Schrödinger-Bopp-Podolsky system with steep potential well *

Zhu Qiutong[a)], Chunfang Chen[a)], Chenggui Yuan[b)]

[a)]Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China

[b)]Department of Mathematics, Swansea University, Bay Campus, SA1 8EN, UK

2296350469@qq.com, chenchunfang@ncu.edu.cn, c.yuan@swansea.ac.uk

Abstract

In this paper, we study the following Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + \lambda V(x)u + q^2\varphi u = f(u) \\ -\Delta\varphi + a^2\Delta^2\varphi = 4\pi u^2, \end{cases}$$

where $u, \varphi : \mathbb{R}^3 \to \mathbb{R}$, $a > 0$, $q > 0$, λ is real positive parameter, f satisfies supper 2 lines growth. Let $V \in C(\mathbb{R}^3, \mathbb{R})$ and suppose that $V(x)$ represents a potential well with the bottom $V^{-1}(0)$. There are no results of solutions for the system with steep potential well in the current literature because of the presence of the nonlocal term. By using the truncation technique and the parameter-dependent compactness lemma, we obtain a positive energy solution $u_{\lambda,q}$ for λ large and q small. Moreover, the asymptotic behavior as $q \to 0$, $\lambda \to +\infty$ is investigated.

Keywords: Schrödinger-Bopp-Podolsky system; Truncation technique; Parameter-dependent compactness lemma; Asymptotic behavior.

1 Introduction

Let us study the following system

$$\begin{cases} -\Delta u + \lambda V(x)u + q^2\varphi u = f(u) \\ -\Delta\varphi + a^2\Delta^2\varphi = 4\pi u^2, \end{cases} \tag{1.1}$$

where $u, \varphi : \mathbb{R}^3 \to \mathbb{R}$, $a > 0$, $q > 0$, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R})$. We suppose that the potential $V(x)$ satisfies the conditions as follows:

*Corresponding author: Chunfang Chen

$(V_1)\, V \in \mathcal{C}\left(\mathbb{R}^3, \mathbb{R}\right)$ and $V(x) \geq 0$;
(V_2) there exists $b > 0$, such that $\mathcal{V}_b := \{x \in \mathbb{R}^3 : V(x) < b\}$ is nonempty and has finite measure;
(V_3) $\Omega = \text{int}\, V^{-1}(0)$ is a nonempty open set with locally Lipschitz boundary and $\bar{\Omega} = V^{-1}(0)$.

This kind of hypotheses was first introduced by Bartsch and Wang [1] in the study of Schrödinger equations, and has attracted the attention of many domestic scholars, see e.g. [2-6]. Note that, the assumptions $(V_1) - (V_3)$ imply that λV represents a potential well with the bottom $V^{-1}(0)$ and its steepness is controlled by the parameter λ. As a result, λV is often known as the steep potential well if λ is sufficiently large, and we expect to find solutions which are localized near the bottom of the potential V.

To state our results in the paper, we give the following assumptions:
$(f_1)\, f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, there exists $p \in (2, 4]$, such that

$$|f(u)| \leq 1 + |u|^{p-1};$$

(f_2) there exists $\kappa \in (2, 4]$, such that

$$0 < F(u) := \int_0^u f(t)dt \leq \frac{1}{\kappa} f(u)u, \ \forall u \in \mathbb{R}\backslash\{0\};$$

$(f_3)\, f(u) = o(|u|)$ as $|u| \to 0$.

When we couple a Schrödinger field $\psi = \psi(t, x)$ with its electromagnetic field in the Bopp-Podolsky electromagnetic theroy, especially, in the case of localized oscillating sources, the system that we study will appear. The problem (1.1) has a physical meaning especially in the Bopp-Podolsky theory which is a second order gauge theory for the electromagnetic field, and its development is drived by Bopp [7] and Podolsky [8]. As the Mie [9] theory and its generalizations given by Born and Infeld [10-13], it was introduced to solve the infinity problems, which appear in the classical Maxwell theory. In reality, from the Gauss law, the electrostatic potential φ for a given charge distribution whose density is ρ satisfies the following equation:

$$-\Delta\varphi = \rho \text{ in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\pi\delta_{x_0}$ with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $\chi(x - x_0)$, where

$$\chi(x) = \frac{1}{|x|}, \tag{1.3}$$

and the electrostatic energy is

$$\varepsilon_M(\chi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\chi|^2 = +\infty. \tag{1.4}$$

Hence, equation is replaced by

$$-\text{div}\left(\frac{\nabla\varphi}{\sqrt{1 - |\nabla\varphi|^2}}\right) = \rho \tag{1.5}$$

in the Born-Infeld theory and by

$$-\Delta\varphi + a^2\Delta^2\varphi = \rho \tag{1.6}$$

in the Bopp-Podolsky theory. In these both cases, if $\rho = 4\pi\delta_{x_0}$, we can write explicitly the solutions of the respective equations and see that energy is finite. In particular, when we consider the operator $-\Delta + a^2\Delta^2$, we have that $\chi(x - x_0)$ with

$$\chi(x) := \frac{1 - e^{-|x|/a}}{|x|}, \tag{1.7}$$

is the fundamental solution of the equation

$$-\Delta\varphi + a^2\Delta^2\varphi = 4\pi\delta_{x_0}. \tag{1.8}$$

Then χ has no singularity in x_0 since it satisfies

$$\lim_{x\to x_0} \chi(x - x_0) = \frac{1}{a}, \tag{1.9}$$

and its energy is

$$\varepsilon_{BP}(\chi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\chi|^2 + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\chi|^2 < +\infty. \tag{1.10}$$

Moreover the Bopp-Podolsky theory may be interpreted as an effective theory for short distances (see [14]) and for large distances which is experimentally indistinguishable from the Maxwell. Thus, the Bopp-Podolsky parameter $a > 0$ which has dimension of the inverse of mass, can be interpreted as a cut-off distance or can be linked to an effective radius for the electron. For more physical details we refer the reader to the recent parers [15-20] and references therein.

In the recent paper [21], Chen and Tang considered the following Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + V(x)u + \varphi u = \mu f(u) + u^5, & \text{in } \mathbb{R}^3, \\ -2\Delta\varphi + a^2\Delta^2\varphi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0$, $V \in C(\mathbb{R}^3, [0, \infty))$. By using new analytic techniques, they proved the existence of ground state solutions for all $\mu > 0$, if $p \in (4, 6)$; for all $\mu > \mu_0$, if $p \in (2, 4]$, where μ_0 is a positive constant determined by a, V_∞, and p. In [22], Yang, Yuan and Liu studied the existence of ground states for a nonlinear Schrödinger-Bopp-Podolsky system with asymptotically periodic potentials, where $V(x)$ has a positive lower bound. And they proved the existence of ground states for the nonlinear Schrödinger-Bopp-Podolsky system with periodic potentials. In [23], Li, Pucci and Tang studied the existence of ground state solutions for the following nonlinear Schrödinger-Bopp-Podolsky system with critical Sobolev exponent:

$$\begin{cases} -\Delta u + V(x)u + q^2\varphi u = \mu|u|^{p-1}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\varphi + a^2\Delta^2\varphi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$ is a parameter and $p \in (2,5)$. Under certain assumptions on V, they proved the existence of a nontrivial ground state solution by using the Pohozaev-Nehari manifold, the arguments of Arzis-Nirenberg, the monotonicity trick and a global compactness lemma. In [24], under certain assumptions on f and V, by using minimizations arguments and generalized subdifferential, Bahrouni and Missaoui got the existence of a ground state with a fixed sign and a least energy nodal solutions for the following Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + V(x)u + q^2\varphi u &= f(x,u), \quad \text{in } \mathbb{R}^3, \\ -\Delta\varphi + a^2\Delta^2\varphi &= 4\pi u^2, \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0,\ q \neq 0$. Moreover, they proved that the energy of the nodal solution is twice as large as that of the ground state solution. In [25], Figueiredo and Siciliano proved the existence of solutions for a Schrödinger-Bopp-Podolsky system under positive potentials. By using the Ljusternick-Schnirelmann and Morse Theory, they got multiple solutions with a priori prescribed interaction energy. In [26], Zhang and Du considered the following Kirchhoff type problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + \lambda V(x)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a > 0$ is a constant, b and λ are positive parameters, and $2 < p < 6$. They supposed that the nonnegative continuous potential V represents a potential well with the bottom $V^{-1}(0)$. And they proved the existence of positive solutions for b small and λ large in the case $2 < p < 4$ by using the truncation technique and the parameter-dependent compactness lemma. Eventually, they explored the decay rate of the positive solutions as $|x| \to \infty$ and their asymptotic behavior as $b \to 0$ and $\lambda \to \infty$. In [27], Du considered the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \lambda V(x)u + \mu\varphi u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2, \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda,\ \mu > 0$ are real parameters and $2 < p < 6$. He supposed that $V(x)$ represents a potential well with the bottom $V^{-1}(0)$. By using the truncation technique and the parameter-dependent compactness lemma, he proved the existence of positive solutions for λ large and μ small in the case $2 < p < 4$ and proved the nonexistence of nontrivial solutions for λ large and μ large in the case $2 < p \leq 3$. Eventually, he explored the decay rate of the positive solutions as $|x| \to \infty$ and their asymptotic behavior as $\lambda \to \infty$ and $\mu \to 0$.

