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Exploring the Philosophy of Mathematics: Beyond Logicism and Platonism

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Abstract

A perspective in the philosophy of mathematics is developed from a consideration of the strengths and limitations of both logicism and platonism, with an early focus on Frege's work. Importantly, although many set-theoretic structures may be developed each of which offers limited isomorphism with the system of natural numbers, no one of them may be identified with it. Furthermore, the timeless, ever present nature of mathematical concepts and results itself offers direct access, in the face of a platonist account which generates a supposed problem of access. Crucially too, pure mathematics has its own distinctive method of confirming or validating results - mathematical proof - which supplies a higher level of confidence and objectivity than that available elsewhere. The dichotomy of invention and discovery is too jejune a framework for analysing creative mathematical activity. The Gödelian platonist perspective is evaluated and queried through scrutiny of the part played by mathematical resources and constraints in relation to human activity. It appears that there can be non-causal mathematical explanations and mathematical constraint on purely natural processes. Valuable implications of Quine's naturalism are explored, but one must be cautious of his thesis of confirmational holism. The distinction between algebraic and non-algebraic mathematical theories usefully contributes to our understanding of the internally differentiated nature of the subject.

Keywords

Logicism, Platonism, Gödel's Platonism, Quine's Naturalism, Confirmational Holism, Algebraic and Non-Algebraic Mathematical Theories

1. Introduction

Four main schools were prominent in the philosophy of mathematics in the twentieth century, logicism, intuitionism, formalism and predicativism (Horsten, 2019: section 2) and Wittgenstein had his own distinctive approach (Wittgenstein, 1956; Startup, 2020). Each of the first three schools is anti-platonist in character and each for an extended period, but for differing types of reason, came under sustained attack or serious criticism, which led to renewed interest in the possibilities for the historically important platonist approach. Furthermore, in respect of the ontological assessment of mathematics, two views have become particularly prominent in recent decades, platonism and the contrasting position of nominalism. The aim of this article is to make progress in the philosophy of mathematics by developing a position which draws particularly from both logicism and platonism, but also departs from them. It may well be that an accurate appraisal leads to the conclusion that mathematics has an essentially hybrid nature.

Logic is an ancient intellectual discipline, with beginnings going back at least to the 4th century BCE, but being very actively pursued today; it has always been intimately connected with philosophy and mathematics. It was revolutionised around the turn of the twentieth century through the use of newly developed mathematical techniques. (Priest, 2000: Preface; for an account of classical logic, see Shapiro & Kouri Kissel, 2024) The thesis of logicism has two main elements: that the concepts of mathematics can be defined in terms of logical concepts; that the theorems of mathematics may be derived from logical axioms through purely logical deduction (Savitt, 1986: p. 26). The early Wittgenstein seems inclined in this same direction when he strikingly affirms: "Mathematics is a method of logic" (Wittgenstein, 1922: 6.234). It is widely agreed, however, following the contributions particularly of Frege (1974) and Whitehead & Russell (1956) that the logicist programme substantially succeeds in respect of arithmetic, but finds the remainder of the discipline beyond its scope. This striking partial success rather points up the strategic importance of logic in relation to mathematics while also suggesting that it is in need of supplementation.

Platonism in general is the view that "there exist such things as abstract objects—where an abstract object is an object that does not exist in space or time and which is therefore entirely non-physical and non-mental." (Balaguer, 2016: Introduction) While related to Plato's ideas, it is unclear that he fully endorsed this view. Regarding mathematics, a central idea of platonist ontology is that mathematical objects, together with mathematical relations and structures, exist and are abstract, in the sense that they are not located in space and time and have no causal connection with us (Linnebo, 2018). It may be readily judged that this way of putting it tends to make problematical the issue of how we can gain access to mathematical entities. A further recurrent idea of platonism is that an analogy may be drawn between mathematical entities and physical objects, particularly in respect of their objectivity and the constraint they impose upon us. Against this, but again with many versions, nominalists posit that mathematical objects, mathematical relations and structures either do not exist, or need not exist for mathematics to make sense as a discipline (Bueno, 2020). Hence there is a startling divergence between the two positions. To make progress in the evaluation of platonism there is evidently a need to scrutinise the idea of a mathematical *object*, and clarify the sense in which mathematical results are *constraining*. This task is addressed here while also establishing links between that approach and logicism.

2. Enumeration, Arithmetic and Geometry

A useful first step in clarifying issues concerns the understanding of numbers. There are two very different aspects to consider: empirical propositions of enumeration and statements of arithmetic. The latter are not empirical propositions, although the validity of simple statements may be suggested empirically to an observer (e.g. a child). Indeed, one may well tend to agree with the view attributed to Plato himself that, "arithmetic, and pure mathematics generally, is not derived from perception" (Russell, 1946: p. 177). On the other hand, a statement of enumeration, such as 'I have four books' does involve perception while also involving conceptualisation. The indicated statement says something about what exists. The formalist approach within the philosophy of mathematics holds that the propositions of mathematics can be considered to be propositions about the consequences of the manipulation of strings—sequences of symbols, usually equations—using established rules of manipulation, Hilbert being a major proponent of this position (Weir, 2022). The symbols may even be taken simply as marks on paper. However, as far as enumeration is concerned, Ramsey offers an effective rebuttal: "...I do not see that it can be seriously held that a cardinal number which answers the question 'How many?' is merely a mark on paper" (Ramsey, 1990: p. 235).

Just as the experience of discrete entities is the source of arithmetic, the experience of spatial relations is the source of geometry and also gives rise to mensuration. The statement that a building has a pyramidal structure involves perception but it also involves conceptualisation. Yet geometry - both planar and solid - as it develops is not derived from perception, although the understanding of the subject is powerfully assisted by the systematic use of appropriate diagrams, pictures and models. A reason for the latter is that a model of a pyramid on a desk forms part of empirical reality and is no more or less a pyramid than is one of those at Giza; similarly a picture of a square is as much a part of reality as is Trafalgar Square. There is an important difference, however, between the development of arithmetic and geometry - in the form of Euclidean geometry.

