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Multivariate polynomial splines on generalized oranges

Maritza Sirvent[a], Tatyana Sorokina[b], Nelly Villamizar[c,*,1], Beihui Yuan[d,1]

[a] *Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, OH, United States*

[b] *Department of Mathematics, Towson University, 8000 York Road, Towson, MD 21252, MD, United States*

[c] *Department of Mathematics, Swansea University, Fabian Way, Swansea, SA1 8EN, United Kingdom*

[d] *Beijing Institute of Mathematical Sciences and Applications, Huairou, Beijing, 101400, China*

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Abstract

We consider spaces of multivariate splines defined on a particular type of simplicial partitions that we call *(generalized) oranges*. Such partitions are composed of a finite number of maximal faces with exactly one shared *medial* face. We reduce the problem of finding the dimension of splines on oranges to computing dimensions of splines on simpler, lower-dimensional partitions that we call *projected oranges*. We use both algebraic and Bernstein–Bézier tools.

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1. Introduction and preliminaries

For an integer $k \geqslant 0$, let Δ be a k-dimensional simplicial complex. That is, Δ is a collection of k-dimensional simplices such that if a simplex $\beta \in \Delta$, then all the simplices which are faces of β are also in Δ, and if $\beta_1, \beta_2 \in \Delta$, then $\beta_1 \cap \beta_2$ is either empty or a proper face of both β_1 and β_2. We refer to the simplices of Δ as the faces of Δ. (For a comprehensive treatment of

* Corresponding author.
E-mail addresses: sirventleon.1@osu.edu (M. Sirvent), tsorokina@towson.edu (T. Sorokina), n.y.villamizar@swansea.ac.uk (N. Villamizar), beihuiyuan@bimsa.cn (B. Yuan).

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simplicial complexes, we direct interested readers to [12].) To simplify the notation, we use the term Δ for both the union of all its simplices and the simplicial complex itself. Furthermore, we always consider Δ to be embedded in a k-dimensional real space $\mathbb{R}^k$. For integers $0 \leqslant r \leqslant d$, the space of splines $S_d^r(\Delta)$ is defined as the set of C^r-smooth piecewise polynomial functions of degree at most d over Δ. The space $S_d^r(\Delta)$ is a real vector space. These spline spaces are used throughout numerical analysis and approximation theory to solve diverse problems such as the interpolation and approximation of data, design of curves and surfaces, and, in the finite element method, for the solution of differential equations, among others. A fundamental question is to determine the dimension of such spaces. The problem has proven to be difficult due to the dependency of the dimension on the specific geometry of Δ.

In 1973, Strang [16], published a famous conjecture on the *generic* dimension of $S_d^1(\Delta)$ for the bivariate case. A simplicial partition Δ is said to be generic (with respect to r and d) provided that for all sufficiently small perturbations of the coordinates of the vertices of Δ in $\mathbb{R}^k$, the resulting partition $\tilde{\Delta}$ satisfies $\dim S_d^r(\tilde{\Delta}) = \dim S_d^r(\Delta)$, see [6] where this definition was explicitly coined. Although the term generic dimension was not explicitly mentioned by Strang, he was aware of the fact that the dimension might increase for some particular configurations. In [3], Alfeld and Schumaker prove a dimension formula for $S_d^r(\Delta)$, when the polynomial degree $d \geqslant 4r + 1$, and Δ is an arbitrary triangulation. This result was later extended to $d \geqslant 3r + 2$ by Hong in [13], and to $d \geqslant 3r + 1$ for generic triangulations by Alfeld and Schumaker in [4]. For $r < d < 3r + 2$, the only dimension known explicitly is the case when $r = 1$ and $d = 4$, see [2].

In 1988, Billera, see [7], pioneered the use of algebraic homological methods to study multivariate splines, and proved the conjecture given by Strang.

For the trivariate case, Alfeld, Schumaker, and Whiteley, see [6], use the Bernstein–Bézier analysis, algebraic homology, and rigidity theory to give an explicit expression for the generic dimension of $S_d^1(\Delta)$, for $d \geqslant 8$.

For the spatial dimension $k \geqslant 3$, no explicit dimension formula is known for spline spaces on arbitrary tetrahedral partitions. There are results on upper and lower bounds and exact dimensions for some specific partitions. For example, in [9], DiPasquale and Villamizar proved a lower bound for the dimension for splines on tetrahedral stars of a vertex. A natural generalization of a star of a vertex is a star of a simplex. A star of a simplex $\tau \in \Delta$ is the set of simplices in Δ that contain τ. If a simplicial complex Δ equals the star of τ, we just call Δ a star of τ. In this paper, we study stars of a simplex of arbitrary dimension. We call such stars *generalized oranges* or *oranges*.

Recall that a k-dimensional simplicial complex is *pure* if all its maximal faces are of dimension k.

Definition 1.1 (*Generalized Orange*)**.** For integers $0 \leqslant i \leqslant k$, a (k, i)-*orange* is a pure k-dimensional simplicial complex $\mathcal{O}$ composed of n maximal faces and with exactly one face τ of dimension $k - i$ such that every maximal face of $\mathcal{O}$ contains τ. We say that τ is a *medial simplex*.

Note that the two extremal cases are $i = 0$ and $i = k$. For $i = 0$, $\Delta = \tau$ is a simplex. For $i = k$, the medial face τ is a vertex, and Δ is a star of a vertex. Examples of $(3, 2)$, $(2, 1)$, and $(3, 1)$-oranges are shown in Figs. 1–3, respectively.

Historically, the term "orange" was introduced in [5] to describe a tetrahedral partition with exactly one interior edge which is common to all tetrahedra in the partition, see Fig. 1. This matches our definition of a $(3, 2)$-orange except that for our purposes the medial edge does

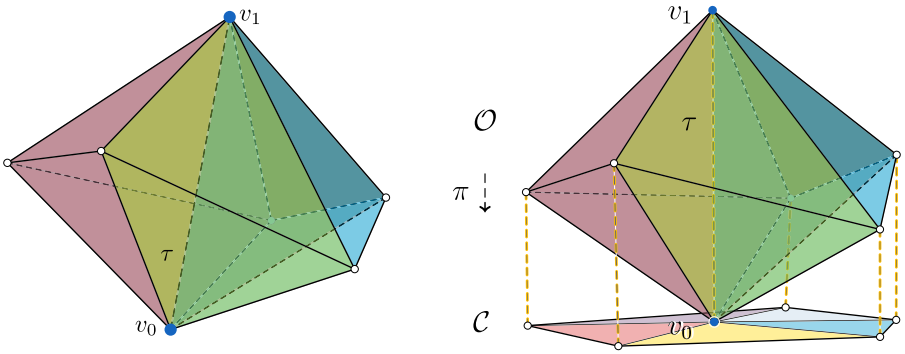

Fig. 1. Example of a $(3,2)$-orange $\mathcal{O}$ (on the left), and its projection $\mathcal{C}$ (on the right). The orange $\mathcal{O}$ is a 3-dimensional simplicial complex with medial edge τ, which has vertices v_0 and v_1, and it is common to all tetrahedra in $\mathcal{O}$. The map π projects $\mathcal{O}$ onto $\mathcal{C}$, the image of τ is the vertex v_0.

not have to be interior. The dimension for the spline spaces on a $(3,2)$-orange with interior medial edge was found in [5] using the trivariate cofactor method, see [14] for the proof as well. In [15], it was noted that the dimension for the spline spaces on a $(3,2)$-orange with interior medial edge can be computed by adding up the dimensions of bivariate splines on a planar vertex star obtained by projecting the orange along the common interior edge.

