# ANALYSIS ON THE CONE OF DISCRETE RADON MEASURES

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Abstract. We study analysis on the cone of discrete Radon measures over a locally compact Polish space $X$. We discuss probability measures on the cone and the corresponding correlation measures and correlation functions on the sub-cone of finite discrete Radon measures over $X$. For this, we consider on the cone an analogue of the harmonic analysis on the configuration space developed in [12]. We also study elements of finite-difference calculus on the cone: we introduce discrete birth-and-death gradients and study the corresponding Dirichlet forms; finally, we discuss a system of polynomial functions on the cone which satisfy the binomial identity.

*In memory of Prof. Anatoly Vershik*

## 1. Introduction

Let $X$ be a locally compact Polish space and $\mathcal{B}(X)$ be the corresponding Borel $\sigma$-algebra on $X$. Let $\mathbb{M}(X)$ be the space of real-valued Radon measures on $(X, \mathcal{B}(X))$.

A random measure on $X$ is a random element of the space $\mathbb{M}(X)$. The theory of random measures plays an essential role in the modern studies of probability and point processes, see e.g. [4, 10]. The (marked) point processes can be interpreted as random *discrete* (non-negative) measures. The cone of non-negative discrete Radon measures on $\mathcal{B}(X)$ is the set

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) : s_i > 0, x_i \in X \right\}, \tag{1.1}$$

where $\delta_{x_i}$ denotes the Dirac measure with unit mass at $x_i \in X$. Here the atoms $x_i$ are assumed to be distinct and their total number is at most countable. By convention, $\mathbb{K}(X)$ contains the zero measure $\eta = 0$, which is represented by the sum over the empty set of indices $i$.

The probability measures on the cone $\mathbb{K}(X)$, in particular, the Gamma measure (see Section 2.2 below), are important e.g. for the respresentation theory of big groups [8], study of infinite-dimensional analogue of Lebesgue

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measure [25], constructions of orthogonal polynomials in the generalisations of the white noise analysis [16, 19], etc.

The *support* of a $\eta \in \mathbb{K}(X)$ is defined by

$$\tau(\eta) := \{x \in X : s_x(\eta) := \eta(\{x\}) > 0\}.$$

In particular, $\tau(0) = \emptyset$. If $\eta \in \mathbb{K}(X)$ is fixed, we just write $s_x := s_x(\eta)$. Thus, each $\eta \in \mathbb{K}(X)$ can be rewritten in the form

$$\eta = \sum_{x \in \tau(\eta)} s_x \delta_x.$$

For each $\eta \in \mathbb{K}(X)$ and a compact $\Lambda \in \mathcal{B}(X)$, we have that

$$\sum_{x \in \tau(\eta) \cap \Lambda} s_x = \eta(\Lambda) < \infty.$$

Hence, if we interpret $s_x$ as the weight at a point $x \in X$, the total weight in any compact region is finite. Nevertheless, for a typical measure on $\mathbb{K}(X)$,

$$|\tau(\eta) \cap \Lambda| = \infty \tag{1.2}$$

for almost all $\eta \in \mathbb{K}(X)$ (henceforth, $|\cdot|$ denotes the cardinality of a discrete set); in other words, the configuration of points constituting the support of $\eta$ is not locally finite. The crucial assumption for (1.2) is that the measure that provides the distribution of weights $s_x > 0$ is infinite on $(0, \infty)$. This makes analysis on the cone very different from the case of weights (marks) with a finite distribution, the so-called marked configurations with a finite measure on marks, where the results are essentially pretty similar to the case without marks (when all $s_x = 1$), see e.g. [15, 20, 18].

The further studies of the analysis and probability on the cone were provided in several directions. The Gibbs perturbation of the Gamma measure was constructed and studied in [9], the Laplace operator and the corresponding diffusion process on the cone were developed in [14], see also [3], a moment problem on the cone was analysed in [13], and some non-equilibrium birth-and-death dynamics on the cone were studied in [5].

The present paper deals with two other important topics of the analysis on the cone of discrete Radon measures: we discuss an analogue of the harmonic analysis on the cone, similar to [12] and we introduce elements of finite-difference calculus on the cone, influenced by [6].

The paper is organised as follows. In Section 2, we define the basic structures on the cone $\mathbb{K}(X)$ and discuss probability measures on the cone. In particular, in Proposition 2.5, we describe a class of measures for which (1.2) holds almost everywhere. In Section 3, we discuss harmonic analysis on the cone. For this, we consider an analogue of the $K$-transform (Definition 3.1), cf. [12], which maps functions defined on the sub-cone $\mathbb{K}_0(X)$ of discrete measures with finite support to the cone $\mathbb{K}(X)$, and study properties of the $K$-transform. Next, we discuss the correlation measures and correlation functions of probability measures on the cone. Finally, in Section 4, we study discrete gradients on the cone, derive several properties of the corresponding Dirichlet forms (Propositions 4.3 and 4.4), and describe properties of certain polynomial functions on the cone which satisfy, in particular, the binomial formula (Proposition 4.7).

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This article was written with the enthusiasm, commitment and deep scientific knowledge of our collaborator, co-author and friend Yuri Kondratiev. He passed away on September 5th, 2023.

## 2. Framework

Let $\mathbb{M}(X)$ denote the space of real-valued Radon measures on $X$. We equip $\mathbb{M}(X)$ with the vague topology, that is the coarsest topology on $\mathbb{M}(X)$ such that, for each continuous function $f : X \to \mathbb{R}$ with compact support (shortly $f \in C_{\mathrm{c}}(X)$), the following mapping is continuous:

$$\mathbb{M}(X) \ni \nu \mapsto \langle \nu, f\rangle := \int_X f(x)\, d\nu(x) \in \mathbb{R}.$$

We consider the corresponding Borel $\sigma$-algebra $\mathcal{B}(\mathbb{M}(X))$. Let $\mathbb{K}(X) \subset \mathbb{M}(X)$ be defined by (1.1). We endow $\mathbb{K}(X)$ with the vague topology induced by $\mathbb{M}(X)$ and denote the corresponding Borel $\sigma$-algebra by $\mathcal{B}(\mathbb{K}(X))$; that is then the trace $\sigma$-algebra of $\mathcal{B}(\mathbb{M}(X))$. Note that, for each $f \in C_{\mathrm{c}}(X)$,

$$\langle \eta, f\rangle = \sum_{x\in\tau(\eta)} s_x f(x), \qquad \eta \in \mathbb{K}(X).$$

Let $\mathbb{R}_+^* := (0, +\infty)$ be endowed with the logarithmic metric

$$d_{\mathbb{R}_+^*}(s_1, s_2) := |\ln(s_1) - \ln(s_2)|, \quad s_1, s_2 > 0.$$

It is easy to see that $\mathbb{R}_+^*$ is a locally compact Polish space, and any set of the form $[a, b]$, with $0 < a < b < \infty$, is compact. Then, $\widehat{X} := \mathbb{R}_+^* \times X$ is also a locally compact Polish space. Hence, we can define $\mathbb{M}(\widehat{X})$ and $\mathcal{B}_{\mathrm{c}}(\widehat{X})$ as before. Clearly, any compact set in $\widehat{X}$ is a subset of $[a, b] \times \Lambda$ for some $0 < a < b < \infty$ and $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$; henceforth, $\mathcal{B}_{\mathrm{c}}(X)$ denotes the family of sets in $\mathcal{B}(X)$ with compact closure.

We define the space of *locally finite configurations* over $\widehat{X}$ as the set

$$\Gamma(\widehat{X}) := \Big\{\gamma \subset \widehat{X} : \big|\gamma \cap ([a, b] \times \Lambda)\big| < \infty \ \ \forall\, \Lambda \in \mathcal{B}_{\mathrm{c}}(X), b > a > 0\Big\}.$$

As usual, each configuration $\gamma \in \Gamma(\widehat{X})$ can be identified with the Radon measure $\sum_{y\in\gamma} \delta_y \in \mathbb{M}(\widehat{X})$. Thus, the inclusion $\Gamma(\widehat{X}) \subset \mathbb{M}(\widehat{X})$ holds, which allows to endow $\Gamma(\widehat{X})$ with the vague topology induced by $\mathbb{M}(\widehat{X})$ and the corresponding Borel $\sigma$-algebra $\mathcal{B}(\Gamma(\widehat{X}))$.

Let $\Gamma_p(\widehat{X}) \subset \Gamma(\widehat{X})$ be the set of *pinpointing configurations*, which consists of all configurations $\gamma$ such that if $(s_1, x_1), (s_2, x_2) \in \gamma$ with $x_1 = x_2$, then

$s_1 = s_2$. For each $\gamma \in \Gamma_p(\widehat{X})$ and each $\Lambda \in \mathcal{B}_c(X)$, we define the local mass of $\Lambda$ by

$$\gamma(\Lambda) := \int_{\widehat{X}} s \mathbb{1}_\Lambda(x)\, d\gamma(s,x) = \sum_{(s,x)\in\gamma} s\mathbb{1}_\Lambda(x) \in [0,+\infty]\,,$$

where $\mathbb{1}_\Lambda$ denotes the indicator function of the set $\Lambda$. The set of pinpointing configurations with finite local mass is then defined by

$$\Pi(\widehat{X}) := \{\gamma \in \Gamma_p(\widehat{X}) : \gamma(\Lambda) < \infty \ \forall \Lambda \in \mathcal{B}_c(X)\}.$$

One has $\Pi(\widehat{X}) \in \mathcal{B}(\Gamma(\widehat{X}))$ and we fix the trace $\sigma$-algebra of $\mathcal{B}(\Gamma(\widehat{X}))$ on $\Pi(\widehat{X})$, denoted by $\mathcal{B}(\Pi(\widehat{X}))$.

Thus, we have defined a bijective mapping

$$\begin{aligned} \mathcal{R} : \Pi(\widehat{X}) &\longrightarrow \mathbb{K}(X) \\ \gamma = \sum_{(s,x)\in\gamma} \delta_{(s,x)} &\longmapsto \sum_{(s,x)\in\gamma} s\delta_x. \end{aligned} \tag{2.1}$$

As shown in [9], both $\mathcal{R}$ and its inverse mapping $\mathcal{R}^{-1}$ are measurable with respect to $\mathcal{B}(\Pi(\widehat{X}))$ and $\mathcal{B}(\mathbb{K}(X))$ and $\mathcal{R}$ is a $\sigma$-isomorphism. Moreover,

$$\mathcal{B}(\mathbb{K}(X)) = \{\mathcal{R}(A) : A \in \mathcal{B}(\Pi(\widehat{X}))\},$$

and thus $\mathcal{B}(\Pi(\widehat{X})) = \{\mathcal{R}^{-1}(B) : B \in \mathcal{B}(\mathbb{K}(X))\}$.