By way of context, Euclidean geometry is the body of work in *Elements*, a textbook on geometry attributed to the ancient Greek mathematician Euclid. Within the volume a small set of intuitively appealing axioms is assumed and many other theorems are deduced from these. Many of its results were previously indicated but Euclid was the first to organize the set of propositions into a logical system whereby each result is proved from axioms and previously proved results. *Elements* starts with plane geometry, still taught in some secondary schools today as an axiomatic system with associated proofs, and proceeds to consider the solid geometry of three dimensions. (For issues raised by Euclid see Gray & Ferreirós, 2021: section 1)

A key difference from arithmetic is that Euclidean geometry is understood to apply to ideal elements. Thus Pythagoras' theorem is understood to apply exactly to a Euclidean right-angled triangle but it may not apply exactly to one drawn on paper. This particular assertion may seem to have the effect of distancing geometrical results from empirical reality. However, an important part of our understanding is that when we set about drawing triangles with ever greater accuracy on smoother and flatter plain surfaces we are entitled to expect the theorem to apply to an ever closer approximation. Had this been found not to be the case, the development of geometry would never have proceeded as it did. What is also true is that at a much later date in the subject's development there emerged understanding that there could be alternative geometries - besides the Euclidean - which the physicist might employ because they have more exact application to spatial reality.

3. Reasoning and Mathematics

Mathematics is primarily an activity and its product; to understand that activity one must foreground the fact and nature of human reasoning. A key distinction in this respect is between theoretical and practical reason (Blackburn, 2009: pp. 48-52). The former is concerned with cognition or knowledge of circumstance. We make use of it so that our beliefs and actions may be brought into line with the actualities of the situation. By contrast, we use practical reason to choose actions taking appropriate account of both cognition and our desires. The traditional division made within theoretical reasoning is between the *a priori* and the *a posteriori*. Regarding the former it should be the case that anyone understanding the issue should conclude that it holds. Thus the *a priori* inference that if four people live in a house, there are more than three, should readily find assent, without it even being necessary to evaluate the conditional claim. The affinity of mathematics and logic is indicated by the fact that they are the spheres of *a priori* inferences. By contrast, the hallmark of an *a posteriori* inference is that it depends upon knowledge of particular circumstance.

Two points tend to arise regarding *a priori* inferences, which hinge on the issue as to how we can know something without using actual experience (Blackburn, 2009: pp. 50-51). In this connection it may be claimed that we are 'hardwired' i.e. that some ability to make these inferences is innate—which is not to say that we could never be mistaken. Reflecting on the context of human evolution it seems plausible that some ability along these lines could have survival value. A further, rather different, idea is that these types of inference are essentially trivial. This would be because they are matters of convention or follow from linguistic rules that humans lay down and teach children. However, this suggestion seems totally inadequate in respect of logic and mathematics. We can change conventions or linguistic rules but feel confident—even certain—that three plus four is still going to equal seven. Conventionalism does not explain why we have no choice about arithmetical sums. It must be noted, however, that there are minor but strategically important exceptions here—as when the rule that $(-1) \times (-1) = +1$ may be understood as conventional and adhered to so as to maintain the rules governing the use of the rational arithmetical operations. Further examples of this type of important conventional rule within mathematics are $n^0 = 1$, where n is an integer, and $0! = 1$.

In general, however, when the seemingly *a priori* is successfully challenged, far from being a relatively minor occurrence, it is an intellectual upheaval. Thus when Euclidean geometry was challenged in the 19th century by the idea that the angles of a triangle might not total two right angles, this was not felt to have arisen simply from a change of convention or linguistic rule; it had something of the character of an intellectual revolution. Indeed, examples such as this may even give rise to the thought that *a priori* inference may be unreliable; even that what we call the *a priori* is essentially what we are most reluctant to give up. Hence we must remain aware of the possibility that what we take to be 'self-evident' at a particular time and place may be mistaken, that what we find difficult to imagine may be correct.

4. Gottlob Frege's Impact on Logicism

In seeking to evaluate logicism and platonism, it is vital to situate the work of Gottlob Frege (1848-1925); he and Bertrand Russell (1872-1970) were centrally important early contributor's to the analytical tradition in philosophy (Beaney, 2017: p. 4). Frege addresses the question as to what numbers are and how we gain knowledge of arithmetic. He views arithmetic as essentially a form of logic and numbers as a kind of logical object. In developing this approach he is a major proponent of *logicism*—defined earlier—in short the view that mathematics may in some sense be reduced to logic.

Centrally important is his work *The Foundations of Arithmetic*, published in 1884, in which a key assertion is that statements about numbers are assertions about concepts (Beaney, 2017: p. 9). For Frege, the street has 50 houses means that the concept 'house of the street' has 50 instances. He sees the property of having an instance as a logical property, in that it can be defined purely logically. Indeed, Frege views an object falling under a concept as the most basic element needed to understand what numbers are. As Beaney (2017: p. 10) puts it, "It is not things in themselves to which numbers are assigned but the concepts by means of which we think of things." He proceeds to develop an account involving distinct levels. Thus to say that a street has 50 houses is to say that the first-level concept of house of the street falls within the second-level concept of 'has 50 instances'. Furthermore, for every concept there is a class or set of things which falls under it, which is referred to as the extension of the concept. In this connection, concrete objects belong to the empirical world, while abstract objects belong within out rational thought.

Turning specifically to the natural numbers, these may be defined relatively straightforwardly using the logical concepts of identity, negation and disjunction. (See Beaney, 2017: pp. 13-14; as Beaney points out, in what immediately follows Frege is simplified.) The first step is to form the concept 'not identical with itself'. This is true of nothing, so the corresponding class has no members; this is called the 'null class' and the number 0 may be identified with it. The next step is to form the concept 'is identical to 0 (i.e. the null class)'. The number 1 may be identified with this class i.e. the class of things identical with 0. Proceeding further, one can form the concept 'is identical with 0 or 1' (using 'or', the logical concept of disjunction): this generates a class which may be identified with the number 2; again, the class generated by the concept 'is identical with 0 or 1 or 2' may be identified with three. This process may be successively iterated to define the other natural numbers. There is little reason to doubt that, given any natural number however large in magnitude, a definition in these terms may in principle be provided for it; also given any number one could form its successor i.e. the next greater one. Proceeding further, it proves possible to define addition and multiplication operations so that the familiar mathematical statements involving them come out to be true.

Taking stock of these basic elements of Frege's contribution, there are some sound reasons to accept his central claim that statements about numbers are statements about concepts. However, the status of his attempt to define the natural numbers is more problematical and in need of further comment. Consider, for instance, the definition just given for the number 2: 'the class of things identical with 0 or 1'. Given near-universal recognition of the number 2 and our familiarity with elementary sums such as 2 + 2 = 4 and 2 x 2 = 4, it is a striking fact that almost no-one would recognise this as a definition of 2. At least one may say that there is a substantial gulf between that definition and the everyday understanding of a very familiar—if abstract—idea.

