Stochastic differential equations with critically irregular drift coefficients

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Abstract This paper is concerned with stochastic differential equations (SDEs for short) with irregular coefficients. By utilising a functional analytic approximation approach, we establish the existence and uniqueness of strong solutions to a class of SDEs with critically irregular drift coefficients in a new critical Lebesgue space, where the element may be only weakly integrable in time. To be more precise, let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable, where $T > 0$ is arbitrarily fixed and $d \geqslant 1$. We consider the following SDE

$$X_t = x + \int_0^t b(s, X_s)ds + W_t, \quad t \in [0, T],\, x \in \mathbb{R}^d,$$

where $\{W_t\}_{t\in[0,T]}$ is a d-dimensional standard Wiener process. For $p, q \in [1, +\infty)$, we denote by $\mathcal{C}_{[q]}([0, T]; L^p(\mathbb{R}^d))$ the space of all Borel measurable functions f such that $t^{\frac{1}{q}} f(t) \in \mathcal{C}([0, T]; L^p(\mathbb{R}^d))$. If $b = b_1 + b_2$ such that $|b_1(T - \cdot)| \in \mathcal{C}_{[q]}([0, T]; L^p(\mathbb{R}^d))$ with $2/q + d/p = 1$ and $\|b_1(T - \cdot)\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}$ is sufficiently small, and that b_2 is bounded and Borel measurable, then we show that there exists a weak solution to the above equation, and if in addition $\lim_{t\downarrow 0} \|t^{\frac{1}{q}} b(T-t)\|_{L^p(\mathbb{R}^d)} = 0$, the pathwise uniqueness holds. Furthermore, we obtain the strong Feller property of the semigroup and the existence of density associated with the above SDE. Besides, we extend the classical results concerning partial differential equations (PDEs) of parabolic type with $L^q(0, T; L^p(\mathbb{R}^d))$ coefficients to the case of parabolic PDEs with $L^\infty_{[q]}(0, T; L^p(\mathbb{R}^d))$ coefficients, and derive the Lipschitz regularity for solutions of second order parabolic PDEs (see Theorem 3.1). Our results extend

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Krylov-Röckner and Krylov's profound results of SDEs with singular time dependent drift coefficients [20, 23] to the critical case of SDEs with critically irregular drift coefficients in a new critical Lebesgue space.

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1 Introduction

Let $T \in (0, +\infty)$ be arbitrarily fixed. For a Borel measure function $h : [0, T] \to \mathbb{R}$, we set the notation $\mathcal{I}_T h(t) := h(T-t),\ t \in [0, T]$. Furthermore, for a (joint) Borel measurable function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, we denote $\mathcal{I}_T f(t, x) := f(T-t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d$. We are concerned with the following SDE in $\mathbb{R}^d$:

$$\begin{cases} dX_t(x) = b(t, X_t(x))dt + dW_t,\ t \in (0, T], \\ X_0(x) = x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\{W_t\}_{t\in[0,T]} = \{(W_{1,t}, W_{2,t,\dots}, W_{d,t})\}_{t\in[0,T]}$ is a d-dimensional standard Wiener process defined on a given stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and the drift coefficient $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable such that $b \in L^1(0, T; L^1_{loc}(\mathbb{R}^d; \mathbb{R}^d))$.

When b is Lipschitz continuous in $x \in \mathbb{R}^d$ uniformly for $t \in [0, T]$, the existence and uniqueness for strong solutions of (1.1) can be followed by the classical Cauchy-Lipschitz theorem. This result was firstly extended by Veretennikov in the seminal work [36] to the bounded Borel measurable function b. Since then, Veretennikov's result was strengthened in different forms under the same assumption on b. For instance, Mohammed, Nilssen, Proske in [25] not only showed the existence and uniqueness of strong solutions, but also obtained that the unique strong solution forms a Sobolev differentiable stochastic flow; Davie showed in [6] that for almost every Wiener path W, there is a unique continuous X satisfying the integral equation (also see [9]); Wei, Lv and Wang in [39] further proved that the unique strong solution forms a stochastic flow of quasi-diffeomorphisms if b is Dini continuous in the spatial variable.

For integrable drift coefficient, i.e.

$$b \in L^q(0, T; L^p(\mathbb{R}^d; \mathbb{R}^d)) \tag{1.2}$$

with some $p, q \in [2, +\infty)$ such that

$$\frac{2}{q} + \frac{d}{p} < 1, \tag{1.3}$$

by applying Girsanov's theorem and Krylov's estimate, Krylov and Röckner [20] showed the existence and uniqueness of strong solutions for SDE (1.1). On the other hand, under the same conditions (1.2) and (1.3), Fedrizzi and Flandoli [8] proved the α-Hölder continuity of $x \in \mathbb{R}^d \mapsto X_t(x) \in \mathbb{R}^d$ for every $\alpha \in (0,1)$. Some further interesting extensions for non-constant diffusion coefficients can be found in Zhang [44, 45]. Recently, Menoukeu-Pamen and Mohammed in [24] studied Malliavin regularity of SDE (1.1) with unbounded drift coefficient b but fulfilling linear growth condition, and further proved the existence of Sobolev differentiable flows. More recently, Yang and Zhang [43] established strong well-posedness for time-dependent Kato class drifts.

It is known that solutions of Navier-Stokes equations can be analysed by probabilistic representations based on SDEs with rough vector fields b, see, e.g., Rezakhanlou [28, 29], Constantin and Iyer [5], and from the viewpoint of Navier-Stokes equations b can be taken in the critical case, i.e.

$$\frac{2}{q} + \frac{d}{p} = 1, \quad p, q \in [2, +\infty). \tag{1.4}$$

Therefore, the study of the qualitative properties of solutions of SDEs in the critical case is of very high importance. However, it has been a long-standing problem whether SDE (1.1) is well-posed or not in the strong or weak sense under the critical case.

Inspired by Ambrosio [2], by introducing a notation of Lagrangian flow, Beck, Flandoli, Gubinelli and Maurelli in [4] derived the existence and uniqueness, in the present setting, for SDE (1.1) for every $\omega \in \Omega$ being fixed. Recently, Kinzebulatov and Semenov [15] constructed a weak solution for (1.1) with time-independent drift and $p = d$. Later, Kinzebulatov and Madou [17] generalized this result and established the weak well-posedness for time-dependent drifts (also see [16, 40]). For more details about the weak well-posedness, we refer to [21, 22, 30]. More recently, Krylov in [23] established the strong well-posedness for time-independent critical drift, and his result was then extended by Röckner and Zhao in [31] to the time-dependent drift. For the drift in the critical Lorentz space, the unique strong solvability for (1.1) was derived by Nam in [26], and for square integrable (in the time variable) drift, the strong well-posedness was proved by Tian, Ding and Wei in [35].

All the above research works are concerned with the Lebesgue integrable or weakly integrable (with respect to the spatial variables) drift, which satisfies the critical condition (1.4), there is no investigation to deal with the weakly integrable (with respect to the time variable) ones. In this paper, we shall discuss the existence and uniqueness of strong solutions to SDE (1.1) with the weakly integrable (with respect to the time variable) drift under another critical condition, which is analogue of, but weaker than, (1.2) and (1.4). Before giving our main result, we first introduce the following notion.

Definition 1.1 *For $q \geqslant 1$, we denote by $L^{\infty}_{[q]}(0,T)$ the space of all Borel measurable functions $h : [0,T] \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that ess $\sup_{t \in [0,T]} (t^{\frac{1}{q}} |h(t)|) < +\infty$, and the norm is specified by*

$$\|h\|_{L^{\infty}_{[q]}(0,T)} := ess \sup_{t \in [0,T]} (t^{\frac{1}{q}} |h(t)|).$$

In case that $q=+\infty$, *we then set* $L^{\infty}_{[\infty]}(0,T)=:L^{\infty}(0,T)$. *We use* $B([0,T])$ *to denote the subspace of* $L^{\infty}_{[\infty]}(0,T)$ *such that for every* $h\in B([0,T])$

$$\|h\|_{B([0,T])}:=\sup_{t\in[0,T]}|h(t)|<+\infty.$$

Similarly, for $p\geqslant 1$, *we define* $L^{\infty}_{[q]}(0,T;L^p(\mathbb{R}^d))$ *to be the set of all* $L^p(\mathbb{R}^d)$*-valued* $L^{\infty}_{[q]}(0,T)$ *functions* f *such that*

$$\|f\|_{L^{\infty}_{[q]}(0,T;L^p(\mathbb{R}^d))}:=ess\sup_{t\in[0,T]}(t^{\frac{1}{q}}\|f(t)\|_{L^p(\mathbb{R}^d)})<+\infty.$$

Let $B([0,T];L^p(\mathbb{R}^d))$ *be the set of all* $L^{\infty}_{[\infty]}(0,T;L^p(\mathbb{R}^d))$ *such that for every* $f\in B([0,T];L^p(\mathbb{R}^d))$,

$$\|f\|_{B([0,T];L^p(\mathbb{R}^d))}:=\sup_{t\in[0,T]}\|f(t)\|_{L^p(\mathbb{R}^d)}<+\infty. \tag{1.5}$$

Analogously, we define $\mathcal{C}_{[q]}([0,T])$ *to be the subspace of* $L^{\infty}_{[q]}(0,T)$ *such that for every* $h\in L^{\infty}_{[q]}(0,T)$, $t^{\frac{1}{q}}h(t)\in\mathcal{C}([0,T])$. *The norm of* h *in* $\mathcal{C}_{[q]}([0,T])$ *is defined by*

$$\|h\|_{\mathcal{C}_{[q]}([0,T])}:=\sup_{t\in[0,T]}(t^{\frac{1}{q}}|h(t)|).$$

$\mathcal{C}^0_{[q]}([0,T])$ *is the space consisting by all the functions* h *in* $\mathcal{C}_{[q]}([0,T])$ *such that* $\lim_{t\downarrow 0}(t^{\frac{1}{q}}|h(t)|)=0$. *Respectively, for* $p\geqslant 1$, *we define* $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ *and* $\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d))$, *and the norms are specified by (1.5).*

Remark 1.1 *Clearly, for* $p,q\in[1,+\infty]$ *and* $T\in(0,+\infty)$, *spaces* $L^{\infty}_{[q]}(0,T)$, $L^{\infty}_{[q]}(0,T;L^p(\mathbb{R}^d))$, $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ *and* $\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d))$ *are Banach spaces.*

Our main result is the following

Theorem 1.1 *Assume that* $d\geqslant 1$. *Let* $b=b_1+b_2$ *such that* $\mathcal{I}_Tb_1\in\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$ *with* p,q *satisfying*

$$\frac{2}{q}+\frac{d}{p}=1,\quad p,q\in[1,+\infty), \tag{1.6}$$

and b_2 *is bounded, Borel measurable. Suppose that* $\|\mathcal{I}_Tb_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}$ *is sufficiently small, then we have the following consequences*

(i) There is a filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]},\tilde{\mathbb{P}})$ *on which there are two processes* $\{\tilde{X}_t\}_{t\in[0,T]}$ *and* $\{\tilde{W}_t\}_{t\in[0,T]}$ *such that* $\{\tilde{W}_t\}_{t\in[0,T]}$ *is a* d*-dimensional* $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$*-Wiener process and* $\{\tilde{X}_t\}_{t\in[0,T]}$ *is an* $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$*-adapted, continuous,* d*-dimensional process for which*

$$\tilde{\mathbb{P}}\Big(\int_0^T|b(t,\tilde{X}_t)|dt<+\infty\Big)=1 \tag{1.7}$$

and the equation

$$\tilde{X}_t = x + \int_0^t b(s, \tilde{X}_s)ds + \tilde{W}_t, \quad t \in [0,T], \quad \tilde{\mathbb{P}} - a.s. \tag{1.8}$$

holds.