Let $\pi_{\nu\otimes\sigma}$ be the *Poisson measure* on $\mathcal{B}(\Gamma(\widehat{X}))$ (see e.g. [1] for the details) with intensity measure $\nu \otimes \sigma$, where $\nu$ and $\sigma$ are two non-atomic positive Radon measures on the Borel $\sigma$-algebras $\mathcal{B}(\mathbb{R}_+^*)$ and $\mathcal{B}(X)$, respectively, and $\nu$ has finite first moment, i.e.,

$$\int_{\mathbb{R}_+^*} s\, d\nu(s) < \infty. \tag{2.2}$$

In terms of the Laplace transform of $\pi_{\nu\otimes\sigma}$, for each continuous function $f : \widehat{X} \to [0,\infty)$ with compact support, we have

$$\int_{\Gamma(\widehat{X})} e^{-\langle\gamma,f\rangle} d\pi_{\nu\otimes\sigma}(\gamma) = \exp\left(\int_{\widehat{X}} \left(e^{-f(s,x)} - 1\right) d(\nu\otimes\sigma)(s,x)\right).$$

**Proposition 2.1.** *For $\nu$ and $\sigma$ as above, $\pi_{\nu\otimes\sigma}(\Pi(\widehat{X})) = 1$.*

*Proof.* As easily seen, $\Gamma_p(\widehat{X}), \Pi(\widehat{X}) \in \mathcal{B}(\Gamma(\widehat{X}))$. Moreover, using the distribution of configurations of the form $\gamma \cap ([a,b]\times\Lambda)$, $\gamma \in \Gamma(\widehat{X})$, $0 < a < b$, $\Lambda \in \mathcal{B}_c(X)$, under $\pi_{\nu\otimes\sigma}$ (see e.g. [11]), we conclude that, for each $0 < a < b$, $\Lambda \in \mathcal{B}_c(X)$ fixed,

$$\pi_{\nu\otimes\sigma}\Big(\{\gamma \in \Gamma(\widehat{X}) : \exists\, (s_1,x_1),(s_2,x_2) \in \gamma \cap ([a,b]\times\Lambda)\ \text{s.t.}\ x_1 = x_2, s_1 \neq s_2\}\Big) = 0.$$

Thus $\pi_{\nu\otimes\sigma}(\Gamma_p(\widehat{X})) = 1$. The rest of the proof is a consequence of the Mecke identity [23], which states that for every measurable function $H : \Gamma(\widehat{X}) \times \widehat{X} \to [0,+\infty)$ the following equality holds:

$$\begin{aligned}(2.3)\qquad &\int_{\Gamma(\widehat{X})}\int_{\widehat{X}} H(\gamma,s,x)\,d\gamma(s,x)d\pi_{\nu\otimes\sigma}(\gamma)\\ &= \int_{\Gamma(\widehat{X})}\int_{\widehat{X}} H(\gamma\cup\{(s,x)\},s,x)\,d(\nu\otimes\sigma)(s,x)d\pi_{\nu\otimes\sigma}(\gamma).\end{aligned}$$

In particular, for each $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ fixed and for the measurable function $H(\gamma,s,x) = s1\!\!1_\Lambda(x)$, the latter leads to

$$\begin{aligned}\int_{\Gamma_p(\widehat{X})}\gamma(\Lambda)d\pi_{\nu\otimes\sigma}(\gamma) &= \int_{\Gamma_p(\widehat{X})}\int_{\widehat{X}} s1\!\!1_\Lambda(x)\,d(\nu\otimes\sigma)(s,x)d\pi_{\nu\otimes\sigma}(\gamma)\\ &= \sigma(\Lambda)\int_{\mathbb{R}_+^*} s\,d\nu(s) < \infty, \qquad (2.4)\end{aligned}$$

which implies that $\gamma(\Lambda) < \infty$ for $\pi_{\nu\otimes\sigma}$-a.a. $\gamma \in \Gamma_p(\widehat{X})$. Hence, $\pi_{\nu\otimes\sigma}(\Pi(\widehat{X})) = 1$. $\square$

**Definition 2.2.** By Proposition 2.1, we may view $\pi_{\nu\otimes\sigma}$ as a probability measure on $\mathcal{B}(\Pi(\widehat{X}))$. Thus, we consider the push-forward of the measure $\pi_{\nu\otimes\sigma}$ under $\mathcal{R}$ to $\mathbb{K}(X)$, which we denote by $\pi_{\mathbb{K},\nu\otimes\sigma}$.

Then, by (2.4),

$$\int_{\mathbb{K}(X)}\eta(\Lambda)d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) < \infty.$$

Note that, for each continuous bounded function $g \in C_{\mathrm{b}}(\mathbb{R}^N)$, $N \in \mathbb{N}$, and for every functions $\varphi_1,\dots,\varphi_N \in C_{\mathrm{c}}(X)$, we have

$$\begin{aligned}&\int_{\mathbb{K}(X)} g\left(\langle\eta,\varphi_1\rangle,\dots,\langle\eta,\varphi_N\rangle\right)d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &\qquad= \int_{\Pi(\widehat{X})} g\left(\langle\gamma,\mathrm{id}\otimes\varphi_1\rangle,\dots,\langle\gamma,\mathrm{id}\otimes\varphi_N\rangle\right)d\pi_{\nu\otimes\sigma}(\gamma),\end{aligned}$$

where $(\mathrm{id}\otimes\varphi)(s,x) := s\varphi(x)$. In terms of the Laplace transform of $\pi_{\mathbb{K},\nu\otimes\sigma}$, for each $f \in C_{\mathrm{c}}(X)$ one finds

$$\int_{\mathbb{K}(X)} e^{-\langle\eta,f\rangle}d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) = \exp\left(\int_X\int_{\mathbb{R}_+^*}\left(e^{-sf(x)}-1\right)d\nu(s)d\sigma(x)\right).$$

**Proposition 2.3** (Mecke-type identity)**.** *For each measurable function $F : \mathbb{K}(X)\times\widehat{X} \to [0,+\infty)$, the following equality holds*

$$\begin{aligned}(2.5)\qquad &\int_{\mathbb{K}(X)}\int_X F(\eta,s_x,x)\,d\eta(x)d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)}\int_X\int_{\mathbb{R}_+^*} s\,F(\eta+s\delta_x,s,x)\,d\nu(s)d\sigma(x)d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta).\end{aligned}$$

*Proof.* Formula (2.5) is a direct consequence of the Mecke identity (2.3) and Proposition 2.1:

$$\begin{aligned}
&\int_{\mathbb{K}(X)}\int_X F(\eta, s_x, x)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\
&= \int_{\Pi(\widehat{X})}\int_{\widehat{X}} sF(\mathcal{R}(\gamma), s, x)\, d\gamma(s,x) d\pi_{\nu\otimes\sigma}(\gamma)\\
&= \int_{\Pi(\widehat{X})}\int_{\widehat{X}} sF(\mathcal{R}(\gamma\cup\{(s,x)\}), s, x)\, d(\nu\otimes\sigma)(s,x) d\pi_{\nu\otimes\sigma}(\gamma)\\
&= \int_{\mathbb{K}(X)}\int_X\int_{\mathbb{R}_+^*} sF(\eta + s\delta_x, s, x)\, d\nu(s) d\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta).
\end{aligned}$$
□

A special case concerns $\nu = \nu_\theta$, where

$$\nu_\theta(ds) := \frac{\theta}{s} e^{-s} ds$$

for some $\theta > 0$. In this case, the corresponding Poisson measure is called a Gamma–Poisson measure and the push-forward to $\mathbb{K}(X)$ is called a *Gamma measure* with intensity $\theta$ and denoted by $\mathcal{G}_\theta$. Alternatively, a Gamma measure can be characterized by its Laplace transform [2]: for each $-1 < f \in C_c(X)$,

$$\int_{\mathbb{K}(X)} e^{-\langle \eta, f\rangle} d\mathcal{G}_\theta(\eta) = \exp\left(-\theta\int_X \ln(1+f(x))\, d\sigma(x)\right).$$

The next result states a Mecke-type characterization result for Gamma measures. There, $\mathbb{M}_+(X) \subset \mathbb{M}(X)$ denotes the cone of all non-negative measures.

**Proposition 2.4** ([9])**.** *Let $\mu$ be a probability measure defined on $\mathbb{M}_+(X)$ which has finite first local moments, i.e., for each $\Lambda \in \mathcal{B}_c(X)$,*

$$\int_{\mathbb{M}_+(X)} \eta(\Lambda)\, d\mu(\eta) < \infty.$$

*Then, $\mu = \mathcal{G}_\theta$ if and only if for any measurable function $G : \mathbb{M}_+(X) \times X \to [0, +\infty)$ we have*

$$\begin{aligned}
&\int_{\mathbb{M}_+(X)}\int_X G(\eta, x)\, d\eta(x) d\mu(\eta)\\
&\qquad = \int_{\mathbb{M}_+(X)}\int_X\int_{\mathbb{R}_+^*} sG(\eta + s\delta_x, x)\, d\nu_\theta(s) d\sigma(x) d\mu(\eta).
\end{aligned}$$

Measure $\nu_\theta$ is an example of an infinite measure on $\mathbb{R}_+^*$. The next statement shows that, for any such $\nu$ (with finite first moment), the support $\tau(\eta)$ of $\pi_{\mathbb{K},\nu\otimes\sigma}$-a.a. $\eta \in \mathbb{K}(X)$ is not a locally finite subset of $X$.

