CORRELATED EQUILIBRIUM STRATEGIES WITH MULTIPLE INDEPENDENT RANDOMIZATION DEVICES

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ABSTRACT. A primitive assumption underlying Aumann (1974,1987) is that all players of a game may correlate their strategies by agreeing on a common *single* 'public roulette'. A natural extension of this idea is the study of correlated strategies when the assumption of a single random device common to all the players (public roulette) is dropped and (arbitrary) disjoint subsets of players forming a coalition structure are allowed to use *independent* random devices (private roulette) a la Aumann. Under multiple independent random devices, the coalitions mixed strategies form an equilibrium of the induced non-cooperative game played across the coalitions–the 'partitioned game'–when the profile of such coalitions' strategies is a profile of correlated equilibria. These correlated equilibria which are the mutual joint best responses of the coalitions are called the *Nash coalitional correlated equilibria* (NCCEs) of the game. The paper identifies various classes of finite and infinite games where there exists a non-empty set of NCCEs lying outside the regular correlated equilibrium distributions of the game. We notably relate the class of NCCEs to the 'coalitional equilibria' introduced in Ray and Vohra (1997) to construct their 'Equilibrium Binding Agreements'. In a ' coalitional equilibrium', coalitions' best responses are defined by Pareto dominance and their existence are not guaranteed in arbitrary games without the use of correlated mixed strategies. We characterize a family of games where the existence of a non-empty set of non-trivial NCCEs is guaranteed to coincide with a subset of coalitional equilibria. Most of our results are based on the characterization of the induced non-cooperative 'partitioned game' played across the coalitions.

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INTRODUCTION

A key motivation for the introduction of the correlated equilibrium (CE) solution concept (Aumann 1974, 1987) has been that correlated equilibrium strategies could improve upon Nash equilibrium outcomes (see e.g., Aumann 1974, Moulin and Vial 1978, Ray (1996) and Moulin et al., 2014).[1] For arbitrary n-player games, the mechanics to achieve these better outcomes is based on the introduction of a unique public lottery for all the players: There is a (unique) mediator who informs each player of his own recommendation, without revealing the recommendation to any other player. The key underlying assumption of such Aumann's class of correlated strategies is thus that *all* the players of a game can correlate their strategies via a single common random device or 'public roulette'. Under this assumption, the class of Aumann correlated strategies forms a set of (canonical) correlated equilibria of the original game if and only if the players' strategy profile of deviation plans is trivial (i.e. each player follows the mediator's recommendations) and forms a Nash equilibrium of the extended game.

The initial motivation for this paper is the remark that in more general settings, the source of random signals may not be common to all players, because different disjoint subsets of players will typically have access to stochastically independent randomization devices. When this happens, the different coalitions of players are only allowed to use independent '*private roulettes*'. This paper explores this natural extension of Aumann's original definition by relating the possible coalitions of (disjoint) subsets of players that can be formed to the existence of independent randomization devices (one for each coalition). In this extended framework, the class of correlated strategies becomes induced by a tuple of correlation devices, one assigned to each coalition of players so that correlation may now take place within some subsets of players i.e., coalitions only, while there is stochastic independence across the strategic choices of the coalitions. *Analysing this extended setting, we study the associated expanded class of correlated equilibrium strategies by demanding that the tuple of correlated mixed strategies used by each coalition is a tuple of correlated equilibria which forms itself a mixed Nash equilibrium of the induced non-cooperative game played between the coalitions. For an arbitrary coalition structure a Nash coalitional correlated equilibrium (NCCE) of the original game is then a tuple of correlated strategies such that, given the strategic choice of the other coalitions,*

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[1] Forges (2012) provides a concise overview of the literature.

no player in a coalition can deviate from its recommendation. The set of all the the possible NCCEs of a game is defined for all the coalition structures. It is represented by the class of correlated strategies inducing a tuple of correlated equilibria within some coalitions of players corresponding to a Nash equilibrium of a game played across the coalitions.

This class of correlated equilibrium strategies (for an arbitrary coalition structure) is the main object of study of this paper.

Examples where disjoint subsets of players make their decisions by observing some independent random signals coming from stochastically independent randomization devices abound. This goes from the existence of independent mediating institutions, like government agencies or international bodies (see e.g., Arce, 1995, 1997) to companies adhering to different alliances or countries regrouped in different international agreements. In all these examples, the strategic decisions of the individual players is differently affected by the different randomization devices that they may receive from their mediator representing their '*coalition*'.

We provide a formal analysis of these scenarios and ask what happens when such disjoint coalitions of players have access to stochastically independent randomization devices. We shall answer this question for the class of normal form games with finite strategies and certain classes of games with infinite space of actions. We analyse this set of games by defining the *Nash coalitional correlated equilibria* (NCCE) solution concept which allows disjoint subsets of players to independently correlate their strategies inside their coalition given the profile of correlated strategies used by the other coalitions of players. More precisely, a NCCE is a profile of correlated mixed strategies one for each coalition of players that no player and no coalition of players has incentives to change unilaterally. Because such correlated equilbrium strategies are generated by a tuple of independent randomizations across some coalitions of players rather than by a single public roulette common to all players, the resulting distributions over actions will typically differ from what can be achieved by a regular Aumann correlated equilibrium. For a fixed arbitrary coalition structure, a NCCE captures scenarios wherein there are some self-enforcing equilibria a la Aumann occurring simultaneously *within* and *across* some arbitrary disjoint subsets of players forming disjoint coalitions. At a conceptual level, the notion of NCCE provides a solution concept that (i) requires self-enforcing agreements *within* each coalition of players (ii) imposes the resulting profile of non-binding agreements to be self-enforcing *between* all coalitions. Hence, from this perspective the NCCE permits to encapsulate two notions of rationality. The first is a notion of "group rationality", where a coalition of players is represented by a mediator who selects optimally a profile of recommendations for his group members. The second, is a standard notion of individual rationality, where each player cannot benefit from deviating unilaterally from the recommendation of the mediator of the coalition.