Motivated by the above works, in our paper, we will try to obtain a positive energy solution for λ large and q small for the Schrödinger-Bopp-Podolsky system with steep potential well. Meanwhile, we will explore the asymptotic behavior of $q \to 0,\ \lambda \to +\infty$. Compared with [22], in our paper, $V(x)$ is deep well potential, which satisfies weaker conditions. Compared with [23], we assume $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ rather than $V \in \mathcal{C}^1(\mathbb{R}^3)$. Compared with [24], we assume that f is superquadratic rather than superquartic in our paper.

The differential operator $-\Delta + \Delta^2$ appears in various interesting mathematical and physical situations; see [14, 28] and the references therein. Before stating our results, few preliminaries will be introduced in order. We introduce here the space $\mathcal{D}$ as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\sqrt{\|\nabla\varphi\|_2^2 + a^2\|\Delta\varphi\|_2^2}$.

For fixed $a > 0$ and $q \neq 0$, we say that a pair $(u, \varphi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a solution of problem (1.1) if

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)\, dx + q^2 \int_{\mathbb{R}^3} \varphi uv dx = \int_{\mathbb{R}^3} f(u)v dx,\ \ v \in H^1(\mathbb{R}^3),$$

$$\int_{\mathbb{R}^3} \nabla\varphi\nabla\xi + a^2 \int_{\mathbb{R}^3} \Delta\varphi\Delta\xi dx = 4\pi \int_{\mathbb{R}^3} \varphi u^2 dx,\ \ \xi \in \mathcal{D}.$$

And a solution (u, φ) is nontrivial if $u \neq 0$. To solve problem (1.1) is equivalent to solve

$$-\Delta u + \lambda V(x)u + q^2\left(\frac{1 - e^{-|x|/a}}{|x|} * u^2\right)u = f(u),\ \text{ in } \mathbb{R}^3, \tag{1.11}$$

whose solution is the critical point of the associated energy functional

$$I_{\lambda,q}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) dx + \frac{q^2}{4}\int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

In our paper, we aim to verify the existence of nontrivial solutions.

Theorem 1.1. *Assume that* $(V_1) - (V_3)$ *and* $(f_1) - (f_3)$ *hold. Then there exist* $\lambda^* > 1$ *and* $q_* > 0$, *such that (1.11) has at least a nontrivial solution* $u_{\lambda,q} \in E_\lambda$. *Moreover, there exist two constants* $\tau,\ T > 0$ *(independent of* λ *and* q *), such that*

$$0 < \tau \leq \|u_{\lambda,q}\|_\lambda \leq T \quad \text{and} \quad I_{\lambda,q}(u_{\lambda,q}) > 0, \tag{1.12}$$

for each $\lambda \in (\lambda^*, \infty),\ q \in (0, q_*)$.

Now we shall give the idea to verify Theorem 1.1, and this idea goes back to [26, 27]. By applying the Mountain Pass Theorem to the energy functional $I_{\lambda,q}$ in direct, we can get a Cerami sequence for $q > 0$ enough small. But the boundedness of the Cerami sequence is a key difficulty. Thus we shall use the truncation technique as e.g. in [26, 27, 29]. More precisely, for each $T > 0$, we study the truncated functional $I_{\lambda,q}^T : E_\lambda \to \mathbb{R}$ defined by

$$I_{\lambda,q}^T(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) dx + \frac{q^2}{4}\eta\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

In this case, we give the summarization of the proof of Theorem 1.1. Firstly, we verify that the truncated functional $I_{\lambda,q}^T$ satisfies the mountain pass geometry for $q > 0$ enough small, and then get a Cerami sequence $\{u_n\}$ of $I_{\lambda,q}^T$ at the mountain pass level $c_{\lambda,q}^T$. Subsequently, we observe that $c_{\lambda,q}^T$ has an upper bound independent of T, λ, q. From this observation, we may use the standard truncation argument to infer that for a given $T > 0$ correctly, after passing to a subsequence, $\|u_n\|_\lambda \leq T$ for all $n \in N$ by restricting $q > 0$ sufficiently small. Therefore $\{u_n\}$ is a bounded Cerami sequence of $I_{\lambda,q}$, i.e.,

$$\sup_{n\in N}\|u_n\|_\lambda \leq T,\ I_{\lambda,q} \to c_{\lambda,q} \text{ and } (1 + \|u_n\|_\lambda)\|I'_{\lambda,q}(u_n)\|_{E_\lambda^*} \to 0,$$

where E_λ^* is the dual space of E_λ. Finally, by using the parameter-dependent compactness lemma, we may pass to a subsequence of $\{u_n\}$ which converges to $u_{\lambda,q}$ in E_λ for $\lambda > 0$ enough large. Therefore, $u_{\lambda,q}$ is a positive solution of (1.11) with $\|u_{\lambda,q}\|_\lambda \leq T$ and $I_{\lambda,q}(u_{\lambda,q}) = c_{\lambda,q}^T$.

In the final part, we study the asymptotic behavior of the positive energy solutions which are obtained by Theorem 1.1 as $\lambda \to \infty$ and $q \to 0$. We give the theorems as follows.

Theorem 1.2. *Assume that $(V_1)-(V_3)$ and $(f_1)-(f_3)$ hold. If $u_{\lambda,q}$ is a nontrivial solution of (1.11) obtained by Theorem 1.1, then for each fixed $q \in (0, q_*)$, and any sequence $\{\lambda_n\} \subset (\lambda^*, \infty)$, $u_{\lambda_n,q} \to u_q$ in E as $\lambda_n \to +\infty$ up to a subsequence, where $u_q \in H_0^1(\Omega)$ is a nontrivial solution of*

$$\begin{cases} -\Delta u + q^2\varphi_u u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

Theorem 1.3. *Let $u_{\lambda,q}$ be a nontrivial solution of (1.11) obtained by Theorem 1.1. Then for each fixed $\lambda \in (\lambda^*, \infty)$, and any sequence $\{q_n\} \subset (0, q_*)$, $u_{\lambda,q_n} \to u_\lambda$ in E_λ with $q_n \to 0$ as $n \to \infty$ up to a subsequence, where $u_\lambda \in E_\lambda$ is a nontrivial solution of*

$$\begin{cases} -\Delta u + \lambda V(x)u = f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.14}$$

Theorem 1.4. *Let $u_{\lambda,q}$ be a nontrivial solution of (1.11) obtained by Theorem 1.1. Then for any sequence $\{\lambda_n\} \subset (\lambda^*, \infty)$ and $\{q_n\} \in (0, q_*)$, $u_{\lambda,q} \to u_0$ in $H^1(\mathbb{R}^3)$ as $q_n \to 0$ and $\lambda_n \to \infty$ up to a subsequence, where $u_0 \in H_0^1(\Omega)$ is a nontrivial solution of*

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Remark. (i) Let $q > 0$ be a small fixed-parameter, Theorem 1.2 shows that the nontrivial solutions $u_{\lambda,q}$ are well localized near the bottom of the potential as $\lambda \to \infty$.
(ii) Let $\lambda > 0$ be a large fixed-parameter, Theorem 1.3 shows that the nontrivial solutions of (1.11) may converge in E_λ to a nontrivial solution of (1.14) as $q \to 0$ up to a subsequence.
(iii) Theorem 1.4 shows that the nontrivial solutions of (1.11) may converge in $H^1(\mathbb{R}^3)$ to a nontrivial solution of (1.15) as $\lambda \to \infty$ and $q \to 0$ up to a subsequence.

The rest of this paper is organized in this way. In Section 2, we set up the variational framework of (1.11) and some preliminary results. In Section 3, we give the proof of Theorem 1.1. Finally, we will explore the asymptotic behavior as $q \to 0$, $\lambda \to +\infty$ and complete the proofs of Theorems 1.2, 1.3 and 1.4 in Section 4.

Throughout this paper, we make use of the following nations:
♣ $L^s(\mathbb{R}^3), 1 \leq s \leq \infty$, denotes the usual Lebesgue space with the norm $|\cdot|_s$.
♣ $|M|$ is the Lebesgue measure of the set M.
♣ X^* denotes the dual space of X.
♣ The weak convergence is denoted by $\rightharpoonup$, and $\to$ denotes the strong convergence.
♣ S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$.
♣ $C_0,\ C_1,\ C_2, ...$ denote positive constants.

2 Variational setting and preliminary results

In this section, we give some preliminary results and introduce the variational framework of equation (1.11).