In this article we generalize the idea of the projection to an i-dimensional star of a vertex, and relate the dimension of the spline spaces over a (k,i)-orange $\mathcal{O}$ to those over a lower dimensional simplicial complex $\mathcal{C}$, which we call *projected orange*, see Section 2 for the exact definition, by proving the following result.

Theorem 1.1. *Let $\mathcal{C} \subseteq \mathbb{R}^i$ be the projected orange of a (k,i)-orange $\mathcal{O} \subseteq \mathbb{R}^k$. Then*

$$\dim S_d^r(\mathcal{O}) = \sum_{j=0}^{d} \binom{d+k-i-j-1}{k-i-1} \dim S_j^r(\mathcal{C}). \tag{1}$$

The paper is organized as follows. In Section 2, we construct a projection $\mathcal{C}$ of a (k,i)-orange $\mathcal{O}$ along the medial face. We prove that the projection $\mathcal{C}$ is also a simplicial complex. In Section 3, we analyze the relation between the space of splines on $\mathcal{O}$ and that on $\mathcal{C}$ by using the so-called *cofactor criterion*. Finally, we prove our main result, Theorem 1.1, by two methods: the algebraic method in Section 4, and the Bernstein–Bézier method in Section 5. The algebraic method is based on Hilbert series of spline spaces viewed as filtered vector spaces, while the Bernstein–Bézier method uses minimal determining sets and smoothness conditions in Bernstein–Bézier form.

2. Geometry

Since the dimension of the spline space is affinely invariant, without loss of generality, we embed a (k,i)-orange $\mathcal{O}$ in $\mathbb{R}^k$, and we assume that the medial face τ in $\mathcal{O}$ of dimension $(k-i)$ is in the $(k-i)$-dimensional coordinate subspace of $\mathbb{R}^k$ defined by $x_1 = \cdots = x_i = 0$. As a

simplifying assumption, we identify $\mathcal{O}$ with its embedding in $\mathbb{R}^k$. For this choice of coordinates, we define the following projection:

$$\pi : \mathbb{R}^k \to \mathbb{R}^i,$$
$$(x_1, \ldots, x_k) \mapsto \begin{cases} (x_1, \ldots, x_i), & \text{if} \quad i \geqslant 1; \\ (0) \in \mathbb{R}^0, & \text{if} \quad i = 0. \end{cases} \tag{2}$$

Additionally, we write $\mathcal{C}$ for the collection of sets $\pi(\sigma) \subseteq \mathbb{R}^i$ of images under the projection π of the faces $\sigma \in \mathcal{O}$. Namely,

$$\mathcal{C} := \pi(\mathcal{O}) = \big\{\pi(\sigma) : \sigma \in \mathcal{O}\big\}. \tag{3}$$

We shall refer to $\mathcal{C}$ as a *projected orange*. Note that the projection of the medial face $\pi(\tau)$ is the origin O of $\mathbb{R}^i$. We use the following notation throughout this paper.

Definition 2.1. For any $X, Y \subseteq \mathbb{R}^k$, denote by $X * Y$ the *join* of X and Y which is the union of all line segments joining the points in X to the points in Y, that is,

$$X * Y = \big\{tx + (1-t)y \ : x \in X, \ \ y \in Y, \text{ and } t \in [0,1]\big\}.$$

For any n subsets $X_1, \ldots, X_n \subseteq \mathbb{R}^k$, the join $X_1 * \cdots * X_{n-1} * X_n$ is defined as $(X_1 * \cdots * X_{n-1}) * X_n$. If $X = \{v\}$ and $Y = \{w\}$, we simply write $v * w$ instead of $X * Y$.

Remark 1. If both X and Y are simplices, the join $X * Y$ is the convex hull of $X \cup Y$, this property does not hold in general. In fact, one may check that the following properties hold:

- $X * Y = Y * X$, for any $X, Y \subseteq \mathbb{R}^k$,
- for simplices $\alpha, \beta, \zeta \subseteq \mathbb{R}^k$, we have $(\alpha * \beta) * \zeta = \alpha * (\beta * \zeta)$, and
- if π is the projection map defined in (2), then

$$\pi(X * Y) = \pi(X) * \pi(Y), \tag{4}$$

for any $X, Y \subseteq \mathbb{R}^k$.

Definition 2.2. An *affine subspace* $W \subseteq \mathbb{R}^k$ is a translate of a vector subspace $V \subseteq \mathbb{R}^k$ by a point $p \in \mathbb{R}^k$, i.e, $W = \{p\} + V = \big\{p + t(v - p) : v \in V, \text{ and } t \in \mathbb{R}\big\}$. The dimension of W is defined as $\dim W = \dim V$. A set of points $x_0, \ldots, x_s \in \mathbb{R}^k$ is called *affinely independent* if $x_1 - x_0, \ldots, x_s - x_0$ are linearly independent in $\mathbb{R}^k$. If $X \subseteq \mathbb{R}^k$, the *affine span* $\mathrm{aff}(X)$ of X in $\mathbb{R}$ is the smallest affine subspace of $\mathbb{R}^k$ containing X.

Recall that the set of points $\{p_1, \ldots, p_j\} \subseteq \mathbb{R}^k$ is said to be in *general position* if its affine span is an affine subspace of $\mathbb{R}^k$ of dimension $j - 1$. Note that every m-simplex $\sigma \in \mathcal{O}$ can be written as the join of $m + 1$ points in general position.

Remark 2. Notice that for a set of points $\{p_1, \ldots, p_j\} \subseteq \mathbb{R}^k$, the relation $p_1 * p_2 * \cdots * p_j \subseteq \mathrm{aff}(p_1, \ldots, p_j)$ is strict unless $j = 1$. The join corresponds to the convex hull of the points $p_1, \ldots, p_j$.

Proposition 2.1. *If $\mathcal{O}$ is a (k, i)-orange, then the projected orange $\mathcal{C}$ defined in (3) is a simplicial complex. Moreover, $\mathcal{C}$ is a star of a vertex.*

Before we prove Proposition 2.1, we need some preliminary lemmas that will help us verify that every element in $\mathcal{C}$ is a simplex, every face of a simplex in $\mathcal{C}$ is also in $\mathcal{C}$, and that the intersection of any two elements in $\mathcal{C}$ is a face of each of them.

Lemma 2.2. *Let $\mathcal{O}$ be a (k,i)-orange with the medial simplex τ. The image $\pi(\sigma)$ of a simplex $\sigma \in \mathcal{O}$ is a simplex. Moreover, if σ and τ do not intersect, then $\pi(\sigma)$ and σ have the same dimension. Otherwise,* $\dim(\pi(\sigma)) = \dim(\sigma) - \dim(\sigma \cap \tau)$.

