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Solutions to a class of nonlinear Schrödinger equations involving a nonlocal term

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Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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Abstract

This thesis is dedicated to the study of the following nonlinear and non-local Schrödinger equation:

$$-\Delta u + u + \lambda^2 \left(\frac{1}{\omega |x|^{N-2}} \star \rho u^2 \right) \rho(x) u = |u|^{q-1} u \quad x \in \mathbb{R}^N,$$

where $\omega = (N-2) \left| \mathbb{S}^{N-1} \right|$, $\lambda > 0$, $q \in (1, 2^* - 1)$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, locally bounded, and possibly non-radial, $N = 3, 4, 5$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. We look for the existence and multiplicity of nontrivial solutions to the above under the following assumptions on ρ:

(ρ_1) $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$\left| x \in \mathbb{R}^N : \rho(x) \leq \overline{M} \right| < \infty;$$

(ρ_2) for every $M > 0$,

$$\left| x \in \mathbb{R}^N : \rho(x) \leq M \right| < \infty.$$

The variational properties of our problem require analysis of suitable functional spaces and their properties, such as separability and compactness, which play key roles in the variational tools which we use to prove existence and multiplicity of solutions. Under both assumptions our work is strongly inspired by that of Bartsch and Wang [7]. In the latter case (ρ_2) we find the existence of least energy solutions for a wide range of $q \in (2, 2^* - 1)$ using techniques related to the work of Jeanjean-Tanaka [29]. Furthermore, multiplicity results under this assumption are also obtained using the Lusternik-Schnirelman theory combined with results of Ambrosetti-Rabinowitz [4] and Ambrosetti-Ruiz [5].

In the former case (ρ_1) it is unclear whether compactness exists for our variational formulation, therefore we define a priori bounds, due to the relations between λ and $\overline{M}$, to show that there do not exist Palais-Smale sequences weakly convergent to 0 for λ large in order to prove the existence of a solution.

We also show nonexistence of nontrivial solutions in the critical case ($q = 2^* - 1$) and the case when $q \in (1, 2]$.

Contents

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1 Introduction

This thesis is dedicated to existence, nonexistence and multiplicity of solutions and their variational and qualitative properties, together with the functional properties and questions related to compactness of the following non-local Schrödinger equation:

$$-\Delta u + u + \lambda^2 \left(\frac{1}{\omega |x|^{N-2}} \star \rho u^2 \right) \rho(x) u = |u|^{q-1} u \quad x \in \mathbb{R}^N, \tag{$\mathcal{SP}$}$$

where $\omega = (N-2)\left|\mathbb{S}^{N-1}\right|$, $\lambda > 0$, $q \in (1, 2^* - 1)$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, locally bounded, and possibly non-radial, $N = 3, 4, 5$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

Under various assumptions on ρ, we are interested in addressing problems related to selecting a suitable functional setting and its relevant properties, such as those related to separability and compactness. In particular, the variational formulation of ($\mathcal{SP}$) requires a functional setting different from the standard Sobolev space $H^1(\mathbb{R}^N)$. Namely, situations where the right hand of the classical Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}} \,\mathrm{d}x\,\mathrm{d}y \lesssim ||\rho u^2||^2_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}, \tag{HLS}$$

is not necessarily finite for some $u \in H^1(\mathbb{R}^N)$. More precisely, taking inspiration from the work of Bartsch and Wang [7] we consider both of the following two assumptions on ρ:

(ρ_1) $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$\left| x \in \mathbb{R}^N : \rho(x) \leq \overline{M} \right| < \infty;$$

(ρ_2) for every $M > 0$,

$$\left| x \in \mathbb{R}^N : \rho(x) \leq M \right| < \infty.$$

We can see from the above assumptions that given a $u \in H^1(\mathbb{R}^N)$, the (HLS) may not be bounded in cases when, for example, $\rho(x) \to +\infty$ as $|x| \to +\infty$. This leads us to the study of the following space. Throughout the thesis these two assumptions will be described as (ρ_1) the *vanishing case* and (ρ_2) the *coercive case*.

Due to the above considerations, it is clear we will not be working in $H^1(\mathbb{R}^N)$, therefore we define $E(\mathbb{R}^N) \subseteq H^1(\mathbb{R}^N)$ as

$$E(\mathbb{R}^N) \coloneqq \left\{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) \,:\, \|u\|_{E(\mathbb{R}^N)} < +\infty \right\},$$

with norm

$$\|u\|_{E(\mathbb{R}^N)} \coloneqq \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,\mathrm{d}x \right. \\ \left. + \lambda \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \right)^{1/2}.$$

Variants of this space have been studied since the work of P.L. Lions [34], other works include [47], and [9], [16], [40]. Formally, solutions to ($\mathcal{SP}$) are the critical points in $E(\mathbb{R}^N)$ of the $C^1(E(\mathbb{R}^N);\mathbb{R})$ energy functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}. \tag{1.1}$$

One could regard ($\mathcal{SP}$) as formally equivalent to a nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda^2\rho(x)\phi u = |u|^{q-1}u, & x \in \mathbb{R}^N, \\ -\Delta\phi = \rho(x)u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

In fact, it is well-known from classical potential theory (Proposition 1 details this further) that if $u^2\rho \in L^1_{\text{loc}}(\mathbb{R}^N)$ is such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y < +\infty, \tag{1.3}$$

then,

$$\phi_u(x) = \int_{\mathbb{R}^N} \frac{\rho(y)u^2(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}y \tag{1.4}$$

is the unique weak solution in $D^{1,2}(\mathbb{R}^N)$ of the Poisson equation

$$-\Delta\phi = \rho(x)u^2 \tag{1.5}$$

and it holds that

$$\int_{\mathbb{R}^N} |\nabla\phi_u|^2 = \int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.6}$$

Here we set

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}, \tag{1.7}$$

equipped with norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

By elliptic regularity, the local boundedness of ρ implies that any pair $(u, \phi) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solution to (1.2) is such that u and ϕ are both of class $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. In particular, if $u \geq 0$ is nontrivial, it holds that $u, \phi > 0$. Note that $\inf I_\lambda = -\infty$, however it is an easy exercise to see that I_λ is bounded below on the set of its nontrivial critical points by a positive constant. It therefore makes sense to define a solution $u \in E(\mathbb{R}^N)$ to ($\mathcal{SP}$) as a *groundstate* if it is nontrivial, and if it holds that $I_\lambda(u) \leq I_\lambda(v)$ for every nontrivial critical point $v \in E(\mathbb{R}^N)$ of I_λ.

The methodology contained in this thesis is inspired by the work of Ambrosetti-Rabinowitz [4] and their pioneering analysis of nonlinear elliptic partial differential equations which show the existence of unique bound states to non-autonomous Schrödinger problems in bounded domains under certain conditions on the nonlinearities, which we will refer to as the Ambrosetti-Rabinowitz condition, and suitable compactness conditions. Since the classical work of Ambrosetti-Rabinowitz, considerable advances have been made in the understanding of several classes of nonlinear elliptic PDE's in the absence of either the so-called Palais-Smale or the Ambrosetti-Rabinowitz conditions, yet achieving in the spirit of [4] existence and multiplicity results, some examples include [2, 3, 51, 55]. A core theme when working on unbounded domains such as $\mathbb{R}^N$ are problems related to compactness, namely the lack of compactness phenomena which occur. These sort of problems have been tackled by Strauss [49] by means of radial functions and by Berestycki-Lions [13], which have been a breakthrough in the study of autonomous scalar field equations on the whole of $\mathbb{R}^N$. A great deal of work, certainly inspired by that of Floer and Weinstein [24], has been devoted to the study of nonlinear Schrödinger equations with nonradial potentials and involving various classes of nonlinearities:

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.8}$$

The classical works of Rabinowitz [45] and Benci-Cerami [10] have provided a penetrating analysis on equations like (1.8), and inspired the work on various remarkable variants of it, under different hypotheses on V and f which may allow loss of compactness phenomena to occur. Authors have contributed to understand these phenomena in a min-max setting, in analogy to what had been discovered and highlighted in the context of minimisation problems by P.L. Lions in [35] and related papers. An interesting case has been considered by Bartsch and Wang [7] who take a novel approach by using restrictions on the potential $V(x)$ in purely local Schrödinger equations to recover compactness in a non-radial setting. Their work has been a major inspiration to much of our content. In particular they proved existence and multiplicity of solutions to (1.8) for $V(x) = 1+\lambda^2\rho(x)$, with $\rho(x)$ satisfying the measure conditions (ρ_1) or (ρ_2), the condition (ρ_1) is particularly interesting as no compactness occurs. Years later Jeanjean and Tanaka in [29] and related papers, have looked into cases where $f(x,u)$ may violate the Ambrosetti-Rabinowitz condition. Remarkably, they have been able to overcome the possible unboundedness of the Palais-Smale sequences, with an approach which is reminiscent of the 'monotonicity trick' introduced for a different problem by Struwe [50].

To give insight into the physical properties of $(\mathcal{SP})$, some variants of this equation appear in the study of the quantum many-body problem, some examples can be found in [6, 18, 37]. The convolution term represents a repulsive interaction between particles, whereas the local nonlinearity $|u|^{q-1}u$ is a generalisation of the $u^{5/3}$ term introduced by Slater [48] as local approximation of the exchange term in Hartree-Fock type models, further examples of this may be found in [14, 38]. In the last few decades, nonlocal equations like $(\mathcal{SP})$ have received increasing attention on questions related to existence, nonexistence, variational setting and singular limit in the presence of a parameter. We draw the reader's attention to [1, 11, 18] and references therein, for a broader mathematical picture on questions related to Schrödinger-Poisson type systems. Relevant contributions to the existence of positive solutions, mostly for $q > 3 = N$, such as [19, 20], are based on the classification of positive solutions given by Kwong [31] to

$$-\Delta u + u = u^q, \qquad x \in \mathbb{R}^3,$$

regarded as a 'limiting' PDE when $\rho(x) \to 0$, as $|x| \to \infty$. Recently in [41, 52], in the case $\rho(x) \to 1$, as $|x| \to \infty$, the relation between (1.2) and

$$\begin{cases} -\Delta u + u + \lambda^2 \phi u = |u|^{q-1}u, & \mathbb{R}^3 \\ -\Delta \phi = u^2 & \mathbb{R}^3 \end{cases} \tag{1.9}$$

as a limiting problem, has been studied, though a full understanding of the set of positive solutions to (1.9) has not yet been achieved.

Considerably fewer results have been obtained in relation to the multiplicity of solutions. It is worth mentioning [5] whose (radial) approach is suitable in the presence of constant potentials. More precisely Ambrosetti-Ruiz [5] have studied the problem (1.9) with $\lambda > 0$ and $1 < q < 5$. When $q \in (1, 2) \cap (3, 5)$ their approach relies on the symmetric version of the Mountain-Pass Theorem [4], whereas for $q \in (2, 3]$ and in the spirit of [29, 50], they develop a min-max approach to the multiplicity which in fact improves upon [4] and is based on the existence of bounded Palais-Smale sequences at specific levels associated with the perturbed functional

$$I_{\mu,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{\omega|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \frac{\mu}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}\,\mathrm{d}x,$$

for a dense set of values $\mu \in \left[\frac{1}{2}, 1\right)$.

1.1 Main Results

In this section we collect the main results of this thesis under both assumptions $(\boldsymbol{\rho_1})$ and $(\boldsymbol{\rho_2})$ which appear in [23]. Our new results are about existence, nonexistence and multiplicity of solutions.

Working in the functional setting $E(\mathbb{R}^N)$ causes several problems to arise which are not shared with more common spaces such as $H^1(\mathbb{R}^N)$. We may not have a Hilbert space structure, as anticipated earlier notice that the assumption $(\boldsymbol{\rho_1})$ is compatible with a situation where $\rho(x) \to \rho_\infty > 0$ as $|x| \to \infty$, in which the space $E(\mathbb{R}^N) \simeq H^1(\mathbb{R}^N)$, as well as with the case $\rho(x) \to \infty$ as $|x| \to \infty$, in which $E(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$. We tackle the case of vanishing ρ with a unified approach for these particular sub-cases.

Showing the boundedness of Palais-Smale sequences when $q < 3$ is a well known open problem for this system, discussions on this topic can be found in [46]. Furthermore, a possible lack of compactness due to the invariance by translations in unbounded domains such as $\mathbb{R}^N$ is a common issue in our work. Methods by Strauss [49] for radial functions can overcome these issues or P.L. Lions [35] in nonradial settings. In our case it is not obvious if compactness can be regained under the assumption $(\boldsymbol{\rho_1})$, however, under the assumption $(\boldsymbol{\rho_2})$ we can use methods similar to those on weighted Sobolev spaces to regain compactness (Lemma 4.4).

In contrast with the local Schrödinger problem (1.8), where the infimum over the Nehari manifolds can be shown to coincide with the mountain-pass level for a wide range of nonlinearities f, as is shown in the book Minimax Theorems by

Willem [55, p. 73]. In our problem ($\mathcal{SP}$) it is not standard to prove, or disprove, that these levels coincide when $q \le 3$.

In the case of pure power nonlinearities and $q \in (1, 2]$, and unlike for the action functional associated with (1.8), the variational properties of I_λ are particularly sensitive to λ, yielding existence, multiplicity (of a local minimiser and at the same time of a mountain-pass solution) and nonexistence results, for example [39, 46] and [47].

1.1.1 Existence Results

The Theorems labelled in this section are mirrored in Chapter 4 under the labels Theorems 4.1, 4.2, 4.3, 4.4 respectively in order of presentation here.

We begin with the following existence result in the case (ρ_1) and analysing the behaviour of ($\mathcal{SP}$) as λ varies.

Theorem 1.1 (**Groundstates for** $q \ge 3$ **under** (ρ_1))**.** *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 2^* - 1)$. There exists a positive constant $\lambda_* = \lambda_*(q, \overline{M})$ such that for every $\lambda \ge \lambda_*$, ($\mathcal{SP}$) admits a positive groundstate solution $u \in E(\mathbb{R}^3)$. For $q > 3$, u is a mountain-pass solution.*

To prove the above theorem we use a constraint minimisation approach over the Nehari manifold. Special consideration is taken in the delicate case of $q = 3$ as it is not clear whether the mountain-pass level is critical for this exponent.

By construction $\lambda_* = \max\{\lambda_0, \lambda_1\}$, where λ_0 and λ_1 are defined as in Propositions 6 and 7, provide uniform lower thresholds for λ to ensure that certain Palais-Smale sequences have non-zero weak limits and that these weak limits have a precise variational characterisation.

In a situation, as mentioned earlier, where $E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$, with equivalent norms by (HLS), a sufficient condition at infinity for certain Palais-Smale sequences to be compact is given by Propositions 9 and 10.

The following theorems consider the assumption (ρ_2). We separate the cases depending on the conditions on q, namely, we look at $q \ge 3$ and $q < 3$ individually due to differences in the variational characteristics of the functional I_λ. It is worth noting that unlike in the case above, under the assumption (ρ_2) $\lambda > 0$ is a fixed constant.

Theorem 1.2 (**Groundstates for** $q \ge 3$ **under** (ρ_2))**.** *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ be nonnegative, satisfying (ρ_2), and $q \in [3, 2^* - 1)$. Then, for any fixed $\lambda > 0$, ($\mathcal{SP}$) has both a positive mountain-pass solution and a positive groundstate solution in $E(\mathbb{R}^3)$, whose energy levels coincide for $q > 3$.*

The main advantage when working under assumption (ρ_2) compared to (ρ_1) in the prior Theorem 1.1 comes from the result provided by Lemma 4.4 giving a compact embedding yielding a variationally stronger result than in the (ρ_1) case. Under this condition we can show the mountain-pass level is indeed critical since we have the Palais-Smale condition for a wide range of q. At this stage it is unclear whether the groundstate solutions and the mountain-pass level coincide when $q = 3$.

In the range $q \in (2, 3)$ it is not standard to show the boundedness of Palais-Smale sequences for the functional I_λ. We take inspiration from the approach of Jeanjean and Tanaka [29] and tools developed in [23, 41]. This approach concerns constructing a sequence of critical points $(u_n)_{n\in\mathbb{N}}$ of the following perturbed functional

$$I_{\mu_n,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho(x)\phi_u u^2 - \frac{\mu_n}{q+1}\int_{\mathbb{R}^N}|u|^{q+1},$$

which we find converge to our desired solution as $\mu_n \to 1^-$.

Theorem 1.3 (**Groundstates for $q < 3$ under (ρ_2)**). *Let $N = 3, 4, 5$, $q \in (2, 3)$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Let $\lambda > 0$, and assume $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative and satisfies (ρ_2). Moreover suppose that $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(q-2)}{(q-1)}$. Then, ($\mathcal{SP}$) has a mountain-pass solution $u \in E(\mathbb{R}^N)$. Moreover, there exists a groundstate solution.*

Remark 1.1. The same proof when working instead with the functional

$$I_{\lambda,+}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + u^2\right) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho(x)\phi_u u^2 - \frac{1}{q+1}\int_{\mathbb{R}^N}u_+^{q+1},$$

allows to show that mountain-pass and groundstate critical points exist for this functional, and are positive by construction.

The question of whether the mountain-pass and groundstate critical points coincide, as in Theorem 1.2 for $q = 3$, remain unclear. We obtain some more insight into this problem when we consider a homogeneity condition on ρ, in particular we further assume ρ is homogeneous of order $\bar{k} > 0$, the exact value of this lower bound to be explained later.

Theorem 1.4 (**Homogeneous case for $q \leq 3$: mountain-pass solutions vs. groundstates**). *Let $N = 3, 4, 5$, $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\lambda > 0$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, satisfies*

(ρ_2), and is homogeneous of degree $\bar{k}$, namely $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some

$$\bar{k} > \left(\max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1\right)_+ .$$

Then, the mountain-pass solutions that we find in Theorem 1.2 ($q = 3$) and Theorem 1.3 ($q < 3$) are groundstates.

The proof of this Theorem comes from the analysis shown in Proposition 2. This proposition gives us a characterisation of the mountain-pass level over a specific manifold, closely related to, or could be thought of as a combination of, the Nehari and Pohozaev identities. More detail on the lower bound assumption on $\bar{k}$ is given by Remark 3.1.

1.1.2 Multiplicity Results

The two Theorems in this section are also mirrored in Chapter 6 under the labels Theorems 6.2, 6.3 respectively in order of presentation here.

In the spirit of Ambrosetti-Rabinowitz [4] and under (ρ_2) we show that (1.2) possesses infinitely many high energy solutions. In our context it seems appropriate to distinguish the cases $q \in (3, 5)$ and $q \in (2, 3]$ when working within the Lusternik-Schnirelman theory. Since for $q \in (3, 5)$ Lemma 4.4 implies that the Palais-Smale condition is satisfied, we can use the $\mathbb{Z}_2$-equivariant Mountain-Pass theorem, adapting to $E(\mathbb{R}^N)$ arguments similar to those developed for a different functional setting by Szulkin; see [53]. To this aim, in Lemma 3.1 we prove that for $N \geq 3$, $E(\mathbb{R}^N)$ is a separable Banach space, by constructing a suitable linear isometry of $E(\mathbb{R}^N)$ onto the Cartesian product of $H^1(\mathbb{R}^N)$ with some of the mixed norm Lebesgue spaces studied by Benedek and Panzone [12], namely $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$. As a consequence of this identification, we can show that $E(\mathbb{R}^N)$ admits a Markushevic basis, that is a set of elements $\{(e_m, e_m^*)\}_{m\in\mathbb{N}} \subset E(\mathbb{R}^N)\times E^*(\mathbb{R}^N)$ such that the duality product $< e_n, e_m^* >= \delta_{nm}$ for all $n, m \in \mathbb{N}$, the e_m's are linearly dense in $E(\mathbb{R}^N)$, and the weak*-closure of $\text{span}\{e_m^*\}_{m\in\mathbb{N}}$ is $E^*(\mathbb{R}^N)$. We use this, combined with Lemma 4.4 to obtain lower bounds on the energy which allow us to show the divergence of a sequence of min-max critical levels defined by means of the classical notion of Krasnoselskii genus; see Lemma 6.1 below. This yields the following

Theorem 1.5 (**Infinitely many high energy solutions for** $q > 3$)**.** *Let $N = 3$, $q \in (3, 2^* - 1)$ and $\lambda > 0$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ is nonnegative and satisfies (ρ_2). Then, there exists infinitely many distinct pairs of critical points $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, for I_λ such that $I_\lambda(u_m) \to +\infty$ as $m \to +\infty$.*

As in the existence case, when $q < 3$ we face the possible unboundedness of the Palais-Smale sequences. The work of Ambrosetti-Ruiz [5] contains a deformation Lemma in the spirit of Jeanjean and Tanaka which is suitable for Lusternik-Schnirelman type results. As before we require the homogeneity condition on ρ which allow us to define certain classes of admissable subsets of $E(\mathbb{R}^N)$ as in [5]. Lemmas 3.6 and 6.4, together with the Pohozaev type inequality (Lemma 3.9), form the basis of the following Theorem.

Theorem 1.6 (**Infinitely many high energy solutions for** $q \leq 3$)**.** *Let $N = 3, 4, 5$. Assume $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\lambda > 0$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, satisfies (ρ_2), and is homogeneous of degree $\bar{k}$, namely, $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \left(\max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1\right)_+ .$$

Then, there exist infinitely many distinct pairs of critical points, $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, for I_λ such that $I_\lambda(u_m) \to +\infty$ as $m \to +\infty$.

1.2 Further Research

The following open questions are still unknown and are worthy of further research:

- Much of the work under assumption (ρ_1) is done in the framework of $q \geq 3$. The question of whether a result when $q < 3$ is possible remains.
- We only show existence under (ρ_1) but could multiplicity be shown when $q \in [3, 2^* - 1)$, and possibly for lower values of q.
- We use some strict conditions on the homogenity of ρ in the coercive case (ρ_2) when working in the "low q" case in both the existence and multiplicity results. It is possible that this assumption could be further relaxed.
- Nonexistence in the case $q \in (1, 2]$ (Proposition 5) when $\rho(x)$ is nonnegative instead of $\rho(x) \geq 1$ is still open.

1.3 Organisation of thesis

This thesis is split into multiple Chapters to give the reader a coherent flow regarding the main topic and previous work in this field.

We begin with a background chapter 2 detailing some of the many techniques used in critical point theory for time independent Schrödinger type problems and also a discussion about the Poisson equation and how it relates to our system. Many of the techniques used for this type of problems are applicable in our setting with some modifications, we end this chapter with a brief discussion on the differences between the local and nonlocal problems.

Chapter 3 introduces some preliminary results found in [23] regarding properties of the space E and characteristics of the functional (1.1) which will be used throughout the latter chapters. Discussions regarding the homogeneity of ρ and why we make this assumption are contained in this chapter. We also state some standard results regarding regularity and positivity of weak solutions and end with two nonexistence results, separating the critical case $q = 2^* - 1$ and when $q \in (1, 2]$.

Chapter 4 deals with our various existence results and is organised into multiple sections. The results in this chapter are found in [23]. Sections 4.1 and 4.2 discuss the situation when ρ is under assumption (ρ_1), i.e. vanishing on a region. It is unclear under this assumption whether we may recover compactness, we present a number of estimates and sufficient conditions to show existence of mountain-pass solutions in a certain range of q. A brief discussion on recovering the Palais-Smale condition is included, we use an approach similar to [41] for the "problem at infinity" where we set ρ in such a way which satisfies (ρ_1). Sections 4.3 and 4.4 discuss what happens when ρ is under the assumption (ρ_2), namely when ρ is coercive. Here the lack of compactness phenomena is overcome by Lemma 4.4 and hence we are able to show our functional I satisfies the Palais-Smale condition. Furthermore we show existence of solutions for a wide range of q.

Chapter 5 gives some background for the techniques we use to obtain multiplicity results. This ranges from an introduction of Brouwer degree, to the Lusternik-Schnirelmann theory and its relations to deformations. We end this chapter with an example to a generic problem and finding the existence of infinitely many solutions.

We end with Chapter 6 which details our multiplicity results under the assumption (ρ_2). The results in this chapter are found in [23]. In this chapter we perform a separate analysis between the "high q" and "low q" case as these require different techniques due to complexities involving the boundedness of Palais-Smale sequences.

2 Background

Before we delve into the details of our Schrödinger-Poisson an introduction to the techniques which will be used will be given in this chapter. We aim to cover Schrödinger type problems and the Poisson equation separately in order to give a flavour of techniques used to find solutions of these problems in the current literature and to reflect on how these methods are adapted to the nonlocal Schrödinger-Poisson case.

All of the problems covered here will be in $\mathbb{R}^N$. This choice of domain aims to show the lack of compactness phenomenon when working in unbounded domains and will give the reader an insight of a few techniques developed by earlier mathematicians [13, 28, 29, 35] to combat this issue.

Much of the content presented in this chapter is a review of work done by myself presented in my Master's Thesis at Swansea University, namely sections 2.1 and 2.2, to show a natural progression to the current research discussed in the bulk of this thesis.

2.1 Solutions to a local Schrödinger problem

The aim of this section is to provide the reader with an introduction to finding solutions of Schrödinger type problems when working on unbounded domains, in particular we work in $\mathbb{R}^N$. We highlight the difficulties faced in this scenario and the techniques we use to overcome them.

We begin with an introduction to the concentration compactness lemma by Pierre Louis Lions [35, 36]. When working on unbounded domains such as $\mathbb{R}^N$, compactness is lost due to the invariance by translations of the functionals. Namely, we can no longer construct subsequences that are not weakly convergent to 0.

The space we will be working in is the standard weighted Hilbert space defined as follows.

Definition 1 (The space H^1_V)**.** Let $V(x) \in \mathcal{C}(\mathbb{R}^N)$ and $V(x) \geq c > 0$, then we define the weighted space with norm

$$\|f\|_{H^1_V(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} |\nabla f|^2 + V(x) f^2 \Big)^{\frac{1}{2}} < \infty.$$

It is easy to see that $\|f\|_{H^1_V(\mathbb{R}^N)} \geq c\|f\|_{H^1(\mathbb{R}^N)}$ and hence $H^1_V \hookrightarrow H^1$.

The following theorem uses the Lagrange multiplier rule to apply a constrained minimisation technique to obtain least energy solutions. The main difficulties faced here are due to the fact we are working in $\mathbb{R}^N$ with non-radial functions.

Theorem 2.1. *Let $N \geq 2$, $V(x) \in \mathcal{C}(\mathbb{R}^N)$, $V(x) \geq c > 0$, $V(x) \leq \liminf_{|y| \to \infty} V(y)$ for all $x \in \mathbb{R}^N$ and $V(x) < C$, where C is a finite constant. Then, for $1 < p < 2^* - 1$,*

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-1} u, & x \in \mathbb{R}^N. \\ u > 0, \end{cases} \tag{2.1}$$

has a nontrivial weak solution $u \in H^1(\mathbb{R}^N)$. In the case of $N = 2$ the result holds for all $p > 1$.

Weak solutions, as mentioned in the above theorem, are defined as follows

Definition 2. We call $u \in H^1_V(\mathbb{R}^N)$ a weak solution to (2.1) if

$$\int_{\mathbb{R}^N} \nabla u \nabla \phi + V(x) u \phi - \int_{\mathbb{R}^N} |u|^{p-2} u \phi = 0$$

holds for all $\phi \in H^1_V(\mathbb{R}^N)$.

Before we prove Theorem 2.1 we require the following lemma by Lions [36, Lemma 1.1]. The usefulness of this lemma will become apparent in the proof of Theorem 2.1 where it will be used to recover compactness.

Lemma 2.1. *Let $2 \leq q < 2^*$, $r > 0$. Suppose $(u_n)_n \subset H^1(\mathbb{R}^N)$ bounded and*

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n|^q \to 0, \quad n \to \infty.$$

Then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*)$.