$H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with the standard scalar product and norm $\|\cdot\|_{H^1}$. Let

$$E=\left\{u\in H^1\left(\mathbb{R}^3\right):\int_{\mathbb{R}^3}V(x)u^2dx<\infty\right\}$$

be equipped with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^3}(\nabla u\nabla v+V(x)uv)dx,\quad \|u\|=\langle u,u\rangle^{1/2}.$$

For $\lambda>0$, we also need the following inner product and norm

$$\langle u,v\rangle_\lambda=\int_{\mathbb{R}^3}(\nabla u\nabla v+\lambda V(x)uv)dx,\quad \|u\|_\lambda=\langle u,u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\|\le\|u\|_\lambda$ for $\lambda\ge1$. Set $E_\lambda=(E,\|\cdot\|_\lambda)$.

Lemma 2.1. *Assume that (V_1) and (V_2) hold. Then the embedding $E\hookrightarrow H^1(\mathbb{R}^3)$ is continuous for $\lambda\ge1$ and $2\le s\le6$, and there exists $d_s>0$ (independent of $\lambda\ge1$), such that*

$$|u|_s\le d_s\|u\|\le d_s\|u\|_\lambda,\ for\ u\in E. \tag{2.1}$$

Proof. This proof is well-known, see e.g. [27]. For the reader's convenience, we provide the proof here. It follows from (V_1), Hölder inequality and Sobolev inequality that

$$\begin{aligned}\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx&=\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}u^2dx\\&=\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathcal{V}_b}|u|^2dx+\int_{\mathbb{R}^3\setminus\mathcal{V}_b}|u|^2dx\\&\le\int_{\mathbb{R}^3}|\nabla u|^2dx+\left(\int_{V_b}1^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\left(\int_{V_b}(u^2)^3dx\right)^{\frac{1}{3}}+\int_{\mathbb{R}^3\setminus V_b}u^2dx\\&\le\int_{\mathbb{R}^3}|\nabla u|^2dx+|\mathcal{V}_b|^{\frac{2}{3}}\left(\int_{\mathcal{V}_b}|u|^6dx\right)^{\frac{1}{3}}+b^{-1}\int_{\mathbb{R}^3\setminus\mathcal{V}_b}V(x)u^2dx\\&\le\int_{\mathbb{R}^3}|\nabla u|^2dx+|\mathcal{V}_b|^{\frac{2}{3}}S^{-1}\left(\int_{\mathcal{V}_b}|\nabla u|^2dx\right)+b^{-1}\int_{\mathbb{R}^3\setminus\mathcal{V}_b}V(x)u^2dx\\&\le\max\left\{1+|\mathcal{V}_b|^{\frac{2}{3}}S^{-1},b^{-1}\right\}\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)u^2)dx,\end{aligned}$$

which implies that $E\hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Thus, for each $s\in[2,6]$, there exists $d_s>0$ (independent of $\lambda\ge1$), such that

$$|u|_s\le d_s\|u\|\le d_s\|u\|_\lambda,\quad \text{for } u\in E.$$

The lemma is completed. ∎

Next, let us consider the nonlinear Schrödinger Lagrangian density

$$\pounds_{S_c} = i\hbar\overline{\psi}\partial_t\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + 2F(\psi),$$

where i is an imaginary unit, Gs is the unit of electromagnetic field, J/m^3 is the unit of Lagrange density, and $\hbar, m > 0$, and let (φ, A) be the gauge potential of the electromagnetic field (E, B), namely $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfy

$$E = -\nabla\varphi - \frac{1}{c}\partial_t A, B = \nabla \times A.$$

The coupling of the field ψ with the electromagnetic field (E, B) through the minimal coupling rule, namely the study of the interaction between ψ and its own electromagnetic field can be obtained by replacing in $\pounds_{S_c}$ the derivatives ∂_t and ∇ respectively with the covariant ones

$$D_t = \partial_t + \frac{iq}{\hbar}\varphi, D = \nabla - \frac{iq}{\hbar c}A,$$

q being a couple constant. These results are valid only when a nonpermeable medium is considered for which $\mu = 1$. This let us consider

$$\pounds_{CS_c} = i\hbar\psi D_t\psi - \frac{\hbar^2}{2m}|D\psi|^2 + 2F(\psi) = i\hbar\varphi(\partial_t + \frac{iq}{\hbar}\varphi)\psi - \frac{\hbar^2}{2m}|(\nabla - \frac{iq}{\hbar c}A)\psi|^2 + 2F(\psi).$$

Now, to get the total Lagrangian density, we have to add to $\pounds_{S_c}$ the Lagrangian density of the electromagnetic field.

The Bopp-Podolsky Lagrangian density is

$$\begin{aligned} \mathcal{L}_{\text{BP}} =& \frac{1}{8\pi}\left\{|E|^2 - |B|^2 + a^2\left[(\operatorname{div} E)^2 - \left|\nabla \times B - \frac{1}{c}\partial_t E\right|^2\right]\right\} \\ =& \frac{1}{8\pi}\left\{\left|\nabla\varphi + \frac{1}{c}\partial_t A\right|^2 - |\nabla \times A|^2\right. \\ & \left. + a^2\left[\left(\Delta\varphi + \frac{1}{c}\operatorname{div}\partial_t A\right)^2 - \left|\nabla \times \nabla \times A + \frac{1}{c}\partial_t\left(\nabla\varphi + \frac{1}{c}\partial_t A\right)\right|^2\right]\right\}. \end{aligned} \tag{2.2}$$

Thus, the total action is

$$S(\psi, \varphi, A) = \int_{\mathbb{R}^3} \pounds dxdt$$

where $\pounds := \pounds_{CS_c} + \pounds_{BP}$ is the total Lagrangian density.

Let $\mathcal{D}$ be the completion of $C_c(\mathbb{R}^3)$ with the respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle\varphi, \psi\rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\varphi\nabla\psi dx + a^2\int_{\mathbb{R}^3} \Delta\varphi\Delta\psi dx.$$

Then $\mathcal{D}$ is a Hilbert space continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$.

We notice the following auxiliary properties.

Lemma 2.2. *(See [30]) The space $\mathcal{D}$ is continuously embedded in $L^{\infty}(\mathbb{R}^3)$.*

The next property gives a useful characterization of the space $\mathcal{D}$.

Lemma 2.3. *(See [30]) The space $C_c^{\infty}(\mathbb{R}^3)$ is dense in*

$$\mathcal{A} := \left\{\phi \in D^{1,2}\left(\mathbb{R}^3\right) : \Delta\phi \in L^2\left(\mathbb{R}^3\right)\right\}$$

normed by $\sqrt{\langle\varphi,\varphi\rangle_{\mathcal{D}}}$ and, thus, $\mathcal{D} = \mathcal{A}$.

For every fixed $u \in H^1(\mathbb{R}^3)$, the Riesz representation theorem implies that there is a unique solution $\varphi_u \in \mathcal{D}$ of the second equation in (1.1). To write explicitly such a solution, we consider

$$\chi(x) = \frac{1 - e^{-|x|/a}}{|x|}.$$

We have the following fundamental properties.

Lemma 2.4. *(See [30]) For all $y \in \mathbb{R}^3$, $\chi(\cdot - y)$ solves*

$$-\triangle\varphi + a^2\Delta^2\varphi = 4\pi\delta_y$$

in the sense of distributions. Moreover,

(i) if $g \in L^1_{loc}(\mathbb{R}^3)$ and the fact that for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \longmapsto g(y)/|x-y|$ is summable, then $\chi * g \in L^1_{loc}(\mathbb{R}^3)$;

(ii) if $f \in L^s\left(\mathbb{R}^3\right)$ with $1 \leq s < 3/2$, then $\chi * g \in L^q\left(\mathbb{R}^3\right)$ for $q \in (3s/(3-2s), +\infty]$. In both cases, $\chi * g$ solves

$$-\triangle\varphi + a^2\Delta^2\varphi = 4\pi g$$

in the sense of distributions, and we have the following distributional derivatives:

$$\nabla(\chi * g) = (\nabla\chi) * g \quad \text{and} \quad \Delta(\chi * g) = (\Delta\chi) * g, \quad \text{a.e.} \quad \text{in}(\mathbb{R}^3).$$

Fixed $u \in H^1(\mathbb{R}^3)$, the unique solution in $\mathcal{D}$ of the second equation in (1.1) is

$$\varphi_u := \chi * u^2.$$

Actually the following useful properties hold.

