^n)$ by formulas (2.27) and (2.28), which now hold pointwise. For each $n \geq 2$, we denote

$$\mathfrak{S}\mathfrak{F}\left(X^n\right) = \{f^{(n)} \in \mathfrak{F}\left(X^n\right) \mid P_n f^{(n)} = f^{(n)}\},$$

and for any $m, n \geq 1$,

$$\mathfrak{S}\mathfrak{F}^{(m,n)}\left(X^{m+n}\right) = \{f^{(m+n)} \in \mathfrak{F}\left(X^{m+n}\right) \mid P_m \otimes P_n f^{(m+n)} = f^{(m+n)}\}.$$

It easily follows from (3.4) that, if $\sharp_1 = \cdots = \sharp_m$, then for any $f^{(m+n)} \in \mathfrak{F}(X^{m+n})$,

$$\Phi^{(m+n)}\left(f^{(m+n)}; \sharp_1, \ldots, \sharp_{m+n}\right) = \Phi^{(m+n)}\left(P_m \otimes \mathbf{1}_n f^{(m+n)}; \sharp_1, \ldots, \sharp_{m+n}\right), \tag{3.7}$$

and if $\sharp_{m+1} = \cdots = \sharp_{m+n}$, then

$$\Phi^{(m+n)}\left(f^{(m+n)}; \sharp_1, \ldots, \sharp_{m+n}\right) = \Phi^{(m+n)}\left(\mathbf{1}_m \otimes P_n f^{(m+n)}; \sharp_1, \ldots, \sharp_{m+n}\right), \tag{3.8}$$

where the operators $\mathbf{1}_m \otimes P_n$ and $P_m \otimes \mathbf{1}_n$ have the obvious meaning.

Due to the commutation relation (3.5), each element of the Q-ACR algebra can be represented as a finite sum of $c\mathbf{1}$, $c \in \mathbb{C}$, and elements of the form

$$W^{(m,n)}\left(f^{(m+n)}\right) = \Phi^{(m+n)}\left(f^{(m+n)}; \sharp_1, \ldots, \sharp_{m+n}\right),$$

where $m, n \in \mathbb{N}_0$, $m + n \geq 1$, $f^{(m+m)} \in \mathfrak{F}\left(X^{m+n}\right)$ and

$$\sharp_1 = \cdots = \sharp_m = +, \quad \sharp_{m+1} = \cdots = \sharp_{m+n} = -.$$

We call $W^{(m,n)}\left(f^{(m+n)}\right)$ *Wick-ordered elements of the Q-ACR algebra.* By (3.7) and (3.8),

$$W^{(m,n)}\left(f^{(m,n)}\right) = W\left(P_m \otimes P_n f^{(m+n)}\right),$$

where $P_1 = \mathbf{1}$. Thus, we have proved the following proposition.

Proposition 3.7. *Each element of the Q-ACR algebra can be represented in the form*

$$c\mathbf{1} + \sum_{m,n \in \mathbb{N}_0,\, m+n \geq 1} W^{(m,n)}\left(f^{(m,n)}\right), \tag{3.9}$$

where $c \in \mathbb{C}, f^{(m,n)} \in \mathfrak{S}\mathfrak{F}^{(m,n)}\left(X^{m+n}\right)$*, the sum in* (3.9) *being finite.*

Remark 3.8. In view of Proposition 3.7, we may think of the Q-ACR algebra as a $*$-algebra allowing the Wick (or normal) ordering, compare with [17].

From (2.130), we formally have

$$a^+(x)a^-(y) = Q(x,y)a^-(y)a^+(x) - \varkappa\,\delta(x,y), \tag{3.10}$$

where $\varkappa = Q(x,x) = +1$ or -1. The following crucial proposition is a rigorous formulation of (3.10).

Proposition 3.9. *If* $f^{(n)} \in \mathfrak{F}(X^n)$, $n \geq 2$, $i \in \{1,\dots,n-1\}$, $\sharp_i = +$, $\sharp_{i+1} = -$, *then*

$$\Phi^{(n)}\left(f^{(n)};\sharp_1,\dots,\sharp_n\right) = \Phi^{(n)}\left(Q(x_{i+1},x_i)f^{(n)}(x_1,\dots,x_{i+1},x_i,\dots,x_n);\sharp_1,\dots,\sharp_{i+1},\sharp_i,\dots,\sharp_n\right)$$
$$-\varkappa\Phi^{(n-2)}\left(\int_X f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx);\sharp_1,\dots,\sharp_{i-1},\sharp_{i+2},\dots,\sharp_n\right),$$

where $\varkappa = Q(x,x)$ *for* $x \in X$.

Proof. Choose

$$g^{(n)}(x_1,\dots,x_n) = Q(x_{i+1},x_i)f^{(n)}(x_1,\dots,x_{i+1},x_i,\dots,x_n).$$

Then by formula (3.5), we get

$$\begin{aligned}
&\Phi^{(n)}\left(g^{(n)}(x_1,\dots,x_n);\sharp_1,\dots,\sharp_{i+1},\sharp_i,\dots,\sharp_n\right)\\
&\quad= \Phi^{(n)}\left(Q(x_{i+1},x_i)g^{(n)}(x_1,\dots,x_{i+1},x_i,\dots,x_n);\sharp_1,\dots,\sharp_n\right)\\
&\qquad+\Phi^{(n-2)}\left(\int_X g^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_n)\sigma(dx);\sharp_1,\dots,\sharp_{i-1},\sharp_{i+2},\dots,\sharp_n\right)\\
&\quad= \Phi^{(n)}\left(Q(x_{i+1},x_i)Q(x_i,x_{i+1})f^{(n)}(x_1,\dots,x_n);\sharp_1,\dots,\sharp_n\right)\\
&\qquad+\Phi^{(n-2)}\left(\int_X Q(x,x)f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx);\sharp_1,\dots,\sharp_{i-1},\sharp_{i+2},\dots,\sharp_n\right)\\
&\quad= \Phi^{(n)}\left(f^{(n)};\sharp,\dots,\sharp_n\right)\\
&\qquad+\varkappa\,\Phi^{(n-2)}\left(\int_X f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx);\sharp_1,\dots,\sharp_{i-1},\sharp_{i+2},\dots,\sharp_n\right).
\end{aligned}$$

□

3.2 Complex-valued Radon measures

To discuss states on the Q-ACR algebra, we will now briefly recall some key facts related to complex-valued Radon measures. For more detail, we refer the reader to Chapter IV of [3] or Chapter 7 of [26].

Let $\mathfrak{X}$ be a locally compact Polish space, let $\mathcal{B}(\mathfrak{X})$ be the Borel σ- algebra on $\mathfrak{X}$, and let $\mathcal{B}_0(\mathfrak{X})$ be the collection of all sets from $\mathcal{B}_0(\mathfrak{X})$ that have compact closure.

Recall that a (positive) Radon measure on X is a measure m on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$ that has the property $m(K) < \infty$ for each compact set K in $\mathfrak{X}$.

A *complex-valued Radon measure* m on $\mathfrak{X}$ is a map of the form

$$\mathcal{B}_0(\mathfrak{X}) \ni \Delta \mapsto m(\Delta) \in \mathbb{C}, \tag{3.11}$$

where m admits a representation

$$m = m_1 - m_2 + i(m_3 - m_4) \tag{3.12}$$

in which m_1, m_2, m_3, m_4 are positive Radon measures. We denote by $M(\mathfrak{X})$ the set of all complex-valued Radon measures on $\mathfrak{X}$, and by $M_+(\mathfrak{X})$ the set of all positive Radon measures.

Denote by $C_0(\mathfrak{X})$ the space of all continuous complex-valued functions on $\mathfrak{X}$ with compact support. For each $f \in C_0(\mathfrak{X})$ and each $m \in M(\mathfrak{X})$, using representation (3.11), we define

$$\int_{\mathfrak{X}} f(x)m(dx) = \int_{\mathfrak{X}} f(x)m_1(dx) - \int_{\mathfrak{X}} f(x)m_2(dx) + i\left(\int_{\mathfrak{X}} f(x)m_3(dx) - \int_{\mathfrak{X}} f(x)m_4(dx)\right). \tag{3.13}$$

Thus, each $m \in M(\mathfrak{X})$ determines, through formula (3.13), a complex-valued linear functional on $C_0(\mathfrak{X})$.

A linear functional $L : C_0(\mathfrak{X}) \to \mathbb{C}$ is called *positive* if, for each $f \in C_0(\mathfrak{X})$, $f \geq 0$, we have $Lf \geq 0$. Obviously, each positive Radon measure $m \in M_+(\mathfrak{X})$ determines a positive linear functional on $C_0(\mathfrak{X})$.

In fact, $M_+(\mathfrak{X})$ can be identified with the set of all continuous positive functionals on $C_0(\mathfrak{X})$, and $M(\mathfrak{X})$ can be identified with the dual space of $C_0(\mathfrak{X})$.

To this end, we introduce a topology in $C_0(\mathfrak{X})$ as follows. Choose any sequence $(K_n)_{n=1}^{\infty}$ of compact sets K_n in $\mathfrak{X}$ such that

$$K_1 \subset K_2 \subset K_3 \subset \cdots, \quad \bigcup_{n=1}^{\infty} K_n = \mathfrak{X}.$$

Denote by B_n the space of all $f \in C_0(\mathfrak{X})$ that satisfy $f(x) = 0$ for all $x \notin K_n$. We equip B_n with the topology of the supremum norm,

$$\|f\|_n = \sup_{x \in K_n} |f(x)| = \sup_{x \in \mathfrak{X}} |f(x)|,$$

then B_n becomes a Banach space.

Obviously, each B_n is a subset of B_{n+1} and for each $f_n \in B_n$,

$$\|f\|_n = \|f\|_{n+1}. \tag{3.14}$$

Since $C_0(\mathfrak{X}) = \bigcup_{n=1}^{\infty} B_n$, we equip $C_0(\mathfrak{X})$ with the inductive limit topology of the spaces B_n, i.e., the finest locally convex topology on $C_0(\mathfrak{X})$ for which each embedding of B_n into $C_0(\mathfrak{X})$ is continuous. Furthermore, by (3.14), this inductive limit is strict, see e.g. page 57 in [25].

By Section 6.6 of [25], $C_0(\mathfrak{X})$ is a complete locally convex topological vector space. In particular, the inductive limit topology on $C_0(\mathfrak{X})$ is finer than the topology of convergence in the supremum norm. (Note that the completion of $C_0(\mathfrak{X})$ in the supremum norm $\|f\| = \sup_{x \in \mathfrak{X}} |f(x)|$ is $C_{\mathrm{b}}(X)$, the space of all bounded continuous functions on $\mathfrak{X}$.)

In fact, a sequence $(f_k)_{k=1}^{\infty}$ converges to f in $C_0(\mathfrak{X})$ if and only if there exists $n \in \mathbb{N}$ such that $f_k \in B_n$ for all $k \in \mathbb{N}$, $f \in B_n$, and $f_k \to f$ in B_n.

A linear functional $F : C_0(\mathfrak{X}) \to \mathbb{C}$ is continuous if and only if the restriction of F to each B_n is continuous on B_n, e.g. see Section 6.1 of [25].

Obviously, each $m \in M(\mathfrak{X})$ defines through (3.13) a continuous linear functional on $C_0(\mathfrak{X})$, and for each $m \in M_+(\mathfrak{X})$ this continuous linear functional is positive.

In fact, $M_+(\mathfrak{X})$ coincides with the set of all positive continuous linear functionals on $C_0(\mathfrak{X})$, see e.g. Section 2.9 in [3] or Section 2.2 of Chapter III in [4].

Every continuous linear functional $F : C_0(\mathfrak{X}) \to \mathbb{C}$ can be represented in the form

$$F = F_1 - F_2 + i(F_3 - F_4),$$

where F_1, F_2, F_3, F_4 are continuous linear positive functionals on $C_0(\mathfrak{X})$. Hence, the space of all continuous linear functionals on $C_0(\mathfrak{X})$ can be identified with $M(\mathfrak{X})$, see e.g. Chapter 7 in [26].

3.3 States on the Q-ACR algebra

We first recall the definition of a state on a unital $*$-algebra.

Definition 3.10. Let $\mathbb{A}$ be a unital $*$-algebra. A *state on* $\mathbb{A}$ is a map $\tau : \mathbb{A} \to \mathbb{C}$ that satisfies

(i) $\tau(aa^*) \geq 0$ for each $a \in \mathbb{A}$;

(ii) $\tau(\mathbf{1}) = 1$, where $\mathbf{1}$ is the identity element in $\mathbb{A}$.

Let $\mathbb{A}$ be the Q-ACR algebra and let $\tau : \mathbb{A} \to \mathbb{C}$ be a state on $\mathbb{A}$. Then, for any $\sharp_1, \dots, \sharp_n \in \{+, -\}$, we define a linear functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n} : C_0\left(Y^n\right) \times \mathcal{G}^n \to \mathbb{C}$ by

$$\tau^{(n)}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) = \tau\left(\Phi^{(n)}(\varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n); \sharp_1, \dots, \sharp_n)\right). \quad (3.15)$$

Obviously these linear functionals uniquely identify the state τ.

From now on, we assume that each functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n) \times \mathcal{G}^n$. We note that, for any fixed $\sharp_1, \dots, \sharp_n \in \{+, -\}$ and $g_1, \dots, g_n \in \mathcal{G}$, $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ determines a continuous linear functional

$$C_0(Y^n) \ni \varphi^{(n)} \mapsto \tau^{(n)}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) \in \mathbb{C}.$$

Hence, there exists a complex-valued Radon measure $m^{(n)}_{\sharp_1,\dots,\sharp_n}[g_1, \dots, g_n]$ on Y^n that satisfies

$$\tau^{(n)}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) = \int_{Y^n} \varphi^{(n)}(y_1, \dots, y_n)\, m^{(n)}_{\sharp_1,\dots,\sharp_n}[g_1, \dots, g_n](dy_1 \cdots dy_n). \quad (3.16)$$

For $f \in \mathfrak{F}(X)$ and $\sharp \in \{+, -\}$, we denote $A^{\sharp}(f) = \Phi^{(1)}(f; \sharp)$.

Proposition 3.11. *Assume that, for any $\sharp_1 \dots, \sharp_n \in \{+, -\}$, the functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0\left(Y^n\right) \times \mathcal{G}^n$. Then the following statements hold:*

(i) The state τ is completely determined by its values on $A^{\sharp_1}(f_1) \cdots A^{\sharp_{m+n}}(f_{m+n})$, where $f_k = \varphi_k \otimes g_k$ with $\varphi_k \in C_0(Y)$ and $g_k \in \mathcal{G}$ $(k = 1, \dots, m+n)$, and $\sharp_1 = \cdots = \sharp_m = +$, $\sharp_{m+1} = \cdots = \sharp_{m+n} = -$, where $m, n \geq 0$, $m + n \geq 1$.

(ii) Define

$$B(f) = A^+(f) + A^-(\bar{f}), \quad f \in \mathfrak{F}(X). \tag{3.17}$$

The state τ is completely determined by its values on $B(f_1)\cdots B(f_n)$, where $f_k = \varphi_k \otimes g_k$ with $\varphi_k \in C_0(Y \to \mathbb{R})$ and $g_k \in \mathcal{G}$, $k = 1, \ldots, n$. Here $C_0(Y \to \mathbb{R})$ denotes the subspace of $C_0(Y)$ consisting of all real-valued continuous functions on Y with compact support.

Proof. By (3.16),

$$\tau\big(A^{\sharp_1}(\varphi_1 \otimes g_1)\cdots A^{\sharp_n}(\varphi_n \otimes g_n)\big) = \int_{Y^n} \varphi_1(y_1)\cdots\varphi_n(y_n)\, m^{(n)}_{\sharp_1,\ldots,\sharp_n}[g_1,\ldots,g_n](dy_1\cdots dy_n),$$

where $\varphi_k \in C_0(Y)$ and $g_k \in \mathcal{G}$. Since the measure $m^{(n)}_{\sharp_1,\ldots,\sharp_n}[g_1,\ldots,g_n]$ is completely determined by its values on functions of the form $\varphi_1(y_1)\cdots\varphi_n(y_n)$, with $\varphi_k \in C_0(Y)$, we conclude that the state τ is completely determined by its values on $A^{\sharp_1}(f_1)\cdots A^{\sharp_n}(f_n)$, where $f_k = \varphi_k \otimes g_k$ with $\varphi_k \in C_0(Y)$ and $g_k \in \mathcal{G}$. From here and Proposition 3.7, statement (i) follows.

As for statement (ii), let us first prove it in the case where each φ_k is allowed to be taken from the space $C_0(Y)$ rather than $C_0(Y \to \mathbb{R})$. But then this statement follows from statement (i) and the formulas

$$\begin{aligned} A^+(f) &= \frac{1}{2}\left(B(f) - iB(if)\right), \\ A^-(f) &= \frac{1}{2}\left(B(\bar{f}) + iB(i\bar{f})\right). \end{aligned} \tag{3.18}$$

Now assume that $\varphi_1, \ldots, \varphi_n \in C_0(Y \to \mathbb{R})$. For each $k \in \{1, \ldots, n\}$,

$$\begin{aligned} &\overline{f}_k = \varphi_k \otimes \overline{g}_k, \quad \overline{g}_k \in \mathcal{G}, \\ &if_k = i\varphi_k \otimes g_k = \varphi \otimes (ig_k), \quad ig_k \in \mathcal{G}. \end{aligned}$$

Hence, by (3.18), for any $\sharp_1, \ldots, \sharp_n \in \{+, -\}$ and f_k with $\varphi_1, \ldots, \varphi_n \in C_0(Y \to \mathbb{R})$, we know $\tau\left(A^{\sharp_1}(f_1)\cdots A^{\sharp_n}(f_n)\right)$. But for each $\varphi \in C_0(Y)$, we have $\varphi = \varphi_1 + i\varphi_2$, where $\varphi_1, \varphi_2 \in C_0(Y \to \mathbb{R})$, hence, for each $g \in \mathcal{G}$, we have

$$\varphi \otimes g = (\varphi_1 + i\varphi_2) \otimes g = \varphi_1 \otimes g + i\varphi_2 \otimes g,$$

and by linearity, for $\sharp \in \{+, -\}$, we get

$$A^\sharp(\varphi \otimes g) = A^\sharp(\varphi_1 \otimes g) + iA^\sharp(\varphi_2 \otimes g).$$

From this and the statement (i), the statement (ii) follows. □

Lemma 3.12. *Let a state τ satisfy the assumption of Proposition 3.11.*

(i) For any $g_1, g_2 \in \mathcal{G}$, there exists a complex-valued Radon measure $\gamma^{(2)}[g_1, g_2]$ on Y^2 that satisfies

$$\tau\left(A^+(\varphi_1 \otimes g_1)A^-(\varphi_2 \otimes g_2)\right) = \int_{Y^2} \varphi_1(y_1)\varphi_2(y_2)\gamma^{(2)}[g_1, g_2](dy_1\, dy_2), \tag{3.19}$$

for all $\varphi_1, \varphi_2 \in C_0(Y)$.

(ii) For any $g_1, g_2 \in \mathcal{G}$, there exists a complex-valued Radon measure $\lambda^{(2)}[g_1, g_2]$ on Y^2 that satisfies

$$\tau\left(B(\varphi_1 \otimes g_1)B(\varphi_2 \otimes g_2)\right) = \int_{Y^2} \varphi_1(y_1)\varphi_2(y_2)\lambda^{(2)}[g_1, g_2](dy_1\, dy_2), \tag{3.20}$$

for all $\varphi_1, \varphi_2 \in C_0(Y \to \mathbb{R})$.

Proof. (i) This statement is obvious. Indeed, by (3.16), $\gamma^{(2)}[g_1, g_2] = m^{(2)}_{+,-}[g_1, g_2]$.

(ii) We define, for each $g_1, g_2 \in \mathcal{G}$,

$$\lambda^{(2)}[g_1, g_2] = m^{(2)}_{+,+}[g_1, g_2] + m^{(2)}_{+,-}[g_1, \overline{g}_2] + m^{(2)}_{-,+}[\overline{g}_1, g_2] + m^{(2)}_{-,-}[\overline{g}_1, \overline{g}_2]. \tag{3.21}$$

Then (3.20) folllows from (3.16) and the observation that

$$B(\varphi_k \otimes g_k) = A^+(\varphi_k \otimes g_k) + A^-(\varphi_k \otimes \overline{g}_k)$$

for any $\varphi_1, \varphi_2 \in C_0(Y \to \mathbb{R})$. □

3.4 Fock state on the Q-ACR algebra

We will now see that our considerations in Chapter 2 allow us to construct the so-called Fock state on the Q-ACR algebra.

For $\varphi \in C_0(Y)$ and $g \in \mathcal{G}$ denote

$$A^+(\varphi \otimes g) = a^+(\varphi \otimes g), \quad A^-(\varphi \otimes g) = a^-(\varphi \otimes g), \tag{3.22}$$

where the operators $a^+(\varphi \otimes g)$ and $a^-(\varphi \otimes g)$ are defined as the usual creation and annihilation operators in the Q-symmetric Fock space $\mathcal{F}(\mathcal{H})$. In particular, $A^+(\varphi \otimes g)$ and $A^-(\varphi \otimes g)$ act continuously on $\mathcal{F}_{\text{fin}}(\mathcal{H})$.

Recall that the linear span of functions of the form $\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n$ with $\varphi_1, \dots, \varphi_n \in C_0(Y)$ is dense in $C_0(Y^n)$. It is easy to see from Chapter 2 that, for arbitrary $\sharp_1, \dots, \sharp_n \in \{+, -\}$, the map

$$C_0(Y^n) \times \mathcal{G}^n \ni (\varphi_1, \dots, \varphi_n, g_1, \dots, g_n) \mapsto A^{\sharp_1}(\varphi_1 \otimes g_1) \cdots A^{\sharp_n}(\varphi_n \otimes g_n) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$$

can be uniquely extended to the map

$$\begin{aligned} &C_0(Y^n) \times \mathcal{G}^n \ni \left(\varphi^{(n)}, g_1, \dots, g_n\right) \longmapsto \\ &\qquad \Phi^{(n)}\left(\varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n); \sharp_1, \dots, \sharp_n\right) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})) \end{aligned}$$

such that

(i) if $\varphi^{(n)}(y_1, \dots, y_n) = \varphi_1(y_1) \cdots \varphi_n(y_n)$ with $\varphi_i \in C_0(Y)$, then

$$\Phi^{(n)}\left(\varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n); \sharp_1, \dots, \sharp_n\right) = A^{\sharp_1}(\varphi_1 \otimes g_1) \cdots A^{\sharp_n}(\varphi_n \otimes g_n);$$

(ii) for each $F \in \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$, the map

$$C_0(Y^n) \times \mathcal{G} \ni \left(\varphi^{(n)}, g_1, \dots, g_n\right) \mapsto \Phi^{(n)}\left(\varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n)\right) F \in \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$$

is continuous.

In particular, by Proposition 2.60, we have, for any $\varphi_1, \dots, \varphi_{m+n} \in C_0(Y)$ and $g_1, \dots, g_{m+n} \in \mathcal{G}$,

$$\begin{aligned} &\left(a^+(\varphi_1 \otimes g_1) \cdots a^+(\varphi_m \otimes g_m) a^-(\varphi_{m+1} \otimes g_{m+1}) \cdots a^-(\varphi_{m+n} \otimes g_{m+n}) u^{(k)}\right)(x_1, \dots, x_l) \\ &\quad = P_l \Bigg[(\varphi_1 \otimes g_1)(x_1) \cdots (\varphi_m \otimes g_m)(x_m) \int_{X^n} \sigma^{\otimes n}(dx_1' \cdots dx_n') \\ &\qquad (\varphi_{m+1} \otimes g_{m+1})(x_n')(\varphi_{m+2} \otimes g_{m+2})(x_{n-1}') \cdots (\varphi_{m+n} \otimes g_{m+n})(x_1') \\ &\qquad \times u^{(k)}(x_1', \dots, x_n', x_{m+1}, \dots, x_l) \Bigg], \end{aligned}$$

where $u^{(k)} \in \mathcal{F}_k(\mathcal{H})$, $k \geq n$, and $l = k - n + m$. Hence, for $f^{(m+n)} \in \mathfrak{F}(X^{m+n})$, we have

$$\left(W^{(m,n)}(f^{(m+n)}) u^{(k)}\right)(x_1, \dots, x_l) = (\Phi^{(m+n)}(f^{(m+n)}; \underbrace{+, \dots, +}_{m \text{ times}}, \underbrace{-, \dots, -}_{n \text{ times}}) u^{(k)})(x_1, \dots, x_l)$$

$$= P_l\Big[\int_{X^n}\sigma^{\otimes n}(dx_1'\cdots dx_n')f^{(m+n)}(x_1,\dots,x_m,x_n',x_{n-1}',\dots,x_1')$$
$$\times u^{(k)}(x_1',\dots,x_n',x_{m+1},\dots,x_l)\Big]. \tag{3.23}$$

This gives a representation of the Q-ACR algebra.

Definition 3.13. Let $\mathbb{A}$ be the Q-ACR algebra as described above. The state τ on $\mathbb{A}$ defined by

$$\tau(a) = (a\Omega,\Omega)_{\mathcal{F}(\mathcal{H})} \quad \text{for each } a\in\mathbb{A} \tag{3.24}$$

is called the *Fock state*.

Lemma 3.14. *Let τ be the Fock state on the Q-ACR algebra $\mathbb{A}$. For any $\sharp_1,\dots,\sharp_n \in \{-,+\}$, the linear functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n} : C_0(Y^n)\times\mathcal{G}^n \to \mathbb{C}$ defined by (3.15) is continuous.*

Proof. Assume $\sharp_1 = \cdots = \sharp_m = +$, $\sharp_{m+1} = \cdots = \sharp_{m+n} = -$ and $m+n \ge 1$. Then $\tau^{(n)}_{\sharp_1,\dots,\sharp_{m+n}} = 0$. Indeed, if $n \ge 1$, then the annihilation operators map the vacuum vector to zero, therefore

$$\Phi^{(m+n)}\left(f^{(m+n)};\sharp_1,\dots,\sharp_{m+n}\right)\Omega = 0.$$

If $n = 0$ and $m \ge 1$, then

$$\Phi^{(m)}(f^{(m)};\underbrace{+,\dots,+}_{m \text{ times}})\Omega = f^{(m)} \in \mathcal{F}_m(\mathcal{H}).$$

This implies that

$$(\Phi^{(m)}(f^{(m)};\underbrace{+,\dots,+}_{m \text{ times}})\Omega,\Omega)_{\mathcal{F}(\mathcal{H})} = 0.$$

In the case, where $\sharp_1,\dots,\sharp_n \in \{-,+\}$ are not Wick ordered, i.e, for some $i \in \{1,\dots,n-1\}$ we have $\sharp_i = -$, $\sharp_{i+1} = +$, we apply formula (3.5) to bring $\sharp_1,\dots,\sharp_n \in \{-,+\}$ to a Wick order. From here we easily conclude that the functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n} : C_0(Y^n)\times\mathcal{G}^n \to \mathbb{C}$ is indeed continuous. □

Remark 3.15. It follows from the proof of Lemma 3.14 that, for the Fock state τ, we have

$$\tau\big(A^+(f_1)\cdots A^+(f_m)A^-(f_{m+1})\cdots A^-(f_{m+n})\big) = 0 \tag{3.25}$$

if $m+n \ge 1$. Here $f_1,\dots,f_{m+n} \in \mathfrak{F}(X^{m+n})$. In particular, the measure $\gamma^{(2)}[g_1,g_2]$ from Lemma 3.12 (i) is equal to zero.

Remark 3.16. For the Fock state τ, the measure $\lambda^{(2)}[g_1, g_2]$ from Lemma 3.12 (ii) is determined by

$$\int_{Y^2} \varphi^{(2)}(y_1, y_2)\, \lambda^{(2)}[g_1, g_2](dy_1\, dy_2) = \int_Y \varphi^{(2)}(y, y)\nu(dy) \int_Z \overline{g_1(z)} g_2(z)\mu(dz)$$
$$= \int_Y \varphi^{(2)}(y, y)\nu(dy)(g_2, g_1)_{\mathcal{G}},$$

i.e.,

$$\lambda^{(2)}[g_1, g_2](dy_1\, dy_2) = (g_2, g_1)_{\mathcal{G}}\, \nu^{(2)}(dy_1\, dy_2), \tag{3.26}$$

where $\nu^{(2)}$ is the measure on Y^2 given by

$$\int_{Y^2} \varphi^{(2)}(y_1, y_2)\nu^{(2)}(dy_1\, dy_2) = \int_Y \varphi^{(2)}(y, y)\nu(dy). \tag{3.27}$$

Our next aim is to calculate $\tau\left(B(f_1)\cdots B(f_n)\right)$ for a general n. It is obvious that if n is odd, then $\tau\left(B(f_1)\cdots B(f_n)\right) = 0$, therefore we only need to deal with the case of an even n.