A central property of the NCCE solution is to simultaneously model the noncooperative interaction across coalitions in the spirit of Nash while allowing players to only use self-enforcing deviations from the recommendations inside their coalition a la Aumann. These two characteristics permit to view the NCCE solution as a simultaneous extensions of the Nash solution–at the level of coalitions–and a natural extension of the regular Aumann correlated equilibrium concept–with coalitions using stochastically independent random devices. To the best of our knowledge, Ray and Vohra (1997) notion of '*coalitional equilibrium*' is the only existing solution concept in the literature that models the noncooperative interaction across coalitions *a la* Nash. The notion of '*coalitional equilibrium*' has been introduced in the construction of Ray and Vorha's equilibrium Binding Agreements (EBA). The idea of a '*coalitional equilibrium* is that the Nash equilibria of a fixed non-cooperative game are analyzed by dividing the players into disjoint subsets of players forming a partition of the players set'– a 'coalition structure'. However, unlike a NCCE, a *coalitional equilibrium*' does not initially impose to coalitions the use correlated strategies a la Aumann. In fact, Ray and Vohra 's original definition only demands that the joint optimal strategies are based on the possibility for players inside a coalition to form some *binding agreements* by coordinating their moves on the existing *undominated Pareto strategy profiles* (given the tuple of optimal mixed strategies used by the other coalitions). However, as noted by Haeringer (2004), the existence of an equilibrium across the coalitions, (hence fixed point of the combined joint best response set of all the coalitions), may require that each coalition of players mutually attains their efficient outcomes inside their coalition (via binding agreements) by using some *correlated strategies*–rather than mixed strategies based on independent randomizations.[2] Since one of the motivations for the study of Aumann correlated equilibrium strategies is their potential to improve the welfare of players upon Nash equilibrium outcomes, one of our leading question will be to characterize some classes of games where all the *coalitional equilibria* of Ray and Vohra and all the NCCEs of the game being played coincide. When this happens,

[2] When the quasi-concavity of payoffs is not assumed, the existence of coalitional equilibria is not guaranteed unless coalitions are allowed to use correlated mixed strategies (see Haeringer (2004)).

each correlation device of a coalition which acts as a 'public roulette' for the players *inside* a coalition models the 'binding agreement' assumed by Ray and Vohra: Hence, for the set of games where the class of coalitional equilibria (forming EBAs) and NCCEs coincide, we will have the appealing property that the profile of the coalition mixed strategies is simultaneously Pareto improving and self-enforcing inside and across the coalitions of players.

Our results. The first series of results established in this paper is on the identification of the class of games where the existence of *non-trivial NCCEs* which do *not* coincide with the regular correlated equilibria of the game is guaranteed. Taken together, the combinations of our various results pin down some sets of games where there exists a non-empty set of non-trivial NCCEs which are guaranteed to coincide with the coalitional equilibria of the game being played. To achieve this identification we need to answer two separate questions, which form the two main streams of our central results.

We first need to identify the class of games where some non-trivial NCCEs which are not simply some regular correlated equilibria of the original game exist. Second, we need to characterize the class of games where there is coincidence between the coalitional equilibria of Ray and Vohra and and NCCEs. As shown in this paper, one way to achieve these two characterizations is to analyze the non-cooperative game played between the coalitions of players. While this non-cooperative game which we refer to as '*partitioned game*' is implicit in Ray and Vohra (1997), it is formally undefined and several of our results are geared towards its characterization. In particular, since NCCEs in pure strategies are trivial, the class of games that possesses a non trivial subset of NCCEs must have partitioned games admitting some properly mixed Nash equilibria. Hence, some of our results are based on the characterization of partitioned games wherein the coalitions are playing some profiles of non-degenerate correlated equilibrium distributions which do not induce some regular mixed Nash equilibria of the original game. This class of games may notably include those where the '*sub-games*' played by the players inside each coalition–given the profile of correlated strategies used by the other coalitions–belong to that class of (anti)-coordination games where completely mixed equilibrium payoffs may be strictly Pareto dominated by some correlated equilibria (see Moulin and Vial, 1978). The characterization of the 'partitioned games' is also crucial to determine the supports of the correlated equilibrium distributions used by the coalitions in the NCCEs : even for games which have partitioned games with properly mixed Nash equilibria, we need to find some conditions that guarantee that the set optimal undominated Pareto correlated strategies used by a coalition in a EBA will be inside the support of the correlated equilibrium distributions used by the coalitions.

Our analysis of the existence of non-trivial NCCEs for the class of infinite convex partitioned games exploits the property of potential games (Monderer and Shapley, 1996). The hallmark of this class of games is that every maximizer of a potential function is a pure-strategy Nash equilibrium (PSNE). Here we apply a much more generalized form of the potential techniques by considering the '*partition potentials*' introduced by Uno (2007, 2010). In this class of games, the single global function—the potential—for all the players is replaced by a tuple of 'local potentials', whose maximizers only give the optimal strategies for a certain subsets of players. This technique notably allows to obtain some conditions for the existence of those non-trivial NCCEs in which each coalition of players plays into a *non-degenerate* correlated equilibrium distribution.

Examples of Nash coalitional correlated equilibria.