(ii) If in addition $\mathcal{I}_T b_1 \in \mathcal{C}^0_{[q]}([0,T]; L^p(\mathbb{R}^d;\mathbb{R}^d))$*, then the pathwise uniqueness of solutions for SDE (1.1) holds.*

(iii) Assume that the condition in (ii) is fulfilled. Let $\mathbb{P}_x$ *be the unique probability law of the solution* $\{X_t\}_{t\in[0,T]}$ *on d-dimensional classical Wiener space* $(W^d([0,T]), \mathcal{B}(W^d([0,T])))$ *corresponding to the initial value* $x \in \mathbb{R}^d$*. For every* $f \in L^\infty(\mathbb{R}^d)$*, we define*

$$P_t f(x) := \mathbb{E}^{\mathbb{P}_x} f(w(t)), \ t > 0,$$

where $w(t)$ *is the canonical realization of the solution* X_t *with initial data* $X_0 = x \in \mathbb{R}^d$ *on* $(W^d([0,T]), \mathcal{B}(W^d([0,T])))$*. Then, the semigroup* $\{P_t\}_{t\in[0,T]}$ *has strong Feller property, i.e. each* P_t *maps a bounded function to a bounded and continuous function. Moreover,* P_t *admits a density* $p(t,x,y)$ *for almost all* $t \in [0,T]$*. Besides, for every* $t_0 > 0$ *and for every* $r \in [1, +\infty)$*,*

$$\int_{t_0}^T \int_{\mathbb{R}^d} |p(t,x,y)|^r dydt < +\infty. \tag{1.9}$$

Remark 1.2 *By Theorem 1.1 (i) and (ii), and with the help of Yamada-Watanabe's principle (see [41]), there is a unique strong solution to SDE (1.1). Let* $L^{q,1}(0,T;L^p(\mathbb{R}^d))$ $(1 < q, p < \infty)$ *be the Lorentz-Lebesgue space. If* b *lies in* $L^{q,1}(0,T;L^p(\mathbb{R}^d))$ *and condition (1.6) holds true, by utilizing Girsanov's theorem, Nam [26] also established the existence and uniqueness of strong solutions for (1.1). We notice that Nam's method follows exactly the approach developed in Krylov-Röckner [20] wherein it required that* $p \geq 2$ *(remarked [20, p160] after inequality (3.3)) so Nam's main result is valid for* $p \geqslant 2$*, i.e. (1.4) holds. With a different start of point, in the present paper, we use an approximation approach to establish the existence and uniqueness of strong solutions, as well as the strong Feller property for SDE (1.1) with requirement that* $p > 1$*, i.e. (1.6) holds. Therefore, our condition is obviously more general than Nam's condition in the case of* $d = 1$*.*

Remark 1.3 *(i) Recently, for time independent drift Krylov [23] proved the unique strong solvability for (1.1) by assuming* $b \in L^d(\mathbb{R}^d;\mathbb{R}^d)$ *and* $d \geqslant 3$*. More recently, Röckner and Zhao [31] generalized Krylov's result to the time dependent drift and proved the unique strong solvability by assuming that* $p > 3$*. In the present paper, we consider time dependent drift for* $p > 1$*, so it is different from that in [23, 31]. Moreover, our methods and main results are very different from Krylov's and Röckner-Zhao's as well.*

(ii) In case of $q = 2$*, if one assumes further that* $\mathcal{I}_T b_1 \in \mathcal{C}^0_{[2]}([0,T]; \mathcal{C}_u(\mathbb{R}^d;\mathbb{R}^d))$*, where* $\mathcal{C}_u(\mathbb{R}^d)$ *is the space of functions that are bounded and uniformly continuous, all derivations in the proof of Theorem 1.1 are valid, thus Theorem 1.1 remains true in this setting.*

The novelties of Theorem 1.1 are twofold: p is only assumed to be greater than d and the space $\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$ contains a class of weakly Lebesgue integrable (with respect to the time variable) functions that are in $L^q_w([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$, where $L^q_w([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$ is the space consisting by all the $L^p(\mathbb{R}^d;\mathbb{R}^d)$-valued q-th order weakly integrable functions. And in this sense, our "critical" is stronger than the "critical" in Krylov-Röckner [20]. To illustrate the novelty of our assumptions on the drift, let us consider the following example.

Example 1.1 *Let (1.6) hold. Suppose $\tilde{b}\in\mathcal{C}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$ with $T=1/2$, so that $\|\tilde{b}\|_{\mathcal{C}([0,T];L^p(\mathbb{R}^d))}$ is small enough. We set*

$$b(t,x)=(\frac{1}{2}-t)^{-\frac{1}{q}}|\log(\frac{1}{2}-t)|^{-\frac{1}{q}}\tilde{b}(t,x), \tag{1.10}$$

then

$$t^{\frac{1}{q}}b(\frac{1}{2}-t,x)=|\log t|^{-\frac{1}{q}}\tilde{b}(\frac{1}{2}-t,x),$$

which indicates that $\mathcal{I}_{\frac{1}{2}}b\in\mathcal{C}^0_{[q]}([0,\frac{1}{2}];L^p(\mathbb{R}^d;\mathbb{R}^d))$. Let b be given by (1.10) in SDE (1.1). By Theorem 1.1, there exists a unique strong solution to SDE (1.1). On the other hand, from the explicit form (1.10), $b\in L^q_w(0,\frac{1}{2};L^p(\mathbb{R}^d;\mathbb{R}^d))\setminus L^q(0,\frac{1}{2};L^p(\mathbb{R}^d;\mathbb{R}^d))$, the existing results do not imply the existence and uniqueness of strong solutions to (1.1) with b given by (1.10). From this point of view, it is clear that we extend the existing results on $L^q(0,\frac{1}{2};L^p(\mathbb{R}^d;\mathbb{R}^d))$ with $2/q+d/p=1$ partially.

The existence and uniqueness of strong solutions here is only for constant diffusion coefficient. We do not know in the present setting for general d, whether the strong solutions do exist and further, if they would exist, whether the uniqueness holds for non-constant and non-degenerate diffusion coefficients. But for $d=1$, we can give a positive answer.

Theorem 1.2 *Let $\sigma:\mathbb{R}\to\mathbb{R}$ be Borel measurable. Suppose that there are positive constants δ_1 and δ_2 such that $\delta_1\leqslant\sigma\leqslant\delta_2$. Consider the following SDE with non-constant diffusion in $\mathbb{R}$*

$$dX_t=b(t,X_t)dt+\sigma(X_t)dW_t,\quad X_0=x\in\mathbb{R},\quad t\in(0,T]. \tag{1.11}$$

Let p and q be given in Theorem 1.1, that $b=b_1+b_2$ such that $\mathcal{I}_Tb_1\in\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}))$ and $\|\mathcal{I}_Tb_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}))}$ is sufficiently small, b_2 is bounded Borel measurable. Moreover, for this p, we assume in addition that $\sigma'=\tilde{\sigma}_1+\tilde{\sigma}_2$, with $\tilde{\sigma}_1\in L^p(\mathbb{R})$ and $\|\tilde{\sigma}_1\|_{L^p(\mathbb{R})}$ is small enough, $\tilde{\sigma}_2\in L^\infty(\mathbb{R})$. Then there exists a unique strong solution to SDE (1.11).

The rest of this paper is arranged as follows. In Section 2, we present some preliminaries. Section 3 is devoted to establishing the well-posedness for parabolic PDEs. In Sections 4, we derive a new Krylov type estimate. Section 5 is devoted to the proof of existence result of Theorem 1.1 and in Section 6, we prove the uniqueness, strong Feller property as well as the existence of the density. In Section 7, the final section, we show Theorem 1.2.

When there is no ambiguity, we use C to denote a constant whose true value may vary from line to line. As usual, $\mathbb{N}$ stands for the set of all natural numbers.

2 Preliminaries

Let us start by showing that functions in $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ possess certain fine approximation properties, which will be used in Section 5.

Proposition 2.1 *Suppose that $p,q \in [1,+\infty)$. Given a function f in $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$, we set $f_n(t,x) = (f(t,\cdot) * \rho_n)(x), n \in \mathbb{N}$, where $*$ stands for the usual convolution and*

$$\rho_n := n^d \rho(n\cdot) \text{ with } 0 \leqslant \rho \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \text{support}(\rho) \subset B_0(1), \tag{2.1}$$

and $\int_{\mathbb{R}^d} \rho(x)dx = 1$. Then

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \left(t^{\frac{1}{q}} \|f_n(t) - f(t)\|_{L^p(\mathbb{R}^d)} \right) = 0. \tag{2.2}$$

Proof. Let $f \in \mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ and set $g(t,x) = t^{\frac{1}{q}} f(t,x)$. Then $g \in \mathcal{C}([0,T];L^p(\mathbb{R}^d))$, if one defines the value at 0 by its right limit. Thus, to prove (2.2), it is sufficient to show that for $f \in \mathcal{C}([0,T];L^p(\mathbb{R}^d))$

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \|f_n(t) - f(t)\|_{L^p(\mathbb{R}^d)} = 0. \tag{2.3}$$

By virtue of properties of the convolution, for every fixed $t \in [0,T]$, then

$$\lim_{n\to\infty} \|f_n(t) - f(t)\|_{L^p(\mathbb{R}^d)} = 0. \tag{2.4}$$

On the other hand for $t_1, t_2 \in [0,T]$, by utilizing Young's inequality,

$$\begin{aligned} \|f_n(t_1) - f_n(t_2)\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |(f(t_1,\cdot) - f(t_2,\cdot)) * \rho_n(x)|^p dx \\ &\leqslant \int_{\mathbb{R}^d} |f(t_1,x) - f(t_2,x)|^p dx. \end{aligned} \tag{2.5}$$

From (2.5), for any $\epsilon > 0$, there exists $\delta > 0$ such that for $|t_1 - t_2| \leqslant \delta$, then one has uniformly in n the following

$$\|f_n(t_1) - f_n(t_2)\|_{L^p(\mathbb{R}^d)} \leqslant \|f(t_1) - f(t_2)\|_{L^p(\mathbb{R}^d)} < \frac{\epsilon}{2}. \tag{2.6}$$

Let $t \in [0,T]$ be given, then (2.4) holds. With the aid of (2.5) and (2.6), then

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{s\in[t-\delta,t+\delta]\cap[0,T]} \|f_n(s) - f(s)\|_{L^p(\mathbb{R}^d)} \\ \leqslant\ &\limsup_{n\to\infty} \sup_{s\in[t-\delta,t+\delta]\cap[0,T]} \|f_n(s) - f(s) - f_n(t) + f(t)\|_{L^p(\mathbb{R}^d)} + \limsup_{n\to\infty} \|f_n(t) - f(t)\|_{L^p(\mathbb{R}^d)} \\ <\ &\epsilon. \end{aligned}$$

Since $\epsilon > 0$ and $t \in [0,T]$ are arbitrary, we conclude that (2.3) holds. □

Remark 2.1 *We claim that the above approximation property is not true if one takes the function in $L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$ instead of in $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$. For simplicity, we assume that $T=d=1$ and $p=2$. For $k\in\mathbb{N}$, we define $f_k(x)$ by the following*

$$f^k(x) := k1_{[k,k+\frac{1}{k^2})}(x),$$

and further set

$$f(t,x) := \sum_{k=1}^{\infty} 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) f^k(x) = \sum_{k=1}^{\infty} 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) k1_{[k,k+\frac{1}{k^2})}(x).$$