**Proposition 2.5.** *Let $\nu(\mathbb{R}_+^*) = \infty$. Then, for any $\Lambda \in \mathcal{B}_c(X)$ with $\sigma(\Lambda) > 0$ and for $\pi_{\mathbb{K},\nu\otimes\sigma}$-a.a. $\eta \in \mathbb{K}(X)$, the set $\tau(\eta) \cap \Lambda$ has an infinite number of elements.*

*Proof.* By e.g. [1], for any $\Lambda \in \mathcal{B}_c(X)$, $0 < a < b$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,
(2.6)
$$\pi_{\nu\otimes\sigma}\Big\{\gamma \in \Gamma(\widehat{X})\colon \big|\gamma \cap ([a,b]\times\Lambda)\big| = n\Big\} = \frac{\big(\sigma(\Lambda)\nu([a,b])\big)^n}{n!} e^{-\sigma(\Lambda)\nu([a,b])}.$$
Therefore, if $\sigma(\Lambda) > 0$ and $\nu(\mathbb{R}_+^*) = \infty$ then, passing $a \to 0$, $b \to \infty$, we obtain, by the definition of $\pi_{\mathbb{K},\nu\otimes\sigma}$, that
$$\pi_{\mathbb{K},\nu\otimes\sigma}(\{\eta \in \mathbb{K}(X) : |\tau(\eta) \cap \Lambda| = n\}) = 0,$$
that implies the statement. □

**Remark 2.6.** Note that by (2.6), for any $\Lambda \in \mathcal{B}_c(X)$ with $\sigma(\Lambda) = 0$,
$$\pi_{\mathbb{K},\nu\otimes\sigma}(\{\eta \in \mathbb{K}(X) : \tau(\eta) \cap \Lambda = \emptyset\}) = 1.$$

**Definition 2.7.** Let $\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ denote the space of all probability measures $\mu$ on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X))$ which have finite local moments of all orders, that is, for all $n \in \mathbb{N}$ and all $\Lambda \in \mathcal{B}_c(X)$,
$$\int_{\mathbb{K}(X)} (\eta(\Lambda))^n \, d\mu(\eta) < \infty.$$

**Proposition 2.8.** $\pi_{\mathbb{K},\nu\otimes\sigma} \in \mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ *if and only if* $\nu$ *has finite moments of all orders, that is, for all* $n \in \mathbb{N}$,
$$\int_{\mathbb{R}_+^*} s^n \, d\nu(s) < \infty. \tag{2.7}$$

*Proof.* For each $n \in \mathbb{N}$ and each $\Lambda \in \mathcal{B}_c(X)$ we have
$$\begin{aligned}\int_{\mathbb{K}(X)} (\eta(\Lambda))^n \, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) &= \int_{\Pi(\widehat{X})} (\gamma(\Lambda))^n \, d\pi_{\nu\otimes\sigma}(\gamma)\\ &= \int_{\Gamma(\widehat{X})} \langle\gamma, \mathrm{id}\otimes 1\!\!1_\Lambda\rangle^n \, d\pi_{\nu\otimes\sigma}(\gamma),\end{aligned}$$
where, as shown e.g. in [7, Theorem 6.1], the latter integral is equal to
$$\sum_{k=1}^n \frac{(\sigma(\Lambda))^k}{k!} \sum_{\substack{(i_1,\dots,i_k)\in\mathbb{N}^k\\ i_1+\dots+i_k=n}} \binom{n}{i_1 \dots i_k} \prod_{j=1}^k \left(\int_{\mathbb{R}_+^*} s^{i_j}\, d\nu(s)\right).$$
The required necessary and sufficient condition then follows. □

**Remark 2.9.** By Proposition 2.8, $\mathcal{G}_\theta \in \mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ for every intensity $\theta > 0$.

Let $\Lambda \in \mathcal{B}_c(X)$ and $0 < a < b$ be fixed. We consider
$$\Gamma([a,b]\times\Lambda) := \{\gamma \in \Gamma(\widehat{X})\colon \gamma \subset [a,b]\times\Lambda\}$$
and let $\mathcal{B}_{[a,b]\times\Lambda}(\Gamma(\widehat{X}))$ be the corresponding $\sigma$-algebra on $\Gamma(\widehat{X})$ which is $\sigma$-isomorphic to the trace $\sigma$-algebra $\mathcal{B}(\Gamma([a,b]\times\Lambda))$, see [12, Section 2.2] for details. Let $\Pi([a,b]\times\Lambda) = \Gamma_p([a,b]\times\Lambda)$ be the corresponding measurable

subset of pinpointing configurations (which are finite and, hence, have finite local mass), and let $\mathcal{B}_{[a,b]\times\Lambda}(\Pi(\widehat{X}))$ be the corresponding $\sigma$-algebra on $\Pi(\widehat{X})$. Then

$$\begin{aligned}\mathbb{K}([a,b]\times\Lambda) &:= \mathcal{R}(\Pi([a,b]\times\Lambda))\\ &= \{\eta\in\mathbb{K}(X)\colon \tau(\eta)\subset\Lambda,\ s_x(\eta)\in[a,b]\ \forall x\in\tau(\eta)\}\in\mathcal{B}(\mathbb{K}(X)),\end{aligned}$$

and we can consider the corresponding $\sigma$-algebra $\mathcal{B}_{[a,b]\times\Lambda}(\mathbb{K}(X))$. We also consider the measurable mapping $\hat{p}_{[a,b]\times\Lambda} : \Pi(\widehat{X}) \to \Pi([a,b]\times\Lambda)$ given by $\hat{p}_{[a,b]\times\Lambda}(\gamma) := \gamma\cap([a,b]\times\Lambda)$. We then define the measurable mapping $p_{\Lambda,a,b} : \mathbb{K}(X)\to\mathbb{K}([a,b]\times\Lambda)$ by

$$p_{\Lambda,a,b}(\eta) := \mathcal{R}\,\hat{p}_{[a,b]\times\Lambda}\,\mathcal{R}^{-1}\eta,\quad \eta\in\mathbb{K}(X).$$

Then, for each $\eta\in\mathbb{K}(X)$,

$$p_{\Lambda,a,b}(\eta) = \sum_{x\in\tau(\eta)\cap\Lambda} 1\!\!1_{[a,b]}(s_x)s_x\delta_x,$$

and

$$|\tau(p_{\Lambda,a,b}(\eta))| = \sum_{x\in\tau(\eta)\cap\Lambda} 1\!\!1_{[a,b]}(s_x) \le \frac{1}{a}\sum_{x\in\tau(\eta)\cap\Lambda} s_x = \frac{1}{a}\eta(\Lambda)<\infty. \tag{2.8}$$

**Definition 2.10.** A measure $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ is called locally absolutely continuous with respect to $\pi_{\mathbb{K},\nu\otimes\sigma}$ if, for all $\Lambda\in\mathcal{B}_{\mathrm{c}}(X)$ and $0<a<b$, the measure $\mu^{\Lambda,a,b} := \mu\circ p^{-1}_{\Lambda,a,b}$ is absolutely continuous with respect to $\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma} := \pi_{\mathbb{K},\nu\otimes\sigma}\circ p^{-1}_{\Lambda,a,b}$. Equivalently, the push-forward of the measure $\mu$ under $\mathcal{R}^{-1}$ to $\Pi(\widehat{X})$ is locally absolutely continuous with respect to $\pi_{\nu\otimes\sigma}$ (see [12] for details).

## 3. Harmonic analysis on the cone

### 3.1. Discrete Radon measures with finite support.

We consider the sub-cone of all non-negative Radon measures with finite support

$$\mathbb{K}_0(X) := \{\eta\in\mathbb{K}(X) : |\tau(\eta)|<\infty\} = \bigsqcup_{n=0}^{\infty}\mathbb{K}_0^{(n)}(X),$$

where $\mathbb{K}_0^{(0)}(X) := \{0\}$ is the set consisting of the zero measure and, for each $n\in\mathbb{N}$, $\mathbb{K}_0^{(n)}(X) := \{\eta\in\mathbb{K}_0(X) : |\tau(\eta)| = n\}$.

We consider also the space $\Pi_0(\widehat{X}) := \{\gamma\in\Pi(\widehat{X}) : |\gamma|<\infty\}$ of pinpointing finite configurations on $\widehat{X}$. Since $\Pi_0(\widehat{X})\subset\Gamma_0(\widehat{X}) := \{\gamma\in\Gamma(\widehat{X}) : |\gamma|<\infty\}$, we can endow $\Pi_0(\widehat{X})$ with the topology induced by the topology defined on $\Gamma_0(\widehat{X})$ [12] and the corresponding Borel $\sigma$-algebra, denoted by $\mathcal{B}(\Pi_0(\widehat{X}))$. As easily seen, $\Pi_0(\widehat{X})\in\mathcal{B}(\Gamma_0(\widehat{X}))$, thus, $\mathcal{B}(\Pi_0(\widehat{X})) = \mathcal{B}(\Gamma_0(\widehat{X}))\cap\Pi_0(\widehat{X})$. Also, $\mathbb{K}_0(X) = \mathcal{R}(\Pi_0(\widehat{X}))$, which allows to endow $\mathbb{K}_0(X)$ with the $\sigma$-algebra

$$\mathcal{B}(\mathbb{K}_0(X)) := \{\mathcal{R}(A) : A\in\mathcal{B}(\Pi_0(\widehat{X}))\}$$

with respect to which the restriction $\mathcal{R}_{|\Pi_0(\widehat{X})} : \Pi_0(\widehat{X}) \to \mathbb{K}_0(X)$ is a $\sigma$-isomorphism.

A set $A \in \mathcal{B}(\mathbb{K}_0(X))$ is called *bounded* if there exist $0 < a < b$, $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ and an $N \in \mathbb{N}$ such that, for all $\eta \in A$,

$$\tau(\eta) \subset \Lambda, \qquad |\tau(\eta)| \le N, \qquad s_x \in [a,b] \quad \forall\, x \in \tau(\eta). \tag{3.1}$$

We denote by $\mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$ the set of all bounded sets in $\mathcal{B}(\mathbb{K}_0(X))$.

Let $L^0(\mathbb{K}_0(X))$ be the class of all $\mathcal{B}(\mathbb{K}_0(X))$-measurable functions $G : \mathbb{K}_0(X) \to \mathbb{R}$. Assume that $G = 1\!\!1_A G$, where a $A \in \mathcal{B}(\mathbb{K}_0(X))$ is such that, for some $0 < a < b$ and $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$, we have, for all for all $\eta \in A$,

$$\tau(\eta) \subset \Lambda, \qquad s_x \in [a,b] \quad \forall\, x \in \tau(\eta). \tag{3.2}$$

Then we will say that $G$ has a *local support* on $\mathbb{K}_0(X)$. The class of all measurable functions with local support is denoted by $L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$. Similarly, if $G = 1\!\!1_A G$ for a bounded $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$, we will say that $G$ has *bounded support* on $\mathbb{K}_0(X)$. We consider a subclass of all *bounded* measurable functions with bounded support, denoted by $B_{\mathrm{bs}}(\mathbb{K}_0(X))$. Thus, $B_{\mathrm{bs}}(\mathbb{K}_0(X)) \subset L^0_{\mathrm{ls}}(\mathbb{K}_0(X)) \subset L^0(\mathbb{K}_0(X))$. We will denote the pre-images of these classes under $(\mathcal{R}_{|\Pi_0(\widehat{X})})^{-1}$ by $B_{\mathrm{bs}}(\Pi_0(\widehat{X}))$, $L^0_{\mathrm{ls}}(\Pi_0(\widehat{X}))$, $L^0(\Pi_0(\widehat{X}))$, respectively.

3.2. ***K*-transform.** In the sequel, for $\eta, \xi \in \mathbb{K}(X)$, we write $\xi \subset \eta$ if $\tau(\xi) \subset \tau(\eta)$ and $s_x(\xi) = s_x(\eta)$ for all $x \in \tau(\xi)$. If, additionally, $\xi \in \mathbb{K}_0(X)$, we write $\xi \Subset \eta$. Note that, for $\xi, \eta \in \mathbb{K}(X) \subset \mathbb{M}(X)$ with $\xi \subset \eta$, the measure $\eta - \xi \in \mathbb{M}(X)$ is well-defined and $\eta - \xi \in \mathbb{K}(X)$ with $\tau(\eta - \xi) = \tau(\eta) \setminus \tau(\xi)$ and $s_x(\eta - \xi) = s_x(\eta)$ for each $x \in \tau(\eta) \setminus \tau(\xi)$.