Example 1: trivial Nash coalitional correlated equilibria

A_3

	H_2	T_2
H_1	0,1,3	0,0,0
T_1	1,1,1	1,0,0

B_3

	H_2	T_2
H_1	2,2,2	0,0,0
T_1	2,2,0	2,2,2

C_3

	H_2	T_2
H_1	0,1,0	0,0,0
T_1	1,1,1	1,0,3

In this game, player 1 is the row, player 2 the column and player 3 chooses a matrix. The pure strategy space of each player is a two-point set $\Theta_i, i = 1,2,3$. Let $\{S_{12}, S_3\}$ defines a *coalition structure* for the player set $N = \{1,2,3\}$ with $S_{12} \equiv \{1,2\}$ and $S_3 \equiv \{3\}$. An example of a correlated equilibrium of this game is when players 1 and 2 condition their actions by jointly observing the outcome of a fair coin as follows: If heads comes up they play $(\mathbf{H}_1, \mathbf{H}_2)$ and if tail occurs, they pick $(\mathbf{T}_1, \mathbf{T}_2)$, while player 3's best reply is to always plays matrix $\mathbf{B}_3$. The resulting correlated equilibrium distribution, $p = p_{12} \otimes p_3$, where $p_3 = \delta_{\mathbf{B}_3}$ is the Dirac probability measure onto $\mathbf{B}_3$, forms what we shall refer to as a $\mathscr{C}(1,2,3) = \{S_{12}, S_3\}$-**trivial Nash coalitional correlated equilibrium** (NCCE) of the game Γ. In general games, this coincidence between a NCCE and a regular correlated equilibrium distribution will fail

because unlike the Aumann assumption, for arbitrary games and arbitrary coalition structures the NCCE class of correlated strategies requires a *different* correlating device to *each* coalition of players (rather than a single one for all the players). The terminology 'trivial' will be used throughout this paper when a NCCE forms a regular correlated equilibrium of the original game. In this particular example, the NCCE (p_{12}, p_3) corresponds to a regular correlated equilibrium of the game because the induced correlated equilibrium distribution $p^* = p_{12} \otimes p_3$ only involves a proper randomization for one of the coalitions with one player (player 3) with a constant best reply (always choose $\mathbf{B}_3$). The NCCE (p_{12}, p_3) is a particular (degenerated) case which is compatible with the use of a single 'public roulette' d for the entire coalition structure $\mathscr{C}(1,2,3)$. Indeed, one may equivalently view d as generated by a pair of mediators using a pair of correlating devices $(d^{'}_{12}, d^{'}_{3})$ such that player 3 makes his constant choice following his extraneous constant signal $\mathbf{B}_3$ coming from $d^{'}_{3}$, while players in coalition $S_{1,2}$ make their choices as a function of extraneous random signals coming from $d^{'}_{12}$. As illustrated in the following examples below, the coincidence between NCCEs and regular generally fails, thereby showing why correlated equilibria are a strict subset of the class of all the NCCEs of a game.
This example also features the most characteristic property of the class of NCCE correlated strategies : In Table 1, the correlated strategies $p^* = p^*_{12} \otimes p^*_3$, induces a *Nash equilibrium* of the $\mathbf{B}_3$-non-cooperative game (we shall refer to it as a ' *partitioned coalitional game*') played between the *subsets of players* $S_{12} = \{1,2\}$ and player 3. The general notion of *partitioned coalitional game*' is formally defined below and a characterization of these games is at the core of our main results. Compared to a regular correlated equilibrium, a NCE possesses another key property since for the class of canonical NCCEs —the class of correlated equilibria used by each coalition of players is based on a *canonical correlation device* whose signals are defined by the coalition joint strategies— the profile of correlating devices becomes *endogenously* determined: Even in the simple game of example 1, the randomized strategy p^*_{12} over the joint strategy profiles $(\mathbf{H}_1, \mathbf{H}_2)$ and $(\mathbf{T}_1, \mathbf{T}_2)$ depends on the degenerate strategy $p_3 = \delta_{\mathtt{C}}$ of player 3. For an arbitrary mixed strategy choice $p^{'}_3$ (which puts some mass onto C), the new correlated equilibrium strategy $p^{*'}_{12}$ will typically require the mediator of $S_{1,2}$ to randomize with a different probability distribution $p^{*'}_{12} \neq p^*_{12}$. $\Gamma_{12}(p^{'}_3)$ correlated strategy $p^{*'}_{12}$. Hence, for an NCCE to happen, it must be that the mediators have a 'more active role' than what they are assumed in a regular correlated equilibrium: Given the coalition structure, $\{S_{12}, S_3\}$, the the randomized strategy p^*_{12} of coalition S_{12} *induces* the distribution p^*_3 according to which the mediator of player 3 should send a his recommendations. Symmetrically, the randomized strategy p^*_3 of coalition S_3 *induces* the distribution p^*_{12} according to which the mediator of players S_{12} should send a his recommendations. Hence, for the class of arbitrary NCCE, the correlation devices $d^{'}_{12}$ and $d^{'}_{3}$ become some *mutually dependent* objects. This property is another clear break with the exogenously given single correlation device d_{123} assumed to define the class of regular correlated equilibria. It is also useful to note how the NCCE relates to the Nash solution concept. For the finest partition of players $\mathscr{C}^*(N)$, a $\mathscr{C}^*(N)$-NCCE boils down to a Nash equilibrium of the game being played. So, one can also view the class of NCCEs as being the extension of the Nash idea to an arbitrary coalition structure $\mathscr{C}(N)$ of disjoint subsets of players: each coalition of players S plays an optimal correlated strategy whenever it is playing into a correlated equilibrium of the non-cooperative game induced by the profile of (independent) correlated strategies used by the other coalitions.

Example 2: Intersection of NCCEs and Ray and Vohra's 'Coalitional Equilibria'.

