


# Naive Fracterm Calculus


**Jan Bergstra**

(University of Amsterdam, The Netherlands)

 <https://orcid.org/0000-0003-2492-506X>, [j.a.bergstra@uva.nl](mailto:j.a.bergstra@uva.nl), [janaldertb@gmail.com](mailto:janaldertb@gmail.com))

**John V. Tucker**

(Swansea University, Swansea, Wales, United Kingdom)

 <https://orcid.org/0000-0003-4689-8760>, [j.v.tucker@swansea.ac.uk](mailto:j.v.tucker@swansea.ac.uk))

**Abstract:** An outline is provided of a new perspective on elementary arithmetic, based on addition, multiplication, subtraction and division, which is informal and unique and may be considered naive when contrasted with a plurality of algebraic and logical, axiomatic formalisations of elementary arithmetic.

**Keywords:** elementary arithmetic, division by zero, fracterm calculus, common meadow, abstract data type

**Categories:** F.4.2, F.4.1, D.3.1

**DOI:** 10.3897/jucs.87563

## 1 Introduction

Our ideas about numbers and arithmetical practices arise from our desire and need for measurement. The development of current notations and rules for calculation is the work of centuries. By the 16th Century, arithmetic was recognisable with its operations of

$$+, -, \cdot, \div$$

and calculations with expressions containing variables and the equality sign  $=$ .<sup>1</sup> So it remains, as can be seen in school textbooks and in the buttons of pocket calculators (in hardware and software).

It took further centuries to develop our current theories of arithmetic as mathematics itself changed in matters of rigour and foundations. Foremost are the theories of rings and fields, with their axioms, structures, morphisms, and polynomials etc. Although beautiful and deep, the algebra and logic of rings and fields are incomplete as theories of arithmetic because they rather neglect division  $\div$ . This neglect leads to division problems in fields that can be seen in calculating with rational numbers – the primary number system of measurement and practical arithmetic – in the misunderstanding of fractions in school arithmetic and the need to avoid partial operators in computer arithmetic.

In field theory, division is not understood as a primitive operation, so it is not on a par with the addition, subtraction and multiplication. The working of the operations of

<sup>1</sup> See, for example, the English vernacular works of Robert Recorde, where  $=$  first appears in 1557; actually, for division, Recorde uses fractions with numerators and divisors, which were well established [Williams 2014]. The notation  $\div$  becomes a sign for division later – after some years serving as one of several signs for subtraction.

$+$ ,  $-$ ,  $\cdot$  are specified axiomatically, and, in particular, there is a plurality of options for axioms that are equations. In the practical arithmetic of the schools, equations rule and division is omnipresent! Surprisingly, however, writing axioms for division – unlike for addition, multiplication, and subtraction – has hardly any tradition in arithmetic.

In any case, fields with an explicit multiplicative division, or inverse, cannot be axiomatised by equations.<sup>2</sup> The stumbling block is division by zero, namely  $\frac{x}{0}$ . Fields are axiomatised by avoiding division by zero

$$\forall x \exists y [x \neq 0 \rightarrow x \cdot y = 1]$$

or, on adding a partial inverse operation  $x^{-1}$ , by

$$\forall x [x \neq 0 \rightarrow x \cdot x^{-1} = 1].$$

The axioms involve negation  $\neq$ . Starting in [Bergstra and Tucker 2007], we have studied several ways of enriching fields with division to form algebras we call *meadows* and ways to axiomatise reasoning about meadows using equations, e.g., [Bergstra et al. 2009, Bethke and Rodenburg 2010, Bergstra and Ponse 2021, Bergstra and Tucker 2021, Bergstra and Tucker 2022].

The purpose of this paper is to propose and develop ideas of informal calculi for division that are close to practical arithmetic as taught in schools, and to contrast these informal calculi with a plurality of formal calculi, close to theories of arithmetic of interest to mathematicians and computer scientists.

### 1.1 On formal versus informal calculi

Practical arithmetic, even at the elementary levels of the schools, is not easy to discuss in an explicit and systematic way – as an extensive literature on teaching arithmetic largely confirms. Certainly, an account that would qualify as an ‘informal calculus’ seems remote. The term ‘practical’ refers to practices by people, which are inherently diverse and can be incomparable. By ‘calculus’ we have in mind operations, expressions, equations, and methods for their application as found in school textbooks; note that a calculus is *not* necessarily a formal object.<sup>3</sup> Thus, our task is anthropomorphic as well as mathematical.

At the outset, we conceive of the task in four stages which we name and sketch as follows:

1. *Raw Arithmetic*. Examine arithmetical practices ‘in the wild’.
2. *Naive Arithmetic*. Formulate a description of what seems to be a consensus on what ideas and conventions are agreed, disputed or remain ambiguous in peoples’ practices.
3. *Synthetic Arithmetic*. Tighten and refine the informal description that is Naive Arithmetic to resolve ambiguities and arbitrate some disputed ideas and conventions and make it fit for systematic logical reasoning.

<sup>2</sup> Structures axiomatised by equations are closed under products, but products of fields have zero divisors.

<sup>3</sup> In mathematical logic and computing, calculi are formal systems with carefully defined notations, rules and meanings capable of metamathematical investigation and machine implementations. It’s nice to remember the etymology of ‘calculus’ in Roman literature: pebble or stone used in arithmetic reckoning on counting boards.

4. *Formal Calculi*. Propose formal calculi for elementary arithmetic based on the informal analysis of the Naive and Synthetic.

In this paper, we focus on building a calculus that captures Naive Arithmetic and contrast it with the various designs for formal calculi. Now, explorations of Raw Arithmetic have been undertaken already, e.g., [Bergstra 2020]; indeed these partially motivate the current problem of describing a consensus. Our technical development of several formal calculi, motivated by computer arithmetic, we have mentioned earlier and will return to later. The creation of an intermediate Synthetic Arithmetic is for another paper.

The technical needs of division require special attention to fractions. The explorations of Raw Arithmetic reveal considerable conceptual difficulties with this term that we have decided to highlight in Naive Arithmetic by coining and using the word ‘fracterm’ (see section 3.1 below). Thus, our *naive fracterm calculus* of the title is an attempt at an informal requirements analysis of peoples’ practices when actually doing arithmetic, especially divisions.

## 1.2 On fracterm calculi

The axioms of our formal calculi for arithmetic are based on equations and the calculations on term rewriting. The central concepts are division and the fracterm, which is any arithmetical expression with division as its leading function symbol [Bergstra 2020]. Usually, we will use the horizontal bar notation for denoting the division function. Since division is the least familiar operation, certainly from a logical perspective, following [Bergstra and Tucker 2022], a calculus with these primitives is referred to as a *fracterm calculus*. Thus, a formal fracterm calculus is a calculus with syntax generated by the operations for a meadow (a field together with division). Assertions of a formal fracterm calculus first of all take the form of equations  $t = r$  where  $t$  and  $r$  are terms that may contain variables; more involved assertions require logical connectives and quantifiers.

This paper is focussed on designing an *informal* fracterm calculus that we call a *naive* fracterm calculus in order to contrast with the formal fracterm calculi. A word about ‘naive’. Although focussed on the most elementary part of mathematics, we are influenced by one of the most logically subtle parts, *set theory*. Mathematical practitioners need a reliable set theory that can be used *without* technical knowledge of its logical foundations [Halmos 1960]. Just as *naive set theory* acquires its name on the basis of the existence of formal axiomatic set theories (e.g.,  $ZF$ ,  $ZFC$ ,  $NF$ , ...), to which it stands in contrast, so *naive fracterm calculus* acquires its name in contrast with formal fracterm calculi. The parallels are closer than simply naming, as we will see from time to time.

## 1.3 Methodology

Guiding our design of fracterm calculi are ideas about abstract data types for arithmetic. Clearly, the theory of equational specification and reasoning with abstract data types is ideally suited to developing *formal* axiomatic calculi for arithmetic, as we will demonstrate later. Interestingly, arithmetical data types also provide intuitions and means to explore what elementary school arithmetic is about, what is essential in its use, and how it can be organised.

The theory of abstract data types offers an approach to the formalisation of arithmetical data types with equations centre stage. Formalising arithmetic for rational numbers leads directly to a plurality of fracterm calculi. Each formal fracterm calculus comes with

a commitment to the principles of data types, abstract data types, and formal reasoning as well as a contrast between syntax and semantics. (We provide a limited survey of formal fracterm calculi later.)

The idea of an arithmetical data type also provides some guidance on how elementary arithmetic can be further developed in less conventional directions. The informal calculus we propose here is intended to lie in a spectrum: it improves on arithmetic as expressed in natural language, but remains informal. It is intended as one source and reference for the formal calculi.

Our methodology makes explicit the process of designing the informal calculus by making explicit our objectives, claims about potential users, our design options and our design decisions. What drives the development are arguments and perceptions about *meaningfulness*, the *plausibility* of decisions, and their *relevance* to users. We use the intuitions of abstract data type theory to source and choose technical options to adopt, reject, or leave open. For this, we have a terminology: the options that make up NFTC are *committed*, *adversely committed*, or *uncommitted*, respectively.

An arithmetical data type provides syntax by means of a signature that lists names of kinds of numbers, constants and functions; and it provides meaning in terms of idealised entities which serve as abstractions. Intuitions emerge for distinctions between expressions and entities, such as the expression  $(2 + (3 \cdot 7)) - 1$  which denotes 22. Of course, more subtle points about numbers surface related to signification and semiotics. One may imagine that a class of numbers constitutes a collection of mental, ideal, entities. Decimal notations such as 22 are names which refer to these underlying entities. The actual occurrence of the name 22 on a backboard is itself a sign. The same number may come under various names and the same name may be represented by a plurality of signs at any moment of time. The number denoted by 22 is then understood as an abstraction of its name.