Lemma 2.5. *(See [30]) For every $u \in H^1(\mathbb{R}^3)$, we have:*

(1) for every $y \in (\mathbb{R}^3)$, $\varphi_{u(\cdot+y)} = \varphi_u(\cdot + y)$;

(2) $\varphi_u \geq 0$;

(3) for every $s \in (3, +\infty]$, $\varphi_u \in L^s(\mathbb{R}^3) \bigcap C_0(\mathbb{R}^3)$;

(4) for every $s \in (3/2, +\infty]$, $\nabla\varphi_u = \nabla\chi * u^2 \in L^s(\mathbb{R}^3) \bigcap C_0(\mathbb{R}^3)$;

(5) $\varphi_u \in \mathcal{D}$;

(6) $\|\varphi_u\|_6 \leq C\|u\|^2$;

(7) φ_u is the unique minimizer of the functional

$$E(\varphi) = \frac{1}{2}\|\nabla\varphi\|_2^2 + \frac{a^2}{2}\|\triangle\varphi\|_2^2 - \int_{\mathbb{R}^3} \varphi u^2 dx, \quad \varphi \in \mathcal{D}.$$

Moreover, if $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\varphi_{v_n} \rightharpoonup \varphi_v$ in $\mathcal{D}$.

The energy functional defined in $H^1(\mathbb{R}^3)\times\mathcal{D}$ by

$$S(u,\varphi)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda V(x)u^2\right)dx+\frac{q^2}{2}\int_{\mathbb{R}^3}\varphi u^2dx-\frac{q^2}{16\pi}\|\nabla\varphi\|_2^2-\int_{\mathbb{R}^3}F(u)dx \tag{2.3}$$

is continuously differentiable and its critical points correspond to the weak solutions of problem (1.1). Indeed, if $(u,\varphi)\in H^1(\mathbb{R}^3)\times\mathcal{D}$ is a critical point of S, then

$$0=\partial_u S(u,\varphi)[v]=\int_{\mathbb{R}^3}\left(\nabla u\nabla v+\lambda V(x)uv\right)dx+q^2\int_{\mathbb{R}^3}\varphi uvdx-\int_{\mathbb{R}^3}f(u)vdx,\quad v\in H^1(\mathbb{R}^3)$$

and

$$0=\partial_\varphi S(u,\varphi)[\xi]=\frac{q^2}{2}\int_{\mathbb{R}^3}u^2\xi dx-\frac{q^2}{8\pi}\int_{\mathbb{R}^3}\nabla\varphi\nabla\xi dx-\frac{a^2q^2}{8\pi}\int_{\mathbb{R}^3}\Delta\varphi\Delta\xi dx,\quad \xi\in\mathcal{D}. \tag{2.4}$$

In order to avoid the difficulty generated by the strongly indefiniteness of the functional S, we apply a reduction procedure. Noting that $\partial_\varphi S$ is a C^1 functional, if G_φ is the graph of the map $\Phi: u\in H^1(\mathbb{R}^3)\mapsto\phi_u\in\mathcal{D}$, an application of the implicit function theorem gives

$$G_\Phi=\left\{(u,\phi)\in H^1\left(\mathbb{R}^3\right)\times\mathcal{D}:\partial_\phi\mathcal{S}(u,\phi)=0\right\}\quad\text{and}\quad\Phi\in\mathcal{C}^1\left(H^1\left(\mathbb{R}^3\right),\mathcal{D}\right).$$

From (2.3) and (2.4), the functional $I(u):=S(u,\varphi_u)$ has reduced from

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda V(x)u^2\right)dx+\frac{q^2}{4}\int_{\mathbb{R}^3}\left(\frac{1-e^{-\frac{|x|}{a}}}{|x|}*u^2\right)u^2dx-\int_{\mathbb{R}^3}F(u)dx, \tag{2.5}$$

which is of class C^1 on $H^1(\mathbb{R}^3)$ and, for all $u,v\in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned}I'(u)[v]&=\partial_u S(u,\varphi(u))[v]+\partial_\varphi S(u,\varphi(u))\circ\varphi'(u)[v]\\&=\partial_u S(u,\varphi(u))[v]\\&=\int_{\mathbb{R}^3}\left(\nabla u\nabla v+\lambda V(x)uv\right)dx+q^2\int_{\mathbb{R}^3}\varphi uvdx-\int_{\mathbb{R}^3}f(u)vdx.\end{aligned} \tag{2.6}$$

Moreover, the following statements are equivalent:

(i) the pair $(u,\varphi)\in H^1(\mathbb{R}^3)\times\mathcal{D}$ is a critical point of S, that is, (u,φ) is a solution of problem (1.1);

(ii) u is a critical point of I and $\varphi=\varphi_u$.

Hence, if $u\in H^1(\mathbb{R}^3)$ is a critical point of I, then the pair (u,φ_u) is a solution of (1.1). For the sake of simplicity, in many cases we just say $u\in H^1(\mathbb{R}^3)$, instead of $(u,\varphi_u)\in H^1(\mathbb{R}^3)\times\mathcal{D}$, is a solution of (1.1). Consequently, the functional $I_{\lambda,q}:E_\lambda\to\mathbb{R}$ of (1.11) given by

$$I_{\lambda,q}(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda V(x)u^2\right)dx+\frac{q^2}{4}\int_{\mathbb{R}^3}\varphi_u u^2dx-\int_{\mathbb{R}^3}F(u)dx$$

is well defined, and it is of class C^1 with derivative

$$\left\langle I'_{\lambda,q}(u),v\right\rangle=\int_{\mathbb{R}^3}\left(\nabla u\nabla v+\lambda V(x)uv\right)dx+q^2\int_{\mathbb{R}^3}\varphi_u uvdx-\int_{\mathbb{R}^3}f(u)vdx$$

for all $u,v\in E_\lambda$. It is easy to deduce that the weak solutions of (1.11) is the critical points of the functional $I_{\lambda,q}$.

3 Existence of nontrivial solution to (1.11)

In this part, we shall study the existence of nontrivial solution for (1.11) and give the proof of Theorem 1.1. We follow the argument in [26, 27]. Firstly, we introduce a cut-off function $\eta \in C^1([0,\infty),\mathbb{R})$ which can help us to overcome the difficulty of getting bounded Palais-Smale sequences for the functional $I_{\lambda,q}$. Then we give the definitions of it.

$$
\begin{cases} \eta(t) = 1, & 0 \le t \le 1, \\ \eta(t) = 0, & t \ge 2, \\ 0 \le \eta(t) \le 1, & t > 0, \\ \max_{t>0} |\eta'(t)| \le 2, & t > 0, \\ \eta'(t) \le 0, & t > 0. \end{cases} \tag{3.1}
$$

Using η, for each $T > 0$, we give the truncated functional $I_{\lambda,q}^T : E_\lambda \to \mathbb{R}$ defined by

$$
I_{\lambda,q}^T(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) dx + \frac{q^2}{4}\eta\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx, \tag{3.2}
$$

where η is a smooth cut-off function such that

$$
\eta\left(\frac{\|u\|_\lambda^2}{T^2}\right) = \begin{cases} 1, & \|u\|_\lambda \le T, \\ 0, & \|u\|_\lambda \ge \sqrt{2}T. \end{cases}
$$

We find that $I_{\lambda,q}^T$ is of class $\mathcal{C}^1$. Meanwhile, for each $u, v \in E_\lambda$, we have

$$
\begin{aligned} \left\langle \left(I_{\lambda,q}^T\right)'(u), v \right\rangle =& \langle u, v\rangle_\lambda + q^2\eta\left(\|u\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3} \varphi_u uv dx \\ &+ \frac{q^2}{2T^2}\eta'\left(\|u\|_\lambda^2/T^2\right)\langle u, v\rangle_\lambda \int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} f(u)v dx. \end{aligned} \tag{3.3}
$$

In this case, by choosing an appropriate $T > 0$ and restricting $q > 0$ enough small, we may obtain a Cerami sequence $\{u_n\}$ of $I_{\lambda,q}^T$ satisfying $\|u_n\|_\lambda \le T$, and so $\{u_n\}$ is also a Cerami sequence $\{u_n\}$ of $I_{\lambda,q}$ satisfying $\|u_n\|_\lambda \le T$.

Then we prove that the truncated functional $I_{\lambda,q}^T$ has the mountain pass geometry.

Lemma 3.1. *Assume that* $(V_1) - (V_3)$, $(f_1) - (f_3)$ *hold. Then for each* $T,\ q > 0$ *and* $\lambda \ge 1$, *there exist* $\alpha,\ \rho > 0$ *(independent of* $T,\ q$ *and* λ *), such that* $I_{\lambda,q}^T(u) \ge \alpha$ *for all* $u \in E_\lambda$ *with* $\|u\|_\lambda = \rho$.