Proof. We prove this lemma for a specific exponent

$$\bar{p} := \frac{2(N+q)}{N} \in (q, 2^*).$$

Consider the cube $C(y,r)$ centered at y and with vertices of distance r from its center. It holds, by the interpolation inequality, that

$$\begin{aligned} \|u\|_{L^{\bar{p}}(C(y,r))} &\leq \|u\|^{\alpha}_{L^q(C(y,r))} \|u\|^{1-\alpha}_{L^{2^*}(C(y,r))} \\ &\leq c \|u\|^{\alpha}_{L^q(C(y,r))} \|u\|^{1-\alpha}_{H^1(C(y,r))}. \end{aligned}$$

Set $S_y(u) = \|u\|^q_{L^q(C(y,r))}$, then the previous inequality implies

$$\|u\|^{\bar{p}}_{L^{\bar{p}}(C(y,r))} \leq c S_y(u)^{\frac{\alpha \bar{p}}{q}} \|u\|^{(1-\alpha)\bar{p}}_{H^1(C(y,r))}$$

$$\leq cS_y(u)^{\frac{\alpha\bar{p}}{q}}\|u\|^2_{H^1(C(y,r))},$$

as $(1-\alpha)\bar{p} = 2$. Note that $\mathbb{R}^N = \bigcup_n C(y_n, r)$ disjoint. Hence

$$\|u\|^{\bar{p}}_{L^{\bar{p}}(\mathbb{R}^N)} \leq c \sup_{y\in\mathbb{R}^N} S_y(u)^{\frac{\alpha\bar{p}}{q}}\|u\|^2_{H^1(R^N)}.$$

Now using the above inequality with u_n, our assumption on the supremum of $S_y(u_n)$ and $\|u_n\|_{H^1(\mathbb{R}^N)} < C$ gives

$$\|u_n\|^{\bar{p}}_{L^{\bar{p}}(\mathbb{R}^N)} \leq c \cdot o(1) \cdot C \to 0.$$

If $2 < p < \bar{p}$, then

$$\|u_n\|^p_{L^p(\mathbb{R}^N)} \leq \|u_n\|^\alpha_{L^2(\mathbb{R}^N)}\|u_n\|^{1-\alpha}_{L^{\bar{p}}(\mathbb{R}^N)}.$$

If $\bar{p} < p < 2^*$, then

$$\|u_n\|^p_{L^p(\mathbb{R}^N)} \leq \|u_n\|^\alpha_{L^{2^*}(\mathbb{R}^N)}\|u_n\|^{1-\alpha}_{L^{\bar{p}}(\mathbb{R}^N)}.$$

□

Now we can move on to the proof of Theorem 2.1. The proof uses various results that are used as a preliminary to allow the use of Lemma 2.1 which is then used to show the existence of solutions.

Proof of Theorem 2.1. We prove 2.1 by using the standard Langrange multiplier rule showing

$$\alpha := \inf_{\substack{u\in H^1_V(\mathbb{R}^N)\\ \|u\|^{p+1}_{p+1}=1}} I(u) \tag{2.2}$$

is achieved, where

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2,$$

and by the Sobolev embedding theorem, $\alpha > 0$. We start with some remarks on the spaces used in this proof and a brief discussion on the regularity and positivity of u.

Under the assumptions on $V(x)$ it is possible to show that the weighted space H^1_V is isomorphic to H^1, we have

$$c\int_{\mathbb{R}^N} u^2 \leq \int_{\mathbb{R}^N} V(x)u^2 \leq C\int_{\mathbb{R}^N} u^2,$$

which, when adding the gradient terms back in becomes,

$$\min(c,1)\int_{\mathbb{R}^N}|\nabla u|^2+u^2\leq\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2\leq(C+1)\int_{\mathbb{R}^N}|\nabla u|^2+u^2.$$

From here it is clear we can construct the following bounds:

$$c\|u\|_{H^1(\mathbb{R}^N)}\leq\|u\|_{H^1_V(\mathbb{R}^N)}\leq C\|u\|_{H^1(\mathbb{R}^N)},$$

and

$$\tilde{c}\|u\|_{H^1_V(\mathbb{R}^N)}\leq\|u\|_{H^1(\mathbb{R}^N)}\leq\tilde{C}\|u\|_{H^1_V(\mathbb{R}^N)}.$$

Hence, $H^1_V\simeq H^1$ with equivalent norms and we proceed to work in $H^1(\mathbb{R}^N)$ for the remainder of the proof.

A quick note on the regularity and positivity of the solution, if u is a solution to (2.2) then so is $|u|$, also since u is real valued then $|\nabla|u||=|\nabla u|$ almost everywhere on $\mathbb{R}^N$. As a consequence $I(u)=I(|u|)$. Furthermore it can be shown [33, 51] that $u\in C^1$ and u is a nontrivial solution of (2.2). Hence by Vázquez [54] it can be shown that $u>0$.

Moving back to the original problem. By the definition of α in (2.2), we can write

$$\alpha=\inf_{0\neq u\in H^1(\mathbb{R}^N)}I\Big(\frac{u}{\|u\|_{L^{p+1}(\mathbb{R}^N)}}\Big)=\inf_{0\neq u\in H^1(\mathbb{R}^N)}\frac{I(u)}{\Big(\int_{\mathbb{R}^N}|u|^{p+1}\Big)^{\frac{2}{p+1}}}.\tag{2.3}$$

We solve (2.3) by considering a minimising sequence:

$$\begin{cases}u_n\in H^1(\mathbb{R}^N),\\ \|u_n\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}=1,\\ I(u_n)\to\alpha.\end{cases}\tag{2.4}$$

where $\alpha=\inf\limits_{0\neq u\in H^1(\mathbb{R}^N)}\frac{I(u)}{\|u\|^2_{L^{p+1}(\mathbb{R}^N)}}$. Since u_n is minimising, $\|u_n\|_{H^1(\mathbb{R}^N)}\leq C$ therefore by Banach-Alaoglu, $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$. For $u\in H^1(\mathbb{R}^N)$ we define $I(u)=\frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^N)}$ and

$$\begin{aligned}I(u_n)=&I(u_n-u+u)=\frac{1}{2}\|u_n-u+u\|^2_{H^1_V(\mathbb{R}^N)}\\=&\frac{1}{2}\big[\|u_n-u\|^2_{H^1_V(\mathbb{R}^N)}+\|u\|^2_{H^1_V(\mathbb{R}^N)}-2(u_n-u,u)\big]=I(u_n-u)+I(u)+o(1).\end{aligned}$$

Which, by our assumption, means that

$$I(u_n - u) + I(u) \to \alpha. \tag{2.5}$$

We break solving (2.3) into 3 steps.
Step 1: *The following alternative holds:*

- $\|u\|_{L^{p+1}(\mathbb{R}^N)} = 1$.
- $u \equiv 0$.

By the definition of α we have

$$\alpha\Big[\int_{\mathbb{R}^N} |u|^{p+1}\Big]^{\frac{2}{p+1}} \leq I(u),$$

for all $u \in H^1(\mathbb{R}^N)$. From this we can use the result in (2.5) to get

$$\alpha \leftarrow I(u_n - u) + I(u) \geq \alpha\Big[\int_{\mathbb{R}^N} |u_n - u|^{p+1}\Big]^{\frac{2}{p+1}} + \alpha\Big[\int_{\mathbb{R}^N} |u|^{p+1}\Big]^{\frac{2}{p+1}}. \tag{2.6}$$

Set $\beta = \int_{\mathbb{R}^N} |u|^{p+1}$, by Fatou, $\beta \in [0, 1]$. Here we can assume, if passing to a subsequence, we have almost everywhere convergence in $\mathbb{R}^N$. We have by Brezis-Lieb lemma,

$$o(1) + \int_{\mathbb{R}^N} |u|^{p+1} + \int_{\mathbb{R}^N} |u_n - u|^{p+1} = \int_{\mathbb{R}^N} |u_n|^{p+1} = 1.$$

As $n \to \infty$ in (2.6), this implies $\alpha \geq \alpha(1-\beta)^{\frac{2}{p+1}} + \alpha\beta^{\frac{2}{p+1}}$. Dividing by α we obtain

$$1 \geq (1-\beta)^{\frac{2}{p+1}} + \beta^{\frac{2}{p+1}}, \quad \beta \in [0, 1].$$

On the RHS we have

$$f(\beta) = (1-\beta)^{\frac{2}{p+1}} + \beta^{\frac{2}{p+1}}.$$

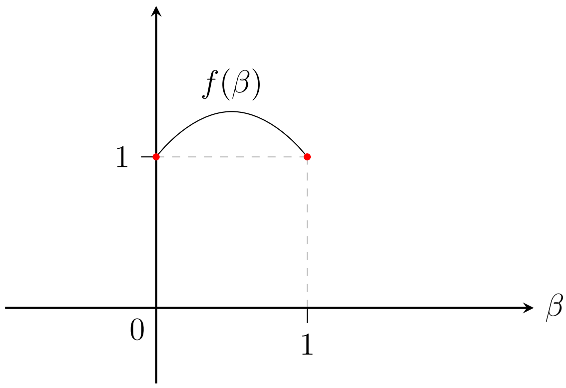

Figure 1: Sketch of $f(\beta)$.

Observing the behaviour of this function we can see in Figure 1 that if $f(\beta) \leq 1$, this is only possible if $\beta = 0, 1$.

Step 2: *If $\beta = 1$ we have a solution to the minimum problem. If $\beta = 0$ there exists a sequence $y_n \in \mathbb{R}^N$ such that the translated sequence $v_n := u_n(\cdot - y_n)$ does not contain any subsequence weakly converging to zero.*

If $\beta = 1$ it follows by the weakly lower semicontinuity of the norm $I(u) = \alpha$. If $\beta = 0$, recall that $\|u_n\|_{L^{p+1}(\mathbb{R}^N)} = 1$. By Lions Lemma 2.1

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n|^2 \nrightarrow 0,$$

implies there exists $\rho > 0$ and a subsequence $(y_n)_n \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,1)} |u_n|^2 \geq \rho > 0.$$

Here we use in particular $r = 1$. If we set $v_n := u_n(\cdot - y_n)$ we get

$$\int_{B(0,1)} v_n^2 \geq \rho > 0, \quad \forall n \in \mathbb{N},$$

which implies that $(v_n)_n \in \mathbb{R}^N$ does not contain any subsequence weakly converging to zero.

Step 3: *If $\beta = 0$, construct v_n as above. v_n is also a minimising sequence. Note that, because of the first two steps:*

$$\begin{cases} v_n \rightharpoonup 0, & \textit{does not hold by step 2,} \\ v_n \rightharpoonup v, & \|v\|_{L^{p+1}(\mathbb{R}^N)} = 1, \quad \textit{we have a solution.} \end{cases}$$

Note that:

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 = \int_{\mathbb{R}^N} |\nabla u_n|^2,$$

and

$$\int_{\mathbb{R}^N} |v_n|^{p+1} = \int_{\mathbb{R}^N} |u_n|^{p+1} = 1.$$

Also, $\|v_n\|^2_{L^2(\mathbb{R}^N)} = \|u_n\|^2_{L^2(\mathbb{R}^N)} < C$ and this implies that $\|v_n\|^2_{L^2(\mathbb{R}^N)} \to k$ by Bolzano-Weierstrass. We are left to show that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} V(x)v_n^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2,$$

as this would show that v_n is also a minimising sequence. Set

$$\gamma := \liminf_{|y|\to\infty} V(y) \geq V(x),$$

then

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} V(x)v_n^2 \leq \limsup_{n\to\infty} \gamma \int_{\mathbb{R}^N} v_n^2 = \gamma k.$$

If we can show that

$$\gamma k \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2,$$

then we are done. We split the integral to the integral over a ball and its complement,

$$\int_{\mathbb{R}^N} V(x)u_n^2 = \int_{B_R} V(x)u_n^2 + \int_{B_R^c} V(x)u_n^2,$$

however

$$\int_{B_R} V(x)u_n^2 \leq c \int_{B_R} u_n^2 \to 0.$$

Thus,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2 = \liminf_{n\to\infty} \int_{|x|>R} V(x)u_n^2,$$

for all $R > 0$. By definition of γ it follows that for all $\varepsilon > 0$, there exists R_ε such that $V(x) > \gamma - \varepsilon$ on $B^c_{R_\varepsilon}$ and

$$\int_{|x|>R_\varepsilon} V(x)u_n^2 \geq (\gamma - \varepsilon) \int_{|x|>R_\varepsilon} |u_n|^2,$$

implying, from the invariance by translations,

$$\liminf_{n\to\infty} \int_{|x|>R_\varepsilon} V(x)u_n^2 \geq (\gamma - \varepsilon)k,$$

which concludes that v_n is also minimising.

Conclusion: Let $(u_n)_n$ be a minimising sequence for (2.2). Then considering if necessary the modified sequence $v_n := u_n(\cdot - y_n)$, we have up to a subsequence

$$\begin{aligned} &v_n \to v \neq 0 \quad \text{in } L^{p+1}(\mathbb{R}^N),\\ &v_n \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^N),\\ &v_n \text{ is minimising.} \end{aligned}$$

This and the weakly lower semicontinuity of the norm imply

$$\alpha = \liminf_{n\to\infty} \frac{I(v_n)}{\left(\int_{\mathbb{R}^N} |v_n|^{p+1}\right)^{\frac{2}{p+1}}} \geq \frac{I(v)}{\left(\int_{\mathbb{R}^N} |v|^{p+1}\right)^{\frac{2}{p+1}}} \geq \alpha.$$

Which implies $\lim_{n\to\infty} I(v_n) = I(v)$, and $\lim_{n\to\infty} \|v_n\|_{H^1(\mathbb{R}^N)} = \|v\|_{H^1(\mathbb{R}^N)} \Rightarrow \|v_n - v\|_{H^1(\mathbb{R}^N)} \to 0$.

Hence, all the minimising sequences for (2.2) are relatively compact in $H^1(\mathbb{R}^N)$ up to translations. We then use the Lagrange multiplier rule to show the existence of a solution. □

We turn our attention to a similar problem to Theorem 2.1 which has a coercive potential. When dealing with such a potential the following compactness result emerges.

Lemma 2.2. *If* $\lim_{|x|\to\infty} V(x) = +\infty$ *and* $V(x) \geq c > 0$ *then* $H^1_V(\mathbb{R}^N)$ *is compactly embedded into* $L^{p+1}(\mathbb{R}^N)$ *for* $1 \leq p < 2^* - 1$ *if* $N > 2$ *and for* $p \geq 1$ *if* $N = 2$.

Proof. Assume $(u_n)_n \subset H^1_V(\mathbb{R}^N)$ and $\|u_n\|_{H^1_V(\mathbb{R}^N)} < C$ for some finite constant C. This implies, passing if necessary to a subsequence, there exists $u_{n_k} \to u$ in $L^{p+1}_{loc}(\mathbb{R}^N)$ by Rellich and also $u_{n_k} \to u$ almost everywhere on $\mathbb{R}^N$. We want to show that $\|u_{n_k} - u\|^2_{L^2(\mathbb{R}^N)} \to 0$. Writing the norm explicitly we have

$$\int_{\mathbb{R}^N} |u_{n_k} - u|^2 = \int_{B_R} |u_{n_k} - u|^2 + \int_{B_R^c} |u_{n_k} - u|^2. \tag{2.7}$$

Notice that by Rellich we already have the first integral over B_R goes to zero, also by our assumption on $V(x)$, namely that it is never zero, we can re-write (2.7) as

$$\int_{\mathbb{R}^N} |u_{n_k} - u|^2 = \int_{B_R^c} |u_{n_k} - u|^2 \frac{V(x)}{V(x)} dx. \tag{2.8}$$

By definition, $\lim_{|x|\to\infty} V(x) = +\infty$ holds if, and only if, for all $M > 0$ there exists $R > 0$ such that for $|x| > R$, $V(x) > M$. Or, inversely, $V(x)^{-1} < M^{-1}$. Therefore, by Fatou and since the sequence is bounded in H^1_V it follows that

$$\int_{B_R^c} |u_{n_k} - u|^2 \frac{V(x)}{V(x)} dx \le \varepsilon \int_{B_R^c} V(x) |u_{n_k} - u|^2 \le C\varepsilon.$$

We extend this to a general p term. We want $\|u_{n_k} - u\|_{L^{p+1}(\mathbb{R}^N)} \to 0$. By interpolation we have

$$\|u_{n_k} - u\|_{L^{p+1}(\mathbb{R}^N)} \le \|u_{n_k} - u\|^{\alpha}_{L^2(\mathbb{R}^N)} \|u_{n_k} - u\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^N)}. \tag{2.9}$$

By Sobolev's inequality, $\|u_{n_k} - u\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^N)} < C$. Thus, as we have already shown that $\|u_{n_k} - u\|^{\alpha}_{L^2(\mathbb{R}^N)} \le C\varepsilon$,

$$\|u_{n_k} - u\|_{L^{p+1}(\mathbb{R}^N)} \le \bar{C}\varepsilon.$$

Hence, $H^1_V(\mathbb{R}^N)$ is compactly embedded into $L^{p+1}(\mathbb{R}^N)$ for $1 \le p < 2^* - 1$. In the case of $N = 2$ we can see that (2.9) becomes

$$\|u_{n_k} - u\|_{L^{p+1}(\mathbb{R}^N)} \le \|u_{n_k} - u\|^{\alpha}_{L^2(\mathbb{R}^N)} \|u_{n_k} - u\|^{1-\alpha}_{L^{q}(\mathbb{R}^N)}, \tag{2.10}$$

where $q > p + 1$. Thus, from the Sobolev embedding, it follows that (2.10) holds for all $p \ge 1$. □

As a consequence of this Lemma we can obtain the following result.

Theorem 2.2. *Let $N \ge 2$, $V(x) \in \mathcal{C}(\mathbb{R}^N)$, $V(x) \ge c > 0$, $V(x) \le \lim_{|y|\to\infty} V(y) = +\infty$ for all $x \in \mathbb{R}^N$. Then, for $1 < p < 2^* - 1$,*

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-1} u, & x \in \mathbb{R}^N, \\ u > 0, \end{cases} \tag{2.11}$$

has a nontrivial weak solution $u \in H^1_V(\mathbb{R}^N)$. In the case of $N = 2$ the result holds for all $p > 1$.

Proof. We apply the Lagrange Multiplier rule meaning for $I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2$ we want to find some α such that

$$\alpha := \inf_{\substack{u \in H^1_V(\mathbb{R}^N) \\ \|u\|^{p+1}_{p+1} = 1}} I(u), \tag{2.12}$$

We construct a minimising sequence such that

$$\begin{cases} u_n \in H_V^1(\mathbb{R}^N), \\ \|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = 1, \\ I(u_n) \to \alpha. \end{cases}$$

Indeed, by the weakly lower semicontinuity of the norm and the compact embedding of $H_V^1(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$ given by Lemma 2.2, it holds that

$$\frac{I(u)}{\left(\int_{\mathbb{R}^N} |u|^{p+1}\right)^{\frac{2}{p+1}}} \leq \liminf_{n\to\infty} \frac{I(u_n)}{\left(\int_{\mathbb{R}^N} |u_n|^{p+1}\right)^{\frac{2}{p+1}}} = \alpha,$$

which can only be true if the above is a strict equality, thus $\alpha = I(u)$. Hence, there exists a minimiser for α and the rest of the proof, including the remarks about the regularity and positivity of u, are identical to the proof of Theorem 2.1. □

2.2 The Mountain Pass Theorem

Many of the techniques contained in this thesis rely heavily on the Mountain Pass Theorem (MPT). The pioneering work by Ambrosetti-Rabinowitz in 1973 [4] involves analysing the family of curves around the well centered at the origin to find minimax type solutions to corresponding PDEs under certain conditions. Before we can state the Mountain Pass Theorem we have a few preliminaries. Firstly the functional we are working with needs to have a certain geometry, known as the Mountain Pass geometry.

Definition 3 (Mountain Pass Geometry)**.** Let X be a Banach space. A functional $I \in C^1(X, \mathbb{R})$ is said to have the Mountain Pass Geometry if

- $I(0) = 0$.
- There exists $r, a > 0$ such that $I(u) \geq a$ for all $u \in S_r$, $S_r := \{u \in X : \|u\|_X = r\}$.
- There exists $v \in X$ where $\|v\|_X > r$ such that $I(v) \leq 0$.

As a consequence if we have a functional I that has these geometric properties then we can define the family of paths joining $u = 0$ and $u = v$ as

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = v\}. \tag{2.13}$$

We set

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{2.14}$$

It is not automatic that the functional at c has a critical point and we use the Palais-Smale condition (Definition 4) to overcome this problem, an example of why this is so is given in [3, p. 121].

By the above definition we can conclude that for each $\varepsilon > 0$ there exists $u \in X$ such that $c - 2\varepsilon \leq I(u) \leq c + 2\varepsilon$ and $\|I'(u)\|_* < 2\varepsilon$. We can construct a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$, then there exists $\{u_n\}_{n\in\mathbb{N}} \in X$ such that, $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$, i.e. $I'(u_n) \to 0$, as $n \to \infty$. Such a sequence $\{u_n\}_{n\in\mathbb{N}}$ is known as a **Palais-Smale sequence** at level c.

The following compactness condition is required to show that the levels c are in fact critical.

Definition 4 (Palais-Smale condition)**.** Let X be a Banach space, $\psi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function ψ satisfies the $(PS)_c$ condition if any sequence $\{u_n\}_n \subset X$ such that

$$\psi(u_n) \to c, \quad \psi'(u_n) \to 0$$

has a convergent subsequence.

With the notion of Mountain Pass geometry (Definition 3) and the above compactness condition on Palais-Smale sequences (also known as the Palais-Smale condition, $(\mathrm{PS})_c$ for short) we can state the Mountain Pass Theorem of Ambrosetti-Rabinowitz [4].

Theorem 2.3 (Mountain Pass Theorem)**.** *Under the assumption of definition 3, if I satisfies the $(PS)_c$ condition (Definition 4), then c is a critical value of I.*

The proof of the above Theorem is omitted for the sake of brevity but may be found in [55]. The discussions contained in Chapter 5, namely section 5.2.1 (Relation to deformations), offer an insight as to how critical levels are obtained using the Mountain Pass Theorem.

2.3 Mountain Pass solutions to a local Schrödinger problem

We look at a problem similar to one which will be discussed in the latter part of this Thesis where difficulties arising from a low power of our nonlinearity create a situation where the boundedness of Palais-Smale sequences might not be achievable. In this case we look at the scenario where $p \leq 3$ and the so called Ambrosetti-Rabinowitz condition is not satisfied.

Consider the following PDE:

$$-\Delta u + V(x)u = f(u). \tag{2.15}$$

Studying the variational form of (2.15) yields

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} F(u), \quad F(s) := \int_0^s f(t)\,\mathrm{d}t.$$

Since I is a C^1 functional, we can show that critical points of I are solutions to (2.15). To begin we first show that I has the Mountain Pass Geometry (Definition 3). Namely we want to show

$$\Gamma := \{\gamma \in C([0,1], H) \mid \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$$

is nonempty, and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > 0.$$

Methods such as Ekelands variational principle can show that I indeed has the Mountain Pass geometry. This implies the existence of a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

As described in the previous section, at this stage we would like to show I satisfies the Palais-Smale condition. To begin this analysis we need to show u_n is bounded. This fails due to the assumptions placed on f, i.e. due to the absence of the Ambrosetti-Rabinowitz condition: There exists $\epsilon > 2$ such that

$$0 < \epsilon \int_0^s f(\tau)\,\mathrm{d}\tau \le sf(s), \tag{2.16}$$

for all $s \in \mathbb{R}$.

Louis Jeanjean and Kazunaga Tanaka developed a method to overcome these difficulties [28, 29] taking advantage of the so called "monotonicity trick" by Struwe [50]. Their approach is as follows, for $\mu \in [\frac{1}{2}, 1]$, consider the family of functionals $I_\mu : H \to \mathbb{R}$ defined as

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \mu\int_{\mathbb{R}^N} F(u).$$

Thanks to the perturbation by μ it can be shown that for any $\mu \in [\frac{1}{2}, 1]$, there exist bounded Palais-Smale sequences for I_μ. Namely the following Lemma was proved by Jeanjean [28] for generic interval $\mathcal{I}$ for μ,

Lemma 2.3. *Let X be a normed Banach space and let $\mathcal{I} \subset \mathbb{R}^+$ be an interval. Consider the family of C^1-functionals $(I_\mu)_{\mu\in\mathcal{I}}$ on X of the form*

$$I_\mu(w) = A(w) - \mu B(w), \quad \forall \mu \in \mathcal{I},$$

where $B(w) \geq 0$ for all $w \in X$ and such that either, $A(w) \to +\infty$ or $B(w) \to +\infty$ as $\|w\|_X \to +\infty$. Assume there are two points (v_0, v_1) in X such that setting

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_0, \gamma(1) = v_1\}$$

the level

$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(v_0), I_\mu(v_1)\}$$

holds for all $\mu \in \mathcal{I}$. Then, for almost every $\mu \in \mathcal{I}$, there exists a sequence $\{v_n\} \subset X$ such that

1. *$\{v_n\}$ is bounded,*
2. *$I_\mu(v_n) \to c_\mu$,*
3. *$I'_\mu(v_n) \to 0$ in X^{-1} (The dual of X).*

It is important that Γ is defined independent of μ and $\mu \mapsto c_\mu$ be monotone, hence c'_μ exists almost everywhere to be able to prove the existence of the Palais-Smale sequences. The results of Jeanjean [28] yielded continued research by Jeanjean and Tanaka [29] resulting in the following Theorem. In this case the above Lemma 2.3 shows the existence of bounded Palais-Smale sequences for almost every $\mu \in \mathcal{I}$ by means of Struwe's [51] "monotonicity trick". The sequel shows the sequence of critical points obtained from the family of perturbed functionals form a bounded Palais-Smale sequence for an unperturbed functional I in the case $\mu \in [\frac{1}{2}, 1]$.

Theorem 2.4. *Suppose $f \in C(\mathbb{R}^+, \mathbb{R})$ satisfying:*

(f_1) $f(0) = 0$ *and* $f'(0)$*, defined as* $\lim_{s\to 0^+} f(s)s^{-1}$*, exists,*

(f_2) there is $p < \infty$ if $N = 2$ or $p < (N+2)/(N-2)$ if $N \geq 3$ such that $\lim_{s\to+\infty} f(s)s^{-p} = 0$,

(f_3) $\lim_{s\to+\infty} = f(s)s^{-1} + \infty.$

Suppose further the following conditions on $V(x)$:

(V_1) $f'(0) < \inf \sigma(-\Delta + V(x))$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, i.e.,

$$\inf \sigma(-\Delta + V(x)) = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x},$$

(V_2) $V(x) \to V(\infty) \in \mathbb{R}$ as $|x| \to +\infty$,

(V_3) $V(x) \le V(\infty)$, a.e., $x \in \mathbb{R}^N$,

(V_4) there exists a function $\varphi \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$|x| \, |\nabla V(x)| \le \varphi(x)^2, \quad \forall \; x \in \mathbb{R}^N.$$

Then there exists a nontrivial positive solution to

$$-\Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N)$$

for $N \ge 2$.

The full proof will not be given and can be found in [29], instead a brief sketch will be provided.

Sketch of proof for Theorem 2.4. The core concept involved is showing that the sequence of critical points obtained from Lemma 2.3 $\{u_j\} \subset H^1(\mathbb{R}^N)$ of I_{μ_j} where $\mu \in [\frac{1}{2}, 1]$ and $\mu_j \nearrow 1$. The proof involves the use of a Pohozaev type identity (Similar to the one we provide in Lemma 3.9) which is used to show that the sequence $\{u_j\} \subset H^1(\mathbb{R}^N)$ is bounded. By the boundedness of $\{u_j\}$ and the fact the critical levels of I_μ, given by c_μ in Lemma 2.3 are monotone, Jeanjean and Tanaka prove that this sequence is infact a bounded Palais-Smale sequence for I which satisfy $\sup_{j\to\infty} I(u_j) \le c$ and $\|u_j\|_{H^1(\mathbb{R}^N)} \not\to 0$. This means there exists a bounded Palais-Smale sequence for I at the level c which does not converge to 0, hence by the Mountain Pass theorem there exists a nontrivial solution to (2.15). $\square$

The above describes an interesting situation where even the boundedness of the Palais-Smale sequences is not obvious. To look at how one might find Mountain Pass solutions in a setting which the Ambrosetti-Rabinowitz condition is satisfied we can look at the problem described in Theorem 2.2. In this case we look at the problem

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \\ u > 0. \end{cases} \tag{2.17}$$

where $N \geq 2$, $V(x) \in \mathcal{C}(\mathbb{R}^N)$ such that $V(x) \geq c > 0$ and $V(x) \leq \lim_{|y|\to\infty} V(y) = +\infty$ for all $x \in \mathbb{R}^N$, $p \in (1, 2^* - 1)$.

The above has the corresponding energy functional

$$I(u) = \frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}.$$

To apply the Mountain Pass Theorem 2.3 we have three requirements; To show that I satisfies the Mountain Pass geometry (Definition 3), show that the Palais-Smale sequences for I are bounded, show that I satisfies the Palais-Smale condition (Definition 4).

To show that I satisfies the MPG, consider $u_t := tu(x)$, then

$$I(u_t) = \frac{t^2}{2}\|u\|^2_{H^1_V(\mathbb{R}^N)} - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}$$

It is clear that 0 is a local minimum and $I(u_t) \to -\infty$ as $t \to \infty$. Moreover, $I(u_t) > 0$ when $t \in (0, r)$, where r is such that, for all $t > r$, $I(u_t) \leq 0$. Hence, I satisfies the MPG.