Proof. For a simplex $\sigma \in \mathcal{O}$, the vertices of σ can always be divided into two sets, one of which contains all vertices that are in τ, the other contains those not in τ. This means σ can be written in the form

$$\sigma = \alpha * \beta, \tag{5}$$

where α, β are faces of σ such that none of the vertices in α is a vertex of τ, and that $\beta = \sigma \cap \tau$. Then by (4), we have

$$\pi(\sigma) = \begin{cases} \pi(\alpha), & \text{if } \beta = \emptyset, \\ \pi(\alpha) * O, & \text{if } \beta \neq \emptyset. \end{cases} \tag{6}$$

Let $z_{i+1}, \ldots, z_{k+1}$ be the vertices of τ. Without loss of generality assume that $z_{k+1} = O$. Let $p_1, \ldots, p_j$ be the vertices of α, where $j \leqslant i$. Then $p_1, \ldots, p_j, z_{i+1}, \ldots, z_{k+1}$ are in general position and $p_1, \ldots, p_j, z_{i+1}, \ldots, z_k$ are affinely independent in $\mathbb{R}^k$. We claim that then $\{\pi(p_1), \ldots, \pi(p_j)\}$ is also affinely independent. Note that in general the projection of a affinely independent set of points is not necessarily affinely independent. Let $V = \operatorname{span}\{p_1, \ldots, p_j, z_{i+1}, \ldots, z_k\}$, and let $U = \operatorname{span}\{z_{i+1}, \ldots, z_k\}$. Then $\dim V = k + j - i$, and $\dim U = k - i$. Since U is the $(k-i)$-subspace containing τ, we have $\ker(\pi) = U$, and $\dim(\pi(V)) = j$. Therefore, the affine span of the set $\{\pi(p_1), \ldots, \pi(p_j)\}$ is $\pi(V)$. This space is affinely independent.

This shows that both $\pi(\alpha)$ and $\pi(\alpha) * O$ are simplices. Moreover, if $\beta = \emptyset$, then $\dim \pi(\sigma) = \dim \pi(\alpha)$. Otherwise, $\dim(\pi(\sigma)) = \dim(\sigma) - \dim(\beta)$, and the proof is complete. □

Corollary 2.3. *The projected orange $\mathcal{C}$ is a collection of simplices. A face of an element in $\mathcal{C}$ is also in $\mathcal{C}$.*

Proof. It immediately follows from Lemma 2.2 that every element of $\mathcal{C}$ is a simplex. Now if a simplex $\omega \in \mathcal{C}$, then by Eq. (3), there is $\sigma \in \mathcal{O}$ such that $\pi(\sigma) = \omega$. We may assume $\sigma = v_1 * \cdots * v_j$, for vertices $v_s \in \mathbb{R}^k$ and $j \leqslant k$. Then $\pi(v_1 * \cdots * v_j) = \pi(v_1) * \cdots * \pi(v_j) = \omega$. Thus any face of ω can be written as $\pi(v_{j_1}) * \cdots * \pi(v_{j_\ell}) = \pi(v_{j_1} * \cdots * v_{j_\ell})$ and, hence, by Eq. (3), this face is in $\mathcal{C}$. □

Lemma 2.4. *Let v_1, v_2 be two vertices in $\mathcal{O}$. If neither v_1 nor v_2 belongs to τ, then $\pi(v_1) = \pi(v_2)$ implies $v_1 = v_2$.*

Proof. Let $\alpha_1 = v_1 * \tau$ and $\alpha_2 = v_2 * \tau$. Both α_1 and α_2 are $(k - i + 1)$-faces in $\mathcal{O}$ and $\tau \subseteq \alpha_1 \cap \alpha_2$. Since $\mathcal{O}$ is a simplicial complex and $\dim \tau = k - i$, then there are only two possibilities:

$$\tau = \alpha_1 \cap \alpha_2 \quad \text{or} \quad \alpha_1 = \alpha_2. \tag{7}$$

Let $z_{i+1}, \dots, z_{k+1}$ be the vertices of τ, where we again assume that z_{k+1} is the origin. Since $\pi(v_1) = \pi(v_2)$, we note that $v_1 - v_2$ is in the $(k-i)$-dimensional coordinate space spanned by τ. Thus

$$v_1 - v_2 = \beta_1 z_{i+1} + \cdots + \beta_{k-i} z_k. \tag{8}$$

Without loss of generality, we assume that the first j coefficients $\{\beta_\ell\}_{\ell=1}^{j}$ are negative, while the remaining $k-i-j$ coefficients $\{\beta_\ell\}_{\ell=j+1}^{k-i}$ are non-negative. We next rewrite (8) as follows:

$$v_1 - \beta_1 z_{i+1} - \cdots - \beta_j z_{i+j} = v_2 + \beta_{j+1} z_{i+j+1} + \cdots + \beta_{k-i} z_k = v. \tag{9}$$

Let $m = \left(1 + \sum_{\ell=1}^{k-i} |\beta_\ell|\right)^{-1}$, and let $u = mv$.

Since u can be written as two different convex combinations, one in the vertices of α_1 and a second in the vertices of α_2, then $u \in \alpha_1 \cap \alpha_2$. From (9), it follows that $u \notin \tau$, so by (7) we must have $\alpha_1 = \alpha_2$, and hence $v_1 = v_2$. □

Corollary 2.5. *Let $\omega \in \mathcal{C}$ such that the origin O is not contained in ω. Then there exists a unique $\alpha \in \mathcal{O}$ such that $\pi(\alpha) = \omega$.*

Proof. The existence is immediate from the definition of $\mathcal{C}$. Assume there are two simplices α and α' in $\mathcal{O}$ such that $\pi(\alpha) = \pi(\alpha') = \omega$. Since $O \notin \omega$, we have $\alpha \cap \tau = \emptyset$, and $\alpha' \cap \tau = \emptyset$. Then by Lemma 2.2, $\dim(\alpha) = \dim(\alpha') = \dim(\omega) =: j$. We may assume $\alpha = p_1 * \cdots * p_j$ and $\alpha' = p'_1 * \cdots * p'_j$ for vertices p_s's and p'_t's in $\mathcal{O}$. Hence $\pi(p_1) * \cdots * \pi(p_j) = \pi(p'_1) * \cdots * \pi(p'_j) = \omega$. Therefore, possibly after a permutation, $\pi(p_\ell) = \pi(p'_\ell)$ for all $\ell = 1, \dots, j$. By Lemma 2.4, this implies $p_\ell = p'_\ell$ for all $\ell = 1, \dots, j$, and $\alpha = \alpha'$. □

Corollary 2.6. *Let $\omega \in \mathcal{C}$, and let the origin O be contained in ω. Then there is a unique face $\sigma \in \mathcal{O}$ such that $\pi(\sigma) = \omega$ and $\tau \subseteq \sigma$.*

Proof. By Corollary 2.3, every face ω of $\mathcal{C}$ is a simplex, so we may assume $\omega = \theta * O$, where $O \notin \theta$ and θ is also a face of $\mathcal{C}$. By Corollary 2.5, there is a face $\alpha \in \mathcal{O}$ such that $\pi(\alpha) = \theta$. Hence, $\pi(\alpha * \tau) = \omega$. So we can take $\sigma = \alpha * \tau$. To prove the uniqueness, assume there is a face σ' such that $\pi(\sigma') = \omega$ and $\tau \subseteq \sigma'$. Then we can write $\sigma' = \alpha' * \tau$ for some $\alpha' \cap \tau = \emptyset$. Then $\pi(\alpha') = \theta$. Corollary 2.5 implies $\alpha = \alpha'$. Hence, $\sigma = \sigma'$. □

The next corollary is an immediate consequence of Lemma 2.2 and Corollary 2.6. We are going to use it in Section 3.