#### 1.4 Contributions and structure of the paper

Remarkably, in spite of the ancient and familiar collection of operations, fracterm calculi have not widely been studied before our research programme on meadows.

The contributions of this paper are:

1. To design rigorously an *informal* calculus NFTC for arithmetic focussed on division.
2. To make explicit our methods in developing the calculus – design objectives, options and decisions for and against technical features for the calculus.
3. To introduce and compare some *formal* calculi FTC for arithmetic and contrast them with NFTC.

Our paper aims to offer new ideas to those interested in: teaching elementary arithmetic; mathematical pedagogy; semantics of computer arithmetics; and the nature and design of formalisations.

In Section 2, we clarify why we wish to pursue arithmetic at this elementary and fundamental level. In Section 3 we introduce the basics of fracterms and our naive fracterm calculus NFTC. In Section 4 we develop a more comprehensive list of commitments and non-commitments for NFTC. In Section 5 we look at rewriting expressions in NFTC. Then we turn our attention to the formal calculi. In Section 6 we look at formal calculi for partial meadows; in Section 7 we look at formal calculi for common meadows in which division is made total by an ‘error’ flag  $0^{-1} = \perp$ ; and in Section 8 we look at formal calculi based on algebras in which division is made total in other ways. We conclude with some reflections.

## 2 Why naive fracterm calculi?

Most arithmetical practitioners will be unaware of *any* of fracterm calculus. Upon being made aware of a particular FTC (such as those introduced later) they may well be unconvinced of the advantages of adopting such an FTC as a point of departure for elementary arithmetic. Fundamentally, a fairly clear pattern of aversion to, and dismissal of, formalisation seems to exist with regard to elementary arithmetical practice.

Shortly, we will collect such viewpoints and attitudes in a list of claims concerning elementary arithmetic that we suppose those with a negative attitude to formalising arithmetic would share. Our hypothesis is that these attitudes can be taken together and used as a package of attitudes and views with which to guide the design of a naive fracterm calculus NFTC. NFTC is a perspective on arithmetic which comes about from this package.

**Design Objective 2.1.** *Adopting NFTC implicitly – i.e., adopting its tenets – constitutes a majority position among persons interested in elementary mathematics, including the majority of school teachers.*

Although intuitions about abstract data types are in our mind as we analyse elementary arithmetic, let us confirm it that our focus is *elementary arithmetic as it is practiced*:

**Claim 2.1.** *Once aware of the notion of a data type most practitioners of elementary arithmetic prefer not to understand the classic portfolio of arithmetics (of the naturals, integers, rationals, etc.) as a portfolio of data types. In particular, having to think syntactically, in terms of signatures, will create resistance.*

**Claim 2.2.** *If a formal basis for arithmetic is to be adopted, then a person who is initially disinclined to adopt any formalisation of elementary arithmetic is likely to prefer an FTC presenting division as a partial function over any of the plurality of FTC's that incorporate division as a total function.*

Logics of partial functions abound, but are non-trivial, e.g., [Robinson 1989, Jones and Middelburg 1994]. NFTC as outlined below may be considered naive or even inadequate from the perspective of logic. However, adopting NFTC may come with endorsing the following working hypothesis:

**Claim 2.3.** *Arithmetic is based in natural language, taking the form of an extension of core natural language with certain notations and conventions.*

*Arithmetic is conceptually prior to logic and, therefore, there is no need to look for a logical basis of arithmetic. Any proposal for a logical basis for arithmetic will turn out to be both defective and artificial.*

For the acquisition of basic competence in a natural language, awareness of its grammar is not essential. This idea may be extended to a claim about arithmetic as follows:

**Claim 2.4.** *Arithmetic is conceptually prior to syntax, and therefore there is no need to look for a syntactic basis of arithmetic. Any proposal for a syntactic basis for arithmetic will turn out to be both defective and artificial.*

The situation in elementary arithmetic is comparable to the situation with the classical paradoxes. The famous liar paradox is intriguing while it constitutes no impediment to

the practical use of natural language. Knowledge of the liar paradox is useful because it indicates which kind of sentences may be hard to understand and are better avoided in practice. To some extent the liar paradox is merely a side-effect of a particular approach towards the formalisation of natural language, however, implicit. Similar observations may apply to arithmetic:

**Claim 2.5.** *Logical complications do not stand in the way of a proper understanding of arithmetic, even if some logical complications may prove rather hard to settle.*

**Claim 2.6.** *Elementary arithmetic is not protected against inconsistency and error on the basis of a package of assumptions and reasoning patterns that have been deliberately designed to avoid inconsistency and error as much as possible, while still covering sufficient ground (such as ZF set theory does).*

*On the contrary, elementary arithmetic acquires its stability and reliability from the daily practice of a community of users who will eventually correct one another if needed.*

### 3 Fracterms and a naive fracterm calculus

NFTC is to be understood as naive from the perspective of more formal characterisations. Yet NFTC is supposed to be a stand alone description of elementary arithmetic which is not in need of any foundations provided by formalisation.

#### 3.1 Fracterms

As argued in [Bergstra 2020], ‘fraction’ is an ambiguous notion for which there is no consensus. The meaning of fraction ranges from an expression, where different interpretations of fraction consider different classes of expressions, to a value and to a rational or related form of number. Remarkably, there is no obvious majority position in this spectrum of explanations of fraction. Thus, the purpose of the word *fracterm* is to restrict the ambiguity of fraction by always opting to mean an expression in circumstances where a distinction between expression and value is made. Complementary to *fracterm*, *quotient* also restricts the ambiguity of fraction by opting for a value whenever a distinction between value and expression is made.

Fracterm and quotient are notions which are both meaningful in various formal settings and for which a corresponding interpretation can be used in NFTC. Where fracterm has (or may have in the perception of some) a syntactic bias, quotient has a complementary semantic bias.

**Definition 3.1.** *A fracterm is a structured entity involving a numerator, a denominator and a leading function symbol for division.*

A fracterm comprises the inputs of a division, irrespective of whether or not the operation can actually be performed. A quotient, on the other hand, specifies the output of a division; a quotient implies that division must have been enabled. In a context where expressions and values are distinguished, a fracterm is an expression and is not a value. In a context where some expressions also serve as values, the rule may occasionally be compromised.

**Definition 3.2.** *A quotient is the result of performing division on a pair of arguments. The respective arguments of a quotient are sometimes called dividend and divisor.*

**Definition 3.3.** *A sum is the result of performing addition on two (or more) arguments. The respective arguments of a sum are called summands.*

**Definition 3.4.** *A sumterm of length  $n$ , with natural  $n \geq 2$ , is an expression of the form  $t_1 + \dots + t_n$ . Here  $t_i$  is called the  $i$ -th summand. A sumterm is by default a sumterm of length 2.*

**Definition 3.5.** *A product is the result of performing multiplication on two (or more) arguments. The respective arguments of a product are called factors.*

**Definition 3.6.** *A producterm of length  $n$ , with natural  $n \geq 2$ , is an expression of the form  $t_1 \cdot \dots \cdot t_n$ . Here  $t_i$  is the  $i$ -th factor of the producterm.*

**Design Choice 3.1.** *In the context of NFTC, quotient and fracterm are not distinguished.*

**Design Choice 3.2.** *In the context of NFTC, sum and sumterm are not distinguished, and product and producterm are not distinguished. However, both 'sumterm' and 'producterm' are not used in NFTC.*

The reason to treat division differently from addition and multiplication lies in the observation that fracterms as fractions play a special role in elementary arithmetic:

(i) There is a significant nomenclature for the classification of fracterms: *simple* fracterms, *proper* fracterms, *unit* fracterms, *flat* fracterms, *simplified* fracterms, etc. A survey of terminology on fracterms is given in [Bergstra 2020].

(ii) Calculating with fracterms is a technical theme of its own.

(iii) Simplification of fracterms is a key transformation in calculation.

(iv) Unlike with sumterms and producterms, speaking of evaluating fracterms is less common, as evaluation is often understood as being merely the final stage of simplification.

There can be a plurality of different approaches to the formalisation of calculation with fracterms, whereas there seems to be a rather uniform informal or 'no-nonsense' approach to the subject which underlies much of today's practice regarding elementary arithmetic, including educational practice. Now we will describe a naive (i.e., non-formal) approach to calculation with fracterms.

### 3.2 Naive Fracterm Calculus (NFTC)

NFTC comes about from the following combination of design objectives, design choices, and underlying general definitions.

**Design Objective 3.1.** *Many judgements of elementary arithmetic are uncontroversial. These judgements together constitute what may be called naive fracterm calculus (NFTC).*

NFTC focuses on the large body of uncontroversial assertions in and about elementary arithmetic.

**Design Objective 3.2.** *NFTC is 'by design' intended to be immune from being split into a plurality of views on details.*

In naive set theory, the equation  $\{x|x \notin x\} = \{x|x \notin x\}$  is seemingly obvious as it equates two identical entities. But asserting this equality is problematic because at closer inspection both sides of the equation contain an expression that does not denote

anything (cf. Russell's Paradox). Similarly, in naive fracterm calculus, the equation  $\frac{1}{0} = \frac{1}{0}$  features complications: although the idea that  $\frac{1}{0}$  is non-denoting is not coming about from an important paradox or a contradiction, in both cases, the subtle question arises to what extent one is committed to a two-valued classical logic.