The game below is taken from Haeringer (2004) in his discussion of the existence of the first step of Ray and Vorha's Equilibrium Binding Agreement (1997). Player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

$\mathbf{H}_3$

	$\mathbf{H}_2$	$\mathbf{T}_2$
$\mathbf{H}_1$	2,2,0	0,0,0
$\mathbf{T}_1$	0,0,0	-1,-1,2

$\mathbf{T}_3$

	$\mathbf{H}_2$	$\mathbf{T}_2$
$\mathbf{H}_1$	1,1,0	0,0,0
$\mathbf{T}_1$	0,0,0	2,2,0

Consider the coalition structure defined by: $S_{12} = \{\text{Row}, \text{Column}\}$ and $S_3 = \{\text{Matrix}\}$. A class of $\{S_{12}, S_3\}$-NCCE of this game is induced when the coalition formed by Row and Column plays in a correlated equilibrium by randomizing onto the pure Nash equilibrium strategy profile $(\mathbf{H}_1, \mathbf{H}_2)$ with a probability p_{12} and pure Nash equilibrium strategy profile $(\mathbf{T}_1, \mathbf{T}_2)$ with a probability $1 - p_{12}$ for $p_{12} = 1/2$ and while 'Matrix' S_3 picks $\mathbf{H}_3$ with probability $p_3 = 1/2$. The $\{S_{12}, S_3\}$ mixed strategy profile $\mathbf{p}_{12} = ((1/2 \circledast (\mathbf{H}_1, \mathbf{H}_2), 1/2 \circledast (\mathbf{T}_1, \mathbf{T}_2); (1/2 \circledast (\mathbf{H}_3), 1/2 \circledast (\mathbf{T}_3))$

which forms a $\{S_{12}, S_3\}$-NCCE coincides with an *coalitional equilibrium* as defined in Ray and Vorha (1997) in the construction of their 'binding equilibrium agreement'. However, unlike example 1, such a NCCE is *not* a regular Aumann correlated equilibrium: Both coalitions of players are now using a *non-degenerate mixed strategy* (corresponding to a correlated equilibrium for S_{12}) rather than a *pure strategy* for the singleton coalition S_3. The fundamental reason for the departure from the Aumann class of correlated strategies is that any partial non-trivial correlation that takes place within a non singleton coalition of players breaks the Aumann's definition of correlated equilibria of a single correlating device.

So, we note that the *coalitional equilibrium* of Ray and Vohra intersects with the class of NCCEs with the following proviso: While NCCE requires players inside the coalition to randomize in a correlated equilibrium over some joint strategies profiles of their coalitional game (induced by the correlated strategies used by the other coalitions), in a coalitional equilibrium, the joint best replies of a coalition of players are defined via Pareto dominance. The coincidence between some *coalitional equilibria* and *non trivial* NCCEs for some coalition structures of a given game arises whenever the (coalitional) game played by each coalition have some *pure strategy Nash equilibria* onto which the players can jointly randomize. This is exactly what happens in the case in example 3 below.

Example 3 :Non-trivial NCCEs and coalition-proof equilibria.

The next example shows that the class of non-trivial NCCEs have the potential to be used to characterize several other solutions concepts of the literature. The game below is taken from Moreno and Wooders (1996) and discussed in Heller (2008). Player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

H_3

	H_2	T_2
H_1	1,1,-2	-1,-1,2
T_1	-1,-1,2	-1,-1,2

T_3

	H_2	T_2
H_1	-1,-1,2	-1,-1,2
T_1	-1,-1,2	1,1,-2

The tuple of probability distributions wherein players 1 and 2 correlate their strategies against player 3 given by correlated strategy profile

$$p_{S_{12},S_3} = (p_{12}, p_3) = ((\frac{1}{2} \circledast (\mathbf{T}_1\mathbf{T}_2), \frac{1}{2} \circledast (\mathbf{H}_1\mathbf{H}_2)), (\frac{1}{2} \circledast \mathbf{H}_3, \frac{1}{2} \circledast \mathbf{T}_3)),$$

forms a *non-trivial* $\{S_{12}, S_3\}$ Nash coalitional correlated equilibrium, which does not form a regular correlated equilibrium. Indeed, unlike the previous example 1, this NCCE required a pair of correlating devices (d'_{12}, d'_3) whose associated pair of mediators select a pair of correlated pure strategies (θ_{12}, θ_3) according to a pair of non-degenerate probability distributions $(p_{S_{12}}, p_{S_3})$. Specifically, the mediator of S_{12} privately recommends a pure strategy T_1 or H_1 to player 1 and T_2 or H_2 to player 2, according to p_{12} ,while the mediator of S_3 privately recommends a pure strategyT_3 or H_3 to player 3 according to p_3. The coincidence between this NCCE and a correlated equilibrium of the game breaks down in this example because *both coalitions of players are now using non-degenerate mixed strategies* thereby making impossible to obtain a profile $(p_{S_{12}}, p_{S_3})$ by using a single correlating device $d_{123} \neq (d'_{12}, d'_3)$ with a single mediator.

In fact, it turns out that this particular $\{S_{12}, S_3\}$-NCCE coincides with a correlated strategy forming a *coalition-proof correlated equilibrium* (CPCE)(see Moreno and Wooders, 1996): It is a self-enforcing agreement for each player i inside the coalition S_{12} and player j of coalition S_3 in the sense that neither of these player wants to unilaterally deviate from the agreement. As remarked by Moreno and Wooders (1996), this is indeed the unique CPCE of this game. As in example 1, the correlated strategy profile (p_{12}, p_3) is outside the set of the regular Aumann's correlated equilibria. For two-player games the set of CPCEs is the set of correlated equilibria which are not Pareto dominated by other correlated equilibria. Hence, if they exist, the set of $\mathscr{C}(N)$-NCCEs in any game Γ defined for the coalition structures $\mathscr{C}(N) = \{S_K\}_{K=1}^m$ such that $|S_K| = 2$ for all $K = 1, ..., m$, are identified by a tuple correlated equilibria $(p_{S_K})_{K=1}^m$ which coincide with the CPCEs of the family of coalitional games $\{\Gamma_{S_K}(p_{S_{-K}})\}_{K=1}^m$. [3]

The hallmark of all the above examples is that non-trivial NCCEs which are *not* forming a correlated equilibrium of the original n player game are not guaranteed to exist. For nontrivial coalition structures made up of at least two non singleton coalitions of players, a tuple Nash equilibrium of the game played between the coalitions may not correlated play into a properly mixed correlated strategy (against the other coalitions) is guaranteed to exist

[3]The set of correlated equilibria which are not Pareto dominated by other correlated equilibria is convex and compact. Hence, applying standard Kakutany fixed point theorem shows that this set of $\mathscr{C}(N)$-NCCEs is always non-empty.