Corollary 2.7. *There is a one-to-one correspondence between the maximal faces of an orange $\mathcal{O}$ and the maximal faces of its projection $\mathcal{C}$.*

Proof. If $\sigma \in \mathcal{O}$ is a maximal face then the medial simplex $\tau \subseteq \sigma$, and so Lemma 2.2 implies that $\pi(\sigma)$ is a simplex of dimension i. The existence and uniqueness of $\sigma \in \mathcal{O}$ for every maximal face of $\mathcal{C}$ follow from Corollary 2.6. □

Lemma 2.8. *For any two faces $\sigma_1, \sigma_2 \in \mathcal{O}$ containing τ, we have*

$$\pi(\sigma_1) \cap \pi(\sigma_2) = \pi(\sigma_1 \cap \sigma_2). \tag{10}$$

In particular, (10) implies that $\pi(\sigma_1) \cap \pi(\sigma_2) \in \mathcal{C}$ in this case.

Proof. We only need to prove $\pi(\sigma_1) \cap \pi(\sigma_2) \subseteq \pi(\sigma_1 \cap \sigma_2)$. Let $b \in \pi(\sigma_1) \cap \pi(\sigma_2)$, and let $b \neq O$. Then there exist $a_1 \in \sigma_1$ and $a_2 \in \sigma_2$ such that $b = \pi(a_1) = \pi(a_2)$. If either $a_1 \in \sigma_1 \cap \sigma_2$ or $a_2 \in \sigma_1 \cap \sigma_2$ then $b \in \pi(\sigma_1 \cap \sigma_2)$, and we are done. Suppose that neither a_1 nor a_2 is in $\sigma_1 \cap \sigma_2$. Then $(a_1 * \tau) \cap \sigma_2 = \tau$ and $(a_2 * \tau) \cap \sigma_1 = \tau$. Since $(a_1 * \tau) \cap (a_2 * \tau) \subseteq \sigma_1 \cap \sigma_2$, we have

$$(a_1 * \tau) \cap (a_2 * \tau) = (a_1 * \tau) \cap (a_2 * \tau) \cap \sigma_1 = \tau.$$

Without loss of generality, we may assume that $x_1(b) = 1$ and $x_2(b) = \cdots = x_i(b) = 0$. Note that $a_1 * \tau$ and $a_2 * \tau$ are in the same $(k-i+1)$-subspace $V \subseteq \mathbb{R}^k$ defined by $x_2 = \cdots = x_i = 0$. By a similar reasoning as in the proof of Lemma 2.4, we get $\tau \neq (a_1 * \tau) \cap (a_2 * \tau)$, which is a contradiction. Therefore, the hypothesis that neither a_1 nor a_2 is in $\sigma_1 \cap \sigma_2$ is false. □

Corollary 2.9. *For any two faces $\alpha, \sigma \in \mathcal{O}$ such that $\alpha \cap \tau = \emptyset$, we have*

$$\pi(\alpha) \cap \pi(\sigma) = \pi(\alpha \cap \sigma).$$

In particular, $\pi(\alpha) \cap \pi(\sigma) \in \mathcal{C}$ in this case.

Proof. We only need to show that $\pi(\alpha) \cap \pi(\sigma) \subseteq \pi(\alpha \cap \sigma)$. Note that $\sigma = \alpha' * \beta'$ such that $\alpha' \cap \tau = \emptyset$ and $\beta' \subseteq \tau$. Since $\mathcal{O}$ is a simplicial complex, then $\alpha \cap \sigma = \alpha \cap \alpha'$, and $(\alpha * \tau) \cap (\alpha' * \tau) = (\alpha \cap \alpha) * \tau$. By Lemma 2.8, we have that $\pi(\alpha \cap \alpha') * \pi(\tau) = (\pi(\alpha) * \pi(\tau)) \cap (\pi(\alpha') * \pi(\tau))$. This means that for any $b \in \pi(\alpha) \cap \pi(\sigma)$, it holds $b \in \pi(\alpha \cap \alpha') * O$. So we can assume that b is on the line segment $\overline{b'O}$, where $b' \in \pi(\alpha \cap \alpha')$. This implies that both b and b' are in $\pi(\alpha)$. Since $O \notin \pi(\alpha)$, then $b = b'$. This shows $\pi(\alpha) \cap \pi(\sigma) = \pi(\alpha \cap \sigma)$. □

Lemma 2.10. *For any $\omega_1, \omega_2 \in \mathcal{C}$, we have $\omega_1 \cap \omega_2 \in \mathcal{C}$.*

Proof. If $O \notin \omega_1$, then by Corollary 2.5, there exists a unique $\alpha \in \mathcal{O}$ such that $\pi(\alpha) = \omega_1$. Let σ be a face in $\mathcal{O}$ such that $\pi(\sigma) = \omega_2$. By Corollary 2.9 we have that $\omega_1 \cap \omega_2 = \pi(\alpha \cap \sigma)$, and so $\omega_1 \cap \omega_2 \in \mathcal{C}$.

If O is in both ω_1 and ω_2, then by Corollary 2.6, there exists $\sigma_l \in \mathcal{O}$ such that $\pi(\sigma_l) = \omega_l$ and $\tau \subseteq \sigma_l$, for $l = 1, 2$ respectively. By Lemma 2.8, it follows $\omega_1 \cap \omega_2 = \pi(\sigma_1 \cap \sigma_2) \in \mathcal{C}$. □

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. The fact that $\mathcal{C}$ is a simplicial complex follows from Corollary 2.3 and Lemma 2.10. Note that O is the only interior vertex of $\mathcal{C}$ and every facet of $\mathcal{C}$ contains O, so $\mathcal{C}$ must be the star of the vertex O. □

3. Cofactors

In this section and thereafter, we use the notion of filtered vector spaces. Recall (for example, from [11]) that a *filtered vector space* is a vector space V with a nested sequence of subspaces $\{V_d \subseteq V : d = 0, 1, 2, \dots\}$ such that

$$\{0\} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

and that

$$V = \bigcup_{d \geqslant 0} V_d.$$

In this article, we always assume V_d is finite dimensional for each d. Let Δ be a fixed finite k-dimensional triangulation of a domain $\Omega \subseteq \mathbb{R}^k$. If $r \geqslant 0$ is an integer, the *total spline space* over Δ is defined as

$$S^r(\Delta) = \bigcup_{d \geqslant 0} S^r_d(\Delta), \tag{11}$$

where $V_d = S^r_d(\Delta)$. In particular, $S^r(\Delta)$ is a filtered vector space. Denote by Δ_j the set of j-dimensional faces (or j-faces) of Δ, for $0 \leqslant j \leqslant k$. We say that two k-faces $\sigma, \sigma' \in \Delta_k$ are *maximal adjacent faces*, or simply *adjacent*, if their intersection is a $(k-1)$-face of Δ i.e., if σ and σ' share (or have in common) a $(k-1)$-face of Δ.