For each even $n \in \mathbb{N}$, we denote by $\mathcal{P}_2^{(n)}$ the collection of all partitions of the set $\{1, \dots, n\}$ into n parts, each of which has exactly two elements.

Proposition 3.17. *Let τ be the Fock state on the Q-ACR algebra $\mathbb{A}$. Then, for any $f_1, \dots, f_n \in \mathfrak{F}(X)$, we have, for an odd n,*

$$\tau\left(B(f_1)\cdots B(f_n)\right) = 0,$$

and for an even n,

$$\tau\left(B(f_1)\cdots B(f_n)\right) = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{X^n} \bigg(\bigotimes_{\{i,j\}\in\xi,\, i<j} \sigma^{(2)}(dx_i\, dx_j)\, \overline{f_i(x_i)} f_j(x_j) \bigg) Q(\xi; x_1, \dots, x_n), \tag{3.28}$$

where $\sigma^{(2)}$ is the measure on X^2 given by

$$\int_{X^2} f^{(2)}(x_1, x_2)\sigma^{(2)}(dx_1\, dx_2) = \int_X f^{(2)}(x, x)\sigma(dx)$$

and

$$Q(\xi; x_1, \dots, x_n) = \prod_{\substack{\{i,j\},\,\{k,l\}\in\xi \\ i<k<j<l}} Q(x_k, x_j).$$

Remark 3.18. In the case where the functions $f_1, \dots, f_n$ are real-valued, the statement of Proposition 3.17 follows from Corollary 4.8 in [23].

Proof. For each $f \in \mathfrak{F}(X)$, we define continuous linear operators $\mathcal{A}^+(f), \mathcal{A}^-(f)$ on $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$ as follows: $\mathcal{A}^+(f) = a^+_{\mathrm{free}}(f)$ and $\mathcal{A}^-(f)$ acts as follows:

$$\big(\mathcal{A}^-(f)g^{(n)}\big)(x_1, \dots, x_{n-1}) = \int_X \sigma(dx')f(x')\big[g^{(n)}(x', x_1, \dots, x_{n-1}) \\ + \sum_{k=1}^{n-1} Q(x', x_1)Q(x', x_2)\cdots Q(x', x_k)g^{(n)}(x_1, \dots, x_k, x', x_{k+1}, \dots, x_{n-1})\big], \tag{3.29}$$

where $g^{(n)} \in \mathcal{H}^{\otimes(n)}$.

Lemma 3.19. *For any $f_1, \dots, f_n \in \mathfrak{F}(X)$ and $\sharp_1, \dots, \sharp_n \in \{+, -\}$, we have*

$$\big(A^{\sharp_1}(f_1)\cdots A^{\sharp_n}(f_n)\Omega, \Omega\big)_{\mathcal{F}(\mathcal{H})} = \big(\mathcal{A}^{\sharp_1}(f_1)\cdots \mathcal{A}^{\sharp_n}(f_n)\Omega, \Omega\big)_{\mathbb{F}(\mathcal{H})}.$$

Proof. For each $u^{(n)} \in \mathcal{H}^{\otimes n}$, formula (2.14) implies

$$a^+(f)P_n u^{(n)} = P_{n+1}\left(f \otimes u^{(n)}\right) = P_{n+1}\left(\mathcal{A}^+(f)u^{(n)}\right). \tag{3.30}$$

By Proposition 2.60 (ii),

$$a^-(f)P_n u^{(n)} = \frac{1}{n} a^-_{\mathrm{free}}(f)\left(\mathbf{1}_{\mathcal{H}} \otimes P_{n-1}\right)\left[\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}\right]u^{(n)}. \tag{3.31}$$

We observe that, for $u^{(n)} \in \mathcal{H}^{\otimes n}$,

$$a^-_{\mathrm{free}}(f)(\mathbf{1}_{\mathcal{H}} \otimes P_{n-1})u^{(n)} = P_{n-1}a^-_{\mathrm{free}}(f)u^{(n)}. \tag{3.32}$$

Hence, formula (3.31) becomes

$$a^-(f)P_n u^{(n)} = P_{n-1}\frac{1}{n}a^-_{\mathrm{free}}(f)\big[\mathbf{1} + U_1 + U_1U_2 + \cdots + U_1U_2\cdots U_{n-1}\big]u^{(n)}.$$

Hence, by (2.78), we get

$$a^-(f)P_n u^{(n)} = P_{n-1}\mathcal{A}^-(f)u^{(n)}. \tag{3.33}$$

By (3.30) and (3.33), the lemma follows. □

For $f \in \mathfrak{F}(X)$, we define

$$\mathcal{B}(f) = \mathcal{A}^+(f) + \mathcal{A}^-(\overline{f}).$$

Then, by Lemma 3.19,

$$\tau\left(B(f_1)\cdots B(f_n)\right) = \left(\mathcal{B}(f_1)\cdots\mathcal{B}(f_n)\Omega,\Omega\right)_{\mathbb{F}(\mathcal{H})}. \tag{3.34}$$

Recall formula (3.29) and set $g^{(n)} = f_1 \otimes f_2 \otimes \cdots \otimes f_n$, $f_i \in \mathfrak{F}(X)$. We will say that the term

$$\int_X \sigma(dx')\, Q(x',x_1)\cdots Q(x',x_k) f(x') f_{k+1}(x') f_1(x_1)\cdots f_k(x_k) f_{k+2}(x_{k+1})\cdots f_n(x_{n-1})$$

describes how *the operator* $\mathcal{A}^-(f)$ *annihilates the function* f_{k+1}. Note that the term $Q(x',x_1)\cdots Q(x',x_k)$ corresponds to the vectors $f_1,\dots,f_k$ that appear on the left of f_{k+1} in the tensor product $f_1\otimes f_2\otimes\cdots\otimes f_k\otimes f_{k+1}\otimes\cdots\otimes f_n$. Thus, the annihilation operator $\mathcal{A}^-(f)$ *crosses all living vectors* $f_1,\dots,f_k$ when it annihilates f_{k+1}, and each time when $\mathcal{A}^-(f)$ *crosses a living vector* f_i, this contributes the factor $Q(x',x_i)$.

It immediately follows that, when evaluating the right-hand side of formula (3.34), one can describe the obtained result by using partitions ξ from $\mathcal{P}_2^{(n)}$. Indeed, each $\{k,l\}\in\xi$, $k<l$, says that, in the respective term, the operator $\mathcal{A}^-(\overline{f_k})$ annihilates the vector f_l.

Let $\{k,l\}\in\xi$, $k<l$, and consider any $\{i,j\}\in\xi$, $i<j$, such that $i<k<j<l$. Then the operator $\mathcal{A}^-(\overline{f_k})$ annihilates f_l and later on the operator $\mathcal{A}^-(\overline{f_i})$ annihilates f_j. Therefore, at the time when the operator $\mathcal{A}^-(\overline{f_k})$ annihilates f_l, f_j is a *living vector* and this contributes the factor $Q(x_k,x_j)$. This proves the proposition. □

Corollary 3.20. *For any $g_1,g_2\in\mathcal{G}$, let the measure $\lambda^{(2)}[g_1,g_2](dy_1\,dy_2)$ on Y^2 be given by (3.26), (3.27). Let τ be the Fock state on the Q-ACR algebra $\mathbb{A}$. Let $\varphi_1,\dots,\varphi_n\in C_0(Y\to\mathbb{R})$ and $g_1,\dots,g_n\in\mathcal{G}$. Then*

$$\begin{aligned}&\tau\left(B(\varphi_1\otimes g_1)\cdots B(\varphi_n\otimes g_n)\right)\\ &\quad=\sum_{\xi\in\mathcal{P}_2^{(n)}}\int_{Y^n}\bigotimes_{\{i,j\}\in\xi,\,i<j}\lambda^{(2)}[g_i,g_j](dy_i\,dy_j)\varphi_1(y_1)\cdots\varphi_n(y_n)Q(\xi;y_1,\dots,y_n),\end{aligned}$$

where

$$Q(\xi;y_1,\dots,y_n) = \prod_{\substack{\{i,j\},\,\{k,l\}\in\xi\\ i<k<j<l}} Q(y_k,y_j) \tag{3.35}$$

Proof. The statement follows immediately from Proposition 3.17 and formulas (3.1), (3.26), (3.27). □

Remark 3.21. As easily seen from the proof of Proposition 3.17, its statement allows for the following generalization: For any $e_1, \dots, e_n, f_1, \dots, f_n \in \mathfrak{F}(X)$, we have, for an odd n,

$$\tau\left(\left(A^+(f_1)+A^-(e_1)\right)\cdots\left(A^+(f_n)+A^-(e_n)\right)\right)=0,$$

and for an even n,

$$\begin{aligned}&\tau\left(\left(A^+(f_1)+A^-(e_1)\right)\cdots\left(A^+(f_n)+A^-(e_n)\right)\right)\\&\quad=\sum_{\xi\in\mathcal{P}_2^{(n)}}\int_{X^n}\left(\bigotimes_{\{i,j\}\in\xi,\, i<j}\sigma^{(2)}(dx_i\,dx_j)e_i(x_i)f_j(x_j)\right)Q(\xi;x_1,\dots,x_n).\end{aligned}\tag{3.36}$$

In particular, for any $\varphi_1, \dots, \varphi_{2n} \in C_0(Y \to \mathbb{R})$ and $g_1, \dots, g_{2n} \in \mathcal{G}$, we have

$$\begin{aligned}&\tau\left(A^-(\varphi_1\otimes g_1)\cdots A^-(\varphi_n\otimes g_n)A^+(\varphi_{n+1}\otimes g_{n+1})\cdots A^+(\varphi_{2n}\otimes g_{2n})\right)\\&\quad=\sum_{\xi\in\mathcal{P}_2^{(2n)}}\int_{Y^{2n}}\left(\bigotimes_{\substack{\{i,j\}\in\xi\\1\le i\le n\\n+1\le j\le 2n}}\nu^{(2)}(dy_i\,dy_j)\int_Z g_i(z)g_j(z)\mu(dz)\right)\\&\qquad\times\varphi_1(y_1)\cdots\varphi_{2n}(y_{2n})Q(\xi;y_1,\cdots,y_{2n}),\end{aligned}\tag{3.37}$$

where the function $Q(\xi; y_1, \dots, y_{2n})$ is defined by (3.35).

3.5 Quasi-free states on the Q-ACR algebra

In the classical case of bose and fermi statistics, one defines a quasi-free state on the CCR/CAR algebra by generalizing formula (3.28) with $Q \equiv \pm 1$, which is valid for the Fock state. An important subclass of quasi-free states is given by gauge-invariant quasi-free states. In the case of anyon statistics, gauge-invariant quasi-free states were studied in [22]. However, we will see below that, in the anyon setting, gauge-invariant quasi-free states do not satisfy a reasonable generalization of the definition of a quasi-free state, hence the class of gauge-invariant quasi-free states is smaller than the class of what we will call strongly quasi-free states.

In view of our results from Sections 3.1 and 3.4 we now give the following

Definition 3.22. Let $\mathbb{A}$ be a Q-ACR algebra and let τ be a state on it. Assume that the state τ satisfies the assumption of Proposition 3.11. For each $g_1, g_2 \in \mathcal{G}$, let the complex-valued measure $\lambda^{(2)}[g_1, g_2]$ on Y^2 be determined by formula (3.20). We say that τ is a *strongly quasi-free state* if for any $\varphi_1, \dots, \varphi_n \in C_0(Y \to \mathbb{R})$ and $g_1, \dots, g_n \in \mathcal{G}$, we have, for an odd n,

$$\tau\big(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\big) = 0 \tag{3.38}$$

and for an even n,

$$\begin{aligned}&\tau\big(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\big)\\ &\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{Y^n} \bigotimes_{\{i,j\} \in \xi,\, i<j} \lambda^{(2)}[g_i, g_j](dy_i\, dy_j) \varphi_1(y_1) \cdots \varphi_n(y_n) Q(\xi; y_1, \dots, y_n).\end{aligned} \tag{3.39}$$

Here $Q(\xi; y_1, \dots, y_n)$ is given by (3.35). In particular, the state τ is completely determined by the measures $\lambda^{(2)}[g_1, g_2]$ $(g_1, g_2 \in \mathcal{G})$ and the function $Q(y_1, y_2)$.

Corollary 3.20 implies:

Proposition 3.23. *The Fock state on the Q-ACR algebra is strongly quasi-free, and in this case the measure* $\lambda^{(2)}[g_1, g_2]$ *on* Y^2 *is given by* (3.26), (3.27)*.*

Consider a Q-ACR algebra $\mathbb{A}$. Let us fix a constant $c \in \mathbb{C}$, $|c| = 1$, and let us define the operators

$$\widetilde{A}^+(f) = A^+(cf), \quad \widetilde{A}^-(f) = A^-(\bar{c}f).$$

As easily seen, these operators also satisfy the Q-ACR. Furthermore,

$$\big(\widetilde{A}^+(f)\big)^* = \big(A^+(cf)\big)^* = A^-(\bar{c}\bar{f}) = \widetilde{A}^-(\bar{f}).$$

Hence, we can define a new Q-ACR algebra $\widetilde{\mathbb{A}}$ by setting, for any $\sharp_1, \dots, \sharp_n \in \{+, -\}$ and $f^{(n)} \in \mathfrak{F}(X^n)$,

$$\widetilde{\Phi}(f^{(n)}; \sharp_1, \dots, \sharp_n) = \Phi(c^{k-l} f^{(n)}; \sharp_1, \dots, \sharp_n), \tag{3.40}$$

where k is the number of pluses among $\sharp_1, \dots, \sharp_n$ and l is the number of minuses among $\sharp_1, \dots, \sharp_n$.

Next, we note that $\widetilde{\mathbb{A}}$ and $\mathbb{A}$ coincide as sets ($\mathbb{A} = \widetilde{\mathbb{A}}$). Furthermore, the $*$-operations in the algebras $\widetilde{\mathbb{A}}$ and $\mathbb{A}$ coincide. Therefore, we can consider the state τ on $\mathbb{A}$ as the mapping $\tau : \widetilde{\mathbb{A}} \to \mathbb{C}$ and this map determines a state $\widetilde{\tau}$ on $\widetilde{\mathbb{A}}$.

Definition 3.24. We will say that the state τ on the Q-ACR algebra $\mathbb{A}$ is *gauge-invariant* if, for each constant c as above, the states τ and $\widetilde{\tau}$ coincide.

In view of Proposition 3.11 (i), the state τ being gauge-invariant means that

$$\widetilde{\tau}\left(\widetilde{A}^+(\varphi_1 \otimes g_1)\cdots \widetilde{A}^+(\varphi_m \otimes g_m)\widetilde{A}^-(\varphi_{m+1} \otimes g_{m+1})\cdots \widetilde{A}^-(\varphi_{m+n} \otimes g_{m+n})\right)$$
$$= \tau\left(A^+(\varphi_1 \otimes g_1)\cdots A^+(\varphi_m \otimes g_m)A^-(\varphi_{m+1} \otimes g_{m+1})\cdots A^-(\varphi_{m+n} \otimes g_{m+n})\right).$$

But formula (3.40) implies

$$\widetilde{\tau}\left(\widetilde{A}^+(\varphi_1 \otimes g_1)\cdots \widetilde{A}^+(\varphi_m \otimes g_m)\widetilde{A}^-(\varphi_{m+1} \otimes g_{m+1})\cdots \widetilde{A}^-(\varphi_{m+n} \otimes g_{m+n})\right)$$
$$= c^{m-n}\tau\left(A^+(\varphi_1 \otimes g_1)\cdots A^+(\varphi_m \otimes g_m)A^-(\varphi_{m+1} \otimes g_{m+1})\cdots A^-(\varphi_{m+n} \otimes g_{m+n})\right).$$

For $m = n$, $c^{m-n} = c^0 = 1$, whereas if $m \neq n$, $c^{m-n} \neq 1$ and $c^{m-n} \neq 0$ for some $c \in \mathbb{C}$, $|c| = 1$. Hence, the state τ is gauge-invariant if and only if, for all $m \neq n$,

$$\tau\left(A^+(\varphi_1 \otimes g_1)\cdots A^+(\varphi_m \otimes g_m)A^-(\varphi_{m+1} \otimes g_{m+1})\cdots A^-(\varphi_{m+n} \otimes g_{m+n})\right) = 0.$$

Definition 3.25. Let $\mathbb{A}$ be the Q-ACR algebra and let τ be a state on it. Assume that the state τ satisfies the assumption of Proposition 3.11. For any $g_1, g_2 \in \mathcal{G}$, let the complex-valued measure $\gamma^{(2)}[g_1, g_2]$ on Y^2 be determined by (3.19). We say that the state τ is *gauge-invariant quasi-free* if for any $m, n \in \mathbb{N}_0$, $\varphi_1, \dots, \varphi_{m+n} \in C_0(Y)$ and $g_1, \dots, g_{m+n} \in \mathcal{G}$, we have

$$\tau\left(A^+(\varphi_1 \otimes g_1)\cdots A^+(\varphi_m \otimes g_m)A^-(\varphi_{m+1} \otimes g_{m+1})\cdots A^-(\varphi_{m+n} \otimes g_{m+n})\right)$$
$$= \delta_{m,n} \sum_{\xi \in \mathcal{P}_2^{(2n)}} \int_{Y^{2n}} \bigotimes_{\substack{\{i,j\} \in \xi,\, 1 \le i \le n \\ n+1 \le j \le 2n}} \gamma^{(2)}[g_i, g_j](dy_i\, dy_j)\, \varphi_1(y_1)\cdots \varphi_{2n}(y_{2n}) Q(\xi; y_1, \dots, y_{2n}). \tag{3.41}$$

Here $Q(\xi; y_1, \cdots, y_{2n})$ is defined by (3.35). In particular, the state τ is completely determined by the measure $\gamma^{(2)}[g_1, g_2]$ ($g_1, g_2 \in \mathcal{G}$) and the function $Q(\xi; y_1, \dots, y_{2n})$.

By analogy with [22], we will now construct a class of gauge-invariant quasi-free states on the Q-ACR algebra. To this end, we will

- construct the vacuum state on a certain $\mathbf{Q}$-ACR algebra over a larger space $\mathbf{X}$;
- consider a properly chosen sub-algebra of the $\mathbf{Q}$-ACR algebra, which will be identified with the Q-ACR algebra, and the vacuum state on the $\mathbf{Q}$-ACR algebra will yield a quasi-free state on the sub-Q-ACR algebra of the $\mathbf{Q}$-ACR algebra.

Denote by $X_1 = Y_1 \times Z_1$, $X_2 = Y_2 \times Z_2$ two copies of the set X, and consider their disjoint union

$$\mathbf{X} = X_1 \sqcup X_2.$$

Note that

$$\mathbf{X}^2 = (X_1 \times X_1) \sqcup (X_1 \times X_2) \sqcup (X_2 \times X_1) \sqcup (X_2 \times X_2)\,.$$

Let $\mathcal{B}(\mathbf{X})$ be the σ-algebra on $\mathbf{X}$ such that, for each $\mathbf{A} \in \mathcal{B}(\mathbf{X})$, $\mathbf{A} \cap X_1 \in \mathcal{B}(X_1)$ and $\mathbf{A} \cap X_2 \in \mathcal{B}(X_2)$.

We consider the measure on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ that satisfies that its restriction to $(X_i, \mathcal{B}(X_i))$, $i = 1, 2$, coincides with the measure σ, considered as a measure on $(X_i, \mathcal{B}(X_i))$. We will keep the notation σ for this measure on $\mathbf{X}$. Then we obviously have

$$L^2\left(\mathbf{X}, d\sigma\right) = L^2\left(X_1, d\sigma\right) \oplus L^2\left(X_2, d\sigma\right).$$

We denote $\mathbf{H} = L^2\left(\mathbf{X}, d\sigma\right)$.

We define a (continuous) function $\mathbf{Q} : \mathbf{X}^2 \to \mathbb{C}$ by

$$\mathbf{Q}(x_1, x_2) = \begin{cases} Q(x_1, x_2), & \text{if } x_1, x_2 \in X_1 \text{ or } x_1, x_2 \in X_2, \\ Q(x_2, x_1), & \text{if } x_1 \in X_1,\ x_2 \in X_2 \text{ or } x_1 \in X_2,\ x_2 \in X_1. \end{cases} \tag{3.42}$$

Note that $|\mathbf{Q}(x_1, x_2)| = 1$ and $\mathbf{Q}(x_2, x_1) = \overline{\mathbf{Q}(x_1, x_2)}$ for all $(x_1, x_2) \in \mathbf{X}^2$.

We denote by $\mathbf{A}$ the $\mathbf{Q}$-ACR algebra corresponding to the function $\mathbf{Q}$. Let $\boldsymbol{\tau}$ denote the Fock state on $\mathbf{Q}$-ACR $\mathbf{A}$.

Remark 3.26. Denote

$$\mathbf{Y} = Y_1 \sqcup Y_2, \quad \mathbf{Z} = Z_1 \sqcup Z_2.$$

We obviously do not have the representation of $\mathbf{X}$ as $\mathbf{Y}\times\mathbf{Z}$, instead $\mathbf{X}$ is a subset of $\mathbf{Y}\times\mathbf{Z}$. So strictly speaking we cannot apply the results of Sections 3.3, 3.4 to this case. Nevertheless these results still hold in the current setting after one has done a trivial modification. In particular, the counterpart of formula (3.2) should be read as follows. We define $\mathfrak{F}(\mathbf{X}^n)$ as the linear span of functions $\mathbf{f}^{(n)}:\mathbf{X}^n\to\mathbb{C}$ such that, for each $(i_1,\dots,i_n)\in\{1,2\}^n$, the restriction of $\mathbf{f}^{(n)}$ to $X_{i_1}\times X_{i_2}\times\dots\times X_{i_n}$, denoted by $\mathbf{f}^{(n)}_{i_1,\dots,i_n}$, is of the form

$$\mathbf{f}^{(n)}_{i_1,\dots,i_n}(x_1,\dots,x_n)=\varphi^{(n)}(y_1,\dots,y_n)g_1(z_1)\dots g_n(z_n),$$

where $\varphi^{(n)}\in C_0(Y^n)$, $g_1,\dots,g_n\in\mathcal{G}$ and the choice of functions $\varphi^{(n)},g_1,\dots,g_n$ depends on $(i_1,\dots,i_n)$.

In accordance with the intuitive formula (3.3), we will denote elements of the $\mathbf{Q}$-ACR algebra by

$$\int_{\mathbf{X}^n}\mathbf{f}^{(n)}(x_1,\dots,x_n)\mathbf{a}^{\sharp_1}(x_1)\cdots\mathbf{a}^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n) \tag{3.43}$$

for $\mathbf{f}^{(n)}\in\mathfrak{F}(\mathbf{X}^n)$ and $\sharp_1,\dots,\sharp_n\in\{+,-\}$.

Let $(i_1,\dots,i_n)\in\{1,2\}^n$ and assume that

$$\mathbf{f}^{(n)}(x_1,\dots,x_n)=\begin{cases} f^{(n)}(x_1,\dots,x_n), & \text{if } (x_1,\dots,x_n)\in X_{i_1}\times X_{i_2}\times\dots\times X_{i_n},\\ 0, & \text{if } (x_1,\dots,x_n)\notin X_{i_1}\times X_{i_2}\times\dots\times X_{i_n}.\end{cases} \tag{3.44}$$

where $f^{(n)}\in\mathfrak{F}(X^n)$. In this case, we will denote the corresponding element (3.43) of the $\mathbf{Q}$-ACR algebra $\mathbf{A}$ by

$$\int_{X^n}f^{(n)}(x_1,\dots,x_n)\mathbf{a}^{\sharp_1}_{i_1}(x_1)\cdots\mathbf{a}^{\sharp_n}_{i_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n). \tag{3.45}$$

So, informally, for $\sharp\in\{+,-\}$, $i\in\{1,2\}$, and $x\in X$, $\mathbf{a}^\sharp_i(x)$ is equal to $\mathbf{a}^\sharp(x)$, where x is identified with an element of X_i.

Recall that $\varkappa\in\{-1,+1\}$ is the value of the Q function on the diagonal, i.e., $Q(y,y)=\varkappa$ for all $y\in Y$. In view of (2.130), (3.10) and (3.42), the operators $\mathbf{a}^\sharp_i(x)$ with $i\in\{1,2\}$ and $\sharp\in\{+,-\}$ satisfy the following formal commutation relations:

$$\mathbf{a}^\sharp_i(x_1)\mathbf{a}^\sharp_i(x_2)=Q(x_2,x_1)\mathbf{a}^\sharp_i(x_2)\mathbf{a}^\sharp_i(x_1),\quad i\in\{1,2\},\ \sharp\in\{+,-\},$$

$$\mathbf{a}_i^\sharp(x_1)\mathbf{a}_j^\sharp(x_2) = Q(x_1,x_2)\mathbf{a}_j^\sharp(x_2)\mathbf{a}_i^\sharp(x_1), \quad i,j\in\{1,2\},\ i\neq j,\ \sharp\in\{+,-\},$$
$$\mathbf{a}_i^-(x_1)\mathbf{a}_i^+(x_2) = \delta(x_1,x_2)+Q(x_1,x_2)\mathbf{a}_i^+(x_2)\mathbf{a}_i^-(x_1), \quad i\in\{1,2\},$$
$$\mathbf{a}_i^-(x_1)\mathbf{a}_j^+(x_2) = Q(x_2,x_1)\mathbf{a}_j^+(x_2)\mathbf{a}_i^-(x_1), \quad i,j\in\{1,2\},\ i\neq j,$$
$$\mathbf{a}_i^+(x_1)\mathbf{a}_i^-(x_2) = -\varkappa\delta(x_1,x_2)+Q(x_1,x_2)\mathbf{a}_i^-(x_2)\mathbf{a}_i^+(x_1), \quad i\in\{1,2\},$$
$$\mathbf{a}_i^+(x_1)\mathbf{a}_j^-(x_2) = Q(x_2,x_1)\mathbf{a}_j^-(x_2)\mathbf{a}_i^+(x_1), \quad i,j\in\{1,2\}, i\neq j. \tag{3.46}$$

Note that, for $x_1 \in X_i$ and $x_2 \in X_j$ with $i \neq j$, the term $\delta(x_1, x_2)$ vanishes.

Let us fix a bounded linear operator $K \in \mathcal{L}(\mathcal{G})$. In the case $\varkappa = 1$, we assume that $K \geq 0$, and in the case $\varkappa = -1$, we assume that $0 \leq K \leq 1$. Let

$$K_1 = \sqrt{K}, \quad K_2 = \sqrt{1+\varkappa K}.$$

For a bounded linear operator $B \in \mathcal{L}(\mathcal{G})$, we denote by B' its complex conjugate operator, i.e.,

$$B' = JBJ, \tag{3.47}$$

where $(Jg)(z) = \overline{g(z)}$ is the complex conjugation in $\mathcal{G}$.

Let $f = \varphi \otimes g$ with $\varphi \in C_0(Y)$ and $g \in \mathcal{G}$. We define

$$A^+(f) = \int_X \varphi(y)\,(K_2 g)\,(z)\mathbf{a}_2^+(x)\sigma(dx) + \int_X \varphi(y)\,(K_1 g)\,(z)\mathbf{a}_1^-(x)\sigma(dx),$$
$$A^-(f) = \int_X \varphi(y)(K_2' g)(z)\mathbf{a}_2^-(x)\sigma(dx) + \int_X \varphi(y)(K_1' g)(z)\mathbf{a}_1^+(x)\sigma(dx). \tag{3.48}$$

Remark 3.27. Note that the representation $f = \varphi \otimes g$ is not unique. Indeed, choose an arbitrary constant $c \in \mathbb{C}$, $c \neq 0$. Then $f = (c\varphi) \otimes (\frac{1}{c}\,g)$, where $c\varphi \in C_0(Y)$, $\frac{1}{c}\,g \in \mathcal{G}$. But, for each operator $B \in \mathcal{L}(\mathcal{G})$, we have $(c\varphi) \otimes (B\frac{1}{c}\,g) = \varphi \otimes (Bg)$. Hence, the definition of $A^+(f)$ and $A^-(f)$ does not depend on the representation of f as $\varphi \otimes g$.