**Design Choice 3.3.** *NFTC is committed to accepting  $\frac{1}{0}$  as a fracterm, but it is a 'bad' fracterm, and bad fracterms must not be used in written or spoken language about elementary arithmetic.*

*The description 'let  $g(x) = \frac{1}{x}$  with  $x > 0$ ' is acceptable, although reading from left to right the fragment  $\frac{1}{x}$  is at risk of being non-denoting when first read. The occurrence of  $\frac{1}{x}$  is potentially bad but not necessarily bad, and in an adequate context (such as the context just mentioned where  $x > 0$ ) it is acceptable.*

**Design Choice 3.4.** *NFTC is uncommitted to a judgement of the validity of  $\frac{1}{0} = \frac{1}{0}$ . Furthermore, NFTC is not committed to reflecting on the validity of the 'badly' formulated statement  $\frac{1}{0} = \frac{1}{0}$ .*

**Design Choice 3.5.** *NFTC is uncommitted to two-valued classical logic.*

**Design Choice 3.6.** *NFTC makes use of the word 'fracterm' where readers might expect 'fraction'.*

The idea is that fracterm is used in NFTC in such a way that it acquires the same meaning as fraction in school arithmetic, where the presence of ambiguities and of various biases are accommodated.

**Definition 3.7.** *In the context of NFTC, a fraction is a fracterm.*

Thus, in view of Definition 3.7, in NFTC fracterm and fraction are the same notion. However, we prefer not to constrain this work by any claim about the ontology of fractions – that being a difficult subject – and will use fracterm within NFTC. Definition 3.7 states our own preference for defining a fraction. The constraint 'in the context of NFTC' is critical for Definition 3.7. Without that constraint, i.e., by simply defining a fraction as a fracterm, an overly syntactical bias concerning fractions will result. A strong syntactic bias in the conception of the notion of fraction is undesirable and imposing such a bias is avoided by introducing the constraint.

**Design Objective 3.3.** *NFTC is uncommitted to the belief that  $\frac{1}{2}$  and  $\frac{2}{4}$  are different fracterms, while acknowledging that both denote the same quotients; at the same time NFTC is committed to the belief that  $\frac{1}{2}$  is a simplification of  $\frac{2}{4}$ .*

**Design Objective 3.4.** *NFTC is committed to the validity of*

$$\phi(x) \equiv (x \neq 0 \rightarrow \frac{x}{x} = 1).$$

*This assertion is true with the 'short-circuit reading' of implication: if the condition fails (i.e., if  $x = 0$ ) then  $\phi(x)$  is considered valid and no attention is paid to the conclusion at all.*

The short circuit semantics of logical connectives plays a role in our development and we adopt a notation to make it explicit when needed. In the following claim we use the notation  $\circ \rightarrow$  for short-circuit implication; similar notations exist for all the connectives, e.g.,  $\circ \wedge$ ,  $\circ \vee$ ,  $\circ \neg$ ,  $\circ \rightarrow$  and so on [Bergstra et al. 1995].



**Design Choice 3.7.** *NFTC acknowledges that short-circuit implication plays a central role so that it may warrant its own notation (for which, currently, no generally agreed candidate is available) and then positively affirms that:*

$$x \neq 0 \circ \rightarrow \frac{x}{x} = 1.$$

NFTC acknowledges (or at least does not reject) differences in plausibility between various assertions, each of which is neither clearly true nor clearly false – compare Design Choice 3.5.

**Design Choice 3.8.** *NFTC is uncommitted to three valued logic while being compatible with three valued logic. NFTC is compatible with perceiving subtle distinctions of validity which are not rigorously used or explained.*

For instance, NFTC accommodates a view where one assigns  $\frac{1}{0} = 1$  a lower plausibility than  $\frac{1}{0} = \frac{1}{0}$ , which in turn is considered to be less plausible than  $\text{true} \vee (\frac{1}{0} = \frac{1}{0})$  which then in turn is less plausible than  $\text{true} \overset{\circ}{\vee} (\frac{1}{0} = \frac{1}{0})$ , the latter being still problematic because the fracterm  $\frac{1}{0}$  should preferably not be used in writing. At the same time, NFTC accommodates a view where each of these assertions has a single third truth value.

Now  $\frac{1}{0}$  is a fracterm, but NFTC rejects the sentence ‘ $\frac{1}{0}$  is a fracterm’, not because it is supposed to be wrong or invalid but because  $\frac{1}{0}$  is bad and considered meaningless to such an extent that there is insufficient justification for its use within NFTC. Existence in NFTC is understood as meaningful existence. What can be said and written in NFTC instead is that ‘dividing 1 by 0 is not possible because it cannot deliver an adequate result’.

**Design Choice 3.9.** *NFTC is committed to the existence of various kinds of numbers: natural, integer, rational, real and complex.*

However:

**Design Choice 3.10.** *NFTC is adversely committed to attempts to define the various numbers in detail, with the idea that these are self-supporting intuitions that come about from natural language as much as from any ideology on how to design mathematical theory and the corresponding notational conventions and patterns of reasoning.*

Questions like ‘is division a function name’ are outside the scope of interest of NFTC, more generally:

**Design Choice 3.11.** *NFTC is adversely committed to the idea that naturals, integers, and rationals are domains of data types that in turn serve as plausible representatives of corresponding abstract data types. In particular, the notion of a signature is foreign to NFTC.*

(Recall the claims of Section 2.)

## 4 More commitments and non-commitments of NFTC

The design objectives and choices above may be understood as ‘meta-axioms’ for NFTC, asserting that NFTC should satisfy certain properties. We think of NFTC as a package

of opinions that are shared by many people. Consistency of the package of opinions is not our major concern, rather the stability and the practicality of the package is. A rationale for the package of views is that it allows to minimize the time and energy spent on matters deemed to be of little importance. Stability and practicality take their meaning from experience.

Someone who adopts NFTC will have opinions about many more issues that will need to be established. Opinions about technical options will need to be accepted, rejected or left open. We will progress the design of NFTC with a list of options that we are not committed to and then with a list of options we are committed to; these listings are not meant to be complete in any rigorous sense.

#### 4.1 Non-commitments of NFTC

##### *Issues to do with syntax and semantics*

1. For numerals, NFTC is uncommitted to the distinction between a value and an expression. For instance, 251 is an expression as well as a value. In essence, 251 is itself as a number and whether or not it is an expression as well is deemed an irrelevant question.
2. More generally, NFTC is uncommitted to making a distinction between syntax and semantics. However, if for some concept, say fraction, a distinction between syntax and semantics is put forward then the concept will become ambiguous and, thereupon, the same concept will refer to both syntactic forms and semantic entities. Resolution of such ambiguities is considered a matter of natural language processing which is not to be considered a mathematical subject, and which for that reason merits little or no attention.  
For instance, in NFTC  $\frac{4}{2}$  and 2 are the same number, yet  $\frac{4}{2}$  has a nominator and 2 has no nominator. The underlying logic of which properties are shared by the same entities is not made explicit. When reasoning about properties of numbers care must be taken to avoid wrong conclusions about equality and difference of numbers. For instance, it is not the case that if  $a$  and  $b$  are the same number and if  $a$  and  $b$  both have a numerator, say  $a_n$  and  $b_n$  respectively, that  $a_n = b_n$  must hold.
3. The distinction between fracterm and fraction is considered immaterial (and by consequence the phrase fracterm calculus is endorsed with some hesitation). In NFTC, fracterm is not considered a mathematical notion (following [Fandino Pinilla 2007]); it is a *word* in English intended to resolve the ambiguities of the term fraction.
4. NFTC adopts no commitment to any systematic notion of legality for expressions as the class of expressions is not a relevant notion, nor is any property of expressions. (Below we will discuss legality in the context of the formal FTC for partial meadows.)
5. For identities (or formulas) with free variables (e.g.,  $x + \frac{5 \cdot y}{3} = y + x + \frac{2 \cdot y}{3}$ ), the scope of the variables depends on the context, which is cast in terms of natural language.

*Issues to do with logical formulae*

6. No distinction is made between, say

$$\forall x \in V(x \neq a \rightarrow \phi(x))$$

and the possibly infinitary conjunction

$$\bigwedge_{x \in V - \{a\}} \phi(x)$$

where  $a \in V$ .

For a formal FTC, the rationale of making a distinction between both statements follows from this observation: in case of universal quantification, it is expected or required that  $\phi(a)$  is well-formed or well-defined, whereas for the conjunction that requirement need not be imposed.

Another way of stating the lack of distinction is to claim that NFTC adopts the convention that  $\forall x \in V(x \neq a \rightarrow \phi(x))$  is understood as  $\forall x \in V(x \neq a \circ \rightarrow \phi(x))$ .

7. NFTC acknowledges no commitment to any of the following distinctions:
- (i) the distinction between total and partial functions as being of relevance for reasoning;
  - (ii) the distinction between a formal proof and an informal proof;
  - (iii) the distinction between naive set theory and any of the formalised set theories that provide protection against various paradoxes;
  - (iv) the distinction between constructive and non-constructive reasoning.
8. For NFTC no knowledge of classical logic, be it propositional logic or first order logic, is presupposed. Logics of any relevance for mathematics are considered to constitute a separate subject, which one may or may not have any desire to study in more detail. Knowledge of various logics is considered immaterial for the acquisition of a proper understanding of arithmetic which originated several millennia ago unlike mathematic logic, which is a creation of the nineteenth century.
9. In NFTC no commitment is made to any technical explanation about what it means that all natural numbers are integers, and that all integers are rationals. In particular, no set-theoretic explanation of these inclusions based on definitions of the various number classes is contemplated. These inclusions are intentional rather than empirical and can be used in a context where strictly speaking the various set-theoretic definitions of these number classes do not comply with the intended inclusions.