This is not for a moment to contest the magnitude of Frege's overall contribution—developed further by Russell and others—towards showing that mathematics may in some sense be reduced to logic. As is well-known, Frege's work fed into Whitehead and Russell's [1910] (1956) *Principia Mathematica* which *inter alia* led on to Gödel's seminal contribution to mathematical logic and metamathematics. Potter (2021) gives this contemporary assessment: "Logicism is a brilliant view but it does not work because of Gödel. Regarding logicism, Hume's principle together with second order logic delivers arithmetic - but not much more." (Hume's principle says that the number of *F*s is equal to the number of *G*s if and only if there is a one-one correspondence between the *F*s and the *G*s.) Nevertheless, despite the partial nature of the achievement, this line of work has proved incredibly fertile in its illumination of the relation between logic on the one hand and number and arithmetic on the other. Prior to Frege that relation was very imperfectly understood. (That body of work also stimulated the development of the computer.)

In point of fact, however, a further line of analysis due to (Benacerraf (1965;

1973); see also Benacerraf & Putnam, 1983: pp. 272-294) is strongly suggestive of the idea that the natural numbers are not sets at all (Horsten, 2019: section 4.1). This is an implication of the fact that there is an indefinitely large number of ways of identifying the natural numbers with pure sets of which that given above is simply one. Benacerraf draws particular attention to the following alternative set, which differs from the above iterated pattern from the number 2 onwards. Instead of 2 being 'the class of things identical with 0 or 1', it is 'the class of things identical with the class of things identical with 0'; following on in this same vein 3 is 'the class of things identical with the class of things identical with the class of things identical with 0', and the other natural numbers may be similarly generated successively. Again, addition and multiplication operations may be so defined as to yield the expected outcomes.

So given two differing structures of identification, one may reasonably inquire as to which of them consists solely of true identity statements. In this connection, what does seem clear is that one is evaluating two *isomorphic* structures. What is meant by the italicised mathematical and set-theoretic word is that the items in the two structures may be put into one-one correspondence i.e. to each item in the first structure there corresponds a single item in the second and vice versa; also there is appropriate correspondence of outcomes when the operations of addition and multiplication appropriate to each structure are put to use on corresponding items.

Despite the seeming arithmetical equivalence of the two structures, however, problems arise once one asks extra-arithmetical questions, such as does 1 belong to 3 (Horsten, 2019: section 4.1)? Perusal of the above two definitions of 3 provided within each structure yields different answers. Since each structure provides an item whose identity is supposedly that of the natural number 3, it follows that they should be identical with each other, but in actuality what is generated is a set-theoretic falsehood (Horsten, 2019: section 4.1). This is so given a context where neither structure is superior to the other; yet the accounts cannot both be correct. The issue here is referred to as *Benacerraf's identification problem*. There is good reason to conclude that neither structure fully does the job for which it was designed. The overall conclusion is that, although many set-theoretic structures may be developed each of which offers limited isomorphism with the system of natural numbers, no one of them may be identified with it.

5. Frege's Significance for Platonism

At this point attention may usefully be directed at a strategically important concept that Frege deploys in his account: that of *an abstract object* which is understood to belong to our rational thought. The reference is to logical objects such as classes and sets and is plainly to be sharply distinguished from the notion of a physical object. It is an idea that fits neatly into his system of thought and was certainly taken up by others within the wider framework of the subject of logic.

However, there is reason to be wary of too readily accepting an implied assignment of ontological status leading to incorporation into a platonist account.

Frege's own thinking indeed rather points in this platonist direction. As Brown (2008: pp. 9-10) points out Frege distinguishes among our *ideas* (which are psychological in their nature), *thoughts* and the sentences used to express them. Seemingly *thoughts* are neither 'things of the outer world' i.e. physical entities nor are they ideas of a psychological nature. He points to the need for us to recognise a third realm and says: "What belongs to this corresponds with ideas, in that it cannot be perceived by the senses, but with things, in that it needs no bearer to the contents of whose consciousness to belong. Thus the thought, for example, which we express in the Pythagorean theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no bearer." (Frege, 1974: p. 523) The point may be well taken that the truth of Pythagoras' theorem is in no way qualified by considerations of time and it will be readily agreed that it is true irrespective of its place within any individual consciousness. However, one must be more wary of the suggestion that thoughts and the sentences used to express them are to be considered as 'things', if that mode of expression supposedly bears on their ontological status.

To situate the topic it is helpful briefly to refer to the way the notion of existence is employed in standard semantics and to note some examples. The key point is that the objects denoted by singular terms in true sentences are taken to exist. Thus if we formulate the true sentence, 'Fred loves his bike.', 'Fred' refers to an identifiable person, 'bike' refers to a physical object and 'loves' refers to a particular relation. Given that the sentence is true, it follows that Fred exists. Where the subject of such a true sentence is a physical object of some type - and often it is a person - the point is well taken. In passing, one notes, however, the difficulty in analysing in similar terms the fictional but *prima facie* true sentence, 'Cinderella had difficulty loving her two step-sisters.' But what then of sentences involving numbers? To conform to the same pattern, if we take the true sentence, '13 is greater than 12', then 13 (and seemingly also 12) would be taken to exist. But, one is inclined to ask: exist as what? In this case certainly not as a physical object; conceivably the response might be: it exists as a number. None of this clarifies its ontological status. The most appropriate response is to say that '13' may, and here does, constitute a meaningful abstract subject of a true sentence. That, however, is a point of no small importance: after all most mathematical works consist of a series of sentences.

The Frege quotation above also refers to the Pythagorean theorem. That theorem itself may figure in such a true sentence as: 'The Pythagorean theorem may be proved within pure geometry or analytical geometry.' The theorem may evidently constitute the subject of a true sentence; so do we therefore conclude that it is an object and exists? Again, one may say it exists as a theorem or result within mathematics. In point of fact, by the Pythagorean theorem people sometimes mean the summary result and sometimes the whole proof culminating in that result. The first essentially consists of one sentence, while the second consists of a series of sentences doubtless accompanied by diagrams.

As regards the issue of ontological status the crucial point is as follows. Were one to say that numbers and mathematical theorems and results exist as abstract (non-physical) objects, it tends to lead on to the question: How do we gain access to them? Hence that way of structuring our philosophical thought leads in a platonist direction. It may be said, for instance, that we intuit them or in some way interact with them, which tends to come across as mystifying. On the other hand, if we focus instead on the central point that numbers may constitute the subjects of true meaningful sentences and that theorems consist of a series of sentences, many taking the form of true meaningful ones, then a different response to the supposed problem of access suggests itself: to gain access we need simply to understand the sentence or sentences! In practice of course things are made more difficult because an intense process of reasoning is needed to understand and follow the series of sentences constituting a mathematical proof.