Proposition 3.28. *We have* $\big(A^+(f)\big)^* = A^-(Jf)$ *and the operators* $A^+(f)$, $A^-(f)$ *defined by* (3.48) *satisfy the Q-ACR.*

Proof. The statement easily follows from the commutation relations (3.46). We only need to note that for any $\varphi_1 \otimes g_1$, $\varphi_2 \otimes g_2$ with $\varphi_1, \varphi_2 \in C_0(Y)$ and $g_1, g_2 \in \mathcal{G}$, we have

$$\int_X \varphi_1(y)(K_2' g_1)(z)\varphi_2(y)(K_2 g_2)(z)\sigma(dx) - \varkappa \int_X \varphi_1(y)(K_1' g_1)(z)\varphi_2(y)(K_1 g_2)(z)\sigma(dx)$$

$$= \big(\varphi_2 \otimes (K_2 g_2), \overline{\varphi_1} \otimes (K_2 \overline{g_1})\big)_{L^2(X,\sigma)} - \varkappa\big(\varphi_2 \otimes (K_1 g_2), \overline{\varphi_1} \otimes (K_1 \overline{g_1})\big)_{L^2(X,\sigma)}$$
$$= \big(\varphi_2 \otimes (K_2^2 g_2), \overline{\varphi_1} \otimes \overline{g_1}\big)_{L^2(X,\sigma)} - \varkappa\big(\varphi_2 \otimes (K_1^2 g_2), \overline{\varphi_1} \otimes \overline{g_1}\big)_{L^2(X,\sigma)}$$
$$= \big(\varphi_2 \otimes (1+\varkappa K) g_2, J f_1\big)_{L^2(X,\sigma)} - \varkappa\big(\varphi_2 \otimes (K g_2), J f_1\big)_{L^2(X,\sigma)}$$
$$= (f_2, J f_1)_{L^2(X,\sigma)} = \int_X f_1(x) f_2(x)\sigma(dx). \qquad \square$$

We formally introduce operators $A^+(x)$, $A^-(x)$, $x \in X$, such that, for each $f = \varphi \otimes g$ with $\varphi \in C_0(Y)$ and $g \in \mathcal{G}$,

$$A^+(f) = \int_X f(x) A^+(x)\sigma(dx),$$
$$A^-(f) = \int_X f(x) A^-(x)\sigma(dx), \tag{3.49}$$

where $A^+(f)$, $A^-(f)$ are given by (3.48).

By Proposition 3.28, $A^+(x)$, $A^-(x)$ satisfy the Q-ACR, i.e., formulas (2.130) and (3.10) hold with $a^+(\cdot)$, $a^-(\cdot)$ being replaced with $A^+(\cdot)$, $A^-(\cdot)$.

Let $\varphi^{(n)} \in C_0(Y^n)$ and $g_1, \dots, g_n \in \mathcal{G}$ and let

$$f^{(n)}(x_1, \dots, x_n) = \varphi^{(n)}(y_1, \dots, y_n) g_1(z_1) \cdots g_n(z_n) \in \mathfrak{F}(X^n).$$

We now want to define, for $\sharp_1, \dots, \sharp_n \in \{-, +\}$,

$$\int_{X^n} f^{(n)}(x_1, \dots, x_n) A^{\sharp_1}(x_1) \cdots A^{\sharp_n}(x_n)\sigma(dx_1) \cdots \sigma(dx_n).$$

To do this, we define the operators

$$K(+,1) = K_1, \quad K(+,2) = K_2,$$
$$K(-,1) = K_1', \quad K(-,2) = K_2'.$$

We also define

$$\gamma(+,1) = -, \quad \gamma(+,2) = +,$$
$$\gamma(-,1) = +, \quad \gamma(-,2) = -.$$

Note that, using these notations, we can re-write formulas (3.48), as follows:

$$A^+(f) = \int_X \varphi(y)\, (K(+,1)g)\,(z) \mathbf{a}_1^{\gamma(+,1)}(x)\sigma(dx) + \int_X \varphi(y)\, (K(+,2)g)\,(z) \mathbf{a}_2^{\gamma(+,2)}(x)\sigma(dx),$$

$$A^{-}(f)=\int_X \varphi(y)(K(-,1)g)(z)\mathbf{a}_1^{\gamma(-,1)}(x)\sigma(dx)+\int_X \varphi(y)(K(-,2)g)(z)\mathbf{a}_2^{\gamma(-,2)}(x)\sigma(dx). \tag{3.50}$$

Then, in view of formulas (3.49) and (3.50), we define

$$\begin{aligned}&\int_{X^n} f^{(n)}(x_1,\dots,x_n)A^{\sharp_1}(x_1)\cdots A^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n)\\ &\quad=\sum_{(i_1,\dots,i_n)\in\{1,2\}^n}\int_{X^n}\varphi^{(n)}(y_1,\dots,y_n)\\ &\quad\quad\times\left(K(\sharp_1,i_1)g_1\right)(z_1)\cdots\left(K(\sharp_n,i_n)g_n\right)(z_n)\mathbf{a}_{i_1}^{\gamma(\sharp_1,i_1)}(x_1)\cdots\mathbf{a}_{i_n}^{\gamma(\sharp_n,i_n)}(x_n)\sigma(x_1)\cdots\sigma(dx_n).\end{aligned} \tag{3.51}$$

In particular, if $\varphi^{(n)}(y_1,\dots,y_n)=\varphi_1(y_1)\cdots\varphi_n(y_n)$ with $\varphi_1,\dots,\varphi_n\in C_0(Y)$, we have

$$\int_{X^n} f^{(n)}(x_1,\dots,x_n)A^{\sharp_1}(x_1)\cdots A^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n)=A^{\sharp_1}(f_1)\cdots A^{\sharp_n}(f_n), \tag{3.52}$$

where $f_i(x)=\varphi_i(y)g_i(z)$.

Next, we extend our definition of

$$\int_{X^n} f^{(n)}(x_1,\dots,x_n)A^{\sharp_1}(x_1)\cdots A^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n), \tag{3.53}$$

by linearity to the case where $f^{(n)}$ is an arbitrary element of $\mathfrak{F}(X^n)$.

Obviously, each element of the form (3.53) belongs to the **Q**-ACR algebra **A**. Let $\mathbb{A}$ be the unital $*$-algebra generated by elements of the form (3.53). Thus, $\mathbb{A}$ is a sub-$*$-algebra of **A**. Hence, the restriction to $\mathbb{A}$ of the Fock state $\boldsymbol{\tau}$ on **A** is a state on $\mathbb{A}$. We will keep the notation $\boldsymbol{\tau}$ for this state on $\mathbb{A}$.

Theorem 3.29. *For $\sharp_1,\dots,\sharp_n\in\{-,+\}$ and $f^{(n)}\in\mathfrak{F}(X^n)$, denote*

$$\Phi^{(n)}\left(f^{(n)};\sharp_1,\dots,\sharp_n\right)=\int_{X^n} f^{(n)}(x_1,\dots,x_n)A^{\sharp_1}(x_1)\cdots A^{\sharp_n}(x_n)\sigma(dx_1)\cdots\sigma(dx_n). \tag{3.54}$$

Then the $\Phi^{(n)}\left(f^{(n)};\sharp_1,\dots,\sharp_n\right)$ satisfy conditions (3.4), (3.5), *and* (3.6). *Hence, the $*$-algebra $\mathbb{A}$ can be considered as a Q-ACR algebra. Furthermore, the state $\boldsymbol{\tau}$ on $\mathbb{A}$ is gauge-invariant quasi-free, and the corresponding measure $\gamma^{(2)}[g_1,g_2]$ is given by*

$$\gamma^{(2)}[g_1,g_2](dy_1dy_2)=\nu^{(2)}(dy_1,dy_2)\int_Z (Kg_1)(z)g_2(z)\mu(dz). \tag{3.55}$$

Proof. It follows from formulas (3.50), (3.51), (3.52), (3.54) and our considerations in Section 3.4 that Proposition 3.28 implies that conditions (3.4), (3.5), and (3.6) are satisfied. Thus, $\mathbb{A}$ can be considered as a Q-ACR algebra. □

Lemma 3.30. *For any $n \in \mathbb{N}$ and $\sharp_1, \dots, \sharp_n \in \{-,+\}$, the functional*

$$\boldsymbol{\tau}^{(n)}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) = \boldsymbol{\tau}\left(\Phi^{(n)}(\varphi^{(n)}(y_1,\dots,y_n)g_1(z_1)\cdots g_n(z_n); \sharp_1,\dots,\sharp_n)\right)$$

is continuous on $C_0(Y^n) \times \mathcal{G}^n$.

Proof. Let us fix arbitrary $\sharp_1, \dots, \sharp_n \in \{-,+\}$ and $i_1, \dots, i_n \in \{1,2\}$. Consider the functional

$$\begin{aligned}\boldsymbol{\tau}^{i_1,\dots,i_n}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) = \boldsymbol{\tau}\bigg(\int_{X^n} &\varphi^{(n)}(y_1,\dots,y_n)g_1(z_1)\cdots g_n(z_n)\\ &\times \mathbf{a}_{i_1}^{\gamma(\sharp_1,i_1)}(x_1)\cdots \mathbf{a}_{i_n}^{\gamma(\sharp_n,i_n)}(x_n)\sigma(x_1)\cdots\sigma(dx_n)\bigg)\end{aligned}$$

on $C_0(Y^n) \times \mathcal{G}^n$. It follows from Lemma 3.14 that this functional is continuous.

Since the operators K_1, K_2, K_1', K_2' act continuously on $\mathcal{G}$, the functional

$$C_0(Y^n) \times \mathcal{G}^n \ni (\varphi^{(n)}, g_1, \dots, g_n) \mapsto \boldsymbol{\tau}^{i_1,\dots,i_n}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, K(\sharp_1, i_1)g_1, \dots, K(\sharp_n, i_n)g_n\right)$$

is continuous.

By formulas (3.51) and (3.54), we have

$$\boldsymbol{\tau}^{(n)}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, g_1, \dots, g_n\right) = \sum_{(i_1,\dots,i_n)\in\{1,2\}^n} \boldsymbol{\tau}^{i_1,\dots,i_n}_{\sharp_1,\dots,\sharp_n}\left(\varphi^{(n)}, K(\sharp_1, i_1)g_1, \dots, K(\sharp_n, i_n)g_n\right).$$

Hence, $\boldsymbol{\tau}^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n) \times \mathcal{G}^n$. □

For any $\varphi_1, \varphi_2 \in C_0(Y)$ and $g_1, g_2 \in \mathcal{G}$, we have, by (3.48),

$$\begin{aligned}&\boldsymbol{\tau}\left(A^+(\varphi_1 \otimes g_1)A^-(\varphi_2 \otimes g_2)\right)\\ &\quad = \boldsymbol{\tau}\left(\int_X \varphi_1(y_1)(K_1 g_1)(z_1)\boldsymbol{a}_1^-(x_1)\sigma(dx_1)\int_X \varphi_2(y_2)(K_1' g_2)(z_2)\boldsymbol{a}_1^+(x_2)\sigma(dx_2)\right)\\ &\quad = \int_Y \varphi_1(y)\varphi_2(y)\nu(dy)\int_Z (K_1 g_1)(z)(K_1' g_2)(z)\mu(dz)\\ &\quad = \int_Y \varphi_1(y_1)\varphi_2(y_2)\nu^{(2)}(dy_1 dy_2)(K_1 g_1, K_1 J g_2)_{L^2(Z,d\mu)}\end{aligned}$$

$$= \int_Y \varphi_1(y_1)\varphi_2(y_2)\nu^{(2)}(dy_1dy_2)(Kg_1, Jg_2)_{L^2(Z,d\mu)}$$
$$= \int_Y \varphi_1(y_1)\varphi_2(y_2)\nu^{(2)}(dy_1dy_2)\int_Z (Kg_1)(z)g_2(z)\mu(dz)$$
$$= \int_Y \varphi_1(y_1)\varphi_2(y_2)\gamma^{(2)}[g_1,g_2](dy_1dy_2), \tag{3.56}$$

where the measure $\gamma^{(2)}[g_1,g_2]$ is given by (3.55).

Similarly, we have for $\varphi_1,\cdots,\varphi_{m+n}\in C_0(Y)$ and $g_1,\dots,g_{m+n}\in\mathcal{G}$,

$$\begin{aligned}
&\boldsymbol{\tau}\left(A^+(\varphi_1\otimes g_1)\cdots A^+(\varphi_m\otimes g_m)A^-(\varphi_{m+1}\otimes g_{m+1})\cdots A^-(\varphi_{m+n}\otimes g_{m+n})\right)\\
&\quad=\delta_{m,n}\,\boldsymbol{\tau}\bigg(\int_X \varphi_1(y_1)(K_1g_1)(z_1)\boldsymbol{a}_1^-(x_1)\sigma(dx_1)\cdots\int_X \varphi_n(y_n)(K_1g_n)(z_n)\boldsymbol{a}_1^-(x_n)\sigma(dx_n)\\
&\qquad\times\int_X \varphi_{n+1}(y_{n+1})(K_1'g_{n+1})(z_{n+1})\boldsymbol{a}_1^+(x_{n+1})\sigma(dx_{n+1})\\
&\qquad\times\cdots\int_X \varphi_{2n}(y_{2n})(K_1'g_{2n})(z_{2n})\boldsymbol{a}_1^+(x_{2n})\sigma(dx_{2n})\bigg)\\
&\quad=\delta_{m,n}\sum_{\xi\in\mathcal{P}_2^{(2n)}}\int_{Y^{2n}}\bigotimes_{\substack{\{i,j\}\in\xi,\,1\le i\le n\\ n+1\le j\le 2n}}\gamma^{(2)}[g_i,g_j](dy_idy_j)\varphi_1(y_1)\cdots\varphi_{2n}(y_{2n})Q(\xi;y_1,\dots,y_{2n}),
\end{aligned}$$

where the last equality follows from (3.37) and (3.42).

Theorem 3.31. *(i) Let the function Q be not real-valued. If $K\neq 0$ and for $\varkappa=-1$, if additionally $K\neq 1$, the state $\boldsymbol{\tau}$ is not strongly quasi-free.*

(ii) Let the function Q be real-valued, i.e., $Q(y_1,y_2)=\varkappa$ for all $y_1,y_2\in Y$. Then the state $\boldsymbol{\tau}$ is strongly quasi-free and the corresponding measure $\lambda^{(2)}[g_1,g_2]$ is given by

$$\lambda^{(2)}[g_1,g_2](dy_1,dy_2)=\nu^{(2)}(dy_1,dy_2)\Big((g_2,g_1)_{L^2(Z,\mu)}+(Kg_1,g_2)_{L^2(Z,\mu)}+\varkappa\overline{(Kg_1,g_2)}_{L^2(Z,\mu)}\Big). \tag{3.57}$$

In particular, for $\varkappa=1$, we have

$$\lambda^{(2)}[g_1,g_2](dy_1,dy_2)=\nu^{(2)}(dy_1,dy_2)\Big((g_2,g_1)_{L^2(Z,\mu)}+2\operatorname{Re}(Kg_1,g_2)_{L^2(Z,\mu)}\Big), \tag{3.58}$$

and for $\varkappa=-1$,

$$\lambda^{(2)}[g_1,g_2](dy_1,dy_2)=\nu^{(2)}(dy_1,dy_2)\Big((g_2,g_1)_{L^2(Z,\mu)}+2i\operatorname{Im}(Kg_1,g_2)_{L^2(Z,\mu)}\Big). \tag{3.59}$$

Remark 3.32. As mentioned above, part (ii) of Proposition 3.31 is of course well-known.

Proof. For any $\varphi \in C_0(Y \to \mathbb{R})$ and $g \in \mathcal{G}$, we have, by (3.17),

$$B(\varphi \otimes g) = A^+(\varphi \otimes g) + A^-(\varphi \otimes (Jg)). \tag{3.60}$$

We define

$$\begin{aligned}
\mathcal{A}^+(\varphi \otimes g) &= \int_X \varphi(y)(K_2 g)(z)\mathbf{a}_2^+(x)\sigma(dx) + \int_X \varphi(y)(JK_1 g)(z)\mathbf{a}_1^+(x)\sigma(dx) \\
&= \int_{\mathbf{X}} \varphi(y)(\mathbf{K}^+ g)(z)\mathbf{a}^+(x)\sigma(dx) \\
\mathcal{A}^-(\varphi \otimes g) &= \int_X \varphi(y)(K_1 g)(z)\mathbf{a}_1^-(x)\sigma(dx) + \int_X \varphi(y)(JK_2 g)(z)\mathbf{a}_2^-(x)\sigma(dx) \\
&= \int_{\mathbf{X}} \varphi(y)(\mathbf{K}^- g)(z)\mathbf{a}^-(x)\sigma(dx)
\end{aligned} \tag{3.61}$$

where

$$\begin{aligned}
(\mathbf{K}^+ g)(z) &= \chi_{Z_1}(z)(JK_1 g)(z) + \chi_{Z_2}(z)(K_2 g)(z), \\
(\mathbf{K}^- g)(z) &= \chi_{Z_1}(z)(K_1 g)(z) + \chi_{Z_2}(z)(JK_2 g)(z).
\end{aligned} \tag{3.62}$$

Then, by (3.48), (3.60) and (3.61),

$$B(\varphi \otimes g) = \mathcal{A}^+(\varphi \otimes g) + \mathcal{A}^-(\varphi \otimes g). \tag{3.63}$$

Let us now prove part (i). For any $\varphi_1, \dots, \varphi_4 \in C_0(Y \to \mathbb{R})$ and $g_1, \dots, g_4 \in \mathcal{G}$, consider

$$\begin{aligned}
&\boldsymbol{\tau}\big(B(\varphi_1 \otimes g_1) \cdots B(\varphi_4 \otimes g_4)\big) \\
&\quad = \boldsymbol{\tau}\big(\big(\mathcal{A}^+(\varphi_1 \otimes g_1) + \mathcal{A}^-(\varphi_1 \otimes g_1)\big)\big(\mathcal{A}^+(\varphi_2 \otimes g_2) + \mathcal{A}^-(\varphi_2 \otimes g_2)\big) \\
&\qquad \times \big(\mathcal{A}^+(\varphi_3 \otimes g_3) + \mathcal{A}^-(\varphi_3 \otimes g_3)\big)\big(\mathcal{A}^+(\varphi_4 \otimes g_4) + \mathcal{A}^-(\varphi_4 \otimes g_4)\big)\big).
\end{aligned} \tag{3.64}$$

To calculate the right-hand side of (3.64), we use formula (3.36). Consider the term corresponding to the partition $\xi = \big\{\{1,3\},\{2,4\}\big\} \in \mathcal{P}^{(4)}$:

$$\begin{aligned}
&\int_{\mathbf{X}^4} \sigma^{(2)}(dx_1\, dx_3)\sigma^{(2)}(dx_2\, dx_4)\varphi_1(y_1)\varphi_2(y_2)\varphi_3(y_3)\varphi_4(y_4) \\
&\qquad \times (\mathbf{K}^- g_1)(z_1)(\mathbf{K}^- g_2)(z_2)(\mathbf{K}^+ g_3)(z_3)(\mathbf{K}^+ g_4)(z_4)\mathbf{Q}(y_2, y_3).
\end{aligned}$$

Note that, for $y_2, y_3 \in Y_i$, $i \in \{1,2\}$, we indeed have

$$\mathbf{Q}(y_2, y_3) = Q(y_2, y_3)$$

but for $y_2 \in Y_i$, $y_3 \in Y_j$ with $i \neq j$,

$$\mathbf{Q}(y_2, y_3) = \overline{Q(y_2, y_3)}.$$

But to have a strongly quasi-free state, we must only have $Q(y_2, y_3)$ for this term. Hence, unless the function Q is real-valued or one of the operators, K_1 or K_2, is equal to zero, the state $\boldsymbol{\tau}$ is not strongly quasi-free.

We now prove part (ii). By (3.36), (3.60), (3.61) and (3.63), the state $\boldsymbol{\tau}$ is strongly quasi-free and the corresponding measure $\lambda^{(2)}[g_1, g_2]$ is given by

$$\lambda^{(2)}[g_1, g_2](dy_1\, dy_2) = \nu^{(2)}(dy_1\, dy_2) \int_{\mathbf{Z}} (\mathbf{K}^- g_1)(z)(\mathbf{K}^+ g_2)(z)\mu(dz).$$

By (3.62),

$$\begin{aligned}
&\int_{\mathbf{Z}} (\mathbf{K}^- g_1)(z)(\mathbf{K}^+ g_2)(z)\mu(dz) \\
&\quad = \int_Z (K_1 g_1)(z)(JK_1 g_2)(z)\mu(dz) + \int_Z (JK_2 g_1)(z)(K_2 g_2)(z)\mu(dz) \\
&\quad = (K_1 g_1, K_1 g_2)_{\mathcal{G}} + (K_2 g_2, K_2 g_1)_{\mathcal{G}} \\
&\quad = (K g_1, g_2)_{\mathcal{G}} + ((1 + \varkappa K) g_2, g_1)_{\mathcal{G}} \\
&\quad = (g_2, g_1)_{\mathcal{G}} + (K g_1, g_2)_{\mathcal{G}} + \varkappa (g_2, K g_1)_{\mathcal{G}}. \qquad (3.65)
\end{aligned}$$

□

Chapter 4

Gauge-invariant and strongly quasi-free states on the Wick algebra of multicomponent commutation relations

The aim of this chapter is to define and study gauge-invariant quasi-free and strongly quasi-free free states on the Wick algebra of multicomponent commutation relations. Unlike the case of the Q-ACR, our construction will be done under additional, rather restrictive assumptions on the function $C(x_1, x_2)$ taking values in $\mathcal{L}(V^{\otimes 2})$. Nevertheless, when the dimension of V is two (i.e., for a two-component quantum system), we will show that the class of functions $C(x_1, x_2)$, that was proposed in [9, Example 4.9], does satisfy our assumptions. Hence, we are able to construct a gauge-invariant quasi-free state on the corresponding Wick algebra. Furthermore, for a non-trivial subclass of this class, the corresponding states appear to be strongly quasi-free. This happens in stark difference with the case of the Q-ACR, where we could not find non-trivial examples of strongly quasi-free states.

4.1 The algebra of multicomponent commutation relations (C-MCR)

Just as in Chapter 3, we assume that $X = Y \times Z$ and $\sigma = \nu \otimes \mu$. Let V be a complex Hilbert space, and this chapter we will assume that V is finite-dimensional. We will denote by d the dimension of V.

Remark 4.1. Let us explain why we now need V to be finite-dimensional. As well-known, if V is a separable Hilbert space, then the space $V^{\otimes 2}$ can be identified with Hilbert–Schmidt operators in V. Indeed, let us fix an arbitrary orthonormal basis $(e_n)_{n\geq 1}$ in V. Then each $f^{(2)} \in V^{\otimes 2}$ is of the form

$$f^{(2)} = \sum_{m,n\geq 1} (f^{(2)}, e_m \otimes e_n)_{V^{\otimes 2}}\, e_m \otimes e_n,$$

and

$$\|f^{(2)}\|^2_{V^{\otimes 2}} = \sum_{m,n\geq 1} |(f^{(2)}, e_m \otimes e_n)|^2.$$

Then we can identify $f^{(2)}$ with the Hilbert-Schmidt operator A in V whose matrix in the $(e_n)_{n\geq 1}$ basis is $A = \big[(f^{(2)}, e_m \otimes e_n)\big]_{m,n\geq 1}$ and every Hilbert–Schmidt operator in V is of this form. If we additionally assume that V is finite-dimensional, then each linear operator A in V is of trace class, and the trace of A is given by

$$\operatorname{Tr} A = \sum_{n=1}^{d} (Ae_n, e_n)_V = \sum_{n=1}^{d} (f^{(2)}, e_n \otimes e_n)_{V^{\otimes 2}}.$$

Here we identified A with $f^{(2)} \in V^{\otimes 2}$. Hence, we can define the linear functional

$$V^{\otimes 2} \ni f^{(2)} \mapsto \operatorname{Tr} f^{(2)} = \sum_{n=1}^{d} (f^{(2)}, e_n \otimes e_n)_{V^{\otimes 2}} \in \mathbb{C}$$

and this definition does not depend on the choice of a basis $(e_n)_{n=1,\dots,d}$ in V. Note that, for any $v_1, v_2 \in V$,

$$\operatorname{Tr}(v_1 \otimes v_2) = \langle v_1, v_2 \rangle_V.$$

Here $\langle v_1, v_2 \rangle_V = (v_1, Jv_2)_V$.

We fix a map $C : Y^2 \to \mathcal{L}\,(V^{\otimes 2})$ that satisfies the following

Assumption 4.2. (i) The map $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ is continuous;

(ii) For each $(y_1, y_2) \in Y^2$, $C(y_1, y_2)$ is a unitary operator in $V^{\otimes 2}$;

(iii) For each $(y_1, y_2) \in Y^2$, $C^*(y_1, y_2) = C(y_2, y_1)$;

(iv) The functional Yang–Baxter equation is satisfied point-wise, i.e.,

$$C_1(y_1, y_2)C_2(y_1, y_3)C_1(y_2, y_3) = C_2(y_2, y_3)C_1(y_1, y_3)C_2(y_1, y_2), \quad (y_1, y_2, y_3) \in Y^3. \quad (4.1)$$

We will also consider C as a function on X^2 by setting $C(y_1, z_1, y_2, z_2) = C(y_1, y_2)$.

We have

$$\mathcal{H} = L^2(X \to V, \sigma) = L^2(X, \sigma) \otimes V = L^2(Y, \nu) \otimes L^2(Z, \mu) \otimes V.$$

Hence, we can interpret $\mathcal{H}$ as

$$\mathcal{H} = L^2(Y \to V, \nu) \otimes L^2(Z, \mu).$$

Just as in Section 3.1, we denote $\mathcal{G} = L^2(Z, \mu)$. Next we define $C_0(Y^n \to V^{\otimes n})$ to be the space of all continuous maps $\varphi^{(n)} : Y^n \to V^{\otimes n}$ with compact support.

Similarly to (3.2), we define

$$\begin{aligned}\mathfrak{F}(X^n \to V^{\otimes n}) &= C_0(Y^n \to V^{\otimes n}) \otimes_{\mathrm{alg}} \mathcal{G}^{\otimes_{\mathrm{alg}} n} \\ &= \mathrm{l.\,s.}\big\{f^{(n)}(x_1, \dots, x_n) = \varphi^{(n)}(y_1, \dots, y_n) g_1(z_1), \dots, g_n(z_n) \mid \\ &\qquad \varphi^{(n)} \in C_0(Y^n \to V^{\otimes n}),\ g_1, \dots, g_n \in \mathcal{G}\big\}.\end{aligned}$$

We fix a complex conjugation $J : V \to V$, which determines the complex conjugation $J : V^{\otimes n} \to V^{\otimes n}$. This can be done by fixing an orthonormal basis $(e_i)_{i=1,\dots,d}$ in V, and setting $Je_i = e_i$ for all i. For $u^{(n)}, v^{(n)} \in V^{\otimes n}$, we denote

$$\langle u^{(n)}, v^{(n)} \rangle_{V^{\otimes n}} = (u^{(n)}, Jv^{(n)})_{V^{\otimes n}}.$$

For $n \geq 2$ and $i \in \{1, \dots, n-1\}$, we define

$$\mathrm{Tr}_i : V^{\otimes n} \to V^{\otimes (n-2)}, \quad \mathrm{Tr}_i = \mathbf{1}_{i-1} \otimes \mathrm{Tr} \otimes \mathbf{1}_{n-1-i}.$$

Lemma 4.3. *For $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$, $n \geq 2$, and $i \in \{1, \dots, n-1\}$, define*

$$h^{(n-2)} : X^{n-2} \to V^{\otimes (n-2)}$$

by

$$h^{(n-2)}(x_1,\dots,x_{n-2}) = \int_X \operatorname{Tr}_i f^{(n)}(x_1,\dots,x_{i-1},x,x,x_i,\dots,x_{n-2})\sigma(dx).$$

Then $h^{(n-2)} \in \mathfrak{F}(X^{n-2} \to V^{\otimes(n-2)})$.