#### 4.2 Commitments of NFTC

The following list of commitments adds further details to the view of arithmetic that NFTC is supposed to comprise.

*Issues to do with algebra and calculation*

1. Fracterm is considered a mathematically relevant notion, which, however, need not be rigorously defined as if it were a mathematical concept proper; fracterm is like proof, definition, theorem, which are notions belonging to mathematical practice rather than to mathematical substance and which, at least working at the informal level of NFTC, can be left without proper mathematical definitions.
2. NFTC is committed to the existence of a bundle of widely agreed upon closed identities of the form  $t = r$ , as well as to a bundle of universally agreed upon inequations of the form  $t \neq r$ . Examples are:

$$1 + 2 = 3, \frac{1}{2} \neq \frac{4}{3}, \frac{1}{2} + \frac{4}{3} = \frac{11}{6}, \text{ and } \frac{x}{2} + \frac{y}{3} = \frac{3 \cdot x + 2 \cdot y}{6}.$$

An informal theory is supposedly available – in the form of a plurality of rules and algorithms – concerning how to distinguish between  $t = r$  and  $t \neq r$  for closed expressions  $t$  and  $r$ .

3. NFTC adopts a fairly systematic interpretation of variables. Variables in an equation may be implicitly universally quantified in which case one may speak of a law regarding such variables. Alternatively, a variable  $x$  may be supposed to have a fixed but arbitrary value in some domain. The latter form of quantification is often used in formulating and proving results.

However, in an equation (e.g.,  $a \cdot x^2 + b \cdot x + c = 0$ ), one assumes that at some level  $a, b, c$  are universally quantified while  $x$  denotes one or more specific values depending on  $a, b, c$ . In the latter case no universal quantification over  $x$  is implied, rather  $x$  plays the role of an additional constant which is further specified by the mentioned equation.

The natural language in which formulas are embedded determines the use of a variable.

*Issues to do with arithmetic and calculation*

4. Arithmetic is primarily embedded in natural language for which 100% precision is unachievable as well as unnecessary. There is in practice no disagreement about the status of facts in the language of fracterm calculus, not even between persons or groups who have adopted significantly different positions regarding the possible commitments and non-commitments we are listing.
5. A commitment to calculation is understood as a mechanical procedure. Calculation is not understood as an instance of mathematical or logical reasoning but as an independent competence.
6. NFTC maintains a commitment to calculation with decimal naturals and decimal integers.
7. NFTC is based on intuitions of raw arithmetic, unmediated by logic, set theory, or computability theory. Such themes are considered to be of secondary importance in comparison to arithmetic which constitutes the starting point of mathematical

thinking. Indeed, NFTC stands on its own feet without a need, or even a use, for any preparatory mathematical, logical, or philosophical preliminaries.

Defining expressions (even for a finite semantic world like the Boolean data type) requires an inductive definition and comes with the ability to count sizes of an expression. It is therefore implausible to consider the notion of an expression to be prior to the notion and practice of counting. So it was historically for millennia before the emergence of formulae with variables.

An attempt to define the class of expressions for an FTC will be confronted with marginal cases which may uncover a certain lack of precision of the definition. For instance: which of the following “expressions” are valid (legal):

$()$ ,  $(0)$ ,  $((0))$ ,  $((0), (-0))$ ,  $0 \cdot (1 + -3)$ ,  $--7$ ,  $[1 + 3 \cdot (7 + 7)] \cdot 8$ ,  $007 + 1$ ,  $000 + 1$

#### *Issues to do with relevance*

8. For each equation, condition, or more involved formula of fracterm calculus broadly conceived, the first question to ask is about its *relevance* and not about its meaning. When confronted with a fragment of text the primary focus it merits is about: why has it been produced and what is its origin, what do we want to know about it, and why does that matter?
9. NFTC suggests rating relevance over truth and consistency in the following sense: *only once potential relevance has been established do questions about truth matter.*
10. There is no proof system for relevance, there are no axioms for it, though there are some rules of thumb: for instance, if a panel of teachers agrees that a certain formula makes good sense as part of an exercise then that formula is relevant for that reason; and if an expression occurs in an application, that fact having been confirmed by various readers, then that expression is relevant, etc.

Given the minimal commitments of NFTC, it is hard to imagine that it would be possible to claim any inconsistency in its approach.

## 5 Calculation as rewriting in NFTC

We suppose that understanding and knowledge regarding NFTC comes about directly from experience – inductively rather than deductively. For example, once someone has become convinced that, say

$$\frac{5 \cdot 2}{7 \cdot 2} = \frac{5}{7}, \frac{8 \cdot 3}{13 \cdot 3} = \frac{8}{13}, \frac{1 \cdot 5}{15 \cdot 5} = \frac{1}{15},$$

it becomes plausible to guess that for all  $x, y$  and positive natural  $n$ ,

$$\frac{x \cdot n}{y \cdot n} \text{ may be replaced by } \frac{x}{y}.$$

Such observations suggest a particular form of *rewriting* of expressions that underlies much of the calculating practices of elementary arithmetic. In general:

**Definition 5.1.** Given a collection of rewrite rules  $R$ ,  $t \Rightarrow_R^* s$  asserts that with 0 or more steps allowed by the rules in  $R$ ,  $t$  can be rewritten to  $s$ . In particular, if  $t$  has been obtained using the rules in  $R$  then also  $r$  can be obtained and  $t = r$  holds.

Thus, the command ‘simplify the expression’, common in student exercises, means try to apply rules to rewrite the expression to make an equivalent expression that is simpler or special in some way.

### 5.1 Rewriting by example

Substitutions into equations and rewriting are not quite the same. For instance,

$$\frac{x \cdot 7}{y \cdot 7} \Rightarrow \frac{x}{y}$$

is a valid rewrite rule for NFTC while the corresponding equation

$$\frac{x \cdot 7}{y \cdot 7} = \frac{x}{y}$$

fails because  $y = 0$  has not been excluded! Why exactly?

Now, applying the rule  $\frac{x \cdot 7}{y \cdot 7} \Rightarrow \frac{x}{y}$  is only enabled if for some  $r, s$ , a fracterm  $\frac{r \cdot 7}{s \cdot 7}$  has been obtained previously, from which it follows that  $s \neq 0$ . Thus, it follows that as a rewrite rule  $\frac{x \cdot 7}{y \cdot 7} \Rightarrow \frac{x}{y}$  is correct without any need to require that  $y \neq 0$ .

Many more rules can be proposed. For instance:

$$\frac{x \cdot y}{u \cdot v} \Rightarrow \frac{x}{u} \cdot \frac{y}{v} \text{ and its converse } \frac{x}{u} \cdot \frac{y}{v} \Rightarrow \frac{x \cdot y}{u \cdot v}.$$

In NFTC, it is considered unproblematic to write  $t = r$  if  $t \Rightarrow r$  is actually meant. However, the cost of this convention is significant as rewriting collides with symmetry of the equality relation  $=$ . Indeed, while  $\frac{x \cdot z}{y \cdot z} \Rightarrow \frac{x}{y}$  is valid,  $\frac{x}{y} \Rightarrow \frac{x \cdot z}{y \cdot z}$  fails for  $z = 0$  so that dropping the orientation of arrows is unwarranted in this case.

Division by 1 takes the form of the rule  $\frac{x}{1} \Rightarrow x$ . For numerator 0 one finds the rule  $\frac{0}{x} \Rightarrow 0$ . Now, taking 0 for  $x$  will be outside NFTC as it involves contemplating a term  $\frac{0}{0}$ . Remarkably, there is no logical problem with the rule  $\frac{0}{x} \Rightarrow 0$  in case  $x$  is 0 because it will *not* be the case that  $\frac{0}{0}$  is obtained earlier. Thus, by material implication from the hypothesis that  $\frac{0}{0}$  has been obtained, none whatever conclusion may be drawn, including the that (i)  $\frac{0}{0}$  rewrites to 0 and (ii)  $\frac{0}{0} = 0$ .

Nevertheless, this application of material implication lies outside the scope of NFTC. Seen from outside NFTC, say from a viewpoint where use of the fracterm  $\frac{0}{0}$  is not rejected, the rule  $\frac{0}{x} \Rightarrow 0$  becomes a valid counterfactual about NFTC.

### 5.2 Fracfree and flat terms

**Definition 5.2.** A term or expression is *fracfree* if it has no fracterm as a subterm. A *flat fracterm* is a fracterm of the form  $\frac{p}{q}$  with  $p$  and  $q$  both *fracfree*.

It is immediate that fracfree terms are closed under addition, subtraction and multiplication.

Now, the familiar rule

$$\frac{x}{y} + \frac{u}{v} \Rightarrow \frac{x \cdot v + u \cdot y}{y \cdot v}$$

allows the following conclusion: *a sum of flat fracterms can be replaced by a single flat fracterm (with the same value).*

Using rewriting from ‘given’ terms and by phrasing results as being about fracfree entities (perhaps referred to as fracfree numbers, which would be acceptable in NFTC) lots of patterns of calculation that are encountered in teaching elementary arithmetic can be correctly paraphrased without becoming trivial and in such a manner as to avoid the explicit use of conditional logic.