Nevertheless a possible, if provisional, anti-platonist conclusion suggests itself. The feature of mathematical results that their validity is beyond time - and also place - has a striking implication. It is that potentially one can have access to them at any time and place. For instance, here and now one can rehearse in one's mind the theorem that there are an infinite number of primes; so too can one consider and present a proof of the Pythagorean theorem, which as indicated primarily takes the form of a series of sentences. The timeless, ever present nature of mathematical concepts and results itself offers direct access, in the face of a platonist account which generates a supposed philosophical problem of access.

6. Proof and the Truth of Mathematical Statements

So there is reason to query a platonist ontology focused on the notion of mathematical *objects*. Does this, however, have implications for the claim that mathematical results are objectively true and may be said to be true independently of whether they are known by us? Let it be said that there is no intention here to query the validity of any established mathematical theory or conclusion. To sort out the issue, it can help to distinguish between the language in which mathematicians make their claims and the language in which philosophers - including platonists - make theirs.

Linnebo (2018: section 1.4) points up this distinction while inviting consideration of the mathematical statement:

'There are prime numbers between 10 and 20.'

This statement is clearly true. However, a statement of that form or affirming that 'such prime numbers exist' is itself made in the language of mathematics, while the controversy as to whether or not numbers are abstract objects is conducted in the language of philosophy.

In this case one has an assertion easily established by scrutiny of each number successively. One must, however, give full importance to the fact that pure mathematics has its own distinctive method of confirming or validating results -

mathematical proof - which is totally different from that applying in the empirical sciences. Mathematical proof consists of a series of sentences sometimes taking the form of mathematical equations, perhaps also accompanied by diagrams. Here is an example:

Theorem. There are infinitely many prime numbers.

Proof.

1) Suppose, contrary to the theorem, that there are only a finite number of primes.

2) Then there will be a largest, which we will call p.

3) Consider the number n which is 1 more than the product of all the primes: $n = (2 \times 3 \times 5 \times 7 \times \ldots \times p) + 1$

4) Is n prime or composite? ('composite' means it can be divided without remainder into primes)

5) If it is prime, then the original supposition is false, since n is larger than p.

6) Consider it composite.

7) Then it must be divisible without remainder by prime numbers.

8) However, no prime up to p does divide n, so any number which does divide n must be greater than p.

9) This means that there is a prime number greater than p after all.

10) Hence, whether n is prime or composite, our supposition concerning a largest prime is false.

11) Therefore the set of primes is infinite.

One's acceptance of the theorem flows from one's following a chain of reasoning. There are differing types of sentence in the series; it includes questions and invitations to 'consider' something. The proof establishes the theorem beyond doubt; it also gives it the character of an objective proof. Proof is an essential element of mathematics. Our querying of platonist ontology in no way undermines its role.

7. Mathematics: Invented or Discovered?

According to the view attributed to Plato, the abstract universe contains all mathematical entities, including numbers, axioms, theorems and geometrical figures, which are eternal and unchanging. Consequently, humans do not invent mathematics - in the sense of devising or designing it for the first time; instead they gradually discover it - in the sense of finding it or finding it out for the first time. This kind of view is also maintained by modern platonists. Now, while it is undoubtedly the case that the notion of discovery has extensive application, this particular dichotomy does not do justice to the development of the subject. This is because the creative acts involved break out of this somewhat jejune framework.

Mathematics is all to do with communication. Given its esoteric concepts and with proof taking the form of a series of sentences, it is not surprising that it is likened to a language. (In actuality it takes the form of a standardised specialised element belonging to any modern language.) Children now acquire facility in elementary arithmetic relatively easily but that is all to do with the nature of our modern number system, which is decimal, employs positional value in number presentation, and employs 0 as a number which may also be used to distinguish, for instance, 930, 903 and 93 (Higgins, 2011: pp. 2-8). The base ten positional idea has also been extended into fractional parts within the decimal number system. The layout of work is important as when children are encouraged to place equal signs directly under one another. All this was not so much invented as developed over centuries. Generally, effective mathematics has a lot to do with the use of notations and methods of presentation and layout which aid thought.

Newton and Leibniz developed calculus, but they used differing notations and methods of presentation. Leibniz's notation was the more widely adopted for it was found to assist and guide thinking; Newton's notation was taken up more in dynamics. Newton—as discussed further below—used his method in working out the implications of the theory of gravitation. The various approaches, when modified and 'fine-tuned', proved their value overtime but it was only later that the foundations for the use of calculus were secured in mathematical analysis. It is evident you cannot usefully summarise these developments simply by talking about invention and discovery. Furthermore, in the eighteenth century particularly, mathematicians such as Euler—an early influential user of functional notation—developed some results which were only rigorously proved later; again, a pattern resisting easy use of the same dichotomy.

Today we are only two familiar with the axiomatic method but a modern presentation may be utterly misleading as to the pattern of development of the ideas. Given a modern presentation of Euclidean geometry, which, in fact, owes a lot to a much later reworking by the French mathematician Legendre (1794), you might think postulates or axioms were first set up which do not require proof and then people set about proving theorems i.e. 'discovering' the logical implications of axioms, but there is every reason to judge that the development of the system was much more ragged than that. It is well-known, for instance, that centuries were devoted to trying to prove the parallel postulate before its status was properly understood. Indeed, following Lakatos (1976) there is good reason to believe that concept and rule establishment in mathematics is not done on a 'once and for all' basis, but proceeds at least partly through stages of conjecture and refutation or counterexample.

8. Internal and External Resources and Constraints

In an extended evaluation of platonism a useful focus is the way in which mathematical entities may be said to resource and constrain activity. A powerful mind taking a platonist perspective was that of Gödel who held that "there is a strong parallelism between plausible theories of mathematical objects and concepts on the one hand, and plausible theories of physical objects and properties on the other hand. Like physical objects and properties, mathematical objects and concepts are not constructed by humans. Like physical objects and properties, mathematical objects and concepts are not reducible to mental entities. Mathematical objects and concepts are as objective as physical objects and properties. Mathematical objects and concepts are, like physical objects and properties, postulated in order to obtain a good satisfactory theory of our experience." (Horsten, 2019: section 3.1) So Gödel draws an analogy between the mathematical and physical spheres but it is important to test its limits; where there are analogies one may also find significant differences or disanalogies.

Considering platonism in general Brown (2008: pp. 11-15) draws out the following core points: mathematical objects are perfectly real and exist independently of us; mathematical objects are outside of space and time, they are not concrete or physical; we can intuit mathematical objects and grasp mathematical truths; mathematics is *a priori*, not empirical; even though mathematics is *a priori*, it need not be certain; platonism, more than any other account of mathematics, is open to the possibility of an endless variety of investigative techniques.