To show boundedness of the Palais-Smale sequences for I we can see by definition of a Palais-Smale sequence:

$$C + o(1)\|u\|_{H^1_V(\mathbb{R}^N)} \geq (p+1)I(u_n) - I'(u_n)u_n = \frac{p-1}{2}\|u_n\|^2_{H^1_V(\mathbb{R}^N)}.$$

Assuming $\|u_n\|_{H^1_V(\mathbb{R}^N)} \to +\infty$ as $u_n \to \infty$ we can see there is a contradiction as the RHS will grow faster than the LHS. Thus, u_n is bounded in $H^1_V(\mathbb{R}^N)$.

Finally the compactness of the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is given by Lemma 2.2. Therefore we can apply the Mountain Pass Theorem 2.3 to (2.17) giving the existence of a weak solution $u^* \in H^1_V(\mathbb{R}^N)$ such that $I(u^*) = c$ and $I'(u^*) = 0$ where c is given by (2.14).

2.4 Poisson Equation

In this section, we discuss some preliminary results on the Poisson equation. The space we use is the space of distributions $D^{1,2}(\mathbb{R}^N)$ as defined in (1.7). The results of this section are from a critical review from lectures presented to myself by Carlo Mercuri during his time at Swansea University.

Proposition 1. *Assume $N \geq 3$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function and $\rho u^2 \in L^1_{loc}(\mathbb{R}^N)$, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho(x)u^2(x)\rho(y)u^2(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Then,

$$\phi_u := \frac{1}{\omega\,|x|^{N-2}} \star \rho u^2$$

is the unique weak solution to $-\Delta\phi = \rho u^2$ in $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid \partial_i u \in L^2(\mathbb{R}^N)\}$. Moreover, it holds that

$$\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \phi_u \rho u^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho(x)u^2(x)\rho(y)u^2(y)}{\omega\,|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. Firstly we will show the uniqueness.
Uniqueness

$$\begin{cases} -\Delta\phi_1 = \rho u^2 \\ -\Delta\phi_2 = \rho u^2 \end{cases} \quad \text{in } D^{1,2}(\mathbb{R}^N),$$

with $\phi_1 - \phi_2$ an subtracting we obtain

$$\int_{\mathbb{R}^N} |\nabla(\phi_1 - \phi_2)|^2 = 0,$$

hence, by Sobolev's inequality, $\phi_1 \equiv \phi_2$.
Observe also that the identity

$$\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \phi_u \rho u^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho(x)u^2(x)\rho(y)u^2(y)}{\omega\,|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y$$

easily follows testing $-\Delta\phi_u = \rho u^2$ with ϕ_u.

Finally, we are left to show that $\phi_u \in D^{1,2}(\mathbb{R}^N)$ and that $\int_{\mathbb{R}^N} \nabla\phi_u \nabla\Psi = \int_{\mathbb{R}^N} \rho u^2 \Psi$ for all $\Psi \in D^{1,2}(\mathbb{R}^N)$. Namely, that ϕ_u is the weak solution to $-\Delta\phi = \rho u^2$ in $D^{1,2}(\mathbb{R}^N)$. We break the proof into multiple steps.

Step 1: Set $f_n = \min(\rho^{\frac{1}{2}}u, n)\chi_{\{|x|<n\}}$. We note that:

(i) f_n has compact support,

(ii) f_n is nondecreasing,

(iii) $f_n \nearrow \rho^{\frac{1}{2}}u$ almost everywhere,

(iv) $f_n \in L^\infty(\mathbb{R}^N)$.

Claim: Then $\phi_n := \int_{\mathbb{R}^N} \frac{f_n^2(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}y$ *is such that*

$$-\int_{\mathbb{R}^N} \phi_n \Delta\Psi = \int_{\mathbb{R}^3} f_n^2 \Psi, \tag{2.18}$$

for all $\Psi \in C_c^\infty(\mathbb{R}^N)$. *Moreover* $\phi_n \in C^1(\mathbb{R}^N)$ *and* $\|\phi_n\|_{D^{1,2}(\mathbb{R}^N)} < C$ *for all* n.

In fact (2.18) follows by Theorem 6.21 in [33], from which

$$\partial_i \phi_n = \int_{\mathbb{R}^N} \frac{\partial G_y}{\partial x_i}(x) f_n^2(y)\,\mathrm{d}y$$

where

$$G_y(x) := \frac{1}{(N-2)\,|\mathbb{S}^{N-1}|\,|x-y|^{N-2}} = \frac{1}{\omega\,|x-y|^{N-2}}$$

is the Greens function for $N \geq 3$,

$$\frac{\partial G_y}{\partial x_i} = \frac{2-N}{\omega}\,|x-y|^{-N}\,(x_i - y_i)$$

and

$$\int_{\mathbb{R}^N} G_y(x)\Delta\Psi(x)\,\mathrm{d}x = -\Psi(y)$$

for all $\Psi \in C_c^\infty(\mathbb{R}^N)$. By (iv) and Theorem 10.2 in [33] $\phi_n \in C^1(\mathbb{R}^N)$. Moreover

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla\phi_n|^2 &= \sum_{i=1}^N \int_{\mathbb{R}^N} (\partial_i \phi_n(z))^2\,\mathrm{d}z \\
&= \sum_{i=1}^N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{1}{\omega}\frac{z_i - y_i}{|z-y|^N} f_n^2(y)\,\mathrm{d}y\right)\left(\int_{\mathbb{R}^N} \frac{1}{\omega}\frac{z_i - x_i}{|z-x|^N} f_n^2(x)\,\mathrm{d}x\right)\mathrm{d}z \\
&\leq \sum_{i=1}^N \int_{\mathbb{R}^N} f_n^2(y)\,\mathrm{d}y \int_{\mathbb{R}^N} f_n^2(x)\,\mathrm{d}x \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-1}}\frac{1}{|z-x|^{N-1}}\,\mathrm{d}z.
\end{aligned}$$

Using the substitution $w = z - y$, the final integral becomes

$$\int_{\mathbb{R}^N} \frac{1}{|w|^{N-1}} \cdot \frac{1}{|w-(x-y)|^{N-1}}\,\mathrm{d}w.$$

By the semigroup property of Riesz potentials [32, p.45], it holds that

$$\int_{\mathbb{R}^N} \frac{1}{|w|^{N-1}} \cdot \frac{1}{|w-(x-y)|^{N-1}}\,\mathrm{d}w = C \cdot \frac{1}{|x-y|^{N-2}}$$

for some constant $C > 0$. Hence

$$\int_{\mathbb{R}^N} |\nabla \phi_n|^2 \lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\rho(x)u^2(x)\rho(y)u^2(y)}{|x-y|^{N-2}} \,\mathrm{d}x\,\mathrm{d}y$$

is the claimed uniform bound.

Step 2: Since $\{\phi_n\}_{n\in\mathbb{N}} \subset D^{1,2}(\mathbb{R}^N)$ is uniformly bounded in $D^{1,2}(\mathbb{R}^N)$, $\phi_n \rightharpoonup \phi \in D^{1,2}(\mathbb{R}^N)$ by Banach-Alaoglu.

Claim: It holds that $\phi = \phi_u := \frac{1}{\omega|x|^{N-2}} \star \rho u^2$.

Indeed by Theorem 8.6 in [33], $\phi_n \to \phi$ in $L^p_{loc}(\mathbb{R}^N)$, $p \leq 2^*$ and a.e. hence, by monotone convergence,

$$\phi_n = \int_{\mathbb{R}^N} \frac{f_n^2(y)}{\omega\,|x-y|^{N-2}} \,\mathrm{d}y \to \phi_u$$

a.e. and we conclude by uniqueness of the limit. Roughly speaking $\{\phi_n\}$ approximate ϕ_u weakly in $D^{1,2}(\mathbb{R}^N)$, a.e., $\phi_u \in D^{1,2}(\mathbb{R}^N)$.

Step 3: We claim that

$$\int_{\mathbb{R}^N} \nabla\phi_u \nabla\Psi = \int_{\mathbb{R}^N} \rho u^2 \Psi$$

for all $\Psi \in D^{1,2}(\mathbb{R}^N)$. Note that $\int_{\mathbb{R}^N} \nabla\phi_n\nabla\Psi = \int_{\mathbb{R}^N} f_n^2\Psi$ for all $\Psi \in C_c^\infty(\mathbb{R}^N)$. Indeed, if $\Psi \in C_c^\infty(\mathbb{R}^N)$ by Fubini, integration by parts and by $-\Delta G_y = \delta_y$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\phi_n\nabla\Psi &= \int_{\mathbb{R}^N} \sum_{i=1}^N \left(\int_{\mathbb{R}^N} \frac{\partial G_y}{\partial x_i}(x) f_n^2(y)\,\mathrm{d}y \right) \partial_i\Psi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} G_y(x)(-\Delta\Psi(x))\,\mathrm{d}x \right) f_n^2(y)\,\mathrm{d}y.\end{aligned}$$

From this, $\int_{\mathbb{R}^N} \nabla\phi_n\nabla\Psi = \int_{\mathbb{R}^N} f_n^2\Psi$ follows for all $\Psi \in C_c^\infty(\mathbb{R}^N)$. Hence, since $\phi_n \rightharpoonup \phi_u$ we have $\int_{\mathbb{R}^N} \nabla\phi_u\nabla\Psi = \int_{\mathbb{R}^N} \rho u^2\Psi$ for all $\Psi \in C_c^\infty(\mathbb{R}^N)$. Since $D^{1,2}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\nabla\cdot\|_{L^2(\mathbb{R}^N)}}$ note that $\int_{\mathbb{R}^N} \nabla\phi_n\nabla\Psi = \int_{\mathbb{R}^N} f_n^2\Psi$ holds for all $\Psi \in D^{1,2}(\mathbb{R}^N)$.

Pick $\Psi_k \xrightarrow{D^{1,2}(\mathbb{R}^N)} \Psi$, since it holds that

$$\int_{\mathbb{R}^N} \nabla\phi_u\nabla\Psi_k = \int_{\mathbb{R}^N} \rho u^2 \Psi_k.$$

and that $\int_{\mathbb{R}^N} \nabla\phi_u\nabla\Psi_k \to \int_{\mathbb{R}^N} \nabla\phi_u\nabla\Psi$, we are left to show $\int_{\mathbb{R}^N} \rho u^2\Psi_k \to \int_{\mathbb{R}^N} u^2\Psi$. Namely, we shall prove that $\Psi \mapsto \int_{\mathbb{R}^N} \rho u^2\Psi$ is a linear and continuous functional in $D^{1,2}(\mathbb{R}^N)$. To show this test $-\Delta\phi_n = f_n^2$ with $|\Psi_k| \in D^{1,2}(\mathbb{R}^N)$, (this is an admissible test function by Stampacchia's classical result):

$$\int_{\mathbb{R}^N} f_n^2 |\nabla \Psi_k| = \int_{\mathbb{R}^N} \nabla \phi_n \nabla |\Psi_k| \leq \|\nabla \phi_n\|_{L^2(\mathbb{R}^N)} \|\nabla \Psi_k\|_{L^2(\mathbb{R}^N)}.$$

Hence by Fatou, as $n \to \infty$

$$\int_{\mathbb{R}^N} \rho u^2 |\Psi_k| \leq c \|\nabla \Psi_k\|_{L^2(\mathbb{R}^N)}.$$

Hence by Fatou again and $\Psi_k \to \Psi$ as $k \to \infty$ we have

$$\left| \int_{\mathbb{R}^N} \rho u^2 \Psi \right| \leq \left| \int_{\mathbb{R}^N} \rho u^2 |\Psi| \right| \leq c \|\Psi\|_{D^{1,2}(\mathbb{R}^N)}$$

and this completes the proof. □

2.5 Relations between Local Schrödinger and Schrödinger-Poisson

The most obvious difference between these two type of problems is the existence of a nonlocal term in the Schrödinger-Poisson case. One might ask how this effects our repertoire of tools highlighted above as the most common approach is to tackle the Schrödinger-Poisson problem as a single Schrödinger type equation by substituting the solution of the Poisson equation into the Schrödinger problem as a nonlocal term.

The initial problem that arises here, as may have been noticed by the reader after reading the previous section, is the space in which we work. Obtaining solutions to the Poisson equation in section 2.4 requires us to work in the space $D^{1,2}(\mathbb{R}^N)$ whereas all our examples on the Schrödinger problem involved working in $H^1(\mathbb{R}^N)$. The key issue that arises is finding a suitable functional space. From the analysis on the Poisson equation we see that the boundedness of the double integral term is vital, when working in other spaces it is not obvious whether this term is bounded when using the Hardly-Littlewood-Sobolev inequality as shown in (HLS), instead we work in the space $E(\mathbb{R}^N) \subseteq H^1(\mathbb{R}^N)$ to ensure this bound exists. The space $E(\mathbb{R}^N)$ is good enough for us find solutions to the Schrödinger-Poisson system, however, some key properties of Hilbert spaces are not retained, for instance we do not have the inner product structure that is so convenient for Hilbert spaces.

Many of the techniques used in the Schrödinger equation can be applied to Schrödinger-Poisson albeit with some caveats. While the introduction of a nonlocal term does not necessarily mean we no longer satisfy the Ambrosetti-Rabinowitz condition (2.16), in cases where $q \in (2,3)$ the condition is no longer satisfied, instead the method described in section 2.3 by Jeanjean-Tanaka [29] is required as we can no longer show Palais-Smale sequences bounded in this scenario.

In our work we look to Ambrosetti-Ruiz [5]. Their generalisation of results shown by Jeanjean-Tanaka to a more general Schrödinger-Poisson system is effective at overcoming the problems we face when $q \in (2,3)$ and in search of possible multiplicity of solutions within this range.

3 Preliminaries

Some preliminaries are in order before we may discuss the intricacies of the Poisson-Schrödinger system described in ($\mathcal{SP}$). In this chapter many of the properties can be found in [23] and will lay out the foundations of the functional setting in which we will be working and the properties of our functional I_λ (1.1). We end this chapter with some remarks on positivity and regularity of solutions to ($\mathcal{SP}$) and a discussion on what happens when the exponent of our nonlinearity $q \leq 2$, namely, we can show nonexistence of nontrivial solutions.

3.1 Functional Setting

We introduce some properties of our functional setting $E(\mathbb{R}^N)$. The following Lemma is vital for proving multiplicity results for (1.2).

Lemma 3.1 (Properties of $E(\mathbb{R}^N)$). *Assume $N \geq 3$, and $\rho \geq 0$ is in $L^\infty_{loc}(\mathbb{R}^N)$. The space $E(\mathbb{R}^N)$ is a separable Banach space that admits a Markushevic basis, that is a fundamental and total biorthogonal system, $\{(e_m, e_m^*)\}_{m\in\mathbb{N}} \subset E(\mathbb{R}^N) \times E^*(\mathbb{R}^N)$. Namely, $< e_n, e_m^* >= \delta_{nm}$ for all $n, m \in \mathbb{N}$, the e_m's are linearly dense in $E(\mathbb{R}^N)$, and the weak*-closure of span$\{e_m^*\}_{m\in\mathbb{N}}$ is $E^*(\mathbb{R}^N)$.*

Proof. Following [40], we note that we can equip $E(\mathbb{R}^N)$ with the equivalent norm

$$\|u\|_{E^{1,2}(\mathbb{R}^N)} = \left(\|u\|^2_{H^1(\mathbb{R}^N)} + \lambda\left(\int_{\mathbb{R}^N}\left|I_1 \star (\sqrt{\rho}|u|)^2\right|^2\right)^{1/2}\right)^{1/2}. \tag{3.1}$$

Here, we have set $\alpha = 1$ in $I_\alpha : \mathbb{R}^N \to \mathbb{R}$, the Riesz potential of order $\alpha \in (0, N)$, defined for $x \in \mathbb{R}^N \setminus \{0\}$ as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha},$$

and the choice of normalisation constant A_α ensures that the kernel I_α enjoys the semigroup property

$$I_{\alpha+\beta} = I_\alpha \star I_\beta \text{ for each } \alpha, \beta \in (0, N) \text{ such that } \alpha + \beta < N.$$

We first notice that the operator $T : E(\mathbb{R}^N) \to H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ defined by

$$(Tu)(x_0, x_1, x_2) = [u(x_0), (\lambda I_1(x_2 - x_1)\rho(x_1))^{\frac{1}{2}} u(x_1)],$$

is a linear isometry from $E(\mathbb{R}^N)$ into the product space $H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$, endowed with the norm

$$\|[u,v]\|_\times = \left(\|u\|^2_{H^1(\mathbb{R}^N)} + \|v\|^2_{L^4(\mathbb{R}^N;L^2(\mathbb{R}^N))}\right)^{1/2}.$$

Here $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ is the mixed norm Lebesgue space of functions $v : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that

$$||v||_{L^4(\mathbb{R}^N;L^2(\mathbb{R}^N))} = \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |v(x_1,x_2)|^2 \, dx_1\right)^2 dx_2\right)^{1/4} < +\infty,$$

see [12]. Since $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ is a separable (see e.g. [44, p. 107]) Banach space (see e.g. [12]), it follows that the linear subspace $T(E(\mathbb{R}^N)) \subseteq H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$, and hence $E(\mathbb{R}^N)$, also satisfies each of these properties. Since every separable Banach space admits a Markushevic basis (see e.g. [27]), the proof is complete. □

Reasoning as in [47] and [40] it is easy to see that $C^\infty_c(\mathbb{R}^N)$ is dense in $E(\mathbb{R}^N)$ and that the unit ball in $E(\mathbb{R}^N)$ is weakly compact, moreover, this space is uniformly convex and hence is reflexive. The following variant to the classical Brezis-Lieb lemma will be useful to study the convergence of bounded sequences in $E(\mathbb{R}^N)$, some examples can be found in [8] and [40].

Lemma 3.2 (**Nonlocal Brezis-Lieb lemma**). *Assume $N \geq 3$ and $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. Let $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ be a bounded sequence such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Then it holds that*

$$\lim_{n\to\infty} \left[\|\nabla\phi_{u_n}\|^2_{L^2(\mathbb{R}^N)} - \|\nabla\phi_{(u_n-u)}\|^2_{L^2(\mathbb{R}^N)}\right] = \|\nabla\phi_u\|^2_{L^2(\mathbb{R}^N)}.$$

Proof. The proof follows [40, Proposition 4.2] and [8] replacing u with $u\sqrt{\rho}$. □

The next simple estimate is based on an observation of P.-L. Lions, given in [37] for $\rho \equiv 1$; other forms of this estimate can be found in [47], and [9], [40].

Lemma 3.3 (**Coulomb-Sobolev inequality**). *Assume $N \geq 3$, $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. Then the following inequality holds for all $u \in E(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \rho(x)\,|u|^3 \leq \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla\phi_u|^2\right)^{\frac{1}{2}}. \tag{3.2}$$

Proof. Testing the Poisson equation (1.5) with $|u|$, the statement follows immediately by Cauchy-Schwarz inequality. □

3.2 Properties of I

The present section looks at the min-max properties of I_λ. Here we will discuss the Mountain-Pass geometry for our functional and its relation with groundstate solutions. We also include some relevant uniform lower bounds on Palais-Smale sequences.

Lemma 3.4 (Mountain-Pass Geometry for I_λ). *Assume $N = 3, 4, 5$, $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative and $q \in (2, 2^* - 1]$. Then, it holds that*

- *(i) $I_\lambda(0) = 0$ and there exist constants $r, a > 0$ such that $I_\lambda(u) \geq a$ if $\|u\|_{E(\mathbb{R}^N)} = r$;*
- *(ii) there exist $v \in E(\mathbb{R}^N)$ with $\|v\|_{E(\mathbb{R}^N)} > r$ such that $I_\lambda(v) \leq 0$.*

Proof. Statement (i) follows reasoning as in Lemma 3.8. To show (ii), pick $u \in C^1(\mathbb{R}^N)$, supported in the unit ball, B_1. Setting $v_t(x) := t^2 u(tx)$ we find that

$$\begin{aligned} I_\lambda(v_t) = \frac{t^{6-N}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{4-N}}{2} \int_{\mathbb{R}^N} u^2 \\ + \frac{t^{6-N}}{4} \lambda^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{\omega|x-y|^{N-2}} \,\mathrm{d}y \,\mathrm{d}x \\ - \frac{t^{(2q+2-N)}}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}. \end{aligned} \tag{3.3}$$

Since for every $t \geq 1$ and for almost every $x \in B_1$ we have $\rho(x/t) \leq ||\rho||_{L^\infty(B_1)}$, the fact that $2q + 2 > 6$ in (3.3) yields $I_\lambda(v_t) \to -\infty$ as $t \to +\infty$, and this is enough to conclude the proof. □

To prove our results for $q < 3$, we will need to work with a perturbed functional, $I_{\mu,\lambda} : E(\mathbb{R}^N) \to \mathbb{R}^N$, defined by

$$I_{\mu,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{\lambda^2}{4} \int_{\mathbb{R}^N} \rho \phi_u u^2 - \frac{\mu}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}, \tag{3.4}$$

for $\mu \in \left[\frac{1}{2}, 1\right]$. As in Lemma 3.4, $I_{\mu,\lambda}$ has the mountain-pass geometry in $E(\mathbb{R}^N)$ for all $\mu \in \left[\frac{1}{2}, 1\right]$. This, as well as the monotonicity of $I_{\mu,\lambda}$ with respect to μ, imply that we can define the min-max level associated with $I_{\mu,\lambda}$ as

$$c_{\mu,\lambda} = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_{\mu,\lambda}(\gamma(t)), \quad \mu \in \left[\frac{1}{2}, 1\right] \tag{3.5}$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], E(\mathbb{R}^N)) : \gamma(0) = 0,\ I_{\frac{1}{2},\lambda}(\gamma(1)) < 0\}. \tag{3.6}$$

Since the mapping $[1/2, 1] \ni \mu \mapsto c_{\mu,\lambda}$ is non-increasing and left-continuous in μ (this has been discussed in 2.3 briefly and may also be found in [5, Lemma 2.2]) and the non-perturbed functional I_λ has the mountain-pass geometry by Lemma 3.4, we are now in position to define the min-max level associated with I_λ for all $q \in (2, 2^* - 1)$.

Definition 5 (**Definition of mountain-pass level for I_λ**)**.** We set

$$c_\lambda = \begin{cases} c_{1,\lambda}, & q \in (2,3), \\ \inf\limits_{\gamma \in \bar{\Gamma}_\lambda} \max\limits_{t \in [0,1]} I_\lambda(\gamma(t)), & q \in [3, 2^* - 1), \end{cases} \tag{3.7}$$

where $c_{1,\lambda}$ is given by (3.5) and $\bar{\Gamma}_\lambda$ is the family of paths defined as

$$\bar{\Gamma}_\lambda = \left\{ \gamma \in C([0,1]; E(\mathbb{R}^N)) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \right\}. \tag{3.8}$$

In order to show the relation between mountain-pass solutions and ground-state solutions we provide the following Lemma from [55]. First we define

$$\mathcal{N}_\lambda := \left\{ u \in E(\mathbb{R}^N) \setminus \{0\} \ : \ I'_\lambda(u)u = 0 \right\}, \tag{3.9}$$

as the Nehari manifold.

Lemma 3.5. *Let $q \in (3, 2^* - 1)$, $N = 3$. Suppose $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonegative and assume there exists $t_0 =: t(u) > 0$ such that $t(u) \cdot u \in \mathcal{N}_\lambda$ and $t(u)$ is the maximum of $g(t) := I_\lambda(t(u))$. Furthermore, assume*

$$c_{\mathcal{N}} := \inf_{\mathcal{N}_\lambda} I_\lambda(u),$$
$$c_t := \inf_{\substack{u \in E(\mathbb{R}^N) \\ u \neq 0}} \max_{t \geq 0} I_\lambda(t(u)).$$

is such that $u \mapsto t(u)$ is continuous and the map $u \to t(u) \cdot u$ defines a homeomorphism of the unit sphere of E with $\mathcal{N}_\lambda$. Then, we have that

$$c_{\mathcal{N}} = c_t = c_\lambda > 0,$$

where c_λ is defined by (3.7).

Proof. The proof may be found in [55] and is omitted here. □

The next few Lemmas are devoted to further characterisations of the min-max level c_λ for $q \leq 3$. This case is extremely delicate and depends largely on some homogeneity conditions on $\rho(x)$. We first require the following technical lemma.

Lemma 3.6. *Suppose $N \geq 3$, $q > 2$ and $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$. Let $\bar{k} \in \left(\frac{\nu(3-q)-2}{2}, \frac{4\nu-N-2}{2}\right)$. Define $f : \mathbb{R}_0^+ \to \mathbb{R}$ as*

$$f(t) = at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t \geq 0,$$

where $a, b, c, d \in \mathbb{R}$ are such that $a, b, d > 0$, $c \geq 0$. Then, f has a unique critical point corresponding to its maximum.

Remark 3.1. We point out that our range of parameters ensures that $f(t) \to -\infty$ as $t \to +\infty$ and it holds that

$$\left(\frac{\nu(3-q)-2}{2}, \frac{4\nu-N-2}{2}\right) \bigcap \left(\frac{(\nu+1)(3-q)-2}{2}, \frac{4(\nu+1)-N-2}{2}\right) \neq \emptyset.$$

In Theorem 4.4 and Theorem 6.3, we use Lemma 3.6, assuming

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\}(3-q) - 1$$

for $\bar{k}$ to belong to one of these intervals.

Proof of Lemma 3.6. Note that by our assumptions, we can write

$$f(t) = \sum_{i=1}^{k} a_i t^{p_i} - t^p,$$

where $a_i \geq 0$, $0 \leq p_i < p$ and both a_i, $p_i \neq 0$ for some i. Setting $s = t^p$, we find

$$f(s) = \sum_{i=1}^{k} a_i s^{\frac{p_i}{p}} - s.$$

It follows that $f(s)$ is strictly concave and has a unique critical point, which is a maximum. Since our assumptions ensure that $f(t) \to -\infty$ as $t \to +\infty$, we can conclude. □

Lemma 3.5 is only possible for very specific cases for our problem, namely when $N = 3$ and $q \in (3, 2^* - 1)$, therefore we need another result to show this relation in cases when $q \in (2, 3]$ or higher dimensions. To state our next result, for any $\nu \in \mathbb{R}$, we set

$$\bar{\mathcal{M}}_{\lambda,\nu} = \left\{u \in E(\mathbb{R}^N) \setminus \{0\} : J_{\lambda,\nu}(u) = 0\right\}, \tag{3.10}$$

where $J_{\lambda,\nu} : E(\mathbb{R}^N) \to \mathbb{R}^N$ is defined as

$$J_{\lambda,\nu}(u) = \frac{2\nu+2-N}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{2\nu-N}{2}\int_{\mathbb{R}^N}u^2 \\ + \frac{4\nu-N-2-2\bar{k}}{4}\cdot\lambda^2\int_{\mathbb{R}^N}\rho\phi_u u^2 - \frac{\nu(q+1)-N}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}. \tag{3.11}$$

Notice that, if ρ is homogeneous of order $\bar{k}$, $J_{\lambda,\nu}(u)$ is the derivative of the polynomial $f(t) = I_\lambda(t^\nu u(t\cdot))$ at $t = 1$.

Proposition 2 (Mountain-pass characterisation of groundstates). *Let $N = 3, 4, 5$, $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative and is homogeneous of degree $\bar{k}$, namely $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\}(3-q) - 1.$$

Then, there exists $\nu > \max\{\frac{N}{2}, \frac{2}{q-1}\}$ such that

$$c_\lambda = \inf_{u\in\bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u) = \inf_{u\in E(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)),$$

where c_λ and $\bar{\mathcal{M}}_{\lambda,\nu}$ are defined in (3.7) and (3.10), respectively.

Proof. We first note that under the assumptions on the parameters, it holds that

$$\frac{4\nu-N-2}{2} > \frac{(\nu+1)(3-q)-2}{2}.$$

It follows from this and the lower bound assumption on $\bar{k}$ that we can always find at least one interval

$$\left(\frac{\nu(3-q)-2}{2}, \frac{4\nu-N-2}{2}\right), \quad \text{with } \nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\},$$

that contains $\bar{k}$. We fix ν corresponding to such an interval. We break the remainder of the proof into a series of claims.