Avoiding triviality does require some care: for instance, asserting that a sum of fracterms can be rewritten into a single fracterm is trivial because of the validity of the rewrite rule  $x + y \Rightarrow \frac{x+y}{1}$ .

Closed fracfree terms can be rewritten to numerals ( $n$  or  $-m$  with  $n, m$  decimal naturals and  $m > 0$ ), but doing so requires a sufficiently large collection of rules  $R$ . At this point, the rewrite rules needed may start looking more imposing and less convincing. For addition, the relevant rule reads: “if decimal number  $a$  results from addition of decimal numbers  $b$  and  $c$  then  $b + c \Rightarrow a$ ”. Alternatively, one may consider the rule “if binary number  $a$  results from addition of binary numbers  $b$  and  $c$  then  $b + c \Rightarrow a$ ”. Mixing such rules is not a good idea, however, and when collecting rewrite rules for calculation a choice between decimal and binary notation must be made beforehand.

## 6 Formal fracterm calculi I: FTC for partial meadows

In summary, NFTC intends to capture an explicit picture of a most conventional and unpretentious understanding of elementary arithmetic. In spite of the disrespect within NFTC for formalisation, thanks to a background of logic or computing, we may make formalisations FTCs that try to meet the requirements of NFTC. When formalising fracterm calculi for meadows, a plurality of informal and formal options soon appear, each of which has different and incompatible merits. Indeed, there are quite some refinements to NFTC that can constitute an informal calculus in the gap between NFTC and FTCs (recall Synthetic Arithmetic in 1.1).

A meadow is a field with division and a *partial meadow* is a meadow in which division is a partial operation. First, we will discuss the FTC for partial meadows, which is relatively close to conventional practice in arithmetic and, for that reason, is close to meeting NFTC.

If someone paints their house, the house is still the same house thereafter in spite of having been marginally changed. For a change to change an identity it must be a significant change, and it must be a destructive change to some extent. For mathematics destructive change is not a common intuition. If a proof is hard to read it can be gradually improved, with each step keeping the proof the same in essence but improved qua presentation. However, upon a serious error being found, the proof may be changed to a proof of the negated assertion. Now the proof has been destructively changed. Nevertheless, the idea that a mathematical proof is a potential candidate for destructive change into a proof of the negated result is unfamiliar. Instead one assumes that at closer inspection a proof may be considered somehow unconvincing or manifestly defective.

Now, a fundamental transformation in elementary arithmetic is so-called *simplification*, e.g.,  $\frac{2}{4}$  can be simplified to  $\frac{1}{2}$ . Is this simplification a significant change which changes the identity of the entity at hand? Yes and no: we will say that  $\frac{2}{4}$  and  $\frac{1}{2}$  are different fracterms, and that simplification significantly changes a fracterm. Unlike fraction, fracterm does not have the flexibility to be understood as ‘the number denoted by a fracterm’, which remains unchanged with simplification.

FTC for partial meadows adopts  $\frac{1}{0}$  as a fracterm, in spite of it having no value; however, it is not case that  $\frac{1}{0} = \frac{1}{0}$ , neither is it the case that  $\frac{1}{0} \neq \frac{1}{0}$ . These observations motivate a discussion of the ‘legality’ of expressions and assertions. This serves as an example of an issue which is left implicit in NFTC and which can be made explicit in a formal FTC for partial meadows.

### 6.1 Partial meadows

The signature  $\Sigma_m$  of meadows is obtained by extending the signature of unital rings with a two place division, denoted or  $x/y$  or  $\frac{x}{y}$  or, when the operator is needed without arguments,  $\div$ . The sort name of numbers involved is named Number.

**Definition 6.1.** *A partial meadow is a structure with signature  $\Sigma_m$  that is obtained by expanding a field  $F$  with a partial division operator (with the usual definition), thereby obtaining  $F(\div)$ .*

One may provide a traditional Tarski semantics adapted to a *three valued logic* for assertions with semantics  $\{\text{true, false, m}\}$ ; here are the details.

Let  $\sigma$  range over valuations of variables that assign to each variable an element of  $F(\div)$ . We will use  $=_{\text{pm}}$  for equality in partial meadows. The idea is that for every valuation  $\sigma$ , and for each term  $t$  over  $\Sigma_m$ , either the evaluation  $(F(\div), \sigma \models t)$  has a value or it has no value. But, for equality between terms  $t$  and  $r$  there are three cases:

- (i)  $F(\div), \sigma \models t =_{\text{pm}} r$
- (ii)  $F(\div), \sigma \models t \neq_{\text{pm}} r$
- (iii)  $F(\div), \sigma \not\models t =_{\text{pm}} r$  and  $F(\div), \sigma \not\models t \neq_{\text{pm}} r$

so that in case (iii) both equality and inequality of  $t$  and  $r$  have truth value m. Case (iii) applies precisely if at least one of  $t, r$  has no value under valuation  $\sigma$ .

Partial meadow equality  $=_{\text{pm}}$  may be explored informally by means of examples:

1.  $\frac{1}{0}$  is undefined, i.e., it has no value; more generally,
2.  $\frac{x}{y}$  is undefined if, and only, if  $y =_{\text{pm}} 0$ ;
3.  $\frac{1+1}{1-1} =_{\text{pm}} 1$  has no truth value because it involves a term (on the LHS of the equation) with an undefined value;
4.  $\frac{1}{0} =_{\text{pm}} \frac{1}{0}$  has no truth value for the same reason; nevertheless,
5. for all  $x$ ,  $x =_{\text{pm}} x$  is true, because a valuation must assign a value to each variable including  $x$  (in other words,  $x$  ranges over existing entities only);
6. it follows that substituting a closed term for a variable in a valid identity may result in an identity without a truth value (as a consequence when formalising reasoning



---

$(x + y) + z =_{\text{pm}} x + (y + z)$	(1)
$x + y =_{\text{pm}} y + x$	(2)
$x + 0 =_{\text{pm}} x$	(3)
$x + (-x) =_{\text{pm}} 0$	(4)
$x \cdot (y \cdot z) =_{\text{pm}} (x \cdot y) \cdot z$	(5)
$x \cdot y =_{\text{pm}} y \cdot x$	(6)
$1 \cdot x =_{\text{pm}} x$	(7)
$x \cdot (y + z) =_{\text{pm}} (x \cdot y) + (x \cdot z)$	(8)

---

*Table 1: CRpm: axioms for commutative rings with equality renamed to  $=_{\text{pm}}$*

---

$\text{import CRpm}$	(9)
$0 \neq_{\text{pm}} 1$	(10)
$\neg x =_{\text{pm}} 0 \circ \rightarrow x \cdot \frac{y}{x} =_{\text{pm}} y$	(11)

---

*Table 2: FCTpm: axioms of the fracterm calculus of partial meadows*

about partial meadows, either some logic of partial functions is needed, or some translation to a data type with only total functions is required, so that first order logic for the latter can be applied);

7.  $x \neq 0 \circ \rightarrow \frac{x}{x} = 1$  is true in all partial meadows; for nonzero  $x$  that is immediate, while upon substituting 0 for  $x$  one finds  $0 \neq 0 \circ \rightarrow \frac{0}{0} = 1$  which is equivalent to  $\text{false} \circ \rightarrow \frac{0}{0} = 1$ , which thanks to the short circuit logic, is just true.

Given a partial meadow  $F(\div)$ , a satisfaction relation  $F(\div), \sigma \models \phi$  can be inductively defined in the usual manner.

The implication  $\phi \rightarrow \psi$  is more restrictive than  $\phi \circ \rightarrow \psi$  because in case of  $\neg \phi$  it is now required that either  $\psi$  or  $\neg \psi$  holds for  $\phi \circ \rightarrow \psi$  to be true (otherwise  $\phi \circ \rightarrow \psi$  is considered neither true nor false).

## 6.2 FTC for partial meadows

**Definition 6.2.** *The theory of assertions of FTC for partial meadows consists of the collection of universally quantified propositions*

$$\psi \equiv \forall x_1 \dots \forall x_k \phi$$

where

$$\frac{x}{1} =_{\text{pm}} x \quad (12)$$

$$\neg y =_{\text{pm}} 0 \circ \rightarrow -\frac{x}{y} =_{\text{pm}} \frac{-x}{y} \quad (13)$$

$$(\neg y =_{\text{pm}} 0 \delta \wedge \neg v =_{\text{pm}} 0) \circ \rightarrow \frac{x}{y} \cdot \frac{u}{v} =_{\text{pm}} \frac{x \cdot u}{y \cdot v} \quad (14)$$

$$(\neg y =_{\text{pm}} 0 \delta \wedge \neg v =_{\text{pm}} 0) \circ \rightarrow \frac{x}{y} + \frac{u}{v} =_{\text{pm}} \frac{(x \cdot v) + (y \cdot u)}{y \cdot v} \quad (15)$$

$$(\neg u =_{\text{pm}} 0 \delta \wedge \neg v =_{\text{pm}} 0 \delta \wedge \neg y =_{\text{pm}} 0) \circ \rightarrow \frac{\left(\frac{x}{y}\right)}{\left(\frac{u}{v}\right)} =_{\text{pm}} \frac{x \cdot v}{y \cdot u} \quad (16)$$

$$\neg x =_{\text{pm}} 0 \circ \rightarrow \frac{x}{x} =_{\text{pm}} 1 \quad (17)$$

Table 3: Valid assertions of the fracterm calculus of partial meadows

(i)  $\phi$  is quantifier free, and is made from equations  $t =_{\text{pm}} r$  and inequations  $t \neq_{\text{pm}} r$ , and

(ii)  $\psi$  is valid in all partial meadows, i.e., for each partial meadow  $F(\div)$  and for all valuations  $\sigma$ ,  $F(\div), \sigma \models \phi$ .