**Definition 3.1.** For each $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$, we define a function $KG : \mathbb{K}(X) \to \mathbb{R}$, called the $K$-transform of $G$, by

$$(KG)(\eta) := \sum_{\xi \Subset \eta} G(\xi), \quad \eta \in \mathbb{K}(X). \tag{3.3}$$

Note that since $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$, there exist $0 < a < b$ and $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ such that

$$\sum_{\xi \Subset \eta} G(\xi) = \sum_{\xi \Subset \eta} 1\!\!1_{\substack{\tau(\xi) \subset \Lambda \\ s_x \in [a,b], x \in \tau(\xi)}}(\xi) G(\xi) = \sum_{\xi \subset p_{\Lambda,a,b}(\eta)} G(\xi),$$

and hence, by (2.8), the sum in (3.3) is well-defined (note that, by (2.8), $p_{\Lambda,a,b}(\eta) \in \mathbb{K}_0(X)$). Moreover, we have shown that, for $F := KG$, $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$,

$$F(\eta) = F(p_{\Lambda,a,b}(\eta)), \quad \eta \in \mathbb{K}(X), \tag{3.4}$$

for some $0 < a < b$ and $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ (dependent on $F$).

**Proposition 3.2.** *Let $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$ and $KG$ be defined by (3.3). The $KG$ is a $\mathcal{B}(\mathbb{K}(X))$-measurable function.*

*Proof.* Let $F \in L^0_{\mathrm{ls}}(\Pi_0(\widehat{X}))$. We define the $\mathcal{B}(\Gamma_0(\widehat{X}))$-measurable function $\widetilde{F} : \Gamma_0(\widehat{X}) \to \mathbb{R}$ given by

$$\widetilde{F}(\eta) := \begin{cases} F(\eta), & \text{if } \eta \in \Pi_0(\widehat{X}), \\ 0, & \text{otherwise.} \end{cases}$$

By [12], the function $K_{\widehat{X}}\widetilde{F} : \Gamma(\widehat{X}) \to \mathbb{R}$, given by

$$(K_{\widehat{X}}\widetilde{F})(\gamma) := \sum_{\substack{\zeta \subset \gamma \\ |\zeta| < \infty}} \widetilde{F}(\zeta), \quad \gamma \in \Gamma(\widehat{X}),$$

is well-defined and $\mathcal{B}(\Gamma(\widehat{X}))$-measurable. Therefore, the restriction

$$(K_\Pi F)(\gamma) := \Big( (K_{\widehat{X}}\widetilde{F})\big|_{\Pi(\widehat{X})} \Big)(\gamma) = \sum_{\substack{\zeta \subset \gamma \\ |\zeta| < \infty}} F(\zeta), \quad \gamma \in \Pi(\widehat{X})$$

is $\mathcal{B}(\Pi(\widehat{X}))$-measurable.

Now, let $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$ be given and consider $F = G \circ \mathcal{R}\big|_{\Pi_0(\widehat{X})} \in L^0_{\mathrm{ls}}(\Pi_0(\widehat{X}))$. Note that

$$KG = \sum_{n=0}^{\infty} K(G 1\!\!1_{\mathbb{K}_0^{(n)}(X)}). \tag{3.5}$$

For an arbitrary set of indices $I \subseteq \mathbb{N}_0$, let $\eta = \sum_{i \in I} s_i \delta_{x_i} \in \mathbb{K}(X)$. Hence, by (3.5),

$$\begin{aligned} (KG)(\eta) &= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\} \subseteq I} G\left( \sum_{i=1}^{n} s_{i_k} \delta_{x_{i_k}} \right) \\ &= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\} \subseteq I} G\left( \mathcal{R}\left( \sum_{i=1}^{n} \delta_{(s_{i_k}, x_{i_k})} \right) \right) \\ &= K_\Pi (G \circ \mathcal{R}_{|\Pi_0(\widehat{X})})(\mathcal{R}^{-1}\eta) = (K_\Pi F)(\mathcal{R}^{-1}\eta) \end{aligned} \tag{3.6}$$

with $\mathcal{R}^{-1}\eta = \sum_{i \in I} \delta_{(s_i, x_i)} \in \Pi(\widehat{X})$.

Since $K_\Pi F$ is $\mathcal{B}(\Pi(\widehat{X}))$-measurable, we have that $KG$ is $\mathcal{B}(\mathbb{K}(X))$-measurable. □

Let $\mathcal{F}L^0(\mathbb{K}(X))$ be the class of all $\mathcal{B}(\mathbb{K}(X))$-measurable functions $F : \mathbb{K}(X) \to \mathbb{R}$ such that (3.4) holds (for some $0 < a < b$ and $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ dependent on $F$). By (3.4) and Proposition 3.2, $K : L^0_{\mathrm{ls}}(\mathbb{K}_0(X)) \to \mathcal{F}L^0(\mathbb{K}(X))$.

**Proposition 3.3.** *1. The mapping $K : L^0_{\mathrm{ls}}(\mathbb{K}_0(X)) \to \mathcal{F}L^0(\mathbb{K}(X))$ is linear, positivity-preserving and invertible, with the inverse mapping given by, for $F \in \mathcal{F}L^0(\mathbb{K}(X))$,*

$$(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\tau(\eta)| - |\tau(\xi)|} F(\xi), \qquad \eta \in \mathbb{K}_0(X). \tag{3.7}$$

*2. For each $G \in B_{\mathrm{bs}}(\mathbb{K}_0(X))$, there exist $C > 0$, $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ and $N \in \mathbb{N}$ such that*

$$|(KG)(\eta)| \le C(1+\eta(\Lambda))^N, \quad \eta \in \mathbb{K}(X).$$

*Proof.* 1. The linearity and positivity-preserving properties follow directly from the definition (3.3). Next, for $G \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$, we set $F = KG$ and denote the right-hand side of (3.7) by $K^{-1}F$. Then

$$\begin{aligned}(K^{-1}F)(\eta) &= \sum_{\xi\subset\eta}(-1)^{|\tau(\eta)|-|\tau(\xi)|}\sum_{\zeta\subset\xi}G(\zeta) = \sum_{\zeta\subset\eta}G(\zeta)\sum_{\substack{\xi\subset\eta:\\ \zeta\subset\xi}}(-1)^{|\tau(\eta)|-|\tau(\xi)|}\\ &= \sum_{\zeta\subset\eta}G(\zeta)\sum_{\xi\subset\eta-\zeta}(-1)^{|\tau(\eta-\zeta)|-|\tau(\xi)|} = \sum_{\zeta\subset\eta}G(\zeta)0^{|\tau(\eta-\zeta)|} = G(\eta).\end{aligned}$$

On the other hand, let $F \in \mathcal{F}L^0(\mathbb{K}(X))$ and let $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ and $0 < a < b$ be such that (3.4) holds. We have

$$\begin{aligned}(K^{-1}F)(\eta) &= \sum_{\xi\subset\eta}(-1)^{|\tau(\eta)|-|\tau(\xi)|}F(\xi)\\ &= \sum_{\substack{\xi_1\subset\eta:\\ \tau(\xi_1)\subset\Lambda,\\ s_x\in[a,b]\ \forall x\in\tau(\xi_1)}}\ \sum_{\substack{\xi_2\subset\eta:\\ \exists x\in\tau(\xi_2):\, x\in X\setminus\Lambda \text{ or } s_x\notin[a,b]}}(-1)^{|\tau(\eta)|-|\tau(\xi_1)|-|\tau(\xi_2)|}F(\xi_1+\xi_2)\\ &= \sum_{\substack{\xi_1\subset\eta:\\ \tau(\xi_1)\subset\Lambda,\\ s_x\in[a,b]\ \forall x\in\tau(\xi_1)}}F(\xi_1)(-1)^{|\tau(\eta)|-|\tau(\xi_1)|}\sum_{\substack{\xi_2\subset\eta:\\ \exists x\in\tau(\xi_2):\, x\in X\setminus\Lambda \text{ or } s_x\notin[a,b]}}(-1)^{-|\tau(\xi_2)|};\end{aligned}$$

and since

$$\sum_{\substack{\xi_2\subset\eta:\\ \exists x\in\tau(\xi_2):\, x\in X\setminus\Lambda \text{ or } s_x\notin[a,b]}}(-1)^{-|\tau(\xi_2)|} = 0^{|\{x\in\tau(\eta):\, x\in X\setminus\Lambda \text{ or } s_x\notin[a,b]\}|},$$

we obtain

$$(K^{-1}F)(\eta) = \mathbb{1}_{\tau(\eta)\subset\Lambda,\, s_x\in[a,b]\ \forall x\in\tau(\eta)}(\eta)(K^{-1}F)(\eta),$$

thus $K^{-1}F \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$. We then have

$$\begin{aligned}(KK^{-1}F)(\eta) &= \sum_{\xi\Subset\eta}\mathbb{1}_{\tau(\xi)\subset\Lambda,\, s_x\in[a,b]\ \forall x\in\tau(\xi)}(\xi)\sum_{\zeta\subset\xi}(-1)^{|\tau(\xi)|-|\tau(\zeta)|}F(\zeta)\\ &= \sum_{\xi\subset p_{\Lambda,a,b}(\eta)}\sum_{\zeta\subset\xi}(-1)^{|\tau(\xi)|-|\tau(\zeta)|}F(\zeta)\\ &= \sum_{\zeta\subset p_{\Lambda,a,b}(\eta)}F(\zeta)\sum_{\substack{\xi\subset p_{\Lambda,a,b}(\eta):\\ \zeta\subset\xi}}(-1)^{|\tau(\xi)|-|\tau(\zeta)|}\\ &= \sum_{\zeta\subset p_{\Lambda,a,b}(\eta)}F(\zeta)0^{|\tau(p_{\Lambda,a,b}(\eta))|-|\tau(\zeta)|} = F(p_{\Lambda,a,b}(\eta)) = F(\eta),\end{aligned}$$

by (3.4).