Claim 1. $\inf_{u\in E(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) \leq \inf_{u\in\bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u)$

To see this, let $u \in E(\mathbb{R}^N) \setminus \{0\}$ be fixed and consider the function

$$\begin{aligned} g(t) &= I_\lambda(t^\nu u(t\cdot)) \\ &= at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t \geq 0, \end{aligned} \tag{3.12}$$

where

$$a = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2, \ b = \frac{1}{2}\int_{\mathbb{R}^N} u^2, \ c = \frac{\lambda^2}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2, \ d = \frac{1}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}.$$

By Lemma 3.6, it holds that g has a unique critical point, $t = \tau_u$, corresponding to its maximum. Moreover, we can see that

$$\begin{aligned} g'(t) &= \frac{\mathrm{d}I_\lambda(t^\nu u(t\cdot))}{\mathrm{d}t} \\ &= \frac{2\nu+2-N}{2} \cdot t^{2\nu+1-N}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2\nu-N}{2} \cdot t^{2\nu-N-1}\int_{\mathbb{R}^N} u^2 \\ &\quad + \frac{4\nu-N-2-2\bar{k}}{4} \cdot t^{4\nu-N-3-2\bar{k}} \cdot \lambda^2 \int_{\mathbb{R}^N} \rho\phi_u u^2 \\ &\quad - \frac{\nu(q+1)-N}{q+1} \cdot t^{\nu(q+1)-N-1}\int_{\mathbb{R}^N} |u|^{q+1}, \end{aligned}$$

and so

$$g'(t) = 0 \iff t^\nu u(t\cdot) \in \bar{\mathcal{M}}_{\lambda,\nu}.$$

Taken together, we have shown that for any $u \in E(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t = \tau_u$ such that $\tau_u^\nu u(\tau_u\cdot) \in \bar{\mathcal{M}}_{\lambda,\nu}$ and the maximum of $I_\lambda(t^\nu u(t\cdot))$ for $t \geq 0$ is achieved at τ_u. Thus, it holds that

$$\inf_{u\in E(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) \leq \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) = I_\lambda(\tau_u^\nu u(\tau_u\cdot)), \quad \forall u \in E(\mathbb{R}^N) \setminus \{0\},$$

from which we can deduce that the claim holds.

Claim 2. $c_\lambda \leq \inf_{u\in E(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot))$.

By the assumptions on our parameters, we can deduce that $\nu(q+1) - N > 2\nu + 2 - N$ and $\nu(q+1) - N > 4\nu - N - 2 - 2\bar{k}$. It follows that $I_\lambda(t^\nu u(t\cdot)) < 0$ for every $u \in E(\mathbb{R}^N) \setminus \{0\}$ and t large. Similarly, $I_{\frac{1}{2},\lambda}(t^\nu u(t\cdot)) < 0$ for every $u \in E(\mathbb{R}^N) \setminus \{0\}$ and t large. Therefore, we obtain

$$c_\lambda \leq \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)), \quad \forall u \in E(\mathbb{R}^N) \setminus \{0\},$$

and the claim follows.

Claim 3. $\inf_{u\in\bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u) \le c_\lambda$.

We define

$$A_{\lambda,\nu} = \left\{u \in E(\mathbb{R}^N) \setminus \{0\} : J_{\lambda,\nu}(u) > 0\right\} \cup \{0\},$$

and first note that $A_{\lambda,\nu}$ contains a small ball around the origin. Indeed, arguing as in the proof of Lemma 3.8, we can show that for every $u \in E(\mathbb{R}^N) \setminus \{0\}$ and any $\beta > 0$, we have

$$\begin{aligned} J_{\lambda,\nu}(u) \ge \frac{2\nu - N}{2}||u||^2_{H^1(\mathbb{R}^N)} &- \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4}\right)||u||^4_{H^1(\mathbb{R}^N)} \\ &+ \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} \\ &- \frac{S_{q+1}^{-(q+1)}(\nu(q+1) - N)}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}. \end{aligned}$$

We now pick $\delta = \left(\frac{(2\nu-N)(q+1)S_{q+1}^{q+1}}{4(\nu(q+1)-N)}\right)^{1/(q-1)}$ and note that since $\nu > \frac{N}{2}$, it follows that $\delta > 0$. We assume $||u||_{E(\mathbb{R}^N)} < \delta$ and choosing $\beta > 1$ sufficiently near 1 we obtain

$$\begin{aligned} J_{\lambda,\nu}(u) &\ge \left[\frac{2\nu - N}{4} - \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4}\right)\delta^2\right]||u||^2_{H^1(\mathbb{R}^N)} \\ &\quad + \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} \\ &\ge \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)}, \end{aligned}$$

which is strictly positive by our choice of ν. This is enough to prove that $A_{\lambda,\nu}$ contains a small ball around the origin. Now, notice that if $u \in A_{\lambda,\nu}$, then $g'(1) > 0$, where g is defined in (3.12). Since $g(0) = 0$ and we showed in Claim 1 that τ_u is the unique critical point of g corresponding to its maximum, it follows that $1 < \tau_u$. Using the facts that $I_\lambda(0) = 0$ and $g'(t) = \frac{\mathrm{d}I_\lambda(t^\nu u(t\cdot))}{\mathrm{d}t} \ge 0$ for all $t \in [0, \tau_u]$, we obtain that $I_\lambda(t^\nu u(t\cdot)) \ge 0$ for all $t \in [0, \tau_u]$ and, in particular, at $t = 1$. Thus, we have shown $I_\lambda(u) \ge 0$, which also implies that $I_{\frac{1}{2},\lambda}(u) \ge 0$, for every $u \in A_{\lambda,\nu}$. Therefore, every $\gamma \in \Gamma_\lambda$ and every $\gamma \in \bar{\Gamma}_\lambda$, where Γ_λ and $\bar{\Gamma}_\lambda$ are given by (3.6) and (3.8) respectively, has to cross $\bar{\mathcal{M}}_{\lambda,\nu}$, and so the claim holds.

Conclusion. Putting the claims together, it is clear that the statement holds.

□

We recall that a sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is said to be a Palais-Smale sequence for I_λ at some level $c \in \mathbb{R}$ if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

If any such a sequence is relatively compact in the $E(\mathbb{R}^N)$ topology, then we say that the functional I_λ satisfies the Palais-Smale condition at level c.

Lemma 3.7 (**Boundedness of Palais-Smale sequences).** *Assume $N = 3, 4$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, $q \in [3, 2^* - 1]$, and $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is a Palais-Smale sequence for I_λ at any level $c > 0$. Then, for any fixed $\lambda > 0$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E(\mathbb{R}^N)$.*

We stress that our assumption on N yields $3 \le 2^* - 1$.

Proof. For convenience, set

$$\begin{aligned} a_n &= ||u_n||_{H^1(\mathbb{R}^N)}, \\ b_n &= \lambda \left(\int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x) \right)^{\frac{1}{2}}, \\ c_q &= \min \left\{ \left(\frac{q-1}{2} \right), \left(\frac{q-3}{4} \right) \right\} \end{aligned}$$

and note that, as $n \to +\infty$,

$$\begin{aligned} C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} &\ge (q+1) I_\lambda(u_n) - I'_\lambda(u_n)(u_n) \\ &= \left(\frac{q-1}{2} \right) a_n^2 + \left(\frac{q-3}{4} \right) b_n^2 \end{aligned} \tag{3.13}$$

for some $C_1 > 0$. Assuming $||u_n||_{E(\mathbb{R}^N)} \to +\infty$, we show a contradiction in each of the cases:

(i) $a_n, b_n \to +\infty$,

(ii) a_n bounded and $b_n \to +\infty$,

(iii) $a_n \to +\infty$ and b_n bounded.

First consider $q > 3$. If $b_n \to +\infty$, for large n we have $b_n^2 \ge b_n$ and by (3.13) we get

$$C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} \ge c_q ||u_n||^2_{E(\mathbb{R}^N)},\ n \to +\infty,$$

a contradiction in case (i) and (ii). If $a_n \to +\infty$ and b_n is bounded, then $||u_n||_{E(\mathbb{R}^N)} \sim a_n$, hence

$$C_1 + o(1)a_n \geq c_q a_n^2,\ n \to +\infty,$$

a contradiction in case (iii). This makes the proof complete for $q > 3$.
Consider now $q = 3$. By Sobolev inequality we have

$$C_2 \geq I_\lambda(u_n) \geq \frac{1}{2}a_n^2 + \frac{1}{4}b_n^2 - C_3 a_n^4,$$

for some C_2, $C_3 > 0$, which yields a contradiction in case (ii). On the other hand if $a_n \to +\infty$, from the same estimate we have

$$b_n \lesssim a_n^2,\ n \to +\infty. \tag{3.14}$$

Note that (3.13) yields

$$C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} \geq a_n^2,\ n \to +\infty. \tag{3.15}$$

Dividing by $||u_n||_{E(\mathbb{R}^N)} = (a_n^2 + b_n)^{\frac{1}{2}}$, we get $\frac{a_n^4}{a_n^2+b_n} = o(1)$, $n \to +\infty$, hence

$$b_n \gtrsim a_n^4,\ n \to +\infty,$$

a contradiction in case (iii). This and (3.14), give

$$a_n^4 \lesssim a_n^2,\ n \to +\infty,$$

a contradiction in case (i). This completes the proof. □

Lemma 3.8 (Lower bound uniform in λ for PS sequences at level c_λ). *Assume $N = 3, 4, 5$, $\lambda > 0$, $q \in (2, 2^* - 1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. There exists a universal constant $\alpha = \alpha(q) > 0$ independent of λ such that for any Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ for I_λ at level c_λ, it holds that*

$$\liminf_{n\to\infty} \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq \alpha.$$

Proof. For every $u \in E(\mathbb{R}^N)$, denoting S_{q+1} the best constant such that $S_{q+1}\|u\|_{L^{q+1}(\mathbb{R}^N)} \leq \|u\|_{H^1(\mathbb{R}^N)}$, we have

$$I_\lambda(u) \geq \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} + \frac{\lambda^2}{4}\int_{\mathbb{R}^3} \rho\phi_u u^2 - \frac{S_{q+1}^{-(q+1)}}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}.$$

Since $\omega\lambda^2 \int_{\mathbb{R}^N} \rho\phi_u u^2 = \left(||u||^2_{E(\mathbb{R}^N)} - ||u||^2_{H^1(\mathbb{R}^N)}\right)^2$, estimating the term $||u||^2_{E(\mathbb{R}^N)}||u||^2_{H^1(\mathbb{R}^N)}$ with Young's inequality, we have for any $\beta > 0$

$$I_\lambda(u) \geq \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{1}{\omega}\left(\frac{\beta-1}{4}\right)||u||^4_{H^1(\mathbb{R}^N)} + \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} - \frac{S_{q+1}^{-(q+1)}}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}. \tag{3.16}$$

We now pick $\delta = \left(\frac{(q+1)S_{q+1}^{q+1}}{4}\right)^{1/(q-1)}$ and assume $||u||_{E(\mathbb{R}^N)} < \delta$, which also implies that $||u||_{H^1(\mathbb{R}^N)} < \delta$. Then, choosing $\beta > 1$ sufficiently near 1 we obtain

$$\begin{aligned} I_\lambda(u) &\geq \left[\frac{1}{4} - \frac{1}{\omega}\left(\frac{\beta-1}{4}\right)\delta^2\right]||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} \\ &\geq \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)}. \end{aligned}$$

We note here that both δ and β depend on q but not on λ. Thus, we have shown that if $||u||_{E(\mathbb{R}^N)} = \delta/2$, then $I_\lambda(u) \geq \underline{c}$, for some $\underline{c} > 0$ independent of λ. So, since every path connecting the origin to where the functional I_λ is negative crosses the sphere of radius $\delta/2$, it follows that

$$c_\lambda \geq \underline{c} \text{ for every } \lambda \geq 0.$$

For convenience, set

$$a_n = ||u_n||_{H^1(\mathbb{R}^N)}, \qquad b_n = \lambda\left(\int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x)\right)^{\frac{1}{2}},$$

where $(u_n)_{n\in\mathbb{N}}$ is an arbitrary Palais-Smale sequence at the level c_λ. It holds that

$$\begin{aligned} c_\lambda + o(1) - \|I'_\lambda(u_n)\|_{E'(\mathbb{R}^N)}\|u_n\|_{E(\mathbb{R}^N)} &\leq I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &= \left(\frac{1}{2} - 1\right)a_n^2 + \left(\frac{1}{4} - 1\right)b_n^2 \\ &\quad + \left(1 - \frac{1}{q+1}\right)\|u_n\|^{q+1}_{q+1}. \end{aligned}$$

By concavity note that $\|u_n\|_{E(\mathbb{R}^N)} \leq a_n + b_n^{\frac{1}{2}}$, hence the above yields

$$\underline{c} + o(1) \underbrace{-\|I'_\lambda(u_n)\|_{E'(\mathbb{R}^N)}(a_n + b_n^{\frac{1}{2}}) + \frac{1}{2}\left(a_n^2 + b_n^2\right)}_{c_n} \leq \|u_n\|^{q+1}_{q+1},$$

and it is easy to see that $\liminf c_n \geq 0$. The conclusion follows then with $\alpha := \underline{c}$. ☐

3.3 Remarks on Regularity and Positivity

This section uses standard elliptic regularity theory and the maximum principle to provide a result yielding the regularity and positivity of solutions to the following Schrödinger-Poisson system.

Proposition 3. *[**Regularity and positivity**] Let $N \in [3, 6]$ and $q \in [1, 2^* - 1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative and $\rho(x) \not\equiv 0$ and $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a nontrivial weak solution to*

$$\begin{cases} -\Delta u + bu + c\rho(x)\phi u = d|u|^{q-1}u, & x \in \mathbb{R}^N, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^N, \end{cases} \tag{3.17}$$

with $b, c, d \in \mathbb{R}_+$. Then, u, $\phi_u \in W^{2,s}_{loc}(\mathbb{R}^N)$, for every $s \geq 1$, and so u, $\phi_u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$; moreover $\phi_u > 0$. If, in addition, $u \geq 0$, then $u > 0$ everywhere.

Proof. Under the hypotheses of the proposition, both u and ϕ_u have weak second derivatives in $L^s_{\rm loc}(\mathbb{R}^N)$ for all $s < +\infty$. To show this, note that from the first equation in (3.17), we have that $-\Delta u = g(x, u)$, where

$$\begin{aligned} |g(x, u)| &= |(-bu - c\rho(x)\phi_u u + d|u|^{q-1}u| \\ &\leq C(1 + |\rho\phi_u| + |u|^{q-1})(1 + |u|) \\ &:= h(x)(1 + |u|). \end{aligned}$$

Using our assumptions on ρ, ϕ_u, u, and that $q \leq 2^* - 1$, we can show that $h \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$, which implies that $u \in L^s_{\rm loc}(\mathbb{R}^N)$ for all $s < +\infty$ (see e.g. [51, p.270]). Note that here the restriction on the dimension implies that $\phi_u \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$. Since $u^2\rho \in L^s_{\rm loc}(\mathbb{R}^N)$ for all $s < +\infty$, then by the second equation in (3.17) and the Calderón-Zygmund estimates, we have that $\phi_u \in W^{2,s}_{\rm loc}(\mathbb{R}^N)$ (see e.g. [26]). This then enables us to show that $g \in L^s_{\rm loc}(\mathbb{R}^N)$ for all $s < +\infty$, which implies, by Calderón-Zygmund estimates, that $u \in W^{2,s}_{\rm loc}(\mathbb{R}^N)$ (see e.g. [26]). The $C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ regularity of both u, ϕ_u is a consequence of Morrey's embedding theorem. Finally, the strict positivity is a consequence of the strong maximum principle with $L^\infty_{\rm loc}(\mathbb{R}^N)$ coefficients [43], and this concludes the proof. ☐

3.4 Nonexistence

In this section we look specifically at nonexistence results. Nonexistence of solutions can be shown in the critical case and when the power of the nonlinearity is small. We use a Pohozaev type condition to show this but it is not the only use case of this condition.

Lemma 3.9. [**Pohozaev-type condition**] *Assume $N \in [3,6]$, $q \in [1, 2^* - 1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, and $k\rho(x) \leq (x, \nabla\rho)$ for some $k \in \mathbb{R}$. Let $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a weak solution to (3.17). Then, it holds that*

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \\ + \frac{(N+2+2k)c}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \,\mathrm{d}x \leq 0.\end{aligned} \tag{3.18}$$

In particular the above is an identity, provided $k\rho(x) = (x, \nabla\rho)$ (by Euler's theorem, this is the case if ρ is homogeneous of order k, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\rho(tx) = \frac{\mathrm{d}}{\mathrm{d}t}t^k\rho(x) = k\rho(x)$).

Proof. With the regularity remarks of Proposition 3 in place, we now multiply the first equation in (3.17) by $(x, \nabla u)$ and integrate on $B_R(0)$ for some $R > 0$. We will compute each integral separately. We first note that

$$\begin{aligned}\int_{B_R} -\Delta u(x, \nabla u)\,\mathrm{d}x = \frac{2-N}{2}\int_{B_R} |\nabla u|^2 \,\mathrm{d}x \\ - \frac{1}{R}\int_{\partial B_R} |(x, \nabla u)|^2 \,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma.\end{aligned} \tag{3.19}$$

Fixing $i = 1, \ldots, N$, integrating by parts and using the divergence theorem, we then see that,

$$\begin{aligned}\int_{B_R} bu(x_i\partial_i u)\,\mathrm{d}x &= b\left[-\frac{1}{2}\int_{B_R} u^2 \,\mathrm{d}x + \frac{1}{2}\int_{B_R} \partial_i(u^2 x_i)\,\mathrm{d}x\right] \\ &= b\left[-\frac{1}{2}\int_{B_R} u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R} u^2 \frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right].\end{aligned}$$

So, summing over i, we get

$$\int_{B_R} bu(x, \nabla u)\,\mathrm{d}x = b\left[-\frac{N}{2}\int_{B_R} u^2 \,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R} u^2 \,\mathrm{d}\sigma\right]. \tag{3.20}$$

Again, fixing $i = 1, \ldots, N$, integrating by parts and using the divergence theorem, we find that,

$$\begin{aligned}\int_{B_R} c\rho\phi_u u x_i(\partial_i u)\,\mathrm{d}x &= c\Big[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad - \frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\partial_i(\rho\phi_u u^2 x_i)\,\mathrm{d}x\Big] \\ &= c\Big[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad - \frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R}\rho\phi_u u^2\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\Big].\end{aligned}$$

Thus, summing over i, we get

$$\begin{aligned}\int_{B_R} c\rho\phi_u u(x,\nabla u)\,\mathrm{d}x = c\Big[&-\frac{N}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x \\ &- \frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\Big].\end{aligned} \tag{3.21}$$

Finally, once more fixing $i = 1, \ldots, N$, integrating by parts and using the divergence theorem, we find that,

$$\int_{B_R} d|u|^{q-1}u(x_i\partial_i u)\,\mathrm{d}x = d\left[\frac{-1}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x + \frac{1}{q+1}\int_{\partial B_R}|u|^{q+1}\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right],$$

and so, summing over i, we see that

$$\begin{aligned}\int_{B_R} d|u|^{q-1}u(x,\nabla u)\,\mathrm{d}x = d\Big[&\frac{-N}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x \\ &+ \frac{R}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma\Big].\end{aligned} \tag{3.22}$$

Putting (3.19), (3.20), (3.21) and (3.22) together, we see that

$$
\begin{aligned}
&\frac{2-N}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma\\
&\quad+b\Big[-\frac{N}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\Big]\\
&\qquad+c\Big[-\frac{N}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x-\frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x\\
&\qquad\quad-\frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\Big]\\
&\qquad\qquad-d\left[\frac{-N}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x+\frac{R}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma\right]=0.
\end{aligned}
\tag{3.23}
$$

We now multiply the second equation in (3.17) by $(x,\nabla\phi_u)$ and integrate on $B_R(0)$ for some $R>0$. By a simple calculation we see that

$$
\begin{aligned}
\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x&=\int_{B_R}-\Delta\phi_u(x,\nabla\phi_u)\,\mathrm{d}x\\
&=\frac{2-N}{2}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma\\
&\quad+\frac{R}{2}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma.
\end{aligned}
$$

Substituting this into (3.23) and rearranging, we get

$$
\begin{aligned}
&\frac{N-2}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{Nb}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{(N+k)c}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x\\
&\quad+\frac{c(2-N)}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{Nd}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x\\
&\le\frac{N-2}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{Nb}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{Nc}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x\\
&\quad+\frac{c}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x+\frac{c(2-N)}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{Nd}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x\\
&=-\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma+\frac{bR}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\\
&\quad+\frac{cR}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma+\frac{c}{2R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma\\
&\qquad-\frac{cR}{4}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma-\frac{dR}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma,
\end{aligned}
\tag{3.24}
$$

where we have used the assumption $k\rho(x) \leq (x, \nabla\rho)$ for some $k \in \mathbb{R}$ to obtain the first inequality. We now call the right hand side of (3.24) I_R, namely

$$\begin{aligned}I_R := -\frac{1}{R}\int_{\partial B_R} |(x,\nabla u)|^2 \,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2 \,\mathrm{d}\sigma \\ + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2 \,\mathrm{d}\sigma + \frac{c}{2R}\int_{\partial B_R} |(x,\nabla\phi_u)|^2 \,\mathrm{d}\sigma \\ - \frac{cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2 \,\mathrm{d}\sigma - \frac{dR}{q+1}\int_{\partial B_R} |u|^{q+1} \,\mathrm{d}\sigma.\end{aligned}$$

We note that $|(x,\nabla u)| \leq R|\nabla u|$ and $|(x,\nabla\phi_u)| \leq R|\nabla\phi_u|$ on ∂B_R, so it holds that

$$\begin{aligned}|I_R| \leq \frac{3R}{2}\int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2 \,\mathrm{d}\sigma \\ + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2 \,\mathrm{d}\sigma + \frac{3cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2 \,\mathrm{d}\sigma + \frac{dR}{q+1}\int_{\partial B_R} |u|^{q+1} \,\mathrm{d}\sigma.\end{aligned}$$

Now, since $|\nabla u|^2$, $u^2 \in L^1(\mathbb{R}^N)$ as $u \in E(\mathbb{R}^N) \subseteq H^1(\mathbb{R}^N)$, $\rho\phi_u u^2$, $|\nabla\phi_u|^2 \in L^1(\mathbb{R}^N)$ because $\int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla\phi_u|^2 \,\mathrm{d}x$ and $\phi_u \in D^{1,2}(\mathbb{R}^N)$, and $|u|^{q+1} \in L^1(\mathbb{R}^N)$ because $E(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [2, 2^*]$, then it holds that $I_{R_n} \to 0$ as $n \to +\infty$ for a suitable sequence $R_n \to +\infty$. Moreover, since (3.24) holds for any $R > 0$, it follows that

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x + \frac{(N+k)c}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x \\ + \frac{c(2-N)}{4}\int_{\mathbb{R}^N} |\nabla\phi_u|^2 \,\mathrm{d}x - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \,\mathrm{d}x \leq 0,\end{aligned}$$

and so, we obtain

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \\ + \frac{(N+2+2k)c}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x \\ - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \,\mathrm{d}x \leq 0,\end{aligned}$$

using the fact that $\int_{\mathbb{R}^N} |\nabla\phi_u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x$. This completes the proof. □

Although we will use the above necessary condition mainly for existence purposes, this also allows us to find a family of nonexistence results in a certain range of the parameters N, q, λ, k. The following result looks at the so called "critical case", i.e. when the power of the nonlinearity is equal to the critical Sobolev exponent.

Proposition 4 (**Nonexistence: the critical case** $q = 2^* - 1$)**.** *Assume* $N \in [3, 6]$, $q = 2^* - 1$, $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ *nonnegative,* $k\rho(x) \leq (x, \nabla\rho)$ *for some* $k \geq \frac{N-6}{2}$, *and* $\lambda > 0$. *Let* $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ *be a weak solution to* (1.2). *Then,* $(u, \phi_u) = (0, 0)$.

Proof. Combining the Nehari identity $I'_\lambda(u)(u) = 0$ with Lemma 3.9 yields

$$\Big(\frac{N-2}{2} - \frac{N}{q+1}\Big) \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \Big(\frac{N}{2} - \frac{N-2}{2}\Big) \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + \Big(\frac{2k+6-N}{4}\Big)\lambda^2 \int_{\mathbb{R}^N} \rho\phi_u u^2 \, \mathrm{d}x \leq 0.$$

Hence,

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x \leq 0,$$

and this concludes the proof. □

The following proposition looks at the case when q is "small", we note that the following is stated to cover also the dimensions $N > 2\left(\frac{q+1}{q-1}\right)$, namely the supercritical cases $3 \geq q + 1 > 2^*$ where $E(\mathbb{R}^N)$ does not embed in $L^{q+1}(\mathbb{R}^N)$.

Proposition 5 (**Nonexistence: the case** $q \in (1, 2]$)**.** *Assume* $N \geq 3$, $q \in (1, 2]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ *and* $\rho(x) \geq 1$ *almost everywhere and* $\lambda \geq \frac{1}{2}$. *Let* $u \in E(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ *satisfy*

$$-\Delta u + u + \lambda^2 \left(\frac{1}{\omega|x|^{N-2}} \star \rho u^2\right) \rho(x) u = |u|^{q-1}u, \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.25}$$

Then, $u \equiv 0$.

Proof. By density we can test (3.25) by u and so we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 + \lambda^2 \rho(x)\phi_u u^2 - |u|^{q+1} = 0. \tag{3.26}$$

Following [46, Theorem 4.1], by Lemma 3.3 and Young's inequality we have

$$\int_{\mathbb{R}^N} \rho(x) |u|^3 \leq \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4} \int_{\mathbb{R}^N} \rho(x)\phi_u u^2. \tag{3.27}$$

Combining (3.26) and (3.27), we have for all $\lambda \geq \frac{1}{2}$

$$0 \geq \int_{\mathbb{R}^N} u^2 + \rho(x)\,|u|^3 - |u|^{q+1} \geq \int_{\mathbb{R}^N} f(u),$$

where $f(u) = u^2 + |u|^3 - |u|^{q+1}$ is positive except at zero. Hence $u \equiv 0$, and this concludes the proof. □

Remark 3.2. Similar nonexistence results have been obtained in the case of constant potentials and for $N = 3$, in [22]. We point out that the in the above proposition $\lambda > 0$ is arbitrary and the condition on ρ is compatible with (ρ_1), as well as with (ρ_2). It is interesting to note that for $N = 6$ we have $q = 2^* - 1 = 2$, namely nonexistence occurs in a 'low-q' regime, under both conditions (ρ_1) and (ρ_2). The proof shows also that for supercritical exponents $q+1 > 2^*$ and higher dimensions, under further regularity assumptions required for Lemma 3.9 to hold, nonexistence also occurs.

4 Existence Results

This chapter is split in two to focus on our main assumptions on ρ.

$(\boldsymbol{\rho_1})$ $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$\left| x \in \mathbb{R}^N : \rho(x) \leq \overline{M} \right| < \infty,$$

$(\boldsymbol{\rho_2})$ For every $M > 0$,

$$\left| x \in \mathbb{R}^N : \rho(x) \leq M \right| < \infty.$$

In both cases we suffer from lack of compactness phenomenon, in the latter a more standard approach to recover compactness can be made when $q \in (1, 2^* - 1)$ culminating in showing that the Palais-Smale condition is satisfied for $q \in [3, 2^* - 1)$. In the former a different approach is needed. Here we show that when $\lambda \geq \lambda_*$ we see that certain Palais-Smale sequences possess weak limits, furthermore this allows us to show a precise variational characterisation of our problem where the Nehari manifold and these weak limits coincide in cases where $q > 3$. The results of this chapter are from [23].

4.1 Preliminaries for ρ vanishing on a region

Throughout this section we will make the assumption that

$(\boldsymbol{\rho_1})$ $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$\left| x \in \mathbb{R}^N : \rho(x) \leq \overline{M} \right| < \infty.$$

In what follows it is convenient to set

$$A(R) = \{x \in \mathbb{R}^N : |x| > R, \ \rho(x) \geq \overline{M}\},$$

$$B(R) = \{x \in \mathbb{R}^N : |x| > R, \ \rho(x) < \overline{M}\},$$

for any $R > 0$.

We begin with the following Lemma that shows a key vanishing property under assumption $(\boldsymbol{\rho_1})$. This Lemma is vital when splitting integrals over $\mathbb{R}^N$ into integrals over balls and outside.

Lemma 4.1 (Key vanishing property). *Suppose ρ is a measurable function and that for some $\overline{M} \in \mathbb{R}$ it holds that*

$$\overline{B} := \left| x \in \mathbb{R}^N : \rho(x) < \overline{M} \right| < \infty.$$

Then

$$\lim_{R\to\infty} |B(R)| = 0.$$

Proof. The conclusion follows by the dominated convergence theorem as $B(R) \subseteq \overline{B}$ yields

$$|B(R)| = \int_{\overline{B}} \chi_{B(R)}(x)\,\mathrm{d}x \le |\overline{B}|.$$

□

It is worth having a brief discussion on the dimensions of the domain $\mathbb{R}^N$. The reader may notice that we begin working in $N = 3, 4$, with some parts valid for an even larger range of N, later we reduce this to simply $N = 3$. The issues we face is to do with our range of q, ultimately it is not possible to work in a larger dimension than $N = 3$ due to the critical Sobolev exponent.