for arbitrary coalition structures even in 'well-behaved' (compact and convex) class of games. If they exist, such NCCEs where a tuple of correlated equilibria is simultaneously played within some subsets of players does not form a regular Aumann correlated equilibrium. The bulk of the paper consists in identifying those 'extended correlated equilibria ' wherein there is a simultaneous profile of joint best responses for each coalition corresponding to a profile of correlated equilibria induced by a tuple of correlating devices only tailored for each coalition. As discussed in the next example below, it turns out that even in the case of a simple coalition structure of a three player game, a NCCE which only involves a correlation between player 1 and 2 is not in general forming a regular Aumann correlated equilibrium. [4]

1. Nash coalitional correlated equilibrium: Definition and basic properties

Consider a finite set of players $N = \{1, 2, ..., n\}$ playing a normal-form game, $\Gamma = \langle N, (\Theta_i, U_i)_{i\in N}\rangle$, where Θ_i is is a nonempty compact set $\Theta_i \subset \mathbb{R}^{m_i}$ (of finite dimension m_i) for each player i. Given a subset $S \subset N$, the joint strategy set $\Theta_S = \prod_{i\in S} \Theta_i$, of subset of players S is assumed to be a compact normed space and $U_i : \prod_{i\in N} \Theta_i \to \mathbb{R}$ designates the continuous and measurable payoff function of player i. We denote the set of Borel probability measures over a set $M \subset \mathbb{R}^m$ by $\Delta(M)$ and endow $\Delta(M)$ with the topology of weak convergence. [5] When M is a finite set, $\Delta(M)$ is the unit simplex on M in $\mathbb{R}^m$. We will provide the characterization of various NCCEs for **finite games** i.e., games in which the spaces of pure strategies is *finite* and also establish some existence results for **continuous games** i.e., games with compact and convex strategy spaces and continuous payoff functions. A **coalition structure**, $\mathscr{C}(N)$, is a *partition* of N, and its elements are called coalitions. We denote $\mathscr{C} = \{\mathscr{C}(N)\}$ as the set of all the partitions of a set of N players. To simplify exposition, the next series of definitions leading to the class of NCCEs is given for finite games (we discuss below why the finiteness assumption of signals is without loss of generality when we deal with some classes of continuous games where the mediators of the coalitions use canonical correlation devices).

For any non-singleton coalition S, an element p_S of $\Delta(\Theta_S)$ is called a *correlated strategy distribution* for S. For any coalition structure $\mathscr{C}(N)$, let $p_{-S} \equiv (p_{S'})_{S'\in\mathscr{C}(N)\setminus\{S\}}$. Now, take a finite collection of probability spaces $(\Theta_S, p_S)_{S\in\mathscr{C}(N)}$, and denote the associated product measure p in $\prod_{S\in\mathscr{C}(N)} \Delta(\Theta_S)$ by $p = \bigotimes_{S\in\mathscr{C}(N)} p_S$.

A *generic correlation device* $(\Omega_S, q_S, (\mathscr{P}^i_S)_{i\in S}) \equiv d_S$, for a coalition S is described by a *finite* set of signals Ω_S, a probability distribution q_S over Ω_S and a partition $\mathscr{P}^i_S$ of Ω_S for every player $i \in S$. Since Ω_S is finite, the probability distribution q_S is just a real vector $q_S = (q_S(w))_{w\in\Omega_S}$. In the following we shall express our definitions of a NCCE and results in terms the '*canonical representation*' of the Aumann correlated equilibrium: For a (finite) games, with a coalition structure $\mathscr{C}(N)$, and an arbitrary profile of correlated strategies $q = (q_S)_{S\in\mathscr{C}(N)}$ we denote the tuple of *canonical correlation devices* under q by $d_{\mathscr{C}(N)}(q) = (\Omega_S, q_S, \mathscr{P}_S)_{S\in\mathscr{C}(N)}$ where the finite set of signals of each coalition S is given the joint strategy space of pure strategies $\Omega_S = \Theta_S$. [6] From Γ and $(d_S)_{S\in\mathscr{C}(N)} \equiv d_{\mathscr{C}(N)}$, we define the extended game $\Gamma_{\mathscr{C}(N)}$ as follows:

- for each coalition $S \in \mathscr{C}(N)$, $w \equiv (w_i)_{i\in S}$ is chosen in Ω_S according to q_S
- every player $i \in S$ is informed of the element $P^i_S(w)$ of $\mathscr{P}^i_S$ which contains w.
- Γ is played: every player i chooses a strategy θ_i in Θ_i and gets the utility $U_i(\theta_N)$ where $\theta_N = (\theta_i)_{i\in N}$.

A (pure) strategy for player $i \in S$ in $\Gamma_{\mathscr{C}(N)}$ is a mapping $\tau^i_S : \Omega_S \to \Theta_i$ which is $\mathscr{P}^i_S$-measurable.[7] Let $\tau_S = (\tau^i_S)_{i\in S}$ be a strategy profile in game $\Gamma_{\mathscr{C}(N)}$ and $\phi = (\phi_i)_{i\in S}$ be the corresponding behavioral strategy profile with

$$\phi_i : w_i \mapsto \phi_i(\cdot\,|\,w_i).$$

The interpretation is that in, $\Gamma_{\mathscr{C}(N)}$, every player i in a coalition S chooses θ_i (or more generally a probability distribution ϕ_i) as a function of his private information on the random signal $w \in \Omega_S$ which is selected before the

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[4]Interestingly, in this example, the NCCE induced by the distribution μ also corresponds to the unique (ex-ante) strong correlated equilibrium of Moreno and Wooders (1996) i.e. a correlated strategy profile that is immune to joint deviations. More precisely, an ex-ante strong correlated equilibrium (Moreno and Wooders, 1996) is immune to deviations that are planned before receiving the recommendations. The correlated distribution in example 1 is also reported to be the only equilibrium played in experiments (see Moreno and Wooders (1998)).