Proof. By linearity, it suffices to check that the statement holds for $f^{(n)}$ of the form

$$f^{(n)}(x_1,\dots,x_n) = \varphi^{(n)}(y_1,\dots,y_n)g_1(z_1)\cdots g_n(z_n),$$

with $\varphi^{(n)} \in C_0(Y^n \to V^{\otimes n})$ and $g_1,\dots,g_n \in \mathcal{G}$. But then

$$\begin{aligned} h^{(n-2)}(x_1,\dots,x_{n-2}) &= \int_Y \operatorname{Tr}_i \varphi^{(n)}(y_1,\dots,y_{i-1},y,y,y_i,\dots,y_{n-2})\nu(dy) \\ &\quad\times \left(\int_Z g_i(z)g_{i+1}(z)\mu(dz)\right) g_1(z_1)\cdots g_{i-1}(z_{i-1})g_{i+2}(z_i)\cdots g_n(z_{n-2}), \end{aligned}$$

which obviously belongs to $\mathfrak{F}(X^{n-2} \to V^{\otimes(n-2)})$. □

Intuitively, elements of the C-MCR algebra can be represented by operator-valued integrals

$$\int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), a^{\sharp_1}(x_1)\otimes\cdots\otimes a^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n) \tag{4.2}$$

where $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$ and $\sharp_1,\dots,\sharp_n \in \{+,-\}$. These operator-valued integrals are subject to the commutation relation of Corollary 2.94. More precisely, if $i \in \{1,\dots,n-1\}$ and $\sharp_i = \sharp_{i+1} = +$, then

$$\begin{aligned} &\int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), a^{\sharp_1}(x_1)\otimes\cdots\otimes a^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n) \\ &= \int_{X^n} \langle C_i(x_i,x_{i+1})f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n), a^{\sharp_1}(x_1)\otimes\cdots\otimes a^{\sharp_n}(x_n)\rangle_{V^{\otimes n}} \\ &\qquad\times\sigma(dx_1)\cdots\sigma(dx_n). \end{aligned}$$

Similarly, for $\sharp_i = \sharp_{i+1} = -$, we get

$$\begin{aligned} &\int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), a^{\sharp_1}(x_1)\otimes\cdots\otimes a^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n) \\ &= \int_{X^n} \langle \widehat{C}_i(x_{i+1},x_i)f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n), a^{\sharp_1}(x_1)\otimes\cdots\otimes a^{\sharp_n}(x_n)\rangle_{V^{\otimes n}} \\ &\qquad\times\sigma(dx_1)\cdots\sigma(dx_n). \end{aligned}$$

If $\sharp_i = -$, $\sharp = +$, then

$$\int_{X^n} \langle f^{(n)}(x_1, \dots, x_n), a^{\sharp_1}(x_1) \otimes \cdots \otimes a^{\sharp_n}(x_n) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n)$$
$$= \int_{X^n} \langle \widetilde{C}_i(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_n), a^{\sharp_1}(x_1) \otimes \cdots a^{\sharp_{i-1}}(x_{i-1}) \otimes a^{\sharp_{i+1}}(x_{i+1}) \otimes a^{\sharp_i}(x_i)$$
$$\otimes a^{\sharp_{i+2}}(x_{i+2}) \otimes \cdots \otimes a^{\sharp_n}(x_n) \rangle_{V^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n)$$
$$+ \int_{X^{n-2}} \Big\langle \int_X \mathrm{Tr_i}\, f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n) \sigma(dx),$$
$$a^{\sharp_1}(x_1) \otimes \cdots a^{\sharp_{i-1}}(x_{i-1}) \otimes \otimes a^{\sharp_{i+2}}(x_{i+2}) \otimes \cdots \otimes a^{\sharp_n}(x_n) \Big\rangle_{V^{\otimes n}}$$
$$\times \sigma(dx_1) \cdots \sigma(dx_{i-1}) \sigma(dx_{i+2}) \cdots \sigma(dx_n)$$
$$= \int_{X^n} \langle \widetilde{C}_i(x_{i+1}, x_i) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots x_n),$$
$$a^{\sharp_1}(x_1) \otimes \cdots \otimes a^{\sharp_{i+1}}(x_i) \otimes a^{\sharp_i}(x_{i+1}) \otimes \cdots \otimes a^{\sharp_n}(x_n) \rangle_{V^{\otimes (n-2)}} \, \sigma(dx_1) \cdots \sigma(dx_n)$$
$$+ \int_{X^n} \Big\langle \int_X \mathrm{Tr_i}\, f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \sigma(dx),$$
$$a^{\sharp_1}(x_1) \otimes \cdots \otimes a^{\sharp_{i-1}}(x_{i-1}) \otimes a^{\sharp_{i+2}}(x_i) \otimes \cdots \otimes a^{\sharp_n}(x_{n-2}) \rangle_{V^{\otimes (n-2)}}$$
$$\times \sigma(dx_1) \cdots \sigma(dx_{i-1}) \sigma(dx_{i+2}) \cdots \sigma(dx_n).$$

We also note that the adjoint of

$$\int_{X^n} \langle f^{(n)}(x_1, \dots, x_n), a^{\sharp_1}(x_1) \otimes \cdots \otimes a^{\sharp_n}(x_n) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n)$$

should be

$$\int_{X^n} \langle \mathbb{S}^{(n)} f^{(n)}(x_1, \dots, x_n), a^{-\sharp_n}(x_n) \otimes \cdots \otimes a^{-\sharp_1}(x_1) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n)$$
$$= \int_{X^n} \langle \mathbb{S}^{(n)} f^{(n)}(x_n, \dots, x_1) a^{-\sharp_n}(x_1) \otimes \cdots \otimes a^{-\sharp_1}(x_n) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n). \tag{4.3}$$

Here similarly to (2.84), we used the continuous antilinear operator $\mathbb{S}^{(n)} : V^{\otimes n} \to V^{\otimes n}$ that is defined by

$$\mathbb{S}^{(n)} v_1 \otimes \cdots \otimes v_n = (Jv_n) \otimes \cdots \otimes (Jv_1). \tag{4.4}$$

Defining the operator

$$\mathbb{G}^{(n)} : \mathfrak{F}(X^n \to V^{\otimes n}) \to \mathfrak{F}(X^n \to V^{\otimes n})$$

by

$$\left(\mathbb{G}^{(n)} f^{(n)}\right)(x_1, \dots, x_n) = \mathbb{S}^{(n)} f^{(n)}(x_n, \dots, x_1).$$

Therefore, formula (4.3) becomes

$$\int_{X^n} \langle (\mathbb{G}^{(n)} f^{(n)})(x_1, \dots, x_n) a^{-\sharp_n}(x_1) \otimes \cdots \otimes a^{-\sharp_1}(x_n) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n).$$

Similarly to the Q-ACR case, we will denote the integrals

$$\int_{X^n} \langle f^{(n)}(x_1, \dots, x_n), a^{\sharp_1}(x_1) \otimes \cdots \otimes a^{\sharp_n}(x_n) \rangle_{V^{\otimes n}} \, \sigma(dx_1) \cdots \sigma(dx_n)$$

by $\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right)$.

Definition 4.4. The C-MCR algebra is defined as the unital $*$-algebra that is generated by elements of the form $\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right)$ for $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$ and $\sharp_1, \dots, \sharp_n \in \{+, -\}$. These elements satisfy the following relations:

(i) If $i \in \{1, \dots, n-1\}$ and $\sharp_i = \sharp_{i+1} = +$, then

$$\begin{aligned} &\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right) \\ &\quad = \Phi^{(n)}\left(C_i(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n); \sharp_1, \dots, \sharp_n\right). \end{aligned} \tag{4.5}$$

(ii) If $i \in \{1, \dots, n-1\}$ and $\sharp_i = \sharp_{i+1} = -$, then

$$\begin{aligned} &\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right) \\ &\quad = \Phi^{(n)}\left(\widehat{C}_i(x_{i+1}, x_i) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n); \sharp_1, \dots, \sharp_n\right). \end{aligned} \tag{4.6}$$

(iii) If $i \in \{1, \dots, n-1\}$ and $\sharp_i = -, \sharp_{i+1} = +$, then

$$\begin{aligned} &\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right) \\ &= \Phi^{(n)}\left(\widetilde{C}_i(x_{i+1}, x_i) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n); \sharp_1, \dots, \sharp_{i+1}, \sharp_i, \dots, \sharp_n\right) \\ &+ \Phi^{(n-2)}\left(\int_X \operatorname{Tr}_i f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \, \sigma(dx); \sharp_1, \dots, \sharp_{i-1}, \sharp_{i+2}, \dots, \sharp_n\right). \end{aligned} \tag{4.7}$$

The multiplication in the C-MCR algebra is given by

$$\Phi^{(m)}\left(f^{(m)};\sharp_1,\dots,\sharp_m\right)\Phi^{(n)}\left(g^{(n)};\sharp_{m+1},\dots,\sharp_{m+n}\right)=\Phi^{(m+n)}\left(f^{(m)}\otimes g^{(n)};\sharp_1,\dots,\sharp_{m+n}\right),$$

where $f^{(m)}\in\mathfrak{F}(X^m\to V^{\otimes m})$, $g^{(n)}\in\mathfrak{F}(X^n\to V^{\otimes n})$ and

$$\left(f^{(m)}\otimes g^{(n)}\right)(x_1,\dots,x_{m+n})=f^{(m)}(x_1,\dots,x_m)\otimes g^{(n)}(x_{m+1},\dots,x_{m+n}). \tag{4.8}$$

(On the right-hand side of (4.8), the symbol $\otimes$ denotes the tensor product in $V^{\otimes(m+n)}$.)

The addition in the C-MCR algebra satisfies, for any $\lambda,\mu\in\mathbb{C}$ and $f^{(n)},g^{(n)}\in\mathfrak{F}(X^n\to V^{\otimes n})$,

$$\lambda\Phi^{(n)}(f^{(n)};\sharp_1,\dots,\sharp_n)+\mu\Phi^{(n)}(g^{(n)};\sharp_1,\dots,\sharp_n)=\Phi^{(n)}(\lambda f^{(n)}+\mu g^{(n)};\sharp_1,\dots,\sharp_n).$$

The $*$- operation in the C-MCR algebra is defined by

$$\Phi^{(n)}(f^{(n)};\sharp_1,\dots,\sharp_n)^*=\Phi^{(n)}\left(\mathbb{G}^{(n)}f^{(n)};-\sharp_n,\dots,-\sharp_1\right). \tag{4.9}$$

We define

$$P_n:\mathfrak{F}(X^n\to V^{\otimes n})\to\mathfrak{F}(X^n\to V^{\otimes n})$$

by formulas (2.4), (2.8), (2.10) with

$$\left(U_if^{(n)}\right)(x_1,\dots,x_n)=C_i(x_i,x_{i+1})f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n),$$

for all $(x_1,\dots,x_n)\in X^n$. It follows from our previous considerations that the image of P_n, i.e., the set $P_n\,\mathfrak{F}(X^n\to V^{\otimes n})$, consists of all functions $f^{(n)}\in\mathfrak{F}(X^n\to V^{\otimes n})$ that satisfy

$$f^{(n)}(x_1,\dots,x_n)=C_i(x_i,x_{i+1})f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n),$$

for all $(x_1,\dots,x_n)\in X^n$ and $i\in\{1,\dots,n-1\}$.

We define

$$\widehat{P}_n:\mathfrak{F}(X^n\to V^{\otimes n})\to\mathfrak{F}(X^n\to V^{\otimes n}),\quad \widehat{P}_n=\mathbb{G}^{(n)}P_n\mathbb{G}^{(n)}.$$

It is easy to see that the operator $\widehat{P}_n$ can be constructed similarly to the operator P_n, by starting with the operators $\widehat{U}_i$ instead of U_i. Here the operators $\widehat{U}_i$ are defined by

$$\left(\widehat{U}_if^{(n)}\right)(x_1,\dots,x_n)=\widehat{C}_i(x_{i+1},x_i)f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n).$$

The image of $\widehat{P}_n$, i.e., the set $\widehat{P}_n\,\mathfrak{F}(X^n \to V^{\otimes n})$, consists of all functions $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$ that satisfy

$$f^{(n)}(x_1,\dots,x_n) = \widehat{C}_i(x_{i+1},x_i)f^{(n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n)$$

for all $(x_1,\dots,x_n) \in X^n$ and $i \in \{1,\dots,n-1\}$.

We also define

$$\begin{aligned}\mathfrak{S}\mathfrak{F}^{(m,n)}\left(X^{m+n} \to V^{\otimes(m+n)}\right) &= P_m \otimes \widehat{P}_n\,\mathfrak{F}(X^{m+n} \to V^{\otimes(m+n)}) \\ &= \left\{f^{(m+n)} \in \mathfrak{F}(X^n \to V^{\otimes n}) \mid P_m \otimes \widehat{P}_n f^{(m+n)} = f^{(m+n)}\right\}.\end{aligned}$$

Thus, $\mathfrak{S}\mathfrak{F}^{(m,n)}\left(X^{m+n} \to V^{\otimes(m+n)}\right)$ consists of all functions $f^{(m+n)} \in \mathfrak{F}(X^{m+n} \to V^{\otimes(m+n)})$ that satisfy, for all $(x_1,\dots,x_{m+n}) \in X^{m+n}$,

$$f^{(m+n)}(x_1,\dots,x_{m+n}) = C_i(x_i,x_{i+1})f^{(m+n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_{m+n})$$

for all $i \in \{1,\dots,m-1\}$ and

$$f^{(m+n)}(x_1,\dots,x_{m+n}) = \widehat{C}_i(x_{i+1},x_i)f^{(m+n)}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_{m+n})$$

for all $i \in \{m+1,\dots,m+n-1\}$.

For $m,n \geq 0,\ m+n \geq 1,\ f^{(m+n)} \in \mathfrak{F}(X^{m+n} \to V^{\otimes(m+n)})$, we define

$$W^{(m,n)}(f^{(m+n)}) = \Phi^{(m+n)}\left(f^{(m+n)};\sharp_1,\dots,\sharp_{m+n}\right),$$

where $\sharp_1 = \dots, = \sharp_m = +,\ \sharp_{m+1} = \dots = \sharp_{m+n} = -.$ We call $W^{(m,n)}(f^{(m+n)})$ Wick-ordered elements of the C-MCR algebra.

By the commutation relations, we have

$$W^{(m,n)}(f^{(m+n)}) = W^{(m+n)}\left(P_m \otimes \widehat{P}_n f^{(m+n)}\right).$$

Thus, we have the following proposition.

Proposition 4.5. *Each element of the C-MCR algebra can be represented in the form*

$$\boldsymbol{c1} + \sum_{m,n\in\mathbb{N}_0,\ m+n\geq 1} W^{(m,n)}(f^{(m,n)}) \tag{4.10}$$

where $c \in \mathbb{C}$ and $f^{(m,n)} \in \mathfrak{S}\mathfrak{F}^{(m,n)}\left(X^{(m+n)} \to V^{\otimes(m+n)}\right)$. The sum in (4.10) is finite.

Proposition 4.6. *Assume that, for all $(x_1, x_2) \in X^2$, the operator $\widetilde{C}(x_1, x_2) : V^{\otimes 2} \to V^{\otimes 2}$ is invertible. Furthermore, assume that there exists a constant $\varkappa \in \mathbb{R}$ such that, for all $x \in X$ and $v^{(2)} \in V^{\otimes 2}$,*

$$\operatorname{Tr}\left(\widetilde{C}(x,x)^{-1} v^{(2)}\right) = \varkappa \operatorname{Tr} v^{(2)}. \tag{4.11}$$

Then, for all $f^{(n)} \in \mathfrak{F}\left(X^n \to V^{\otimes n}\right)$ and $i \in \{1, \dots, n-1\}$, if $\sharp_i = +$, $\sharp_{i+1} = -$, then

$$\begin{aligned}
&\Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right) \\
&= \Phi^{(n)}\left(\widetilde{C}_i(x_i, x_{i+1})^{-1} f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n); \sharp_1, \dots, \sharp_{i+1}, \sharp_i, \dots, \sharp_n\right) \\
&- \varkappa \Phi^{(n-2)}\left(\int_X \operatorname{Tr}_i f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \sigma(dx); \sharp_1, \dots, \sharp_{i-1}, \sharp_{i+2}, \dots, \sharp_n\right).
\end{aligned}$$

Proof. Choose

$$g^{(n)}(x_1, \dots, x_n) = \widetilde{C}_i(x_i, x_{i+1})^{-1} f^{(n)}(x_1, \dots, x_{i+1}, x_i, \dots, x_n).$$

Then, by formula (4.7),

$$\begin{aligned}
&\Phi^{(n)}(g^{(n)}(x_1, \dots, x_n); \sharp_1, \dots, \sharp_{i+1}, \sharp_i, \dots, \sharp_n) \\
&\quad = \Phi^{(n)}\left(\widetilde{C}_i(x_{i+1}, x_i) \widetilde{C}_i(x_{i+1}, x_i)^{-1} f^{(n)}(x_1, \dots, x_n); \sharp_1, \dots, \sharp_n\right) \\
&\qquad + \Phi^{(n-2)}\left(\int_X \operatorname{Tr}_i \widetilde{C}_i(x,x)^{-1} f^{(n)}(x_1 \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \sigma(dx); \sharp_1, \dots, \sharp_{i-1}, \sharp_{i+2}, \dots, \sharp_n\right) \\
&\quad = \Phi^{(n)}\left(f^{(n)}; \sharp_1, \dots, \sharp_n\right) + \varkappa \Phi^{(n-2)}\Bigg(\int_X \operatorname{Tr}_i f^{(n)}(x_1, \dots, x_{i-1}, x, x, x_i, \dots, x_{n-2}) \sigma(dx); \\
&\qquad \sharp_1, \dots, \sharp_{i-1}, \sharp_{i+2}, \dots, \sharp_n\Bigg).
\end{aligned}$$

□

Remark 4.7. Let $(e_i)_{i=1,\dots,d}$ be an orthonormal basis of V. Denote

$$b_{ijkl}(x) = \left(\widetilde{C}(x,x)^{-1} e_i \otimes e_j, e_k \otimes e_l\right)_{V^{\otimes 2}}.$$

Then, for $v^{(2)} \in V^{\otimes 2}$,

$$\begin{aligned}
\operatorname{Tr}\left(\widetilde{C}(x,x)^{-1} v^{(2)}\right) &= \sum_{k=1}^{d} \left(\widetilde{C}(x,x)^{-1} v^{(2)}, e_k \otimes e_k\right)_{V^{\otimes 2}} \\
&= \sum_{k=1}^{d} \left(\widetilde{C}(x,x)^{-1} \sum_{i,j=1,\dots,d} (v^{(2)}, e_i \otimes e_j)_{V^{\otimes 2}}\, e_i \otimes e_j, e_k \otimes e_k\right)_{V^{\otimes 2}}
\end{aligned}$$

$$= \sum_{i,j=1,\dots,d} (v^{(2)}, e_i \otimes e_j)_{V^{\otimes 2}} \sum_{k=1}^{d} \left(\widetilde{C}(x,x)^{-1} e_i \otimes e_j, e_k \otimes e_k \right)_{V^{\otimes 2}}.$$

Hence, the condition (4.11) can be written in the following equivalent form: For any $i, j \in \{1, \dots, d\}$, $i \neq j$,

$$\sum_{k=1}^{d} \left(\widetilde{C}(x,x)^{-1} e_i \otimes e_j, e_k \otimes e_k \right)_{V^{\otimes 2}} = 0, \tag{4.12}$$

and for any $i, j \in \{1, \dots, d\}$, $i = j$

$$\sum_{k=1}^{d} \left(\widetilde{C}(x,x)^{-1} e_i \otimes e_i, e_k \otimes e_k \right)_{V^{\otimes 2}} = \varkappa. \tag{4.13}$$

4.2 States on the C-MCR algebra

Let $\mathbb{A}$ be a C-MCR algebra and let $\tau : \mathbb{A} \to \mathbb{C}$ be a state on $\mathbb{A}$. For any $\sharp_1, \dots, \sharp_n \in \{+, -\}$, we define a linear functional

$$\tau^{(n)}_{\sharp_1,\dots,\sharp_n} : C_0 \left(Y^n \to V^{\otimes n} \right) \times \mathcal{G}^n \to \mathbb{C}$$

by the formula (3.15) in which $\varphi^{(n)} \in C_0 \left(Y^n \to V^{\otimes n} \right)$. Obviously these linear functionals uniquely identify the state τ.

We equip $C_0 \left(Y^n \to V^{\otimes n} \right)$ with the topology of the uniform convergence on compact sets in Y^n. We will assume that each functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0 \left(Y^n \to V^{\otimes n} \right) \times \mathcal{G}^n$. Then, similarly to Proposition 3.16, we conclude that, for any $g_1, \dots, g_n \in \mathcal{G}$, there exist $V^{\otimes n}$-valued measures $m^{(n)}_{\sharp_1,\dots,\sharp_n}[g_1, \dots, g_n]$ on Y^n such that, for all $\varphi^{(n)} \in C_0 \left(Y^n \to V^{\otimes n} \right)$,

$$\tau^{(n)}_{\sharp_1,\dots,\sharp_n} \left(\varphi^{(n)}, g_1, \dots, g_n \right) = \int_{Y^n} \left\langle \varphi^{(n)}(y_1, \dots, y_n), m^{(n)}_{\sharp_1,\dots,\sharp_n}[g_1, \dots, g_n](dy_1 \cdots dy_n) \right\rangle_{V^{\otimes n}}.$$

For $v \in V$, denote

$$\mathrm{Re}(v) = \frac{v + Jv}{2}, \quad \mathrm{Im}(v) = \frac{v - Jv}{2i},$$

so that

$$v = \mathrm{Re}(v) + i\,\mathrm{Im}(v).$$

Denote

$$V_{\mathbb{R}} = \{v \in V \mid \mathrm{Im}(v) = 0\} = \{v \in V \mid v = Jv\}.$$

Note that, for each $v \in V$, $\mathrm{Re}(v)$, $\mathrm{Im}(v) \in V_{\mathbb{R}}$, hence

$$v = v_1 + iv_2, \quad \text{where } v_1, v_2 \in V_{\mathbb{R}}. \tag{4.14}$$

As easily seen the representation (4.14) of $v \in V$ is unique.

For $f \in \mathfrak{F}(X \to V)$ and $\sharp \in \{+,-\}$, we denote $A^{\sharp}(f) = \Phi^{(1)}(f;\sharp)$. Similarly to Proposition 3.11, we prove the following

Proposition 4.8. *Assume that, for any $\sharp_1, \dots, \sharp_n \in \{+,-\}$, the functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n \to V^{\otimes n}) \times \mathcal{G}^n$. Then the following statements hold:*

(i) The state τ is completely determined by its values on $A^{\sharp_1}(f_1)\cdots A^{\sharp_{m+n}}(f_{m+n})$, where $f_k = \varphi_k \otimes g_k$ with $\varphi_k \in C_0(Y \to V)$ and $g_k \in \mathcal{G}$ $(k = 1, \dots, m+n)$, and $\sharp_1 = \cdots = \sharp_m = +$, $\sharp_{m+1} = \cdots = \sharp_{m+n} = -$, where $m, n \geq 0$, $m + n \geq 1$.

(ii) Define

$$B(f) = A^+(f) + A^-(Jf), \quad f \in \mathfrak{F}(X \to V).$$

Then the state τ is completely determined by its values on $B(f_1)\cdots B(f_n)$, where $f_k = \varphi_k \otimes g_k$ with $\varphi_k \in C_0(Y \to V_{\mathbb{R}})$ and $g_k \in \mathcal{G}$ $(k = 1, \dots, n)$.

Similarly to Lemma 3.12, we have:

Lemma 4.9. *Let a state τ satisfy the assumption of Proposition 4.8.*

(i) For any $g_1, g_2 \in \mathcal{G}$, there exists a $V^{\otimes 2}$-valued Radon measure $\gamma^{(2)}[g_1, g_2]$ on Y^2 that satisfies

$$\tau\left(A^+(\varphi_1 \otimes g_1) A^-(\varphi_2 \otimes g_2)\right) = \int_{Y^2} \left\langle \varphi_1(y_1) \otimes \varphi_2(y_2), \gamma^{(2)}[g_1, g_2](dy_1\, dy_2)\right\rangle_{V^{\otimes 2}}, \tag{4.15}$$

for all $\varphi_1, \varphi_2 \in C_0(Y \to V)$.

(ii) For any $g_1, g_2 \in \mathcal{G}$, there exists a $V^{\otimes 2}$-valued measure $\alpha^{(2)}[g_1, g_2]$ on Y^2 that satisfies

$$\tau\big(B(\varphi_1 \otimes g_1) B(\varphi_2 \otimes g_2)\big) = \int_{Y^2} \left\langle \varphi_1(y_1) \otimes \varphi_2(y_2), \alpha^{(2)}[g_1, g_2](dy_1 dy_2)\right\rangle_{V^{\otimes 2}}$$

for all $\varphi_1, \varphi_2 \in C_0(Y \to V_{\mathbb{R}})$.

4.3 The Fock state on the C-MCR algebra

We consider the U-deformed Fock space $\mathcal{F}(\mathcal{H})$ with $\left(Uf^{(2)}\right)(x,y)=C(x,y)f^{(2)}(y,x)$.

For $f\in\mathfrak{F}(X\to V)$, we define

$$A^+(f)=a^+(f),\quad A^-(f)=a^-(f),$$

where $a^+(f)$ and $a^-(f)$ are the creation and annihilation operators in $\mathcal{F}(\mathcal{H})$. In particular, $a^+(f)$ and $a^-(f)$ act continuously on $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$.

Note that the linear span of functions of the form $\varphi_1(y_1)\otimes\varphi_2(y_2)\otimes\cdots\otimes\varphi_n(y_n)$ with $\varphi_1,\dots,\varphi_n\in C_0(Y\to V)$ is dense in $C_0\left(Y^n\to V^{\otimes n}\right)$. Then, similarly to Section 3.4, we will now define

$$\Phi^{(n)}(\varphi^{(n)}(y_1,\dots,y_n)g_1(z_1)g_2(z_2)\cdots g_n(z_n))\in\mathcal{L}\left(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})\right)$$

for $\varphi^{(n)}\in C_0\left(Y^n\to V^{\otimes n}\right)$ and $g_1,\dots,g_n\in\mathcal{G}$.

For $\varphi_1,\dots,\varphi_{m+n}\in C_0(Y\to V)$, $g_1,\dots,g_{m+n}\in\mathcal{G}$ and $u^{(k)}\in\mathcal{F}_k(\mathcal{H})$, we have, with $l=k-n+m$,

$$\begin{aligned}&\left(a^+(\varphi_1\otimes g_1)\cdots a^+(\varphi_m\otimes g_m)a^-(\varphi_{m+1}\otimes g_{m+1})\cdots a^-(\varphi_{m+n}\otimes g_{m+n})u^{(k)}\right)(x_1,\dots,x_l)\\&\quad=P_l\Bigg[(\varphi_1\otimes g_1)(x_1)\otimes\cdots\otimes(\varphi_m\otimes g_m)(x_m)\\&\qquad\times\int_{X^n}\sigma^{\otimes n}(dx_1'\cdots dx_n')\big\langle(\varphi_{m+n}\otimes g_{m+n})(x_n')\otimes\cdots\otimes(\varphi_{m+1}\otimes g_{m+1})(x_1'),\\&\qquad\quad u^{(k)}(x_n',\dots,x_1',x_{m+1},\dots,x_{m+n})\big\rangle_{V^{\otimes n}}\Bigg].\end{aligned}\tag{4.16}$$

Here the pairing $\langle\cdot,\cdot\rangle_{V^{\otimes n}}$ is taken in the first n 'variables' of $V^{\otimes k}$, i.e., for any $v_1,\dots,v_n$, $v_1',\dots,v_k'\in V$,

$$\langle v_1\otimes\cdots\otimes v_n,v_1'\otimes\cdots\otimes v_k'\rangle_{V^{\otimes n}}=\langle v_1,v_1'\rangle_V\cdots\langle v_n,v_n'\rangle_V\,v_{n+1}'\otimes\cdots\otimes v_k'.$$

We define a continuous linear functional $\mathbb{T}^{(2n)}:V^{\otimes(2n)}\to\mathbb{C}$ by

$$\mathbb{T}^{(2n)}\,v_1\otimes\cdots\otimes v_{2n}=\langle v_n,v_{n+1}\rangle_V\langle v_{n-1},v_{n+2}\rangle_V\cdots\langle v_1,v_{2n}\rangle_V.\tag{4.17}$$

Then we can write formula (4.16) as follows:

$$\left(a^+(\varphi_1\otimes g_1)\cdots a^+(\varphi_m\otimes g_m)a^-(\varphi_{m+1}\otimes g_{m+1})\cdots a^-(\varphi_{m+n}\otimes g_{m+n})u^{(k)}\right)(x_1,\dots,x_l)$$

$$= P_l\Big[\int_{X^n}\sigma^{\otimes n}(dx_1'\cdots dx_n')(\mathbf{1}_m\otimes\mathbb{T}^{(2n)}\otimes\mathbf{1}_{k-n})\big((\varphi_1\otimes g_1)\otimes$$

$$\cdots\otimes(\varphi_{m+n}\otimes g_{m+n})\big)(x_1,\dots,x_m,x_1',x_2',\dots,x_n')\otimes u^{(k)}(x_n',\dots,x_1',x_{m+1},\dots,x_l)\Big].$$

Hence, similarly to (3.23), we get, for $f^{(m+n)}\in\mathfrak{F}\left(X^{m+n}\to V^{(m+n)}\right)$ and $u^{(k)}\in\mathcal{F}_k(\mathcal{H})$,

$$\left(W^{(m,n)}\left(f^{(m+n)}\right)u^{(k)}\right)(x_1,\dots,x_l)=P_l\Big[\int_{X^n}\sigma^{\otimes n}(dx_1'\cdots dx_n')\left(\mathbf{1}_m\otimes\mathbb{T}^{(2n)}\otimes\mathbf{1}_{k-n}\right)$$
$$f^{(m+n)}(x_1,\dots,x_m,x_1',\dots,x_n')\otimes u^{(k)}(x_n',\dots,x_1',x_{m+1},\dots,x_l)\Big]. \tag{4.18}$$

Thus we get a representation of the C-MCR algebra. Similarly to Definition 3.13, we define the Fock state τ on the C-MCR algebra as in (3.24).