We recall from [7] that $f = (f_\sigma : \sigma \in \Delta_k) \in S^r(\Delta)$ if and only if for every pair $\sigma, \sigma' \in \Delta_k$ such that $\sigma \cap \sigma' = \varepsilon \in \Delta_{k-1}$ there exists a polynomial c_ε such that $f_\sigma - f_{\sigma'} = c_\varepsilon \cdot \ell_\varepsilon^{r+1}$, where ℓ_ε is the linear polynomial defining the hyperplane containing ε. This result is called the *cofactor criterion* for C^r-splines.

As before, we denote by $\mathcal{O}$ a (k,i)-orange. We embed $\mathcal{O}$ in $\mathbb{R}^k$, and up to a change of coordinates assume that the medial simplex τ of $\mathcal{O}$ satisfies $x_1 = \cdots = x_i = 0$. In these coordinates, we consider the projection π along τ given in (2). We denote by $\mathcal{C}$ the image of $\mathcal{O}$ by π, as defined in (3).

The orange $\mathcal{O}$ is by definition a k-dimensional simplicial complex (see Definition 1.1), and Proposition 2.1 implies that $\mathcal{C}$ is an i-dimensional simplicial complex. Following the notation above, we denote by $\mathcal{O}_k$ the set of k-dimensional (or maximal) faces of $\mathcal{O}$.

The following proposition shows how the total C^r-spline spaces over $\mathcal{O}$ and $\mathcal{C}$ are related. As usual, we denote by $\mathbb{R}[x_1, \ldots, x_n]$ the set of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{R}$.

Proposition 3.1. *For any integer $r \geqslant 0$, the total spline space over $\mathcal{O}$ satisfies*

$$S^r(\mathcal{O}) \simeq S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \ldots, x_k], \tag{12}$$

where "$\simeq$" means the two sides are isomorphic as filtered vector spaces, that is, for each degree d

$$S^r_d(\mathcal{O}) \simeq (S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \ldots, x_k])_{\leqslant d},$$

where "$\simeq$" is an isomorphism between vector spaces, and the space $(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \ldots, x_k])_{\leqslant d}$ is spanned by

$$\left\{(g,h) \in S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \ldots, x_k] : \deg g + \deg h \leqslant d\right\}.$$

Proof. By Corollary 2.7, the projection π establishes a bijective correspondence between the maximal faces of $\mathcal{O}$ and those of $\mathcal{C}$. Hence, every spline $g \in S^r(\mathcal{C})$ can be written as the tuple $g = \left(g_{\pi(\sigma)} : \sigma \in \mathcal{O}_k\right)$, where $g_{\pi(\sigma)} = g|_{\pi(\sigma)} \in \mathbb{R}[x_1, \ldots, x_i]$. Let $R = \mathbb{R}[x_1, \ldots, x_k]$, and consider the map

$$\begin{aligned} \varphi : (S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \ldots, x_k])_{\leqslant d} &\to \bigoplus_{\sigma \in \mathcal{O}_k} R, \\ (g,h) &\mapsto \left(h g_{\pi(\sigma)} : \sigma \in \mathcal{O}_k\right). \end{aligned}$$

Notice that $S^r_c(\mathcal{O}) \subseteq \bigoplus_{\sigma \in \mathcal{O}_k} R$. We want to show that φ is an isomorphism of vector spaces and $\operatorname{Im} \varphi = S^r_d(\mathcal{O})$.

It is clear that φ is $\mathbb{R}$-linear and it is injective because $S^r(\mathcal{C}) \subseteq \mathbb{R}[x_1, \dots, x_i]$, so none of the variables x_t for $t > i$ is involved in the polynomials $g_\pi(\sigma)$. First we show that $\operatorname{Im}\varphi \subseteq S^r_d(\mathcal{O})$.

Let $g \in S^r(\mathcal{C})$. Then, for any $h \in \mathbb{R}[x_{i+1}, \dots, x_k]$, and any pair of adjacent faces $\sigma, \sigma' \in \mathcal{O}_k$ such that $\sigma \cap \sigma' = \varepsilon \in \mathcal{O}_k$, the cofactor criterion implies $hg_{\pi(\sigma)} - hg_{\pi(\sigma')} = hc_{\pi(\varepsilon)} \cdot \ell^{r+1}_{\pi(\varepsilon)}$, for some polynomial $c_{\pi(\varepsilon)} \in \mathbb{R}[x_1, \dots, x_i]$, where $\ell_{\pi(\varepsilon)}$ is the linear polynomial vanishing on $\pi(\varepsilon)$. Notice that by construction, the $(k-1)$-face ε contains the medial simplex τ of $\mathcal{O}$. Consequently, the linear polynomial ℓ_ε vanishing on ε is in $\mathbb{R}[x_1, \dots, x_i]$, and hence $\ell_{\pi(\varepsilon)} = \ell_\varepsilon$. Therefore, $\varphi(g, h) \in S^r(\mathcal{O})$. By filtering on the degree we have that $\deg \varphi(g, h) \leqslant d$. This shows that $\operatorname{Im}(\varphi) \subseteq S^r_d(\mathcal{O})$.

We now prove that $\operatorname{im}\varphi \supseteq S^r_d(\mathcal{O})$. Let $f = (f_\sigma : \sigma \in \mathcal{O}_k) \in S^r_d(\mathcal{O})$, where $f|_\sigma = f_\sigma \in R$ and $\deg f_\sigma \leqslant d$. Notice that we can rewrite each f_σ as a polynomial in $x_{i+1}, \dots, x_k$ with coefficients in $\mathbb{R}[x_1, \dots, x_k]$. More precisely, we have

$$f_\sigma = \sum_{\boldsymbol{j} \in \mathbb{Z}^{k-i}_{\geqslant 0},\ |\boldsymbol{j}| \leqslant d} a_{\boldsymbol{j},\sigma}\, \boldsymbol{y}^{\boldsymbol{j}}, \quad \text{for } a_{\boldsymbol{j},\sigma} \in \mathbb{R}[x_1, \dots, x_i],$$

where $\boldsymbol{y} = (x_{i+1}, \dots, x_k)$, and for a tuple of non-negative integers $\boldsymbol{j} = (j_{i+s})_{s=1}^{k-i}$ we write $|\boldsymbol{j}| = \sum_{s=1}^{k-i} j_s$ and $\boldsymbol{y}^{\boldsymbol{j}} = x_{i+1}^{j_{i+1}} \cdots x_k^{j_k}$. Then, if $a_{\boldsymbol{j}} = \left(a_{\boldsymbol{j},\sigma} : \sigma \in \mathcal{O}_k\right)$, we can write