2. For $G \in B_{\mathrm{bs}}(\mathbb{K}_0(X))$, we have $|G| \le c1\!\!1_A$ for some $c > 0$, $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$. Hence, for some $0 < a < b$, $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$, $N \in \mathbb{N}$, (3.1) holds for all $\eta \in A$. Therefore, for each $\eta \in \mathbb{K}(X)$,

$$|(KG)(\eta)| \le c(K1\!\!1_A)(\eta) = c\sum_{k=0}^{N} \sum_{\substack{\xi \subset p_{\Lambda,a,b}(\eta) \\ |\tau(\xi)|=k}} 1\!\!1_A(\xi)$$

$$\le c\sum_{k=0}^{N} \binom{|\tau(p_{\Lambda,a,b}(\eta))|}{k} \le C(1+\eta(\Lambda))^N, \tag{3.8}$$

for $C = c(\max\{1, \frac{1}{a}\})^N$, where we used the estimate (2.8). □

We can also extend the $K$-transform to the class $\mathcal{F}_{\exp}(\mathbb{K}_0(X))$ of measurable functions $G : \mathbb{K}_0(X) \to \mathbb{R}$ such that, for some $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ and $C > 0$,

$$|G(\xi)| \le 1\!\!1_{\{\tau(\xi)\subset\Lambda\}}(\xi)\, C^{|\tau(\xi)|} \prod_{x\in\tau(\xi)} s_x, \qquad \xi \in \mathbb{K}_0(X). \tag{3.9}$$

Indeed, for each $\eta \in \mathbb{K}(X)$, we have then

$$|(KG)(\eta)| \le \sum_{\xi \Subset \eta} |G(\xi)| \le \sum_{\substack{\xi\Subset\eta: \\ \tau(\xi)\subset\tau(\eta)\cap\Lambda}} C^{|\tau(\xi)|} \prod_{x\in\tau(\xi)} s_x = \prod_{x\in\tau(\eta)\cap\Lambda} (1+Cs_x) < \infty,$$

since $\eta \in \mathbb{K}(X)$ and hence, $\sum_{x\in\tau(\eta)\cap\Lambda} Cs_x = C\eta(\Lambda) < \infty$.

**Example 3.4.** Let $f : X \to \mathbb{R}$ be a bounded measurable function with compact support.

1. For $G \in \mathcal{F}_{\exp}(\mathbb{K}_0(X))$ defined by

$$G(\eta) := \begin{cases} sf(x), & \text{if } \eta = \{s\delta_x\} \in \mathbb{K}_0^{(1)}(X) \\ 0, & \text{otherwise} \end{cases}, \quad \eta \in \mathbb{K}_0(X),$$

the $K$-transform of $G$ is given by

$$(KG)(\eta) = \sum_{x\in\tau(\eta)} s_x f(x) = \langle \eta, f\rangle, \quad \eta \in \mathbb{K}(X).$$

2. For the so-called Lebesgue–Poisson exponent $e_{\mathbb{K}}(f) \in \mathcal{F}_{\exp}(\mathbb{K}_0(X))$ corresponding to $f$,

$$e_{\mathbb{K}}(f,\eta) := \prod_{x\in\tau(\eta)} s_x f(x), \quad \eta \in \mathbb{K}_0(X), \tag{3.10}$$

its $K$-transform is equal to

$$(Ke_{\mathbb{K}}(f))(\eta) = \prod_{x\in\tau(\eta)} (1 + s_x f(x)), \quad \eta \in \mathbb{K}(X). \tag{3.11}$$

Given $G_1,\ G_2 \in L^0(\mathbb{K}_0(X))$, let us define the $\star$-convolution between $G_1$ and $G_2$,

$$(G_1 \star G_2)(\eta) := \sum_{\substack{\xi_1+\xi_2+\xi_3=\eta \\ \tau(\xi_i)\cap\tau(\xi_j)=\emptyset,\, i\neq j}} G_1(\xi_1+\xi_2)G_2(\xi_2+\xi_3), \quad \eta \in \mathbb{K}_0(X),$$

where the sum is over all $\xi_1, \xi_2, \xi_3 \subset \eta$ such that $(\tau(\xi_1), \tau(\xi_2), \tau(\xi_3))$ is a partition of $\tau(\eta)$. As easily seen, under this product $L^0(\mathbb{K}_0(X))$ has a commutative algebraic structure with unit element $e_{\mathbb{K}}(0)$.

**Proposition 3.5.** *For all $G_1, G_2 \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$ we have $G_1 \star G_2 \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$ and*

$$K(G_1 \star G_2) = (KG_1) \cdot (KG_2). \tag{3.12}$$

*Proof.* Given $G_1, G_2 \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$ we have $G_i = G_i 1\!\!1_A$ for some $A \in \mathcal{B}(\mathbb{K}_0(X))$ such that (3.2) holds. Then

$$(G_1 \star G_2)1\!\!1_A = ((G_1 1\!\!1_A) \star (G_2 1\!\!1_A))1\!\!1_A = ((G_1 1\!\!1_A) \star (G_2 1\!\!1_A)) = G_1 \star G_2.$$

This shows that $G_1 \star G_2 \in L^0_{\mathrm{ls}}(\mathbb{K}_0(X))$. Concerning the right-hand side of (3.12),

$$(KG_1)(\eta) \cdot (KG_2)(\eta) = \left(\sum_{\xi \Subset \eta} G_1(\xi)\right)\left(\sum_{\zeta \Subset \eta} G_2(\zeta)\right), \ \eta \in \mathbb{K}(X), \tag{3.13}$$

observe that, for each $\eta \in \mathbb{K}(X)$ fixed, there is a one-to-one correspondence between pairs $\xi \Subset \eta$, $\zeta \Subset \eta$ and groups $\vartheta \Subset \eta$, $\xi_1, \xi_2, \xi_3 \subset \vartheta$ with $\xi_1 + \xi_2 + \xi_3 = \vartheta$, hence, $(\tau(\xi_1), \tau(\xi_2), \tau(\xi_3))$ forms a partition of $\tau(\vartheta)$. This one-to-one correspondence is defined by the following rule: $\tau(\vartheta) = \tau(\xi) \cup \tau(\zeta)$, $\tau(\xi_1) = \tau(\xi) \setminus \tau(\zeta)$, $\tau(\xi_2) = \tau(\xi) \cap \tau(\zeta)$, $\tau(\xi_3) = \tau(\zeta) \setminus \tau(\xi)$. In this way, product (3.13) can be rewritten as

$$\sum_{\vartheta \Subset \eta} \sum_{\xi_1+\xi_2+\xi_3=\vartheta} G_1(\xi_1+\xi_2)G_2(\xi_2+\xi_3),$$

which completes the proof. □

3.3. **Correlation measures and correlation functions on $\mathbb{K}_0(X)$.** A measure $\rho$ on $\big(\mathbb{K}_0(X), \mathcal{B}(\mathbb{K}_0(X))\big)$ is said to be *locally finite* if $\rho(A) < \infty$ for each $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$.

**Example 3.6.** An example of a locally finite measure is the Lebesgue–Poisson measure $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$, where $\nu$ and $\sigma$ are non-atomic positive Radon measures on $\mathcal{B}(\mathbb{R}^*_+)$ and $\mathcal{B}(X)$, respectively, and $\nu$ has a finite first moment

(2.2). The measure $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$ is defined so that, for each $G \in B_{\mathrm{bs}}(\mathbb{K}_0(X))$,

$$\int_{\mathbb{K}_0(X)} G(\eta)\, d\lambda_{\mathbb{K}_0,\nu\otimes\sigma}(\eta) \tag{3.14}$$
$$= G(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\widehat{X}^n} G\left(\sum_{k=1}^{n} s_i \delta_{x_i}\right) d\nu(s_1)\cdots d\nu(s_n) d\sigma(x_1)\cdots d\sigma(x_n).$$

Note that the sum in (3.14) is finite for $G \in B_{\mathrm{bs}}(\mathbb{K}_0(X))$, in particular, for $G = 1\!\!1_A$ with $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$. Moreover, it is easily seen that $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$ is the push-forward of $\lambda_{\nu\otimes\sigma}\big|_{\mathcal{B}(\Pi_0(\widehat{X}))}$ under $\mathcal{R}\big|_{|\Pi_0(\widehat{X})}$ to $\big(\mathbb{K}_0(X), \mathcal{B}(\mathbb{K}_0(X))\big)$, where $\lambda_{\nu\otimes\sigma}$ is the Lebesgue–Poisson on $\big(\Gamma_0(\widehat{X}), \mathcal{B}(\Gamma_0(\widehat{X}))\big)$, see [12]. In particular, we can extend (3.14) to measurable functions on $\mathbb{K}_0(X)$ integrable with respect to $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}(\eta)$ for which the right-hand side of (3.14) is well-defined. For example, if $f \in L^1(X,\sigma)$ and $e_{\mathbb{K}}(f)$ is defined as in (3.10) for $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$-a.a. $\eta \in \mathbb{K}_0(X)$, then $e_{\mathbb{K}}(f) \in L^1(\mathbb{K}_0(X), \lambda_{\mathbb{K}_0,\nu\otimes\sigma})$ with

$$\int_{\mathbb{K}_0(X)} e_{\mathbb{K}}(f,\eta)\, d\lambda_{\mathbb{K}_0,\nu\otimes\sigma}(\eta) = \exp\left(\int_{\mathbb{R}_+^*} s\, d\nu(s) \int_X f(x)\, d\sigma(x)\right).$$

**Definition 3.7.** Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ be given. We (uniquely) define a measure $\rho_\mu$ on on $(\mathbb{K}_0(X), \mathcal{B}(\mathbb{K}_0(X))$ by requiring that, for all $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$,

$$\rho_\mu(A) = \int_{\mathbb{K}(X)} (K 1\!\!1_A)(\eta)\, d\mu(\eta).$$

Then $\rho_\mu$ is called the correlation measure corresponding to $\mu$.

**Remark 3.8.** Note that, by (3.8), the assumption $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ ensures that $\rho_\mu(A) < \infty$ for all $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$.

**Remark 3.9.** It can be easily derived from [21, Theorem 2 and formula (3.9)] that, under a very week assumption, the correlation measure $\rho_\mu$ in Definition 3.7 uniquely determines the measure $\mu$.