Similarly to Lemma 3.14, we conclude that each corresponding functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$: $C_0\left(Y^n\to V^{\otimes n}\right)\times\mathcal{G}^n\to\mathbb{C}$ is continuous. Furthermore, for the Fock state τ, the measure $\gamma^{(2)}[g_1,g_2]$ from Lemma 4.9 (i) is equal to zero, while the measure $\alpha^{(2)}[g_1,g_2]$ from Lemma 4.9 (ii) is given by

$$\alpha^{(2)}[g_1,g_2](dy_1\,dy_2)=(g_2,g_1)_{\mathcal{G}}\,\nu^{(2)}(dy_1\,dy_2)\sum_{i,j=1}^{d}\delta_{ij}\,e_i\otimes e_j$$
$$=\lambda^{(2)}[g_1,g_2](dy_1\,dy_2)\sum_{i,j=1}^{d}\delta_{ij}\,e_i\otimes e_j.$$

Here $\nu^{(2)}(dy_1\,dy_2)$ is the $\mathbb{C}$-valued measure on Y^2 defined by (3.27), the the $\mathbb{C}$-valued measure $\lambda^{(2)}[g_1,g_2](dy_1\,dy_2)$ on Y^2 is defined by (3.26), $(e_i)_{i=1,\dots,d}$ is an arbitrary orthonormal basis of V such that $Je_i=e_i$ and δ_{ij} is the Kronecker delta.

For each $f\in\mathcal{F}(X\to V)$ we define continuous linear operators $\mathcal{A}^+(f)$, $\mathcal{A}^-(f)$ on $\mathbb{F}_{\mathrm{fin}}(\mathcal{H})$ as follows: $\mathcal{A}^+(f)=a^+_{\mathrm{free}}(f)$ and $\mathcal{A}^-(f)$ acts as follows: for any $u^{(n)}\in\mathcal{H}^{\otimes n}$,

$$\left(\mathcal{A}^-(f)u^{(n)}\right)(x_1,\dots,x_{n-1})=\int_X\sigma(dx')\langle f(x'),u^{(n)}(x',x_1,\dots,x_{n-1})$$
$$+\sum_{k=1}^{n-1}C_1(x',x_1)C_2(x',x_2)\cdots C_k(x',x_k)u^{(n)}(x_1,\dots,x_k,x',x_{k+1},\dots,x_{n-1}\rangle_V.$$

Our next aim is to generalize Proposition 3.17 to the case of a multicomponent system. We first need some preparation. Let $n\in\mathbb{N}$ and fix a partition $\xi\in\mathcal{P}_2^{(2n)}$. All the definitions below depend on the choice of ξ, nevertheless for simplicity we will not show this dependence in our notation.

Let

$$\xi = \big\{\{i_1, j_1\}, \dots, \{i_n, j_n\}\big\} \tag{4.19}$$

with

$$i_1 < j_1,\ i_2 < j_2, \dots,\ i_n < j_n \quad \text{and} \quad i_1 > i_2 > \dots > i_n = 1. \tag{4.20}$$

We define the sets

$$I = \{i_1, i_2, \dots, i_n\}, \quad J = \{j_1, j_2, \dots, j_n\}.$$

For $k = 1, 2, \dots, n$, we define the sets $J^{(k)}$ and $\mathbf{J}^{(k)}$ as follows. Let

$$J^{(1)} = \{j \in J \mid i_1 < j \le j_1\}, \quad \mathbf{J}^{(1)} = \{j \in J \mid i_1 < j\},$$

and for $k = 2, \dots, n$,

$$J^{(k)} = \big\{j \in J \mid i_k < j \le j_k,\ j \ne j_1,\ j \ne j_2, \dots,\ j \ne j_{k-1}\big\}, \tag{4.21}$$

$$\mathbf{J}^{(k)} = \big\{j \in J \mid i_k < j,\ j \ne \min \mathbf{J}^{(1)},\ j \ne \min \mathbf{J}^{(2)}, \dots,\ j \ne \min \mathbf{J}^{(k-1)}\big\}. \tag{4.22}$$

We write

$$J^{(k)} = \left\{j_1^{(k)}, j_2^{(k)}, \dots, j_{l_k}^{(k)}\right\}, \quad \mathbf{J}^{(k)} = \left\{\mathbf{j}_1^{(k)}, \mathbf{j}_2^{(k)}, \dots, \mathbf{j}_{m_k}^{(k)}\right\},$$

with

$$j_1^{(k)} < j_2^{(k)} < \dots < j_{l_k}^{(k)}, \quad \mathbf{j}_1^{(k)} < \mathbf{j}_2^{(k)} < \dots < \mathbf{j}_{m_k}^{(k)}.$$

Remark 4.10. Note that $j_{l_k}^{(k)} = j_k$ and $m_k \ge l_k$ for all k.

Remark 4.11. Note that

$$\xi' = \left\{\{i_1, \mathbf{j}_1^{(1)}\}, \{i_2, \mathbf{j}_1^{(2)}\}, \dots, \{i_n, \mathbf{j}_1^{(n)}\}\right\} \tag{4.23}$$

belongs to $\mathcal{P}_2^{(2n)}$. As easily seen, the partition ξ' has no crossings, i.e., there are no pairs $\{i_k, \mathbf{j}_1^{(k)}\}, \{i_m, \mathbf{j}_1^{(m)}\}$ in ξ' such that

$$i_k < i_m < \mathbf{j}_1^{(k)} < \mathbf{j}_1^{(m)}. \tag{4.24}$$

Indeed, assume that (4.24) holds for some k and m. Since $i_k < i_m$, by (4.20), we have $k > m$. Hence, by (4.22),

$$\mathbf{j}_1^{(k)} \ne \min \mathbf{J}^{(1)},\ \mathbf{j}_1^{(k)} \ne \min \mathbf{J}^{(2)}, \dots, \mathbf{j}_1^{(k)} \ne \min \mathbf{J}^{(m-1)}.$$

Since we also have $\mathbf{j}_1^{(k)} > i_m$, this implies $j_1^{(k)} \in \mathbf{J}^{(m)}$. But $\mathbf{j}_1^{(m)} = \min \mathbf{J}^{(m)}$, hence we must have $\mathbf{j}_1^{(m)} \le \mathbf{j}_1^{(k)}$, which is a contradiction.

Recall that, for a linear operator $A : V^{\otimes 2} \to V^{\otimes 2}$ and $i \in \{1, 2, \dots, m-1\}$ we defined a linear operator $A_i : V^{\otimes m} \to V^{\otimes m}$ by

$$A_i = \mathbf{1}_{V^{\otimes(i-1)}} \otimes A \otimes \mathbf{1}_{V^{\otimes(m-i-1)}},$$

i.e., A_i acts by the operator A on the i^{th} and $(i+1)^{\text{th}}$ 'variables' of the tensor product $V^{\otimes m}$. Now similarly, for $1 \le i < j \le m$, we define a linear operator $A[i,j] : V^{\otimes m} \to V^{\otimes m}$ that acts as the operator A on the i^{th} and j^{th} 'variables' of $V^{\otimes m}$.

Let us now fix an arbitrary $(x_1, x_2, \dots, x_{2n}) \in X^{2n}$. For $k = 1, 2, \dots, n$, we define a linear operator $C^{(k)}(\xi; x_1, x_2, \dots, x_{2n}) : V^{\otimes 2n} \to V^{\otimes 2n}$ as follows. If $J^{(k)} = \{j_k\}$, i.e., $l_k = 1$, then $C^{(k)}(\xi; x_1, x_2, \dots, x_{2n})$ is the identity operator. If $l_k > 1$, then we define

$$\begin{aligned} &C^{(k)}(\xi; x_1, x_2, \dots, x_{2n}) \\ &\quad = C(x_{i_k}, x_{j_1^{(k)}})\big[\mathbf{j}_1^{(k)}, \mathbf{j}_2^{(k)}\big]\, C(x_{i_k}, x_{j_2^{(k)}})\big[\mathbf{j}_2^{(k)}, \mathbf{j}_3^{(k)}\big] \cdots C(x_{i_k}, x_{j_{l_k-1}^{(k)}})\big[\mathbf{j}_{l_k-1}^{(k)}, \mathbf{j}_{l_k}^{(k)}\big]. \end{aligned} \tag{4.25}$$

Next we define a linear operator

$$\begin{aligned} &C(\xi; x_1, x_2, \dots, x_{2n}) \\ &\quad = C^{(k)}(\xi; x_1, x_2, \dots, x_{2n}) C^{(k-1)}(\xi; x_1, x_2, \dots, x_{2n}) \cdots C^{(1)}(\xi; x_1, x_2, \dots, x_{2n}). \end{aligned} \tag{4.26}$$

Recall Remark 4.11. similarly to (4.17), we define a linear functional

$$\mathbb{T}^{(2n)}(\xi) : V^{\otimes 2n} \to \mathbb{C}$$

by

$$\mathbb{T}^{(2n)}(\xi) v_1 \otimes v_2 \otimes \cdots \otimes v_{2n} = \langle v_{i_1}, v_{\mathbf{j}_1^{(1)}} \rangle_V \langle v_{i_2}, v_{\mathbf{j}_1^{(2)}} \rangle_V \cdots \langle v_{i_n}, v_{\mathbf{j}_1^{(n)}} \rangle_V. \tag{4.27}$$

Theorem 4.12. *Let τ be the Fock state on the C-MCR algebra $\mathbb{A}$. Let $n \in \mathbb{N}$ and $f_1, g_1 \dots, f_{2n}, g_{2n} \in \mathcal{F}(X \to V)$. Then*

$$\tau\big((A^+(f_1) + A^-(g_1)) \cdots (A^+(f_{2n-1}) + A^-(g_{2n-1}))\big) = 0 \tag{4.28}$$

and

$$\tau\big((A^+(f_1) + A^-(g_1)) \cdots (A^+(f_{2n}) + A^-(g_{2n}))\big)$$

$$= \sum_{\xi \in \mathcal{P}_2^{(2n)}} \int_{X^{2n}} \bigotimes_{\substack{\{i,j\}\in\xi \\ i<j}} \sigma^{(2)}(dx_i\, dx_j)$$

$$\times\, \mathbb{T}^{(2n)}(\xi) C(\xi; x_1, \dots, x_{2n}) h_1(x_1) \otimes h_2(x_2) \otimes \cdots \otimes h_{2n}(x_{2n}), \tag{4.29}$$

where

$$h_k(x) = \begin{cases} g_k(x), & \text{if } k \in I, \\ f_k(x), & \text{if } k \in J. \end{cases}$$

Proof. For each $f \in \mathcal{F}(X \to V)$, we define continuous linear operators $\mathcal{A}^+(f)$, $\mathcal{A}^-(f)$ on $\mathbb{F}_{\text{fin}}(\mathcal{H})$ as follows: $\mathcal{A}^+(f) = a^+_{\text{free}}(f)$ and for any $u^{(n)} \in \mathcal{H}^{\otimes n}$,

$$\begin{aligned}
&\left(\mathcal{A}^-(f) u^{(n)}\right)(x_1, \dots, x_{n-1}) \\
&\quad = \int_X \sigma(dx') \Big\langle f(x'), u^{(n)}(x', x_1, \dots, x_{n-1}) \\
&\qquad + \sum_{k=1}^{n-1} C_1(x', x_1) C_2(x', x_2) \cdots C_k(x', x_k) u^{(n)}(x_1, \dots, x_k, x', x_{k+1}, \dots, x_{n-1}) \Big\rangle_V \\
&\quad = \int_X \sigma(dx') \Big[\operatorname{Tr}_1 f(x') \otimes u^{(n)}(x', x_1, \dots, x_{n-1}) \\
&\qquad + \sum_{k=1}^{n-1} \operatorname{Tr}_1 f(x') \otimes C_1(x', x_1) C_2(x', x_2) \cdots C_k(x', x_k) u^{(n)}(x_1, \dots, x_k, x', x_{k+1}, \dots, x_{n-1}) \Big] \\
&\quad = \int_X \sigma(dx') \Big[\operatorname{Tr}_1 f(x') \otimes u^{(n)}(x', x_1, \dots, x_{n-1}) \\
&\qquad + \sum_{k=1}^{n-1} \operatorname{Tr}_1 C_2(x', x_1) C_3(x', x_2) \cdots C_{k+1}(x', x_k) f(x') \otimes u^{(n)}(x_1, \dots, x_k, x', x_{k+1}, \dots, x_{n-1}) \Big].
\end{aligned} \tag{4.30}$$

Lemma 4.13. *For any $f_1, \dots, f_n \in \mathcal{F}(X \to V)$ and $\sharp_1, \dots, \sharp_n \in \{+, -\}$ we have*

$$\left(A^{\sharp_1}(f_1) \cdots A^{\sharp_n}(f_n)\Omega, \Omega\right)_{\mathcal{F}(\mathcal{H})} = \left(\mathcal{A}^{\sharp_1}(f_1) \cdots \mathcal{A}^{\sharp_n}(f_n)\Omega, \Omega\right)_{\mathbb{F}(\mathcal{H})}.$$

Proof. The proof is similar to that of Lemma 3.19. We only need to note that, for each $u^{(n)} \in \mathcal{H}^{\otimes n}$,

$$\begin{aligned}
&\left(U_1 U_2 \cdots U_k u^{(n)}\right)(x', x_1, \dots, x_{n-1}) \\
&\quad = C_1(x', x_1) C_2(x', x_2) \cdots C_k(x', x_k) u^{(n)}(x_1, \dots, x_k, x', x_{k+1}, \dots, x_{n-1}).
\end{aligned}$$

□

Formula (4.28) trivially holds, so we only need to prove (4.29). By Lemma 4.13,

$$\begin{aligned}
&\tau\big((A^+(f_1)+A^-(g_1))\cdots(A^+(f_{2n})+A^-(g_{2n}))\big)\\
&\quad=\big((A^+(f_1)+A^-(g_1))\cdots(A^+(f_{2n})+A^-(g_{2n}))\Omega,\Omega\big)_{\mathcal{F}(\mathcal{H})}\\
&\quad=\big((\mathcal{A}^+(f_1)+\mathcal{A}^-(g_1))\cdots(\mathcal{A}^+(f_{2n})+\mathcal{A}^-(g_{2n}))\Omega,\Omega\big)_{\mathcal{F}(\mathcal{H})}. \qquad (4.31)
\end{aligned}$$

For each $f\in\mathcal{F}(X\to V)$ and $k\in\mathbb{N}$, we define the annihilation operator $\mathcal{A}_k^-(f)$ as follows: if $u^{(n)}\in\mathbb{F}_n(\mathcal{H})$ and $n<k$, then $\mathcal{A}_k^-(f)u^{(n)}=0$, and if $n\geq k$, then

$$\begin{aligned}
&\left(\mathcal{A}_k^-(f)u^{(n)}\right)(x_1,\dots,x_{n-1})\\
&\quad=\int_X\sigma(dx')\operatorname{Tr}_1 C_2(x',x_1)C_3(x',x_2)\cdots C_k(x',x_{k-1})\\
&\qquad f(x')\otimes u^{(n)}(x_1,\dots,x_{k-1},x',x_k,\dots,x_{n-1}). \qquad (4.32)
\end{aligned}$$

Here, for $k=1$, the operator $C_2(x',x_1)C_3(x',x_2)\cdots C_k(x',x_{k-1})$ is supposed to be the identity operator. Hence, by (4.30),

$$\mathcal{A}^-(f)=\sum_{k\geq1}\mathcal{A}_k^-(f),$$

or equivalently, for each $u^{(n)}\in\mathbb{F}_n(\mathcal{H})$,

$$\mathcal{A}^-(f)u^{(n)}=\sum_{k=1}^n\mathcal{A}_k^-(f)u^{(n)}. \qquad (4.33)$$

Remark 4.14. We may equivalently write formula (4.32) as follows:

$$\begin{aligned}
&\left(\mathcal{A}_k^-(f)u^{(n)}\right)(x_1,\dots,\check{x}_k,\dots,x_n)\\
&\quad=\int_{X^2}\sigma^{(2)}(dx'\,dx_k)\operatorname{Tr}_1 C_2(x',x_1)C_3(x',x_2)\cdots C_k(x',x_{k-1})\\
&\qquad f(x')\otimes u^{(n)}(x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n). \qquad (4.34)
\end{aligned}$$

We will say that the operator $\mathcal{A}_k^-(f)$ annihilates the x_k variable.

By (4.31) and (4.33), we get

$$\begin{aligned}
&\tau\big((A^+(f_1)+A^-(g_1))\cdots(A^+(f_{2n})+A^-(g_{2n}))\big)\\
&\quad=\sum_{\xi\in\mathcal{P}_2^{(2n)}}\mathcal{A}_{l_n}^-(h_{i_n})\mathcal{A}^+(h_{i_n-1})\cdots\mathcal{A}^+(h_{i_{n-1}+1})
\end{aligned}$$

$$\mathcal{A}^-_{l_{n-1}}(h_{i_{n-1}})\mathcal{A}^+(h_{i_{n-1}+1})\cdots\mathcal{A}^+(h_{i_{n-2}+1})\cdots\mathcal{A}^-_{l_1}(h_{i_1})\mathcal{A}(h_{i_1+1})\cdots\mathcal{A}^+(h_{2n})\Omega. \quad (4.35)$$

In words, we interpret the partition (4.19) (see also (4.20)) so that the creation operators are at places $j_1,\dots,j_n$, annihilation operators are at places $i_1,\dots,i_n$, and the annihilation operator at place i_k annihilates the variable created by the creation operator at place j_k.

For a fixed $\xi\in\mathcal{P}_2^{(2n)}$, let us now calculate the value of the expression in the sum appearing in (4.35).

To this end, we introduce the following notations. Assume $\mathbf{R}$ is a subset of $\{1,\dots,2n\}$ and for some $k\in\{1,\dots,2n\}$ let $\mathbf{S}=\{k,k+1,\dots,2n\}\setminus\mathbf{R}$. We write

$$\mathbf{R}=\{\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_l\}.$$
$$\mathbf{S}=\{\mathbf{s}_1,\mathbf{s}_2,\dots,\mathbf{s}_m\},$$

with $\mathbf{s}_1<\mathbf{s}_2<\cdots<\mathbf{s}_m$. Then, for a function $\varphi^{(m)}:X^m\to\mathbb{C}$, we will use the notation

$$\varphi^{(m)}(x_k,x_{k+1},\dots,x_{2n}\setminus x_{\mathbf{r}_1},x_{\mathbf{r}_2},\dots,x_{\mathbf{r}_l}):=\varphi^{(m)}(x_{\mathbf{s}_1},x_{\mathbf{s}_2},\dots,x_{\mathbf{s}_m}).$$

Next let $\zeta_1,\zeta_2,\dots,\zeta_{2l}$ be different numbers from the set $\{k,k+1,\dots,2n\}$. Let

$$\{k,k+1,\dots,2n\}\setminus\{\zeta_1,\zeta_2,\dots,\zeta_{2l}\}=\{\gamma_1,\gamma_2,\dots,\gamma_{2n-k+1-2l}\}$$

with $\gamma_1<\gamma_2<\cdots<\gamma_{2n-k+1-2l}$. We define a functional

$$\mathbb{T}^{(2n-k+1)}\left(k;\,\zeta_1,\zeta_2\mid\zeta_3,\zeta_4\mid\cdots\mid\zeta_{2l-1},\zeta_{2l}\right):V^{\otimes(2n-k+1)}\to V^{\otimes(2n-k+1-2l)}$$

by

$$\begin{aligned}&\mathbb{T}^{(2n-k+1)}\left(k;\,\zeta_1,\zeta_2\mid\zeta_3,\zeta_4\mid\cdots\mid\zeta_{2l-1},\zeta_{2l}\right)v_k\otimes v_{k+1}\otimes\cdots\otimes v_{2n}\\&\quad=\langle v_{\zeta_1},v_{\zeta_2}\rangle\langle v_{\zeta_3},v_{\zeta_4}\rangle\cdots\langle v_{\zeta_{2l-1}},v_{\zeta_{2l}}\rangle\,v_{\gamma_1}\otimes v_{\gamma_2}\otimes\cdots\otimes v_{2n-k+1-2l}.\end{aligned}$$

Recall that, for a linear operator $A:V^{\otimes2}\to V^{\otimes2}$ and $1\le i<j\le m$, we defined a linear operator $A[i,j]:V^{\otimes m}\to V^{\otimes m}$. Now, for $k\ge1$ and $k\le i<j\le k+m-1$, we define an operator $A[k;\,i,j]:V^{\otimes m}\to V^{\otimes m}$ by

$$A[k;\,i,j]:=A[i-k+1,j-k+1].$$

This definition can be interpreted as follows. We enumerate the 'variables' of $V^{\otimes m}$ as $k, k+1, \dots, k+m-1$ and the operator $A[k;\, i, j]$ acts as the operator A on the variables i and j. In particular, $A_1[i,j] = A[i,j]$.

We have

$$\left(\mathcal{A}^+(h_{i_1+1}) \cdots \mathcal{A}^+(h_{2n})\Omega\right)(x_{i_1+1}, \dots, x_{2n}) = h_{i_1+1}(x_{i_1+1}) \otimes \cdots \otimes h_{2n}(x_{2n})$$

and by (4.34),

$$\begin{aligned}
&\left(\mathcal{A}^-_{l_1}(h_{i_1})\mathcal{A}^+(h_{i_1+1})\mathcal{A}^+(h_{2n})\Omega\right)(x_{i_1}, \dots, x_{i_{2n}} \setminus x_{i_1}, x_{j_1}) \\
&\quad = \int_{X^2} \sigma^{(2)}(dx_{i_1}\, dx_{j_1})\, \mathrm{Tr}_1\, C_2(x_{i_1}, x_{j_1^{(1)}}) C_3(x_{i_1}, x_{j_2^{(1)}}) \\
&\qquad \cdots C_{l_1}(x_{i_1}, x_{j^{(1)}_{l_1-1}}) h_{i_1}(x_{i_1}) \otimes h_{i_2}(x_{i_2}) \otimes \cdots \otimes h_{2n}(x_{2n}) \\
&\quad = \int_{X^2} \sigma^{(2)}(dx_{i_1}\, dx_{j_1})\, \mathbb{T}^{(2n-i_1+1)}\left(i_1;\, i_1, \mathbf{j}^{(1)}_1\right) \\
&\qquad C(x_{i_1}, x_{j_1^{(1)}})\left[i_1;\, \mathbf{j}^{(1)}_1, \mathbf{j}^{(1)}_2\right] C(x_{i_1}, x_{j_2^{(1)}})\left[i_1;\, \mathbf{j}^{(1)}_2, \mathbf{j}^{(1)}_3\right] \cdots C(x_{i_1}, x_{j^{(1)}_{l_1-1}})\left[i_1;\, \mathbf{j}^{(1)}_{l_1-1}, \mathbf{j}^{(1)}_{l_1}\right] \\
&\qquad h_{i_1}(x_{i_1}) \otimes h_{i_2}(x_{i_2}) \otimes \cdots \otimes h_{2n}(x_{2n}).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\left(\mathcal{A}^+(h_{i_2+1}) \cdots \mathcal{A}^+(h_{i_1-1})\mathcal{A}^-_{l_1}(h_{i_1})\mathcal{A}^+(h_{i_1+1}) \cdots \mathcal{A}^+(h_{2n})\Omega\right)(x_{i_2+1}, x_{i_2+2} \dots, x_{2n} \setminus x_{i_1}, x_{j_1}) \\
&\quad = \int_{X^2} \sigma^{(2)}(dx_{i_1}\, dx_{j_1}) h_{i_2+1}(x_{i_2+1}) \otimes \cdots \otimes h_{i_1-1}(x_{i_1-1}) \otimes \mathbb{T}^{(2n-i_1+1)}\left(i_1;\, i_1, \mathbf{j}^{(1)}_1\right) \\
&\qquad C(x_{i_1}, x_{j_1^{(1)}})\left[i_1;\, \mathbf{j}^{(1)}_1, \mathbf{j}^{(1)}_2\right] C(x_{i_1}, x_{j_2^{(1)}})\left[i_1;\, \mathbf{j}^{(1)}_2, \mathbf{j}^{(1)}_3\right] \cdots C(x_{i_1}, x_{j^{(1)}_{l_1-1}})\left[i_1;\, \mathbf{j}^{(1)}_{l_1-1}, \mathbf{j}^{(1)}_{l_1}\right] \\
&\qquad h_{i_1}(x_{i_1}) \otimes g_{i_2}(x_{i_2}) \otimes \cdots \otimes g_{2n}(x_{2n}) \\
&\quad = \int_{X^2} \sigma^{(2)}(dx_{i_1}\, dx_{j_1}) \mathbb{T}^{(2n-i_2)}\left(i_2+1;\, i_1, \mathbf{j}^{(1)}_1\right) \\
&\qquad C(x_{i_1}, x_{j_1^{(1)}})\left[i_2+1;\, \mathbf{j}^{(1)}_1, \mathbf{j}^{(1)}_2\right] C(x_{i_1}, x_{j_2^{(1)}})\left[i_2+1;\, \mathbf{j}^{(1)}_2, \mathbf{j}^{(1)}_3\right] \\
&\qquad \cdots C(x_{i_1}, x_{j^{(1)}_{l_1-1}})\left[i_2+1;\, \mathbf{j}^{(1)}_{l_1-1}, \mathbf{j}^{(1)}_{l_1}\right] h_{i_2+1}(x_{i_2+1}) \otimes h_{i_2+2}(x_{i_2+2}) \otimes \cdots \otimes h_{2n}(x_{2n}).
\end{aligned}$$