Lemma 4.2 (Uniform bounds in λ for PS sequences at level c_λ). *Assume $N = 3, 4$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_1), $q \in [3, 2^* - 1]$, $\lambda > 0$. There exists a universal constant $\overline{C} = \overline{C}(q, N) > 0$ independent of λ, such that for any Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ for I_λ at level c_λ it holds that $\|u_n\|_{E(\mathbb{R}^N)} < \overline{C}$.*

Proof. Let $v \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ have support in $\rho^{-1}(0)$. Pick $t_v > 0$ such that $I_0(t_v v) < 0$ and set $v_t = t t_v v$. Then, by definition of c_λ,

$$c_\lambda \le \max_{t\in[0,1]} I_\lambda(v_t) = \max_{t\ge 0} I_0(tv) =: \bar{c}. \tag{4.1}$$

Note that since (u_n) is bounded by Lemma 3.7, it holds that

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty} (I_\lambda(u_n) - \frac{1}{q+1} I'_\lambda(u_n)\cdot u_n) \\ &= \lim_{n\to\infty} \Big(\Big(\frac{1}{2} - \frac{1}{q+1}\Big)\|u_n\|^2_{H^1(\mathbb{R}^N)} + \lambda^2\Big(\frac{1}{4} - \frac{1}{q+1}\Big)\int_{\mathbb{R}^N} \phi_{u_n}\rho(x)u_n^2\Big). \end{aligned}$$

The conclusion follows immediately in the case $q > 3$. For $q = 3$ the above yields a uniform bound independent on λ for the $H^1(\mathbb{R}^N)$ norm and hence for the $L^{q+1}(\mathbb{R}^N)$ norm as well by Sobolev's inequality. Since

$$\lambda^2 \limsup_{n\to\infty} \int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x) \le 4\left(c_\lambda + \limsup_{n\to\infty}\left(\|u_n\|^2_{H^1(\mathbb{R}^N)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)}\right)\right),$$

this concludes the proof. □

It is worth highlighting that (4.1) has weaker restrictions than the assumptions given on Lemma 4.2 and holds in dimensions $N = 3, 4, 5$ and for every $q \in$

$(2, 2^*-1)$. Furthermore, as recovering some form of compactness in this setting is required, the following lemma is essential to control the tails of bounded sequences on $\mathbb{R}^N$.

Lemma 4.3 (Control on the tails of uniformly bounded sequences). *Assume $N = 3, 4, 5$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_1), and $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is bounded uniformly with respect to λ. Then, for every $\beta > 0$ there exists $\lambda_\beta > 0$ and $R_\beta > 0$ such that for $\lambda > \lambda_\beta$ and $R > R_\beta$,*

$$||u_n||^3_{L^3(\mathbb{R}^N \setminus B_R)} < \beta.$$

Proof. By Lemma 3.3 we have

$$\lambda \int_{\mathbb{R}^N} \rho(x)\,|u_n|^3 \leq C\|u_n\|^3_{E(\mathbb{R}^N)} \leq C', \tag{4.2}$$

for some positive constant C' independent of λ. Hence

$$\int_{A(R)} |u_n|^3 \leq \frac{C'}{\lambda \overline{M}}.$$

Also observe that by Hölder's inequality and Lemma 4.1 we have

$$\begin{aligned}\int_{B(R)} |u_n|^3 &\leq \Big(\int_{\mathbb{R}^N} |u_n|^{2^*}\Big)^{\frac{3}{2^*}} \Big(\int_{B(R)} 1\Big)^{\frac{2^*-3}{2^*}} \\ &\leq C''\|u_n\|^3_{E(\mathbb{R}^N)} \cdot |B(R)|^{\frac{2^*-3}{2^*}} \\ &\leq C'''\,|B(R)|^{\frac{2^*-3}{2^*}} \to 0.\end{aligned}$$

as $R \to \infty$, again for some uniform constant $C''' > 0$. Note that our assumption on N yields $3 < 2^*$. This is enough to conclude the proof. $\square$

The following proposition may be thought of a sort of concentration compactness principle (Lemma 2.1) in our setting, namely, we can show that, given a λ large enough, there exist nonzero weak limits of Palais-Smale sequences at the level c_λ.

Proposition 6 (Nonzero weak limits of PS sequences at level c_λ for λ large). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 5)$. There exist universal positive constants $\lambda_0 = \lambda_0(q, \overline{M})$ and $\alpha_0 = \alpha_0(q)$, such that if for some $\lambda \geq \lambda_0$, $u \in E(\mathbb{R}^3)$ is the weak limit of a Palais-Smale sequence for*

I_λ at level c_λ, then it holds that

$$\int_{\mathbb{R}^3} |u|^3 \,\mathrm{d}x > \alpha_0.$$

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ be an arbitrary Palais-Smale sequence at level c_λ. Note that we can pick $\alpha(q) > 0$ independent of λ and of the sequence such that

$$\liminf_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3)} \geq \alpha(q).$$

Indeed by interpolation

$$\int_{\mathbb{R}^3} |u_n|^{q+1} \leq \Big(\int_{\mathbb{R}^3} |u_n|^3\Big)^{\frac{5-q}{3}} \Big(\int_{\mathbb{R}^3} |u_n|^6\Big)^{\frac{q-2}{3}}$$

and the claim follows by Sobolev inequality and the uniform bound given by Lemma 4.2 and by Lemma 3.8. In particular, recall that by Lemma 4.2, there exists a universal constant $\overline{C} = \overline{C}(q, N) > 0$ independent of λ and of the sequence, such that $\|u_n\|_{E(\mathbb{R}^N)} < \overline{C}$. By Lemma 4.3, it follows than that we can pick $\lambda_0(q, \overline{M})$ and $R_\alpha > 0$ such that such that for every $\lambda \geq \lambda_0$ and every $R > R_\alpha$ we have

$$\limsup_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} < \frac{\alpha}{2}.$$

By the classical Rellich theorem, passing if necessary to a subsequence, we can assume that $u_n \to u$ in $L^3_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore, for every $R > R_\alpha$, we have

$$\|u\|^3_{L^3(B_R)} = \lim_{n\to\infty} \|u_n\|^3_{L^3(B_R)} \geq \liminf_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3)} - \limsup_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} > \frac{\alpha}{2}.$$

The conclusion follows with $\alpha_0 = \alpha/2$. □

The following proposition finishes our preliminaries on (ρ_1). Given λ large enough we find the mountain-pass levels c_λ are indeed critical in the range $q \in (3, 5)$ given the sufficient conditions on Palais-Smale sequences in Proposition 6. The case when $q = 3$ is more delicate, instead we find lower estimates of the energy but whether this level is critical remains uncertain.

Proposition 7 (Energy estimates for λ large). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 5)$. Let λ_0 be defined as in Proposition 6. There exists a universal constant $\lambda_1 = \lambda_1(q, \overline{M}) > 0$ such that, if $\lambda \geq \max(\lambda_0, \lambda_1)$ and u is the nontrivial weak limit in $E(\mathbb{R}^3)$ of some Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ for I_λ at level c_λ, then it holds that*

- $I_\lambda(u) = c_\lambda, \quad$ *for* $q \in (3,5)$,
- $\inf_{v\in\mathcal{N}_\lambda} I_\lambda(v) \le I_\lambda(u) \le c_\lambda, \quad$ *for* $q = 3$.

In particular, for all $\lambda \ge \max(\lambda_0, \lambda_1)$, *the mountain-pass level* c_λ *is critical for* $q \in (3,5)$, *as well as the level* $I_\lambda(u)$ *for* $q = 3$.

Proof. By Proposition 6, for every $q \in [3, 2^*-1)$ and $\lambda \ge \lambda_0$, passing if necessary to a subsequence, we can assume that $u_n \rightharpoonup u \in E(\mathbb{R}^3) \setminus \{0\}$ weakly in $E(\mathbb{R}^3)$ and almost everywhere, for some Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ for I_λ at level c_λ. By a standard argument u is a critical point of I_λ. For sake of clarity we break the proof into two steps.

Step 1: We first show that there exists a universal constant $C = C(q) > 0$ such that for every $\lambda \ge \lambda_0$, $R > 0$ and $n \in \mathbb{N}$, it holds that

$$\begin{aligned} I_\lambda(u_n - u) \ge \left(\frac{1}{4} - S_\lambda S^{-1}\left(\int_{A(R)} |u_n - u|^6\right)^{\frac{2}{3}}\right) \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \\ - C\,|B(R)|^{\frac{5-q}{6}} - \frac{1}{q+1}\int_{|x|<R} |u_n - u|^{q+1}, \end{aligned} \tag{4.3}$$

where

$$S_\lambda := (q-2)[3(q+1)]^{\frac{-3}{q-2}} \left(\frac{2(5-q)}{\lambda \overline{M}}\right)^{\frac{5-q}{q-2}},$$

$S = 3(\pi/2)^{4/3}$ is the Sobolev constant, and $\overline{M}$ is defined as in (ρ_1). Reasoning as in Lemma 3.3 and by Lemma 4.3 we obtain,

$$\begin{aligned} I_\lambda(u_n - u) &\ge \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \\ &\qquad + \frac{\lambda^2}{4}\int_{\mathbb{R}^3} \phi_{(u_n-u)}(u_n-u)^2\rho(x) - \frac{1}{q+1}\int_{\mathbb{R}^3} |u_n - u|^{q+1} \\ &\ge \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} \rho(x)\,|u_n - u|^3 - \frac{1}{q+1}\int_{\mathbb{R}^3} |u_n - u|^{q+1} \\ &\ge \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 + \frac{\lambda\overline{M}}{2}\int_{A(R)} |u_n - u|^3 - \frac{1}{q+1}\int_{\mathbb{R}^3} |u_n - u|^{q+1}. \end{aligned} \tag{4.4}$$

Note that

$$\int_{\mathbb{R}^3} |u_n - u|^{q+1} = \int_{|x|<R} \ldots + \int_{A(R)} \ldots + \int_{B(R)} \ldots$$

Using that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $E(\mathbb{R}^3)$ and arguing as in Lemma 4.3

and by Sobolev inequality, we have

$$\int_{B(R)} |u_n - u|^{q+1} \leq C_1 \|u_n - u\|^{q+1}_{L^6(\mathbb{R}^3)} |B(R)|^{\frac{5-q}{6}} \leq C_2 |B(R)|^{\frac{5-q}{6}} . \tag{4.5}$$

By the interpolation and Young's inequalities we obtain for all $\delta > 0$ that

$$\begin{aligned}\frac{1}{q+1}\int_{A(R)} |u_n - u|^{q+1} &\leq \frac{1}{q+1}\left(\int_{A(R)} |u_n - u|^3\right)^{\frac{5-q}{3}} \left(\int_{A(R)} |u_n - u|^6\right)^{\frac{q-2}{3}} \\ &\leq \left(\frac{5-q}{3}\right)\left(\frac{\delta}{q+1}\right)^{\frac{3}{5-q}} \int_{A(R)} |u_n - u|^3 \\ &\quad + \left(\frac{q-2}{3}\right)\delta^{\frac{-3}{q-2}} \int_{A(R)} |u_n - u|^6 .\end{aligned}$$

In particular, we can set

$$\delta = \left(\frac{\lambda \overline{M}}{2} \cdot \frac{3}{5-q}\right)^{\frac{5-q}{3}} (q+1).$$

Hence

$$\begin{aligned}\frac{1}{q+1}\int_{A(R)} |u_n - u|^{q+1} &\leq \frac{\lambda \overline{M}}{2}\int_{A(R)} |u_n - u|^3 + S_\lambda \int_{A(R)} |u_n - u|^6 \\ &\leq \frac{\lambda \overline{M}}{2}\int_{A(R)} |u_n - u|^3 \\ &\quad + S_\lambda S^{-1}\left(\int_{A(R)} |u_n - u|^6\right)^{\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 ,\end{aligned} \tag{4.6}$$

where we have used Sobolev's inequality written as

$$S\left(\int_{A(R)} |u_n - u|^6\right)^{\frac{1}{3}} \leq \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 .$$

Putting together (4.4), (4.5) and (4.6), the claim (4.3) follows.

Step 2: Conclusion. By the classical Brezis-Lieb lemma and Lemma 3.2 we have

$$c_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u) + \lim_{n\to\infty} I_\lambda(u_n - u). \tag{4.7}$$

Note that there exists a positive constant $\lambda_1 = \lambda_1(q, \overline{M})$ such that for every $\lambda \geq \lambda_1$ it holds that

$$\frac{1}{4} - S_\lambda S^{-3}\overline{C}^4 \geq 0, \tag{4.8}$$

where $\overline{C}$ is defined via Lemma 4.2 by the property $\|u_n\|_{E(\mathbb{R}^3)} < \overline{C}$. Note that, again by the Brezis-Lieb lemma, we have

$$\int_{A(R)} |u_n - u|^6 = \int_{A(R)} |u_n|^6 - \int_{A(R)} |u|^6 + o_n(R),$$

with $\lim_{n\to\infty} o_n(R) = 0$ for any fixed $R > 0$; since by Sobolev's inequality it holds that

$$\int_{A(R)} |u_n|^6 \leq S^{-3} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^3 \leq S^{-3}\overline{C}^6,$$

we obtain the estimate

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \int_{A(R)} |u_n - u|^6 \leq S^{-3}\overline{C}^6. \tag{4.9}$$

We conclude, by (4.3), (4.8), (4.9) and the classical Rellich theorem that

$$\begin{aligned}\lim_{n\to\infty} I_\lambda(u_n - u) &\geq \liminf_{R\to\infty} \liminf_{n\to\infty} \left(\frac{1}{4} - S_\lambda S^{-1} \left(\int_{A(R)} |u_n - u|^6 \right)^{\frac{2}{3}} \right) \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 \\ &\geq \left[\frac{1}{4} - S_\lambda S^{-3}\overline{C}^4 \right] \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 \geq 0,\end{aligned}$$

and hence by (4.7) that $I_\lambda(u) \leq c_\lambda$. On the other hand, since $u \in \mathcal{N}_\lambda$, it holds that

$$\inf_{v\in\mathcal{N}_\lambda} I_\lambda(v) \leq I_\lambda(u) \leq c_\lambda,$$

and this completes the proof for $q = 3$. For $q \in (3, 2^* - 1)$ we can use Lemma 3.5 hence,

$$c_\lambda = \inf_{v\in\mathcal{N}_\lambda} I_\lambda(v)$$

and it follows that $I_\lambda(u) = c_\lambda$, and this concludes the proof. □

4.1.1 Some reflections on the Palais-Smale condition

When $q > 3$, the fact that $\lim I_\lambda(u_n - u) = 0$ for λ large suggests that the Palais-Smale condition at the mountain-pass level c_λ can be recovered in some cases. To illustrate this, note that the assumption (ρ_1) is compatible with having, say $\rho(x) \to 2\overline{M}$, as $|x| \to \infty$, namely a situation where lack of compactness phenomena may occur for the system (1.2) as a consequence of the invariance by translations of (1.9), which plays the role of a 'problem at infinity', such an example may be found in the work by Mercuri and Tyler [41]. We stress here that ρ may approach its limit from below as well as from above. To see that in this case the Palais-Smale condition is satisfied for λ large, denote by $I_\lambda^{\rho\equiv 2\overline{M}}$ the functional associated to ($\mathcal{SP}$) with $\rho \equiv 2\overline{M}$, and observe that in this situation $E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$, with equivalent norms by (HLS). We reason as in [41, Proposition 1.6] which we will recall here,

Proposition 8 ([41] Proposition 1.6)**.** *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to \rho_\infty \geq 0$ as $|x| \to +\infty$. Let $q \in (2,5)$ and $\mu \in [1/2, 1]$, and assume $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ is a bounded Palais-Smale sequence for I_μ. Then, there exists $l \in \mathbb{N}$, a finite sequence $(v_0, \dots, v_l) \subset H^1(\mathbb{R}^3)$, and l sequences of points $(y_n^j)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \leq j \leq l$, satisfying, up to a subsequence of $(u_n)_{n\in\mathbb{N}}$,*

- *(i)* v_0 *is a solution of* (4.10),
- *(ii)* v_j *are nonnegative, possibly nontrivial, solutions of* (4.11) *for* $1 \leq j \leq l$,
- *(iii)* $|y_n^j| \to +\infty$, $\left|y_n^j - y_n^{j'}\right| \to +\infty$ *as* $n \to +\infty$ *if* $j \neq j'$,
- *(iv)* $\|u_n - v_0 - \sum_{j=1}^l v_j(\cdot - y_n^j)\|_{H^1(\mathbb{R}^3)} \to 0$ *as* $n \to +\infty$,
- *(v)* $\|u_n\|_{H^1(\mathbb{R}^3)}^2 \to \sum_{j=0}^l \|v_j\|_{H^1(\mathbb{R}^3)}^2$ *as* $n \to +\infty$,
- *(vi)* $I_\mu(u_n) = I_\mu(v_0) + \sum_{j=1}^l I_\mu^\infty(v_j) + o(1)$.

I_μ and I_μ^∞ are the energy functionals corresponding to

$$-\Delta u + u + \rho(x)\phi_u u = \mu\,|u|^{q-1}\,u, \tag{4.10}$$

$$-\Delta u + u + \rho_\infty\phi_u u = \mu\,|u|^{q-1}\,u \tag{4.11}$$

respectively.

The above is primarily used in the case $p \in (2,3)$. In our case, we take $\mu = 1$ and there exist $l \in \mathbb{N} \cup \{0\}$, functions $(v_1, \dots, v_l) \subset H^1(\mathbb{R}^3)$, and sequences of points $(y_n^j)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \leq j \leq l$, such that, passing if necessary to a subsequence,

- v_j are possibly nontrivial critical points of $I_\lambda^{\rho\equiv 2\overline{M}}$ for $1 \leq j \leq l$,
- $|y_n^j| \to +\infty$, $|y_n^j - y_n^{j'}| \to +\infty$ as $n \to +\infty$ if $j \neq j'$,
- $||u_n - u - \sum_{j=1}^l v_j(\cdot - y_n^j)||_{H^1(\mathbb{R}^3)} \to 0$ as $n \to +\infty$,
- $c_\lambda = I_\lambda(u) + \sum_{j=1}^l I_\lambda^{\rho\equiv 2\overline{M}}(v_j)$.

It is standard to see that $I_\lambda^{\rho\equiv 2\overline{M}}$ is uniformly bounded below on the set of its nontrivial critical points by a positive constant, independent of λ. It then follows that for all $\lambda \geq \max(\lambda_0, \lambda_1)$, Proposition 7 and the above yield $c_\lambda = I_\lambda(u)$ and at the same time $l = 0$; as a consequence the Palais-Smale condition is satisfied at the level c_λ. These considerations yield the following

Proposition 9 (**Palais-Smale condition under** (ρ_1)). *Let $N = 3 < q$ and $\rho \geq 0$ be locally bounded such that (ρ_1) is satisfied and such that $\rho(x) \to \rho_\infty > \overline{M}$ as $|x| \to \infty$. Let λ_0 and λ_1 be as in Proposition 7. Then, for all $\lambda \geq \max(\lambda_0, \lambda_1)$, I_λ satisfies the Palais-Smale condition at the mountain-pass level c_λ.*

It is not obvious how to prove the above proposition in the case $q = 3$; nevertheless the same considerations on strong convergence apply instead to approximated critical points of I_λ constrained on the Nehari manifold, see the proof Theorem 4.1 and Proposition 10 below.

4.2 The case of ρ vanishing on a region

Now that we have the necessary preliminaries we present the proof of Theorem 4.1.

Theorem 4.1 (**Groundstates for $q \geq 3$ under** (ρ_1)). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 2^*-1)$. There exists a positive constant $\lambda_* = \lambda_*(q, \overline{M})$ such that for every $\lambda \geq \lambda_*$, ($\mathcal{SP}$) admits a positive groundstate solution $u \in E(\mathbb{R}^3)$. For $q > 3$, u is a mountain-pass solution.*

Proof. We recall the Nehari manifold (3.9),

$$\mathcal{N}_\lambda := \left\{ u \in E(\mathbb{R}^3) \setminus \{0\} : G_\lambda(u) = 0 \right\},$$

where

$$G_\lambda(u) = I'_\lambda(u)(u) = ||u||^2_{H^1(\mathbb{R}^3)} + \lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - ||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)}.$$

Since $q \in [3, 2^*-1)$, it is standard to see that $\mathcal{N}_\lambda$ is nonempty and will be omitted. Moreover, we show that the conditions

(i) $\exists r > 0 : B_r \cap \mathcal{N}_\lambda = \emptyset$,

(ii) $G'_\lambda(u)(u) \neq 0, \quad \forall u \in \mathcal{N}_\lambda$,

are satisfied, which means, by standard arguments, the Nehari manifold $\mathcal{N}_\lambda$ is a natural constraint for our problem (a detailed proof as to why these conditions imply a natural constraint can be found in [3] but will be omitted here for the sake of brevity). Starting with (i), we notice that if $u \in \mathcal{N}_\lambda$, then writing $G_\lambda(u)$ explicitely and

$$0 = ||u||^2_{H^1(\mathbb{R}^3)} + \lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - ||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)} \geq ||u||^2_{H^1(\mathbb{R}^3)} - S_{q+1}^{-(q+1)} ||u||^{q+1}_{H^1(\mathbb{R}^3)},$$

where S_{q+1} is the best constant such that $S_{q+1}\|u\|_{L^{q+1}(\mathbb{R}^N)} \leq \|u\|_{H^1(\mathbb{R}^N)}$. The above gives the following inequality

$$||u||_{E(\mathbb{R}^3)} \geq ||u||_{H^1(\mathbb{R}^3)} \geq S_{q+1}^{(q+1)/(q-1)}, \quad \forall u \in \mathcal{N}_\lambda. \tag{4.12}$$

If we set $r = S_{q+1}^{(q+1)/(q-1)} - \delta$ for some small $\delta > 0$ then we arrive at (i). For (ii), we notice that if $u \in \mathcal{N}_\lambda$, then by the definition of the Nehari manifold, the assumption $q \geq 3$ and (4.12), it holds that

$$\begin{aligned} G'_\lambda(u)(u) &= 2||u||^2_{H^1(\mathbb{R}^3)} + 4\lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - (q+1)||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)} \\ &= (1-q)||u||^2_{H^1(\mathbb{R}^3)} + (3-q)\lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 \\ &\leq (1-q)S_{q+1}^{2(q+1)/(q-1)} \\ &< 0. \end{aligned} \tag{4.13}$$

Thus, the claim holds and so the Nehari manifold is a natural constraint. Now, we focus on the case $q \in (3, 2^* - 1)$, setting $\lambda_* = \max\{\lambda_0, \lambda_1\}$, the conclusion follows immediately from Proposition 7 and the following characterisation of the mountain-pass level,

$$c_\lambda = \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v).$$

On the other hand, assume $q = 3$ and $\lambda \geq \max\{\lambda_0, \lambda_1\}$. We note that

$$c^*_\lambda := \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v)$$

is well-defined since $\mathcal{N}_\lambda$ is nonempty, and so, we take $(\tilde{w}_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$ to be a minimising sequence for I_λ on $\mathcal{N}_\lambda$, namely, $I_\lambda(\tilde{w}_n) \to c^*_\lambda$. As described in [21], by the Ekeland variational principle there exists another minimising sequence $(w_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$ and $\xi_n \in \mathbb{R}$ such that

$$I_\lambda(w_n) \to c^*_\lambda, \tag{4.14}$$

$$I'_\lambda(w_n)(w_n) = 0, \tag{4.15}$$

and

$$I'_\lambda(w_n) - \xi_n G'_\lambda(w_n) \to 0, \qquad \text{in } (E(\mathbb{R}^3))'. \tag{4.16}$$

Now, by Proposition 7, (4.1), (4.14) and (4.15), it holds that

$$\lim_{n\to+\infty}\left(I_\lambda(w_n) - \frac{1}{q+1}I'_\lambda(w_n)(w_n)\right) = c^*_\lambda \le c_\lambda \le \bar{c},$$

for some $\bar{c}$ independent of λ. We can therefore argue as in Lemma 4.2 to show that

$$||w_n||_{E(\mathbb{R}^3)} < \bar{C}, \tag{4.17}$$

where $\bar{C} > 0$ is the same constant independent of λ given by Lemma 4.2. Moreover, since $(w_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$, it follows using (4.12) that

$$||w_n||^4_{L^4(\mathbb{R}^3)} = ||w_n||^2_{H^1(\mathbb{R}^3)} + \lambda^2\int_{\mathbb{R}^3}\rho\phi_{w_n}w_n^2 \ge ||w_n||^2_{H^1(\mathbb{R}^3)} \ge S_4^4 > 0.$$

Thus, by interpolation it holds

$$S_4^4 \le \int_{\mathbb{R}^3}|w_n|^4 \le \left(\int_{\mathbb{R}^3}|w_n|^3\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}|w_n|^6\right)^{\frac{1}{3}},$$

and so, by the Sobolev inequality and (4.17), it follows that we can pick $\alpha > 0$ independent of λ such that

$$\liminf_{n\to\infty}\|w_n\|^3_{L^3(\mathbb{R}^3)} \ge \alpha.$$

Moreover, by Lemma 4.3, we can set $\lambda_* = \max\{\lambda_0, \lambda_1\}$ and $R_\alpha > 0$ such that such that for every $\lambda \ge \lambda_*$ and every $R > R_\alpha$ we have

$$\limsup_{n\to\infty}\|w_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} < \frac{\alpha}{2}.$$

Now, since $(w_n)_{n\in\mathbb{N}}$ is bounded, passing if necessary to a subsequence, we can assume that $w_n \rightharpoonup w$ in $E(\mathbb{R}^3)$ and $w_n \to w$ in $L^3_{\text{loc}}(\mathbb{R}^3)$. It follows that for every $\lambda \ge \lambda_*$ and $R > R_\alpha$,

$$\|w\|^3_{L^3(B_R)} \ge \liminf_{n\to\infty}\|w_n\|^3_{L^3(\mathbb{R}^3)} - \limsup_{n\to\infty}\|w_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} > \frac{\alpha}{2},$$

and so $w \not\equiv 0$. We now notice that by (4.15), (4.16), and (4.17), it holds, up to a constant independent of λ, that

$$\begin{aligned} o(1) &= ||I'_\lambda(w_n) - \xi_n G'_\lambda(w_n)||_{(E(\mathbb{R}^3))'} \\ &\gtrsim |I'_\lambda(w_n)(w_n) - \xi_n G'_\lambda(w_n)(w_n)| \\ &= |\xi_n G'_\lambda(w_n)(w_n)|, \end{aligned}$$

for some $\xi_n \in \mathbb{R}$. Since $(w_n) \subset \mathcal{N}_\lambda$, by (4.13), we have that $G'_\lambda(w_n)(w_n) < -2S_4^4 < 0$, and so the above yields $\xi_n \to 0$. Moreover, using (4.17) and the positivity properties of the Coulomb integral (Theorem 9.8, [33]), we have the inequality

$$|D(f,g)|^2 \leq D(f,f)D(g,g),$$

where we define

$$D(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y,$$

for f, g measurable and nonnegative functions, it follows that $G'_\lambda(w_n)$ is bounded. Taken together, we have that $\xi_n G'_\lambda(w_n) \to 0$, and using this and (4.16), we obtain $I'_\lambda(w_n) \to 0$. Hence, $(w_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for I_λ at level c^*_λ, and so, since we have also shown that $w_n \rightharpoonup w \not\equiv 0$ in $E(\mathbb{R}^3)$, a standard argument yields that w is a nontrivial critical point of I_λ. Namely, $w \in \mathcal{N}_\lambda$, and thus

$$c^*_\lambda \leq I_\lambda(w). \tag{4.18}$$

On the other hand, arguing as in Proposition 7, replacing u_n, u, and c_λ with w_n, w, and c^*_λ, respectively, for every $\lambda \geq \lambda_*$, we obtain

$$I_\lambda(w) \leq c^*_\lambda. \tag{4.19}$$

For convenience we recall that λ_1 is chosen in Proposition 7 so that for every $\lambda \geq \lambda_1$, it holds that $\frac{1}{4} - S_\lambda S^{-3}\bar{C}^4 \geq 0$, where $\bar{C}$ is defined via Lemma 4.2 by the property $||u_n||_{E(\mathbb{R}^3)} < \bar{C}$. Going through the same argument with $(w_n)_{n\in\mathbb{N}}$, since $(w_n)_{n\in\mathbb{N}}$ is bounded by precisely the same uniform constant, namely $||w_n||_{E(\mathbb{R}^3)} < \bar{C}$, we conclude that (4.19) holds for every $\lambda \geq \lambda_*$, as $\lambda_* \geq \lambda_1$ by construction. Putting (4.18) and (4.19) together yields

$$I_\lambda(w) = \inf_{v\in\mathcal{N}_\lambda} I_\lambda(v).$$

Since $I_\lambda(w) = I_\lambda(|w|)$ and $w \in \mathcal{N}_\lambda$ if and only if $|w| \in \mathcal{N}_\lambda$, we can assume $w \geq 0$, and it follows that $w > 0$ by Proposition 3. This completes the proof. □

As a byproduct of the above proof, we have the following

Proposition 10 (Constrained Palais-Smale condition under (ρ_1)). *Let $N = 3 = q$ and $\rho \geq 0$ be locally bounded such that (ρ_1) is satisfied and such that $\rho(x) \to \rho_\infty > \overline{M}$ as $|x| \to \infty$. Let λ_0 and λ_1 be as in Proposition 7. Then, for all $\lambda \geq \max(\lambda_0, \lambda_1)$, the restriction $I_\lambda|_{\mathcal{N}_\lambda}$ satisfies the Palais-Smale condition at the level*

$$c_\lambda^* = \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v).$$

That is, every sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c_\lambda^*, \qquad \nabla I_\lambda(u_n)|_{\mathcal{N}_\lambda} \to 0 \text{ in } H^{-1}(\mathbb{R}^3)$$

is relatively compact.