**Proposition 3.10.** *Let $\nu$ and $\sigma$ be two non-atomic positive Radon measures on $\mathcal{B}(\mathbb{R}_+^*)$ and $\mathcal{B}(X)$, respectively. Assume that $\nu$ has finite moments of all orders (2.7), so that $\pi_{\mathbb{K},\nu\otimes\sigma} \in \mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ by Proposition 2.8. Then, $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$ is the correlation measure of $\pi_{\mathbb{K},\nu\otimes\sigma}$.*

*Proof.* Let $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$ and (3.1) holds. Then, by (3.4), $(K 1\!\!1_A)(\eta) = (K 1\!\!1_A)(p_{\Lambda,a,b}(\eta))$ and hence,

$$\int_{\mathbb{K}(X)} (K 1\!\!1_A)(\eta) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) = \int_{\mathbb{K}([a,b]\times\Lambda)} (K 1\!\!1_A)(\eta) d\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}(\eta).$$

Then, by (3.6) and Proposition 2.1 with $\hat{X}$ replaced by $[a,b]\times\Lambda$, we get that

$$\begin{aligned}
&\int_{\mathbb{K}([a,b]\times\Lambda)}(K1\!\!1_A)(\eta)d\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}(\eta)\\
&=\int_{\Gamma([a,b]\times\Lambda)}(K_{\hat{X}}1\!\!1_{\mathcal{R}^{-1}A})(\gamma)d\pi^{[a,b]\times\Lambda}_{\nu\otimes\sigma}(\gamma)\\
&=e^{-\nu([a,b])\sigma(\Lambda)}\int_{\Gamma([a,b]\times\Lambda)}\sum_{\xi\subset\gamma}1\!\!1_{\mathcal{R}^{-1}A}(\xi)d\lambda_{\nu\otimes\sigma}(\gamma)
\end{aligned}$$

and rewriting $1\!\!1_{\mathcal{R}^{-1}A}(\xi)=1\!\!1_{\mathcal{R}^{-1}A}(\xi)\cdot 1\!\!1_{\Gamma([a,b]\times\Lambda)}(\gamma\setminus\xi)$, we can apply e.g. [12, Lemma A.1]

$$\begin{aligned}
&=e^{-\nu([a,b])\sigma(\Lambda)}\int_{\Gamma([a,b]\times\Lambda)}1\!\!1_{\mathcal{R}^{-1}A}(\xi)d\lambda_{\nu\otimes\sigma}(\xi)\int_{\Gamma([a,b]\times\Lambda)}d\lambda_{\nu\otimes\sigma}(\gamma)\\
&=\lambda_{\nu\otimes\sigma}(\mathcal{R}^{-1}A)=\lambda_{\mathbb{K}_0,\nu\otimes\sigma}(A),
\end{aligned}$$

which proves the statement. □

Let $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$. Then, by (3.8), $K(B_{\mathrm{bs}}(\mathbb{K}_0(X)))\subset L^1(\mathbb{K}(X),\mu)$. Thus, $B_{\mathrm{bs}}(\mathbb{K}_0(X))\subset L^1(\mathbb{K}_0(X),\rho_\mu)$ and standard techniques of the measure theory yield

$$\int_{\mathbb{K}_0(X)}G(\eta)\,d\rho_\mu(\eta)=\int_{\mathbb{K}(X)}(KG)(\eta)\,d\mu(\eta),\quad G\in B_{\mathrm{bs}}(\mathbb{K}_0(X)).\tag{3.15}$$

The density of $B_{\mathrm{bs}}(\mathbb{K}_0(X))$ in $L^1(\mathbb{K}_0(X),\rho_\mu)$ allows to extend the $K$-transform to a bounded operator. We will keep the same notation for the extended operator. More precisely, we have the following result.

**Proposition 3.11.** *Let $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$. Then, there is a bounded operator $K:L^1(\mathbb{K}_0(X),\rho_\mu)\to L^1(\mathbb{K}(X),\mu)$ such that formula* (3.15) *holds for any $G\in L^1(\mathbb{K}_0(X),\rho_\mu)$. Moreover, for each $G\in L^1(\mathbb{K}_0(X),\rho_\mu)$, equality* (3.3) *holds for $\mu$-almost all $\eta\in\mathbb{K}(X)$.*

*Proof.* The proof is based on standard techniques of measure theory and follows similarly to [12, Corollary 4.1 and Theorem 4.1]. □

**Remark 3.12.** In particular, for any $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ and $f:X\to\mathbb{R}$ such that $e_{\mathbb{K}}(f)\in L^1(\mathbb{K}_0(X),\rho_\mu)$, it follows from Proposition 3.11 that (3.11) holds for $\mu$-a.a. $\eta\in\mathbb{K}(X)$.

**Proposition 3.13.** *Let the conditions of Proposition 3.10 hold. Let $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(X))$ be locally absolutely continuous with respect to the measure $\pi_{\mathbb{K},\nu\otimes\sigma}$. Let $\rho_\mu$ be the correlation measure of $\mu$ according to Definition 3.7. Then $\rho_\mu$ is absolutely continuous with respect to $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}$.*

*Proof.* Given $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{K}_0(X))$, assume that $\lambda_{\mathbb{K}_0,\nu\otimes\sigma}(A) = 0$. Hence, for some $0 < a < b$, $N \in \mathbb{N}$, $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$, (3.1) holds for all $\eta \in A$ and we have

$$\begin{aligned}
0 &= \lambda_{\mathbb{K}_0,\nu\otimes\sigma}(A) \\
&= \int_{\mathbb{K}(X)} (K\mathbb{1}_A)(\eta) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&= \int_{\mathbb{K}(X)} (K\mathbb{1}_A)(p_{\Lambda,a,b}(\eta)) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) = \int_{\mathbb{K}([a,b]\times\Lambda)} (K\mathbb{1}_A)(\eta) d\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}(\eta).
\end{aligned}$$

This implies that $K\mathbb{1}_A = 0$ $\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}$-a.e. on $\mathbb{K}([a,b]\times\Lambda)$. As a result,

$$\begin{aligned}
\rho_\mu(A) &= \int_{\mathbb{K}(X)} (K\mathbb{1}_A)(\eta) d\mu(\eta) \\
&= \int_{\mathbb{K}([a,b]\times\Lambda)} (K\mathbb{1}_A)(\eta) \frac{d\mu^{\Lambda,a,b}}{d\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}}(\eta) d\pi^{\Lambda,a,b}_{\mathbb{K},\nu\otimes\sigma}(\eta) = 0.
\end{aligned}$$

□

**Definition 3.14.** Let the conditions of Proposition 3.13 hold. The Radon–Nikodym derivative $k_\mu := \dfrac{d\rho_\mu}{d\lambda_{\mathbb{K}_0,\nu\otimes\sigma}}$ is called the correlation function corresponding to $\mu$.

## 4. Finite-difference calculus on the cone

### 4.1. Discrete gradients on $\mathbb{K}(X)$.

The discrete structure of measures in $\mathbb{K}(X)$ suggests a development of a finite-difference calculus on $\mathbb{K}(X)$. For $\eta \in \mathbb{K}(X)$, the elementary operations on $\eta$ which will be considered are the following ones:

- removing one point $x$ from the support of $\eta$: $\eta \mapsto \eta - s_x\delta_x$;
- adding a new point $x$ with a weight $s$ to $\eta$: $\eta \mapsto \eta + s\delta_x$, $s \in \mathbb{R}^*_+$.

As a set of test functions on $\mathbb{K}(X)$ we will consider the space $\mathscr{F} := \mathcal{F}C_{\mathrm{b}}(C_{\mathrm{c}}(\widehat{X}), \mathbb{K}(X))$ of all functions $F : \mathbb{K}(X) \to \mathbb{R}$ of the form

$$F(\eta) = g\left(\langle \mathcal{R}^{-1}\eta, \varphi_1\rangle, \dots, \langle \mathcal{R}^{-1}\eta, \varphi_N\rangle\right), \quad \eta \in \mathbb{K}(X),$$

where $g \in C_{\mathrm{b}}(\mathbb{R}^N)$, $\varphi_1, \dots, \varphi_N \in C_{\mathrm{c}}(\widehat{X})$, $N \in \mathbb{N}$. We fix two non-atomic positive Radon measures $\nu$ and $\sigma$ on $\mathcal{B}(\mathbb{R}^*_+)$ and $\mathcal{B}(X)$, respectively, such that $\nu$ has finite first moment.

**Definition 4.1.** Let $F \in \mathscr{F}$.

1. A discrete death gradient of $F$ is defined by

$$(D^-_x F)(\eta) := F(\eta - s_x\delta_x) - F(\eta), \quad \eta \in \mathbb{K}(X), x \in \tau(\eta).$$

The corresponding tangent space is chosen to be $T^-_\eta(\mathbb{K}(X)) := L^2(X, \eta)$.

2. A discrete birth gradient of $F$ is defined by

$$(D^+_{(s,x)} F)(\eta) := F(\eta + s\delta_x) - F(\eta), \quad \eta \in \mathbb{K}(X),$$

where $(s,x) \in \widehat{X}$, $x \notin \tau(\eta)$. Here, the corresponding tangent space is chosen to be $T^+_\eta(\mathbb{K}(X)) := L^2(\widehat{X}, s\nu(ds) \otimes \sigma(dx))$.

Observe that, for a fixed $\eta \in \mathbb{K}(X)$, the function $\tau(\eta) \ni x \mapsto (D_x^- F)(\eta)$ is bounded and has compact support in $X$. Thus, $(D_\cdot^- F)(\eta) \in T_\eta^-(\mathbb{K}(X))$. For each $h \in C_{\mathrm{c}}(X)$, we define a directional derivative along $h$ by

$$\begin{aligned}(D_h^- F)(\eta) &:= \langle (D_\cdot^- F)(\eta), h\rangle_{T_\eta^-(\mathbb{K}(X))}\\ &= \int_X (D_x^- F)(\eta) h(x) d\eta(x) = \sum_{x\in\tau(\eta)} s_x h(x)(D_x^- F)(\eta).\end{aligned}$$

As easily seen, for each $\eta \in \mathbb{K}(X)$ fixed, we also have $(D_\cdot^+ F)(\eta) \in T_\eta^+(\mathbb{K}(X))$ and we define a directional derivative along a direction $h \in C_{\mathrm{c}}(X)$ by

$$\begin{aligned}(D_h^+ F)(\eta) &:= \langle (D_\cdot^+ F)(\eta), h\rangle_{T_\eta^+(\mathbb{K}(X))}\\ &= \int_X \int_{\mathbb{R}_+^*} (D_{(s,x)}^+ F)(\eta) s d\nu(s) h(x) d\sigma(x).\end{aligned}$$

The next result states a relation between these two notions of directional derivative.

**Proposition 4.2.** *For any $F, G \in \mathscr{F}$ and $h \in C_{\mathrm{c}}(X)$ we have*

$$\begin{aligned}&\int_{\mathbb{K}(X)} (D_h^- F)(\eta) G(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} F(\eta)(D_h^+ G)(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) - \int_{\mathbb{K}(X)} F(\eta) G(\eta) B_{\nu,\sigma,h}(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta),\end{aligned}$$

*where*

$$B_{\nu,\sigma,h}(\eta) := \int_X h(x)\, d\eta(x) - \int_{\mathbb{R}_+^*} s\, d\nu(s) \int_X h(x)\, d\sigma(x).$$

*Proof.* By formula (2.5),

$$\begin{aligned}&\int_{\mathbb{K}(X)} \int_X F(\eta - s_x\delta_x) h(x) d\eta(x) G(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} \int_X \int_{\mathbb{R}_+^*} s F(\eta) h(x) G(\eta + s\delta_x)\, d\nu(s) d\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} F(\eta)(D_h^+ G)(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &\quad + \int_{\mathbb{K}(X)} F(\eta) G(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \int_{\mathbb{R}_+^*} s\, d\nu(s) \int_X h(x)\, d\sigma(x).\end{aligned}$$

Therefore, by the definition of $D_h^- F$, the required formula follows. $\square$

We proceed to show that a Laplacian-type operator associated with the discrete birth-and-death gradients exists. For each $F, G \in \mathscr{F}$, let $\mathcal{E}$ be the

Dirichlet integral associated with the discrete death gradient:

$$\begin{aligned}\mathcal{E}(F,G) &:= \int_{\mathbb{K}(X)} \langle (D^-_\cdot F)(\eta), (D^-_\cdot G)(\eta)\rangle_{T^-_\eta(\mathbb{K}(X))}\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} \int_X (D^-_x F)(\eta)(D^-_x G)(\eta)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta).\end{aligned}$$