Then

$$\int_{\mathbb{R}} |f(t,x)|^2 dx = \sum_{k=1}^{\infty} 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) \int_{\mathbb{R}} k^2 1_{[k,k+\frac{1}{k^2})}(x)dx = \sum_{k=1}^{\infty} 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) = 1_{[0,1)}(t).$$

Hence $f\in L^\infty(0,1;L^2(\mathbb{R}))$. We estimate (2.3) by the following

$$\begin{aligned}
&\int_{\mathbb{R}} |f_n(t,x)-f(t,x)|^2 dx \\
=\ &\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t,y)\rho_n(x-y)dy - f(t,x)\right|^2 dx \\
=\ &\int_{\mathbb{R}} \sum_{k=1}^{\infty} k^2 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) \left| \int_{\mathbb{R}} 1_{[k,k+\frac{1}{k^2})}(y)\rho_n(x-y)dy - 1_{[k,k+\frac{1}{k^2})}(x)\right|^2 dx \\
=\ &\int_{\mathbb{R}} \sum_{k=1}^{\infty} k^2 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) \left| \int_{k}^{k+\frac{1}{k^2}} \rho_n(x-y)dy - 1_{[k,k+\frac{1}{k^2})}(x)\right|^2 dx \\
\geqslant\ & 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) \int_{k}^{k+\frac{1}{k^2}} k^2 \left| \int_{k}^{k+\frac{1}{k^2}} \rho_n(x-y)dy - 1\right|^2 dx.
\end{aligned}$$

For any fixed n, for sufficiently large k, we have $|\int_k^{k+\frac{1}{k^2}} \rho_n(x-y)dy| < \frac{1}{2}$. Thus

$$\begin{aligned}
&\sup_{t\in[0,1]} \int_{\mathbb{R}} |f_n(t,x)-f(t,x)|^2 dx \\
\geqslant\ &\sup_k \sup_{t\in[0,1]} 1_{[\frac{k-1}{k},\frac{k}{k+1})}(t) \int_{k}^{k+\frac{1}{k^2}} k^2 \left| \int_{k}^{k+\frac{1}{k^2}} \rho_n(x-y)dy - 1\right|^2 dx \geqslant \frac{1}{4}.
\end{aligned}$$

Therefore

$$\liminf_{n\to\infty} \sup_{t\in[0,1]} \int_{\mathbb{R}} |f_n(t,x)-f(t,x)|^2 dx \geqslant \frac{1}{4}.$$

3 Parabolic partial differential equations

Let $g \in L^1(0,T;L^p_{loc}(\mathbb{R}^d;\mathbb{R}^d))$ with $p \geqslant 1$, $f \in L^1(0,T;L^1_{loc}(\mathbb{R}^d))$. Consider the following Cauchy problem for $u:[0,T]\times\mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + g(t,x)\cdot\nabla u(t,x) + f(t,x),\ (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x) = 0,\ x\in\mathbb{R}^d. \end{cases} \tag{3.1}$$

We call $u(t,x)$ a weak solution of (3.1) if it lies in $\mathcal{C}([0,T];W^{1,p}(\mathbb{R}^d))$ such that for every test function $\varphi \in \mathcal{C}_0^\infty([0,T)\times\mathbb{R}^d)$, the identity

$$\begin{aligned} 0 &= \int_0^T\int_{\mathbb{R}^d} u(t,x)\partial_t\varphi(t,x)dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d} u(t,x)\Delta\varphi(t,x)dxdt \\ &\quad + \int_0^T\int_{\mathbb{R}^d} g(t,x)\cdot\nabla u(t,x)\varphi(t,x)dxdt + \int_0^T\int_{\mathbb{R}^d} f(t,x)\varphi(t,x)dxdt \end{aligned} \tag{3.2}$$

holds.

The following proposition is routine and we therefore omit its proof. For more details, the reader is referred to [46, Proposition 3.5].

Proposition 3.1 *Let $p\in[1,+\infty)$ such that $g\in L^1(0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$, $f\in L^1(0,T;L^p(\mathbb{R}^d))$ and $u\in\mathcal{C}([0,T];W^{1,p}(\mathbb{R}^d))$. The following statements are equivalent*

(i) u is a weak solution of (3.1);

(ii) For every $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ and every $t\in[0,T]$, the following identity

$$\begin{aligned} \int_{\mathbb{R}^d} u(t,x)\psi(x)dx &= \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\Delta\psi(x)dxds + \int_0^t\int_{\mathbb{R}^d} g(s,x)\cdot\nabla u(s,x)\psi(x)dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} f(s,x)\psi(x)dxds \end{aligned}$$

holds;

(iii) For every $t\in[0,T]$ and for almost everywhere $x\in\mathbb{R}^d$, u fulfils the following integral equation

$$u(t,x) = \int_0^t K(t-s,\cdot)*(g(s,\cdot)\cdot\nabla u(s,\cdot))(x)ds + \int_0^t (K(t-s,\cdot)*f(s,\cdot))(x)ds, \tag{3.3}$$

where $K(t,x)=(2\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2t}}$, $t>0$, $x\in\mathbb{R}^d$.

We now state our main result of this section.

Theorem 3.1 *Let $d \geqslant 1$ and $p, q \in [1, +\infty)$. Let $g \in L^\infty_{[q]}(0, T; L^p(\mathbb{R}^d; \mathbb{R}^d))$ and $f \in L^\infty_{[q]}(0, T; L^p(\mathbb{R}^d))$ such that (1.6) holds true and $\|g\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}$ is sufficiently small. Then the Cauchy problem (3.1) has a unique weak solution u. Moreover, the unique weak solution lies in $B([0, T]; \mathcal{C}^1_u(\mathbb{R}^d))$ and there is a constant $C_0(p, d)$ such that*

$$\|u\|_{B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))} \leqslant \frac{C_0(p,d)\|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}}{1 - C_0(p,d)\|g\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}}, \tag{3.4}$$

where $\mathcal{C}^1_u(\mathbb{R}^d)$ is the space of functions which are bounded and uniformly continuous and have bounded and uniformly continuous 1 order derivatives, and

$$\|u\|_{B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t,x)| + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla u(t,x)|.$$

Proof. We prove the result by first assuming that $g = 0$. With the help of Proposition 3.1, it suffices to show that

$$u(t,x) = \int_0^t (K(t-s,\cdot) * f(s,\cdot))(x)ds \tag{3.5}$$

is in $\mathcal{C}([0, T]; W^{1,p}(\mathbb{R}^d)) \cap B([0, T]; \mathcal{C}^1_u(\mathbb{R}^d))$. Firstly, by the explicit representation (3.5) and observing from (1.6) that $p > d$, for every $(t, x) \in (0, T) \times \mathbb{R}^d$, we have

$$\begin{aligned} |u(t,x)| &\leqslant \int_0^t \|f(s)\|_{L^p(\mathbb{R}^d)} \|K(t-s)\|_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} dr \\ &\leqslant \|f\|_{L^\infty_{[q]}(0,t;L^p(\mathbb{R}^d))} \int_0^t s^{-\frac{1}{q}} (t-s)^{-\frac{d}{2p}} ds \\ &= t^{\frac{1}{2}} \|f\|_{L^\infty_{[q]}(0,t;L^p(\mathbb{R}^d))} B(1 - \frac{1}{q}, \frac{1}{q} + \frac{1}{2}), \end{aligned}$$

where B is the Beta function.

Therefore $u \in B([0, T] \times \mathbb{R}^d)$ and

$$\|u\|_{B([0,T]\times\mathbb{R}^d)} \leqslant CT^{\frac{1}{2}} \|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}. \tag{3.6}$$

For $x \in \mathbb{R}^d$ and $1 \leqslant i \leqslant d$, by (1.6) it follows that

$$\begin{aligned} \Big|\partial_{x_i} u(t,x)\Big| &= \Big| \int_0^t \int_{\mathbb{R}^d} \partial_{x_i} K(t-r, x-y) f(r,y) dy dr \Big| \\ &\leqslant \frac{1}{(2\pi)^{\frac{d}{2}}} \int_0^t \|f\|_{L^p(\mathbb{R}^d)}(r)(t-r)^{-\frac{d}{2}-1} \Bigg[\int_{\mathbb{R}^d} \Big| e^{-\frac{|x-y|^2}{2(t-r)}} |x_i - y_i| \Big|^{\frac{p}{p-1}} dy \Bigg]^{\frac{p-1}{p}} dr \\ &\leqslant C\|f\|_{L^\infty_{[q]}(0,t;L^p(\mathbb{R}^d))} \int_0^t r^{-\frac{1}{q}} (t-r)^{-1+\frac{1}{q}} dr \\ &= C(p,d)\|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}, \end{aligned} \tag{3.7}$$

where the constant in (3.7) is given by

$$C(p,d)=\pi^{-\frac{d+p-1}{2p}}2^{\frac{p-d}{2p}}\Big[\Gamma\Big(\frac{2p-1}{2p-2}\Big)\Big]^{\frac{p-1}{p}}\Big(\frac{p-1}{p}\Big)^{\frac{(d+1)p-d}{2p}}B(1-\frac{1}{q},\frac{1}{q}),$$

and Γ is the gamma function. Since $1\leqslant i\leqslant d$ is arbitrary, $|\nabla u|\in B([0,T]\times\mathbb{R}^d)$. By (3.6) and (3.7), it remains to check that the derivatives of u in x is uniformly continuous in x.

For $x,h\in\mathbb{R}^d$ and $1\leqslant i\leqslant d$, we estimate the difference of $\partial_{x_i}u(t,x+h)-\partial_{x_i}u(t,x)$ by

$$\begin{aligned}&|\partial_{x_i}u(t,x+h)-\partial_{x_i}u(t,x)|\\ =&\ \Big|\int_0^t\int_{\mathbb{R}^d}\partial_{x_i}K(t-r,y)[f(r,x+h-y)-f(r,x-y)]dydr\Big|\\ \leqslant&\ \Big|\int_0^t\Big[\int_{\mathbb{R}^d}|\partial_{x_i}K(t-r,y)|^{\frac{p}{p-1}}dy\Big]^{\frac{p-1}{p}}\Big[\int_{\mathbb{R}^d}|f(r,x+h-y)-f(r,x-y)|^pdy\Big]^{\frac{1}{p}}dr\Big|\\ \leqslant&\ C\int_0^t(t-r)^{-\frac{1}{2}-\frac{d}{2p}}\Big[\int_{\mathbb{R}^d}|f(r,y+h)-f(r,y)|^pdy\Big]^{\frac{1}{p}}dr. \end{aligned}\tag{3.8}$$

Note that (1.4) is true, from (3.8) and $p>d$, therefore,

$$\begin{aligned}&\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\Big|\partial_{x_i}u(t,x+h)-\partial_{x_i}u(t,x)\Big|\\ \leqslant&\ C\sup_{t\in[0,T]}\int_0^t(t-r)^{-1+\frac{1}{q}}\Big[\int_{\mathbb{R}^d}|f(r,y+h)-f(r,y)|^pdy\Big]^{\frac{1}{p}}dr. \end{aligned}\tag{3.9}$$

We set $h_t(r)=(t-r)^{-1+\frac{1}{q}}r^{-\frac{1}{q}}$, then $h_t\in L^1(0,t)$. Notice that $f\in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$, so for every $t\in(0,T]$, $h_tf\in L^1(0,t;L^p(\mathbb{R}^d))$, and it implies that there exists $T_0\in(0,T]$ such that

$$\begin{aligned}&\sup_{t\in[0,T]}\int_0^t(t-r)^{-1+\frac{1}{q}}\Big[\int_{\mathbb{R}^d}|f(r,y+h)-f(r,y)|^pdy\Big]^{\frac{1}{p}}dr\\ =&\ \int_0^{T_0}(T_0-r)^{-1+\frac{1}{q}}\Big[\int_{\mathbb{R}^d}|f(r,y+h)-f(r,y)|^pdy\Big]^{\frac{1}{p}}dr. \end{aligned}\tag{3.10}$$

With the help of Lebesgue's theorem, from (3.9) and (3.10)

$$\begin{aligned}&\lim_{h\to0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\partial_{x_i}u(t,x+h)-\partial_{x_i}u(t,x)|\\ \leqslant&\ C\lim_{h\to0}\int_0^{T_0}(T_0-r)^{-1+\frac{1}{q}}\Big[\int_{\mathbb{R}^d}|f(r,y+h)-f(r,y)|^pdy\Big]^{\frac{1}{p}}dr=0, \end{aligned}$$

which implies $\partial_{x_i}u\in B([0,T];\mathcal{C}_u(\mathbb{R}^d))$. And $1\leqslant i\leqslant d$ is arbitrary, so ∇u is uniformly continuous in x and uniformly respect to t.