Proof. The proof follows the reasoning discussed in section 4.1.1. We leave out the details. □

4.3 Preliminaries for Coercive ρ

In the present section $\lambda > 0$ is an arbitrary fixed value, and on ρ we make the assumption that

(ρ_2) For every $M > 0$,
$$\left|x \in \mathbb{R}^N : \rho(x) \leq M\right| < \infty.$$

The above assumption on ρ allows for the following compactness property.

Lemma 4.4 (Compactness property). *Let $N = 3, 4, 5$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be non-negative, satisfying (ρ_2), and $q \in (1, 2^* - 1)$. Then, $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$.*

Proof. By Lemma 3.3, multiplying by λ we obtain

$$\lambda \int_{\mathbb{R}^N} \rho(x)\,|u|^3 \leq \left(\frac{1}{\omega}\right)^{\frac{1}{2}} \|u\|^3_{E(\mathbb{R}^N)}. \tag{4.20}$$

Set

$$A(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) \geq M\},$$

$$B(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) < M\}.$$

Without loss of generality, assume that $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is such that $u_n \rightharpoonup 0$. For convenience, write

$$\int_{\mathbb{R}^N\setminus B_R} |u_n|^3 = \int_{A(R)} |u_n|^3 + \int_{B(R)} |u_n|^3$$

where B_R is a ball of radius R centred at the origin. Fix $\delta > 0$ and pick M, r, C, such that $M > \frac{2}{\lambda\delta}\left(\frac{1}{\omega}\right)^{\frac{1}{2}} \sup_n \|u_n\|^3_{E(\mathbb{R}^N)}$, $r = \frac{2^*}{3} > 1$ and

$$C \geq \sup_{u\in E(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^3_{L^{2^*}(\mathbb{R}^N)}}{\|u\|^3_{E(\mathbb{R}^N)}}.$$

Let $\frac{1}{r} + \frac{1}{r'} = 1$. By Lemma 4.1, for every $M > 0$, and every $R > 0$ large enough, it holds that

$$|B(R)| \leq \left[\frac{\delta}{2C\sup_n \|u_n\|^3_{E(\mathbb{R}^N)}}\right]^{r'}. \tag{4.21}$$

Since $N = 3, 4, 5$, we can pick $r = \frac{2^*}{3} > 1$ such that by Hölder inequality it holds that

$$\begin{aligned}\int_{B(R)} |u_n|^3 &\leq \left(\int_{B(R)} |u_n|^{2^*}\right)^{\frac{1}{r}} \left(\int_{B(R)} 1\right)^{\frac{1}{r'}} \\ &\leq \|u_n\|^3_{L^{2^*}(\mathbb{R}^N)} \cdot |B(R)|^{\frac{1}{r'}} \\ &\leq C\|u_n\|^3_{E(\mathbb{R}^N)} \cdot |B(R)|^{\frac{1}{r'}} \leq \frac{\delta}{2},\end{aligned}$$

Moreover, by our choice of M and (4.20), we see that

$$\int_{A(R)} |u_n|^3 \leq \frac{1}{\lambda M}\left(\frac{1}{\omega}\right)^{\frac{1}{2}} ||u_n||^3_{E(\mathbb{R}^N)} \leq \frac{\delta}{2}.$$

By the classical Rellich theorem, and since δ was arbitrary, this is enough to prove our lemma for $q = 2$. By interpolation the case $q \neq 2$ follows immediately, and this concludes the proof. □

Using the above lemma, and for $q \geq 3$, we can show that the Palais-Smale condition holds for I_λ at any level.

Lemma 4.5 (Palais-Smale condition). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_2), and $q \in [3, 2^* - 1)$. Then, I_λ satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.*

Proof. Suppose $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is a Palais-Smale sequence for I_λ. It follows from Lemma 3.7 that u_n is bounded in $E(\mathbb{R}^N)$, and thus, up to a subsequence

$u_n \rightharpoonup u$. By the weak convergence, consider

$$\begin{aligned} o(1) &= I'_\lambda(u_n) \cdot (u_n - u) \\ &= \|u_n\|^2_{H^1(\mathbb{R}^N)} - \|u\|^2_{H^1(\mathbb{R}^N)} + o(1) \\ &\quad + \int_{\mathbb{R}^N} \phi_{u_n} u_n (u_n - u)\rho(x) - \int_{\mathbb{R}^N} |u_n|^{q-1} u_n (u_n - u). \end{aligned} \tag{4.22}$$

We look at each part of the RHS seperately. To begin we know by Lemma 4.4 that $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$ for all $q \in [3, 2^* + 1)$, hence

$$\int_{\mathbb{R}^N} |u_n|^{q-1} u_n (u_n - u) \to 0, \tag{4.23}$$

as $n \to \infty$, i.e $\|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \to \|u\|^{q+1}_{L^{q+1}(\mathbb{R}^N)}$. By Lemma 3.2 we have that $\|\nabla \phi_{u_n}\|_{L^2(\mathbb{R}^N)} \to \|\nabla \phi_u\|_{L^2(\mathbb{R}^N)}$ as $n \to \infty$. Therefore we are left only to show $\|u_n\|_{H^1(\mathbb{R}^N)} \to \|u\|_{H^1(\mathbb{R}^N)}$. From (4.23) we see that (4.22) becomes

$$o(1) = \|u_n\|^2_{H^1(\mathbb{R}^N)} - \|u\|^2_{H^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \phi_{u_n} u_n (u_n - u)\rho(x) + o(1).$$

Splitting the nonlocal integral into the integral over a ball B of radius $R > 0$ and outside,

$$\left| \int_{\mathbb{R}^N} \phi_{u_n} u_n (u_n - u)\rho(x) \right| \leq \left| \int_B \phi_{u_n} u_n (u_n - u)\rho(x) \right| + \left| \int_{\mathbb{R}^N \setminus B} \phi_{u_n} u_n (u_n - u)\rho(x) \right|.$$

Looking at the integral over B, by Hölder and Sobolev inequalities, setting $r = \frac{4N}{N+2}$ we get

$$\begin{aligned} \left| \int_B \phi_{u_n} u_n (u_n - u)\rho(x) \right| &\leq \|\rho(x)\|_{L^\infty(B)} \|\phi_{u_n}\|_{L^{2^*}(B)} \|u_n - u\|_{L^r(B)} \|u_n\|_{L^r(B)} \\ &\leq C \|\phi_{u_n}\|_{D^{1,2}(\mathbb{R}^N)} \|u_n - u\|_{L^r(B)}. \end{aligned}$$

As $\phi_{u_n} \in D^{1,2}(\mathbb{R}^N)$ is a solution to the Poisson equation, $\|\phi_{u_n}\|_{D^{1,2}(\mathbb{R}^N)} < C$ by some constant $C > 0$. By local compactness, $\|u_n - u\|_{L^r(B)} \to 0$ for all $r < 2^*$. Thus $\int_{\mathbb{R}^N} \phi_{u_n} u_n (u_n - u)\rho(x) \to 0$. The integral outside the ball can be bounded similarly to Lemma 3.2 using the bounds found in [16, p. 1077], for every $\delta > 0$ there exists a ball of radius R such that

$$\left| \int_{\mathbb{R}^N \setminus B} \phi_{u_n} u_n (u_n - u)\rho(x) \right| \leq \left| \int_{\mathbb{R}^N \setminus B} \phi_{u_n} u_n^2 \rho(x) \right| + \left| \int_{\mathbb{R}^N \setminus B} \phi_{u_n} u_n u \rho(x) \right| < 2\delta.$$

Therefore, making R large enough this integral tends to 0 and we obtain the result

$$\|u_n\|^2_{H^1(\mathbb{R}^N)} - \|u\|^2_{H^1(\mathbb{R}^N)} = o(1) \tag{4.24}$$

implying, $\|u_n\|^2_{H^1(\mathbb{R}^N)} \to \|u\|^2_{H^1(\mathbb{R}^N)}$. Combining all these into (4.22) concludes the proof. □

4.4 The case of coercive ρ

Now that we have all the necessary preliminaries in place we present the following Theorems 4.2, 4.3, 4.4.

Theorem 4.2 (**Groundstates for $q \geq 3$ under (ρ_2)**). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ be nonnegative, satisfying (ρ_2), and $q \in [3, 2^*-1)$. Then, for any fixed $\lambda > 0$, ($\mathcal{SP}$) has both a positive mountain-pass solution and a positive groundstate solution in $E(\mathbb{R}^3)$, whose energy levels coincide for $q > 3$.*

Proof of Theorem 4.2. Using Lemma 3.4 and Lemma 4.5, the Mountain-Pass Theorem yields the existence a mountain-pass type solution for all $q \in [3, 2^* - 1)$. Namely, there exists $u \in E(\mathbb{R}^N)$ such that $I_\lambda(u) = c_\lambda$ and $I'_\lambda(u) = 0$, where c_λ is given in (3.7). For $q > 3$, by Lemma 3.5 the mountain-pass level c_λ has the characterisation

$$c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \quad \mathcal{N}_\lambda := \{u \in E(\mathbb{R}^N) \setminus \{0\} \mid I'_\lambda(u)(u) = 0\},$$

and it follows that u is a groundstate solution of I_λ. Since $I_\lambda(u) = I_\lambda(|u|)$, we can assume $u \geq 0$, and so $u > 0$ by the strong maximum principle, Proposition 3. For $q = 3$, we can show the existence of a positive mountain-pass solution applying the general min-max principle [55, p.41], and observing that, in our context, we can restrict to admissible curves γ's which map into the positive cone $P := \{u \in E(\mathbb{R}^3) : u \geq 0\}$. In fact, arguing as in [42, p.481], since I_λ satisfies the mountain-pass geometry by Lemma 3.4, it is possible to select a Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ at the level c_λ such that

$$\text{dist}(u_n, P) \to 0,$$

from which it follows that $(u_n)_- \to 0$ in $L^6(\mathbb{R}^3)$, see also [15, Lemma 2.2]. Then, by construction and up to a subsequence, there exists a weak limit $u \geq 0$, and hence, by Lemma 4.5 a nontrivial nonnegative solution, the positivity of which holds by Proposition 3.

The existence of a positive groundstate can be shown with a mild modification to the proof of Theorem 4.1, using here that all the relevant convergence statements hold for any fixed $\lambda > 0$ as a consequence of assumption (ρ_2) and Lemma 4.4. This is enough to conclude. □

Theorem 4.3 (**Groundstates for** $q < 3$ **under** (ρ_2))**.** *Let $N = 3, 4, 5$, $q \in (2, 3)$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Let $\lambda > 0$, and assume $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative and satisfies (ρ_2). Moreover suppose that $k\rho(x) \le (x, \nabla\rho)$ for some $k > \frac{-2(q-2)}{(q-1)}$. Then, $(\mathcal{SP})$ has a mountain-pass solution $u \in E(\mathbb{R}^N)$. Moreover, there exists a groundstate solution.*

Proof of Theorem 4.3. We can argue as in [41, Theorem 1.3], based on [29] and on the compactness of the embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$. The latter is provided in our context by Lemma 4.4. By these, there exists an increasing sequence $\mu_n \to 1$ and $(u_n)_{n\in\mathbb{N}} \in E(\mathbb{R}^N)$ such that $I_{\mu_n,\lambda}(u_n) = c_{\mu_n,\lambda}$ and $I'_{\mu_n,\lambda}(u_n) = 0$, where $I_{\mu_n,\lambda}$ and $c_{\mu_n,\lambda}$ are defined as in (3.4) and (3.5). By Lemma 3.9, we see that

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2) + \left(\frac{N+2+2k}{4}\right)\lambda^2\int_{\mathbb{R}^N}\rho(x)\phi_{u_n}u_n^2 \\ - \frac{N\mu_n}{q+1}\int_{\mathbb{R}^N}|u_n|^{q+1} \le 0.\end{aligned} \tag{4.25}$$

Setting $\alpha_n = \int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2)$, $\gamma_n = \lambda^2\int_{\mathbb{R}^N}\rho(x)\phi_{u_n}u_n^2$, $\delta_n = \mu_n\int_{\mathbb{R}^N}|u_n|^{q+1}$, we can put together the equalities $I_{\mu_n,\lambda}(u_n) = c_{\mu_n,\lambda}$ and $I'_{\mu_n,\lambda}(u_n)(u_n) = 0$ with (4.25) obtaining the system

$$\begin{cases} \alpha_n + \gamma_n - \delta_n = 0, \\ \frac{1}{2}\alpha_n + \frac{1}{4}\gamma_n - \frac{1}{q+1}\delta_n = c_{\mu_n,\lambda}, \\ \frac{N-2}{2}\alpha_n + \left(\frac{N+2+2k}{4}\right)\gamma_n - \frac{N}{q+1}\delta_n \le 0, \end{cases} \tag{4.26}$$

which yields

$$\begin{aligned}\delta_n &\le \frac{c_{\mu_n,\lambda}(6 - N + 2k)(q+1)}{2(q-2) + k(q-1)}, \\ \gamma_n &\le \frac{2c_{\mu_n,\lambda}\big(2(q+1) - N(q-1)\big)}{2(q-2) + k(q-1)}, \\ \alpha_n &= \delta_n - \gamma_n.\end{aligned}$$

We note that $k > \frac{-2(q-2)}{(q-1)} > \frac{N-6}{2}$ since $q < 2^* - 1$, and so since $\alpha_n, \gamma_n, \delta_n$ are all nonnegative, it follows that $\alpha_n, \gamma_n, \delta_n$ are all bounded. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded and there exists $u \in E(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $E(\mathbb{R}^N)$. Using Lemma 4.4 and arguing as in [16, Theorem 1] we obtain that $\|u_n\|^2_{E(\mathbb{R}^N)} \to \|u\|^2_{E(\mathbb{R}^N)}$ and

$$c_{\mu_n,\lambda} = I_{\mu_n,\lambda}(u_n) \to I_\lambda(u). \tag{4.27}$$

It follows that $u_n \to u$ in $E(\mathbb{R}^N)$, which combined with the left-continuity property of the levels [5, Lemma 2.2], namely $c_{\mu_n,\lambda} \to c_{1,\lambda} = c_\lambda$ as $\mu_n \nearrow 1$, yields $I_\lambda(u) = c_\lambda$. Since u is a critical point by the weak convergence, it follows that u is mountain-pass solution. Finally, the existence of a groundstate solution is based on minimising over the set of nontrivial critical points of I_λ, and carrying out an identical argument to the above to show the strong convergence of such a minimising sequence, again using Lemma 4.4. This concludes the proof. □

Under an additional hypotheses on ρ, we now prove that the energy level of the groundstate solutions coincide with the mountain-pass level.

Theorem 4.4 (**Homogeneous case for $q \le 3$: mountain-pass solutions vs. groundstates**). *Let $N = 3, 4, 5$, $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\lambda > 0$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, satisfies (ρ_2), and is homogeneous of degree $\bar{k}$, namely $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \left(\max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1\right)_+ .$$

Then, the mountain-pass solutions that we find in Theorem 4.2 ($q = 3$) and Theorem 4.3 ($q < 3$) are groundstates.

Proof. By Proposition 2, it holds that

$$c_\lambda = \inf_{u \in \bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u),$$

where $\bar{\mathcal{M}}_{\lambda,\nu}$ is defined in (3.10). Since $J_{\lambda,\nu}(u) = 0$ is equivalent to the Pohozaev equation given by Lemma 3.9 minus the equation $\nu I'_\lambda(u)(u) = 0$, it is clear that $\bar{\mathcal{M}}_{\lambda,\nu}$ contains all of the critical points of I_λ, and thus the mountain-pass solutions that we find in Theorem 4.2 ($q = 3$) and Theorem 4.3 ($q < 3$) are groundstates. This completes the proof. □

5 Category Theory

In this chapter we look at certain topological properties of the set of critical points obtained from an invariant functional which satisfies the Palais-Smale condition. In the spirit of Ambrosetti-Rabinowitz [4] and Ambrosetti-Ruiz [5] we analyse the properties of "genus" and look at the role of deformations in a minimax setting. Our goal is to define minimax levels of our functional over a certain closed symmetric set, to do this we take advantage of Lusternik-Schnirelmann theorems, which will be described in the latter half of this chapter. Before these LS-theorems can be discussed an understanding of Brouwer degree and its properties is needed, this will provide context to what we do when working with the concept of "genus". The contents of this chapter were taken from a series of lectures presented by Carlo Mercuri and are based from the content of "Variational Methods in Differential Equations" by Costa [21].

5.1 Brouwer Degree

Before we can discuss the Lusternik-Schnirelmann theory and the concept of "genus" we need to understand the notion of Brouwer degree. This degree theory will form a basis for what will come in the sequel, many of these properties can be applied when working with the notion of category and these properties and their applications (e.g. Borsuk Theorem) will be vital to show existence of multiple critical points.

Definition 6. Consider $N \geq 1$, $\Omega \subset \mathbb{R}^N$ open and bounded. $f \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$, $b \in \mathbb{R}^N$ and $b \notin f(\partial\Omega)$, i.e. $0 < \varepsilon < \operatorname{dist}(b, f(\partial x))$ for $\varepsilon > 0$.

Consider $\varphi \in C((0, +\infty), \mathbb{R}^N)$, *supp* $\varphi \in (0, \varepsilon)$ such that

$$\int_{\mathbb{R}^N} \varphi(|x|)\,\mathrm{d}x = 1.$$

We define the degree as:

$$\deg(f, b.\Omega) := \int_{\mathbb{R}^N} \varphi(|f(x) - b|) J_f(x)\,\mathrm{d}x.$$

The above is said to be the Brouwer degree.

5.1.1 Properties of Brouwer Degree

Under certain assumptions on a generic function f acting on a domain $\Omega \subset \mathbb{R}^N$ we list the following properties of Brouwer degrees.

(BD 1) **Stability:** Consider $f_1, f_2 \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$, $b \in \mathbb{R}^N$ and $b \notin f_1(\partial\Omega) \cup f_2(\partial\Omega)$. Then,

$$0 < \varepsilon < \frac{1}{4}\text{dist}(b, f_1(\partial\Omega) \cup f_2(\partial\Omega)).$$

If $\|f_1 - f_2\|_{L^\infty(\mathbb{R}^N)} < \varepsilon$, this implies

$$\deg(f_1, b, \Omega) = \deg(f_2, b, \Omega).$$

We can drop the $f \in C^1(\Omega, \mathbb{R}^N)$ assumption thanks to the following theorem.

Theorem 5.1. *Let Ω be open and bounded. $f \in C(\overline{\Omega})$, $b \notin f(\partial\Omega)$ and let $(f_k)_{k\in\mathbb{N}} \subset C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ be such that*

$$\|f_k - f\|_{L^\infty(\mathbb{R}^N)} \to 0, \quad \text{as } k \to \infty.$$

One can define

$$\deg(f, b, \Omega) := \lim_{k\to\infty} \deg(f_k, b, \Omega).$$

This allows us to write (BD 1) as follows:

(BD 1.5) **General stability property:** Consider $f_1, f_2 \in C(\overline{\Omega}, \mathbb{R}^N)$, $b \in \mathbb{R}^N$, $b \notin f_1(\partial\Omega) \cup f_2(\partial\Omega)$. Take $0 < \varepsilon < \frac{1}{4}\text{dist}(b, f_1(\partial\Omega) \cup f_2(\partial\Omega))$.

If $\|f_1 - f_2\|_{L^\infty(\mathbb{R}^N)} < \varepsilon$ then, this implies

$$\deg(f_1, b, \Omega) = \deg(f_2, b, \Omega).$$

(BD 2) **Stability with respect to b:** Let Ω be open and bounded. $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and b, b' belong to the same connected component of $f(\partial\Omega)^c$. $f(\partial\Omega)^c$ is an open set, every open set can be written as the disjoint union of its connected components. Then:

$$\deg(f, b, \Omega) = \deg(f, b', \Omega).$$

(BD 3) **Additivity:** Let Ω_1, Ω_2 be open, bounded and disjoint sets. $f \in C(\overline{\Omega}_1 \cup \overline{\Omega}_2, \mathbb{R}^N)$. If $b \notin f(\partial\Omega_1) \cup f(\partial\Omega_2)$, this implies

$$\deg(f, b, \Omega_1 \cup \Omega_2) = \deg(f, b, \Omega_1) + \deg(f, b, \Omega_2).$$

(BD 4) **Excision:** Let Ω be open and bounded. $f \in C(\overline{\Omega}, \mathbb{R}^N)$, $K \subset \Omega$ be compact and $b \notin f(\partial\Omega) \cup f(K)$. Then, this implies

$$\deg(f, b, \Omega) = \deg(f, b, \Omega \setminus K).$$

(BD 5) **Solution property:** Let Ω be open and bounded. $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $b \notin f(\partial\Omega)$. If $\deg(f, b, \Omega) \neq 0$, this implies $f(x) = b$ has a solution in Ω.

(BD 6) **Stability with respect to Homotopy:** If $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^N$ is continuous, $b \notin H(\partial\Omega \times [0, 1])$. Then, for all $t \in [0, 1]$, it holds that

$$\deg(H(\cdot, t), b, \Omega) = \deg(H(\cdot, 0), b, \Omega).$$

(BD 7) **Boundary property:** Let Ω be open and bounded. $f, g \in C(\overline{\Omega}, \mathbb{R}^N)$ and $f = g$ on $\partial\Omega$. If $b \notin f(\partial\Omega)$ (or $g(\partial\Omega)$), this implies

$$\deg(f, b, \Omega) = \deg(g, b, \Omega).$$

Proposition 11. *Let Ω be open and bounded, $b \in \mathbb{R}^N$. Consider $\mathbb{1}$ as the identity map $\mathbb{1} : \mathbb{R}^N \to \mathbb{R}^N$. Then,*

$$\deg(\mathbb{1}, b, \Omega) = \begin{cases} 1, & b \in \Omega, \\ 0, & b \notin \overline{\Omega}, \end{cases}$$

and

$$\deg(-\mathbb{1}, b, \Omega) = \begin{cases} (-1)^N, & b \in \Omega, \\ 0, & b \notin \overline{\Omega}. \end{cases}$$

Definition 7. Let $f, g : X \to Y$ be continuous. We say that f, g are homotopically equivalent (or that f is homotopic to g) if there exists $H : X \times [0, 1] \to Y$ continuous on $X \times [0, 1]$ such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x).$$

In the following, and for the remainder of this chapter, we define $J_f(x)$ as the determinant of the Jacobian matrix of f at the point x.

Theorem 5.2 (Sard's Theorem)**.** *Let Ω be open and bounded, $f \in C^1(\Omega, \mathbb{R}^N)$ and define the singular set*

$$S := \{x \in \Omega \mid J_f(x) = 0\}. \tag{5.1}$$

Then, $f(S)$ is a zero measure set:

$$\mathcal{L}^N(f(S)) = 0.$$

From definition 6 it is difficult to quantify if $\deg(f,b,\Omega) \in \mathbb{R}$ or a subset. Here we claim the degree is indeed an integer.

Claim 1 ($\deg \in \mathbb{Z}$)**.** *Let Ω be open and bounded, $f \in C^1(\Omega,\mathbb{R}^N) \cap C(\overline{\Omega},\mathbb{R}^N)$. Set S as above and assume $b \notin f(\partial\Omega) \cup f(S)$. Then,*

$$\deg(f,b,\Omega) = \sum_{x \in f^{-1}(b)} \operatorname{sgn}(J_f(x)),$$

where sgn *is the sign function defined as*

$$\operatorname{sgn}(r) = \begin{cases} 1, & r > 0, \\ -1, & r < 0. \end{cases}$$

By assumption on b above, definition of S and as a consequence of the inverse function theorem we have that b is a regular value of f and hence the set $f^{-1}(b)$ is finite and the above sum is also finite.

We wish to generalise the claim to functions that are only continuous on $\overline{\Omega}$. We can take advantage of Sard's theorem 5.2 and Theorem 5.1 to drop the C^1 regularity and $f(S)$.

Proposition 12. *Let Ω be open and bounded. $f \in C(\overline{\Omega},\mathbb{R}^N)$, $b \notin f(\partial\Omega)$. Then,*

$$\deg(f,b,\Omega) \in \mathbb{Z}.$$

Sketch of proof. Details for the proof of Proposition 12 can be found in [3, p. 27-28,36], we will provide a sketch. It is possible to construct a sequence of approximating functions $f_k \in C^1(\Omega,\mathbb{R}^N) \cap C(\overline{\Omega},\mathbb{R}^N)$ that converge uniformly on $\overline{\Omega}$ to $f \in C(\overline{\Omega},\mathbb{R}^N)$. Using Theorem 5.1 we can see that $\deg(f,b,\Omega) = \lim_{k\to\infty} \deg(f_k,b,\Omega)$. Similarly, using Sard's Theorem 5.2, we can construct a sequence $b_j \notin f(\partial\Omega) \cup f(S)$ that converge uniformly to $b \notin f(\partial\Omega)$ yielding $\lim_{j\to\infty} \deg(f,b_j,\Omega) = \deg(f,b,\Omega)$. The combination of these concludes

$$\deg(f,b,\Omega) = \lim_{k\to\infty} \deg(f_k,b,\Omega) = \lim_{\substack{k\to\infty \\ j\to\infty}} \deg(f_k,b_j,\Omega) \in \mathbb{Z},$$

by claim 1. ☐

5.1.2 Some applications

We wish to describe some applications of the properties of Brouwer degree, namely the Brouwer fixed point theorem and Borsuk theorem. We begin with the following proposition showing that the unit sphere is not a retract of the unit ball.

Proposition 13. *Consider the unit ball* $B(0,1) \subset \mathbb{R}^N$, $\mathbb{S}^{N-1} = \partial B(0,1)$. *There is no continuous function* $\varphi : \overline{B(0,1)} \to \mathbb{S}^{N-1}$ *such that*

$$\varphi|_{\mathbb{S}^{N-1}} = \mathbb{1}.$$

(i.e. The unit sphere is not a 'retract' of the unit ball).

Proof. Assume the contrary:

$$\deg(\varphi, 0, B) = \deg(\mathbb{1}, 0, B) = 1 \neq 0.$$

Thus, by the solution property (BD 5), $\varphi(x) = 0$ has a solution $x \in B$. This is a contradiction to the fact that $\varphi : \overline{B(0,1)} \to \mathbb{S}^{N-1}$. □

Theorem 5.3 (Brouwer Fixed Point Theorem)**.** *Let* $B(0,1) \subset \mathbb{R}^N$ *be the unit ball in* $\mathbb{R}^N$ *centered at the origin. Assume that* $f : \overline{B(0,1)} \to \overline{B(0,1)}$ *is continuous. Then there exists* $x \in \overline{B(0,1)}$ *such that*

$$f(x) = x.$$

Namely, x is a fixed point of f.