As a response to Gödel particularly, but also to this wider characterisation of platonism, it becomes important to explore both similarities and differences in respect of the place physical objects and mathematical entities occupy in human life. In this connection two ways in which physical objects enter human life are as resources (we use a tool such as a spanner) and as obstacles (our passage is arrested as we come up against a brick wall). Now, as a first point, one would certainly want to say that mathematics is valuable as a resource. However, while a tool like a spanner is an externality for which a person reaches, mathematical resources may sometimes be purely internal to a person. For instance, one checks a bill or bank statement using mental arithmetic. Of course such a check might or might not be correct; a mistake could have unpleasant consequences. On the other hand, a highly trained pure or applied mathematician might well be able to use mathematics to address all manner of problems without leaving his or her desk and even without the help of such external resources as books or the internet.

As noted, the way platonists tend to put it is that, 'mathematical objects are perfectly real and exist independently of us'. One has already queried the notion of a mathematical *object* above. Without doubting that mathematics exists beyond the consciousness of a single individual, it would seem highly strained and artificial to say that mathematical *objects* as opposed to *ideas* may be accessed by a mental process. It is much more natural to say that the person concerned has mastered and can deploy many mathematical concepts and results: these powers have become part of him or her as has mastery of a language. Of course, it is also the case that additional mathematical resources are available externally - in the form of physical books and electronic resources - but these are not the objects to which platonists refer.

Turning next from resources to obstacles, an important difference must be pointed up which is of relevance to examples taken up later. In human life when a physical object is an obstacle there may be no meaningful relation between the intentions of the person concerned and the nature of the physical object. Thus a brick wall may be a barrier to a tottering drunk, to a person trying to escape from a confined space or to someone carrying a message from one place to another. It is all to do with the physical facticity of the wall on the one hand and the moving human body on the other. Now if a mathematical object really were analogous to a physical object some sort of parallelism in this regard might be expected i.e. that a constraint, barrier or obstacle might be encountered which lacks a meaningful relation to the intentions of the person concerned. However, a consideration of examples rather suggests the opposite and that a meaningful relation exists between human objectives and the nature of the obstacle. Given the position developed here as contrasted with platonism, this is to be expected because we have stressed the role of the understanding of meaningful true statements and the use of accurate reasoning. This is precisely what is being deployed when mathematics is being used successfully as a resource and what is not being deployed adequately when obstacles are encountered.

Example 1; *multiplication*. One may ask: does one ever 'come up against' the result that $11 \times 12 = 132$? Yes, one does so within the context of practical human activity. For instance suppose one is tiling a twelve by eleven rectangle and one calculates how many tiles are needed and wrongly calculates that $11 \times 12 = 130$. One proceeds then to acquire 130 tiles and proceeds with the practical tiling task. At the end one will 'come up against' the irritating consequence that one is two tiles short.

Example 2; *Pythagoras' theorem*. Supposing a three yard fence and a four yard fence each meet at one end at right angles. One wishes to construct a fence between their two other ends to create an enclosure. In doing so one will 'come up against' the feature that no fence under five yards in length will be able to complete the enclosure; acquiring an additional four yard fence, for example, will be insufficient.

Example 3; *topology*. The Seven Bridges of Königsberg is a historically notable problem in mathematics. The city was set on both sides of a river which included two islands which were connected to each other and the mainland by seven bridges. Experience showed one could not devise a walk through the city that would cross each of the seven bridges once and only once. The mathematician Euler proved that the problem has no solution, thereby initiating graph theory and topology.

In each example it is apparent that some sort of limit or constraint flows from the mathematical result, but does this 'parallel' the constraint of a brick wall? No, because, as indicated above, the brick wall is a general barrier, while the mathematical results function as a constraint or limitation given the specific task in hand. We saw that a mistake in multiplication created a practical problem, as did a lack of familiarity with Pythagoras' theorem: specifically, in Example 1 constraint is experienced because $130 < 11 \times 12$, while in Example 2 it arises because $4^2 < 4^2 + 3^2$.

In the last example, the task of devising a walk over the bridges without retracing one's steps cannot be followed through because of the topological result. Again, the mathematical result figures as a constraint given the precise defined task in hand - that result is not a constraint upon the use of the bridges to cross the river in general. Contrastingly, a physical object is a constraint external to human activity in its generality.

9. Non-Causal Explanations; Processes Independent of Human Activity

In many areas of science explanation is bound up with an appropriate description of a cause. There is substantial agreement on the point that mathematical tools are an excellent means of representing causes. What needs highlighting for present purposes is the role of mathematics in non-causal explanations (Mancosu, Poggiolesi, & Pincock, 2023). These come into play, for instance, in relation to processes evident in everyday situations. Supposing one asks, for instance, why we cannot divide 13 sweets equally among three friends or why we can display sixty stamps in a rectangular array, the explanations are that 13 is not divisible by 3 without remainder, while 60 is composite with many factors. Hence one has a non-causal explanation of which arithmetic forms part. Simple processes that may (or may not) play out in the world are evidently subject to absolute arithmetical limitation.

So far the discussion of resources and constraints has focused on human activity. To advance understanding of the ontology of mathematics it is important to clarify whether any similar processes operate in the world beyond the context of human activity. There is indeed interesting mathematics in natural processes: two such being the evolution of stars and the evolution of life on Earth by natural selection. A mathematical result in play concerns the relation between the volume and surface area of similarly shaped objects - whether stars or animals - of varying size. The reader is perhaps familiar with the formula for the surface area of a sphere, $4\pi r^2$, and that for its volume, $(4/3)\pi r^3$. It is apparent that surface area varies as the square of radius while for volume it is as the cube of radius; hence the volume builds up more rapidly. Essentially the same result holds for non-spherical objects of identical shape. It is relevant that the mass of an object tends to increase linearly with its volume. Roughly speaking a star (like our sun) has nuclear fuel according to its mass but radiates energy away according to its surface area (and surface temperature). The rapidity of the evolution of stars of differing initial masses can be shown to be variable in unexpected ways (as is their tendency to become supernovae and even black holes). A whole number of factors is involved but the evolution of a star through its various stages is *inter alia* bound up with the ratio of its volume to surface area.

Passing on to the story of animal (and plant) evolution it is important to inquire as to why living organisms are the size they are as opposed to being several times smaller or larger. The ratio of mass to surface area is always one of the factors involved. The energy developed within a warm-bloodied animal is related to its volume, but the tendency for it to lose heat is related to surface area. Hence animals living near the poles benefit from increased size. This is why the polar bear is particularly large within the bear family.