From here, using the commutativity of the functionals, respectively operators acting in different variables, we similarly get:

$$\left(\mathcal{A}^-_{l_2}(h_{i_2})\mathcal{A}^+(h_{i_2+1}) \cdots \mathcal{A}^+(h_{i_1-1})\mathcal{A}^-_{l_1}(h_{i_1})\mathcal{A}^+(h_{i_1+1})\right.$$

$$\cdots \mathcal{A}^{+}(h_{2n})\Omega\big)(x_{i_2}, x_{i_2+1}, \dots, x_{2n} \setminus x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2})$$

$$= \int_{X^4} \sigma^{(2)}(dx_{i_2}\, dx_{j_2}) \sigma^{(2)}(dx_{i_1}\, dx_{j_1}) \mathbb{T}^{(2n-i_2+1)} \left(i_2;\ i_1, \mathbf{j}_1^{(1)} \mid i_2, \mathbf{j}_1^{(2)}\right)$$

$$C(x_{i_2}, x_{j_1^{(2)}}) \left[i_2;\ \mathbf{j}_1^{(2)}, \mathbf{j}_2^{(2)}\right] C(x_{i_2}, x_{j_2^{(2)}}) \left[i_2;\ \mathbf{j}_2^{(2)}, \mathbf{j}_3^{(2)}\right] \cdots C(x_{i_2}, x_{j_{l_2-1}^{(2)}}) \left[i_2;\ \mathbf{j}_{l_2-1}^{(2)}, \mathbf{j}_{l_2}^{(2)}\right]$$

$$C(x_{i_1}, x_{j_1^{(1)}}) \left[i_2;\ \mathbf{j}_1^{(1)}, \mathbf{j}_2^{(1)}\right] C(x_{i_1}, x_{j_2^{(1)}}) \left[i_2;\ \mathbf{j}_2^{(1)}, \mathbf{j}_3^{(1)}\right] \cdots C(x_{i_1}, x_{j_{l_1-1}^{(1)}}) \left[i_2;\ \mathbf{j}_{l_1-1}^{(2)}, \mathbf{j}_{l_1}^{(1)}\right]$$

$$h_{i_2}(x_{i_2}) \otimes h_{i_2+1}(x_{i_2+1}) \otimes \cdots \otimes h_{2n}(x_{2n}).$$

Continuing by analogy, we get at the k^{th} step:

$$\big(\mathcal{A}_{l_k}^{-}(h_{i_k}) \mathcal{A}^{+}(h_{i_k+1}) \cdots \mathcal{A}^{+}(h_{i_{k-1}+1}) \mathcal{A}_{l_{k-1}}^{-}(h_{i_{k-1}}) \mathcal{A}^{+}(h_{i_{k-1}+1}) \cdots A^{+}(h_{i_{k-2}+1})$$

$$\cdots \mathcal{A}_{l_2}^{-}(h_{i_2}) \mathcal{A}^{+}(h_{i_2+1}) \cdots \mathcal{A}^{+}(h_{i_1-1}) \mathcal{A}_{l_1}^{-}(h_{i_1}) \mathcal{A}^{+}(h_{i_1+1}) \cdots \mathcal{A}^{+}(h_{2n})\Omega\big)$$

$$(x_{i_k}, x_{i_k+1}, \dots, x_{2n} \setminus x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, \dots, x_{i_n}, x_{j_n})$$

$$= \int_{X^{2k}} \sigma^{(2)}(dx_{i_1} dx_{j_1}) \sigma^{(2)}(dx_{i_2} dx_{j_2}) \cdots \sigma^{(2)}(dx_{i_k} dx_{j_k})$$

$$\mathbb{T}^{(2n-i_k+1)}(i_k; i_1, \mathbf{j}_1^{(1)} \mid i_2, \mathbf{j}_1^{(2)} \mid \cdots \mid i_k, \mathbf{j}_1^{(k)})$$

$$C(x_{i_k}, x_{j_1^{(k)}}) \left[i_k; \mathbf{j}_1^{(k)}, \mathbf{j}_2^{(k)}\right] C(x_{i_k}, x_{j_2^{(k)}}) \left[i_k; \mathbf{j}_2^{(k)}, \mathbf{j}_3^{(k)}\right] \cdots C(x_{i_k}, x_{j_{l_k-1}^{(k)}}) \left[i_k; \mathbf{j}_{l_k-1}^{(k)}, \mathbf{j}_{l_k}^{(k)}\right]$$

$$C(x_{i_{k-1}}, x_{j_1^{(k-1)}}) \left[i_k; \mathbf{j}_1^{(k-1)}, \mathbf{j}_2^{(k-1)}\right] C(x_{i_{k-1}}, x_{j_2^{(k-1)}}) \left[i_k; \mathbf{j}_2^{(k-1)}, \mathbf{j}_3^{(k-1)}\right]$$

$$\cdots C(x_{i_{k-1}}, x_{j_{l_{k-1}-1}^{(k-1)}}) \left[i_k; \mathbf{j}_{l_{k-1}-1}^{(k-1)}, \mathbf{j}_{l_{k-1}}^{(k-1)}\right]$$

$$\cdots C(x_{i_1}, x_{j_1^{(1)}}) \left[i_k; \mathbf{j}_1^{(1)}, \mathbf{j}_2^{(1)}\right] C(x_{i_1}, x_{j_2^{(1)}}) \left[i_k; \mathbf{j}_2^{(1)}, \mathbf{j}_3^{(1)}\right] \cdots C(x_{i_1}, x_{j_{l_1-1}^{(1)}}) \left[i_k; \mathbf{j}_{l_1-1}^{(1)}, \mathbf{j}_{l_1}^{(1)}\right]$$

$$h_{i_k}(x_{i_k}) \otimes h_{i_{k+1}}(x_{i_k+1}) \otimes \cdots \otimes h_{2n}(x_{2n}).$$

Thus, after n steps, we get, by using the notations (4.25)-(4.27),

$$\big(\mathcal{A}_{l_n}^{-}(h_{i_n}) \mathcal{A}^{+}(h_{i_n-1}) \cdots \mathcal{A}^{+}(h_{i_{n-1}+1}) \mathcal{A}_{l_{n-1}}^{-}(h_{i_{n-1}}) \mathcal{A}^{+}(h_{i_{n-1}+1}) \cdots \mathcal{A}^{+}(h_{i_{n-2}+1})$$

$$\cdots \mathcal{A}_{l_1}^{-}(h_{i_1}) \mathcal{A}(h_{i_1+1}) \cdots \mathcal{A}^{+}(h_{2n})\Omega\big)(x_1, x_2, \dots, x_n) \tag{4.36}$$

$$= \int_{X^{2n}} \sigma^{(2)}(dx_{i_1}\, dx_{j_1}) \sigma^{(2)}(dx_{i_2}\, dx_{j_2}) \cdots \sigma^{(2)}(dx_{i_n}\, dx_{j_n})$$

$$\times \mathbb{T}^{(2n)}(\xi) C(\xi; x_1, x_2, \dots, x_{2n}) h_1(x_1) \otimes h_2(x_2) \otimes \cdots \otimes \widetilde{f}_{2n}(x_{2n}). \tag{4.37}$$

Formulas (4.35) and (4.37) imply (4.29). □

Remark 4.15. We can unify formulas (4.28) and (4.29) into a single formula

$$\tau\big((A^+(f_1)+A^-(g_1))\cdots(A^+(f_n)+A^-(g_n))\big) = \sum_{\xi\in\mathcal{P}_2^{(n)}}\int_{X^n}\bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}}\sigma^{(2)}(dx_i\,dx_j)$$

$$\times\,\mathbb{T}^{(n)}(\xi)C(\xi;x_1,\dots,x_n)h_1(x_1)\otimes h(x_2)\otimes\cdots\otimes h_n(x_n) \tag{4.38}$$

that holds for all $n\in\mathbb{N}$. Indeed, if n is even then this formula becomes (4.29) and if n is odd then the set $\mathcal{P}_2^{(n)}$ is empty, hence the right hand-side of (4.38) is equal to zero.

Recall that the operator $C(\xi;x_1,\dots,x_n)$ was defined through $\xi\in\mathcal{P}_2^{(n)}$ and operators $C(x,x')$. But, for $x=(y,z)$ and $x'=(y',z')$, we have $C(x,x')=C(y,y')$. Therefore, we can write the operator $C(\xi;x_1,\dots,x_n)$ as $C(\xi;y_1,\dots,y_n)$.

Corollary 4.16. *Let τ be the Fock state on the C-MCR algebra $\mathbb{A}$. For any $g_1,g_2\in\mathcal{G}$, we define a complex-valued measure $\lambda^{(2)}[g_1,g_2]$ on Y^2 by*

$$\lambda^{(2)}[g_1,g_2](dy_1\,dy_2)=(g_2,g_1)_{\mathcal{G}}\,\nu^{(2)}(dy_1\,dy_2). \tag{4.39}$$

Then, for any $\varphi_1,\dots,\varphi_n\in C_0(Y\to V_{\mathbb{R}})$ and $g_1,\dots,g_n\in\mathcal{G}$, we have

$$\tau\left(B(\varphi_1\otimes g_1)\cdots B(\varphi_n\otimes g_n)\right)=\sum_{\xi\in\mathcal{P}_2^{(n)}}\int_{Y^n}\bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}}\lambda^{(2)}[g_i,g_j](dy_i\,dy_j)$$

$$\mathbb{T}^{(n)}(\xi)C(\xi;y_1,\dots,y_n)\varphi_1(y_1)\otimes\varphi_2(y_2)\otimes\cdots\otimes\varphi_n(y_n).$$

Proof. Since the functions $\varphi_1,\dots,\varphi_n$ take on values in $V_{\mathbb{R}}$, we get by (4.38)

$$\begin{aligned}&\tau\left(B(\varphi_1\otimes g_1)\cdots B(\varphi_n\otimes g_n)\right)\\&=\tau\big((A^+(\varphi_1\otimes g_1)+A^-(\varphi_1\otimes(Jg_1)))\cdots(A^+(\varphi_1\otimes g_1)+A^-(\varphi_n\otimes(Jg_n)))\big)\\&=\sum_{\xi\in\mathcal{P}_2^{(n)}}\int_{X^n}\bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}}\sigma^{(2)}(dx_i\,dx_j)\\&\quad\mathbb{T}^{(n)}(\xi)C(\xi;y_1,\dots,y_n)\varphi_1(y_1)\widetilde{g}_1(z_1)\otimes\varphi_2(y_2)\widetilde{g}_2(z_2)\otimes\cdots\otimes\varphi_n(y_n)\widetilde{g}_n(z_n)\end{aligned} \tag{4.40}$$

where

$$\widetilde{g}_k(z)=\begin{cases}\overline{g}_k(z), & \text{if } k\in I,\\ g_k(z), & \text{if } k\in J.\end{cases}$$

Since the functions $\widetilde{g}_1(z_1), \dots, \widetilde{g}_n(z_n)$ take on values in $\mathbb{C}$, we continue (4.40) as follows:

$$\tau\left(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\right) = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{X^n} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \sigma^{(2)}(dx_i\, dx_j) \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n)$$
$$\times\, \widetilde{g}_1(z_1) \widetilde{g}_2(z_2) \cdots \widetilde{g}_n(z_n)\, \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n). \tag{4.41}$$

Since $C(\xi; y_1, \dots, y_n)$ is a linear operator in $V^{\otimes n}$ and $\mathbb{T}^{(n)}(\xi)$ is a linear functional on $V^{\otimes n}$, we continue (4.41) as follows:

$$\begin{aligned}
&\tau\left(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\right) \\
&\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{X^n} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \sigma^{(2)}(dx_i\, dx_j) \widetilde{g}_1(z_1) \widetilde{g}_2(z_2) \cdots \widetilde{g}_n(z_n) \\
&\qquad \times \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n) \\
&\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{X^n} \left(\bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \nu^{(2)}(dy_i\, dy_j) \mu^{(2)}(dz_i\, dz_j) \widetilde{g}_i(z_i) \widetilde{g}_j(z_j) \right) \\
&\qquad \times \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n) \\
&\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{Y^n} \left(\bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \nu^{(2)}(dy_i\, dy_j) \int_{Z^2} \mu^{(2)}(dz_i\, dz_j) \overline{g}_i(z_i) g_j(z_j) \right) \\
&\qquad \times \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n) \\
&\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{Y^n} \left(\bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \nu^{(2)}(dy_i\, dy_j) (g_j, g_i)_{\mathcal{G}} \right) \\
&\qquad \times \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n) \\
&\quad = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{Y^n} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \lambda^{(2)}[g_i, g_j](dy_i\, dy_j) \\
&\qquad \times \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n).
\end{aligned}$$

□

Below we will also need a formula for

$$\tau\left(A^-(f_1)\cdots A^-(f_n)A^+(f_{n+1})\cdots A^+(f_{2n})\right), \tag{4.42}$$

where $f_1,\dots,f_{2n}\in\mathcal{F}(X\to V)$. This is, of course, a special case of formula (4.29), which can be simplified in the case of (4.42).

For each permutation $\pi\in S_n$, we define a partition $\xi\in\mathcal{P}_2^{(2n)}$ as follows:

$$\big\{\{1,n+\pi(1)\},\{2,n+\pi(2)\},\dots,\{n,n+\pi(n)\}\big\}.$$

We denote by $\mathcal{S}^{(n)}$ the subset of $\mathcal{P}_2^{(2n)}$ consisting of all such partitions. Equivalently, $\mathcal{S}^{(n)}$ consists of all partitions $\xi\in\mathcal{P}_2^{(2n)}$ such that, for each $\xi\in\mathcal{S}^{(2n)}$ and each $\{i,j\}\in\xi$, with $i<j$, we have $i\le n$ and $j\ge n+1$. The following corollary is immediate.

Corollary 4.17. *Let τ be the Fock state on the C-MCR algebra $\mathbb{A}$.*

(i) For any $f_1,\dots,f_{2n}\in\mathcal{F}(X\to V)$, we have

$$\begin{aligned}&\tau\left(A^-(f_1)\cdots A^-(f_n)A^+(f_{n+1})\cdots A^+(f_{2n})\right)\\ &=\sum_{\xi\in\mathcal{S}^{(2n)}}\int_{X^{2n}}\bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}}\sigma^{(2)}(dx_i\,dx_j)\,\mathbb{T}^{(2n)}(\xi)\,C(\xi;x_1,\dots,x_{2n})f_1(x_1)\otimes f_2(x_2)\otimes\cdots\otimes f_{2n}(x_{2n}).\end{aligned}$$

Here $\mathbb{T}^{(2n)}:V^{\otimes 2n}\to\mathbb{C}$ is the linear functional defined by (4.17) and the linear operator $C(\xi;x_1,\dots,x_{2n}):V^{\otimes 2n}\to\mathbb{C}$ is defined by (4.26) in which

$$C^{(k)}(\xi;x_1,\dots,x_{2n})=C_{n+k}(x_{i_k},x_{j_1^{(k)}})C_{n+k+1}(x_{i_k},x_{j_2^{(k)}})\cdots C_{n+k+l_k-1}(x_{i_k},x_{j^{(k)}_{l_k-1}}),$$

with

$$\begin{aligned}J^{(k)}&=\big\{j\in J\mid j\le j_k,\ j\ne j_1,\dots,j\ne j_{k-1}\big\},\\ J^{(k)}&=\left\{j_1^{(k)},j_2^{(k)},\dots,j_{l_k}^{(k)}\right\},\quad j_1^{(k)}<j_2^{(k)}<\cdots<j_{l_k}^{(k)}.\end{aligned}$$

(ii) For any $\varphi_1,\dots,\varphi_{2n}\in C_0(Y\to V)$ and $g_1,\dots,g_{2n}\in\mathcal{G}$, we have

$$\begin{aligned}&\tau(A^-(\varphi_1\otimes g_1)\cdots A^-(\varphi_n\otimes g_n)A^+(\varphi_{n+1}\otimes g_{n+1})\cdots A^+(\varphi_{2n}\otimes g_{2n}))\\ &\quad=\sum_{\xi\in\mathcal{S}^{(2n)}}\int_{Y^{2n}}\bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}}\lambda^{(2)}[g_i,g_j](dy_i\,dy_j)\,\mathbb{T}^{(2n)}C(\xi;y_1,\dots,y_{2n})\varphi_1(y_1)\otimes\varphi_2(y_2)\otimes\cdots\otimes\varphi_{2n}(y_{2n}),\end{aligned}$$

where the complex-valued measure $\lambda^{(2)}[g_i,g_j]$ is defined by (4.39).

4.4 Quasi-free states on an C-MCR algebra

In view of Corollary 4.16, we now give the following definition.

Definition 4.18. Let $\mathbb{A}$ be an C-MCR algebra and let τ be a state on it. Assume that, for any $\sharp_1, \dots, \sharp_n \in \{+, -\}$, the functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n \to V^{\otimes n}) \times \mathcal{G}^n$. Furthermore, assume that, for any $g_1, g_2 \in \mathcal{G}$ there exists a complex-valued measure $\lambda^{(2)}[g_1, g_2]$ on Y^2 such that, for all $\varphi_1, \varphi_2 \in C_0(Y \to V_{\mathbb{R}})$,

$$\tau\left(B(\varphi_1 \otimes g_1) B(\varphi_2 \otimes g_2)\right) = \int_{Y^2} \lambda^{(2)}[g_1, g_2](dy_1\, dy_2) \langle \varphi_1(y_1), \varphi_2(y_2)\rangle_V.$$

We say that τ is a *strongly quasi-free state*, if for any $\varphi_1, \dots, \varphi_n \in C_0(Y \to V_{\mathbb{R}})$ and $g_1, \dots, g_n \in \mathcal{G}$, we have, for an odd n,

$$\tau\left(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\right) = 0,$$

and for an even n,

$$\tau\left(B(\varphi_1 \otimes g_1) \cdots B(\varphi_n \otimes g_n)\right) = \sum_{\xi \in \mathcal{P}_2^{(n)}} \int_{Y^n} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \lambda^{(2)}[g_i, g_j](dy_i\, dy_j) \mathbb{T}^{(n)}(\xi) C(\xi; y_1, \dots, y_n)$$
$$\times \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_n(y_n).$$

Corollary 4.16 implies to the following proposition.

Proposition 4.19. *The Fock state on the C-MCR algebra is strongly quasi-free and the corresponding measure $\lambda^{(2)}[g_1, g_2](dy_1\, dy_2)$ is given by formula* (4.39).

Similarly to Section 3.5, we give the following

Definition 4.20. Let $\mathbb{A}$ be an C-MCR algebra and let τ be a state on it. Assume that, for any $\sharp_1, \dots, \sharp_n \in \{+, -\}$, the functional $\tau^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n \to V^{\otimes n}) \times \mathcal{G}^n$. Furthermore, assume that, for any $g_1, g_2 \in \mathcal{G}$, there exists a complex-valued measure $\rho^{(2)}[g_1, g_2]$ on Y^2 such that, for all $\varphi_1, \varphi_2 \in C_0(Y \to V_{\mathbb{R}})$,

$$\tau\left(A^+(\varphi_1 \otimes g_1) A^-(\varphi_2 \otimes g_2)\right) = \int_{Y^2} \rho^{(2)}[g_1, g_2](dy_1\, dy_2) \langle \varphi_1(y_1), \varphi_2(y_2)\rangle_V.$$

We say that τ is a *gauge-invariant quasi-free state* if for any $m, n \in \mathbb{N}$, $\varphi_1, \dots, \varphi_{m+n} \in C_0(Y \to V)$ and $g_1, \dots, g_{m+n} \in \mathcal{G}$, we have, for $m \neq n$,

$$\tau\left(A^+(\varphi_1 \otimes g_1) \dots A^+(\varphi_m \otimes g_m) A^-(\varphi_{m+1} \otimes g_{m+1}) \dots A^-(\varphi_{m+n} \otimes g_{m+n})\right) = 0,$$

and for $m = n$,

$$\begin{aligned}
&\tau\left(A^+(\varphi_1 \otimes g_1) \cdots A^+(\varphi_n \otimes g_n) A^-(\varphi_{n+1} \otimes g_{n+1}) \cdots A^-(\varphi_{2n} \otimes g_{2n})\right) \\
&\quad = \sum_{\xi \in \mathcal{S}^{(2n)}} \int_{Y^{2n}} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \rho^{(2)}[g_i, g_j](dy_i\, dy_j) \\
&\qquad \times \mathbb{T}^{(2n)} C(\xi; y_1, \dots, y_{2n}) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_{2n}(y_{2n}).
\end{aligned}$$

We will now construct a class of strongly quasi-free states and gauge-invariant quasi-free states on the C-MCR algebra. We will be able to do this under strong assumptions on the operator-valued function $C(y_1, y_2)$.

Let us first recall the following definitions from Chapter 2.

We define an antilinear operator $\mathbb{S} : V^{\otimes 2} \to V^{\otimes 2}$ by

$$\mathbb{S}\, v_1 \otimes v_2 = (Jv_2) \otimes (Jv_1), \quad v_1, v_2 \in V. \tag{4.43}$$

For a linear operator $A \in \mathcal{L}(V^{\otimes 2})$, we define $\widehat{A} \in \mathcal{L}(V^{\otimes 2})$ by

$$\widehat{A} = \mathbb{S}\, A\, \mathbb{S}. \tag{4.44}$$

For a linear operator $A \in \mathcal{L}(V^{\otimes 2})$, we define $\widetilde{A} \in \mathcal{L}(V^{\otimes 2})$ through the formula

$$(Av_1 \otimes v_2, v_3 \otimes v_4)_{V^{\otimes 2}} = (\widetilde{A} v_3 \otimes v_1, v_4 \otimes v_2)_{V^{\otimes 2}}, \quad v_1, v_2, v_3, v_4 \in V_{\mathbb{R}}. \tag{4.45}$$

Note that the maps

$$\mathcal{L}(V^{\otimes 2}) \ni A \mapsto \widehat{A} \in \mathcal{L}(V^{\otimes 2}), \quad \mathcal{L}(V^{\otimes 2}) \ni A \mapsto \widetilde{A} \in \mathcal{L}(V^{\otimes 2})$$

are linear and continuous.

From now on, we assume that the operator-valued function $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ satisfies the following additional

Assumption 4.21. (i) For all $(y_1, y_2) \in Y^2$, $\widetilde{C}(y_1, y_2)$ is unitary and

$$\widetilde{C}(y_1, y_2)^* = \widetilde{C}(y_2, y_1).$$

(ii) For all $y_1, y_2 \in Y$, $\widetilde{\widetilde{C}}(y_1, y_2) = C(y_1, y_2)$.

(iii) For all $y_1, y_2 \in Y$, we have

$$\widehat{C}(y_1, y_2) = C(y_2, y_1),$$
$$\widehat{\widetilde{C}}(y_1, y_2) = \widetilde{C}(y_2, y_1).$$

(iv) There exists a constant $\varkappa \in \mathbb{R}$ such that, for all $y \in Y$ and $v^{(2)} \in V^{\otimes 2}$,

$$\operatorname{Tr}\left(\widetilde{C}(y, y) v^{(2)}\right) = \varkappa \operatorname{Tr} v^{(2)}.$$

Similarly to Section 3.5, we denote by $X_1 = Y_1 \times Z_1$ and $X_2 = Y_2 \times Z_2$ two copies of the set $X = Y \times Z$ and consider their disjoint union $\mathbf{X} = X_1 \sqcup X_2$.

Let also, $\mathbf{Y}^2 = Y_1 \sqcup Y_2$. We also consider the measure σ on $\mathbf{X}$, and denote $\mathbf{H} = L^2(\mathbf{X}, d\,\sigma)$.

We define a (continuous) function $\mathbf{C} : \mathbf{Y}^2 \to \mathcal{L}(V^{\otimes 2})$ by

$$\mathbf{C}(y_1, y_2) = \begin{cases} C(y_1, y_2), & \text{if } y_1, y_2 \in Y_1 \text{ or } y_1, y_2 \in Y_2, \\ \widetilde{C}(y_2, y_1), & \text{if } y_1 \in Y_1,\ y_2 \in Y_2 \text{ or } y_1 \in Y_2,\ y_2 \in Y_1. \end{cases} \tag{4.46}$$

By Assumption (i), for any $y_1, y_2 \in \mathbf{Y}$, $\mathbf{C}(y_1, y_2)$ is a unitary operator in $V^{\otimes 2}$ and

$$\mathbf{C}^*(y_1, y_2) = \mathbf{C}(y_2, y_1). \tag{4.47}$$

We furthermore assume the following:

Assumption 4.22. The function $\mathbf{C} : \mathbf{Y} \to \mathcal{L}(V^{\otimes 2})$ satisfies the functional Yang-Baxter equation point-wise.

As usual, for any $x_1 = (y_1, z_1)$, $x_2 = (y_2, z_2) \in \mathbf{X}$, we denote $\mathbf{C}(x_1, x_2) = \mathbf{C}(y_1, y_2)$.

Formula (4.47) and Assumption (iv) allow us to consider the $\mathbf{C}$-MCR algebra $\mathbf{A}$. Let $\boldsymbol{\tau}$ denote the Fock state on $\mathbf{A}$.

In accordance with the intuitive formula (4.2), we will denote elements of the **C**-MCR algebra **A** by

$$\int_{\mathbf{X}^n} \langle \mathbf{f}^{(n)}(x_1,\dots,x_n), \mathbf{a}^{\sharp_1}(x_1)\otimes\dots\otimes\mathbf{a}^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n), \tag{4.48}$$

where $\mathbf{f}^{(n)}\in\mathfrak{F}(\mathbf{X}^n\to V^{\otimes n})$ and $\sharp_1,\dots,\sharp_n\in\{+,-\}$.

Similarly to (3.43) and (3.45), for $\mathbf{f}^{(n)}$ as in (3.44) we will denote the corresponding element (4.48) of the **C**-MCR algebra **A** by

$$\int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), \mathbf{a}_{i_1}^{\sharp_1}(x_1)\otimes\dots\otimes\mathbf{a}_{i_n}^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\sigma(dx_1)\cdots\sigma(dx_n).$$

So, informally, for $\sharp\in\{+,-\}$ and $x\in X$, $\mathbf{a}_i^\sharp(x)$ is equal to $a_i^\sharp(x)$, where x is identified with an element of X_i.

In view of Corollary 2.94, Proposition 4.6 and Assumptions (i)–(iv), the operators $\mathbf{a}_i^\sharp(x)$ with $i\in\{1,2\}$ and $\sharp\in\{+,-\}$ satisfy the following commutation relations.

Lemma 4.23. *Let $g^{(2)}\in\mathcal{H}^{\otimes 2}$.*

(i) For $i\in\{1,2\}$ and $\sharp\in\{+,-\}$,

$$\begin{aligned}&\int_{X^2}\langle g^{(2)}(x_1,x_2), \mathbf{a}_i^\sharp(x_1)\otimes\mathbf{a}_i^\sharp(x_2)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2)\\ &\quad=\int_{X^2}\langle C(x_2,x_1)g^{(2)}(x_1,x_2), \mathbf{a}_i^\sharp(x_2)\otimes\mathbf{a}_i^\sharp(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2).\end{aligned}$$

(ii) For $i,j\in\{1,2\}$, $i\ne j$, and $\sharp\in\{+,-\}$,

$$\begin{aligned}&\int_{X^2}\langle g^{(2)}(x_1,x_2), \mathbf{a}_i^\sharp(x_1)\otimes\mathbf{a}_j^\sharp(x_2)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2)\\ &\quad=\int_{X^2}\langle \widetilde{C}(x_1,x_2)g^{(2)}(x_1,x_2), \mathbf{a}_j^\sharp(x_2)\otimes\mathbf{a}_i^\sharp(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2).\end{aligned}$$

(iii) For $i\in\{1,2\}$,

$$\begin{aligned}&\int_{X^2}\langle g^{(2)}(x_1,x_2), \mathbf{a}_i^-(x_1)\otimes\mathbf{a}_i^+(x_2)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2)=\int_X \operatorname{Tr} g^{(2)}(x,x)\sigma(dx)\\ &\quad+\int_{X^2}\langle \widetilde{C}(x_1,x_2)g^{(2)}(x_1,x_2), \mathbf{a}_i^+(x_2)\otimes\mathbf{a}_i^-(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2).\end{aligned}$$

(iv) For $i, j \in \{1,2\}$, $i \neq j$, and $\sharp \in \{+,-\}$,

$$\int_{X^2} \langle g^{(2)}(x_1,x_2), \mathbf{a}_i^-(x_1) \otimes \mathbf{a}_j^+(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)$$
$$= \int_{X^2} \langle C(x_2,x_1) g^{(2)}(x_1,x_2), \mathbf{a}_j^+(x_2) \otimes \mathbf{a}_i^-(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2).$$

(v) For $i \in \{1,2\}$,

$$\int_{X^2} \langle g^{(2)}(x_1,x_2), \mathbf{a}_i^+(x_1) \otimes \mathbf{a}_i^-(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2) = -\varkappa \int_X \operatorname{Tr}\left(g^{(2)}(x,x)\right) \sigma(dx)$$
$$+ \int_{X^2} \langle \widetilde{C}(x_1,x_2) g^{(2)}(x_1,x_2), \mathbf{a}_i^-(x_2) \otimes \mathbf{a}_i^+(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2).$$

(vi) For $i, j \in \{1,2\}$, $i \neq j$, and $\sharp \in \{+,-\}$,

$$\int_{X^2} \langle g^{(2)}(x_1,x_2), \mathbf{a}_i^+(x_1) \otimes \mathbf{a}_j^-(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)$$
$$= \int_{X^2} \langle C(x_2,x_1) g^{(2)}(x_1,x_2), \mathbf{a}_j^-(x_2) \otimes \mathbf{a}_i^+(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2).$$

Let us fix a bounded linear operator $K \in \mathcal{L}(\mathcal{G})$. In case $\varkappa \geq 0$, we assume that $K \geq 0$ and in the case $\varkappa < 0$, we assume $0 \leq K \leq -\frac{1}{\varkappa}$. Let

$$K_1 = \sqrt{K}, \quad K_2 = \sqrt{1 + \varkappa K}.$$

For a bounded linear operator $B \in \mathcal{L}(\mathcal{G})$, we recall the definition (3.47) of the complex conjugate operator B'.