Now we will show that $u \in \mathcal{C}([0,T];W^{1,p}(\mathbb{R}^d))$. To prove this result, for $p \geqslant 1$, $\beta \geqslant 0$, let $H^{\beta,p}(\mathbb{R}^d) := (I-\Delta)^{-\beta/2}(L^p(\mathbb{R}^d))$ be the Bessel potential space with the norm

$$\|h\|_{H^{\beta,p}(\mathbb{R}^d)} = \|(I-\Delta)^{\beta/2}h\|_{L^p(\mathbb{R}^d)}.$$

For $h \in L^p(\mathbb{R}^d)$, we use the notation $\mathcal{T}_t h$ to denote $K(t,\cdot)*h$ with K given in (3.3). Then by a same discussion as [46, Lemma 2.5], we have the following claims:

(i) For $p>1$, $\beta>0$ and every $h \in L^p(\mathbb{R}^d)$, there is a constant $C(p,d,\beta)>0$ such that

$$\|\mathcal{T}_t h\|_{H^{\beta,p}(\mathbb{R}^d)} \leqslant C(p,d,\beta)t^{-\frac{\beta}{2}}\|h\|_{L^p(\mathbb{R}^d)}. \tag{3.11}$$

(ii) For $p>1$, $\theta \in [0,1]$, there is a constant $C(p,d,\theta)>0$ such that for every $h \in H^{\beta,p}(\mathbb{R}^d)$

$$\|\mathcal{T}_t h - h\|_{L^p(\mathbb{R}^d)} \leqslant C(p,d,\theta)t^{\frac{\theta}{2}}\|h\|_{H^{\theta,p}(\mathbb{R}^d)}. \tag{3.12}$$

For every $0 \leqslant s < t \leqslant T$, then

$$\begin{aligned} u(t,x)-u(s,x) &= \int_0^t \mathcal{T}_{t-r}f(r)dr - \int_0^s \mathcal{T}_{s-r}f(r)dr \\ &= \int_s^t \mathcal{T}_{t-r}f(r)dr + \int_0^s [\mathcal{T}_{t-r}-\mathcal{T}_{s-r}]f(r)dr \\ &= \int_s^t \mathcal{T}_{t-r}f(r)dr + \int_0^s \mathcal{T}_{\frac{s-r}{2}}[\mathcal{T}_{t-s}-I]\mathcal{T}_{\frac{s-r}{2}}f(r)dr. \end{aligned} \tag{3.13}$$

Noticing that $W^{1,p}(\mathbb{R}^d) = H^{1,p}(\mathbb{R}^d)$ (see [1]), from (3.13), (3.11) and (3.12), then

$$\begin{aligned} &\|u(t)-u(s)\|_{W^{1,p}(\mathbb{R}^d)} \\ \leqslant\ & \int_s^t \|\mathcal{T}_{t-r}f(r)\|_{H^{1,p}(\mathbb{R}^d)}dr + \int_0^s \|\mathcal{T}_{\frac{s-r}{2}}[\mathcal{T}_{t-s}-I]\mathcal{T}_{\frac{s-r}{2}}f(r)\|_{H^{1,p}(\mathbb{R}^d)}dr \\ \leqslant\ & C\int_s^t (t-r)^{-\frac{1}{2}}\|f(r)\|_{L^p(\mathbb{R}^d)}dr + C\int_0^s (s-r)^{-\frac{1}{2}}\|[\mathcal{T}_{t-s}-I]\mathcal{T}_{\frac{s-r}{2}}f(r)\|_{L^p(\mathbb{R}^d)}dr \\ \leqslant\ & C\int_s^t (t-r)^{-\frac{1}{2}}\|f(r)\|_{L^p(\mathbb{R}^d)}dr + C(t-s)^{\frac{\theta}{2}}\int_0^s (s-r)^{-\frac{1}{2}}\|\mathcal{T}_{\frac{s-r}{2}}f(r)\|_{H^{\theta,p}(\mathbb{R}^d)}dr \\ \leqslant\ & C\int_s^t (t-r)^{-\frac{1}{2}}\|f(r)\|_{L^p(\mathbb{R}^d)}dr + C(t-s)^{\frac{\theta}{2}}\int_0^s (s-r)^{-\frac{1+\theta}{2}}\|f(r)\|_{L^p(\mathbb{R}^d)}dr, \end{aligned} \tag{3.14}$$

for some $\theta \in [0,1]$.

Since $f \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$, if one chooses $\theta=(q-2)/q$, from (3.14) by using Hölder's inequality, it then yields the following

$$\begin{aligned}
&\|u(t)-u(s)\|_{W^{1,p}(\mathbb{R}^d)}\\
\leqslant\ & C\|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}\left[\int_s^t (t-r)^{-\frac{1}{2}}r^{-\frac{1}{q}}dr+(t-s)^{\frac{\theta}{2}}\int_0^s (s-r)^{-\frac{1+\theta}{2}}r^{-\frac{1}{q}}dr\right]\\
\leqslant\ & C\|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}|t-s|^{\frac{\theta}{2}}.
\end{aligned}\tag{3.15}$$

From this, one completes the proof for $g=0$.

For any g, since $u \in B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))$, we conclude that: if $g \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$, then $g\cdot\nabla u \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$. We define a mapping from $B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))$ to itself by

$$Tv(t,x)=\int_0^t K(t-s,\cdot)*(g(s,\cdot)\cdot\nabla v(s,\cdot))(x)ds+\int_0^t (K(t-s,\cdot)*f(s,\cdot))(x)ds.$$

Noticing that $\|g\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}$ is small enough, the mapping is contractive, so there is a unique $u \in B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))$ satisfying $u=Tu$. This fact combining an argument as $g=0$ implies the existence and uniqueness of weak solutions of the Cauchy problem (3.1).

Since (3.3) holds, by virtue of (3.6) and (3.7), there is a constant $C_0(p,d)$ such that

$$\|u\|_{B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))}\leqslant C_0(p,d)\Big[\|g\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}\|u\|_{B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))}+\|f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}\Big],$$

which suggests that (3.4) is valid since $\|g\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}$ is sufficiently small. $\Box$

Remark 3.1 *If $p,q\in(1,+\infty)$, $f\in L^q(0,T;L^p(\mathbb{R}^d))$ and $g\in L^q(0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$, from classical $L^q(L^p)$ theory for second order parabolic PDEs, there is a unique $u\in W^{1,q}(0,T;L^p(\mathbb{R}^d))\cap L^q(0,T;W^{2,p}(\mathbb{R}^d))$ solving the Cauchy problem (3.1). Using the Sobolev embedding theorems (or see [20, Lemma 10.2]), if $2/q+d/p<1$, then ∇u is bounded and continuous in x. But this embedding is not true in general when $2/q+d/p=1$. By extending the Banach space $L^q(0,T;L^p)$ to $L^\infty_{[q]}(0,T;L^p)$, under the critical case $2/q+d/p=1$, we further get the boundedness and continuity for ∇u in x. This result is new and interesting which can also be as a supplement for the classical regularity theory of second order parabolic PDEs.*

In case that both f and g are more regular, we can then obtain the continuity of ∇u in (t,x).

Corollary 3.1 *Let p and q be given in Theorem 3.1, such that $f\in\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d))$ and $g\in\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$ and $\|g\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}$ is small enough. Then $u\in\mathcal{C}([0,T];\mathcal{C}^1_u(\mathbb{R}^d))$.*

Proof. We only need to prove the continuity in t for ∇u and for simplicity, we show the case of $g=0$. We first show the continuity at 0. In view of (3.7), then for $t>0$,

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|\partial_{x_i}u(t,x)|\leqslant\lim_{t\downarrow 0}C[t^{\frac{1}{q}}\|f(t)\|_{L^p(\mathbb{R}^d)}]=0.\tag{3.16}$$

Noting that $u(0,x)=0$, it implies $|\nabla u(0,x)|=0$, so ∇u is continuous in t at 0.

For $t>0$, we only prove the right continuity of ∇u at t since the proof for left continuity is similar. Let $\vartheta>0$, $t\in(0,T)$ such that $t+\vartheta\in(0,T)$, then for every $1\leqslant i\leqslant d$

$$\begin{aligned}
&\Big|\partial_{x_i}u(t+\vartheta,x)-\partial_{x_i}u(t,x)\Big| \\
=&\ \Big|\int_0^{t+\vartheta}\int_{\mathbb{R}^d}\partial_{x_i}K(t+\vartheta-s,x-y)f(s,y)dyds-\int_0^{t}\int_{\mathbb{R}^d}\partial_{x_i}K(t-s,x-y)f(s,y)dyds\Big| \\
\leqslant&\ \Big|\int_0^{\vartheta}\int_{\mathbb{R}^d}\partial_{x_i}K(t+\vartheta-s,x-y)f(s,y)dyds\Big| \\
&+\Big|\int_0^{t}\int_{\mathbb{R}^d}\partial_{x_i}K(t-s,x-y)[f(s+\vartheta,y)-f(s,y)]dyds\Big| \\
=&\ :I_1(t,\vartheta)+I_2(t,\vartheta).
\end{aligned}$$

By using (3.16), then $I_1(t,\vartheta)\to0$ as $\vartheta\to0$. Now let us calculate $I_2(t,\vartheta)$.

$$\begin{aligned}
|I_2(t,\vartheta)| &\leqslant C\int_0^t(t-s)^{-1+\frac{1}{q}}\|f(s+\vartheta)-f(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\leqslant C\int_0^t(t-s)^{-1+\frac{1}{q}}(s+\vartheta)^{-\frac{1}{q}}\|(s+\vartheta)^{\frac{1}{q}}f(s+\vartheta)-s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\quad +C\int_0^t(t-s)^{-1+\frac{1}{q}}(s+\vartheta)^{-\frac{1}{q}}\|(s+\vartheta)^{\frac{1}{q}}f(s)-s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\leqslant C\int_0^t(t-s)^{-1+\frac{1}{q}}s^{-\frac{1}{q}}\|(s+\vartheta)^{\frac{1}{q}}f(s+\vartheta)-s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\quad +C\int_0^t(t-s)^{-1+\frac{1}{q}}s^{-\frac{1}{q}}\|s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}\Big[1-(\frac{s}{s+\vartheta})^{\frac{1}{q}}\Big]ds.
\end{aligned}$$

Noticing that as the functions of s on $(0,t)$, $(t-s)^{-1+\frac{1}{q}}s^{-\frac{1}{q}}\in L^1(0,t)$, $\|(s+\vartheta)^{\frac{1}{q}}f(s+\vartheta)-s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}$ and $\|s^{\frac{1}{q}}f(s)\|_{L^p(\mathbb{R}^d)}\Big[1-(s/(s+\vartheta))^{\frac{1}{q}}\Big]$ are bounded. Applying the dominated convergence theorem, we then have $I_2(t,\vartheta)\to0$ as $\vartheta\to0$. We thus complete the proof. $\square$

4 A Krylov estimate

Let $\{W_t\}_{t\in[0,T]}$ be a d-dimensional standard Wiener process, $X_0 \in \mathcal{F}_0$, $\{\xi_t\}_{t\in[0,T]}$ is a $\{\mathcal{F}_t\}_{t\in[0,T]}$ adapted process, we define

$$X_t = X_0 + \int_0^t \xi_s ds + W_t. \tag{4.1}$$

We now derive a Krylov estimate, which will be pivotal in our proof towards the existence of weak solutions.