[5]A sequence (m) from $\Delta(M)$ is said to converge weakly to $\mu \in \Delta(M)$ if $\int f d\mu' \longrightarrow \int f d\mu$ for all continuous functions $f : M \longrightarrow \mathbb{R}$. If M is a compact metric space, so is $\Delta(M)$ see e.g. Theorem 6.4 in Parthasarathy (1967).

[6]Here, we apply Myerson (1982) to the correlated strategies played inside each coalition S which guarantees the equivalence between a correlated equilibrium associated with an arbitrary correlation device and the corresponding correlated equilibrium distribution, p_S in the coalitional game $\Gamma_S(p_{-S})$ that is induced by the correlated strategy profiles of the other coalitions $-S$, namely, the probability distribution induced over Θ_S by p_S and behavioral strategy $\phi_S = (\phi_i)_{i\in S}$, defines a canonical correlated equilibrium in $\Gamma_S(p_{-S})$.

[7]In other words, $\tau^i_S(w') = \tau^i_S(w)$ if $w' \in P^i_S(w)$.

beginning of Γ.
In this paper our analysis is geared towards the induced equilibrium probability distributions over the action profiles i.e., the correlated equilibrium distributions. Let (d_S, τ_S) be a correlation device for coalition S with strategy profile $\phi = (\phi_i)_{i\in S}$ of a game extended by the correlation device $d_S(\Omega_S, q_S)$. The pair induces a probability distribution $p_S \in \Delta(\Theta_S)$ given by

$$\forall \theta_S \in \Theta_S, p_S(\theta_S) = \sum_{w\in\Omega_S} q_S(w) \prod_{i\in S} \phi_i(\theta_i | w_i).$$

Fix a coalition $S \in \mathscr{C}(N)$, and any strategy profile, p_{-S}. We define, $U_i(\cdot, p_{-S}) : \Theta_S \to \mathbb{R}$, as the payoff function of player $i \in S$. Hence, for each p_{-S} we obtain a p_{-S}-**coalitional game** for coalition S, denoted $\Gamma_S(p_{-S}) = \langle S, (\Theta_i, U_i(\cdot, p_{-S}))_{i\in S}\rangle$. The multi-linear extension of $U_i(\cdot, p_{-S})$ to $\Delta(\Theta_S)$ is still denoted by $U_i(\cdot, p_{-S})$.

The set of correlated equilibria $CE(\Gamma)$ of a N-player game Γ is contained in $\Delta(\Theta_N)$.Thus, in general, a correlated equilibrium will only involve the correlation of *certain subsets of players*. By definition, in the particular extreme case of a correlated equilibrium p that forms a mixed Nash equilibrium, there is *complete independence* of all the players i.e., no correlation occurs within any subset of players.
Let Θ_S be set of pure strategy profiles $\theta_S = (\theta_i : i \in S)$ for a subcoalition of players S of a game Γ. Consider a partition $\mathscr{C}(N)$ of N and take any finitely supported probability distribution p in $\Delta(\Theta_N)$. For a coalition $S \in \mathscr{C}(N)$, let S_i denotes the player $i \in S$ and $S_{-i} \in \Theta_S$, the set of all the other players $j \neq i$ in S i.e., $S_{-i} \equiv S\backslash\{i\}$. A tuple of distributions $p = (p_S, p_{-S})$-coalitional games for a coalition structure $\mathscr{C}(N)$ is denoted $\Gamma_{\mathscr{C}(N)}(p) \equiv (\Gamma_S(p_S), \Gamma_S(p_{-S}))$. Now note that a tuple of action distributions $p = (p_{S'})_{S'_{\mathscr{C}(N)}} \in \prod_{S'\in\mathscr{C}(N)} \Delta(\Theta_S)$ forms a tuple of $\mathscr{C}(N)$-*correlated equilibrium distributions* if each $p_{S'}$ is a correlated equilibrium distribution of the $p_{-S'}$- coalition $\Gamma_S(p_{-S'})$. The Aumann canonical correlated equilibria of a game Γ are fully specified by a probability distribution q over Θ_N. For finite games and an arbitrary coalition structure , a NCE of Γ is fully specified by a tuple probability distribution $p = (p_S)_{S\in\mathscr{C}(N)}$ so that each p_S is a canonical correlated equilibrium of $\Gamma_S(p_{-S}$ over Θ_S in the p_{-S}-coalitional game $\Gamma_S(p_{-S})$. With the definitions given above, we are now in a position to state a natural extension of the original definition of Aumann to an arbitrary coalition structure of players..

Definition 1.1. Fix a finite game Γ and a set of canonical tuple of correlation devices $\{d_{\mathscr{C}(N)}(q) = (\Omega_S, q_S, \mathscr{P}_S)_{S\in\mathscr{C}(N)} | q \in \prod_{S\in\mathscr{C}(N)} \Delta(\Theta_S)\}$. We say that a profile of action distributions $p = (p_S)_{S\in\mathscr{C}(N)} \in \prod_{S\in\mathscr{C}(N)} \Delta(\Theta_S)$ is a $\mathscr{C}(N)$-**Nash coalitional correlated equilibrium** (NCCE) of Γ if p is a tuple of correlated equilibrium distributions of $\Gamma_{\mathscr{C}(N)}(p)$ (induced by a tuple of canonical correlated devices $d_{\mathscr{C}(N)}(p)$) in a collection of coalitional games $\{\Gamma_S(p_{-S}) | S \in \mathscr{C}(N)\}$ Θ_N.[8] Formally:
A tuple $p = (p_S)_{S\in\mathscr{C}(N)}$ forms a $\mathscr{C}(N)$-NCCE of game Γ if for *each* coalition of players $S \in \mathscr{C}(N)$ and each player $i \in S$,

$$\sum_{\theta_{S_{-i}}\in\Theta_{S-i}} p_S(\theta_{S_i}, \theta_{S_{-i}}) U_i(\theta_{S_i}, \theta_{S_{-i}}; p_{-S}) \geqslant \sum_{\theta_{S_{-i}}\in\Theta_{S-i}} p_S(\theta'_{S_i}, \theta_{S_{-i}}) U_i(\theta'_{S_i}, \theta_{S_{-i}}; p_{-S}),$$

for all $\theta_{S_i}, \theta'_{S_{-i}} \in \Theta_S$.