The human understanding of this mathematical result is highly relevant to engineering practice. When building a bridge, it would be madness simply to scale up the dimensions of a model of a bridge sitting on someone's desk. This is because while the weight of the bridge varies as the cube of the scale factor, the strength of the various bonds does not. In the same way, an animal of the shape of a brontosaurus with all its dimensions doubled could never evolve, because an animal of those dimensions would be instantly crushed by its own weight.

Biology gives rise to further striking phenomena which bring mathematical explanations into play. (Potochnik, 2007; Rice, 2015, 2021) This has to do with the detailed implications of the organism-centred Darwinian theory of evolution which focuses on a population model: various optimization processes, sometimes involving maximization or minimization, confer relative fitness on populations. Particularly well-known is the example of the lengths of the life-cycles of three species of periodic cicadas, which are 13 or 17 years (Baker, 2005, 2017). The explanation is that prime-numbered life cycles are advantageous in respect of avoiding predators and in the competition for scarce resources. With respect to this phenomenon, analysis of Wakil and Justus (2017: p. 927) suggests the following argument: "1. A positive correlation exists between the number of divisors of a life cycle period and the predation pressure the respective species experiences. 2. Prime numbers have the fewest divisors possible. Therefore species life cycles cannot minimize predation pressure more than prime life cycles."

In a further example, the hexagonal shape of honeycomb cells may be shown to be optimal (Lyon, 2012; Lyon & Colyvan, 2008; Räz, 2017; Wakil & Justus 2017). Again, so-called Fibonacci numbers figure quite a lot in plant biology. These belong to the sequence 1, 1, 2, 3, 5, 8, 13, 21... where each number is the sum of the previous two. 'In the case of sunflowers, Fibonacci numbers allow for the maximum number of seeds on a seed head, so the flower uses its space to optimal effect. As the individual seeds grow, the centre of the seed head is able to add new seeds, pushing those at the periphery outwards so the growth can continue indefinitely.' (The Flower Council of Holland, 2024) Hence it may be concluded that mathematical results are a condition for purely natural processes.

It is, of course, an important fact that mathematics and its powerful methods and results also figure prominently in the pure and applied sciences and engineering. Arguments concerned with the sciences and bearing upon platonism have been developed by Quine and Putnam. It is important to ask what conclusions about the character of mathematical concepts and results may be drawn from that strategically vital context.

10. The Uses of Mathematics in the Empirical Sciences and Technology

Quine engaged in a methodological critique of traditional philosophy. Arising out of this, he presented a refreshingly different philosophical methodology, which has become known as naturalism (Quine, 1969). This is to the effect that our best theories are our best scientific theories. His key idea was that we can have more confidence in scientific knowledge and understanding than in existing epistemological or metaphysical theories within philosophy. Far from philosophy being needed as a foundation for science, Quine reckons we would be better employed working in the opposite direction.

Crucially - for present purposes - Putnam applied this approach to mathematical ontology (Putnam, 1972; Horsten, 2019: section 3.2.). He was mindful of the fact that developed theories in the empirical sciences are often mathematically expressed; this is evident, for instance, in Newtonian mechanics and Newton's theory of gravitation, also in quantum theory and relativity. Putnam's thought was that it would be a bit odd to claim that mathematical entities do not exist given that they inhere in our best scientific theories; hence we have an ontological commitment to them.

This thought gains strength by reference to another thesis of Quine's, known as confirmational holism. It is a familiar enough fact that scientific theories are confirmed by evidence but Quine wishes to take a further crucial step. He would say that where mathematical theories form part of scientific theories or enter into them, where those scientific theories are confirmed so also are the mathematical ideas which inhere in them. Hence, it is argued, there is empirical confirmation of mathematical theories. However, on this precise point, one must demur. What is undoubtedly confirmed is the value of ideas drawn from mathematics including parts of it designated as theories (such as the theory of functions or of matrices) but this may fall short of the confirmation of a theory as such. Examples may help.

Example 4; *the gravitational attraction of a sphere*. Newton introduced the inverse square law of gravitation and showed that, if it applied, you would expect planetary motion round the sun to be elliptical. As part of this he used the mathematical procedures of calculus - which he himself had developed - to show that a sphere will attract as though its mass were concentrated at the centre. Arising out of this his theories of motion and of gravitation were triumphantly confirmed. Specifically on the mathematical side, one would say that the theory of functions of a real continuous variable was successfully deployed and the usefulness of calculus demonstrated. From a modern perspective it is clear the pure mathematical justification for calculus was lacking at that time, but it has since been secured within the field of mathematical analysis.

Example 5; *the interaction of light and the electron.* It appears that this interaction cannot be represented by any physical model or straightforward diagram but it can be captured mathematically. This was initially done in two different ways - by Schrödinger's wave equation, which uses partial differentiation, and by Heisenberg's use of matrices. Empirical evidence confirms the value of these mathematical representations of the interaction (which are fully compatible with each other). It follows that there is confirmation of the value of the mathematical understanding of partial differentiation and of the theory of matrices.

Further to these pure scientific examples, one may also instance a major technological and scientific triumph: the successfully achieved objective in 1969 of putting men on the moon and returning them to Earth made massive use of mathematical techniques and results embedded in the science and technology. Now the character of these various scientific and technological achievements may be judged to vary but in each case specific mathematical representations and results may be said to inhere in the identified human activity. In each case too it is fair to say that there is a clear sense in which the mathematics is constraining. Had Newton made a mistake in his construction or use of calculus he would have been unable to link his inverse-square law to the observed motion of the planets; had Schrödinger formulated the wrong equation or misunderstood partial differentiation he would not have contributed to understanding the interaction of light and the electron. Had the mathematics used to put men on the moon been incorrect they would not have completed their journey. One is entitled to conclude that in these cases mathematical results are constraining in ways which are internal to specific human activities. Hence the conclusion to be drawn on this point parallels that given earlier for the more mundane tasks.

In connection with the applications of mathematics, however, an important distinction may be made between its role in terms of representation and description. It is plain that elementary arithmetic arises out of our ordinary everyday experience; in that case numbers are contributing descriptively. So too in the field of statistics - of great importance in human social and policy affairs as well as in a scientific field such as natural history - there is description in numerical form. Yet it seems that centrally important applications of mathematics in the sciences are not of this type. In the example of understanding the interaction of light and the electron used earlier Schrödinger and Heisenberg selected particular mathematical tools which might help to represent a particular physical interaction. The very fact that they selected different sophisticated tools makes it clear that they were feeling for an equation or mathematical structure which could represent an aspect of reality; had a particular tool failed to do this another might conceivably have been selected - without there being the implication that the first was deficient or erroneous in a mathematical sense.