Theorem 4.1 *Suppose X_t is given by (4.1) and $X_0 = x \in \mathbb{R}^d$. Let $p, q \in [1, +\infty)$ and $\mathcal{I}_T f \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$ such that (1.6) holds. Let $C_0(p,d)$ be given in Theorem 3.1, then*

$$\mathbb{E}\int_0^T f(t,X_t)dt \leqslant C_0(p,d)\left(1+\mathbb{E}\int_0^T |\xi_t|dt\right)\|\mathcal{I}_T f\|_{L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))}. \tag{4.2}$$

Proof. Let u be given by

$$u(t,x) = \int_0^t (K(t-s,\cdot) * \mathcal{I}_T f(s,\cdot))(x)ds.$$

Noticing that $\mathcal{I}_T f \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$ with p, q satisfying (1.6), by virtue of Theorem 3.1, then $u \in \mathcal{C}([0,T];W^{1,p}(\mathbb{R}^d)) \cap B([0,T];\mathcal{C}^1_u(\mathbb{R}^d))$ and it solves the following Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \mathcal{I}_T f(t,x), \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = 0, \ x \in \mathbb{R}^d, \end{cases}$$

in the sense of (3.2). Moreover, (3.4) holds with $g = 0$.

Noticing that (1.6) holds and $\mathcal{I}_T f \in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$, we conclude that for every $T_0 < T$ and every $m \geqslant 1$, $u(T-\cdot,\cdot) \in L^m(0,T_0;W^{2,p}(\mathbb{R}^d)) \cap W^{1,m}(0,T_0;L^p(\mathbb{R}^d))$. By virtue of Itô's formula (see [20, Theorem 3.7]), for every $t \in [0,T_0]$, we have

$$\begin{aligned} & du(T-t,X_t) \\ = \ & -\partial_t u(T-t,X_t)dt + \frac{1}{2}\Delta u(T-t,X_t)dt + \xi_t \cdot \nabla u(T-t,X_t)dt + \nabla u(T-t,X_t)\cdot dW_t \\ = \ & \xi_t \cdot \nabla u(T-t,X_t)dt - f(t,X_t)dt + \nabla u(T-t,X_t)\cdot dW_t. \end{aligned} \tag{4.3}$$

Since ∇u is bounded, if we integrate the last term in (4.3) on $(0,T_0)$, then the stochastic integral is a martingale, which implies that

$$u(T,x) - u(T-T_0,X_{T_0}) = \mathbb{E}\int_0^{T_0} f(t,X_t)dt - \mathbb{E}\int_0^{T_0} \xi_t\cdot\nabla u(T-t,X_t)dt. \tag{4.4}$$

Hence

$$\mathbb{E}\int_0^{T_0} f(t,X_t)dt \leqslant \sup_{(t,x)\in(0,T)\times\mathbb{R}^d}|u(t,x)| + \sup_{(t,x)\in(0,T)\times\mathbb{R}^d}|\nabla u(t,x)|\mathbb{E}\int_0^T|\xi_t|dt - u(T-T_0,X_{T_0}). \quad (4.5)$$

In view of the fact $u\in\mathcal{C}([0,T];W^{1,p}(\mathbb{R}^d))\hookrightarrow\mathcal{C}([0,T];\mathcal{C}_b(\mathbb{R}^d))$ and by virtue of Theorem 3.1, from (4.5), by letting T_0 tend to T, we obtain that (4.2) holds true. □

Remark 4.1 *Krylov's estimates will play a crucial role in proving the existence of weak solutions for SDE (1.1). Observing that, when verifying a Krylov estimate, the central part is to estimate the boundedness of ∇u (u is the unique solution of a second order parabolic PDE). With this observation, we only assume that $f\in L^\infty_{[q]}(0,T;L^p(\mathbb{R}^d))$ with $2/q+d/p=1$.*

5 An existence result for weak solutions

We now consider SDE (1.1) and our main result is concerning the existence of weak solutions. We have

Theorem 5.1 *Assume that $p,q\in[1,+\infty)$. Let $b=b_1+b_2$ be such that $\mathcal{I}_Tb_1\in\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ with p,q satisfying (1.4), b_2 is bounded and Borel measurable. Let $C_0(p,d)$ be the constant from (4.2) and*

$$\|\mathcal{I}_Tb_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))} < (2C_0(p,d))^{-1}. \quad (5.1)$$

There is a filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]},\tilde{\mathbb{P}})$, two processes $\{\tilde{X}_t\}_{t\in[0,T]}$ and $\{\tilde{W}_t\}_{t\in[0,T]}$ such that $\{\tilde{W}_t\}_{t\in[0,T]}$ is a d-dimensional $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-Wiener process and $\{\tilde{X}_t\}_{t\in[0,T]}$ is an $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-adapted, continuous, d-dimensional process for which (1.7) holds, and almost surely, for all $t\in[0,T]$, (1.8) holds.

Proof. We follow the proof of [19, Theorem 1, p.87]. Firstly, we smooth out b_i ($i=1,2$) using the convolution: $b_1^n(t,x)=(b_1(t,\cdot)*\rho_n)(x)$, $b_2^n(t,x)=(b_2(t,\cdot)*\rho_n)(x)$ with ρ_n given by (2.3).

According to Proposition 2.1 and the properties of convolution, it is clear that, as $n\to\infty$,

$$\|\mathcal{I}_Tb_1^n-\mathcal{I}_Tb_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\to 0, \quad \|b_2^n-b_2\|_{L^q(0,T;L^p_{loc}(\mathbb{R}^d))}\to 0, \quad (5.2)$$

and for every $n\geqslant 1$, by Hausdorff-Young's convolution inequality, we have

$$\left\{\begin{array}{l}\|\mathcal{I}_Tb_1^n\|_{\mathcal{C}_q((0,T];L^p(\mathbb{R}^d))}\leqslant\|\mathcal{I}_Tb_1\|_{\mathcal{C}_q((0,T];L^p(\mathbb{R}^d))},\\ \|b_2^n\|_{L^\infty((0,T)\times\mathbb{R}^d)}\leqslant\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}.\end{array}\right. \quad (5.3)$$

Moreover there is a sequence of integrable functions h_i^n on $[0,T]$, such that

$$|b_i^n(t,x)-b_i^n(t,y)|\leqslant h_i^n(t)|x-y|, \quad \forall\, x,y\in\mathbb{R}^d, \quad i=1,2.$$

Using Cauchy-Lipschitz's theorem, there is a unique $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted, continuous, d-dimensional process X_t^n defined for $[0,T]$ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},\mathbb{P})$ such that

$$X_t^n = x + \int_0^t b^n(s,X_s^n)ds + W_t = x + \int_0^t b_1^n(s,X_s^n)ds + \int_0^t b_2^n(s,X_s^n)ds + W_t. \tag{5.4}$$

With the help of Theorem 4.1 and (5.3),

$$\begin{aligned}\mathbb{E}\int_0^T |b_1^n(t,X_t^n)|dt &\leqslant \Big(1+\mathbb{E}\int_0^T |b^n(t,X_t^n)|dt\Big) C_0(p,d)\|\mathcal{I}_T b_1^n\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\\ &\leqslant \Big(1+T\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}\\ &\quad +\mathbb{E}\int_0^T |b_1^n(t,X_t^n)|dt\Big) C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}.\end{aligned}$$

Noticing that (5.1) holds, then

$$C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))} < \frac{1}{2},$$

therefore

$$\mathbb{E}\int_0^T |b_1^n(t,X_t^n)|dt \leqslant 2\Big(1+T\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}\Big) C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}. \tag{5.5}$$

On the other hand, b_2^n is bounded uniformly in n, with the help of (5.3), one concludes that

$$\mathbb{E}\int_0^T |b_2^n(t,X_t^n)|dt \leqslant T\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}. \tag{5.6}$$

By (5.4)–(5.6), then

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}|X_t^n| \leqslant C < +\infty. \tag{5.7}$$

If one replaces the time interval $[0,T]$ by $[t_1,t_2]$ for every $0\leqslant t_1<t_2\leqslant T$, similar calculations from (4.3) to (4.4) also yields that

$$\begin{aligned}&\mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt\\ \leqslant\ & \mathbb{E}|U_n(T-t_2,X_{t_2}^n)-U_n(T-t_1,X_{t_1}^n)| + \sup_{(t,x)\in(0,T)\times\mathbb{R}^d}\|\nabla U_n\|\mathbb{E}\int_{t_1}^{t_2}|b^n(t,X_t^n)|dt\\ \leqslant\ & \sup_{x\in\mathbb{R}^d}|U_n(T-t_2,x)-U_n(T-t_1,x)|\\ &+\sup_{(t,x)\in(0,T)\times\mathbb{R}^d}\|\nabla U_n\|\Big(\mathbb{E}|X_{t_2}^n-X_{t_1}^n|+\mathbb{E}\int_{t_1}^{t_2}|b^n(t,X_t^n)|dt\Big),\end{aligned} \tag{5.8}$$

where U_n is the unique weak solution of

$$\begin{cases} \partial_t U_n(t,x) = \frac{1}{2}\Delta U_n(t,x) + |\mathcal{I}_T b_1^n(t,x)|, \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ U_n(0,x) = 0, \ x \in \mathbb{R}^d. \end{cases}$$

With the aid of Sobolev's embedding theorem, (3.15) and (5.3), from (5.8)

$$\begin{aligned} & \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt \\ \leqslant\ & C\|U_n(T-t_2) - U_n(T-t_1)\|_{W^{1,p}(\mathbb{R}^d)} + C_0(p,d)\|\mathcal{I}_T b_1^n\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\Bigg(\mathbb{E}|X_{t_2}^n - X_{t_1}^n| \\ & +|t_2-t_1|\|b_2^n\|_{L^\infty((0,T)\times\mathbb{R}^d)} + \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt\Bigg) \\ \leqslant\ & C|t_2-t_1|^{\frac{\theta}{2}}\|\mathcal{I}_T b_1^n\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))} + C_0(p,d)\|\mathcal{I}_T b_1^n\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\Bigg(\mathbb{E}|X_{t_2}^n - X_{t_1}^n| \\ & +|t_2-t_1|\|b_2^n\|_{L^\infty((0,T)\times\mathbb{R}^d)} + \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt\Bigg) \\ \leqslant\ & C|t_2-t_1|^{\frac{\theta}{2}}\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))} + C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\Bigg(\mathbb{E}|X_{t_2}^n - X_{t_1}^n| \\ & +|t_2-t_1|\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)} + \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt\Bigg), \end{aligned}$$

which suggests that

$$\begin{aligned} \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt \ \leqslant\ & \frac{C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}}{1 - C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}}\Big[C|t_2-t_1|^{\frac{\theta}{2}} \\ & +\mathbb{E}|X_{t_2}^n - X_{t_1}^n| + |t_2-t_1|\|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}\Big], \end{aligned} \tag{5.9}$$

where θ is given in (3.15).