It turns out by formula (2.5) that, actually, $\mathcal{E}$ coincides with the Dirichlet integral associated with the discrete birth gradient:

$$\begin{aligned}\mathcal{E}(F,G) &:= \int_{\mathbb{K}(X)} \langle (D^+_\cdot F)(\eta), (D^+_\cdot G)(\eta)\rangle_{T^+_\eta(\mathbb{K}(X))}\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} \int_X \int_{\mathbb{R}^*_+} s(D^+_{(s,x)} F)(\eta)(D^+_{(s,x)} G)(\eta)\, d\nu(s) d\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta).\end{aligned} \tag{4.1}$$

**Proposition 4.3.** *The $(\mathcal{E},\mathscr{F})$ is a well-defined symmetric bilinear form on $L^2(\mathbb{K}(X),\pi_{\mathbb{K},\nu\otimes\sigma})$.*

*Proof.* The symmetry and the bilinear property follow directly. Hence, we just have to show that if $F\in\mathscr{F}$ is $\pi_{\mathbb{K},\nu\otimes\sigma}$-a.e. equal to 0, then $\mathcal{E}(F,G)=0$ for every $G\in\mathscr{F}$. This is a consequence of formula (2.5), because for each $\Lambda\in\mathcal{B}_{\mathrm{c}}(X)$ one then finds

$$\begin{aligned}&\int_{\mathbb{K}(X)} \int_X \int_{\mathbb{R}^*_+} s|F(\eta+s\delta_x)| 1\!\!1_\Lambda(x)\, d\nu(s) d\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta)\\ &= \int_{\mathbb{K}(X)} \int_X |F(\eta)|\eta(\Lambda)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) = 0,\end{aligned}$$

which implies that $F(\eta+s\delta_x)=0$ for $\pi_{\mathbb{K},\nu\otimes\sigma}\otimes\nu\otimes\sigma$-a.a. $(\eta,s,x)\in\mathbb{K}(X)\times\widehat{X}$. Thus, $(D^+_{(s,x)}F)(\eta)=0$ for $\pi_{\mathbb{K},\nu\otimes\sigma}\otimes\nu\otimes\sigma$-a.a. $(\eta,s,x)\in\mathbb{K}(X)\times\mathbb{R}^*_+\times X$. Hence, by (4.1), for each $G\in\mathscr{F}$ we have $\mathcal{E}(F,G)=0$. □

**Proposition 4.4.** *For each $F\in\mathscr{F}$, let*

$$(LF)(\eta) := \int_X (D^-_x F)(\eta)\, d\eta(x) + \int_X \int_{\mathbb{R}^*_+} (D^+_{(s,x)}F)(\eta) s d\nu(s) d\sigma(x).$$

*Then, $(L,\mathscr{F})$ is a symmetric operator on $L^2(\mathbb{K}(X),\pi_{\mathbb{K},\nu\otimes\sigma})$ which verifies the following equality*

$$\mathcal{E}(F,G) = \langle -LF, G\rangle_{L^2(\mathbb{K}(X),\pi_{\mathbb{K},\nu\otimes\sigma})}, \quad F,G\in\mathscr{F}. \tag{4.2}$$

*The bilinear form $(\mathcal{E},\mathscr{F})$ is closable on $L^2(\mathbb{K}(X),\pi_{\mathbb{K},\nu\otimes\sigma})$ and the operator $(L,\mathscr{F})$ has Friedrich's extension, denoted by $(L,D(L))$. Moreover, the extended operator $(L,D(L))$ is the generator of the closed symmetric form, denoted by $(\mathcal{E},D(\mathcal{E}))$.*

*Proof.* First note that, by formula (2.5), for any $F, G \in \mathscr{F}$ we have

$$
\begin{aligned}
\mathcal{E}(F,G) &= \int_{\mathbb{K}(X)} (D_x^- F)(\eta) G(\eta - s_x\delta_x)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&\quad - \int_{\mathbb{K}(X)} (D_x^- F)(\eta) G(\eta)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&= - \int_{\mathbb{K}(X)} \int_X \int_{\mathbb{R}_+^*} sF(\eta)(D_{(s,x)}^+ G)(\eta)\, d\nu(s)\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&\quad - \int_{\mathbb{K}(X)} \int_X F(\eta)(D_x^- G)(\eta)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta),
\end{aligned}
$$

which, due to the symmetry of $\mathcal{E}$, shows that formula (4.2) holds, provided $LF \in L^2(\mathbb{K}(X), \pi_{\mathbb{K},\nu\otimes\sigma})$. In order to prove that $LF \in L^2(\mathbb{K}(X), \pi_{\mathbb{K},\nu\otimes\sigma})$, observe that since $F \in \mathscr{F}$, there are $C \geq 0$, $0 < a < b$, $\Lambda \in \mathcal{B}_c(X)$ such that

$$
\begin{aligned}
&|(D_x^- F)(\eta)| \leq C 1\!\!1_{[a,b]}(s_x) 1\!\!1_\Lambda(x), \quad x \in \tau(\eta), \eta \in \mathbb{K}(X), \\
&|(D_{(s,x)}^+ F)(\eta)| \leq C 1\!\!1_{[a,b]}(s) 1\!\!1_\Lambda(x), \quad s \in \mathbb{R}_+^*, x \in X \setminus \tau(\eta), \eta \in \mathbb{K}(X).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{K}(X)} (LF)^2(\eta)\, d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&\quad \leq 2C^2 \int_{\mathbb{K}(X)} \left( \int_X 1\!\!1_{[a,b]}(s_x) 1\!\!1_\Lambda(x)\, d\eta(x) \right)^2 d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&\quad + 2C^2 \int_{\mathbb{K}(X)} \left( \int_X 1\!\!1_\Lambda(x)\, d\sigma(x) \int_{\mathbb{R}_+^*} s 1\!\!1_{[a,b]}(s)\, d\nu(s) \right)^2 d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta),
\end{aligned}
\tag{4.3}
$$

where the latter integral is finite. Concerning (4.3), three applications of formula (2.5) yield

$$
\begin{aligned}
&\int_{\mathbb{K}(X)} \left( \int_X 1\!\!1_{[a,b]}(s_x) 1\!\!1_\Lambda(x)\, d\eta(x) \right)^2 d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&= \int_{\mathbb{K}(X)} \int_X \int_X 1\!\!1_{[a,b]}(s_x) 1\!\!1_\Lambda(x) 1\!\!1_{[a,b]}(s_y) 1\!\!1_\Lambda(y) d\eta(x) d(\eta - s_x\delta_x)(y) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&+ \int_{\mathbb{K}(X)} \int_X s_x 1\!\!1_{[a,b]}(s_x) 1\!\!1_\Lambda(x)\, d\eta(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&= \int_{\mathbb{K}(X)} \left( \int_X 1\!\!1_\Lambda(x)\, d\sigma(x) \int_{\mathbb{R}_+^*} s 1\!\!1_{[a,b]}(s)\, d\nu(s) \right)^2 d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) \\
&+ \int_{\mathbb{K}(X)} \int_X \int_{\mathbb{R}_+^*} s^2 1\!\!1_{[a,b]}(s) 1\!\!1_\Lambda(x)\, d\nu(s) d\sigma(x) d\pi_{\mathbb{K},\nu\otimes\sigma}(\eta) < \infty.
\end{aligned}
$$

Hence, $LF \in L^2(\mathbb{K}(X), \pi_{\mathbb{K},\nu\otimes\sigma})$. Furthermore, by formula (4.2), $(-L, \mathscr{F})$ is a positive symmetric operator in $L^2(\mathbb{K}(X), \pi_{\mathbb{K},\nu\otimes\sigma})$. It is now standard to prove that the bilinear form $(\mathcal{E}, \mathscr{F})$ is closable and $(L, \mathscr{F})$ has Friedrich's extension (see e.g. [24]). $\square$

**Remark 4.5.** Using techniques of the Dirichlet form theory [22], it is possible to show that there exists an equilibrium Markov process on $\mathbb{K}(X)$ that has the operator $L$ as its generator, compare with [17] and [3].

4.2. **Polynomial functions on $\mathbb{K}(X)$.** Let $n \in \mathbb{N}$. Let $\mathbb{M}^{(n)}(X)$ be the set of all symmetric real-valued Radon measures on $\big(X^n, \mathcal{B}(X^n)\big)$, see [7]. Let $\mathcal{F}(X)$ be the set of all bounded measurable functions on $X$ with compact support, and $\mathcal{F}^{(n)}(X)$ the set of all bounded measurable symmetric functions on $X^n$ with compact support. For $\mu^{(n)} \in \mathbb{M}^{(n)}(X)$ and $f^{(n)} \in \mathcal{F}^{(n)}(X)$, we denote

$$\langle \mu^{(n)}, f^{(n)} \rangle := \int_{X^n} f^{(n)} d\mu^{(n)}, \quad n \in \mathbb{N}.$$

Let $\eta = \sum_i s_{x_i} \delta_{x_i} \in \mathbb{K}(X)$. We set $P^{(0)}(\eta) := 1$ and $P^{(1)}(\eta) := \eta \in \mathbb{M}(X) = \mathbb{M}^{(1)}(X)$, and consider the measure $P^{(n)}(\eta)$ on $(X^n, \mathcal{B}(X^n))$ for $n \geq 2$, given by

$$\begin{aligned} P^{(n)}(\eta)(dx_1 \cdots dx_n) &\\ := \eta(dx_1)(\eta(dx_2) - s_{x_1}\delta_{x_1}(dx_2)) \times \ldots \times &\\ \times (\eta(dx_n) - s_{x_1}\delta_{x_1}(dx_n) - s_{x_2}\delta_{x_2}(dx_n) - \ldots - s_{x_{n-1}}\delta_{x_{n-1}}(dx_n)). \end{aligned} \tag{4.4}$$

It is straightforward to check that (4.4) does not depend on the ordering in $\eta = \sum_i s_{x_i}\delta_{x_i}$, therefore, $P^{(n)}(\eta)$ is a symmetric measure on $X^n$. Moreover,

$$\begin{aligned} P^{(n)}(\eta) &= \sum_{x_1 \in \tau(\eta)} \sum_{x_2 \in \tau(\eta) \setminus \{x_1\}} \cdots \sum_{x_n \in \tau(\eta) \setminus \{x_1, \ldots, x_{n-1}\}} s_{x_1} s_{x_2} \ldots s_{x_n} \\ &\qquad \times \delta_{x_1} \otimes \delta_{x_2} \otimes \ldots \otimes \delta_{x_n} \\ &= n! \sum_{\{x_1, \ldots, x_n\} \subset \tau(\eta)} s_{x_1} s_{x_2} \ldots s_{x_n} \delta_{x_1} \odot \delta_{x_2} \odot \ldots \odot \delta_{x_n}, \end{aligned} \tag{4.5}$$

where $\odot$ denotes the tensor product symmetrization, cf. [7]. Then, for any $f \in \mathcal{F}(X)$,

$$\big|\langle P^{(n)}(\eta), f^{\otimes n}\rangle\big| \leq \langle \eta, |f| \rangle^n < \infty, \quad n \in \mathbb{N}. \tag{4.6}$$

By the polarization identity, any $\mu^{(n)} \in \mathbb{M}^{(n)}(X)$ is uniquely defined by the values of $\langle \mu^{(n)}, f^{\otimes n} \rangle$ for $f \in \mathcal{F}(X)$. Therefore, $P^{(n)}(\eta) \in \mathbb{M}^{(n)}(X)$, $n \in \mathbb{N}$.