As is usual for equational axioms, universal quantification is often left implicit so that quantifiers can be omitted.

Table 1 lists axioms for commutative unital rings, though with equality renamed to  $=_{\text{pm}}$ . Table 2 provides axioms for FTC for partial meadows. These identities seem to be undisputed from the perspective of NFTC.

Assuming one wishes to formalise NFTC, then this FTC for partial meadows may be considered a core for NFTC. The semantics of this notation is clear, but providing a proof system is complex matter that will not be discussed in this paper.

Soundness of the collection of assertions in Table 2 is unproblematic, as one may adopt Suppes-Ono FTC (with division made total by adopting  $x/0 = 0$ ) as the defining equation for the conditional operator, in which case all assertions of Table 4 are satisfied.

As far as completeness is concerned, we find:

**Proposition 6.1.** *The class of models of the axiom system FCT<sub>pm</sub> (i.e., of the assertions collected in Table 2) is precisely the class of partial meadows.*

*Proof.* Soundness is unproblematic, a matter of checking each assertion in a field  $F(\div)$ . Next, let  $R$  be any structure satisfying the assertions of Table 2. First, one notices that  $+$ ,  $\cdot$ , and  $-$  are total functions. In the case of addition,  $x + y =_{\text{pm}} y + x$  implies that for given  $a, b \in R$ ,  $a + b =_{\text{pm}} b + a$  which can only be the case if  $a + b$  has a value. The operations follow similarly. It now follows that  $R$  is a unital ring. Next, one notices that for  $a, b \in R$ ,  $a \neq 0$ ,  $a \cdot \frac{b}{a} =_{\text{pm}} b$ , and so by taking  $b = 1$  that  $\frac{1}{a}$  is an inverse of  $a$ ; hence  $R$  is a field. It then follows that  $\lambda x.y.\frac{x}{y}$  is division in  $R$ .  $\square$

Table 3 displays some consequences of the axioms for the FTC for partial meadows.

### 6.3 Formalising nonsense

FTC for partial meadows is complete in the following sense:

**Claim 6.1.** *The formulas in the notation of FTC for partial meadows, which are valid in all partial meadows in the usual manner, constitute a fracterm calculus which contains all assertions within the framework of its logic that are considered valid from the perspective of NFTC, where NFTC's logical conjunctives are understood as the corresponding left to right sequential conjunctives.*

However, FTC for partial meadows is too liberal, i.e., it is more liberal than NFTC. We hold that the conditional equation

$$0 \neq 0 \rightarrow \frac{0}{0} = \frac{1}{0}$$

would not be endorsed within NFTC. From the perspective of NFTC, that conditional equation is not a legal expression because it contains a fragment – namely,  $\frac{0}{0} =_{\text{pm}} \frac{1}{0}$  – which is not part of any meaningful assertion, because no valuation is either true or false. No-nonsense arithmetical assertions must not contain meaningless fragments.

**Definition 6.3.** (*Meaningless formula.*) *An assertion  $\phi$  over  $\Sigma_m$  is meaningless if for no field  $F$  and for no valuation  $\sigma$  of variables into  $F$  it is the case that either  $F(\div), \sigma \models \phi$  or  $F(\div), \sigma \models \neg\phi$ .*

### 6.4 Legality in FTC for partial meadows

With the word ‘legal’ we refer to the notion of a text being considered correctly or adequately composed as a text. Now, understanding the logical connectives as sequential connectives fails to provide a satisfactory explanation of legality from the NFTC perspective. For instance, one may consider  $x \neq 0 \rightarrow \frac{x}{x} = 1$  a legal text by reading it as  $x \neq 0 \circ \rightarrow \frac{x}{x} = 1$  and at the same time be hesitant about confirming the legality of the particular substitution instance  $0 \neq 0 \rightarrow \frac{0}{0} = 1$  on the grounds that in no conceivable context the subformula  $\frac{0}{0} = 1$  is to be considered meaningful. Reading  $0 \neq 0 \rightarrow \frac{0}{0} = 1$  as  $0 \neq 0 \circ \rightarrow \frac{0}{0} = 1$  makes no difference.

We assume that NFTC can be formalised by means of FTC for partial meadows coupled with an account of legality which is then used to dismiss certain assertions of FTC for partial meadows which are considered non-legal.

**Definition 6.4.** *An equation may be considered non-legal if for no valuation it is assigned a proper truth value (i.e., a truth value that differs from m).*

For instance,  $\frac{1}{0} =_{\text{pm}} 1$  is non-legal. The requirement that a legal assertion may not have non-legal subassertions allows to infer that  $0 \neq 0 \circ \rightarrow \frac{0}{0} = 1$  is not legal. More generally:

**Definition 6.5.** *An assertion in FTC for partial meadows is legal if it has no meaningless subassertions.*

The given definition of legality is far from definitive and it may be challenged as follows: Consider

$$\phi(x) \equiv x^2 =_{\text{pm}} 4 \circ \rightarrow (0 =_{\text{pm}} 0 \wp \left( \frac{x + (-2)}{x + (-2)} =_{\text{pm}} 1 \right)).$$

Now notice that under the condition that  $x^2 =_{\text{pm}} 4$  it is possible for  $\frac{x+(-2)}{x+(-2)} =_{\text{pm}} 1$  to have a proper truth value, in particular, by valuating  $x$  at  $-2$ . It follows that  $\phi(x)$  as a sentence involving implicit universal quantification of  $x$  is legal. In contrast,

$$\phi'(x) \equiv x =_{\text{pm}} 2 \circ \rightarrow (0 =_{\text{pm}} 0 \wp \vee (\frac{x+(-2)}{x+(-2)} =_{\text{pm}} 1))$$

is non-legal because, under the constraint  $x =_{\text{pm}} 2$ , the subformula  $\frac{x+(-2)}{x+(-2)} =_{\text{pm}} 1$  (must) fail to evaluate to a proper truth value (i.e., true or false).

Apparently, a notion legal' can be imagined which takes into account the constraints on variables that are accumulated along a path in an assertion leading to a certain subassertion. Then  $\phi'(x)$  is legal', but not legal. We will not pursue this matter further, the conclusion being that legality matters in principle while demarcation of legality is non-obvious and Definition 6.5 is quite liberal, and may be in need of further refinement.

**6.5 Associativity of sequential conjunction and legality are incompatible**

Let the Dirac-like impulse function  $f(-)$  be given by the two axioms:

$$f(0) =_{\text{pm}} 1 \text{ and } x \neq_{\text{pm}} 0 \circ \rightarrow f(x) =_{\text{pm}} 0$$

and consider  $\phi_1$  and  $\phi_2$  as follows:

$$\phi_1 \equiv 0 =_{\text{pm}} 1 \wp \wedge (\frac{x}{x} =_{\text{pm}} 1 \wp \wedge \frac{f(x)}{f(x)} =_{\text{pm}} 1)$$

$$\phi_2 \equiv (0 =_{\text{pm}} 1 \wp \wedge \frac{x}{x} =_{\text{pm}} 1) \wp \wedge \frac{f(x)}{f(x)} =_{\text{pm}} 1.$$

These are formulae over the extended signature  $\Sigma_m \cup \{f\}$ . Now notice that  $\phi_1$  has a subformula which is meaningless, while none of the subformulas of  $\phi_2$  are meaningless. It follows that  $\phi_1$  is not legal while  $\phi_2$  is legal. Apparently, legality does not respect the associativity of sequential conjunction. We do not know if a similar example exists without extending the signature  $\Sigma_m$ .

**6.6 FTC for partial meadows of rationals**

Meadows in which  $F$  is a prime field of characteristic 0 stand out as having special relevance because school arithmetic is done in such structures  $F(\div)$ . As an additional assertion one may use:

$$\psi_L \equiv \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + 1}{x_1^2 + x_2^2 + x_3^2 + x_4^2 + 1} =_{\text{pm}} 1.$$

This  $\psi_L$  (L for Lagrange) makes use of Lagrange's observation that each natural number is the sum of four squares.

We choose  $\mathbb{Q}$  as a fixed but arbitrary prime field of characteristic 0, then  $\mathbb{Q}(\div) \models \psi_L$ , while  $\psi_L$  is not satisfied in any partial meadow with characteristic  $p > 0$ . Let

$P(x_1, \dots, x_n)$  be a multivariate polynomial. Then

$$\mathbb{Q}(\div) \models \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_n)} =_{\text{pm}} 1$$

if, and only if, the equation  $P(x_1, \dots, x_n) = 0$  has a solution in  $\mathbb{Q}^n$ , a question for which decidability is a long standing open issue. A typical difference between FTC for partial meadows and FTC for partial meadows of rationals is that the latter satisfy:

$$\mathbb{Q}(\div) \models \frac{x^2 + (-2)}{x^2 + (-2)} =_{\text{pm}} 1.$$

## 7 Formal fracterm calculi II: FTC for common meadows

Fracterm calculus for common meadows comes about upon introducing a ‘peripheral’ number or flag  $\perp$  that is absorptive, i.e., if  $\perp$  is an argument to an operator then it returns  $\perp$  as its value; and then by adopting  $\frac{x}{0} = \perp$  for all  $x$ .