Proof. Assume there exists $x \in \mathbb{S}^{N-1}$ such that $f(x) = x$, then we are done. Otherwise, assume $x - f(x) \neq 0$ on $\mathbb{S}^{N-1}$. Define

$$H(x,t) = x - tf(x), \quad t \in [0,1],\ x \in \overline{B(0,1)}.$$

We claim that for all $t \in [0,1]$ and for all $x \in \mathbb{S}^{N-1}$ we have

$$x - tf(x) = b \neq 0.$$

If $t = 1$ this is done. For $t = [0,1)$ assume by contradiction that there exists $x \in \mathbb{S}^{N-1}$ such that $x - tf(x) = 0$. This would imply

$$t\,|f(x)| = |x| = 1 \Rightarrow t\,|f(x)| = 1,$$

however, $|f(x)| < 1$ and $t < 1$, hence a contradiction. Therefore, setting

$$H(x,t) = x - tf(x) \neq 0, \quad \forall x \in \mathbb{S}^{N-1},$$

and by the homotopy invariance (BD 6) we get

$$\deg(H(\cdot,1),0,B) = \deg(H(\cdot,0),0,B) = \deg(\mathbb{1},0,B) = 1 \neq 0.$$

By the solution property (BD 5), the above yields that there exists $x \in B(1,0)$ such that $f(x) = x$. □

Definition 8. Let $A \subset X$ be a vector space and

$$-A := \{x \in X \mid -x \in A\}.$$

We say A is symmetric with respect to the origin $O \in X$ if

$$A = -A.$$

Theorem 5.4 (Borsuk Theorem)**.** *Assume Ω is an open and bounded subset of $\mathbb{R}^N$ that is symmetric with respect to $0 \in \mathbb{R}^N$. $f \in C(\overline{\Omega}, \mathbb{R}^N)$, f is odd, $0 \in f(\partial\Omega)$. Then,*

- *if $0 \in \Omega \Rightarrow \deg(f,0,\Omega)$ is odd.*
- *if $0 \in \overline{\Omega} \Rightarrow \deg(f,0,\Omega)$ is even.*

Corollary 5.1. *Assume the same as above however, assume in addition $f \in C^1(\Omega, \mathbb{R}^N)$ and $0 \notin f(S)$, where S is defined by (5.1). Then the same conclusion follows.*

Proof of Theorem 5.4 and Corollary 5.1. Assume $0 \in \overline{\Omega}$ and $f^{-1}(0) \neq \emptyset$. It follows that $\deg(f,0,\Omega) = 0$ by the solution property (BD 5). If $f^{-1}(0)$ is non-empty, then,

$$f^{-1}(0) = \{x_i, -x_i\}, \quad x_i \in \Omega,\ i = 1,\dots,m.$$

And

$$\begin{aligned}\deg(f,0,\Omega) &= \sum_{i=1}^{m} [\operatorname{sgn}(J_f(x_i)) + \operatorname{sgn}(J_f(-x_i))] \\ &= 2\sum_{i=1}^{m} \operatorname{sgn}(J_f(x_i)) \ \text{ is even.}\end{aligned}$$

If $0 \in \Omega$, then either $f^{-1}(0) = 0$ or

$$f^{-1}(0) = \{0\} \cup \{x_i, -x_i\}, \quad i = 1, \ldots, m.$$

In the former case, $\deg(f, 0, \Omega) = \pm 1$. In the latter case,

$$\begin{aligned}\deg(f, 0, \Omega) &= \pm 1 + \sum_{i=1}^{m} \left[\operatorname{sgn}\left(J_f(x_i)\right) + \operatorname{sgn}\left(J_f(-x_i)\right)\right] \\ &= \pm 1 + 2\sum_{i=1}^{m} \operatorname{sgn}\left(J_f(x_i)\right) \text{ is odd.}\end{aligned}$$

□

5.2 Lusternik-Schnirelmann Theory

The LS-theory was developed by L. Lusternik and L. Schnirelmann in the first half of the 20$^{\text{th}}$ century. The idea was a topological concept of category $\operatorname{cat}(A, X)$ of a closed manifold A on a metric space X. This theory can be applied to variational problems, particularly those involving even functionals φ acting on some closed invariant subset $\mathcal{A}$.

In this section we use the LS-theory to define a minimax characterisation c_k of an even functional φ over a suitable set $\mathcal{A}$ briefly described above, namely we want to define

$$c_k := \inf_{A \in \mathcal{A}_k} \sup_{x \in A} \varphi(x)$$

where $\varphi \in C^1(X, \mathbb{R})$, X is a Banach space and $\{\mathcal{A}_k\}_{k \in \mathbb{N}}$ are closed subsets of X. Here we take advantage of the fact that deformations are odd mappings and have certain properties around critical levels c_k to show that we can obtain at least k pairs of distinct critical points corresponding to these critical levels under some suitable compactness assumption.

First we must state a few definitions for convenience in the latter part of this section.

Definition 9 (Isometric representation). Let X be a Banach space and G be a compact topological group. The set $\{T(g) : g \in G\}$ is an isometric representation of G on X if $T(g) : X \to X$ is an isometry for each $g \in G$ and the following hold:

(i) $T(g_1 + g_2) = T(g_1) \circ T(g_2)$ for all $g_1, g_2 \in G$

(ii) $T(0) = \mathbb{1}$, where $\mathbb{1} : X \to X$ is the identity map on X

(iii) $(g, u) \mapsto T(g)(u)$ is continuous.

Definition 10 (Invariant subset). A subset $A \subset X$ is invariant if, and only if, $T(g)A = A$ for all $g \in G$.

Definition 11 (Invariant functional). A functional $\varphi : X \to \mathbb{R}$ is invariant, if and only if, $\varphi \circ T(g) = \varphi$ for all $g \in G$.

Definition 12 (Equivariant mapping). A mapping R between two invariant subsets A_1 and A_2, namely $R : A_1 \to A_2$, is said to be equivariant if $R \circ T(g) = T(g) \circ R$ for all $g \in G$.

Definition 13 (The class $\mathcal{A}$). We denote the class of all closed and invariant subsets of X by $\mathcal{A}$. Namely,

$$\mathcal{A} := \{A \subset X : A \text{ closed}, \, T(g)A = A \; \forall g \in G\}.$$

Definition 14 (G-index with respect to $\mathcal{A}$). A G-index on X with respect to $\mathcal{A}$ is a mapping $\text{ind} : \mathcal{A} \to \mathbb{N} \cup \{+\infty\}$ such that the following hold:

(i) $\text{ind}(A) = 0$ if and only if $A = \emptyset$.

(ii) If $R : A_1 \to A_2$ is continuous and equivariant, then $\text{ind}(A_1) \leq \text{ind}(A_2)$.

(iii) $\text{ind}(A_1 \cup A_2) \leq \text{ind}(A_1) + \text{ind}(A_2)$.

(iv) If $A \in \mathcal{A}$ is compact, then there exists a neighbourhood N of A such that $N \in \mathcal{A}$ and $\text{ind}(N) = \text{ind}(A)$.

An interesting case of the above definition is the scenario when $G = \mathbb{Z}_2 = \{0, 1\}$. This case yields the following definition of genus and is attributed to Krasnoselksii [30].

Definition 15 (Krasnoselskii Genus)**.** Define $T(0) = \mathbb{1}_x$ as the identity on X, $T(1) = -\mathbb{1}_x$. Given any closed, symmetric w.r.t. origin set $A \subset \mathcal{A}$, define $\varphi(A) = k \in \mathbb{N}$ is the least possible dimension of $\mathbb{R}^k$ such that there exists an odd continuous mapping

$$\Phi : A \to \mathbb{R}^k \setminus \{0\}.$$

Then the mapping $\varphi : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is a $\mathbb{Z}_2$-index. We set $\varphi(\emptyset) = 0$ and $\varphi(A) = \infty$ if there is no such mapping.

Notice that, since $A \subset \mathcal{A}$ implies $T(g)A = A$ for all $g \in G$. Applying this to the above definition for the Krasnoselski genus, $T(1)A = -A \overset{\text{Def } 10}{=} A$.

We define the notion of equivariance for our class of subsets $\mathcal{A}$. We recall $R : A_1 \to A_2$, for $A_1, A_2 \subset \mathcal{A}$ is equivariant if, and only if,

$$R \circ T(g) = T(g) \circ R, \quad \forall g \in G = \{0, 1\}.$$

Namely,

$$\begin{cases} R \circ T(0) = T(0) \circ R \;\Rightarrow\; R \circ \mathbb{1} = \mathbb{1} \circ R, \\ R \circ T(1) = T(1) \circ R \;\Rightarrow\; R \circ (-\mathbb{1}) = (-\mathbb{1}) \circ R. \end{cases}$$

This final line implies

$$\left.\begin{aligned} (R \circ (-\mathbb{1}))(x) = R(-x) \\ ((-\mathbb{1}) \circ R)(x) = -R(x) \end{aligned}\right\} \iff R(-x) = -R(x).$$

which implies an odd mapping. From here on we denote $\gamma(\cdot)$ as the genus.

We verify that Krasnoselski Genus satisfies property (ii) in definition 14, as this will be relevant for future discussions regarding how we use the genus.

Indeed, if $\gamma(A_2) < \infty$, namely, there exists a map $\Phi : A_2 \to \mathbb{R}^k \setminus \{0\}$, continuous and odd. Here $\gamma(A_2) = k$ implies

$$\Phi \circ R : A_1 \to \mathbb{R}^k$$

is odd and continuous. Therefore,

$$\gamma(A_1) \leq k = \gamma(A_2).$$

If $\gamma(A_2) = \infty$ then (ii) is obvious.

The following proposition gives us the genus for some special subsets of a Banach space X.

Proposition 14. *1) If $c \subset X$ is closed and $c \cap (-c) = \emptyset$, then*

$$\gamma(c \cup (-c)) = 1.$$

2) If $A \in \mathcal{A}$ and if there exists an odd homomorphism $h : A \to \mathbb{S}^{k-1}$, then $\gamma(A) = k$.

3) If $A \in \mathcal{A}$, such that $0 \notin A$. Then $\gamma(A) \geq 2$ implies that A has infinitely many points.

Proof. 1) Consider $\Phi : c \cup (-c) \to \mathbb{R} \setminus \{0\}$. Then $\Phi(c) = 1$ and $\Phi(-c) = -1$.

2) Since $h : A \to \mathbb{S}^{k-1} \subset \mathbb{R}^k \setminus \{c\}$ is odd and continuous implies $\gamma(A) \leq k$. We show that if we assume $\gamma(A) < k$ yields a contradiction. If $\gamma(A) = j < k$, then there exists

$$\Phi : A \to \mathbb{R}^j \setminus \{0\}$$

odd and continuous. Then set $\psi := \Phi \circ h^{-1} : \mathbb{S}^{k-1} \to \mathbb{R}^j \setminus \{0\}$ odd and continuous, pick $u_0 = (0, \ldots, x)$, $x \in \mathbb{R}^{k-j}$, such that $|u_0| > 1$ (i.e. $0 = u_0 \in \mathbb{R}^j$, $0 \neq u_0 \in \mathbb{R}^k \setminus \mathbb{R}^j$). Consider the homotopy $\psi_t = (1-t)\psi + tu_0$ for $t \in [0,1]$, then, by the solution property (BD 5), $\psi_1 = u_0$ implies $\deg(u_0, 0, B_k) = 0$. Therefore by the homotopy invariance (BD 6)

$$\deg(\psi, 0, B_k) = \deg(u_0, 0, B_k) = 0$$

Since we are working in a symmetric domain and ψ is an odd mapping this is a contradiction of the Borsuk theorem 5.4, the degree should be odd.

3) Since $0 \notin A$ and A is symmetric w.r.t. the origin, then $A = c \cup (-c)$, if A is a finite set, and such that $c \cap (-c) = \emptyset$ and, by 1), $\gamma(A) = 1$ which is a contradiction.

□

Remark 5.1. A consequence of 2): Set $A = \mathbb{S}^{k-1}$ and $h = \mathbb{1}$ which is an odd mapping. Then, $\gamma(\mathbb{S}^{k-1}) = k$.

5.2.1 Relation to Deformations

In this section we give a flavour of how one might obtain multiplicity results from the above concepts. We consider the role of deformations in critical point theory. Indeed results like the Mountain Pass Theorem by Ambrosetti-Rabinowitz [4] rely heavily on deformations. Here we look at an example problem using the equivariant form of the deformation theorem. To begin, consider the following example in a Hilbert space $\mathcal{H}$.

$$\eta(u,t) \in X = \mathcal{H}$$

and $J : \mathcal{H} \to \mathbb{R}$, $J \in C^1(\mathcal{H}, \mathbb{R})$, $\nabla J(u) : \mathcal{H} \to \mathcal{H}$.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\eta(u,t) = -\nabla J(\eta(u,t)), \\ \eta(u,0) = u. \end{cases}$$

For each u in $\mathcal{H}$ the above has a unique solution $\eta(u, \cdot) \in \mathbb{R}$ and is continuous on $\mathcal{H} \times \mathbb{R}$. This choice of space makes sense as we wish to use the notion of gradient, unfortunately we do not have the gradient for general Banach spaces X. Therefore, we want to show the existence of a pseudo-gradient.

Definition 16 (Equivariant Pseudo-Gradient)**.** Consider an invariant functional $\varphi \in C^1(X, \mathbb{R})$, i.e. we have a G-compact topological group and a $T(g)$-isometric representation, φ possesses an equivariant pseudo-gradient, that is, a locally Lipschitz mapping $v : Y \to X$, where

$$Y = \{u \in X \mid \varphi'(u) \neq 0\},$$

satisfying:

(i) $\|v(u)\|_X \leq 2\|\varphi'(u)\|_*$,

(ii) $\varphi'(u) \cdot v(u) \geq \|\varphi'(u)\|_*^2$,

(iii) v is equivariant.

The following theorem details the properties of equivariant deformations of a functional which satisfies the Palais-Smale condition. These deformations are used heavily as they show the existence of a critical point implies a change in topology and is key in proving that critical points of a functional exist at certain levels.

Theorem 5.5 (Equivariant Deformation)**.** *Let $\varphi \in C^1(X, \mathbb{R})$ be invariant and satisfying the Palais-Smale condition. If U is an invariant open neighbourhood of K_c, where*

$$K_c = \{u \in X \mid \varphi(u) = c, \varphi'(u) = 0\},$$

for $c \in \mathbb{R}$. Then for small $\varepsilon > 0$, there exists $\eta \in C([0,1] \times X, X)$ such that, for any $u \in X$, $t \in [0,1]$, it holds:

(i) $\eta(0, u) = 0$,

(ii) $\eta(t, u) = u$ *if* $u \in \varphi^{-1}[c - 2\varepsilon, c + 2\varepsilon]$,

(iii) $\eta(1, \varphi^{c+\varepsilon} \setminus U) \subset \varphi^{c-\varepsilon}$,

(iv) $\eta(t, \cdot) : X \to X$ *is an equivariant homeomorphism.*

Assume we have $T(g)$ of G, X a Banach space and assume we have a G-index. Set

$$A_j = \{A \subset X \mid A \text{ compact, invariant, } \operatorname{ind}(A) \geq j\}.$$

Set

$$c_j = \inf_{A \in A_j} \max_{u \in A} \varphi(u).$$

Remark 5.2. From our definition of A, we have the following collection of sets, $A_1 \supset A_2 \supset \dots$, and the corresponding for c_j, $-\infty \leq c_1 \leq c_2 \leq \dots$.

Theorem 5.6. *Let $\varphi \in C^1(X, \mathbb{R})$ be invariant and satisfy the Palais-Smale condition for all levels c_j defined above. If $c_j > -\infty$ for some $j \geq 1$, then c_j is a critical value of φ. Moreover, if $c_k = c_j = c$, for $k \geq j$, then ind$(K_c) \geq k - j + 1$.*

Proof. We split the proof into two steps.
Step 1: Since φ is invariant, K_c is invariant and compact by the assumption φ satisfies the Palais-Smale condition. We want to show $-\infty < c_j$ is critical. By definition, pick $A \in \mathcal{A}$, compact, with $\text{ind}(A) \geq j$ such that $\max_A \varphi \leq c_j + \varepsilon$ for some small $\varepsilon > 0$.

If c_j is not critical then we can deform the level into some sublevel. Pick $\eta(\cdot, \cdot)$ as in the equivariant deformation theorem 5.5. Set

$$C_A = \eta(1, A) \subset \varphi^{c_j - \varepsilon},$$

this implies

$$\max_{C_A} \varphi \leq c_j - \varepsilon,$$

and C_A is compact and invariant. Therefore

$$j \leq \text{ind}(A) \leq \text{ind}(C_A) \leq j - 1,$$

a contradiction.
Step 2: We want to show $c_j = c_k$. Consider $N \supset K_c$, closed and invariant neighbourhood such that

$$\text{ind}(N) = \text{ind}(K_c).$$

Set $U = \text{int}(N)$, i.e. the set of interior points of N, open and invariant neighbourhood of K_c. Now setting $c = c_k$ means pick $A \in \mathcal{A}$ such that

$$\max_A \varphi \leq c + \varepsilon,$$

and set $B = A \setminus U$, compact. Then by properties of G-index:

$$k \leq \text{ind}(A) \leq \text{ind}(B) + \text{ind}(N) = \text{ind}(B) + \text{ind}(K_c).$$

Set $c = c_j$. Since $B \subset \varphi^{c+\varepsilon} \setminus U$, then

$$C_B = \eta(1, B) \subset \varphi^{c-\varepsilon},$$

C_B is compact, from B, and invariant, from η equivariance. Since $c = c_j$, then $\max_{C_B} \varphi \leq c - \varepsilon = c_j - \varepsilon$, thus,

$$\text{ind}(C_B) \leq j - 1.$$

Using property (ii), $\text{ind}(B) \leq \text{ind}(C_B) \leq j - 1$, which implies

$$k \leq j - 1 + \text{ind}(K_c).$$

□

Finally, we obtain the following multiplicity result yielding at least k pairs of distinct critical points.

Theorem 5.7. *Let $\varphi \in C^1(X, \mathbb{R})$ be even and satisfy the Palais-Smale condition. Suppose:*

(i) φ is bounded from below.

(ii) There exists a compact set $K \in A$, symmetric, such that

$$\gamma(K) = k, \quad \text{and} \quad \sup_K \varphi < \varphi(0).$$

Then, φ has at least k pairs of distinct critical points corresponding to levels below $\varphi(0)$.

Proof. Set $G = \mathbb{Z}_2$, $A_j = \{A \subset X \mid A \text{ compact}, A = -A, \gamma(A) \geq j\}$. Define c_j as before, i.e. $c_1 \leq c_2 \leq \dots,$

$$-\infty < \inf_X \varphi \leq c_1 \leq c_2 \leq \dots,$$

implying $c_j > -\infty$. If we consider $c_k \leq \varphi(0)$ as a consequence of ii). If all c_j, $j = 1, \dots, k$, are distinct we obtain at least k pairs of critical points corresponding to critical levels

$$-\infty < c_1 < c_2 < \dots < c_k.$$

If $c_i = c_j$ for some $i < j \leq k$ then setting $c = c_i = c_j$, $\gamma(K_c) \geq j - i + 1 \geq 2$, and, as $0 \notin K_c$, then K_c is made up of infinitely many points. □

6 Multiplicity Results

In this final chapter we focus on multiplicity results for (1.2), namely we show the existence of high energy solutions to our PDE in the case where ρ satisfies $(\rho_{\mathbf{2}})$. Under this assumption the role of λ is diminished and can be chosen arbitrarily therefore, we fix $\lambda \equiv 1$ and drop the subscript on our functional to define

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y \\ - \frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}.$$

We follow the work of Ambrosetti-Rabinowitz [4] and Ambrosetti-Ruiz [5] who have developed techniques to study the existence of high energy solutions on special subsets of Banach spaces. The results in this chapter are from [23].

6.1 Divergence of min-max levels

We begin with a preliminary result vital to showing multiplicity in our problem. We follow [4] and define the following set

$$\hat{A}_0 = \{u \in E(\mathbb{R}^N) : I(u) \geq 0\}, \tag{6.1}$$

with the following definition of paths which cross this set:

$$\Gamma^* = \{h \in C(E(\mathbb{R}^N), E(\mathbb{R}^N)) : h(0) = 0, h \text{ is an odd homeomorphism of } E(\mathbb{R}^N) \\ \text{onto } E(\mathbb{R}^N), h(B_1) \subset \hat{A}_0\}. \tag{6.2}$$

In what follows, we invoke Lemma 3.1 which shows us that our space $E(\mathbb{R}^N)$ is separable and emits an orthogonal basis, what we mean by that is, for any $m \in \mathbb{N}$ we can write $E(\mathbb{R}^N)$ as

$$E(\mathbb{R}^N) = \operatorname{span}\{e_1, \dots, e_m\} \oplus \overline{\operatorname{span}}\{e_{m+1}, \dots\}.$$

We define these two components as

$$E_m = \operatorname{span}\{e_1, \dots, e_m\},$$
$$E_m^\perp = \overline{\operatorname{span}}\{e_{m+1}, \dots\},$$

and note that, for any $m \in \mathbb{N}$, E_m and $E_m^\perp$ define algebraically and topologically complementary subspaces of $E(\mathbb{R}^N)$.

The following Lemma shows the min-max levels over the paths described above diverge. This is important for the latter results in sections 6.2 and 6.3 to show that the pairs of critical points found are indeed distinct.

Lemma 6.1 (**Divergence of min-max levels** d_m)**.** *Let $N \geq 3$ and $q > 1$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_2). Define*

$$d_m{}^{1} := \sup_{h \in \Gamma^*} \inf_{u \in \partial B_1 \cap E_{m-1}^\perp} I(h(u)), \tag{6.3}$$

where Γ^* is given by (6.2). Then, $d_m \to +\infty$ as $m \to +\infty$.

Proof. To begin we set

$$T = \left\{ u \in E(\mathbb{R}^N) \setminus \{0\} : ||u||^2_{H^1(\mathbb{R}^N)} = ||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)} \right\}$$

where

$$\tilde{d}_m = \inf_{u \in T \cap E_m^\perp} ||u||_{E(\mathbb{R}^N)},$$

and we claim that $\tilde{d}_m \to +\infty$ as $m \to +\infty$. To see this we reason as in Szulkin [53], assume to the contrary that there exists $u_m \in T \cap E_m^\perp$ and some $d > 0$ such that $||u_m||_{E(\mathbb{R}^N)} \leq d$ for all $m \in \mathbb{N}$. Since $< e_n^*, u_m >= 0$ for all $m \geq n$ and the e_n^*'s are total by Lemma 3.1, then it follows that $u_m \rightharpoonup 0$ in $E(\mathbb{R}^N)$. Since $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$ by Lemma 4.4, it follows that $u_m \to 0$ in $L^{q+1}(\mathbb{R}^N)$. However, since $u_m \in T$, it follows from the Sobolev inequality that

$$||u_m||^{q+1}_{H^1(\mathbb{R}^N)} \geq S^{q+1}_{q+1} ||u_m||^{q+1}_{L^{q+1}(\mathbb{R}^N)} = S^{q+1}_{q+1} ||u_m||^2_{H^1(\mathbb{R}^N)},$$

from which we deduce

$$||u_m||^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq S^{2(q+1)/(q-1)}_{q+1} > 0.$$

This shows that u_m is bounded away from 0 in $L^{q+1}(\mathbb{R}^N)$, a contradiction, and so we have proved that

$$\tilde{d}_m \to +\infty \text{ as } m \to +\infty. \tag{6.4}$$

Now notice that since E_m and $E_m^\perp$ are complementary subspaces, it holds that

[1] d_m's are bounded as shown in the proof of Theorem 6.2, further reference in [5, Thm. 2.8, 2.13]

there exists a $\bar{C} \geq 1$ such that each $u \in B_1$ can be uniquely written as

$$u = v + w, \text{ with } v \in E_m, w \in E_m^{\perp}, \tag{6.5}$$

$$||v||_{E(\mathbb{R}^N)} \leq \bar{C}||u||_{E(\mathbb{R}^N)} \leq \bar{C}, \tag{6.6}$$

$$||w||_{E(\mathbb{R}^N)} \leq \bar{C}||u||_{E(\mathbb{R}^N)} \leq \bar{C}, \tag{6.7}$$

as a consequence of the open mapping theorem, an example of which can be found in [17, p.37]. Define $h_m : E_m^{\perp} \to E_m^{\perp}$ by

$$h_m(u) = (\bar{C}K)^{-1}\tilde{d}_m u,$$

where

$$K > \max\left\{1, \left(\frac{4}{q+1}\right)^{\frac{1}{q-1}}\right\},$$

and note that h_m is an odd homeomorphism of $E_m^{\perp}$ onto $E_m^{\perp}$. Now, for any $u \in E(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\beta(u) > 0$ such that $\beta(u)u \in T$, namely

$$\beta(u) = \left(\frac{||u||^2_{H^1(\mathbb{R}^N)}}{||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)}}\right)^{\frac{1}{q-1}}. \tag{6.8}$$

If we define

$$I_0(u) := \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{1}{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)},$$

then for each $u \in E(\mathbb{R}^N) \setminus \{0\}$, it holds that

$$I_0(tu) = \frac{t^2}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{t^{q+1}}{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)}$$

is a monotone increasing function for $t \in [0, \beta(u)]$ with a maximum at $t = \beta(u)$. Note that for each $u \in (E_m^{\perp} \cap B_{\bar{C}}) \setminus \{0\}$, by the definition of $\tilde{d}_m$ and $\beta(u)$, we have

$$\bar{C}^{-1}\tilde{d}_m \leq \bar{C}^{-1}||\beta(u)u||_{E(\mathbb{R}^N)} \leq \beta(u), \tag{6.9}$$

and so since $K \geq 1$, it holds that

$$(\bar{C}K)^{-1}\tilde{d}_m \leq \bar{C}^{-1}\tilde{d}_m \leq \beta(u), \quad \text{for all } u \in (E_m^{\perp} \cap B_{\bar{C}}) \setminus \{0\}.$$

Putting everything together, it follows that

$$I_0(h_m(u)) = I_0((\bar{C}K)^{-1}\tilde{d}_m u) > 0 \quad \text{for all } u \in (E_m^\perp \cap B_{\bar{C}}) \setminus \{0\}.$$

Moreover,

$$h_m(0) = 0.$$

Therefore,

$$h_m(E_m^\perp \cap B_{\bar{C}}) \subset \left\{ u \in E(\mathbb{R}^N) : I_0(u) \geq 0 \right\}. \tag{6.10}$$

Now, for each $m \in \mathbb{N}$ and some $\delta > 0$, define $\tilde{h}_m : E_m \times E_m^\perp \to E_m \times E_m^\perp$ by

$$\tilde{h}_m([v, w]) = [\delta v,\ (\bar{C}K)^{-1}\tilde{d}_m w].$$

Notice that $\tilde{h}_m$ is an odd homeomorphism of $E_m \times E_m^\perp$ onto $E_m \times E_m^\perp$. Moreover, by (6.5), the function $g_m : E_m \times E_m^\perp \to E(\mathbb{R}^N)$ defined by

$$g_m([v, w]) = v + w,$$

is an odd homeomorphism. Hence, defining $H_m : E(\mathbb{R}^N) \to E(\mathbb{R}^N)$ as

$$H_m = g_m \circ \tilde{h}_m \circ g_m^{-1},$$

we see that H_m is an odd homeomorphism of $E(\mathbb{R}^N)$ onto $E(\mathbb{R}^N)$. By (6.5)-(6.7), it holds that

$$B_1 \subseteq g_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^\perp \cap B_{\bar{C}}\}),$$

and so

$$\begin{aligned} H_m(B_1) &\subseteq H_m(g_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^\perp \cap B_{\bar{C}}\})) \qquad (6.11)\\ &= g_m(\tilde{h}_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^\perp \cap B_{\bar{C}}\})) \\ &= g_m(\{\delta(E_m \cap B_{\bar{C}})\} \times \{\bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^\perp \cap B_{\bar{C}})\}) \\ &= \left\{ u \in E(\mathbb{R}^N) : u = v + w, v \in \delta(E_m \cap B_{\bar{C}}), w \in \bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^\perp \cap B_{\bar{C}}) \right\} \\ &=: Z_{m,\delta}. \end{aligned}$$

Now, fix $m \in \mathbb{N}$. We claim that

$$Z_{m,\delta} \subset \left\{ u \in E(\mathbb{R}^N) : I_0(u) > 0 \right\} \cup \{0\}$$

for some $\delta = \delta(m) > 0$. To see this, assume, by contradiction, that there exists

$\delta_j \to 0$ and $u_j \notin \{u \in E(\mathbb{R}^N) : I_0(u) > 0\} \cup \{0\}$ such that $u_j \in Z_{m,\delta_j}$. Then, by definition of Z_{m,δ_j}, it holds that

$$||u_j||_{E(\mathbb{R}^N)} \le ||v_j||_{E(\mathbb{R}^N)} + ||w_j||_{E(\mathbb{R}^N)} \le \delta_j \bar{C} + K^{-1}\tilde{d}_m,$$

which implies u_j is bounded. Thus, up to a subsequence $u_j \rightharpoonup \bar{u}$ in $E(\mathbb{R}^N)$ and so it follows that $u_j \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$. Moreover, since $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$ by Lemma 4.4, it follows that $u_j \to \bar{u}$ in $L^{q+1}(\mathbb{R}^N)$ and $\|u_j\|_{L^{q+1}(\mathbb{R}^N)} > 0$ by previous arguments. Thus, by the weakly lower semicontinuity of the $H^1(\mathbb{R}^N)$ norm and the strong convergence in $L^{q+1}(\mathbb{R}^N)$, we deduce that

$$\frac{1}{2}||\bar{u}||^2_{H^1(\mathbb{R}^N)} \le \frac{1}{q+1}||\bar{u}||^{q+1}_{L^{q+1}(\mathbb{R}^N)},$$

which implies $\bar{u} \notin \{u \in E(\mathbb{R}^N) : I_0(u) > 0\} \cup \{0\}$. On the other hand, since $\delta_j \to 0$, then $v_j \to 0$. It follows from this and (6.10) that $\bar{u} \in \bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^{\perp} \cap B_{\bar{C}}) \subset \{u \in E(\mathbb{R}^N) : I_0(u) > 0\} \cup \{0\}$. Hence, we have reached a contradiction and so the claim holds. Thus, using this and (6.11), for each $m \in \mathbb{N}$, we pick $\delta = \delta(m) > 0$ so that

$$H_m(B_1) \subset \{u \in E(\mathbb{R}^N) : I_0(u) > 0\} \cup \{0\} \subset \{u \in E(\mathbb{R}^N) : I(u) \ge 0\} = \hat{A}_0,$$

namely $H_m \in \Gamma^*$, where $\hat{A}_0$ and Γ^* are given by (6.1) and (6.2), respectively. We can therefore see that

$$d_{m+1} = \sup_{h \in \Gamma^*} \inf_{u \in \partial B_1 \cap E_m^{\perp}} I(h(u)) \ge \inf_{u \in \partial B_1 \cap E_m^{\perp}} I(H_m(u)). \tag{6.12}$$

Now take $u \in \partial B_1 \cap E_m^{\perp}$. Then, using (6.8), (6.9) and the fact that $\int_{\mathbb{R}^N} \rho\phi_u u^2 = \omega^{-1}(1 - ||u||^2_{H^1(\mathbb{R}^N)})^2$, it holds that

$$\begin{aligned} I(H_m(u)) &= \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{4}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4 \int_{\mathbb{R}^N} \rho\phi_u u^2 \\ &\quad - \frac{1}{q+1}(\bar{C}K)^{-q-1}\tilde{d}_m^{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)} \\ &= \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{4}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4 \int_{\mathbb{R}^N} \rho\phi_u u^2 \\ &\quad - \frac{(\bar{C}K)^{-q-1}\tilde{d}_m^2}{q+1}\left(\frac{\tilde{d}_m}{\beta(u)}\right)^{q-1}||u||^2_{H^1(\mathbb{R}^N)} \\ &\ge \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2\left(1 - \frac{2K^{1-q}}{q+1}\right)||u||^2_{H^1(\mathbb{R}^N)} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{4\omega}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4 \left(1 - ||u||^2_{H^1(\mathbb{R}^N)}\right)^2 \\
&\geq \min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\} \left(||u||^4_{H^1(\mathbb{R}^N)} - ||u||^2_{H^1(\mathbb{R}^N)} + 1\right) \\
&\geq \frac{3}{4}\min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\},
\end{aligned}
$$

where $K_1 \geq \frac{1}{4\bar{C}^2K^2}$ by our choice of K and $K_2 = \frac{1}{4\omega\bar{C}^4K^4}$. Finally, using this, (6.12), and (6.4), we obtain

$$
\begin{aligned}
d_{m+1} &\geq \inf_{u\in\partial B_1\cap E_m^{\perp}} I(H_m(u)) \\
&\geq \frac{3}{4}\min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\} \to +\infty, \qquad \text{as } m \to +\infty.
\end{aligned}
$$

This completes the proof. □

6.2 Multiplicity result in the case of high q

In order to prove Theorem 6.2 the concepts outlined in Chapter 5 will be used, namely the notion of Krasnoselskii-genus (Definition 15) and its properties on special subsets of Banach spaces. For the proof of Theorem 6.2, we recall a classical result of Ambrosetti and Rabinowitz, [4].