Let $f = \varphi \otimes g$ with $\varphi \in C_0(Y \to V)$ and $g \in \mathcal{G}$. We define

$$A^+(f) = \int_X \langle \varphi(y)(K_2 g)(z), \mathbf{a}_2^+(x)\rangle_V\, \sigma(dx) + \int_X \langle \varphi(y)(K_1 g)(z), \mathbf{a}_1^-(x)\rangle_V\, \sigma(dx),$$
$$A^-(f) = \int_X \langle \varphi(y)(K_2' g)(z), \mathbf{a}_2^-(x)\rangle_V\, \sigma(dx) + \int_X \langle \varphi(y)(K_1' g)(z), \mathbf{a}_1^+(x)\rangle_V\, \sigma(dx). \quad (4.49)$$

We denote

$$A^+(f) = \int_X \langle f(x), A^+(x)\rangle_V\, \sigma(dx),$$
$$A^-(f) = \int_X \langle f(x), A^-(x)\rangle_V\, \sigma(dx). \quad (4.50)$$

Proposition 4.24. *We have $(A^+(f))^* = A^-(Jf)$ and the operators $A^+(f)$, $A^-(f)$ defined by (4.49) satisfy the C-MCR.*

Proof. We first show that $(A^+(f))^* = A^-(Jf)$. For $f(y,z) = \varphi(y)g(z)$, we have

$$Jf(x,y) = (J\varphi)(y)(Jg)(z).$$

Hence

$$\begin{aligned}\big(A^+(f)\big)^* &= \int_X \langle (J\varphi)(y)(JK_2 g)(z), \mathbf{a}_2^-(x)\rangle_V \,\sigma(dx) + \int_X \langle (J\varphi)(y)(JK_1 g)(z), \mathbf{a}_1^+(x)\rangle_V \,\sigma(dx)\\ &= \int_X \langle (J\varphi)(y)(K_2' Jg)(z), \mathbf{a}_2^-(x)\rangle_V \,\sigma(dx) + \int_X \langle (J\varphi)(y)(K_1' Jg)(z), \mathbf{a}_1^+(x)\rangle_V \,\sigma(dx)\\ &= A^-(Jf).\end{aligned}$$

Next, we check the C-MCR. For any $f_1 = \varphi_1 \otimes g_1$, $f_2 = \varphi_2 \otimes g_2$, using Lemma 4.23, we have:

$$\begin{aligned}&A^+(f_1)A^+(f_2)\\ &\quad= \left(\int_X \langle \varphi_1(y_1)(K_2 g_1)(z_1), \mathbf{a}_2^+(x_1)\rangle_V \,\sigma(dx_1) + \int_X \langle \varphi_1(y_1)(K_1 g_1)(z_1), \mathbf{a}_1^-(x_1)\rangle_V \,\sigma(dx_1)\right)\\ &\qquad \circ \left(\int_X \langle \varphi_2(y_2)(K_2 g_2)(z_2), \mathbf{a}_2^+(x_2)\rangle_V \,\sigma(dx_2) + \int_X \langle \varphi_2(y_2)(K_1 g_2)(z_2), \mathbf{a}_1^-(x_2)\rangle_V \,\sigma(dx_2)\right)\\ &\quad= \int_{X^2} \langle \varphi_1(y_1)\otimes\varphi_2(y_2)(K_2 g_1)(z_1)(K_2 g_2)(z_2), \mathbf{a}_2^+(x_1)\otimes\mathbf{a}_2^+(x_2)\rangle_{V^{\otimes 2}} \,\sigma(dx_1)\sigma(dx_2)\\ &\qquad+ \int_{X^2} \langle \varphi_1(y_1)\otimes\varphi_2(y_2)(K_2 g_1)(z_1)(K_1 g_2)(z_2), \mathbf{a}_2^+(x_1)\otimes\mathbf{a}_1^-(x_2)\rangle_{V^{\otimes 2}} \,\sigma(dx_1)\sigma(dx_2)\\ &\qquad+ \int_{X^2} \langle \varphi_1(y_1)\otimes\varphi_2(y_2)(K_1 g_1)(z_1)(K_2 g_2)(z_2), \mathbf{a}_1^-(x_1)\otimes\mathbf{a}_2^+(x_2)\rangle_{V^{\otimes 2}} \,\sigma(dx_1)\sigma(dx_2)\\ &\qquad+ \int_{X^2} \langle \varphi_1(y_1)\otimes\varphi_2(y_2)(K_1 g_1)(z_1)(K_1 g_2)(z_2), \mathbf{a}_1^-(x_1)\otimes\mathbf{a}_1^-(x_2)\rangle_{V^{\otimes 2}} \,\sigma(dx_1)\sigma(dx_2)\\ &\quad= \int_{X^2} \langle C(y_2,y_1)(\varphi_1(y_1)\otimes\varphi_2(y_2))(K_2 g_1)(z_1)(K_2 g_2)(z_2), \mathbf{a}_2^+(x_2)\otimes\mathbf{a}_2^+(x_1)\rangle_{V^{\otimes 2}}\\ &\qquad\times \sigma(dx_1)\sigma(dx_2)\\ &\qquad+ \int_{X^2} \langle C(y_2,y_1)(\varphi_1(y_1)\otimes\varphi_2(y_2))(K_2 g_1)(z_1)(K_1 g_2)(z_2), \mathbf{a}_1^-(x_2)\otimes\mathbf{a}_2^+(x_1)\rangle_{V^{\otimes 2}}\\ &\qquad\times \sigma(dx_1)\sigma(dx_2)\\ &\quad+ \int_{X^2} \langle C(y_2,y_1)(\varphi_1(y_1)\otimes\varphi_2(y_2))(K_1 g_1)(z_1)(K_2 g_2)(z_2), \mathbf{a}_2^+(x_2)\otimes\mathbf{a}_1^-(x_1)\rangle_{V^{\otimes 2}}\\ &\qquad\times \sigma(dx_1)\sigma(dx_2)\\ &\quad+ \int_{X^2} \langle C(y_2,y_1)(\varphi_1(y_1)\otimes\varphi_2(y_2))(K_1 g_1)(z_1)(K_1 g_2)(z_2), \mathbf{a}_1^-(x_2)\otimes\mathbf{a}_1^-(x_1)\rangle_{V^{\otimes 2}}\end{aligned}$$

$$\times\, \sigma(dx_1)\sigma(dx_2)$$

$$= \int_{X^2} \langle C(y_2, y_1)(\varphi_1(y_1) \otimes \varphi_2(y_2))\, g_1(z_1)g_2(z_2), A^+(x_2) \otimes A^+(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)$$

$$= \int_{X^2} \langle C(x_2, x_1)\big(f_1(x_1) \otimes f_2(x_2)\big), A^+(x_2) \otimes A^+(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2).$$

The proof of the commutation between $A^-(f_1)$ and $A^-(f_2)$ is similar.

Next, we have

$$
\begin{aligned}
&A^-(f_1)A^+(f_2)\\
&\quad= \bigg(\int_X \langle \varphi_1(y_1)(K_2'g_1)(z_1), \mathbf{a}_2^-(x_1)\rangle_V\, \sigma(dx_1) + \int_X \langle \varphi(y_1)(K_1'g_1)(z_1), \mathbf{a}_1^+(x_1)\rangle_V\, \sigma(dx_1)\bigg)\\
&\qquad \circ \bigg(\int_X \langle \varphi_2(y_2)(K_2g_2)(z_2), \mathbf{a}_2^+(x_2)\rangle_V\, \sigma(dx_2) + \int_X \langle \varphi(y_2)(K_1g_2)(z_2), \mathbf{a}_1^-(x_2)\rangle_V\, \sigma(dx_2)\bigg)\\
&\quad= \int_{X^2} \langle \varphi_1(y_1) \otimes \varphi_2(y_2)(K_2'g_1)(z_1)(K_2g_2)(z_2), \mathbf{a}_2^-(x_1) \otimes \mathbf{a}_2^+(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_{X^2} \langle \varphi_1(y_1) \otimes \varphi_2(y_2)(K_2'g_1)(z_1)(K_1g_2)(z_2), \mathbf{a}_2^-(x_1) \otimes \mathbf{a}_1^-(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_{X^2} \langle \varphi_1(y_1) \otimes \varphi_2(y_2)(K_1'g_1)(z_1)(K_2g_2)(z_2), \mathbf{a}_1^+(x_1) \otimes \mathbf{a}_2^+(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_{X^2} \langle \varphi_1(y_1) \otimes \varphi_2(y_2)(K_1'g_1)(z_1)(K_1g_2)(z_2), \mathbf{a}_1^+(x_1) \otimes \mathbf{a}_1^-(x_2)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)\\
&\quad= \int_{X^2} \langle \widetilde{C}(y_1, y_2)(\varphi_1(y_1) \otimes \varphi_2(y_2))(K_2'g_1)(z_1)(K_2g_2)(z_2), \mathbf{a}_2^+(x_2) \otimes \mathbf{a}_1^-(x_1)\rangle_{V^{\otimes 2}}\\
&\qquad \times \sigma(dx_1)\sigma(dx_2) + \int_X \langle \varphi_1(y), \varphi_2(y)\rangle (K_2'g_1)(z)(K_2g_2)(z)\sigma(dx)\\
&\qquad + \int_{X^2} \langle \widetilde{C}(y_1, y_2)(\varphi_1(y_1) \otimes \varphi_2(y_2))(K_2'g_1)(z_1)(K_1g_2)(z_2), \mathbf{a}_1^-(x_2) \otimes \mathbf{a}_2^-(x_1)\rangle_{V^{\otimes 2}}\\
&\qquad \times \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_{X^2} \langle \widetilde{C}(y_1, y_2)(\varphi_1(y_1) \otimes \varphi_2(y_2))(K_1'g_1)(z_1)(K_2g_2)(z_2), \mathbf{a}_2^+(x_2) \otimes \mathbf{a}_1^+(x_1)\rangle_{V^{\otimes 2}}\\
&\qquad \times \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_{X^2} \langle \widetilde{C}(y_1, y_2)(\varphi_1(y_1) \otimes \varphi_2(y_2))(K_1'g_1)(z_1)(K_1g_2)(z_2), \mathbf{a}_1^-(x_2) \otimes \mathbf{a}_1^+(x_1)\rangle_{V^{\otimes 2}}\\
&\qquad \times \sigma(dx_1)\sigma(dx_2) - \varkappa \int_X \langle \varphi_1(y), \varphi_2(y)\rangle_V (K_1'g_1)(z)(K_1g_2)(z)\sigma(dx)\\
&\quad= \int_{X^2} \langle \widetilde{C}(y_1, y_2)(\varphi_1(y_1) \otimes \varphi_2(y_2))\, g_1(z_1)g_2(z_2), A^+(x_2) \otimes A^-(x_1)\rangle_{V^{\otimes 2}}\, \sigma(dx_1)\sigma(dx_2)\\
&\qquad + \int_Y \langle \varphi_1(y), \varphi_2(y)\rangle_V\, \nu(dy)\bigg(\int_Z (K_2'g_1)(z)(K_2g_2)(z)\mu(dz)
\end{aligned}
$$

$$
\begin{aligned}
&- \varkappa \int_Z (K_1' g_1)(z)(K_1 g_2)(z)\mu(dz)\Big) \\
&= \int_{X^2} \langle \widetilde{C}(x_1,x_2)\big(f_1(x_1)\otimes f_2(x_2)\big), A^+(x_2)\otimes A^-(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2) \\
&\quad + \int_Y \langle \varphi_1(y),\varphi_2(y)\rangle_V\,\nu(dy)\,\big((K_2 g_2, K_2 J g_1)_{L^2(Z,\mu)} - \varkappa (K_1 g_2, K_1 J g_1)_{L^2(Z,\mu)}\big) \\
&= \int_{X^2} \langle \widetilde{C}(x_1,x_2)\big(f_1(x_1)\otimes f_2(x_2)\big), A^+(x_2)\otimes A^-(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2) \\
&\quad + \int_Y \langle \varphi_1(y),\varphi_2(y)\rangle_V \nu(dy)\big((K_2^2 - \varkappa K_1^2)g_2, Jg_1\big)_{L^2(Z,\mu)} \\
&= \int_{X^2} \langle \widetilde{C}(x_1,x_2) f_1(x_1)\otimes f_2(x_2), A^+(x_2)\otimes A^-(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2) \\
&\quad + \int_Y \langle \varphi_1(y),\varphi_2(y)\rangle_V\,\nu(dy)\int_Z g_2(z)g_1(z)\mu(dz) \\
&= \int_{X^2} \langle \widetilde{C}(x_1,x_2)\big(f_1(x_1)\otimes f_2(x_2)\big), A^+(x_2)\otimes A^-(x_1)\rangle_{V^{\otimes 2}}\,\sigma(dx_1)\sigma(dx_2) \\
&\quad + \int_X \langle f_1(x), f_2(x)\rangle_V\,\sigma(dx). \qquad \square
\end{aligned}
$$

Next, similarly to Section 3.5, we define operators

$$\int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), A^{\sharp_1}(x_1)\otimes\cdots\otimes A^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n) \tag{4.51}$$

for each $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$ and $\sharp_1,\dots,\sharp_n \in \{+,-\}$. These operators belong to the **C**-MCR algebra $\mathbf{A}$. Let $\mathbb{A}$ be the unital $*$-algebra generated by the elements of the form (4.51). Thus, $\mathbb{A}$ is a $*$-sub-algebra of $\mathbf{A}$. We will keep the notation $\boldsymbol{\tau}$ for the restriction to $\mathbb{A}$ of the Fock state $\boldsymbol{\tau}$ on $\mathbf{A}$.

Theorem 4.25. *For $\sharp_1,\dots,\sharp_n \in \{+,-\}$ and $f^{(n)} \in \mathfrak{F}(X^n \to V^{\otimes n})$, denote*

$$\Phi^{(n)}(f^{(n)};\sharp_1,\dots,\sharp_n) = \int_{X^n} \langle f^{(n)}(x_1,\dots,x_n), A^{\sharp_1}(x_1)\otimes\cdots\otimes A^{\sharp_n}(x_n)\rangle_{V^{\otimes n}}\,\sigma(dx_1)\cdots\sigma(dx_n).$$

Then the $\Phi^{(n)}(f^{(n)};\sharp_1,\dots,\sharp_n)$ satisfy conditions (4.5), (4.6), (4.7) *and* (4.9). *Hence, the $*$-algebra $\mathbb{A}$ can be considered as a C-MCR algebra.*

Furthermore, the state $\boldsymbol{\tau}$ on $\mathbb{A}$ is gauge-invariant quasi-free and the corresponding complex-valued measure $\rho^{(2)}[g_1,g_2]$ on Y^2 is given by

$$\rho^{(2)}[g_1,g_2](dy_1dy_2) = \nu^{(2)}(dy_1dy_2)\int_Z (Kg_1)(z)g_2(z)\mu(dz). \tag{4.52}$$

Proof. The fact that $\mathbb{A}$ is a C-MCR algebra easily follows from Proposition 4.24.

Next, similarly to Lemma 3.30, we conclude that for each $n \in \mathbb{N}$ and $\sharp_1, \dots, \sharp_n \in \{+,-\}$, the functional $\boldsymbol{\tau}^{(n)}_{\sharp_1,\dots,\sharp_n}$ is continuous on $C_0(Y^n \to V^{\otimes n}) \times \mathcal{G}^n$.

Similarly to (3.56), we have, for any $\varphi_1, \varphi_2 \in C_0(Y \to V)$ and $g_1, g_2 \in \mathcal{G}$,

$$\begin{aligned}
&\boldsymbol{\tau}(A^+(\varphi_1 \otimes g_1)A^-(\varphi_2 \otimes g_2)) \\
&\quad = \boldsymbol{\tau}\bigg(\int_X \langle \varphi_1(y_1)(K_1 g_1)(z_1), \mathbf{a}_1^-(x_1)\rangle_V \, \sigma(dx_1) \int_X \langle \varphi_2(y_2)(K_1' g_2)(z_2), \mathbf{a}_1^+(x_2)\rangle_V \, \sigma(dx_2) \bigg) \\
&\quad = \int_Y \langle \varphi_1, \varphi_2\rangle_V \, \nu(dy) \int_Z (K_1 g_1)(z)(K_1' g_2)(z)\mu(dz) \\
&\quad = \int_{Y^2} \langle \varphi(y_1), \varphi_2(y_2)\rangle_V \, \nu^{(2)}(dy_1 dy_2) \int_Z (K g_1)(z) g_2(z)\mu(dz) \\
&\quad = \int_{Y^2} \langle \varphi(y_1), \varphi_2(y_2)\rangle_V \, \rho^{(2)}[g_1, g_2](dy_1, dy_2),
\end{aligned}$$

where the measure $\rho^{(2)}[g_1, g_2]$ is given by (4.52).

Now, by Corollary 4.17 (ii), we have, for any $\varphi_1, \dots, \varphi_{m+n} \in C_0(Y \to V)$ and $g_1, \dots, g_{m+n} \in \mathcal{G}$,

$$\begin{aligned}
&\boldsymbol{\tau}\left(A^+(\varphi_1 \otimes g_1) \cdots A^+(\varphi_m \otimes g_m) A^-(\varphi_{m+1} \otimes g_{m+1}) \cdots A^-(\varphi_{m+n} \otimes g_{m+n})\right) \\
&\quad = \boldsymbol{\tau}\bigg(\int_X \langle \varphi_1(y_1)(K_1 g_1)(z_1), \mathbf{a}_1^-(x_1)\rangle_V \, \sigma(dx_1) \cdots \int_X \langle \varphi_m(y_m)(K_1 g_m)(z_m), \mathbf{a}_1^-(x_m)\rangle_V \, \sigma(dx_m) \\
&\qquad \int_X \langle \varphi_{m+1}(y_{m+1})(K_1' g_{m+1})(z_{m+1}), \mathbf{a}_1^-(x_{m+1})\rangle_V \, \sigma(dx_{m+1}) \cdots \\
&\qquad \int_X \langle \varphi_{m+n}(y_{m+n})(K_1' g_{m+n})(z_{m+n}), \mathbf{a}_1^-(x_{m+n})\rangle_V \, \sigma(dx_{m+n}) \bigg) \\
&\quad = \delta_{mn} \sum_{\xi \in \mathcal{S}^{(2n)}} \int_{Y^{2n}} \bigotimes_{\substack{\{i,j\} \in \xi \\ i<j}} \rho^{(2)}[g_i, g_j](dy_i, dy_j) \\
&\qquad\qquad \mathbb{T}^{(2n)} C(\xi; y_1, \dots, y_{2n}) \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \cdots \otimes \varphi_{2n}(y_{2n}).
\end{aligned}$$

Hence, the state $\boldsymbol{\tau}$ is gauge-invariant quasi-free. □

Theorem 4.26. *Assume additionally that, for all $y_1, y_2 \in Y$,*

$$\widetilde{C}(y_2, y_1) = C(y_1, y_2).$$

Then the state $\boldsymbol{\tau}$ on the C-MCR-algebra $\mathbb{A}$ is strongly quasi-free and the corresponding complex-valued measure $\lambda^{(2)}[g_1, g_2]$ on Y^2 is given by (3.57).

Proof. For any $\varphi \in C_0(Y \to V_{\mathbb{R}})$ and $g \in \mathcal{G}$, we define

$$\begin{aligned}\mathcal{A}^+(\varphi \otimes g) &= \int_X \langle \varphi(y)(K_2 g)(z), \mathbf{a}_2^+(x)\rangle_V \,\sigma(dx) + \int_X \langle \varphi(y)(JK_1 g)(z), \mathbf{a}_1^+(x)\rangle_V \,\sigma(dx)\\ &= \int_{\mathbf{X}} \langle \varphi(y)(\mathbf{K}^+ g)(z), \mathbf{a}^+(x)\rangle_V \,\sigma(dx),\\ \mathcal{A}^-(\varphi \otimes g) &= \int_X \langle \varphi(y)(K_1 g)(z), \mathbf{a}_1^-(x)\rangle_V \,\sigma(dx) + \int_X \langle \varphi(y)(JK_2 g)(z), \mathbf{a}_2^-(x)\rangle_V \,\sigma(dx)\\ &= \int_{\mathbf{X}} \langle \varphi(y)(\mathbf{K}^- g)(z), \mathbf{a}^-(x)\rangle_V \,\sigma(dx),\end{aligned}$$

where $\mathbf{K}^+ g$ and $\mathbf{K}^- g$ are defined by (3.62). Then

$$B(\varphi \otimes g) = \mathcal{A}^+(\varphi \otimes g) + \mathcal{A}^-(\varphi \otimes g).$$

Hence, by Theorem 4.12, we have, for any $\varphi_1, \dots, \varphi_{2n} \in C_0(Y \to \mathbb{R})$ and $g_1, \dots, g_{2n} \in \mathcal{G}$,

$$\boldsymbol{\tau}\big(B(\varphi_1 \otimes g_1) \cdots B(\varphi_{2n-1} \otimes g_{2n-1})\big) = 0,$$

and

$$\begin{aligned}\boldsymbol{\tau}\big(B(\varphi_1 \otimes g_1) \cdots B(\varphi_{2n} \otimes g_{2n})\big) &= \sum_{\xi \in \mathcal{P}_2^{(2n)}} \int_{\mathbf{X}^{2n}} \bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}} \Big(\sigma^{(2)}(dx_i\,dx_j)(\mathbf{K}^- g_i)(z_i)(\mathbf{K}^+ g_j)(z_j)\Big)\\ &\quad \times \mathbb{T}^{(2n)}(\xi) C(\xi; y_1, \dots, y_{2n}) \varphi_1(y_1) \otimes \cdots \otimes \varphi_{2n}(y_{2n})\\ &= \sum_{\xi \in \mathcal{P}_2^{(2n)}} \int_{Y^{2n}} \bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}} \nu^{(2)}(dy_i\,dy_j) \bigg(\int_{\mathbf{Z}} \mu(dz_i)(\mathbf{K}^- g_i)(z_i)(\mathbf{K}^+ g_j)(z_j)(dz_i)\bigg)\\ &\quad \times \mathbb{T}^{(2n)}(\xi) C(\xi; y_1, \dots, y_{2n}) \varphi_1(y_1) \otimes \cdots \otimes \varphi_{2n}(y_{2n})\\ &= \sum_{\xi \in \mathcal{P}_2^{(2n)}} \int_{Y^{2n}} \bigotimes_{\substack{\{i,j\}\in\xi\\ i<j}} \lambda^{(2)}[g_i, g_j](dy_i\,dy_j) \mathbb{T}^{(2n)}(\xi) C(\xi; y_1, \dots, y_{2n}) \varphi_1(y_1) \otimes \cdots \otimes \varphi_{2n}(y_{2n}),\end{aligned}$$

where in the last equality we used (3.65). □

4.5 Examples

In this section we will consider two classes of gauge-invariant quasi-free and strongly quasi-free states on the C-MCR algebra.

4.5.1 First class of examples

This class of examples is obtained by a 'lifting' of gauge-invariant quasi-free states on a Q-ACR algebra.

Let V be a space of dimension n ($n \in \mathbb{N}$, $n \geq 2$). Let us fix an orthonormal basis $(e_i)_{i=1,\dots,n}$ of V that is real, i.e.,

$$Je_i = e_i, \qquad i = 1, \dots, n. \tag{4.53}$$

Assume that, for each $i, j \in \{1, \dots, n\}$, we fix a continuous function

$$Q(\cdot, \cdot, i, j) : Y^2 \to \mathbb{C}$$

such that

$$|Q(y_1, y_2, i, j)| = 1, \qquad (y_1, y_2) \in Y^2. \tag{4.54}$$

Furthermore, we assume that

$$Q(y_1, y_2, i, j) = \overline{Q(y_2, y_1, j, i)}. \tag{4.55}$$

Remark 4.27. Formulas (4.54) and (4.55) mean that Q is a Hermitian function on

$$(Y \times \{1, 2, \dots, n\})^2$$

of modulus 1.

We also assume that theree exists $\varkappa \in \{-1, 1\}$ such that

$$Q(y, y, i, i) = \varkappa \quad \text{for all } y \in Y \text{ and } i \in \{1, \dots, n\}. \tag{4.56}$$

For any $(y_1, y_2) \in Y^2$, we define an operator $C(y_1, y_2) \in \mathcal{L}(V^{\otimes 2})$ by

$$C(y_1, y_2)\, e_i \otimes e_j = Q(y_1, y_2, i, j)\, e_j \otimes e_i, \qquad i, j \in \{1, \dots, n\}. \tag{4.57}$$

Lemma 4.28. *Let a map $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ be defined by* (4.57). *Then C satisfies Assumption 4.2.*

Proof. Since $Q(\cdot,\cdot,i,j)$ is a continuous function on Y^2 for all $i,j\in\{1,\dots,n\}$, the continuity of C follows from (4.57). That $C(y_1,y_2)$ is a unitary operator in $V^{\otimes 2}$ follows from (4.54) and (4.57).

Next, by (4.57), we have for any $i,j,k,l\in\{1,\dots,n\}$,

$$\begin{aligned}(C(y_1,y_2)e_i\otimes e_j, e_k\otimes e_l) &= Q(y_1,y_2,i,j)(e_j\otimes e_i, e_k\otimes e_l)\\ &= Q(y_1,y_2,i,j)\delta_{jk}\delta_{il}\\ &= Q(y_1,y_2,l,k)\delta_{jk}\delta_{il}\\ &= \left(e_i\otimes e_j, \overline{Q(y_1,y_2,l,k)}e_l\otimes e_k\right)\end{aligned}$$

Hence, by (4.55),

$$\begin{aligned}C(y_1,y_2)^*e_k\otimes e_l &= \overline{Q(y_1,y_2,l,k)}e_l\otimes e_k\\ &= Q(y_2,y_1,k,l)e_l\otimes e_k.\end{aligned}$$

Finally, we check the functional Yang-Baxter equation (4.1). For any $y_1,y_2,y_3\in Y$ and $i,j,k\in\{1,\dots,n\}$, we have

$$\begin{aligned}&C_1(y_1,y_2)C_2(y_1,y_3)C_1(y_2,y_3)e_i\otimes e_j\otimes e_k\\ &\quad= Q(y_2,y_3,i,j)C_1(y_1,y_2)C_2(y_1,y_3)e_j\otimes e_i\otimes e_k\\ &\quad= Q(y_2,y_3,i,j)Q(y_1,y_3,i,k)C_1(y_1,y_2)e_j\otimes e_k\otimes e_i\\ &\quad= Q(y_2,y_3,i,j)Q(y_1,y_3,i,k)Q(y_1,y_2,j,k)e_k\otimes e_j\otimes e_i,\\ &C_2(y_2,y_3)C_1(y_1,y_3)C_2(y_1,y_2)e_i\otimes e_j\otimes e_k\\ &\quad= Q(y_1,y_2,j,k)C_2(y_2,y_3)C_1(y_1,y_3)e_i\otimes e_k\otimes e_j\\ &\quad= Q(y_1,y_2,j,k)Q(y_1,y_3,i,k)C_2(y_2,y_3)e_k\otimes e_i\otimes e_j\\ &\quad= Q(y_1,y_2,j,k)Q(y_1,y_3,i,k)Q(y_2,y_3,i,j)e_k\otimes e_j\otimes e_i.\end{aligned}$$

which proves the statement. ∎

Next, let us calculate the $\widetilde{C}(y_1,y_2)$ and $\widehat{C}(y_1,y_2)$ operators.