$$f = \sum_{\boldsymbol{j} \in \mathbb{Z}^{k-i}_{\geqslant 0},\ |\boldsymbol{j}| \leqslant d} a_{\boldsymbol{j}}\, \boldsymbol{y}^{\boldsymbol{j}}, \tag{13}$$

where $\deg a_{\boldsymbol{j}} = \max\{\deg a_{\boldsymbol{j},\sigma} : \sigma \in \mathcal{O}_k\}$. We prove that $a_{\boldsymbol{j}} \in S^r(\mathcal{C})$ for every tuple $\boldsymbol{j} \in \mathbb{Z}_{\geqslant 0}$ in (13) as follows. Notice that for adjacent faces σ and σ' as above, we have that $f_\sigma - f_{\sigma'} = c_\varepsilon \cdot \ell^{r+1}_\varepsilon$, and we can write

$$c_\varepsilon = \sum_{\boldsymbol{j} \in \mathbb{Z}^{k-i}_{\geqslant 0},\ |\boldsymbol{j}| \leqslant d} b_{\boldsymbol{j},\varepsilon} \cdot \boldsymbol{y}^{\boldsymbol{j}}, \quad \text{where } b_{\boldsymbol{j},\varepsilon} \in \mathbb{R}[x_1, \dots, x_i].$$

Thus,

$$f_\sigma - f_{\sigma'} = \sum_{\boldsymbol{j} \in \mathbb{Z}^{k-i}_{\geqslant 0},\ |\boldsymbol{j}| \leqslant d} (b_{\boldsymbol{j},\varepsilon} \ell^{r+1}_\varepsilon) \cdot \boldsymbol{y}^{\boldsymbol{j}} = \sum_{\boldsymbol{j} \in \mathbb{Z}^{k-i}_{\geqslant 0},\ |\boldsymbol{j}| \leqslant d} (a_{\boldsymbol{j},\sigma} - a_{\boldsymbol{j},\sigma'}) \boldsymbol{y}^{\boldsymbol{j}}.$$

Hence, for every $\boldsymbol{j} \in \mathbb{Z}_{\geqslant 0}$, we have $a_{\boldsymbol{j},\sigma} - a_{\boldsymbol{j},\sigma'} = b_{\boldsymbol{j},\varepsilon} \ell^{r+1}_\varepsilon$, which implies that $a_{\boldsymbol{j}} \in S^r(\mathcal{C})$. Note that $\deg f_\sigma \leqslant d$, so $\deg a_{\boldsymbol{j}} + |\boldsymbol{j}| \leqslant d$. This shows that $\operatorname{Im}\varphi \supseteq S^r_d(\mathcal{O})$, and it completes the proof. □

One immediate consequence of Proposition 3.1 is that the algebraic structure of $S^r_d(\mathcal{O})$ only depends on that of $S^r(\mathcal{C})$ and the $(k-i)$-subspace in which the medial simplex τ lies. Note that the projection map π only depends on the $(k-i)$-subspace containing τ, we have the following corollary.

Corollary 3.2. *Let $\mathcal{O}$ and $\mathcal{O}'$ be two (k, i)-oranges with medial simplices τ and τ', respectively. If τ, τ' lie in the same $(k-i)$-subspace and the projected oranges $\mathcal{C} = \mathcal{C}'$, then for any degree d,*

$$S^r_d(\mathcal{O}) \simeq S^r_d(\mathcal{O}').$$

In particular,

$$\dim S^r_d(\mathcal{O}) = \dim S^r_d(\mathcal{O}').$$

Proof. By Proposition 3.1 it is clear that both $S^r_d(\mathcal{O})$ and $S^r_d(\mathcal{O}')$ are isomorphic to $(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \dots, x_k])_{\leqslant d}$. Therefore, they are isomorphic and have the same dimension. □

4. Hilbert series

In this section we prove the main result in this paper, Theorem 1.1.

For any simplicial complex Δ and any order of smoothness $r \geqslant 0$, we can see $S^r(\Delta)$ as a filtered vector space for the sequence of spline spaces $\{0\} \subseteq S^r_0(\Delta) \subseteq S^r_1(\Delta) \subseteq \cdots$. By definition, see for example [8] or [10], the *Hilbert series* $\mathrm{Hilb}\big(S^r(\Delta), t\big)$ of $S^r(\Delta)$ is given by

$$\mathrm{Hilb}\big(S^r(\Delta), t\big) = \sum_{d=0}^{\infty} \dim S^r_d(\Delta)\, t^d.$$

Following the notation in previous sections, if $0 \leqslant i \leqslant k$ we denote by $\mathcal{O}$ a (k, i)-orange and by $\mathcal{C}$ the image of $\mathcal{O}$ by the projection $\pi : \mathbb{R}^k \to \mathbb{R}^i$ in (2). The following result is a consequence of Proposition 3.1 and relates the Hilbert series of the spline spaces $S^r(\mathcal{O})$ and $S^r(\mathcal{C})$.

Corollary 4.1. *If $\mathcal{O}$ is a (k, i)-orange and $\mathcal{C} = \pi(\mathcal{O})$ as defined in (3), then*

$$\mathrm{Hilb}\big(S^r(\mathcal{O}), t\big) = \frac{1}{(1-t)^{k-i}} \mathrm{Hilb}\big(S^r(\mathcal{C}), t\big). \tag{14}$$

Proof. By Proposition 3.1, we know that $S^r(\mathcal{O}) = S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \dots, x_k]$ and $S^r_d(\mathcal{O}) \cong (S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \dots, x_k])_{\leqslant d}$ for every $d \geqslant 0$. We will prove (14) by induction on the variables x_{i+j} for $1 \leqslant j \leqslant k-i$. Define $\phi : \big(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}]\big)_d \to S^r_d(\mathcal{C})$ by taking $\phi\big(x_{i+1}\big) = 0$. Then, $\ker_d(\phi) = \big(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}]\big)_{d-1} \cdot x_{i+1} \cong \big(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}]\big)_{d-1}$. This holds for every degree $d \geqslant 0$, and therefore it implies

$$\mathrm{Hilb}(\ker(\phi), t) = \mathrm{Hilb}\big(S^r(\mathcal{C}, t) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}], t\big) \cdot t, \tag{15}$$

where $\ker(\phi)$ is the filtered vector of $\ker_d(\phi)$. But ϕ is surjective and linear, hence $\dim \ker_d(\phi) + \dim S^r(\mathcal{C})_d = \dim\big(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}]\big)_d$, and this together with (15) yields

$$\mathrm{Hilb}\big(S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}], t\big)(1-t) = \mathrm{Hilb}(S^r(\mathcal{C})).$$

Since $\mathbb{R}[x_{i+1}, \dots, x_k] \cong \mathbb{R}[x_{i+1}] \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R}[x_k]$, and $S^r(\mathcal{O}) = S^r(\mathcal{C}) \otimes_{\mathbb{R}} \mathbb{R}[x_{i+1}, \dots, x_k]$, we easily see by induction that $\mathrm{Hilb}\big(S^r(\mathcal{O}), t\big)(1-t)^{k-i} = \mathrm{Hilb}\big(S^r(\mathcal{C}), t\big)$, which proves (14). □

Theorem 1.1 can be proved from Corollary 4.1 as follows.