Proof. From (f_1) and (f_3), for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$
|f(u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \tag{3.4}
$$

and

$$
|F(u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p. \tag{3.5}
$$

For all $u \in E_\lambda$,

$$\int_{\mathbb{R}^3} \varphi_u u^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-\frac{|x-y|}{a}}}{|x-y|} u^2(x)u^2(y)dxdy \tag{3.6}$$
$$\geq 0.$$

Let $\varepsilon = \frac{1}{2d_2^2}$, where $d_2 > 0$ is from (2.1), then we have

$$\frac{\varepsilon}{2}|u|_2^2 = \frac{1}{4d_2^2}|u|_2^2 = \frac{1}{4}\left(\frac{|u|_2}{d_2}\right)^2 \leq \frac{1}{4}\|u\|^2 \leq \frac{1}{4}\|u\|_\lambda^2. \tag{3.7}$$

Thus, for each $u \in E_\lambda$, by (2.1), (3.1), (3.2), (3.5)-(3.7), we have

$$\begin{aligned}
I_{\lambda,q}^T(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{q^2}{4}\eta\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\varphi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx \\
&\geq \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^3} F(u)dx \\
&\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon}{2}|u|_2^2 - \frac{C_\varepsilon}{p}|u|_p^p \\
&\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{4}\|u\|_\lambda^2 - \frac{C_\varepsilon}{p}|u|_p^p \\
&\geq \frac{1}{4}\|u\|_\lambda^2 - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^p \\
&= \|u\|_\lambda^2\left(\frac{1}{4} - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^{p-2}\right),
\end{aligned}$$

where $d_p > 0$ and $C_\varepsilon > 0$ are independent of T, q and λ. Since $p > 2$, there exist $\rho > 0$ enough small and $\alpha > 0$ (α, ρ are independent of T, q, λ), such that

$$I_{\lambda,q}^T(u) \geq \alpha > 0,$$

for all $\|u\|_\lambda = \rho$. This lemma is completed. □

Lemma 3.2. *Assume that $(V_1)-(V_3)$ and $(f_1)-(f_3)$ hold. Then there exists $q^* > 0$, such that for each $T, \lambda > 0$ and $q \in (0, q^*)$, we have $I_{\lambda,q}^T(e_0) < 0$ for some $e_0 \in C_0^\infty(\Omega)$ with $|\nabla e_0|_2 > \rho$.*

Proof. Firstly, we define the functional $J_\lambda : E_\lambda \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \lambda V(x)u^2\right)dx - \int_{\mathbb{R}^3} F(u)dx.$$

From $(f_1)-(f_3)$, we have that there exists $C_1 > 0$, such that

$$F(u) \geq C_1(|u|^\kappa - |u|^2), \text{ for all } u \in \mathbb{R}. \tag{3.8}$$

Let $e \in C_0^\infty(\Omega)$ be a positive smooth function, then by (V_3), we have

$$\begin{aligned} J_\lambda(te) &= \frac{t^2}{2}\int_\Omega |\nabla e|^2 dx - \int_\Omega F(te)dx \\ &\leq \frac{t^2}{2}\int_\Omega |\nabla e|^2 dx + C_1 t^2 \int_\Omega |e|^2 dx - C_1 t^\kappa \int_\Omega |e|^\kappa dx \\ &\to -\infty, \end{aligned}$$

as $t \to +\infty$, for $\kappa \in (2,4]$. Thus, there exist $\bar{t} > 0$ large enough and $e_0 := \bar{t}e \in C_0^\infty(\Omega)$ with $|\nabla e_0|_2 > \rho$, such that $J_\lambda(e_0) \leq -1$.

Moreover, from the Hardy-Littlewood-Sobolev inequality (see [31] for details), we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} \varphi_u u^2 dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-\frac{|x-y|}{a}}}{|x-y|} u^2(x)u^2(y)dxdy \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|}{a}}}{|x-y|} u^2(x)u^2(y)dxdy \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy \\ &\leq C_2 |u|_{12/5}^2 |u|_{12/5}^2 \\ &= C_2 |u|_{12/5}^4, \end{aligned} \tag{3.9}$$

for all $u \in L^{12/5}(\mathbb{R}^3)$. Thus, from (3.1) and (3.9), we have

$$\begin{aligned} I_{\lambda,q}^T(e_0) &= J_\lambda(e_0) + \frac{q^2}{4}\eta\left(\|e_0\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3} \varphi_{e_0} e_0^2 dx \\ &\leq -1 + \frac{q^2}{4}\eta\left(\|e_0\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3} \varphi_{e_0} e_0^2 dx \\ &\leq -1 + \frac{q^2}{4} C_2 |e_0|_{12/5}^4. \end{aligned}$$

Then there exists $q^* = \sqrt{\frac{4}{C_2|e_0|_{12/5}^4}} = \frac{C_3}{|e_0|_{12/5}^2} > 0$ (independent of λ and T), such that $I_{\lambda,q}^T(e_0) < 0$ for all $T, \lambda > 0$ and $q \in (0, q^*)$. The lemma is completed. □

The next theorem is a somewhat stronger version of the Mountain Pass Theorem. Throughout it, we can find so-called Cerami sequences instead of Palais-Smale sequences, which can help us to get our results.

Theorem 3.3. *(See [32]) Let X be a real Banach space with its dual space X^*, and suppose that $J \in C^1(X,\mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \mu < \eta \leq \inf_{\|u\|_X = \rho} J(u)$$

for some $\mu < \eta,\ \rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e_0\}$ is the set of continuous paths joining 0 and e_0. Then there exists a sequence $\{u_n \subset X\}$ such that

$$J(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|_X) \|J'(u_n)\|_{X^*} \to 0 \quad \text{as } n \to \infty.$$

Then, define the mountain pass value $c^T_{\lambda,q}$ of $I^T_{\lambda,q}$ by

$$c^T_{\lambda,q} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I^T_{\lambda,q}(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0,\ \gamma(1) = e_0\}.$$

Throughout the Lemmas 3.1, 3.2 and Theorem 3.3, we can deduce that for each $T > 0,\ \lambda \geq 1$ and $q \in (0, q^*)$, there exists a Cerami sequence $\{u_n\} \subset E_\lambda$ (here we do not write the dependence on $T,\ \lambda$ and q), such that

$$I^T_{\lambda,q}(u_n) \to c^T_{\lambda,q} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda) \left\| \left(I^T_{\lambda,q}\right)'(u_n) \right\|_{E^*_\lambda} \to 0. \tag{3.10}$$

Obviously, $c^T_{\lambda,q} \geq \alpha > 0$.

Next, we estimate the upper bound of $c^T_{\lambda,q}$ which is the key ingredient of the truncation technique.

Lemma 3.4. *Assume that $(V_1) - (V_3), (f_1) - (f_3)$ hold. Then for every $T > 0,\ \lambda \geq 1$ and $q \in (0, q^*)$, there exists $M > 0$ (independent of $T,\ q$ and λ), such that $c^T_{\lambda,q} \leq M$.*

Proof. By (3.1), (3.2), (3.8), (3.9) and $e_0 \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
I^T_{\lambda,q}(te_0) &= \frac{t^2}{2}\int_\Omega |\nabla e_0|^2 dx + \frac{q^2}{4}t^4\eta\left(\frac{t^2\|e_0\|^2_\lambda}{T^2}\right)\int_\Omega \varphi_{e_0} e_0^2 dx - \int_\Omega F(te_0)\,dx \\
&\leq \frac{t^2}{2}\int_\Omega |\nabla e_0|^2 dx + \frac{q^2}{4}t^4\int_\Omega \varphi_{e_0} e_0^2 dx - \int_\Omega F(te_0)\,dx \\
&\leq \frac{t^2}{2}\int_\Omega |\nabla e_0|^2 dx + \frac{q^2}{4}t^4\int_\Omega \varphi_{e_0} e_0^2 dx + C_1 t^2\int_\Omega |e_0|^2 dx - C_1 t^\kappa \int_\Omega |e_0|^\kappa dx \\
&< \frac{t^2}{2}\int_\Omega |\nabla e_0|^2 dx + \frac{(q^*)^2}{4}C_2 t^4 |e_0|^4_{12/5} + C_1 t^2\int_\Omega |e_0|^2 dx - C_1 t^\kappa \int_\Omega |e_0|^\kappa dx.
\end{aligned}$$

Thus, there exists a constant $M > 0$ (independent of T, λ and q), such that

$$c^T_{\lambda,q} \leq \max_{t\in[0,1]} I^T_{\lambda,q}(te_0) \leq M.$$

The proof of this lemma is completed. □

In the following crucial lemma, we shall show that for a given $T > 0$ correctly, after passing to a subsequence, the sequence $\{u_n\}$ given by (3.10) satisfies $\|u_n\|_\lambda \leq T$, $\eta\left(\frac{\|u_n\|_\lambda^2}{T^2}\right) = 1$, and so $\{u_n\}$ is also a bounded Cerami sequence of $I_{\lambda,q}$ satisfying $\|u_n\|_\lambda \leq T$.