This is relevant to an evaluation of the comments of Putnam and Quine, who as Brown (2008: p. 56) points out, rather see the role of mathematics in descriptive terms. That viewpoint probably links to their remarks about the applications of mathematics to the sciences possibly leading to the revision of mathematics in the light of experience. Now, while it would be churlish not to view many scientific advances and engineering achievements as triumphs of mathematics as well as empirical science, it is nevertheless the case that no mathematical theorem or result has been revised in this way, which rather points to the representational role of mathematics. By way of illustration attention may usefully be drawn to the way in which Newtonian mechanics was overtaken by relativity: there was nothing wrong with the mathematics of the Newtonian system, but Einstein substituted a new set of equations which more accurately reflected reality. A further aspect was that by this time mathematicians had developed non-Euclidean geometry which could then be employed in the representations used in general relativity. No doubt this type of advance stimulated further pure mathematical work—for instance, in differential geometry—but the validity of the mathematics at all stages was fundamentally secured by the pure mathematical method of proof.

11. Differing Branches of Mathematics

So far, in this evaluation of theories within the philosophy of mathematics, there has been a tendency to address mathematics as a whole. A reason for so doing is that the conceptual interconnections and ramifications within the subject are so complex it would seem unwise to carve it up in some summary way; it has been noted, for instance, that modern conceptions of number and geometry cannot be cleanly conceptually isolated from each other.

A distinction which may, however, appear to have some utility in this respect is that between *algebraic* and *non-algebraic mathematical theories* (Shapiro, 1997; see also Shapiro, 2000). As Horsten (2019: section 4.2) puts it, "Roughly, non-algebraic theories are theories which appear at first sight to be about a unique model: the *intended* model of the theory. We have seen examples of such theories: arithmetic, mathematical analysis… Algebraic theories, in contrast, do not carry a prima facie claim to be about a unique model. Examples are group theory, topology, graph theory…" The suggestion is that these differ in respect of the types of mathematical entities or structures involved.

The key difference is that the non-algebraic theory is directly concerned with mathematical entities, while the algebraic theory is not interested in those entities as such but is only concerned with structural aspects of mathematical entities. The reference to structural aspects in the case of algebraic theories may represent a move towards the adoption of a formalist perspective in respect of those theories i.e. an approach which sees mathematical statements as assertions about the manipulation of strings, or sequences of symbols using established rules of manipulation.

The central issue, however, concerns whether or not the distinction between non-algebraic and algebraic theories bears upon the adequacy of a platonist approach and, in particular, helps to refine or qualify the earlier discussion of constraint encountered internal to activity. The first point to make in this connection is that where you have an algebraic theory which is not about a unique model, it seems to follow that it will not have direct application in that precise form to the empirical sciences; hence it is at least distanced from confirmation in the important way to which Quine and Putnam draw attention. However, creative mathematical work of that type proceeds subject to logical and mathematical constraint. In particular, it is directly relevant to say that although an algebraic theory may not be about a unique theory it will typically have application to one or more such theories. An example may help. There is abstract algebraic work concerned with such structures as groups, rings and fields. These notions are judged to have application among other things to the integers and the real numbers. It follows that theorems at the more abstract level have application at the more specific level, while at least some theorems at the lower level must have implications at the upper level; work at the two levels must absolutely marry up, which represents constraint for all concerned. Indeed the interdependence of levels calls into question the suggestion that a formalist approach might apply to one type of theory, while being judged not to apply to the other.

Topology which was included above within the category of algebraic theories may provide further illustration. Topology is akin to geometry but has as its object the study of the properties of geometrical figures that persist even when the figures are subjected to drastic deformations; it has its own vocabulary of 'simplexes' and 'complexes'. However, a key point is that that elaborate field of mathematics is still understood as rooted in everyday experiences - as illustrated by our *Example 3* above concerned with The Seven Bridges of Königsberg. Constraints internal to topology flow from its roots. Also, the interdependencies of mathematical fields are complex: topology makes use of algebra but topological argument can itself be employed to prove the fundamental theorem of algebra (essentially, that in the field of complex numbers every polynomial equation has a root) (Courant & Robbins, 1958: pp. 269-271). Furthermore, topology has also proved to have application in engineering - so it must not be thought of as a field so abstruse as to be beyond the range of confirmability through scientific application.

At this point it is indeed pertinent to take a sharper look at the work of Quine and Putnam on the implications of the empirical sciences for the confirmation of mathematical theories. That work and argument may be said to apply to a lot of mathematics but not to the whole of it; in particular, it might not apply to parts of set theory. Horsten (2019) puts the position in this way:

For it appears that the natural sciences can get by with (roughly) function spaces on the real numbers. The higher regions of transfinite set theory appear to be largely irrelevant to even our most advanced theories in the natural sciences. Nevertheless, Quine thought (at some point) that the sets that are postulated by [Zermelo-Fraenkel set theory with the Axiom of Choice] *(*ZFC) are acceptable from a naturalistic point of view; they can be regarded as a generous rounding off of the mathematics that is involved in our scientific theories (Horsten, 2019: section 3.2).

Without getting lost in technicalities, at least it does seem possible to identify mathematical work which may not be subject to the same degree of constraint as that experienced elsewhere; Cantor's theory of transfinite numbers is perhaps a clear example. Highly significantly, that work has no known application in the empirical sciences, hence it lacks the kind of confirmation referred to by Quine and Putnam. Significantly too, the other work on the number concept is tightly integrated while that on transfinite numbers is more loosely tied in; perhaps 'semi-detached'. Specifically, the number concept developed from natural numbers through the integers, the system of fractions, the integration implied by the advent of analytical geometry, and the real numbers (including irrational numbers) in a way which was subject to closure under the rational operations; the extension to complex numbers is similarly constrained but fundamentally concerned with the provision of solutions to algebraic equations. By contrast, the extension to transfinite numbers is not one subject fully to closure under the rational operations. Russell (1912: p. 87) puts the position for infinite cardinals thus: "Addition, multiplication, and exponentiation proceed quite satisfactorily, but the inverse operations - subtraction, division, and extraction of roots - are ambiguous, and the notions that depend upon them fail when infinite numbers are concerned." The status of some results concerning transfinite numbers has even got bound up with debates about the adequacy of proof by *reductio ad absurdum* (Startup, 2020: section 6). Those adopting an intuitionist philosophical approach to mathematics in effect claim that the topic of transfinite numbers has developed without sufficient restraint put upon it. If that is going too far, at least it may be said that were one to base the ontological status of number on that of the natural numbers (or integers), it seems that of all numbers, transfinite numbers would have the greatest difficulty sharing in it.