By (5.1), there is a $\delta > 0$, such that

$$C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))} \leqslant (1 - C_0(p,d))\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))})(1-\delta).$$

Combining (5.4) and (5.9), it yields that

$$\begin{aligned} \mathbb{E}|X_{t_2}^n - X_{t_1}^n| \ \leqslant\ & \mathbb{E}\int_{t_1}^{t_2} |b_1^n(t,X_t^n)|dt + \mathbb{E}\int_{t_1}^{t_2} |b_2^n(t,X_t^n)|dt + \mathbb{E}|W_{t_2} - W_{t_1}| \\ \leqslant\ & (1-\delta)\mathbb{E}|X_{t_2}^n - X_{t_1}^n| + C\Big(|t_2-t_1|^{\frac{\theta}{2}} + |t_2-t_1| + |t_2-t_1|^{\frac{1}{2}}\Big) \\ \leqslant\ & (1-\delta)\mathbb{E}|X_{t_2}^n - X_{t_1}^n| + C|t_2-t_1|^{\frac{\theta}{2}}, \end{aligned}$$

which implies that

$$\mathbb{E}|X^n_{t_2}-X^n_{t_1}|\leqslant\frac{C}{\delta}|t_2-t_1|^{\frac{\theta}{2}}\leqslant C|t_2-t_1|^{\frac{\theta}{2}}. \tag{5.10}$$

By (5.7) and (5.10), for every $\epsilon>0$, one concludes that

$$\lim_{c\to\infty}\sup_n\sup_{t\in[0,T]}\mathbb{P}\{|X^n_t|>c\}=0 \tag{5.11}$$

and

$$\lim_{h\downarrow 0}\sup_n\sup_{t_1,t_2\in[0,T],|t_1-t_2|\leqslant h}\mathbb{P}\{|X^n_{t_1}-X^n_{t_2}|>\epsilon\}=0. \tag{5.12}$$

In view of Skorohod's representation theorem (see [19, Lemma 2, p.87]), there is a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random processes $(\tilde{X}^n_t,\tilde{W}^n_t)$, $(\tilde{X}_t,\tilde{W}_t)$ on this probability space such that

(i) the finite-dimensional distributions of $(\tilde{X}^n_t,\tilde{W}^n_t)$ coincide with the corresponding finite-dimensional distributions of the processes same as (X^n_t,W^n_t);

(ii) $(\tilde{X}^n_\cdot,\tilde{W}^n_\cdot)$ converges to $(\tilde{X}_\cdot,\tilde{W}_\cdot)$, $\tilde{\mathbb{P}}$-almost surely.

In particular $\tilde{W}$ is still a Wiener process and

$$\tilde{X}^n_t=x+\int_0^t b^n_1(s,\tilde{X}^n_s)ds+\int_0^t b^n_2(s,\tilde{X}^n_s)ds+\tilde{W}^n_t. \tag{5.13}$$

For any $k\in\mathbb{N}$, by virtue of Theorem 4.1,

$$\begin{aligned}
&\tilde{\mathbb{E}}\Bigg(\int_0^T|b^n_1(s,\tilde{X}^n_s)-b_1(s,\tilde{X}_s)|ds\Bigg)\\
\leqslant\ &\tilde{\mathbb{E}}\Bigg(\int_0^T|b^n_1(s,\tilde{X}^n_s)-b^k_1(s,\tilde{X}^n_s)|ds\Bigg)+\tilde{\mathbb{E}}\Bigg(\int_0^T|b^k_1(s,\tilde{X}^n_s)-b^k_1(s,\tilde{X}_s)|ds\Bigg)\\
&+\tilde{\mathbb{E}}\Bigg(\int_0^T|b^k_1(s,\tilde{X}_s)-b_1(s,\tilde{X}_s)|ds\Bigg)\\
\leqslant\ &C\Big[\|\mathcal{I}_Tb^n_1-\mathcal{I}_Tb^k_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}+\|\mathcal{I}_Tb^k_1-\mathcal{I}_Tb_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))}\Big]\\
&+\tilde{\mathbb{E}}\Bigg(\int_0^T|b^k_1(s,\tilde{X}^n_s)-b^k_1(s,\tilde{X}_s)|ds\Bigg).
\end{aligned} \tag{5.14}$$

By letting $n\to+\infty$ first, $k\to+\infty$ next, from (5.2) and (5.14) we arrive at

$$\lim_{n\to\infty}\int_0^t b^n_1(s,\tilde{X}^n_s)ds=\int_0^t b_1(s,\tilde{X}_s)ds,\quad \tilde{\mathbb{P}}-a.s.. \tag{5.15}$$

Similarly, we obtain

$$\begin{aligned}
&\tilde{\mathbb{E}}\left(\int_0^T |b_2^n(s,\tilde{X}_s^n)-b_2(s,\tilde{X}_s)|ds\right)\\
\leqslant\ &\tilde{\mathbb{E}}\left(\int_0^T |b_2^n(s,\tilde{X}_s^n)-b_2^k(s,\tilde{X}_s^n)|ds\right)+\tilde{\mathbb{E}}\left(\int_0^T |b_2^k(s,\tilde{X}_s^n)-b_2^k(s,\tilde{X}_s)|ds\right)\\
&+\tilde{\mathbb{E}}\left(\int_0^T |b_2^k(s,\tilde{X}_s)-b_2(s,\tilde{X}_s)|ds\right)\\
=\ &: J_1^n+J_2^n+J_3^n. \qquad (5.16)
\end{aligned}$$

For k fixed, as $n\to+\infty$, $J_2^n\to 0$. For J_1^n, we have

$$\begin{aligned}
J_1^n\ &\leqslant\ \|b_2\|_{L^\infty((0,T)\times\mathbb{R}^d)}\tilde{\mathbb{E}}\int_0^T |1-1_{|\tilde{X}_s^n|\leqslant R}|ds+\tilde{\mathbb{E}}\left(\int_0^T 1_{|\tilde{X}_s^n|\leqslant R}|b_2^n(s,\tilde{X}_s^n)-b_2^k(s,\tilde{X}_s^n)|ds\right)\\
&\leqslant\ \frac{C}{R}\tilde{\mathbb{E}}\int_0^T |\tilde{X}_s^n|ds+C\|1_{|x|\leqslant R}|b_2^n-b_2^k|\|_{L^q(0,T;L^p(\mathbb{R}^d))}, \qquad (5.17)
\end{aligned}$$

for every $R>0$.

By taking $n\to+\infty$, $k\to+\infty$, $R\to+\infty$ in turn, then $J_1^n\to 0$ as $n\to+\infty$. And the same conclusion for J_3^n is true by a same discussion. Combining (5.2), (5.16) and (5.17) we arrive at

$$\lim_{n\to\infty}\int_0^t b_2^n(s,\tilde{X}_s^n)ds=\int_0^t b_2(s,\tilde{X}_s)ds,\quad \tilde{\mathbb{P}}-a.s.. \qquad (5.18)$$

From (5.13), (5.15) and (5.18), one reaches at

$$\tilde{X}_t=x+\int_0^t b(s,\tilde{X}_s)ds+\tilde{W}_t.$$

From this one finishes the proof since by using Theorem 4.1, (1.7) holds true obviously. □

6 Uniqueness and the strong Feller property

We first discuss the uniqueness in probability laws. Before proceeding, let us present some useful lemmas.

Consider the following SDE:

$$dX_t=b(t,X_t)dt+\sigma(t,X_t)dW_t,\quad X_0=x\in\mathbb{R}^d,\quad t\in(0,T], \qquad (6.1)$$

where $\sigma(t,x) \in \mathbb{R}^{d\times d}$, $b(t,x) \in \mathbb{R}^d$. The regularity assumptions on σ and b should guarantee the existence of weak solutions to (6.1) (e.g. σ and b are bounded Borel measurable, and σ is uniformly elliptic, see [19, Theorem 1, p.87]). Let $(X_t, W_t)_{t\in[0,T]}$ be a weak solution of (6.1) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a reference family $\{\mathcal{F}_t\}_{t\in[0,T]}$, and let $(\tilde{X}_t, \tilde{W}_t)_{t\in[0,T]}$ be another weak solution of (6.1) on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with a reference family $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$. We denote the probability laws of $\{X_t\}_{t\in[0,T]}$ and $\{\tilde{X}_t\}_{t\in[0,T]}$ on d-dimensional classical Wiener space $(W^d([0,T]), \mathcal{B}(W^d([0,T])))$ by $\mathbb{P}_x = \mathbb{P}\circ X^{-1}$ and $\tilde{\mathbb{P}}_x = \mathbb{P}\circ \tilde{X}^{-1}$, respectively. Then we have

(i) for every $x \in \mathbb{R}^d$, $\mathbb{P}_x(w \in W^d([0,T]); w_0 = x) = \tilde{\mathbb{P}}_x(w \in W^d([0,T]); w_0 = x) = 1$;

(ii) for every $A_1, A_2 \in \mathcal{B}(W^d([0,T]))$, the mappings $x \mapsto \mathbb{P}_x(A_1)$ and $x \mapsto \tilde{\mathbb{P}}_x(A_2)$ are $\mathcal{B}(\mathbb{R}^d)$ measurable.

Moreover, from [13, Theorem 20.1, p.182] (or [14, Proposition 2.1, p.169]), for every $f \in \mathcal{C}_b^2(\mathbb{R}^d)$

(iii) the processes

$$f(X_t) - f(x) - \int_0^t \Big[\nabla f(X_s)\cdot b(s,X_s) + \frac{1}{2}\mathrm{trac}(\sigma(s,X_s)\sigma(s,X_s)^\top \nabla^2(f(X_s)))\Big]ds, \ t\in[0,T],$$

and

$$f(\tilde{X}_t) - f(x) - \int_0^t \Big[\nabla f(\tilde{X}_s)\cdot b(s,\tilde{X}_s) + \frac{1}{2}\mathrm{trac}(\sigma(s,\tilde{X}_s)\sigma(s,\tilde{X}_s)^\top \nabla^2(f(\tilde{X}_s)))\Big]ds, \ t\in[0,T],$$

are locally continuous $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ martingales, respectively.

Combining the above conclusions (i)-(iii) and [13, Theorem 20.1, p.195] (or [14, Corollary, p.206]), we have

Lemma 6.1 *$\mathbb{P}_x = \tilde{\mathbb{P}}_x$ is equivalent to*

$$\int_{W^d([0,T])} f(w(t))\mathbb{P}_x(dw) = \int_{W^d([0,T])} f(w(t))\tilde{\mathbb{P}}_x(dw), \tag{6.2}$$

for every $t\in[0,T]$ and every $f \in \mathcal{C}_b(\mathbb{R}^d)$.

Now we give our uniqueness result on weak solutions.