Lemma 3.5. *Assume that $(V_1) - (V_3)$ and $(f_1) - (f_3)$ hold. Let $T = \sqrt{\frac{2p(M+1)}{p-2}}$. Then there exists $q_* \in (0, q^*)$, such that for each $\lambda \geq 1$ and $q \in (0, q_*)$, if $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.10), then up to a subsequence, there holds*

$$\|u_n\|_\lambda \leq T.$$

Proof. For the sake of our proof, we argue by contradiction. For each $T > 0$, there exists a sub-sequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\|_\lambda > T$. Thus, there are two cases that need to be distinguished:
(1) $\|u_n\|_\lambda > \sqrt{2}T$; (2) $T < \|u_n\|_\lambda \leq \sqrt{2}T$.
Case 1). $\|u_n\|_\lambda > \sqrt{2}T$. By (3.1), we have

$$\eta\left(\|u_n\|_\lambda^2/T^2\right) = 0 \quad \text{and} \quad \eta'\left(\|u_n\|_\lambda^2/T^2\right) \leq 0. \tag{3.11}$$

Moreover, since $\kappa > 2$, from Lemma 3.4, (3.2), (3.3), (3.11), and (f_2), for n enough large, we have

$$\begin{aligned}
M \geqslant c_{\lambda,q}^{\top} &= \lim_{n\to\infty}\left(I_{\lambda,q}^{T}(u_n) - \frac{1}{\kappa}\left\langle \left(I_{\lambda,q}^{T}\right)'(u_n), u_n\right\rangle\right)\\
&= \lim_{n\to\infty}\left(\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|_\lambda^2 - \left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)\eta\left(\|u_n\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx\right.\\
&\quad\left. -\frac{q^2}{2\kappa T^2}\eta'\left(\|u_n\|_\lambda^2/T^2\right)\|u_n\|_\lambda^2\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx - \int_{\mathbb{R}^3}\left(F(u_n)-\frac{1}{\kappa}f(u_n)u_n\right)dx\right)\\
&\geq \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|_\lambda^2\\
&\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)\left(\sqrt{2}T\right)^2\\
&= \frac{\kappa-2}{\kappa}T^2,
\end{aligned}$$

which is a contradiction if we choose T enough large.
Case 2). $T < \|u_n\|_\lambda \leq \sqrt{2}T$.
Since $\kappa \in (2, 4]$, $\eta'(t) \leq 0$, $0 \leq \eta(t) \leq 1$ for each $t > 0$, by Lemma 3.4, (2.1), (3.2), (3.3),

(3.9) and (f_2), for n enough large, we have
(3.12)

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|_\lambda^2-\frac{1}{\kappa}\|\left(I_{\lambda,q}^T\right)'(u_n)\|_{E_\lambda^*}\|u_n\|_\lambda\\
\leq&\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|_\lambda^2+\frac{1}{\kappa}\langle\left(I_{\lambda,q}^T\right)'(u_n),u_n\rangle\\
=&\frac{1}{2}\|u_n\|_\lambda^2+\frac{q^2}{\kappa}\eta\left(\|u_n\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx+\frac{q^2}{2\kappa T^2}\eta'\left(\|u_n\|_\lambda^2/T^2\right)\|u_n\|_\lambda^2\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx-\frac{1}{\kappa}\int_{\mathbb{R}^3}f(u_n)u_ndx\\
=&\frac{1}{2}\|u_n\|_\lambda^2+\frac{q^2}{4}\eta\left(\|u_n\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx-\int_{\mathbb{R}^3}F\left(u_n\right)dx+\left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)\eta\left(\|u_n\|_\lambda^2/T^2\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx\\
&+\frac{q^2}{2\kappa T^2}\eta'\left(\|u_n\|_\lambda^2/T^2\right)\|u_n\|_\lambda^2\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx-\int_{\mathbb{R}^3}\left(\frac{1}{\kappa}f\left(u_n\right)u_n-F\left(u_n\right)\right)dx\\
=&I_{\lambda,q}^T\left(u_n\right)+\left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)\eta\left(\frac{\|u_n\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx+\frac{q^2}{2\kappa T^2}\eta'\left(\|u_n\|_\lambda^2/T^2\right)\|u_n\|_\lambda^2\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx\\
&-\int_{\mathbb{R}^3}\left(\frac{1}{\kappa}f\left(u_n\right)u_n-F\left(u_n\right)\right)dx\\
\leq&I_{\lambda,q}^T\left(u_n\right)+\left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)\eta\left(\frac{\|u_n\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx\\
\leq&I_{\lambda,q}^T\left(u_n\right)+\left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)C_2\|u_n\|_{12/5}^4\\
\leq&I_{\lambda,q}^T\left(u_n\right)+\left(\frac{q^2}{\kappa}-\frac{q^2}{4}\right)C_2d_{12/5}^4\|u_n\|_\lambda^4\\
\leq&I_{\lambda,q}^T\left(u_n\right)+Cq^2\left(\sqrt{2}T\right)^4\\
=&I_{\lambda,q}^T\left(u_n\right)+4Cq^2T^4.
\end{aligned}
$$

Since $I_{\lambda,q}^T(u_n)\to c_{\lambda,q}^T$, and from Lemma 3.4, for n enough large, one has

$$I_{\lambda,q}^T(u_n)\leq 2c_{\lambda,q}^T\leq 2\max_{t\in[0,1]}I_{\lambda,q}^T(te_0)\leq 2M. \tag{3.13}$$

Moreover, for n enough large, we have

$$\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|_\lambda^2-\frac{1}{\kappa}\|(I_{\lambda,q}^T)'(u_n)\|_{E_\lambda'}\|u_n\|_\lambda\geq CT^2-T. \tag{3.14}$$

Thus, by (3.12)-(3.14), we have

$$
\begin{aligned}
CT^2-T\leq&2M+4Cq^2T^4\\
<&2M+4Cq_*^2T^4,
\end{aligned}
$$

which is a contradiction if we choose $q_*:=\frac{1}{2T^2}>0$, $q\in(0,q_*)$, and T sufficiently large. The lemma is finished. □

Remark. From the above lemma, the sequence $\{u_n\}$ obtained in Lemma 3.5 is also a Cerami sequence at level $c^T_{\lambda,q}$ for $I_{\lambda,q}$, i.e.,

$$I_{\lambda,q}(u_n) \to c^T_{\lambda,q} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda) \left\|(I_{\lambda,q})'(u_n)\right\|_{E^*_\lambda} \to 0.$$

Then, we give the useful lemma, which has been proved in [33] and [34]. It can help us to solve the difficulty of dealing with the nonlinearity $f(u)$ in the Lemma 3.8.

Lemma 3.6. *Assume that* (f_1) *and* (f_3) *hold. If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then along a subsequence of* $\{u_n\}$, *we get*

$$\lim_{n\to\infty} \sup_{\phi\in H^1(\mathbb{R}^3), \|\phi\|_{H^1}\le 1} \left| \int_{\mathbb{R}^3} [f(u_n) - f(u_n - u) - f(u)] \phi dx \right| = 0.$$

By $\|v_n + v\|^2 = \|v_n\|^2 + \|v\|^2 + o_n(1)$ and the Brezis-Lieb Lemma, we get the following lemma. Please see the proof of [30, Lemma B.2] for details.

Lemma 3.7. *For every* $v \in H^1(\mathbb{R}^3)$ *and* $v_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^3)$, *we have*

$$\int_{\mathbb{R}^3} \varphi_{v_n+v}(v_n + v)^2 - \int_{\mathbb{R}^3} \varphi_{v_n} v_n^2 dx - \int_{\mathbb{R}^3} \varphi_v v^2 dx \to 0, \text{ as } n \to +\infty.$$

Next, we need to give the compactness conditions for $I_{\lambda,q}$. With the help of the above two lemmas, we establish the key parameter-dependent compactness lemma as follows.

Lemma 3.8. *Assume that* $(V_1)-(V_3)$, *and* $(f_1)-(f_3)$ *hold. Let* $T = \sqrt{\frac{2p(M+1)}{p-2}}$. *Then there exists* $\lambda^* > 1$, *such that for each* $\lambda \in (\lambda^*, \infty)$ *and* $q \in (0, q_*)$, *if* $\{u_n\} \subset E_\lambda$ *is a sequence satisfying (3.10), then* $\{u_n\}$ *has a convergent subsequence in* E_λ.