Let now $n \in \mathbb{N}$ and $f^{(n)} \in \mathcal{F}^{(n)}(X)$. We define the following *polynomial function* on $\mathbb{K}(X)$

$$p_n(\eta) := \langle P^{(n)}(\eta), f^{(n)} \rangle.$$

**Remark 4.6.** We stress that $p_n$ is not the restriction of a polynomial on $\mathbb{M}(X)$ (in the sense of [7]) to $\mathbb{K}(X)$ as the right-hand side of (4.5) may not be even defined for an arbitrary $\eta \in \mathbb{M}(X) \setminus \mathbb{K}(X)$.

We define also the polynomial sequence $\{(\omega)_n : n \geq 0\}$ of *falling factorials* on $\mathbb{M}(\widehat{X})$, cf. [6, 7]. Namely, we set $(\omega)_0 := 1$, $(\omega)_1 := \omega$; and, for $n \geq 2$ and $\hat{y}_i := (s_i, x_i)$, $1 \leq i \leq n$, we define

$$
\begin{aligned}
(4.7) \qquad & (\omega)_n(d\hat{y}_1 \dots d\hat{y}_n) \\
& := \omega(d\hat{y}_1)(\omega(d\hat{y}_2) - \delta_{\hat{y}_1}(d\hat{y}_2)) \\
& \qquad \times \dots \times (\omega(d\hat{y}_k) - \delta_{\hat{y}_1}(d\hat{y}_k) - \delta_{\hat{y}_2}(d\hat{y}_k) - \dots - \delta_{\hat{y}_{n-1}}(d\hat{y}_n)) \\
& = n! \sum_{\{\hat{y}_1, \dots, \hat{y}_n\} \subset \gamma} \delta_{\hat{y}_1} \odot \delta_{\hat{y}_2} \odot \dots \odot \delta_{\hat{y}_n}.
\end{aligned}
$$

By [6, 7], the generating function of falling factorials on $\mathbb{M}(\widehat{X})$ is

$$
(4.8) \qquad \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\omega)_n, \hat{f}^{\otimes n} \rangle = \exp\bigl(\langle \omega, \log(1 + \hat{f}) \rangle\bigr), \qquad \hat{f} \in \mathcal{F}(\widehat{X}),
$$

that is understood as an equality of formal power series, see [6, Subsection 2.2 and Appendix]. Moreover, by [6], the falling factorials are of binomial type, i.e.,

$$
(4.9) \qquad (\omega + \omega')_n = \sum_{k=0}^{n} \binom{n}{k} (\omega)_k \odot (\omega')_{n-k}, \quad \omega, \omega' \in \mathbb{M}(\widehat{X}), n \in \mathbb{N},
$$

and the following lowering property holds: for each $n \in \mathbb{N}_0$, $\hat{y} \in \widehat{X}$, $\omega \in \mathbb{M}(\widehat{X})$,

$$
(4.10) \qquad (\omega + \delta_{\hat{y}})_n - (\omega)_n = n \delta_{\hat{y}} \odot (\omega)_{n-1}.
$$

Let $f \in \mathcal{F}(X)$ and consider, for each $j \in \mathbb{N}$,

$$
\hat{f}_j(s, x) := \mathbb{1}_{[\frac{1}{j}, j]}(s)\, s\, f(x), \quad (s, x) \in \widehat{X}.
$$

Then $\hat{f}_j \in \mathcal{F}(\widehat{X})$ and we can consider $\langle (\omega)_n, \hat{f}_j^{\otimes n} \rangle$, $\omega \in \mathbb{M}(\widehat{X})$. Let $\hat{f}(s, x) := s f(x)$ for $(s, x) \in \widehat{X}$. Then, by (4.7), for each $\gamma \in \Pi(\widehat{X}) \subset \mathbb{M}(\widehat{X})$, we have, cf. (4.6),

$$
\bigl| \langle (\gamma)_n, \hat{f}^{\otimes n} \rangle \bigr| \leq \langle \gamma, |\hat{f}| \rangle^n < \infty.
$$

Since $\hat{f}_j \to \hat{f}$, $j \to \infty$, pointwise, the dominated convergence theorem implies that

$$
\lim_{j \to \infty} \langle (\gamma)_n, \hat{f}_j^{\otimes n} \rangle = \langle (\gamma)_n, \hat{f}^{\otimes n} \rangle, \qquad \gamma \in \Pi(\widehat{X}).
$$

Then, by the polarization identity, for any $f^{(n)} \in \mathcal{F}^{(n)}(X)$, we may also define $\langle (\gamma)_n, \hat{f}^{(n)} \rangle$ for $\gamma \in \Pi(\widehat{X})$ and

$$
\hat{f}^{(n)}\bigl((s_1, x_1), \dots, (s_n, x_n)\bigr) := s_1 \dots s_n f^{(n)}(x_1, \dots, x_n).
$$

Then, by (4.5),

$$
(4.11) \qquad \langle P^{(n)}(\eta), f^{(n)} \rangle = \langle (\mathcal{R}^{-1}\eta)_n, \hat{f}^{(n)} \rangle, \qquad \eta \in \mathbb{K}(X).
$$

**Proposition 4.7.** *Let $f \in \mathcal{F}(X)$. Then, for each $n \in \mathbb{N}$ and for each $\eta$, $\eta' \in \mathbb{K}(X)$ such that $\tau(\eta) \cap \tau(\eta') = \emptyset$, we have*

$$\langle P^{(n)}(\eta + \eta'), f^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle P^{(k)}(\eta), f^{\otimes k}\rangle \langle P^{(n-k)}(\eta'), f^{\otimes(n-k)}\rangle.$$

*Proof.* By (2.1), the assumption $\tau(\eta) \cap \tau(\eta') = \emptyset$ implies that

$$\mathcal{R}^{-1}(\eta + \eta') = \mathcal{R}^{-1}\eta + \mathcal{R}^{-1}\eta'. \tag{4.12}$$

Then the statement follows immediately from (4.9) and (4.11). □

**Corollary 4.8.** *Let $f \in \mathcal{F}(X)$ and $n \in \mathbb{N}$. Then,*
*1. For each $\eta \in \mathbb{K}(X)$, $(s, x) \in \widehat{X}$ such that $x \notin \tau(\eta)$,*

$$(D^+_{(s,x)}\langle P^{(n)}(\cdot), f^{\otimes n}\rangle)(\eta) = nsf(x)\langle P^{(n-1)}(\eta), f^{\otimes(n-1)}\rangle;$$

*2. For each $\eta \in \mathbb{K}(X)$ and $x \in \tau(\eta)$,*

$$(D^-_x\langle P^{(n)}(\cdot), f^{\otimes n}\rangle)(\eta) = -ns_x f(x)\langle P^{(n-1)}(\eta - s_x\delta_x), f^{\otimes(n-1)}\rangle.$$

*Proof.* 1. By (4.12), we get from (4.10) that

$$\begin{aligned}(D^+_{(s,x)}\langle P^{(n)}(\cdot), f^{\otimes n}\rangle)(\eta) &= \langle P^{(n)}(\eta + s\delta_x), f^{\otimes n}\rangle - \langle P^{(n)}(\eta), f^{\otimes n}\rangle \\ &= \langle(\mathcal{R}^{-1}\eta + \delta_{(s,x)})_n, f^{\otimes n}\rangle - \langle(\mathcal{R}^{-1}\eta)_n, \hat{f}^{\otimes n}\rangle \\ &= n\langle(\delta_{(s,x)} \odot (\mathcal{R}^{-1}\eta)_{n-1}), \hat{f}^{\otimes n}\rangle \\ &= n\hat{f}(s, x)\langle(\mathcal{R}^{-1}\eta)_{n-1}, \hat{f}^{\otimes(n-1)}\rangle \\ &= nsf(x)\langle P^{(n-1)}(\eta), f^{\otimes(n-1)}\rangle.\end{aligned}$$

2. By item 1,

$$\begin{aligned}(D^-_x\langle P^{(n)}(\cdot), f^{\otimes n}\rangle)(\eta) &= \langle P^{(n)}(\eta - s_x\delta x), f^{\otimes n}\rangle - \langle P^{(n)}(\eta), f^{\otimes n}\rangle \\ &= -(D^+_{(s_x,x)}\langle P^{(n)}(\cdot), f^{\otimes n}\rangle)(\eta - s_x\delta_x) \\ &= -ns_x f(x)\langle P^{(n-1)}(\eta - s_x\delta_x), f^{\otimes(n-1)}\rangle.\end{aligned}$$ □

**Remark 4.9.** We can also consider the generating function for $P^{(n)}$, $n \geq 0$. Namely, for each $f \in \mathcal{F}(X)$, we have, by (4.11), (4.8) and (3.11),

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\eta), f^{\otimes n} \rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\mathcal{R}^{-1}\eta)_n, \hat{f}^{\otimes n} \rangle \\ &= \exp\bigl(\langle \mathcal{R}^{-1}\eta, \log(1+\hat{f})\rangle\bigr) \\ &= \exp\Biggl( \sum_{(s,x)\in \mathcal{R}^{-1}\eta} \log\bigl(1+sf(x)\bigr)\Biggr) \\ &= \exp\Biggl( \sum_{x\in \tau(\eta)} \log\bigl(1+s_x f(x)\bigr)\Biggr) \\ &= \prod_{x\in\tau(\eta)} \bigl(1+s_x f(x)\bigr) \\ &= (Ke_{\mathbb{K}}(f))(\eta).\end{aligned}$$

More generally, for any sequence $f^{(n)} \in \mathcal{F}^{(n)}(X)$, $n \geq 0$, one can define the function $F \in \mathcal{F}_{\exp}(\mathbb{K}_0(X))$ given by, cf. (3.9),

$$F(\xi) := s_{x_1} \dots s_{x_n} f^{(n)}(x_1, \dots, x_n)$$

for each $\xi = \sum_{i=1}^{n} s_{x_i}\delta_{x_i} \in \mathbb{K}_0(X)$, $n \geq 1$; $F(0) := f^{(0)} \in \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\eta), f^{(n)} \rangle = (KF)(\eta), \quad \eta \in \mathbb{K}(X).$$

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