Lemma 6.2 *Let p, q and b be stated in Theorem 5.1. We assume that $\mathcal{I}_T b_1 \in \mathcal{C}^0_{[q]}([0,T]; L^p(\mathbb{R}^d;\mathbb{R}^d))$ in addition. If there are two weak solutions of (1.1), then the probability laws of them on d-dimensional classical Wiener space $(W^d([0,T]), \mathcal{B}(W^d([0,T])))$ are the same.*

Proof. We show the uniqueness by using Itô-Tanaka's trick (see [10]). Consider the following vector valued Cauchy problem

$$\begin{cases} \partial_t U(t,x) = \frac{1}{2}\Delta U(t,x) + \mathcal{I}_T b_1(t,x)\cdot\nabla U(t,x) \\ \qquad\qquad + \mathcal{I}_T b_1(t,x), \ (t,x) \in (0,T]\times\mathbb{R}^d, \\ U(0,x) = 0, \ x\in\mathbb{R}^d. \end{cases} \tag{6.3}$$

According to Theorem 3.1, there is a unique weak solution U to (6.3). Moreover by Corollary 3.1, $U \in \mathcal{C}([0,T];\mathcal{C}_u^1(\mathbb{R}^d;\mathbb{R}^d))$ and there is a $\delta > 0$ such that

$$\|U\|_{\mathcal{C}([0,T];\mathcal{C}_u^1(\mathbb{R}^d;\mathbb{R}^d))} \leqslant \frac{C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))}}{1 - C_0(p,d)\|\mathcal{I}_T b_1\|_{\mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))}} < 1-\delta,$$

since (5.1) holds.

We define $\Phi(t,x) = x + \mathcal{I}_T U(t,x)$, then

$$\begin{cases} \delta < \|\nabla\Phi\|_{\mathcal{C}([0,T];\mathcal{C}_b(\mathbb{R}^d))} < 2-\delta, \\ \frac{1}{2-\delta} < \|\nabla\Psi\|_{\mathcal{C}([0,T];\mathcal{C}_b(\mathbb{R}^d))} < \frac{1}{\delta}, \end{cases} \tag{6.4}$$

where $\Psi(t,x) = \Phi^{-1}(t,x)$. By the classical Hadamard theorem ([27, p.330]), $\Phi(t,\cdot)$ forms a nonsingular diffeomorphism of class $\mathcal{C}^1$ uniformly in $t \in [0,T]$.

Noticing that $\mathcal{I}_T b_1 \in \mathcal{C}^0_{[q]}([0,T];L^p(\mathbb{R}^d;\mathbb{R}^d))$, thus the unique weak solution of (6.3) also belongs to $L^m(T-T_0,T;W^{2,p}(\mathbb{R}^d;\mathbb{R}^d)) \cap W^{1,m}(T-T_0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$ for every $T_0 < T$ and every $m \geqslant 1$. Therefore, $\mathcal{I}_T U \in L^m(0,T_0;W^{2,p}(\mathbb{R}^d;\mathbb{R}^d)) \cap W^{1,m}(0,T_0;L^p(\mathbb{R}^d;\mathbb{R}^d))$. If $(X_t,W_t)_{t\in[0,T]}$ is a weak solution of (1.1), we can use Itô's formula (see [20, Theorem 3.7]) to $\Phi(t,X_t)$ and obtain

$$d\Phi(t,X_t) = b_2(t,X_t)dt + (I + \nabla\mathcal{I}_T U(t,X_t))dW_t, \quad t \in (0,T_0].$$

By Corollary 3.1, we can take T_0 approaching to T, then get

$$\begin{cases} d\Phi(t,X_t) = b_2(t,X_t)dt + (I + \nabla\mathcal{I}_T U(t,X_t))dW_t, \quad t \in (0,T], \\ \Phi(0,X_0) = x + U(T,x). \end{cases}$$

Denote $Y_t = X_t + \mathcal{I}_T U(t,X_t)$, then

$$dY_t = b_2(t,\Psi(t,Y_t))dt + (I + \nabla\mathcal{I}_T U(t,\Psi(t,Y_t))dW_t, \quad t \in (0,T], \tag{6.5}$$

with $Y_0 = y$.

Now we assume that (X,W) and $(\tilde{X},\tilde{W})$ are weak solutions of (1.1) and the probability laws of X and $\tilde{X}$ on $(W^d([0,T]),\mathcal{B}(W^d([0,T])))$ are $\mathbb{P}_x$ and $\tilde{\mathbb{P}}_x$ respectively. Correspondingly, we denote $\mathbb{P}_y$ and $\tilde{\mathbb{P}}_y$ the probability laws of Y and $\tilde{Y}$. Since $Y_t = \Phi(t,X_t)$ and $\Phi \in \mathcal{C}([0,1];\mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d))$ is a diffeomorphism on $\mathbb{R}^d$ uniformly for every $t \in [0,T]$, the relationships of $\mathbb{P}_x$ and $\mathbb{P}_y$, $\tilde{\mathbb{P}}_x$ and $\tilde{\mathbb{P}}_y$ are given by

$$\mathbb{P}_y = \mathbb{P}_x \circ \Phi^{-1}, \quad \tilde{\mathbb{P}}_y = \tilde{\mathbb{P}}_x \circ \Phi^{-1}, \tag{6.6}$$

where for a given measure μ on a Banach space S, and ϑ a map on S, we use the notation $\mu \circ \vartheta^{-1}$ to denote image measure of μ by the map ϑ, i.e.

$$\int_S \phi d(\mu \circ \vartheta^{-1}) = \int_S \phi(\vartheta) d\mu.$$

Combining (6.4) and noting that b_2 is bounded Borel measurable, $\mathcal{I}_T U$ is continuous in (t,x), and $I+\mathcal{I}_T U$ satisfies uniformly elliptic condition, applying [32, Theorem 5.6] (also see [14, Theorem 3.3, p185] for time independent σ), the uniqueness of probability law for (6.5) is true. So $\mathbb{P}_y = \tilde{\mathbb{P}}_y$.

For every $f \in \mathcal{C}_b(\mathbb{R}^d)$ and every $t \in [0,T]$, by (6.6) then

$$\begin{aligned}\int_{W^d([0,T])} f(w(t))\mathbb{P}_x(dw) &= \int_{W^d([0,T])} f(\Psi(t,\Phi(t,w(t))))\mathbb{P}_x(dw) \\ &= \int_{W^d([0,T])} f(\Psi(t,w(t)))\mathbb{P}_y(dw) \end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\int_{W^d([0,T])} f(w(t))\tilde{\mathbb{P}}_x(dw) &= \int_{W^d([0,T])} f(\Psi(t,\Phi(t,w(t))))\tilde{\mathbb{P}}_x(dw) \\ &= \int_{W^d([0,T])} f(\Psi(t,w(t)))\tilde{\mathbb{P}}_y(dw). \end{aligned} \tag{6.8}$$

Since $\mathbb{P}_y = \tilde{\mathbb{P}}_y$, and for every $t \in [0,T]$, $f \circ \Psi(t,\cdot) \in \mathcal{C}_b(\mathbb{R}^d)$, from (6.7) and (6.8) one ends up with (6.2). By applying Lemma 6.1, it is unique and we finish the proof. $\square$

We are now in the position to give our strong uniqueness result.

Theorem 6.1 *Let p, q, b and b_1 be stated in Lemma 6.2. Then there exists a unique strong solution to SDE (1.1).*

Proof. Clearly, by Yamada-Watanabe's principle (see [41]) and Theorem 5.1, one only needs to prove the pathwise uniqueness. In view of Lemma 6.2 and the fact the uniqueness in probability law implies the pathwise uniqueness for $d=1$ (see [42, Proposition 1.1]), so we need to show the pathwise uniqueness for $d>1$.

Since (1.6) holds and $q<+\infty$, we conclude $p>2$ when $d>1$ and for every $T_0 \in (0,T)$,

$$b_1 \in L^m(0,T_0; L^p(\mathbb{R}^d;\mathbb{R}^d)),$$

with $p>2$ and every $m \geqslant 1$. Therefore, b is in Krylov-Röckner class up to time T_0, and this implies the pathwise uniqueness of solutions for SDE (1.1) up to time $T_0<T$ (see [20]). Particularly, we can choose $T_0 = T-\varepsilon$ ($\varepsilon>0$ is sufficiently small) and it yields the strong uniqueness of solutions on $[0,T-\varepsilon]$. By letting ε tend to 0, we conclude the the strong uniqueness of solutions on $[0,T)$. On the other hand, by virtue of Theorem 5.1 all the solutions of SDE (1.1) are continuous, so the strong uniqueness of solutions holds true on $[0,T]$. $\square$

Remark 6.1 *We divide the proof for the strong uniqueness into two cases: $d=1$ and $d>1$. When $d>1$, Krylov-Röckner's result is adapted here directly. However, when $d=1$, the assumption $2/q+1/p=1$ only implies that $p>1 (\nRightarrow p>2)$, so b is not in Krylov-Röckner class and Krylov-Röckner's* *result (see [20]) can not be used here. Fortunately, when $d = 1$ the uniqueness in probability law and the pathwise uniqueness are equivalent. Therefore, it is sufficient to show the uniqueness in probability. To reach our aim, we introduce a useful lemma (Lemma 6.2) before stating our main theorem.*

Remark 6.2 *(i) When the Banach space $\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}^d))$ is replaced by $L^q(0,T;L^p(\mathbb{R}^d))$, with q,p meeting condition (1.3), there are many elegant works [44, 45]. For example, under the hypothesises that*

(1) $\sigma(t,x)$ is uniformly continuous in $x \in \mathbb{R}^d$ uniformly with respect to t and there is a positive constant δ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\delta|\xi|^2 \leqslant |\sigma(t,x)\xi|^2 \leqslant \frac{1}{\delta}|\xi|^2.$$

(2) $|\nabla_x \sigma(t,x)|$, $|b| \in L^q(0,T;L^p(\mathbb{R}^d))$. Zhang [45] obtained the existence and uniqueness of the strong solution to SDE (6.1.)

(ii) Other topics on SDE (6.1) such as existence, uniqueness of solutions, stochastic homeomorphism, weak differentiability for b and σ in different classes, we refer to [3, 7, 11, 12, 34, 37, 38] and the references cited up there.

Now we discuss the Feller property and the existence of density and initially we give a lemma.

Lemma 6.3 *([33]) Consider the following SDE*

$$\left\{ \begin{array}{l} dX_t(x) = b(t,X_t(x))dt + \sigma(t,X_t(x))dW_t, \ t \in (s,T], \\ X_s(x) = x \in \mathbb{R}^d. \end{array} \right. \tag{6.9}$$

Suppose that b is bounded and Borel measurable, σ is bounded continuous and $(a_{i,j}) = (\sum_k \sigma_{i,k}\sigma_{j,k})/2$ is uniformly continuous and uniformly elliptic. Then there is a unique weak solution of (6.9) which is a strong Markov process. Let $\tilde{P}(s,x,t,dy)$ be its transition probabilities and for every bounded function f, we define

$$\tilde{P}_{s,t}f(x) = \int_{\mathbb{R}^d} f(y)\tilde{P}(s,x,t,dy), \tag{6.10}$$

then we have the following claims:

(i) $\tilde{P}_{s,t}f(x)$ is continuous in s and x for $s < t$.

(ii) $\tilde{P}(s,x,t,dy)$ has a density $\tilde{p}(s,x,t,y)$ for almost all $t \in [0,T]$ which satisfies

$$\int_{t_0}^{T}\int_{\mathbb{R}^d} |\tilde{p}(s,x,t,y)|^r dydt < +\infty, \tag{6.11}$$

for every $r \in [1,+\infty)$ provided $s < t_0$.

Now we give our second result.