Proof. From Lemma 3.5, up to a subsequence, we have $\|u_n\|_\lambda \le T$. Thus we assume that there exists $u \in E_\lambda$, such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } E_\lambda, \\ u_n \to u, & \text{in } L^s_{loc}(\mathbb{R}^3), \quad \forall s \in [2, 6), \\ u_n \to u, & \text{a.e. on } \mathbb{R}^3. \end{cases}$$

Moreover, if u is a critical point of $I_{\lambda,q}$, then $\langle I'_{\lambda,q}(u), v\rangle = 0$, thus we have

$$\langle I'_{\lambda,q}(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x) uv) dx + q^2 \int_{\mathbb{R}^3} \varphi_u uv dx - \int_{\mathbb{R}^3} f(u) v dx = 0. \tag{3.15}$$

Taking $v = u$ in (3.15), we have

$$\begin{aligned} \langle I'_{\lambda,q}(u), u\rangle &= \int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2) dx + q^2 \int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} f(u) u dx \\ &= \|u\|^2_\lambda + q^2 \int_{\mathbb{R}^3} \varphi_u u^2 dx - \int_{\mathbb{R}^3} f(u) u dx \\ &= 0. \end{aligned} \tag{3.16}$$

Next, we shall prove $u_n \to u$ in E_λ. Let $v_n := u_n - u$, then $v_n \rightharpoonup 0$ in E_λ. From (V_2), we get

$$|v_n|_2^2 = \int_{\mathbb{R}^3 \setminus \mathcal{V}_b} v_n^2 dx + \int_{\mathcal{V}_b} v_n^2 dx \leq \frac{1}{\lambda b} \|v_n\|_\lambda^2 + o(1).$$

Then, using the Hölder inequality and Sobolev inequality, we have

$$|v_n|_p \leq |v_n|_2^\sigma |v_n|_6^{1-\sigma} \leq S^{\frac{\sigma-1}{2}} |v_n|_2^\sigma |\nabla v_n|_2^{1-\sigma} \leq S^{\frac{\sigma-1}{2}} (\lambda b)^{-\frac{\sigma}{2}} \|v_n\|_\lambda + o(1), \tag{3.17}$$

where $\sigma = \frac{6-p}{2p} > 0$. Employing Lemma 3.6, by the definition of the operator norm, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} [f(u_n) - f(v_n) - f(u)] u_n dx \right| \\ \leq &\|u_n\|_{H^1(\mathbb{R}^3)} \sup_{\phi \in H^1(\mathbb{R}^3), \|\phi\|_{H^1} \leq 1} \left| \int_{\mathbb{R}^3} [f(u_n) - f(v_n) - f(u)] \phi dx \right| \\ = &o(1). \end{aligned} \tag{3.18}$$

Since $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $v_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $s \in [2, 6)$, so from (3.18), we have

$$\begin{aligned} \int_{\mathbb{R}^3} f(u_n) u_n dx = &\int_{\mathbb{R}^3} f(u) u dx + \int_{\mathbb{R}^3} f(v_n) v_n dx + \int_{\mathbb{R}^3} f(v_n) u dx + \int_{\mathbb{R}^3} f(u) v_n dx \\ &+ \int_{\mathbb{R}^3} [f(u_n) - f(v_n) - f(u)] u_n dx \\ = &\int_{\mathbb{R}^3} f(u) u dx + \int_{\mathbb{R}^3} f(v_n) v_n dx + o(1). \end{aligned} \tag{3.19}$$

By (3.4), for every $\varepsilon^* > 0$, there exists $C_\varepsilon^* > 0$, such that

$$|f(u)u| \leq \varepsilon^* |u|^2 + C_\varepsilon^* |u|^p. \tag{3.20}$$

From (2.1), (3.17), (3.20) and let $\varepsilon^* = \frac{1}{2d_2^2}$, we get

$$\begin{aligned} \int_{\mathbb{R}^3} f(v_n) v_n dx &\leq \varepsilon^* |v_n|_2^2 + C_\varepsilon^* |v_n|_p^{p-2} |v_n|_p^2 \\ &= \frac{|v_n|_2^2}{2d_2^2} + C_\varepsilon^* |v_n|_p^{p-2} |v_n|_p^2 \\ &\leq \frac{1}{2} \|v_n\|_\lambda^2 + C_\varepsilon^* (2T d_p)^{p-2} |v_n|_p^2 \\ &\leq \frac{1}{2} \|v_n\|_\lambda^2 + C_\varepsilon^* (2T d_p)^{p-2} S^{\sigma-1} (\lambda c)^{-\sigma} \|v_n\|_\lambda^2 + o(1). \end{aligned} \tag{3.21}$$

From (3.19), (3.21) and Lemma 3.7, we have

$$
\begin{aligned}
o(1) &= \left\langle I'_{\lambda,q}\left(u_n\right), u_n\right\rangle - \left\langle I'_{\lambda,q}(u), u\right\rangle \\
&= \|u_n\|_\lambda^2 + q^2 \int_{\mathbb{R}^3} \varphi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} f\left(u_n\right) u_n dx \\
&\quad - \|u\|_\lambda^2 - q^2 \int_{\mathbb{R}^3} \varphi_u u^2 dx + \int_{\mathbb{R}^3} f\left(u\right) u dx \\
&= \|v_n\|_\lambda^2 + q^2 \left(\int_{\mathbb{R}^3} \varphi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \varphi_u u^2 dx\right) - \left(\int_{\mathbb{R}^3} f\left(u_n\right) u_n dx - \int_{\mathbb{R}^3} f\left(u\right) u dx\right) \\
&= \|v_n\|_\lambda^2 + q^2 \int_{\mathbb{R}^3} \varphi_{v_n} v_n^2 dx - \int_{\mathbb{R}^3} f\left(v_n\right) v_n dx + o(1) \\
&\geq \|v_n\|_\lambda^2 - \int_{\mathbb{R}^3} f\left(v_n\right) v_n dx + o\left(1\right) \\
&\geq \|v_n\|_\lambda^2 - \left(\frac{1}{2}\|v_n\|_\lambda^2 + C_\varepsilon^*\left(2Td_p\right)^{p-2} S^{\sigma-1}\left(\lambda b\right)^{-\sigma}\|v_n\|_\lambda^2\right) + o\left(1\right) \\
&= \left(\frac{1}{2} - C_\varepsilon^*\left(2Td_p\right)^{p-2} S^{\sigma-1}\left(\lambda b\right)^{-\sigma}\right)\|v_n\|_\lambda^2 + o\left(1\right).
\end{aligned}
$$

Hence, there exists $\lambda^* > 1$, such that $v_n \to 0$ in E_λ for all $\lambda > \lambda^*$. This completes the proof. □

Proof of Theorem 1.1 Let T be defined as in Lemma 3.5. From Lemmas 3.1 and 3.2, there exists $q^* > 0$, such that for every $\lambda \geq 1,\ q \in (0, q^*)$, $I_{\lambda,q}^T$ possesses a Cerami sequence $\{u_n\}$ at the mountain pass level $c_{\lambda,q}^T$. By Lemmas 3.4 and 3.5, we deduce that there exists $q_* \in (0, q^*)$, such that for every $\lambda \geq 1$ and $q \in (0, q_*)$, after passing to a subsequence, $\{u_n\}$ is a Cerami sequence of $I_{\lambda,q}$ satisfying $\|u_n\|_\lambda \leq T$, i.e.,

$$\sup_{n\in\mathbb{N}} \|u_n\|_\lambda \leq T, \quad I_{\lambda,q}\left(u_n\right) \to c_{\lambda,q}^T$$

and

$$(1 + \|u_n\|_\lambda)\|I'_{\lambda,q}(u_n)\|_{E_\lambda^*} \to 0,$$

as $n \to +\infty$. By Lemma 3.8, we show that there exists $\lambda^* > 1$, such that for every $\lambda \in (\lambda^*, \infty)$ and $q \in (0, q_*)$, $\{u_n\}$ has a convergent subsequence in E_λ. Without loss of generality, we suppose that $u_n \to u_{\lambda,q}$ as $n \to \infty$, so

$$0 < \|u_{\lambda,q}\|_\lambda \leq T,\ I_{\lambda,q}(u_{\lambda,q}) = c_{\lambda,q}^T > 0 \text{ and } I'_{\lambda,q}(u_{\lambda,q}) = 0.$$

Therefore, we deduce that $u_{\lambda,q}$ is a nontrivial solution of (1.11) for all $q \in (0,\ q_*)$ and $\lambda \in (\lambda^*,\ \infty)$. The proof is completed.

4 Asymptotic behavior of nontrivial solutions

Proof of Theorem 1.2 Let $q \in (0,\ q_*)$ be fixed. For any sequence $\{\lambda_n\} \subset (\lambda^*,\ +\infty)$ with $\lambda_n \to \infty$, let $u_n := u_{\lambda_n,\ q}$ be the critical point of $I_{\lambda_n,q}$, which is obtained from Theorem 1.1.

From Lemma 3.5, we have

$$0 < \|u_n\|_{\lambda_n} \leq T, \text{ for all } n. \tag{4.1}$$

Then, up to a subsequence, we assume that

$$\begin{cases} u_n \rightharpoonup u_q, & \text{in } E, \\ u_n \to u_q, & \text{in } L^s_{\text{loc}}\left(\mathbb{R}^3\right), \text{ for } s \in [2,6), \\ u_n \to u_q, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{4.2}$$

Since $\lambda_n \to \infty$, by (4.1), (V_1) and Fatou's Lemma, we have

$$\begin{aligned} 0 \leqslant \int_{\mathbb{R}^3 \setminus V^{-1}(0)} V(x) u_q^2 dx &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^3 \setminus V^{-1}(0)} V(x) u_n^2 dx \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{|\nabla u_n|^2 + \lambda_n V(x) u_n^2}{\lambda_n} dx \\ &= \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} \\ &\leq \liminf_{n\to\infty} \frac{T^2}{\lambda_n} = 0. \end{aligned}$$

Thus, we get $u_q = 0$ a.e. in $\mathbb{R}^3 \setminus V^{-1}(0)$, and so $u_q \in H_0^1(\Omega)$ by (V_3).