**Definition 7.1.** *A common meadow is an enlargement  $F_\perp$  of a field  $F$ , which results by first extending the domain with an absorptive element  $\perp$  and then expanding the structure thus obtained with a constant  $\perp$  and a division function which is made total by adopting*

$$\frac{x}{0} = \frac{x}{\perp} = \frac{\perp}{x} = \perp.$$

By introducing an error element, a common meadow provides arguably the most straightforward way to turn division into a total operator.

### 7.1 Equations for FTC for common meadows

The fracterm calculus of common meadows (as first discussed in [Bergstra and Ponse 2021] and in [Bergstra and Ponse 2016]) has many different axiomatisations. Table 4 lists equations for FTC following the presentation of [Bergstra and Tucker 2022], though with some minor modifications. We notice that these equations are not logically independent.

The axioms of common meadows allow fracterm flattening: each expression can be proven equal to a flat fracterm, where a fracterm is flat if it contains precisely one occurrence (i.e., the top level occurrence) of the division operator. Recall Definition 5.2.

## 8 Formal fracterm calculi III: other FTC’s with total division

In this section we will briefly discuss other FTCs with totalised division: FTC for involutive meadows, FTC for transrationals, and FTC for wheels.

### 8.1 FTC for involutive meadows

The best known totalisation of the division function is that of *involutive meadows*, which adopts

$$x/0 = 0$$

---


$$\begin{aligned}
(x + y) + z &=_{\text{cm}} x + (y + z) \\
x + y &=_{\text{cm}} y + x \\
x + 0 &=_{\text{cm}} x \\
x + (-x) &=_{\text{cm}} 0 \cdot x \\
x \cdot (y \cdot z) &=_{\text{cm}} (x \cdot y) \cdot z \\
x \cdot y &=_{\text{cm}} y \cdot x \\
1 \cdot x &=_{\text{cm}} x \\
x \cdot (y + z) &=_{\text{cm}} (x \cdot y) + (x \cdot z) \\
-(-x) &=_{\text{cm}} x \\
0 \cdot (x + y) &=_{\text{cm}} 0 \cdot (x \cdot y) \\
x + \perp &=_{\text{cm}} \perp \\
\\
\frac{x}{1} &=_{\text{cm}} x \\
-\frac{x}{y} &=_{\text{cm}} \frac{-x}{y} \\
\frac{x}{y} \cdot \frac{u}{v} &=_{\text{cm}} \frac{x \cdot u}{y \cdot v} \\
\frac{x}{y} + \frac{u}{v} &=_{\text{cm}} \frac{(x \cdot v) + (y \cdot u)}{y \cdot v} \\
\frac{x}{y + 0 \cdot z} &=_{\text{cm}} \frac{x + 0 \cdot z}{y} \\
\perp &=_{\text{cm}} \frac{1}{0}
\end{aligned}$$


---

Table 4: FTCCm: Equations for common meadows

and takes all consequences of that identity on board. We will also refer to this choice as an FTC for involutive meadows and as a *Suppes-Ono FTC*.

The phrase Suppes-Ono fracterm calculus is motivated by [Anderson and Bergstra 2020], on Suppes' discussion of division by zero in [Suppes1957], and by the observation that [Ono1983] gave a first significant analysis of the logical consequences of adopting  $1/0 = 0$ . Models of Suppes-Ono FTC are called involutive meadows because inverse is an involution (see [Bergstra et al. 2009] and [Bergstra and Middelburg 2011]). The meadows of rational numbers are an early and primary example that motivated theories of meadows – see [Bergstra and Tucker 2007].

In Suppes-Ono FTC, fracterm flattening fails, as was shown in [Bergstra and Middelburg 2016]. In Suppes-Ono FTC, however, according to [Bergstra et al. 2013], fracterms can be rewritten to sums of flat fracterms. Theoretical work on Suppes-Ono FTC can be found in [Bethke and Rodenburg 2010, Bethke et al. 2015]. Suppes-Ono fracterm

$$(x + y) + z =_{\text{im}} x + (y + z) \quad (18)$$

$$x + y =_{\text{im}} y + x \quad (19)$$

$$x + 0 =_{\text{im}} x \quad (20)$$

$$x + (-x) =_{\text{im}} 0 \quad (21)$$

$$x \cdot (y \cdot z) =_{\text{im}} (x \cdot y) \cdot z \quad (22)$$

$$x \cdot y =_{\text{im}} y \cdot x \quad (23)$$

$$1 \cdot x =_{\text{im}} x \quad (24)$$

$$x \cdot (y + z) =_{\text{im}} (x \cdot y) + (x \cdot z) \quad (25)$$

$$\frac{1}{\left(\frac{1}{x}\right)} =_{\text{im}} x \quad (26)$$

$$\frac{x \cdot x}{x} =_{\text{im}} x \quad (27)$$

$$\frac{x}{y} =_{\text{im}} x \cdot \frac{1}{y} \quad (28)$$

Table 5: Equations for the fracterm calculus of involutive meadows

calculus is called Division by Zero Calculus in [Michikawi et al. 2016, Okumura 2018]. Equations of FTC for involutive meadows are listed in Table 5.

## 8.2 FTC for transrationals and for wheels

The FTC for transrationals adopts

$$\frac{1}{0} = +\infty, \frac{1}{+\infty} = \frac{1}{-\infty} = 0.$$

Here  $+\infty$  and  $-\infty$  are peripherals representing signed infinities. FTC for transrationals is inspired by floating point arithmetic (in particular, when following the IEEE 754 standard). FTC for transrationals enjoys no known form of simplification of fracterms. Transrational FTC concerns the rational number substructure of the transreals of [Anderson et al. 2007], adopts  $\frac{0}{0} = \Phi$  with  $\Phi$  (nullity) serving the same role as  $\perp$  for common fracterm calculus. See also [Bergstra and Tucker 2020]. For some technical information on FTC for transrationals we mention [Bergstra 2020].

A fracterm calculus can be specific for wheels ([Setzer 1997, Carlström 2004, Bergstra and Tucker 2021], which have a single, unsigned infinity and adopts

$$\frac{1}{0} = \infty, \frac{1}{\infty} = 0.$$

For more information on these matters we refer to the survey [Bergstra 2019].

A fracterm calculus can alternatively be based on Kleene equality (see e.g., [Andreka et al. 1988, Robinson 1989]).

Some classic literature on arithmetic education comes close to the ideas of fracterm calculus, e.g., [van Engen 1960].

## 9 Concluding remarks

We have tried to express as clearly as possible, perhaps sometimes with a little exaggeration, the tenets of what we call a naive view on elementary arithmetic involving addition, multiplication, subtraction, and division. Naive fracterm calculus captures a (self-proclaimed) “no-nonsense view” of elementary arithmetic. These views, though more often than not left implicit, permeate educational texts on teaching arithmetic, e.g., [Kieren 1976].

Unavoidably, the very objective of formalisation as an analytical tool leads to a ramification of options, and to a plurality of formalisation, here referred to as our formal fracterm calculi. In contrast with the quest for a naive consensus of views on elementary arithmetic, formalisation seeks and finds many differing views.

### 9.1 Fracterm versus fraction

For someone who adopts NFTC there is no incentive to prefer the use of fracterm over fraction. Undeniably, NFTC inherits from the various pre-existing formal FTC’s a terminology which lacks a strong rationale when considered exclusively from the first principles of NFTC.

Thus, NFTC depicts a conventional “no-nonsense” view of elementary arithmetic seen from the perspective of formal FTCs. By working in this manner the following is achieved:

(1) No claim is made that any formal FTC by itself can, or will, provide a workable basis for the practice of elementary arithmetic. If it is possible to ‘do arithmetic’ on the basis of one of the formal FTC’s then that state of affairs has yet to be demonstrated. What may be claimed, however, is that the different formal FTC’s, starting with FTC<sub>pm</sub>, provide a perspective on elementary arithmetic which is compatible with the concepts of data type and abstract data type that are so fundamental to modern computation.

(2) No claim is made about any use of the word ‘fraction’. For an extensive discussion of the ambiguity of ‘fraction’ which motivates the use of ‘fracterm’ see [Bergstra 2020].

(3) The above text can be read by any mathematician and will not contain or promote any unfamiliar claims about elementary arithmetic. In particular, a reader may disagree with the claim that fracterm in NFTC adequately mimics their understanding of fraction. Reading or adopting NFTC allows a reader to maintain their preferred views on fractions for the simple reason that whatever is written or said about fracterms has no compelling implications for fractions.

### 9.2 Potential applications and options for further work

*Computer programs.* The plurality of formal FTC’s plays a role in analysing various conventions, reasoning patterns, and logics in relation to abstract descriptions of computer arithmetics. Regarding computer programs, an NFTC perspective matters in view of the importance of informal communication and understanding by users, whereas



formalised perspectives matter whenever precise definitions are required by designers. The significance of informal considerations in computer programming is visible in the literatures on requirements analysis and on program testing, where most explanations are quite informal. Bridging the gap between informal considerations and formal modelling is not easy and this paper may be read as a case study about that matter.

*Logical reasoning.* We consider short-circuit logic to be essential for FTC for partial meadows. Short-circuit logic is a relatively new theme in logic. We adopt the notation of [Bergstra et al. 1995] for various sequential connectives. A meticulous investigation of short-circuit logic can be found in [Rodenburg 2001, Bergstra and Ponse 2011] and several subsequent papers on so-called Proposition Algebra (e.g., [Bergstra and van de Pol 2011, Ponse and Staudt 2018]). Regarding FTC<sub>pm</sub> and FTC<sub>cm</sub> many questions are still open, in particular, if one focusses on equational proof systems compatible with the given Tarski semantics for three-valued logic.