Theorem 6.2 *Let p, q and b be described in Theorem 6.1. Consider the following SDE:*

$$\begin{cases} dX_t(x) = b(t, X_t(x))dt + dW_t, \ 0 \leqslant s < t \leqslant T, \\ X_s(x) = x \in \mathbb{R}^d. \end{cases} \tag{6.12}$$

Let $P(s, x, t, dy)$ be the transition probabilities associated with the solution of (6.12) and for every bounded function f, we set $P_{s,t}f(x)$ by (6.10). Then $P_{s,t}f(x)$ is continuous in s and x for $s < t$ and $P(s, x, t, dy)$ has a density $p(s, x, t, y)$ for almost all $t \in [0, T]$ which satisfies (6.11).

Proof. In view of Theorems 5.1 and 6.1, there exists a unique strong solution to SDE (6.12). Let $\mathbb{P}_{s,x} = \mathbb{P} \circ X^{-1}$ be the probability laws of $\{X_t\}_{t\in[s,T]}$ on d-dimensional classical Wiener space $(W^d([s,T]), \mathcal{B}(W^d([s,T])))$. Consider SDE (6.5) with initial data $Y_t|_{t=s} = \Phi(s, x)$ and let $\tilde{\mathbb{P}}_{s,\Phi(s,x)} = \mathbb{P} \circ Y^{-1}$ be the probability laws of $\{Y_t\}_{t\in[s,T]}$ on d-dimensional classical Wiener space $(W^d([s,T]), \mathcal{B}(W^d([s,T])))$. By virtue of Lemma 6.3, $\tilde{\mathbb{P}}_{s,\Phi(s,x)}$ is a strong Markov process and then by (6.4) and (6.6), $\mathbb{P}_{s,x}$ is also a strong Markov process. Let $P(s, x, t, dy)$ and $\tilde{P}(s, \Phi(s, x), t, dy)$ be the transition probabilities of $\mathbb{P}_{s,x}$ and $\tilde{\mathbb{P}}_{s,\Phi(s,x)}$, respectively. Then for every $f \in L^\infty(\mathbb{R}^d)$,

$$P_{s,t}f(x) = \mathbb{E}^{\mathbb{P}_{s,x}} f(w(t)) = \int_{\mathbb{R}^d} f(y)P(s, x, t, dy)$$

and

$$\tilde{P}_{s,t}f(x) = \mathbb{E}^{\tilde{\mathbb{P}}_{s,\Phi(s,x)}} f(w(t)) = \int_{\mathbb{R}^d} f(y)\tilde{P}(s, \Phi(s, x), t, dy).$$

With the help of (6.6), it yields that

$$P_{s,t}f(x) = \tilde{P}_{s,t}f(\Psi(t, x)) = \int_{\mathbb{R}^d} f(\Psi(t, y))\tilde{P}(s, \Phi(s, x), t, dy). \tag{6.13}$$

So $P_{s,t}f(x)$ is continuous in s and x for $s < t$. In particular, the semi-group $P_{0,t}(=: P_t)$ has the strong Feller property. And by Lemma 6.3, $\tilde{P}_{s,t}$ has a density $\tilde{p}(s, \Phi(s, x), t, y)$. From (6.13), then

$$P_{s,t}f(x) = \int_{\mathbb{R}^d} f(\Psi(t, y))\tilde{p}(s, \Phi(s, x), t, y)dy = \int_{\mathbb{R}^d} f(y)\tilde{p}(s, \Phi(s, x), t, \Phi(t, y))|\nabla\Phi(t, y)|dy.$$

Hence, for almost all $t \in [0, T]$, $P_{s,t}$ has a density $p(s, x, t, y)$ which is given by

$$p(s, x, t, y) = \tilde{p}(s, \Phi(s, x), t, \Phi(t, y))|\nabla\Phi(t, y)|. \tag{6.14}$$

From (6.14), for every $r \in [1, +\infty)$ and $s < t_0$

$$\begin{aligned} \int_{t_0}^T \int_{\mathbb{R}^d} |p(s, x, t, y)|^r dydt &= \int_{t_0}^T \int_{\mathbb{R}^d} |\tilde{p}(s, \Phi(s, x), t, \Phi(t, y))|\nabla\Phi(t, y)||^r dydt \\ &= \int_{t_0}^T \int_{\mathbb{R}^d} |\tilde{p}(s, \Phi(s, x), t, y)|\nabla\Phi(t, \Psi(t, y))||^r |\nabla\Psi(t, y)|dydt \\ &\leqslant C \int_{t_0}^T \int_{\mathbb{R}^d} |\tilde{p}(s, \Phi(s, x), t, y)|^r dydt < +\infty, \end{aligned}$$

where in the last line we have used (6.4). □

7 SDEs with non-constant diffusion coefficients

Theorems 5.1 and 6.1 can be generalized to the case of non-constant diffusion if the diffusion coefficients are regular enough. For simplicity, we give an extension in this short section and we assume that the diffusion coefficient is time independent and $d = 1$.

Theorem 7.1 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Suppose that there are positive constants δ_1 and δ_2 such that $\delta_1 \leqslant \sigma \leqslant \delta_2$. Consider the following SDE with non-constant diffusion in $\mathbb{R}$*

$$dX_t = b(t, X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}, \quad t \in (0, T]. \tag{7.1}$$

Let p and q be given in Theorem 5.1, that $b = b_1 + b_2$ such that $\mathcal{I}_T b_1 \in \mathcal{C}_{[q]}([0, T]; L^p(\mathbb{R}))$ and $\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}))}$ is sufficiently small, b_2 is bounded Borel measurable. Moreover, for this p, we assume in addition that $\sigma' = \tilde{\sigma}_1 + \tilde{\sigma}_2$, with $\tilde{\sigma}_1 \in L^p(\mathbb{R})$ and $\|\tilde{\sigma}_1\|_{L^p(\mathbb{R})}$ is small enough, $\tilde{\sigma}_2 \in L^\infty(\mathbb{R})$.

(i) There is a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$, two processes $\tilde{X}_t$ and $\tilde{W}_t$ defined for $[0, T]$ on it such that $\{\tilde{W}_t\}_{t\in[0,T]}$ is a 1-dimensional $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-Wiener process and $\{\tilde{X}_t\}_{t\in[0,T]}$ is an $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-adapted, continuous, 1-dimensional process for which (1.7) holds, and almost surely, for all $t \in [0, T]$

$$\tilde{X}_t = x + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t \sigma(\tilde{X}_s)d\tilde{W}_s.$$

(ii) We suppose $\mathcal{I}_T b_1 \in \mathcal{C}^0_{[q]}([0, T]; L^p(\mathbb{R}))$ further, then there exists a unique strong solution to (7.1).

Proof. (i) The proof here is inspired by Zvonkin's transformation. Let us define

$$\Phi(x) = \int_0^x \frac{1}{\sigma(y)}dy,$$

and since $\delta_1 \leqslant \sigma \leqslant \delta_2$, Φ^{-1} exists. Moreover, for every $x, y \in \mathbb{R}$,

$$\begin{cases} \delta_2^{-1}|x - y| \leqslant |\Phi(x) - \Phi(y)| \leqslant \delta_1^{-1}|x - y|, \\ \delta_1|x - y| \leqslant |\Phi^{-1}(x) - \Phi^{-1}(y)| \leqslant \delta_2|x - y|. \end{cases}$$

Let us consider the following SDE

$$Y_t(y) = y + \int_0^t [b(s, \Phi^{-1}(Y_s))\sigma^{-1}(\Phi^{-1}(Y_s)) - \frac{1}{2}\sigma'(\Phi^{-1}(Y_s))]ds + W_t. \tag{7.2}$$

Noting that $\mathcal{I}_T b_1 \in \mathcal{C}_{[q]}([0, T]; L^p(\mathbb{R}))$ and $\|\mathcal{I}_T b_1\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}))}$ is small enough, so $\mathcal{I}_T b_1(\cdot, \Phi^{-1}(\cdot)) \in \mathcal{C}_{[q]}([0, T]; L^p(\mathbb{R}))$, $\|\mathcal{I}_T b_1(\cdot, \Phi^{-1}(\cdot))\|_{\mathcal{C}_{[q]}([0,T];L^p(\mathbb{R}))}$ is sufficiently small too. And this conclusion holds for $\tilde{\sigma}_1(\Phi^{-1})$. Upon using Theorem 5.1, there is a weak solution $(\tilde{Y}_t, \tilde{W}_t)$ of (7.2) on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$ for which $\{\tilde{W}_t\}_{t\in[0,T]}$ is a standard 1-dimensional standard Wiener process to $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$.

Initially, we assume $\sigma \in \mathcal{C}^\infty(\mathbb{R})$, then Φ^{-1} is smooth. By utilizing Itô's formula, one derives that

$$\begin{aligned} d\Phi^{-1}(\tilde{Y}_t) &= [\Phi^{-1}]'(\tilde{Y}_t)d\tilde{Y}_t - \frac{1}{2}[\Phi'(\Phi^{-1}(\tilde{Y}_t))]^{-1}\Phi''(\Phi^{-1}(\tilde{Y}_t))[\Phi^{-1}]'(\tilde{Y}_t)^2 dt \\ &= \sigma(\Phi^{-1}(\tilde{Y}_t))[b(t,\Phi^{-1}(\tilde{Y}_t))\sigma^{-1}(\Phi^{-1}(\tilde{Y}_t)) - \frac{1}{2}\sigma'(\Phi^{-1}(\tilde{Y}_t))]dt \\ &\quad +\sigma(\Phi^{-1}(\tilde{Y}_t))d\tilde{W}_t + \frac{1}{2}\sigma(\Phi^{-1}(\tilde{Y}_t))\sigma'(\Phi^{-1}(\tilde{Y}_t))dt \\ &= b(t,\Phi^{-1}(\tilde{Y}_t))dt + \sigma(\Phi^{-1}(\tilde{Y}_t))d\tilde{W}_t, \end{aligned} \tag{7.3}$$

which implies $(\tilde{X}_t, \tilde{W}_t) = (\Phi^{-1}(\tilde{Y}_t), \tilde{W}_t)$ is a weak solution of (7.1).

For general σ, we smooth it by convolution $\sigma_\varepsilon = \sigma * \varrho_\varepsilon$, where ϱ_ε is a regularising kernel on $\mathbb{R}$, i.e.

$$\varrho_\varepsilon = \frac{1}{\varepsilon}\varrho(\frac{\cdot}{\varepsilon}) \ \text{ with } \ 0 \leqslant \varrho \in \mathcal{C}_0^\infty(\mathbb{R}), \ \ \text{support}(\varrho) \subset (-1,1)$$

and $\int_{\mathbb{R}} \varrho(r)dr = 1$. For Φ_ε^{-1}, one gets an analogue of identity (7.3). With the same argument as in Theorem 5.1, by taking $\varepsilon \to 0$, one derives the conclusion (i).

(ii) Clearly, it suffices to prove the pathwise uniqueness. By the relationship between (7.1) and (7.2), it suffices to show the pathwise uniqueness for (7.2). Since the form of (7.2) is the same as (1.1), we need prove the pathwise uniqueness for (1.1) on $d = 1$. When $d = 1$, the uniqueness in probability law implies the pathwise uniqueness (see [42, Proposition 1.1]) and by Lemma 6.2 the uniqueness in probability law is valid, so we finish the proof. □

Remark 7.1 *(i) The proof for the existence of weak solutions to SDE (7.1) is inspired by Zvonkin's transformation. For more details in this topic, one can consult to [47].*

(ii) Here we do not argue the general case, i.e. σ is time dependent and $d > 1$. As discussed in [18], we may prove the existence and uniqueness for weak solutions, such that the uniqueness holds only in the sense of finite dimensional probability laws. For more details in this topic one can refers to [18] and the references cited up there.

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