Theorem 6.1 ([4]; **Min-max setting high** q**).** *Let* $I \in C^1(E(\mathbb{R}^N), \mathbb{R}^N)$ *satisfy the following:*

(i) $I(0) = 0$ *is a local minimum and there exists constants* $R, a > 0$ *such that* $I(u) \geq a$ *if* $||u||_{E(\mathbb{R}^N)} = R$

(ii) If $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ *is such that* $0 < I(u_n)$*,* $I(u_n)$ *bounded above, and* $I'(u_n) \to 0$*, then* $(u_n)_{n\in\mathbb{N}}$ *possesses a convergent subsequence*

(iii) $I(u) = I(-u)$ *for all* $u \in E(\mathbb{R}^N)$

(iv) For a nested sequence $E_1 \subset E_2 \subset \cdots$ *of finite dimensional subspaces of* $E(\mathbb{R}^N)$ *of increasing dimension, it holds that* $E_i \cap \hat{A}_0$ *is bounded for each* $i = 1, 2, \ldots,$ *where* $\hat{A}_0$ *is given by* (6.1)

Define

$$
b_m = \inf_{K\in\Gamma_m} \max_{u\in K} I(u),
$$

with

$\Gamma_m =\{K \subset E(\mathbb{R}^N) : K$ *is compact and symmetric with respect to the origin* *and for all $h \in \Gamma^*$, it holds that $\gamma(K \cap h(\partial B_1)) \geq m\}$,*

where Γ^* is given by (6.2). Then, for each $m \in \mathbb{N}$, it holds that $0 < a \leq b_m \leq b_{m+1}$ and b_m is a critical value of I. Moreover, if $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r$, where

$$K_b := \{u \in E(\mathbb{R}^N) : I(u) = b,\, I'(u) = 0\},$$

is the set of critical points at any level $b > 0$.

Proof. The full proof can be found in [4, Theorem 2.8], instead here we will provide a sketch. By construction $b_m \geq \alpha > 0$ and since $\Gamma_{m+1} \subset \Gamma_m$, $b_{m+1} \geq b_m$. If $\gamma(K_b) < r$ then by definition 14 (iv) there exists a neighbourhood $N(K_b)$ such that $\gamma(N(K_b)) < r$. By the properties of deformations (Similar to the equivariant deformations we describe in Theorem 5.5) it can be shown that there exists an odd homeomorphism $\eta : E \to E$ such that $\eta(\mathcal{A}_{b+\varepsilon} \setminus N(K_b)) \subset \mathcal{A}_{b-\varepsilon}$ for some $\varepsilon \in (0, \alpha)$. Letting $K \in \Gamma_{m+r}$ such that $\max_{u \in K} I(u) \leq b + \varepsilon$. Setting $Q \in \Gamma_{m+1}$ to be the closure of the above we are left with $b \leq \max_{u \in \eta(Q)} I(u) \leq b - \varepsilon$ which is a contradiction. □

The above yields the min-max characterisation for I necessary to prove our multiplicity results contained in the following theorem.

Theorem 6.2 (**Infinitely many high energy solutions for** $q > 3$)**.** *Let $N = 3$, $q \in (3, 2^* - 1)$ and $\lambda > 0$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ is nonnegative and satisfies (ρ_2). Then, there exists infinitely many distinct pairs of critical points $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, for I_λ such that $I_\lambda(u_m) \to +\infty$ as $m \to +\infty$.*

Proof of Theorem 6.2. We aim to apply Theorem 6.1 and therefore must verify that I satisfies assumptions (i)-(iv) of this theorem. By Lemma 3.4, I satisfies the Mountain-Pass Geometry and thus (i) holds. By Lemma 4.5, (ii) holds. Clearly, (iii) holds due to the structure of the functional I. We now must show that (iv) holds. We first notice by straightforward calculations that for any $u \in \partial B_1$ and any for $t > 0$, it holds that

$$\begin{aligned} I(tu) &= \frac{t^2}{2}||u||^2_{H^1(\mathbb{R}^N)} + \frac{t^4}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2 - \frac{t^{q+1}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \\ &= \frac{t^2}{2}\left(||u||^2_{H^1(\mathbb{R}^N)} + \frac{t^2}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2 - \frac{2t^{q-1}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}\right). \end{aligned}$$

We now set

$$\alpha := ||u||^2_{H^1(\mathbb{R}^N)} > 0, \quad \beta := \frac{1}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2 \geq 0, \quad \gamma := \frac{2}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} > 0,$$

and look for positive solutions of

$$\frac{t^2}{2}(\alpha + \beta t^2 - \gamma t^{q-1}) = 0.$$

Since $q > 3$, it holds that $\alpha + \beta t^2 - \gamma t^{q-1} = 0$ has a unique solution $t = t_u > 0$. That is, we have shown that for each $u \in \partial B_1$, there exists a unique $t = t_u > 0$ such that I satisfies

$$\begin{aligned} &I(t_u u) = 0 \\ &I(tu) > 0, \ \forall t < t_u \\ &I(tu) < 0, \ \forall t > t_u. \end{aligned}$$

Now, for any $m \in \mathbb{N}$, we choose E_m a m-dimensional subspace of $E(\mathbb{R}^N)$ in such a way that $E_m \subset E_{m'}$ for $m < m'$. Moreover, for any $m \in \mathbb{N}$, we set

$$W_m := \{w \in E(\mathbb{R}^N) : w = tu, \ t \geq 0, \ u \in \partial B_1 \cap E_m\}.$$

Then, the function $h : E_m \to W_m$ given by

$$h(z) = t\frac{z}{||z||}, \quad \text{with } t = ||z||$$

defines a homeomorphism from E_m onto W_m, and so $W_1 \subset W_2 \subset \cdots$ is a nested sequence of finite dimensional subspaces of $E(\mathbb{R}^N)$ of increasing dimension. We also notice that

$$T_m := \sup_{u \in \partial B_1 \cap E_m} t_u < +\infty$$

since $\partial B_1 \cap E_m$ is compact. So, for all $t > T_m$ and $u \in \partial B_1 \cap E_m$, it holds that $I(tu) < 0$, and thus $W_m \cap \hat{A}_0$ is bounded, where $\hat{A}_0$ is given by (6.1). Since this holds for arbitrary $m \in \mathbb{N}$, we have shown that (iv) holds. Hence, we have shown that Theorem 6.1 applies to the functional I. If b_m are distinct for $m = 1, \ldots, j$ with $j \in \mathbb{N}$, we obtain j distinct pairs of critical points corresponding to critical levels $0 < b_1 < b_2 < \cdots < b_j$. If $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r \geq 2$. Moreover, $0 \notin K_b$ since $b > 0 = I(0)$. Further, K_b is invariant since I is an invariant functional and K_b is closed since I satisfies the Palais-Smale condition, and so $K_b \in \mathcal{A}$. Therefore, by Proposition 14 (3), K_b possesses infinitely many points. Finally, we note that by [4, Theorem 2.13], for each $m \in \mathbb{N}$, it holds that

$$d_m \leq b_m,$$

where d_m is defined in (6.3). It therefore follows from Lemma 6.1 that

$$b_m \to +\infty, \text{ as } m \to +\infty.$$

This concludes the proof. □

6.3 Multiplicity result in the case of low q

The scenario where $q \leq 3$ becomes more delicate, the setting described above is no longer suitable due to the structure of the functional in this $q \in (2, 3]$ range. Therefore we take an approach in the same vein of Ambrosetti-Ruiz [5], before proving Theorem 6.3 we must establish some preliminary results that we will need to use. Similar issues from before arise in this environment, the boundedness of Palais-Smale sequences is not obvious and hence a mild perturbation is applied to the nonlinearity of the functional. The following min-max setting when working with perturbed functionals by Ambrosetti-Ruiz [5] is required.

Lemma 6.2 ([5]; **Abstract min-max setting for low** q**).** *Consider a Banach space* E*, and a functional* $\Phi_\mu : E \to \mathbb{R}$ *of the form* $\Phi_\mu(u) = \alpha(u) - \mu\beta(u)$*, with* $\mu > 0$*. Suppose that* α*,* $\beta \in C^1$ *are even functions,* $\lim_{||u|| \to +\infty} \alpha(u) = +\infty$*,* $\beta(u) \geq 0$*, and* β*,* β' *map bounded sets onto bounded sets. Suppose further that there exists* $K \subset E$ *and a class* $\mathcal{F}$ *of compact sets in* E *such that:*

($\mathcal{F}$*.1)* $K \subset A$ *for all* $A \in \mathcal{F}$ *and* $\sup_{u \in K} \Phi_\mu(u) < c_\mu$*, where* c_μ *is defined as:*

$$c_\mu := \inf_{A \in \mathcal{F}} \max_{u \in A} \Phi_\mu(u). \tag{6.13}$$

($\mathcal{F}$*.2) If* $\eta \in C([0, 1] \times E, E)$ *is an odd homotopy such that*

- $\eta(0, \cdot) = I$*, where* $I : E \to E$ *is the identity map on* E
- $\eta(t, \cdot)$ *is a homeomorphism*
- $\eta(t, x) = x$ *for all* $x \in K$*,*

then $\eta(1, A) \in \mathcal{F}$ *for all* $A \in \mathcal{F}$*.*

Then, it holds that the mapping $\mu \mapsto c_\mu$ is non-increasing and left-continuous, and therefore is almost everywhere differentiable.

Proof. See [5, Lemma 2.2]. Here the fact $\mathcal{F}$ is independent of μ and $\beta \geq 0$, it follows that $\mu \mapsto c_\mu$ is nonincreasing. Let $\mu_{u_n} \nearrow \mu$, then $c_{\mu_n} \geq c_\mu$. Fix $\varepsilon > 0$, let $A \in \mathcal{F}$ be such that $\max_A \Phi_\mu(u) < c_\mu + \varepsilon$. If μ_n is close enough to μ then, since A is compact we have $\max_{u\in A} |\Phi_{\mu_n}(u) - \Phi_\mu(u)| < \varepsilon$. Then, $c_\mu \leq c_{\mu_n} \leq \max_A \Phi_{\mu_n}(u) \leq c_\mu + 2\varepsilon$. Since ε is arbitrary the proof is complete. □

Under the hypotheses of the previous lemma, we can now define the set of values of $\mu \in \left[\frac{1}{2}, 1\right]$ such that c_μ, given by (6.13), is differentiable. Namely, we define

$$\mathcal{J} := \left\{ \mu \in \left[\frac{1}{2}, 1\right] \; : \text{the mapping } \mu \mapsto c_\mu \text{ is differentiable} \right\}.$$

Corollary 6.1 (On density of perturbation values μ). *The set $\mathcal{J}$ is dense in $\left[\frac{1}{2}, 1\right]$.*

Proof. Fix $x \in \left[\frac{1}{2}, 1\right]$ and $\delta > 0$, and denote by $|\cdot|$ the Lebesgue measure. Since $\left[\frac{1}{2}, 1\right] \setminus \mathcal{J}$ has zero Lebesgue measure by Lemma 6.2, we have

$$|\mathcal{J} \cap (x - \delta, x + \delta)| = \left| \left[\frac{1}{2}, 1\right] \cap (x - \delta, x + \delta) \right| > 0.$$

It follows that $\mathcal{J} \cap (x - \delta, x + \delta)$ is nonempty and so we can choose $y \in \mathcal{J} \cap (x - \delta, x + \delta)$. Since x and δ are arbitrary, this completes the proof. □

With the definition of $\mathcal{J}$ in place, we can also recall another vital result from [5], which will be used to obtain the boundedness of our Palais-Smale sequences.

Lemma 6.3 ([5]; **Boundedness of Palais-Smale sequences at level c_μ).** *For any $\mu \in \mathcal{J}$, there exists a bounded Palais-Smale sequence for Φ_μ at the level c_μ defined by (6.13). That is, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ such that $\Phi_\mu(u_n) \to c_\mu$ and $\Phi'_\mu(u_n) \to 0$.*

Proof. The proof may be found in [5, Proposition 2.3]. □

Moving toward a less abstract setting, for any $\mu \in \left[\frac{1}{2}, 1\right]$, we define the perturbed functional $I_\mu : E(\mathbb{R}^N) \to \mathbb{R}^N$ as

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\mu}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}. \tag{6.14}$$

The next result that we will need in order to prove Theorem 6.3, follows as a result of Lemma 3.6.

Lemma 6.4 (On the sign of the energy level of I_μ along certain curves). *Assume $N = 3, 4, 5$ and $q \in (2, 2^* - 1]$. Suppose further that ρ is homogeneous of degree $\bar{k}$, namely, $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1.$$

Then, there exists $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$ such that for each fixed $\mu \in \left[\frac{1}{2}, 1\right]$ and each $u \in E(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t = t_u > 0$ with the property that

$$\begin{aligned} &I_\mu(t_u^\nu u(t_u\cdot)) = 0, \\ &I_\mu(t^\nu u(t\cdot)) > 0, \ \forall t < t_u, \\ &I_\mu(t^\nu u(t\cdot)) < 0, \ \forall t > t_u, \end{aligned}$$

where I_μ is defined in (6.14).

Proof. We first note that under the assumptions on the parameters, we can show that

$$\frac{4\nu - N - 2}{2} > \frac{(\nu+1)(3-q) - 2}{2}.$$

It follows from this and the lower bound assumption on $\bar{k}$ that we can always find at least one interval

$$\left(\frac{\nu(3-q) - 2}{2}, \frac{4\nu - N - 2}{2}\right), \quad \text{with } \nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\},$$

that contains $\bar{k}$. We pick ν corresponding to such an interval and fix $\mu \in \left[\frac{1}{2}, 1\right]$. Then, for any $u \in E(\mathbb{R}^N) \setminus \{0\}$ and for any $t > 0$, using the assumption that ρ is homogeneous of degree $\bar{k}$, we find that

$$\begin{aligned} I_\mu(t^\nu u(t\cdot)) &= \frac{t^{2\nu+2-N}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{2\nu-N}}{2} \int_{\mathbb{R}^N} u^2 \\ &\quad + \frac{t^{4\nu-N-2}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{\omega|x-y|^{N-2}} - \frac{\mu t^{\nu(q+1)-N}}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} \\ &= \frac{t^{2\nu+2-N}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{2\nu-N}}{2} \int_{\mathbb{R}^N} u^2 + \frac{t^{4\nu-N-2-2\bar{k}}}{4} \int_{\mathbb{R}^N} \rho\phi_u u^2 \\ &\quad - \frac{\mu t^{\nu(q+1)-N}}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}. \end{aligned}$$

We therefore set

$$a = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2, \; b = \frac{1}{2}\int_{\mathbb{R}^N} u^2, \; c = \frac{1}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2, \; d = \frac{\mu}{q+1}\int_{\mathbb{R}^N} |u|^{q+1},$$

and consider the polynomial

$$f(t) = at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t \geq 0.$$

Since $u \in E(\mathbb{R}^N)\backslash\{0\}$, we can deduce that $a, b, d > 0$ and $c \geq 0$, and so, by Lemma 3.6, it holds that f has a unique critical point corresponding to its maximum. Thus, since $I_\mu(t^\nu u(t\cdot)) = f(t)$ and, by assumptions, $\nu(q+1) - N > 2\nu + 2 - N$ and $\nu(q+1) - N > 4\nu - N - 2 - 2\bar{k}$, it follows that there exists a unique $t = t_u > 0$ such that the conclusion holds.

□

With the previous results established, we are finally in position to prove Theorem 6.3.

Theorem 6.3 **(Infinitely many high energy solutions for** $q \leq 3$**).** *Let* $N = 3, 4, 5$. *Assume* $q \in (2, 3]$ *if* $N = 3$ *and* $q \in (2, 2^* - 1)$ *if* $N = 4, 5$. *Suppose* $\lambda > 0$ *and* $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ *is nonnegative, satisfies* (ρ_2), *and is homogeneous of degree* $\bar{k}$, *namely,* $\rho(tx) = t^{\bar{k}}\rho(x)$ *for all* $t > 0$, *for some*

$$\bar{k} > \left(\max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1\right)_+ .$$

Then, there exist infinitely many distinct pairs of critical points, $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, *for* I_λ *such that* $I_\lambda(u_m) \to +\infty$ *as* $m \to +\infty$.

Proof of Theorem 6.3. We first note that by Lemma 6.4, we can choose $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$, so that for each $u \in \partial B_1$, there exists a unique $t = t_u > 0$ such that I_μ with $\mu = \frac{1}{2}$, defined by (6.14), satisfies

$$\begin{aligned} &I_{\frac{1}{2}}(t_u^\nu u(t_u\cdot)) = 0, \\ &I_{\frac{1}{2}}(t^\nu u(t\cdot)) > 0, \; \forall t < t_u, \\ &I_{\frac{1}{2}}(t^\nu u(t\cdot)) < 0, \; \forall t > t_u. \end{aligned} \tag{6.15}$$

Now, for any $m \in \mathbb{N}$, we choose E_m a m-dimensional subspace of $E(\mathbb{R}^N)$ in such a way that $E_m \subset E_{m'}$ for $m < m'$. Moreover, for any $m \in \mathbb{N}$, we set

$$W_m := \{w \in E(\mathbb{R}^N) : w = t^\nu u(t\cdot), \; t \geq 0, \; u \in \partial B_1 \cap E_m\}.$$

Then, the function $h : E_m \to W_m$ given by

$$h(e) = t^\nu u(t\cdot), \quad \text{with } t = ||e||_{E(\mathbb{R}^N)},\ u = \frac{e}{||e||_{E(\mathbb{R}^N)}},$$

defines an odd homeomorphism from E_m onto W_m. We notice that it holds that

$$T_m := \sup_{u \in \partial B_1 \cap E_m} t_u < +\infty, \tag{6.16}$$

since $\partial B_1 \cap E_m$ is compact. So, the set

$$A_m = \{w \in E(\mathbb{R}^N) : w = t^\nu u(t\cdot),\ t \in [0, T_m],\ u \in \partial B_1 \cap E_m\}$$

is compact. We now define

$$H := \{g : E(\mathbb{R}^N) \to E(\mathbb{R}^N) : g \text{ is an odd homeomorphism} \\ \text{and } g(w) = w \text{ for all } w \in \partial A_m\},$$

and

$$G_m := \{g(A_m) : g \in H\}.$$

We aim to verify $(\mathcal{F}.1)$ and $(\mathcal{F}.2)$ of Lemma 6.2. We take G_m as the class $\mathcal{F}$ and $K = \partial A_m$ and define the min-max levels

$$c_{m,\mu} := \inf_{A \in G_m} \max_{u \in A} I_\mu(u).$$

Then, since $T_m \geq t_u$ for all $u \in \partial B_1 \cap E_m$ by definition, it follows from (6.15) that

$$I_\mu(w) \leq I_{\frac{1}{2}}(w) \leq 0, \quad \forall w \in \partial A_m,\ \forall \mu \in \left[\frac{1}{2}, 1\right].$$

Moreover, since $G_m \subset G_{m+1}$ for all $m \in \mathbb{N}$, it holds that $c_{m,\mu} \geq c_{m-1,\mu} \geq \cdots \geq c_{1,\mu} > 0$. Taken together, we have shown that

$$\sup_{w \in \partial A_m} I_\mu(w) \leq 0 < c_{m,\mu}, \tag{6.17}$$

and thus $(\mathcal{F}.1)$ is verified. Moreover, for any η given by $(\mathcal{F}.2)$ and any $g \in H$, it holds that $\tilde{g} = \eta(1, g)$ belongs to H, and so $(\mathcal{F}.2)$ is satisfied. Since $(\mathcal{F}.1)$ and $(\mathcal{F}.2)$ are satisfied, Lemma 6.2 applies. Thus, for any $m \in \mathbb{N}$, we denote by $\mathcal{J}_m$ the set of values $\mu \in \left[\frac{1}{2}, 1\right]$ such that the function $\mu \mapsto c_{m,\mu}$ is differentiable. We then let

$$\mathcal{M} := \bigcap_{m\in\mathbb{N}} \mathcal{J}_m.$$

We note that since

$$\left[\frac{1}{2},1\right] \setminus \mathcal{M} = \bigcup_{m\in\mathbb{N}} \left(\left[\frac{1}{2},1\right] \setminus \mathcal{J}_m\right)$$

and $[\frac{1}{2},1] \setminus \mathcal{J}_m$ has zero Lebesgue measure for each m by Lemma 6.2, then it follows that $\left[\frac{1}{2},1\right] \setminus \mathcal{M}$ has zero Lebesgue measure. Arguing as in the proof of Corollary 6.1, we obtain that $\mathcal{M}$ is dense in $\left[\frac{1}{2},1\right]$. We can now apply Proposition 6.3 with $\Phi_\mu = I_\mu$. Namely, for each fixed $m \in \mathbb{N}$ and $\mu \in \mathcal{M}$ we obtain that there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ such that $I_\mu(u_n) \to c_{m,\mu}$ and $I'_\mu(u_n) \to 0$. The embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$ is compact by Lemma 4.4 so, arguing as in the proof of Theorem 4.3, we can show that the values $c_{m,\mu}$ are critical levels of I_μ for each $m \in \mathbb{N}$ and $\mu \in \mathcal{M}$. We then take m fixed, $(\mu_n)_{n\in\mathbb{N}}$ an increasing sequence in $\mathcal{M}$ such that $\mu_n \to 1$, and $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ such that $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) = c_{m,\mu_n}$. We note that since ρ is homogeneous of degree $\bar{k}$ by assumption, it follows from [25, p. 296] that $\bar{k}\rho(x) = (x, \nabla\rho)$. So, setting $\alpha_n = \int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2)$, $\gamma_n = \int_{\mathbb{R}^N} \rho(x)\phi_{u_n}u_n^2$, $\delta_n = \mu_n \int_{\mathbb{R}^N} |u_n|^{q+1}$ and using the Pohozaev-type condition deduced in Lemma 3.9, we obtain the system

$$\begin{cases} \alpha_n + \gamma_n - \delta_n = 0, \\ \frac{1}{2}\alpha_n + \frac{1}{4}\gamma_n - \frac{1}{q+1}\delta_n = c_{m,\mu_n}, \\ \frac{N-2}{2}\alpha_n + \left(\frac{N+2+2k}{4}\right)\gamma_n - \frac{N}{q+1}\delta_n \leq 0. \end{cases} \tag{6.18}$$

Since the assumptions on $\bar{k}$ guarantee that $\bar{k} > \frac{-2(q-2)}{(q-1)} > \frac{N-6}{2}$ for $q \in (2,3]$ if $N = 3$ and for $q \in (2, 2^* - 1)$ if $N = 4,5$, it follows that we can solve this system and show that $\alpha_n, \gamma_n, \delta_n$ are all bounded as in the proof of Theorem 4.3. Moreover, continuing to argue as in the proof of this theorem and using the compact embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$, we can then prove that for each fixed m there exists $u \in E(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $E(\mathbb{R}^N)$, $I(u) = I_1(u) = c_{m,1}$, and $I'(u) = I'_1(u) = 0$. It therefore remains to show that $I(u) = c_{m,1} \to +\infty$ as $m \to +\infty$. In order to do so, we define

$$\tilde{\Gamma}_m := \{g \in C(E_m \cap B_1, E(\mathbb{R}^N)) : g \text{ is odd, one-to-one,} \\ I(g(y)) \leq 0 \;\forall\, y \in \partial(E_m \cap B_1)\},$$

$$\tilde{G}_m := \left\{ A \subset E(\mathbb{R}^N) : A = g(E_m \cap B_1), g \in \tilde{\Gamma}_m \right\},$$

$$\tilde{b}_m := \inf_{A \in \tilde{G}_m} \max_{u \in A} I(u).$$

We then note that by [4, Corollary 2.16], it holds that

$$d_m \leq \tilde{b}_m,$$

where d_m is given by (6.3). It therefore follows from Lemma 6.1 that

$$\tilde{b}_m \to +\infty, \text{ as } m \to +\infty. \tag{6.19}$$

We will now show $G_m \subseteq \tilde{G}_m$. We take $A \in G_m$. Then, by definition, there exists $g \in H$ such that $A = g(A_m)$. We define an odd homeomorphism $\varphi : E_m \cap B_1 \to A_m$ by

$$\varphi(e) = t^\nu u(t\cdot), \quad \text{with } t = T_m ||e||_{E(\mathbb{R}^N)},\ u = \frac{e}{||e||_E(\mathbb{R}^N)},$$

where T_m is defined in (6.16), and set $\tilde{g} = g \circ \varphi$. Since we can write $A = \tilde{g}(E_m \cap B_1)$, then by the definition of $\tilde{G}_m$ we need only to show that $\tilde{g} \in \tilde{\Gamma}_m$. Clearly, $\tilde{g} \in C(E_m \cap B_1, E(\mathbb{R}^N))$ is odd and one-to-one. Moreover, for every $y \in \partial(E_m \cap B_1)$, setting $w = \varphi(y) \in \partial A_m$, we have $I(\tilde{g}(y)) = I(g(w))$. Since $g \in H$ and $w \in \partial A_m$, then by definition it holds that $g(w) = w$. Putting everything together, we have

$$I(\tilde{g}(y)) = I(g(w)) = I(w) \leq \sup_{w \in \partial A_m} I(w) \leq 0,$$

where the final inequality follows from (6.17). Hence, we have shown $\tilde{g} \in \tilde{\Gamma}_m$ and so $G_m \subseteq \tilde{G}_m$. Therefore, for each $m \in \mathbb{N}$, it follows that

$$\tilde{b}_m = \inf_{A \in \tilde{G}_m} \max_{u \in A} I(u) \leq \inf_{A \in G_m} \max_{u \in A} I(u) = c_{m,1},$$

and so, by (6.19), we conclude that

$$c_{m,1} \to +\infty, \text{ as } m \to +\infty,$$

as required. $\square$

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