Proof (*Proof Theorem 1.1*)**.** Notice that we can rewrite

$$\frac{1}{(1-t)^{k-i}} = \sum_{j=0}^{\infty} \binom{k-i-1+j}{j} t^j.$$

Thus, by Corollary 4.1, we have

$$\sum_{d=0}^{\infty} \dim S^r_d(\mathcal{O}) t^d = \left(\sum_{j=0}^{\infty} \binom{k-i-1+j}{j} t^j \right) \cdot \left(\sum_{j=0}^{\infty} \dim S^r_j(\mathcal{C}) t^j \right),$$

which implies

$$\dim S_d^r(\mathcal{O}) = \sum_{\substack{j+l=d \\ j,l\geqslant 0}} \binom{k-(i+1)+l}{l} \dim S_j^r(\mathcal{C})$$
$$= \sum_{j=0}^{d} \binom{d+k-i-j-1}{k-i-1} \dim S_j^r(\mathcal{C}),$$

for every degree $d \geqslant 0$. □

Note that for the extremal cases, $i = 0$ and $i = k$, Theorem 1.1 trivially holds. Indeed, in the case $i = 0$, the orange $\mathcal{O} = \tau$ is a simplex, $\mathcal{C}$ is a point, and

$$\dim S_d^r(\mathcal{O}) = \sum_{j=0}^{d} \binom{d+k-j-1}{k-1} = \binom{d+k}{k}.$$

For $i = k$, we know that $\mathcal{O} = \mathcal{C}$. In this case, formula (1) is just the trivial identity $\dim S_d^r(\mathcal{O}) = \dim S_d^r(\mathcal{C})$.

5. Bernstein–Bézier techniques

The idea is to use Corollary 3.2 to transform $\mathcal{O}$ into a special $\mathcal{O}'$ constructed from the projection $\mathcal{C}$, so that its geometric structure allows to "lift" the Bernstein–Bézier basis on the projected orange $\mathcal{C}$ to a Bernstein–Bézier basis in $\mathcal{O}'$. The main advantage of Bernstein–Bézier techniques is the tool called *domain points* that essentially replaces basis functions with points located in specified positions in each simplex. We assume familiarity of the reader with basic Bernstein–Bézier concepts, and refer to [14] for a comprehensive treatment of the bivariate and trivariate cases as well as to a survey paper [1].

Recall that by assumption the medial face τ of $\mathcal{O}$ is in the plane $x_1 = \cdots = x_i = 0$. Without loss of generality assume that the first vertex v_0 of τ is at the origin O, and the remaining vertices are $v_1, \dots v_{k-i}$. We also assume that $\mathcal{C}$ is embedded in the subspace $x_{i+1} = \cdots = x_k = 0$, and its only interior vertex is located at O. We construct $\mathcal{O}'$ as follows:

$$\mathcal{O}' := \mathcal{C} * \tau = \{\omega * v_1 * \cdots * v_{k-i} : \omega \in \mathcal{C}\}.$$

It is easy to verify that $\mathcal{O}'$ is also a (k, i)-orange. It has the same medial simplex τ as $\mathcal{O}$, and its projected orange $\mathcal{C}'$ equals $\mathcal{C}$.

The set of all domain points in τ for a polynomial of degree at most d in $(k-i)$ variables is given by:

$$\mathcal{D}^\tau := \{\xi^\tau_{j_0, j_1 \dots, j_{k-i}} = (j_1 v_1 + \cdots + j_{k-i} v_{j_{k-i}})/d : j_0 + j_1 + \cdots + j_{k-i} = d\}.$$

Next we define a family of scaled versions of $\mathcal{C}$ as follows

$$\mathcal{C}_j := (j/d)\mathcal{C}, \text{ for } j = 0, \dots, d,$$

where if $j = 0$, the projected orange $\mathcal{C}$ scales to the origin. Let ω be a maximal simplex in $\mathcal{C}$ with the first vertex u_0 at the origin O, and the remaining vertices $u_1, \dots u_i$. We also define a family of scaled versions of ω as follows

$$\omega_j := (j/d)\,\omega, \text{ for } j = 0, \dots, d,$$

where if $j = 0$, the simplex ω scales to the origin. The set of all domain points in ω_j for a polynomial of degree at most j in i variables is given by:

$$\mathcal{D}_j^{\omega} := \Big\{ \xi^{\omega,j}_{\ell_0,\ell_1\ldots,\ell_i} = \frac{j}{d}(\ell_1 u_1 + \cdots + \ell_i u_i)/j = (\ell_1 u_1 + \cdots + \ell_i u_i)/d : \\ \ell_0 + \ell_1 + \cdots + \ell_i = j \Big\}.$$

The purpose of the following lemma is to show that the domain points in $\mathcal{O}'$ can be split into parallel layers orthogonal to τ, and, thus, the smoothness conditions for a spline in $S_d^r(\mathcal{O}')$ can be also split into independent blocks.

Lemma 5.1. *Let $\mathcal{D}'$ be the set of all domain points for a spline in $S_d^0(\mathcal{O}')$. Then $\mathcal{D}'$ can be lifted from the domain points in $\mathcal{C}$ as follows:*

$$\mathcal{D}' = \bigcup_{j=0}^{d} \big\{ \cup_{\omega\in\mathcal{C}} \mathcal{D}_j^{\omega} + \xi_{j,j_1,j_2,\ldots,j_{k-i}} : \xi_{j,j_1,j_2,\ldots,j_{k-i}} \in \mathcal{D}^{\tau} \big\}.$$

Proof. Let σ be a maximal simplex in $\mathcal{O}'$. Therefore, $\sigma = \omega * \tau$ for some maximal simplex $\omega \in \mathcal{C}$. Let $\eta_{m_0,\ldots,m_k}$, $m_0 + \cdots m_k = d$ be an arbitrary domain point in $\mathcal{D}' \cap \sigma$. Without loss of generality, assume that the first vertex u_0 of ω is at the origin, the next i vertices $u_1, \ldots, u_i$ are in $\mathcal{C}$, and the last $(k-i)$ vertices $v_1, \ldots, v_{k-i}$ are in τ. Then

$$\begin{aligned} \eta_{m_0,\ldots,m_k} &= (m_1 u_1 + \cdots + m_i u_i + m_{i+1} v_1 + \cdots + m_k v_{k-i})/d \\ &= (m_1 u_1 + \cdots m_i u_i)/d + (m_{i+1} v_1 + \cdots + m_k v_{k-i})/d \\ &=: \xi^{\omega,j}_{m_0,m_1,\ldots,m_i} + \xi^{\tau}_{j,m_{i+1},\ldots,m_k}, \end{aligned}$$

where for $j = d - m_{i+1} - \cdots - m_k$, we have $\xi^{\omega,j}_{m_0,m_1,\ldots,m_i} \in \mathcal{D}_j^{\omega}$, and $\xi^{\tau}_{j,m_{i+1},\ldots,m_k} \in \mathcal{D}^{\tau}$, since $j + m_{i+1} + \cdots m_k = d$.