The above definition extends to continuous games in the obvious manner with the appropriate usual requirements. A tuple $p = (p_S)_{S\in\mathscr{C}(N)}$ forms a A (canonical) NCCE of Γ if it is a $\mathscr{C}(N)$-NCCCE induced by a profile of correlation devices $d_{\mathscr{C}(N)}(p) = (\Omega_S, p_S, \mathscr{P}_S)_{S\in\mathscr{C}(N)}$ with $\Omega_S = \Theta_S$. This requires that a tuple of action distributions $p = (p_{S'})_{S'\in\mathscr{C}(N)}$ defines a $\mathscr{C}(N)$-NCCE if each mixed strategy $p_{S'}$ of each coalition $S' \in \mathscr{C}(N)$, simultaneously forms a correlated equilibrium distribution of the p_{-S}- coalitional game $\Gamma_S(p_{-S})$. Intuitively, a n action distribution $p = \prod_{S'} \in \mathscr{C}(N) p_{S'}$ on the original players' choice combinations Θ_N forms a $\mathscr{C}(N)$-NCCE of a game Γ if each $p_{S'}$ is a canonical correlated equilibrium for each coalition of players $S' \in \mathscr{C}(N)$. Equivalently put, in a $\mathscr{C}(N)$-NCCE, every choice θ_{S_i} of i in coalition S that receives positive probability $p_S(\theta_{S_i}) = marg\, p_S(\theta_{S_i}, \cdot) > 0$ is a best response (for player i) given the conditional probability measure $p_S(\cdot|\theta_{S_i})$ on the distribution of choices $\theta_{S_{-i}} \in \Theta_{S_{-i}} = \prod_{j\neq i | j\in S} \Theta_j$ of the other players S_{-i} of the same coalition S *and* given the probability measure

[8] The relation between correlated equilibria associated to an arbitrary correlation device and the induced correlated equilibrium distributions defining the 'canonical correlated equilibria' is discussed in Myerson (1982).

$p_{-S} = \prod_{S' \in \mathscr{C}(N) \backslash S'} p_{S'}$ induced by the action distribution of the other coalitions over Θ_{-S}. Hence, a NCCE induces a probability distribution of actions $p(\theta_i, \theta_{-i}) = \prod_{S \in C(N)} p_S(\theta_S)$ where there is only a correlated equilibrium correlation $p_S \in \Delta(\Theta_S)$ inside each coalition of players $S \in \mathscr{C}(N)$, given the profile of correlated equilibrium distributions $p_{-S} \in \Delta(\Theta_S)$ implemented by the mediators of the other coalitions. Next say that coalition S has a **joint or coalitional best response** p_S when p_S is a correlated equilibrium for the players in S playing Γ given the tuple of correlated strategies p_{-S} used by coalitions $-S$. When this is the case we write $p_S \in BR_S(p_{-S})$ with the understanding that BR_S is the *coalitional best reply* of S. So, an equivalent restatement of the above definition of a NCCE is that when a coalition structure $\mathscr{C}(N)$ forms, the mediators of each coalition $S \in \mathscr{C}(N)$ choose a tuple of correlation devices $d_S(p_S) = (\Theta_S, p_S, \mathscr{P}_S)_{S \in \mathscr{C}(N)}$ choose a tuple of correlation devices with the property that a profile p forms a NCCE if it is a *fixed point* of the $\mathscr{C}(N)$-**combined coalitional best response mapping**

$$BR_{\mathscr{C}(N)} := \prod_{S \in \mathscr{C}(N)} BR_S.$$

Clearly, this definition of a NCCE expands the Aumann' definition. Indeed, by definition, a tuple of action distributions $p = (p_i)_i$ forms a $\mathscr{C}^*(N)$-NCCE of Γ with $\mathscr{C}^*(N)$, the *finest partition* of N i.e., $\mathscr{C}^*(N) = \{\{i\} : i \in N\}$ if it is a mixed Nash equilibrium of Γ. And it is a regular correlated equilibrium if p is a $\mathscr{C}^{**}(N)$-NCCE of Γ with $\mathscr{C}^{**}(N)$, the *coarsest partition* of N i.e., $\mathscr{C}^{**}(N) = \{N\}$. So, the Aumann set of correlated equilibria $CE(\Gamma)$ of a game Γ is in general only a subset of all the $\mathscr{C}(N)$-NCCEs that can be induced in a game for all the other possible coalition structures $\mathscr{C}(N) \neq \mathscr{C}^{**}(N), \mathscr{C}^{**}(N)$. The addition of these $\mathscr{C}(N)$-NCCE can therefore be viewed as the natural complement of the original definition of Aumann. The Aumann original definition of correlated equilibrium demands that action distributions $p \in \Delta(\Theta_N)$ is over the correlated strategies of the *entire* space of players' choices so that for *every* player $i \in N$, every choice θ_i that receives positive probability $marg\, p(\theta_i, \cdot) > 0$ under p is optimal given the conditional probability measure $p(\cdot|\theta_i)$ given i's choice itself:

$$\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i, \theta_{-i}) U_i(\theta_i, \theta_{-i}) \geqslant \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i', \theta_{-i}) U_i(\theta_i', \theta_{-i}),$$

for all $\theta_i, \theta_{-i}' \in \Theta_{-i}$. So, for an arbitrary game, a n action distribution forming a regular Aumann equilibrium distribution, p, does not form a $\mathscr{C}(N) \neq \mathscr{C}^*(N), \mathscr{C}^{**}(N)$ -NCCE. An immediate consequence of the above definition, is thus that for an arbitrary partition of the players in a coalition structure $\mathscr{C}(N) \neq \mathscr{C}^*(N), \mathscr{C}^{**}(N)$ the existence of a non-empty set of non-trivial $\mathscr{C}(N)$-NCCE which does not boil down to some correlated equilibria of the original game is not guaranteed. The next section tackles this issue.