Now we show that $u_n \to u_q$ in $L^s\left(\mathbb{R}^3\right)$ for $2 < s < 6$. Arguing by contradiction, by Lions' vanishing lemma in [35], we assume that there exist $\delta,\ r > 0$ and $x_n \in \mathbb{R}^3$, such that

$$\int_{B_r(x_n)} \left(u_n - u_q\right)^2 dx \geq \delta > 0, \tag{4.3}$$

which implies that $|x_n| \to \infty$ as $n \to \infty$. Thus $|B_r(x_n) \cap \mathcal{V}_b| \to 0$ as $n \to \infty$. Moreover, by Hölder inequality, we get

$$\begin{aligned} \int_{B_r(x_n)\cap\mathcal{V}_b} \left(u_n - u_q\right)^2 dx &\leq \left(\int_{B_r(x_n)\cap\mathcal{V}_b} 1^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} \left(\left(u_n - u_q\right)^2\right)^3 dx\right)^{\frac{1}{3}} \\ &= \left|B_r(x_n) \cap \mathcal{V}_b\right|^{\frac{2}{3}} \left|u_n - u_q\right|_6^2 \\ &\to 0, \qquad \text{as } n \to \infty. \end{aligned} \tag{4.4}$$

Consequently, from (4.3) and (4.4), we get

$$\begin{aligned} \|u_n\|_{\lambda_n}^2 &= \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \lambda_n V(x) u_n^2\right) dx \\ &\geq \lambda_n b \int_{B_r(x_n)\cap\{V\geq b\}} u_n^2 dx \\ &= \lambda_n b \int_{B_r(x_n)\cap\{V\geq b\}} \left(u_n - u_q\right)^2 dx \\ &= \lambda_n b \left(\int_{B_r(x_n)} \left(u_n - u_q\right)^2 dx - \int_{B_r(x_n)\cap\mathcal{V}_b} \left(u_n - u_q\right)^2 dx\right) \\ &\to +\infty, \quad \text{as } n \to \infty, \end{aligned}$$

which contradicts with $0 < \|u_n\|_{\lambda_n} \leq T$, for all n. Thus, we get $u_n \to u_q$ in $L^s(\mathbb{R}^3)$, for all $s \in (2,6)$. Next, we prove $u_n \to u_q$ in E. Since

$$\left\langle I'_{\lambda_n,q}(u_n), u_n \right\rangle = \left\langle I'_{\lambda_n,q}(u_n), u_q \right\rangle = 0.$$

We show that

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} + q^2 \lim_{n\to\infty} \int_{\mathbb{R}^3} \varphi_{u_n} u_n^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n) u_n dx, \tag{4.5}$$

and

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla u_q + \lambda V(x) u_n u_q)\, dx + q^2 \int_{\mathbb{R}^3} \varphi_{u_n} u_n u_q dx = \int_{\mathbb{R}^3} f(u_n)\, u_q dx. \tag{4.6}$$

It follows from $u_q = 0$ a.e. $\mathbb{R}^3 \backslash V^{-1}(0)$ and (4.2) that

$$\|u_q\|^2 + q^2 \int_{\mathbb{R}^3} \varphi_{u_q} u_q^2 dx = \int_{\mathbb{R}^3} f(u_q)\, u_q dx. \tag{4.7}$$

From (3.19), (4.2) and Fatou's Lemma, after passing to subsequence, we have

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^3} \varphi_{u_n} u_n^2 dx \geq \int_{\mathbb{R}^3} \varphi_{u_q} u_q^2 dx, \\ &\text{and} \int_{\mathbb{R}^3} f(u_n)\, u_n dx = \int_{\mathbb{R}^3} f(u_q)\, u_q dx + o(1). \end{aligned} \tag{4.8}$$

By (4.5), (4.7), and (4.8), we infer that

$$\limsup_{n\to\infty} \|u_n\|^2_{\lambda_n} \leq \|u_q\|^2.$$

From the weakly lower semi-continuity of norm, up to a subsequence, we have

$$\|u_q\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2 \leq \limsup_{n\to\infty} \|u_n\|^2 \leq \limsup_{n\to\infty} \|u_n\|^2_{\lambda_n} \leq \|u_q\|^2. \tag{4.9}$$

Thus, we deduce that $u_n \to u_q$ in E. Finally, we shall prove that u_q is a weak solution of (1.13). For any $v \in C_0^\infty(\Omega)$, since $\left\langle I'_{\lambda_n,q}(u_n), v \right\rangle = 0$, it is easy to check that

$$\int_\Omega \nabla u_q \nabla v dx + q^2 \int_\Omega \varphi_{u_q} u_q v dx = \int_\Omega f(u_q) v dx,$$

i.e., u_q is a weak solution of (1.13) by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$. Next, we prove $u_q \neq 0$. Arguing by contradiction, we assume that $u_q = 0$ which implies that $u_n \to 0$ in E. Then by

(2.1), (3.5), (3.9) and (4.9), we deduce that

$$\begin{aligned}
0 &\leq |I_{\lambda_n,q}(u_n)| \\
&\leq \frac{1}{2}\|u_n\|_{\lambda_n}^2 + \frac{q^2}{4}\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx + \int_{\mathbb{R}^3}|F(u_n)|dx \\
&\leq \frac{1}{2}\|u_n\|_{\lambda_n}^2 + \frac{q^2}{4}\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2dx + \int_{\mathbb{R}^3}\left(\frac{\varepsilon}{2}|u_n|^2 + \frac{C_\varepsilon}{p}|u_n|^p\right)dx \\
&\leq \frac{1}{2}\|u_n\|_{\lambda_n}^2 + \frac{q^2}{4}C_2|u_n|_{12/5}^4 + \frac{\varepsilon}{2}|u_n|_2^2 + \frac{C_\varepsilon}{p}|u_n|_p^p \\
&\leq \frac{1}{2}\|u_n\|_{\lambda_n}^2 + \frac{q^2}{4}C_2d_{12/5}^4\|u_n\|^4 + \frac{\varepsilon d_2^2}{2}\|u_n\|^2 + \frac{C_\varepsilon d_p^p}{p}\|u_n\|^p \\
&\to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, we have

$$I_{\lambda_n,q}(u_n)\to 0, \qquad \text{as } n\to\infty. \tag{4.10}$$

Moreover, we get $I_{\lambda_n,q}(u_n) = c_{\lambda,q}^T \geq \alpha > 0$ by virtue of u_n be critical point of $I_{\lambda_n,q}$ obtained by Theorem 1.1. It contradicts (4.10). Thus, u_q is a nontrivial weak solution of (1.13). The proof is completed.

Proof of Theorem 1.3 Let $\lambda \in (\lambda^*, \infty)$ be fixed. Then for each $\lambda \in (\lambda^*, \infty)$ and any sequence $\{q_n\} \subset (0, q_*)$, let $q_n \to 0$ and $u_n := u_{\lambda,q_n}$ be the critical point of I_{λ,q_n} obtained by Theorem 1.1. By Theorem 1.1, we have

$$0 < \|u_n\|_\lambda \leq T, \qquad \text{for all n.} \tag{4.11}$$

Passing to a subsequence if necessary, we may assume that $u_n \to u_\lambda$ in E_λ. Note that $I'_{\lambda,q_n}(u_n) = 0$, we may deduce that $u_n \to u_\lambda$ in E_λ as the proof of Lemma 3.8.

To complete our proof, it suffices to show that u_λ is a weak solution of (1.14). Now for any $v \in E_\lambda$, since $(I_{\lambda,q_n}(u_n), v) = 0$, it is easy to check that

$$\int_{\mathbb{R}^3}(\nabla u_\lambda \nabla v + \lambda V(x)u_\lambda v)\,dx = \int_{\mathbb{R}^3} f(u_\lambda)\,v dx$$

i.e., u_λ is a weak solution of (1.14). Furthermore, similarly to the proof of Theorem 1.2, we know that u_λ is a nontrivial solution of Equation (1.14). So we omit it. This completes the proof.

Proof of Theorem 1.4 The proof of this theorem is similar to the proof of Theorem 1.2. Please verify it by ourselves.

5 Conclusion

The paper studies the Schrödinger-Bopp-Podolsky system with steep potential well, and we get a positive energy solution $u_{\lambda,q}$ for λ large and q small. Moreover, we know the asymptotic

behavior of the Schrödinger-Bopp-Podolsky system with steep potential well is dependent on the parameters q, λ. The nontrivial solutions of the Schrödinger-Bopp-Podolsky system with steep potential well are well localized near the bottom of the potential as $\lambda \to \infty$. The nontrivial solutions of the Schrödinger-Bopp-Podolsky system with steep potential well are approached to the solutions of the Schrödinger-Bopp-Podolsky system as $q \to 0$. In the future, it's possible to have at least two possible solutions of the Schrödinger-Bopp-Podolsky system with steep potential well. And if the nonlinear term is critical, we might get the same result.
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