Conversely, for $0 \leqslant j \leqslant d$, $\ell_0 + \ell_1 + \cdots + \ell_i = j$, and $j + j_1 + \cdots + j_{k-i} = d$, consider

$$\begin{aligned} &\xi^{\omega,j}_{\ell_0,\ell_1,\ldots,\ell_i} + \xi^{\tau}_{j,j_1,\ldots,j_{k-i}} = (\ell_1 u_1 + \cdots + \ell_i u_i)/d + (j_1 v_1 + \cdots + j_{k-1} v_{k-i})/d \\ &= (\ell_1 u_1 + \cdots + \ell_i u_i + j_1 v_1 + \cdots + j_{k-1} v_{k-i})/d \ = \eta_{\ell_0,\ell_1,\ldots,\ell_i,j_1,\ldots,j_{k-i}}, \end{aligned}$$

since $\ell_0 + \ell_1 + \cdots + \ell_i + j_1 + \cdots + j_{k-i} = j + d - j = d$, and the simplex in $\mathcal{O}'$ in $\mathbb{R}^k$ has $(k+1)$ vertices $O, u_1, \ldots, u_i, v_1, \ldots, v_{k-i}$. □

Before proving our next result, we demonstrate the approach on two examples. The first one is a (2, 1)-orange $\mathcal{O}'$ depicted in Fig. 2. The smoothness conditions for a spline in $S_3^r(\mathcal{O}')$ are easily seen to be the same as for a univariate spline of smoothness r on $\mathcal{C}$: a partition of an interval with one node. Moreover, $\dim S_3^r(\mathcal{O}') = \sum_{j=0}^{3} \dim S_j^r(\mathcal{C})$. Our next example is a (3, 1)-orange $\mathcal{O}'$ depicted in Fig. 3. Here again, the smoothness conditions for a spline in $S_3^r(\mathcal{O}')$ are the same as for a univariate spline of smoothness r on $\mathcal{C}$: a partition of an interval with one node. Moreover,

$$\dim S_3^r(\mathcal{O}') = \sum_{j=0}^{3} (4-j) \dim S_j^r(\mathcal{C}).$$

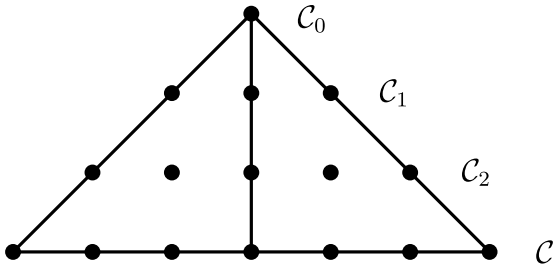

Fig. 2. Four parallel layers of domain points forming $\mathcal{D}'$ for S_3^1 on a $(2,1)$-orange with the associated spaces of univariate splines $S_j^r(\mathcal{C}_j)$, for $j = 0, 1, 2, 3$, and $\mathcal{C} = [-1, 0] \cup [0, 1]$.

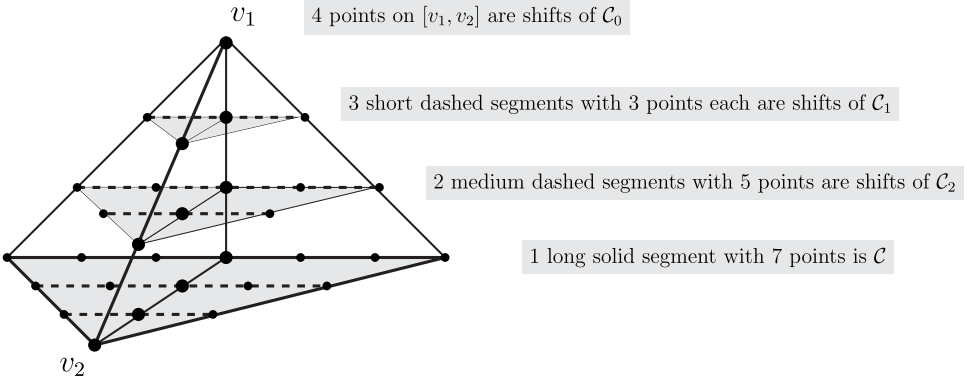

Fig. 3. Domain points for $S_3^0(\mathcal{O}')$: large dots are the domain points in τ; small dots are the domain points on shifts of $\mathcal{C}_j$, $j = 0, \dots, 3$.

Theorem 5.2. *For each $0 \leqslant j \leqslant d$, let $\mathcal{M}_j$ be a minimal determining set for a spline in $S_j^r(\mathcal{C}_j)$. Then the minimal determining set $\mathcal{M}'$ for a spline in $S_d^r(\mathcal{O}')$ is given by*

$$\mathcal{M}' = \bigcup_{j=0}^{d} \{\mathcal{M}_j + \xi_{j,j_1,\dots,j_{k-i}} : \xi_{j,j_1,j_2,\dots,j_{k-i}} \in \mathcal{D}^\tau\}.$$

Moreover, the dimension of $S_j^r(\mathcal{C}_j)$ can be computed by adding the corresponding cardinalities as follows:

$$\sum_{j=0}^{d} \binom{d+k-j-i-1}{k-i-1} |\mathcal{M}_j|. \tag{16}$$

Proof. Lemma 5.1 implies that the domain points in $\mathcal{D}'$ lie on parallel layers that are shifts of $\mathcal{C}_j$ by a vector $\xi_{j,j_1,j_2,\dots,j_{k-i}}$ that is orthogonal to $\mathcal{C}_j$. Thus, all smoothness conditions for a spline in $S_d^r(\mathcal{O}')$ are essentially i-dimensional, not k-dimensional, see Fig. 2 and 3. In fact, they are exactly the same as the ones for $S_j^r(\mathcal{C})$ for all $j = 0, \dots, d$. We now assume that we know Bernstein–Bézier bases for $S_j^r(\mathcal{C})$ for all $j = 0, \dots, d$. If we scale $\mathcal{C}$, these Bernstein–Bézier bases do not change since they are affine invariant. Thus, we know the corresponding minimal determining sets $\mathcal{M}_j$ for a spline in $S_j^r(\mathcal{C}_j)$, for $j = 0, \dots, d$. We assume that if

$j = 0$, we have one point in the corresponding MDS. Thus, we only need to count the number of $\mathcal{C}_j$ layers to complete the proof. Since $j + j_1 + \cdots + j_{k-i} = d$, for each $0 \leqslant j \leqslant d$, we have $\binom{d+k-j-i-1}{k-i-1}$ layers of $\mathcal{C}_j$, and (16) follows. Note that (16) is equal to (1). □

Data availability

No data was used for the research described in the article.

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