*Synthetic calculi.* A necessary challenge is to develop a convincing non-naive, though still informal, detailed perspective on elementary arithmetic – something in the gap between NFTC and useful FTCs, notably FTC<sub>pm</sub>. As described in Section 1.1, we call such a new informal calculus a Synthetic Fracterm Calculus. It could be designed as a basic tool to study specific educational methods and practices about elementary arithmetic. An advantage is that it could exploit the stronger presence of syntactic ideas (as promoted by a use of FTC<sub>pm</sub>) in comparison with the implicit or often absent role of syntactic considerations in traditional approaches preserved in NFTC. Further, it could prepare semantically for pukka logical reasoning about arithmetic, and settle some other unresolved issues in NFCT. That there are many intriguing concepts under the surface of elementary arithmetic is becoming increasingly clear [Bergstra 2022].

## References

- [Anderson and Bergstra 2020] Anderson, J. A., Bergstra, J. A.: “Review of Suppes 1957 proposals for division by zero”; *Transmathematica*. ISSN 2632-9212 (published 06-08-2021), <https://doi.org/10.36285/tm.53>, (2021).
- [Anderson et al. 2007] Anderson, J.A., Völker, N., Adams, A. A.: “Perspecx Machine VIII, axioms of transreal arithmetic”; *Vision Geometry XV*, eds. J. Latecki, D. M. Mount and A. Y. Wu, 649902, (2007).
- [Andreka et al. 1988] Andreka, H., Craig, W., Nemeti, I.: “A system of logic for partial functions under existence-dependent Kleene equality”; *Journal of Symbolic Logic*, 53 (3), (1988), 834–839.
- [Bergstra 2019] Bergstra, J.A.: “Division by zero, a survey of options”; *Transmathematica*, ISSN 2632-9212, (published 2019-06-25), <https://doi.org/10.36285/tm.v0i0.17>, (2019).
- [Bergstra 2020] Bergstra, J.A.: “Arithmetical datatypes, fracterms, and the fraction definition problem”; *Transmathematica*, ISSN 2632-9212, (published 2020-04-30), <https://doi.org/10.36285/tm.33>, (2020).
- [Bergstra 2022] Bergstra, J. A.: “Prospective, retrospective, and formal division: A contribution to philosophical arithmetic”; *Transmathematica*. (published 2022-10-17), <https://doi.org/10.36285/tm.71>, (2022).
- [Bergstra et al. 2013] Bergstra, J.A., Bethke, I., Ponse, A.: “Cancellation meadows: a generic basis theorem and some applications”; *The Computer Journal*, 56 (1), (2013), 3–14. <https://doi.org/10.1093/comjnl/bxs028>.

- [Bergstra et al. 1995] Bergstra, J.A., Bethke, I., Rodenburg, P.H.: “A propositional logic with 4 values: true, false, divergent and meaningless”; *Journal of Applied Non-Classical Logics*, 5 (2), (1995), 199–217.
- [Bergstra and van de Pol 2011] Bergstra, J.A., van de Pol, J.: “A calculus for four-valued sequential logic”; *Theoretical Computer Science*, 412 (28), (2011), 3122–3128.
- [Bergstra et al. 2009] Bergstra, J.A., Hirshfeld, Y., Tucker, J.V.: “Meadows and the equational specification of division”; *Theoretical Computer Science*, 410 (12), (2009), 1261–1271.
- [Bergstra and Middelburg 2011] Bergstra, J.A., Middelburg, C.A.: “Inversive meadows and divisive meadows”; *Journal of Applied Logic*, 9, (2011), 203–220. <https://doi.org/10.1016/j.jal.2011.03.001>
- [Bergstra and Middelburg 2016] Bergstra, J.A., Middelburg, C.A.: “Transformation of fractions into simple fractions in divisive meadows”; *Journal of Applied Logic*, 16, (2015), 92–110. <https://doi.org/10.1016/j.jal.2016.03.001>
- [Bergstra and Ponse 2011] Bergstra, J.A., Ponse, A.: “Proposition Algebra”; *ACM Transactions on Computational Logic*, 12 (3), (2011), 1–36. <https://doi.org/10.1145/1929954.1929958>.
- [Bergstra and Ponse 2021] Bergstra, J.A., Ponse, A.: “Division by zero in common meadows”; In R. de Nicola and R. Hennicker (editors), *Software, Services, and Systems (Wirsing Festschrift)*, *Lecture Notes in Computer Science* 8950, Springer, (2015), 46–61. Also available in improved form (2021) at: [arXiv:1406.6878v4](https://arxiv.org/abs/1406.6878v4) [math.RA] (2021).
- [Bergstra and Ponse 2016] Bergstra, J.A., Ponse, A.: “Fracpairs and fractions over a reduced commutative ring”; *Indagationes Mathematicae*, 27, (2016), 727–748. <http://dx.doi.org/10.1016/j.indag.2016.01.007>.
- [Bergstra and Tucker 2007] Bergstra, J.A., Tucker, J.V.: “The rational numbers as an abstract data type”; *Journal of the ACM*, 54 (2), Article 7, (2007).
- [Bergstra and Tucker 2020] Bergstra, J.A., Tucker, J.V.: “The transrational numbers as an abstract data type”; *Transmathematica*, ISSN 2632-9212, (published 2020-12-16), <https://doi.org/10.36285/tm.47>, (2020).
- [Bergstra and Tucker 2021] Bergstra, J.A., Tucker, J.V.: “The wheel of rational numbers as an abstract data type”. In M. Roggenbach (editor), *Recent Trends in Algebraic Development Techniques. WADT 2020*, *Lecture Notes in Computer Science*, 12669, Springer, 2021, 13–30. [https://doi.org/10.1007/978-3-030-73785-6\\_2](https://doi.org/10.1007/978-3-030-73785-6_2)
- [Bergstra and Tucker 2022] Bergstra, J.A., Tucker, J.V.: “On the axioms of common meadows: Fracterm calculus, flattening and incompleteness”; *The Computer Journal*, 66 (7), 2023, 1565–1572. <https://doi.org/10.1093/comjnl/bxac026>.
- [Bethke and Rodenburg 2010] I. Bethke and P.H. Rodenburg. “The initial meadows”; *Journal of Symbolic Logic*, 75 (3), (2010), 888–895.
- [Bethke et al. 2015] Bethke, I., Rodenburg, P.H., Sevenster, A.: “The structure of finite meadows”; *Journal of Logical and Algebraic Methods in Programming*, 84 (2), (2015), 276–282.
- [Carlström 2004] Carlström, J.: “Wheels – On division by zero”; *Mathematical Structures in Computer Science*, 14 (1), (2004), 143–184.
- [Fandino Pinilla 2007] Fandino Pinilla, M.I.F.: “Fractions: conceptual and didactic aspects”; *Acta Didactica Universitatis Comenianae*, 7, (2007), 82–115.
- [Halmos 1960] Halmos, P.: *Naive Set Theory*. D. Van Nostrand Company, 1960.
- [Jones and Middelburg 1994] Jones, C.B., Middelburg, C.A.: “A typed logic of partial functions, reconstructed classically”; *Acta Informatica*, 31, (1994), 399–430.

- [Kieren 1976] Kieren, T.E.: “On the mathematical, cognitive, and instructional foundations of rational numbers”; In R.A. Lesh and D.A. Bradbart (editors), *Number and Measurement. Papers from a Research Workshop*. ERIC, Columbus Ohio, (1976), 101–144, (available at <http://files.eric.ed.gov/fulltext/ED120027.pdf#page=108>, accessed 1 September, 2023).
- [Michikawi et al. 2016] Michiwaki, H., Saitoh, S., Yamada, N.: “Reality of the division by zero  $z/0 = 0$ ”; *International Journal of Applied Physics and Mathematics*, doi: 10.17706/ijapm.2016.6.1.1–8 (2016).
- [Ono1983] Ono, H.: “Equational theories and universal theories of fields”; *Journal of the Mathematical Society of Japan*, 35 (2), (1983), 289-306.
- [Okumura 2018] Okumura H.: “Is it really impossible to divide by zero?”; *Biostatistics and Biometrics Open Access Journal*. 7 (1), 555703. doi: 10.19080/BBOJ.2018.07.555703, (2018)
- [Ponse and Staudt 2018] Ponse, A., Staudt, D.J.C.: “An independent axiomatisation for free short-circuit logic”; *Journal of Applied Non-Classical Logics*. 28 (1), (2018), 35–71.
- [Robinson 1989] Robinson, A.: “Equational logic of partial functions under Kleene equality: a complete and an incomplete set of rules”; *Journal of Symbolic Logic*, 54 (2), (1989), 354–362.
- [Rodenburg 2001] Rodenburg, P.H.: “A complete system of four-valued logic”; *Journal of Applied Non-classical Logics*, 11 (3/4), (2001), 367–389.
- [Setzer 1997] Setzer, A.: “Wheels (draft)”; <http://www.cs.swan.ac.uk/csetzer/articles/wheel.pdf>, (1997).
- [Suppes1957] Suppes, P.: “Introduction to Logic”, Van Nostrand Reinhold, 1957.
- [van Engen 1960] van Engen, H.: “Rate pairs, fractions, and rational numbers”; *The Arithmetic Teacher*, 7 (8) (1960), 389–399.
- [Williams 2014] Williams, J.: “Robert Recorde. Tudor Polymath, Expositor and Practitioner of Computation”, *History of Computing Series*, Springer, 2014.