Lemma 4.29. *We have, for any $i,j\in\{1,\dots,n\}$,*

$$\widetilde{C}(y_1,y_2)e_i\otimes e_j = Q(y_1,y_2,j,i)e_j\otimes e_i, \tag{4.58}$$

$$\widehat{C}(y_1, y_2)e_i \otimes e_j = Q(y_2, y_1, i, j)e_j \otimes e_i. \tag{4.59}$$

Proof. By (4.45), we have, for any $i, j, k, l \in \{1, \dots, n\}$,

$$\begin{aligned}
(C(y_1, y_2)e_i \otimes e_j, e_k \otimes e_l)_{V^{\otimes 2}} &= Q(y_1, y_2, i, j)(e_j \otimes e_i, e_k \otimes e_l)_{V^{\otimes 2}} \\
&= Q(y_1, y_2, i, j)\, \delta_{jk}\, \delta_{il} \\
&= Q(y_1, y_2, i, k)\, \delta_{jk}\, \delta_{il} \\
&= \left(\widetilde{C}(y_1, y_2)e_k \otimes e_i, e_l \otimes e_j\right)_{V^{\otimes 2}}.
\end{aligned}$$

Hence,

$$\widetilde{C}(y_1, y_2)e_k \otimes e_i = Q(y_1, y_2, i, k)e_i \otimes e_k, \tag{4.60}$$

which implies (4.58).

Next, by (4.43),(4.44), (4.53) and (4.55), we have

$$\begin{aligned}
\widehat{C}(y_1, y_2)e_i \otimes e_j &= \mathbb{S}\, C(y_1, y_2)\, \mathbb{S}\, e_i \otimes e_j \\
&= \mathbb{S}\, C(y_1, y_2)\, e_j \otimes e_i \\
&= \mathbb{S}\, Q(y_1, y_2, j, i)\, e_i \otimes e_j \\
&= \overline{Q(y_1, y_2, j, i)}\, e_j \otimes e_i \\
&= Q(y_2, y_1, i, j)\, e_j \otimes e_i,
\end{aligned}$$

which proves (4.59). □

Lemma 4.30. *The map $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ defined by (4.57) satisfies Assumptions 4.21 and 4.22.*

Proof. Denote

$$\widetilde{Q}(y_1, y_2, i, j) = Q(y_1, y_2, j, i). \tag{4.61}$$

Then, by (4.58),

$$\widetilde{C}(y_1, y_2)e_i \otimes e_j = \widetilde{Q}(y_1, y_2, i, j)e_j \otimes e_i. \tag{4.62}$$

We have, for all $(y_1, y_2) \in Y^2$ and $i, j \in \{1, \dots, n\}$,

$$|\widetilde{Q}(y_1, y_2, i, j)| = 1$$

and by, (4.55),

$$
\begin{aligned}
\overline{\widetilde{Q}(y_2,y_1,j,i)} &= \overline{Q(y_2,y_1,i,j)} \\
&= Q(y_1,y_2,j,i) \\
&= \widetilde{Q}(y_1,y_2,i,j).
\end{aligned}
$$

Hence, part (i) of Assumption 4.21 follows from the proof of Lemma 4.28. Part (ii) of Assumption 4.21 follows from (4.58) and (4.62). Formula $\widehat{C}(y_1,y_2) = C(y_2,y_1)$ follows from (4.55) and (4.59). Similarly, by using (4.61) and (4.62), we see that $\widehat{\widetilde{C}}(y_1,y_2) = \widetilde{C}(y_2,y_1)$.

Next, for each $y \in Y$ and $v^{(2)} \in V^{\otimes 2}$, We have, by (4.56),

$$
\begin{aligned}
\mathrm{Tr}\left(\widetilde{C}(y,y)v^{(2)}\right) &= \sum_{i=1}^{n}\left(\widetilde{C}(y,y)v^{(2)}, e_i\otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{n}\left(v^{(2)}, \widetilde{C}(y,y)^{*}e_i\otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{n}\left(v^{(2)}, \widetilde{C}(y,y)e_i\otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{n}\left(v^{(2)}, Q(y,y,i,i)e_i\otimes e_i\right)_{V^{\otimes 2}} \\
&= \varkappa\sum_{i=1}^{n}\left(v^{(2)}, e_i\otimes e_i\right)_{V^{\otimes 2}} \\
&= \varkappa\,\mathrm{Tr}(v^{(2)}).
\end{aligned}
$$

Hence, $C(y_1,y_2)$ satisfies Assumption 4.21.

Next, to check that Assumption 4.22 is satisfied, we define $\mathbf{C} : \mathbf{Y}^2 \to \mathcal{L}\left(V^{\otimes 2}\right)$ by

$$
\mathbf{C}(y_1,y_2)e_i\otimes e_j = \mathbf{Q}(y_1,y_2,i,j)e_j\otimes e_i,
$$

where

$$
\mathbf{Q}(y_1,y_2,i,j) = \begin{cases} Q(y_1,y_2,i,j), & \text{if } y_1,y_2\in Y_1 \text{ or } y_1,y_2\in Y_2, \\ Q(y_2,y_1,j,i), & \text{if } y_1\in Y_1,\ y_2\in Y_2 \text{ or } y_1\in Y_2,\ y_2\in Y_1 \end{cases}
$$

Obviously, for all $y_1, y_2 \in \mathbf{Y}^2$ and $i,j\in\{1,\dots,n\}$, we have

$$
\mathbf{Q}(y_1,y_2,i,j) = \overline{\mathbf{Q}(y_2,y_1,j,i)}.
$$

Hence, Assumption 4.22 follows from the proof of Lemma 4.28. □

Thus, Theorem 4.25 is applicable to this class of examples. Let us write down the commutation relations in this C-MCR-algebra. To this end, take any $\varphi \in C_0(Y \to V)$ and write it in the form

$$\varphi(y) = \sum_{i=1}^{n} \varphi_i(y) e_i$$

where $\varphi_i \in C_0(Y \to \mathbb{C}) = C_0(Y)$. Let $g \in \mathcal{G}$ and denote $f = \varphi \otimes g$. Then

$$f(x) = \sum_{i=1}^{n} f_i(x) e_i,$$

where $f_i(x) = \varphi_i(y) g(z)$. Then similarly, to (4.50), we write, for $\sharp \in \{+, -\}$,

$$\begin{aligned} A^\sharp(f) &= \int_X \langle f(x), A^\sharp(x) \rangle_V \sigma(dx) \\ &= \sum_{i=1}^{n} \int_X f_i(x) A_i^\sharp(x) \sigma(dx). \end{aligned}$$

In words, $A_i^\sharp(x)$ is the ith coordinate of the operator $A^\sharp(x)$ in the orthonormal basis $(e_i)_{i=1,\dots,n}$.

Then, by (2.132) and (4.57)–(4.59), we have the following proposition.

Proposition 4.31. *Let the map $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ be defined by* (4.57)*. Then Theorem 4.25 is applicable to this choice of $C(y_1, y_2)$. Furthermore, the corresponding commutation relations have the form*

$$\begin{aligned} A_i^+(x_1) A_j^+(x_2) &= Q(x_2, x_1, i, j) A_j^+(x_2) A_i^+(x_1), \\ A_i^-(x_1) A_j^-(x_2) &= Q(x_2, x_1, i, j) A_j^-(x_2) A_i^-(x_1), \\ A_i^-(x_1) A_j^+(x_2) &= \delta_{ij}\, \delta(x_1, x_2) + Q(x_1, x_2, j, i) A_j^+(x_2) A_i^-(x_1). \end{aligned}$$

for $i, j \in \{1, \dots, n\}$ and $x_1, x_2 \in X$. Here $\delta(x_1, x_2)$ is defined by (2.133)*.*

Finally, we note that, by Theorem 4.26, the state $\boldsymbol{\tau}$ is strongly quasi-free if $\widetilde{C}(y_2, y_1) = C(y_1, y_2)$. By (4.55), (4.57) and (4.58), this condition means that for all $y_1, y_2 \in Y$ and $i, j \in \{1, \dots, n\}$, $Q(y_1, y_2, i, j)$ is real, i.e., for all $i, j \in \{1, \dots, n\}$,

$$Q(y_1, y_2, i, j) = q_{ij}, \quad (y_1, y_2) \in Y^2,$$

where $q_{ij} \in \{-1,+1\}$. Thus, in such a case, the statistics is completely determined by the n^2 numbers $(q_{ij})_{i,j\in\{1,\dots,n\}}$ with $q_{ij} \in \{-1,+1\}$ and

$$q_{11} = \dots = q_{nn} = \varkappa.$$

4.5.2 Second class of examples

Let V be a space of dimension 2. Let us fix a real orthonormal basis $\{e_1, e_2\}$ of V. Fix two continuous functions

$$Q_i : Y^2 \to \mathbb{C}, \quad i = 1, 2$$

such that, for all $(y_1, y_2) \in Y^2$,

$$|Q_i(y_1, y_2)| = 1, \tag{4.63}$$

$$Q_i(y_1, y_2) = \overline{Q_i(y_2, y_1)}. \tag{4.64}$$

We assume that, for some $\varkappa \in \{-1,+1\}$ and for all $y \in Y$,

$$Q_2(y, y) = \varkappa. \tag{4.65}$$

(This assumption is automatically satisfied when Y is connected.) Define a permutation $\varphi \in S_2$ by $\varphi(1) = 2$, $\varphi(2) = 1$.

Following [9, Example 4.9], for any $(y_1, y_2) \in Y^2$, we define an operator $C(y_1, y_2) \in \mathcal{L}(V^{\otimes 2})$ by

$$\begin{aligned} C(y_1, y_2)e_i \otimes e_i &= Q_1(y_1, y_2)e_{\varphi(i)} \otimes e_{\varphi(i)}, \\ C(y_1, y_2)e_i \otimes e_{\varphi(i)} &= Q_2(y_1, y_2)e_i \otimes e_{\varphi(i)}, \qquad i = 1, 2. \end{aligned} \tag{4.66}$$

In $V^{\otimes 2}$, fix the orthonormal basis

$$e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2.$$

Then, by (4.66), the matrix of $C(y_1, y_2)$ has the form

$$C(y_1, y_2) = \begin{pmatrix} 0 & 0 & 0 & Q_1(y_1, y_2) \\ 0 & Q_2(y_1, y_2) & 0 & 0 \\ 0 & 0 & Q_2(y_1, y_2) & 0 \\ Q_1(y_1, y_2) & 0 & 0 & 0 \end{pmatrix}. \tag{4.67}$$

Lemma 4.32. *Let a map $C : Y^2 \to \mathcal{L}(Y^2)$ be defined by* (4.66). *Then C satisfies Assumption 4.2.*

Proof. Since $Q_1(\cdot,\cdot)$ and $Q_2(\cdot,\cdot)$ are continuous functions on Y^2, the continuity of C follows from (4.66). That $C(y_1, y_2)$ is a unitary operator in $V^{\otimes 2}$ follows from (4.63) and (4.66). By (4.67), we have

$$C^*(y_1,y_2) = \begin{pmatrix} 0 & 0 & 0 & \overline{Q_1(y_1,y_2)} \\ 0 & \overline{Q_2(y_1,y_2)} & 0 & 0 \\ 0 & 0 & \overline{Q_2(y_1,y_2)} & 0 \\ \overline{Q_1(y_1,y_2)} & 0 & 0 & 0 \end{pmatrix}.$$

Hence, by (4.64), $C^*(y_1, y_2) = C(y_2, y_1)$. The fact that $C(y_1, y_2)$ satisfies the functional Yang-Baxter equation (4.1) will follow from the proof of Lemma 4.35 below. □

Next, we find the $\widetilde{C}(y_1, y_2)$ and $\widehat{C}(y_1, y_2)$ operators.

Lemma 4.33. *We have, for $i = 1, 2$,*

$$\widetilde{C}(y_1,y_2)e_i \otimes e_i = Q_2(y_1,y_2)e_{\varphi(i)} \otimes e_{\varphi(i)}, \tag{4.68}$$

$$\widetilde{C}(y_1,y_2)e_i \otimes e_{\varphi(i)} = Q_1(y_1,y_2)e_i \otimes e_{\varphi(i)}, \quad i = 1,2, \tag{4.69}$$

$$\widehat{C}(y_1,y_2) = C(y_2,y_1). \tag{4.70}$$

Proof. By (4.45), for any $i, j, k, l \in \{1, 2\}$,

$$\left(\widetilde{C}(y_1,y_2)e_i \otimes e_j, e_k \otimes e_l\right)_{V^{\otimes 2}} = (C(y_1,y_2)e_j \otimes e_l, e_i \otimes e_k)_{V^{\otimes 2}}. \tag{4.71}$$

Hence, for $i = 1, 2$,

$$\begin{aligned} \left(\widetilde{C}(y_1,y_2)e_i \otimes e_i, e_{\varphi(i)} \otimes e_{\varphi(i)}\right)_{V^{\otimes 2}} &= \left(C(y_1,y_2)e_i \otimes e_{\varphi(i)}, e_i \otimes e_{\varphi(i)}\right)_{V^{\otimes 2}} \\ &= Q_2(y_1,y_2), \\ \left(\widetilde{C}(y_1,y_2)e_i \otimes e_{\varphi(i)}, e_i \otimes e_{\varphi(i)}\right)_{V^{\otimes 2}} &= \left(C(y_1,y_2)e_{\varphi(i)} \otimes e_{\varphi(i)}, e_i \otimes e_i\right)_{V^{\otimes 2}} \\ &= Q_1(y_1,y_2), \end{aligned}$$

and $\left(\widetilde{C}(y_1,y_2)e_i \otimes e_j, e_k \otimes e_l\right)_{V^{\otimes 2}} = 0$ for all other choices of i, j, k, l.

Next, by (4.43) and (4.44), for $i = 1, 2$,

$$
\begin{aligned}
\widehat{C}(y_1, y_2)e_i \otimes e_i &= \mathbb{S}C(y_1 y_2)e_i \otimes e_i \\
&= \mathbb{S}Q_1(y_1, y_2)e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&= \overline{Q(y_1, y_2)}e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&= Q_1(y_2, y_1)e_{\varphi(i)} \otimes e_{\varphi(i)}, \\
\widehat{C}(y_1, y_2)e_i \otimes e_{\varphi(i)} &= \mathbb{S}C(y_1, y_2)e_{\varphi(i)} \otimes e_i \\
&= \mathbb{S}Q_2(y_1, y_2)e_{\varphi(i)} \otimes e_i \\
&= \overline{Q_2(y_1, y_2)}e_i \otimes e_{\varphi(i)} \\
&= Q_2(y_2, y_1)e_i \otimes e_{\varphi(i)},
\end{aligned}
$$

which proves (4.70). □

Lemma 4.34. *The map $C : Y^2 \to \mathcal{L}\left(V^{\otimes 2}\right)$ defined by* (4.66) *satisfies Assumption 4.21.*

Proof. Part (i) of Assumption 4.21 follows from (4.68) and (4.69) and the proof of Lemma 4.32. Parts (ii) and (iii) of Assumption 4.21 follows from Lemma 4.33. Finally, we prove part (iv) of Assumption 4.21: for all $y \in Y$ and $v^{(2)} \in V^{\otimes 2}$,

$$
\begin{aligned}
\operatorname{Tr}\left(\widetilde{C}(y, y)v^{(2)}\right) &= \sum_{i=1}^{2} \left(\widetilde{C}(y, y)v^{(2)}, e_i \otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{2} \left(v^{(2)}, \widetilde{C}(y, y)^* e_i \otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{2} \left(v^{(2)}, \widetilde{C}(y, y)e_i \otimes e_i\right)_{V^{\otimes 2}} \\
&= \sum_{i=1}^{2} \left(v^{(2)}, Q_2(y, y)e_{\varphi(i)} \otimes e_{\varphi(i)}\right)_{V^{\otimes 2}} \\
&= \varkappa \sum_{i=1}^{2} \left(v^{(2)}, e_{\varphi(i)} \otimes e_{\varphi(i)}\right)_{V^{\otimes 2}} \\
&= \varkappa \sum_{i=1}^{2} \left(v^{(2)}, e_i \otimes e_i\right)_{V^{\otimes 2}} \\
&= \varkappa \operatorname{Tr}\left(v^{(2)}\right).
\end{aligned}
$$

□

Lemma 4.35. *The map $C : Y^2 \to \mathcal{L}\left(V^{\otimes 2}\right)$ defined by* (4.66) *satisfies Assumption 4.22.*

Proof. By (4.46), (4.66), (4.68), and (4.69) we have, for $i = 1, 2$ and $(y_1, y_2) \in \mathbf{Y}^2$,

$$\mathbf{C}(y_1, y_2)e_i \otimes e_i = \mathbf{Q}^{(1)}(y_1, y_2)e_{\varphi(i)} \otimes e_{\varphi(i)},$$
$$\mathbf{C}(y_1, y_2)e_i \otimes e_{\varphi(i)} = \mathbf{Q}^{(2)}(y_1, y_2)e_i \otimes e_{\varphi(i)},$$

where

$$\mathbf{Q}^{(1)}(y_1, y_2) = \begin{cases} Q_1(y_1, y_2) & \text{if } y_1, y_2 \in Y_1 \text{ or } y_1, y_2 \in Y_2, \\ Q_2(y_2, y_1) & \text{if } y_1 \in Y_1,\ y_2 \in Y_2 \text{ or } y_1 \in Y_1,\ y_2 \in Y_1, \end{cases}$$
$$\mathbf{Q}^{(2)}(y_1, y_2) = \begin{cases} Q_2(y_1, y_2) & \text{if } y_1, y_2 \in Y_1 \text{ or } y_1, y_2 \in Y_2, \\ Q_1(y_2, y_1) & \text{if } y_1 \in Y_1,\ y_2 \in Y_2 \text{ or } y_1 \in Y_1,\ y_2 \in Y_1. \end{cases}$$

Then, for any $y_1, y_2, y_3 \in \mathbf{Y}$ and $i = 1, 2$,

$$\begin{aligned}
&\mathbf{C}_1(y_1, y_2)\mathbf{C}_2(y_1, y_3)\mathbf{C}_1(y_2, y_3)e_i \otimes e_i \otimes e_i \\
&\quad = \mathbf{C}_1(y_1, y_2)\mathbf{C}_2(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_i \\
&\quad = \mathbf{C}_1(y_1, y_2)\mathbf{Q}^{(2)}(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_i \\
&\quad = \mathbf{Q}^{(1)}(y_1, y_2)\mathbf{Q}^{(2)}(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_i \otimes e_i \otimes e_i, \\
&\mathbf{C}_2(y_2, y_3)\mathbf{C}_1(y_1, y_3)\mathbf{C}_2(y_1, y_2)e_i \otimes e_i \otimes e_i \\
&\quad = \mathbf{C}_2(y_2, y_3)\mathbf{C}_1(y_1, y_3)\mathbf{Q}^{(1)}(y_1, y_2)e_i \otimes e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&\quad = \mathbf{C}_2(y_2, y_3)\mathbf{Q}^{(2)}(y_1, y_3)\mathbf{Q}^{(1)}(y_1, y_2)e_i \otimes e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&\quad = \mathbf{Q}^{(1)}(y_2, y_3)\mathbf{Q}^{(2)}(y_1, y_3)\mathbf{Q}^{(1)}(y_1, y_2)e_i \otimes e_i \otimes e_i, \\
&\mathbf{C}_1(y_1, y_2)\mathbf{C}_2(y_1, y_3)\mathbf{C}_1(y_2, y_3)e_i \otimes e_i \otimes e_{\varphi(i)} \\
&\quad = \mathbf{C}_1(y_1, y_2)\mathbf{C}_2(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&\quad = \mathbf{C}_1(y_1, y_2)\mathbf{Q}^{(1)}(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_{\varphi(i)} \otimes e_i \otimes e_i \\
&\quad = \mathbf{Q}^{(2)}(y_1, y_2)\mathbf{Q}^{(1)}(y_1, y_3)\mathbf{Q}^{(1)}(y_2, y_3)e_{\varphi(i)} \otimes e_i \otimes e_i, \\
&\mathbf{C}_2(y_2, y_3)\mathbf{C}_1(y_1, y_3)\mathbf{C}_2(y_1, y_2)e_i \otimes e_i \otimes e_{\varphi(i)} \\
&\quad = \mathbf{C}_2(y_2, y_3)\mathbf{C}_1(y_1, y_3)\mathbf{Q}^{(2)}(y_1, y_2)e_i \otimes e_i \otimes e_{\varphi(i)} \\
&\quad = \mathbf{C}_2(y_2, y_3)\mathbf{Q}^{(1)}(y_1, y_3)\mathbf{Q}^{(2)}(y_1, y_2)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_{\varphi(i)} \\
&\quad = \mathbf{Q}^{(1)}(y_2, y_3)\mathbf{Q}^{(1)}(y_1, y_3)\mathbf{Q}^{(2)}(y_1, y_2)e_{\varphi(i)} \otimes e_i \otimes e_i,
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{C}_1(y_1,y_2)\mathbf{C}_2(y_1,y_3)\mathbf{C}_1(y_2,y_3)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{C}_1(y_1,y_2)\mathbf{C}_2(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{C}_1(y_1,y_2)\mathbf{Q}^{(2)}(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{Q}^{(2)}(y_1,y_2)\mathbf{Q}^{(2)}(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_i \otimes e_{\varphi(i)} \otimes e_i,\\
&\mathbf{C}_2(y_2,y_3)\mathbf{C}_1(y_1,y_3)\mathbf{C}_2(y_1,y_2)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{C}_2(y_2,y_3)\mathbf{C}_1(y_1,y_3)\mathbf{Q}^{(2)}(y_1,y_2)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{C}_2(y_2,y_3)\mathbf{Q}^{(2)}(y_1,y_3)\mathbf{Q}^{(2)}(y_1,y_2)e_i \otimes e_{\varphi(i)} \otimes e_i\\
&\qquad = \mathbf{Q}^{(2)}(y_2,y_3)\mathbf{Q}^{(2)}(y_1,y_3)\mathbf{Q}^{(2)}(y_1,y_2)e_i \otimes e_{\varphi(i)} \otimes e_i,\\
&\mathbf{C}_1(y_1,y_2)\mathbf{C}_2(y_1,y_3)\mathbf{C}_1(y_2,y_3)e_{\varphi(i)} \otimes e_i \otimes e_i\\
&\qquad = \mathbf{C}_1(y_1,y_2)\mathbf{C}_2(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_{\varphi(i)} \otimes e_i \otimes e_i\\
&\qquad = \mathbf{C}_1(y_1,y_2)\mathbf{Q}^{(1)}(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_{\varphi(i)}\\
&\qquad = \mathbf{Q}^{(1)}(y_1,y_2)\mathbf{Q}^{(1)}(y_1,y_3)\mathbf{Q}^{(2)}(y_2,y_3)e_i \otimes e_i \otimes e_{\varphi(i)},\\
&\mathbf{C}_2(y_2,y_3)\mathbf{C}_1(y_1,y_3)\mathbf{C}_2(y_1,y_2)e_{\varphi(i)} \otimes e_i \otimes e_i\\
&\qquad = \mathbf{C}_2(y_2,y_3)\mathbf{C}_1(y_1,y_3)\mathbf{Q}^{(1)}(y_1,y_2)e_{\varphi(i)} \otimes e_{\varphi(i)} \otimes e_{\varphi(i)}\\
&\qquad = \mathbf{C}_2(y_2,y_3)\mathbf{Q}^{(1)}(y_1,y_3)\mathbf{Q}^{(1)}(y_1,y_2)e_i \otimes e_i \otimes e_{\varphi(i)}\\
&\qquad = \mathbf{Q}^{(2)}(y_2,y_3)\mathbf{Q}^{(1)}(y_1,y_3)\mathbf{Q}^{(1)}(y_1,y_2)e_i \otimes e_i \otimes e_{\varphi(i)}. \qquad \square
\end{aligned}
$$

Thus, similarly to Subsection 4.5.1, formulas (4.66) and Lemma 4.33 imply

Proposition 4.36. *Let the map $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ be defined by (4.66). Then Theorem 4.25 is applicable to this choice of $C(y_1, y_2)$. Furthermore, the corresponding commutation relations have the form:*

$$
\begin{aligned}
A_i^+(x_1)A_i^+(x_2) &= Q_1(x_2,x_1)A_{\varphi(i)}^+(x_2)A_{\varphi(i)}^+(x_1),\\
A_i^+(x_1)A_{\varphi(i)}^+(x_2) &= Q_2(x_2,x_1)A_i^+(x_2)A_{\varphi(i)}^+(x_1),\\
A_i^-(x_1)A_i^-(x_2) &= Q_1(x_2,x_1)A_{\varphi(i)}^-(x_2)A_{\varphi(i)}^-(x_1),\\
A_i^-(x_1)A_{\varphi(i)}^-(x_2) &= Q_2(x_2,x_1)A_i^-(x_2)A_{\varphi(i)}(x_1),\\
A_i^-(x_1)A_i^+(x_2) &= \delta(x_1,x_2) + Q_2(x_1,x_2)A_{\varphi(i)}^+(x_2)A_{\varphi(i)}^-(x_1),\\
A_i^-(x_1)A_{\varphi(i)}^+(x_2) &= Q_1(x_1,x_2)A_i^+(x_2)A_{\varphi(i)}^-(x_1),
\end{aligned}
$$

for $i = 1, 2$ *and* $(x_1, x_2) \in X^2$.

Corollary 4.37. *Let* $Q : Y^2 \to \mathbb{C}$ *be a continuous function such that*

$$Q(y_1, y_2) = \overline{Q(y_2, y_1)}, \quad |Q(y_1, y_2)| = 1$$

for all $(y_1, y_2) \in Y^2$. *Define* $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ *by*

$$C(y_1, y_2)e_i \otimes e_i = Q(y_1, y_2)e_{\varphi(i)} \otimes e_{\varphi(i)},$$
$$C(y_1, y_2)e_i \otimes e_{\varphi(i)} = Q(y_2, y_1)e_i \otimes e_{\varphi(i)}.$$

Then Theorem 4.25 is applicable to this choice of $C(y_1, y_2)$. *The corresponding commutation relations have the form:*

$$A_i^+(x_1)A_i^+(x_2) = Q(x_2, x_1)A_{\varphi(i)}^+(x_2)A_{\varphi(i)}^+(x_1),$$
$$A_i^+(x_1)A_{\varphi(i)}^+(x_2) = Q(x_1, x_2)A_i^+(x_2)A_{\varphi(i)}^+(x_1),$$
$$A_i^-(x_1)A_i^-(x_2) = Q(x_2, x_1)A_{\varphi(i)}^-(x_2)A_{\varphi(i)}^-(x_1),$$
$$A_i^-(x_1)A_{\varphi(i)}^-(x_2) = Q(x_1, x_2)A_i^-(x_2)A_{\varphi(i)}^-(x_1),$$
$$A_i^-(x_1)A_i^+(x_2) = \delta(x_1, x_2) + Q(x_2, x_1)A_{\varphi(i)}^+(x_2)A_{\varphi(i)}^-(x_1),$$
$$A_i^-(x_1)A_{\varphi(i)}^+(x_2) = Q(x_1, x_2)A_i^+(x_2)A_{\varphi(i)}^-(x_1),$$

for $i = 1, 2$ *and* $(x_1, x_2) \in X^2$. *Furthermore, the corresponding state* $\boldsymbol{\tau}$ *is strongly quasi-free.*

Proof. By Theorem 4.26, formulas (4.66) and Lemma 4.33, the state $\boldsymbol{\tau}$ from Proposition 4.36 is strongly quasi-free when

$$Q_1(y_1, y_2) = Q_2(y_2, y_1).$$

Thus, setting $Q(y_1, y_2) = Q_1(y_1, y_2)$, we get $Q_2(y_1, y_2) = Q(y_2, y_1)$. □

Remark 4.38. It should be noted that, even in the case where the dimension of V is 2, there are plenty of examples of maps $C : Y^2 \to \mathcal{L}(V^{\otimes 2})$ that satisfy Assumption 4.2 but do not satisfy Assumption 4.21. For example, consider the trivial case where $C(y_1, y_2) = \mathbf{1}$ for all $(y_1, y_2) \in Y^2$. It is obvious this $C(y_1, y_2)$ satisfies Assumption 4.2. However, formula (4.71) implies that

$$\widetilde{C}(y_1, y_2) e_1 \otimes e_1 = e_1 \otimes e_1 + e_2 \otimes e_2.$$

Hence, the operator $\widetilde{C}(y_1, y_2)$ is not unitary.

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