12. Conclusion

A position in the philosophy of mathematics is here developed which has an eclectic character drawing as it does from both logicism and platonism, while also departing from each of them. In this connection, although the philosophy of mathematics must ultimately address the whole of that subject including higher mathematics and the theory of sets, it is of the essence that the subject developed - ubiquitously across human societies from the simplest - commencing with the elementary insights concerning the natural numbers together with arithmetic on the one hand, and those concerning space and geometry (taking in elementary mensuration) on the other. Without grasping the *fons et origo* of the subject, the overall philosophical treatment is bound to be distorted.

Crucially, mathematics as an activity is fundamentally dependent upon human reasoning, within which the theoretical and practical may be distinguished. The affinity between mathematics and logic is indicated by the fact that they are the spheres of *a priori* inferences. Where the seemingly *a priori* is successfully challenged - as happened to Euclidean geometry in the nineteenth century - it has the character of an intellectual upheaval.

Frege is a centrally important figure both in his impact on the logicist programme and in his significance for platonism. He analyses statements about numbers and judges that the natural numbers may be defined using logical concepts. However, taking account of Benacerraf's work, it seems that, although many set-theoretic structures may be developed each of which offers limited isomorphism with the system of natural numbers, no one of them may be identified with it.

Frege deploys in his account the notion of an abstract object which is understood to belong to our rational thought. He also makes the suggestion that thoughts and the sentences used to express them are to be considered as 'things'; but one must be wary if that mode of expression supposedly bears on their ontological status. As an alternative one would say that numbers may constitute the subjects or objects of meaningful true sentences in the grammatical sense; so also do theorems consist of a series of meaningful sentences together constituting a proof. To gain access we need simply to understand the sentence or sentences. Thus the timeless, ever present nature of mathematical concepts and results itself offers direct access, in the face of a platonist account which generates a supposed philosophical problem of access.

To take issue with a platonist ontology is not to query mathematical results nor the assertion that they are objectively true. In this connection it can be helpful to distinguish between the language of working mathematicians and the language of working philosophers including platonists. For one thing, the respective criteria for making 'existence' claims differ. Crucially, however, pure mathematics has its own distinctive method of confirming or validating results - mathematical proof - which supplies a higher level of confidence and objectivity than that available elsewhere.

Sometimes platonists assert that mathematical truths are discovered rather than invented. Although the notion of discovery undoubtedly has substantial application, the dichotomy of invention and discovery is too crude and jejune a framework for analysing creative mathematical activity. For instance, creating effective notations, methods of presentation and frameworks which guide thinking is critically important. Also, a tidy modern presentation taking an axiomatic form may be utterly misleading as to the pattern of development of the ideas behind it. Concept and rule establishment in mathematics is rarely done on a 'once and for all' basis, but proceeds at least partly through stages of conjecture and refutation or counterexample.

An attempt is made to evaluate the Gödelian platonist perspective which affirms parallelism between theories of mathematical entities and theories of physical entities. The approach adopted here is to scrutinise the part played by resources and constraints in relation to human activity. Whereas material resources - such as a spanner - are external to the individual (of course know-how in its use is also involved), mathematical resources may be purely internal to a person as in the case of pure or applied mathematical work, but it would certainly sound odd to say that mathematical *objects* as opposed to *ideas* may be accessed by a mental process. As regards constraints on human activity, there does seem to be a difference between the way in which a physical object is a general constraint external to human activity, while a mathematical result constitutes a constraint specifically geared to a precise task in hand. Perhaps these considerations somewhat diminish the sense of parallelism between the two spheres.

It appears that there can be non-causal mathematical explanations and mathematical constraint on processes quite independent of human activity. The latter is particularly evident in respect of the natural processes of cosmology and biology. For instance, the result that volume varies as the cube of radius while surface area varies as the square of radius affects outcomes in stellar evolution and in the evolution of organisms. Indeed within the context of the Darwinian theory of evolution various optimization - maximization and minimization - processes confer relative fitness on populations; this applies to the prime-numbered life cycles of cicadas and the occurrence of Fibonacci numbers within plant biology.

Quine's naturalism as a distinctive philosophical methodology has demonstrable strengths and there is point to Putnam's thought that where scientific theories are mathematically expressed it would be odd to say the mathematical entities do not exist, but one must be cautious of the thesis of confirmational holism. As an approximation one can say confidence in the science (which may involve successful prediction) gives confidence in the associated mathematics. There are instances where this is particularly clear-cut e.g. confidence in the use of calculus would undoubtedly have grown given its successful employment by Newton in relation to the theory of gravitation. However, scrutiny of the main uses of mathematics in the sciences suggests that its role is representational rather than descriptive. Scientists are feeling for an equation or mathematical structure which could represent an aspect of reality and where one is inadequate they may adopt another. There is no known case of a mathematical theory or result being shown as erroneous by empirical scientific inquiry. Pure mathematics is disciplined by proof.

The conceptual interconnections and ramifications within mathematics are so complex one must be wary of carving it up in a summary way, but there clearly are differing specialisms and levels of abstraction. The distinction between algebraic and non-algebraic mathematical theories proves to be of particular significance, however, and it seems that an algebraic theory which is not about a unique model will lack direct application in that precise form to the empirical sciences; hence it is not confirmed in the important way to which Quine and Putnam draw attention. It is also noteworthy that, while all the other work on the number concept is tightly integrated, that on transfinite numbers is more loosely tied in; it being the case that transfinite numbers are not fully subject to closure under the rational operations.

As a wider expression of Quine's naturalistic approach, it would seem we can have more confidence in mathematical knowledge and understanding than in existing epistemological or metaphysical theories within philosophy. Far from philosophy being needed as a foundation for mathematics, we might be better employed working in the opposite direction: in a highly specific sense, this could be said to be Gödel's direction of travel in his important metamathematical proof (Kleene, 1962: p. 206).

Mathematics is *sui generis* and rather escapes the clutches of either logicism or platonism. One cannot identify set-theoretic structures with the number concept; also a possible analogy between mathematical entities and physical objects rather fails because they do not contribute in the same sort of way as resources and constraints on activity and the granite quality of mathematics derives from proof taking the form of accurate sequential reasoning. While not depending upon any individual mind, arithmetic and geometry do depend upon general human attempts to represent the world; and mathematics forms part of the conceptual apparatus with which we address the world.

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