2. Existence of non-trivial coalitional correlated equilibria in finite games

At this stage, it remains unclear when one should expect to find the existence of non-trivial NCCEs which do not simply form regular correlated equilibria of the original game. In this section we provide a first identification of such games when the spaces of actions are finite.

'Partitioned games' and joint best responses of a coalition. : In the introductory examples of this paper, we noted that a non-cooperative game between the coalition is implicit in the definition of a coalitional equilibrium of Ray and Vohra and the class of correlated strategies forming a NCCE: The goal of this section is to precisely characterize this 'partitioned game' induced by the tuple of correlated strategies used by the coalitions. This game will be the main new object of study to establish most of our results. We start by a definition and elementary characterization of these non-cooperative games in general. Our analysis is formulated for *finite* games to avoid unnecessary technicalities.

Given an arbitrary partition structure $\mathscr{C}(N)$, we refer to the game played between the coalitions as the $\mathscr{C}(N)$-*partitioned game* of Γ. In equilibrium, this game contains the *(mixed) Nash equilibria* forming the tuple of correlated strategies for the whole game. Such partitioned games which are by construction implicit in Ray and Vohra's definition of 'coalitional equilibria' , have to the best of our knowledge not been analyzed in the literature. Here, we provide a formal analysis of the NCCEs and 'coalitional equilibria' in terms of these partitioned games. As shown in our main results, these games contain some crucial pieces of information about some structural properties characterizing the existence of certain classes of NCCEs which coincide with the 'coalitional equilibria' of Ray and Vohra. Several of our results seek to identify some class of games which possesses some specific classes of NCEs like e.g., the ones where each coalition use completely mixed (and correlated) strategies by randomizing over some pure Nash equilibria of their (induced) coalitional game. As shown below, the characterization of these games played across the subsets of players allow to characterize some classes of games whose NCCEs are non-trivially forming some regular correlated equilibria of the original game. As such, these partitioned games played across the subsets of players will constitute one of the essential new objects introduced in this paper. To understand why a partitioned game is a crucial element to analyze NCCEs re-consider the induced correlated distribution $p = p_{12} \otimes p_3$ of Example 1. When p_{12} and p_3 are two non-degenerate probability measures, the induced correlated distribution p cannot form a correlated equilibrium distribution of the original game. This is exactly what happens in this example. More formally, if we denote by $\Gamma_{12}(p_3)$ the *coalitional game* played between players 1 and 2 when player 3 chooses a mixed strategy p_3, then the correlated equilibrium distribution,

$$p_{12} = ((\frac{1}{2}(\mathbf{T}_1\mathbf{T}_2), \frac{1}{2}(\mathbf{H}_1\mathbf{H}_2))),$$

of the subcoalition of players $\{\{1,2\}$ in $\Gamma_{12}(p_3)$ is a *best response* to the mixed strategy $p_3 = (\frac{1}{2}\mathbf{H}_3, \frac{1}{2}\mathbf{T}_3)$ played by player 3 and, conversely, the distribution p_3 of player 3 is a *best response* to the correlated equilibrium distribution p_{12} that is played by the subgroup of players 1 and 2 in $\Gamma_{12}(p_3)$. While the induced correlated distribution $p = p_{12} \otimes p_3$ does not form a correlated equilibrium, the resulting profile (p_{12}, p_3) forms a *mixed Nash equilibrium* of a '*partitioned game*' $\Gamma_{\mathscr{C}(N)}$ played between the coalition of players $S_{12} = \{1,2\}$ and $S_3 = \{3\}$. In this game, the set of players are defined by the elements of the partition $\mathscr{C}(N) = \{S_{12}, S_3\}$. This motivates the following definition.

Definition 2.1. Fix a *non-trivial partition* $\mathscr{C}(N)$, of the N players of a game Γ. A **partitioned game** of Γ is a game,

$$\Gamma_{\mathscr{C}(N)} \equiv \left\langle \mathscr{C}(N), \{\Theta_S, U_S\}_{S \in \mathscr{C}(N)} \right\rangle$$

where each coalition's payoff function $U_S, S \in \mathscr{C}(N)$ is derived from Γ and whose **aggregate pure best response correspondence** of the subset of players S denoted $\mathtt{BR}_S(\cdot)$ is non-empty for every profile p_{-S} in Θ_{-S}

The definition of a partitioned game raises in turns two issues:

(1) What is the characterization of the payoff function U_S representing each *subset* of players S ?
(2) Even if each Θ_S is a finite space of pure strategy profiles and there exists an aggregate representation U_S of each subset of players, what is the characterization of the nonempty set of *pure* best reply profiles θ_S of S to conjectures p_{-S} in the coalitional game $\Gamma_S(p_{-S})$? , i.e., $\mathtt{BR}_S(p_{-S}) \bigcap \Theta_S \neq \emptyset, \forall p_{-S}$.

As discussed below, the answers to (1) and (2) rely onto the aggregate deviation function of a normal form game Γ which gives the necessary and sufficient condition for the existence of a correlated equilibrium (see Hart and Schmeidler ,1989) and the notion of 'coherent strategies' introduced by Nau and McCardle (1990). As formally stated in the next result, it follows that the best response correspondence of each subset of players S in the partitioned game is characterized by the parametrized version of the **aggregate deviation function** (also referred to as the **Nikaido-Isoda-function** (1955) or **Ky-Fan function**) (Fan,1972) of the family of coalitional game $\Gamma_S(\cdot)$.
Let $\alpha_i \star \mu$ denote the distribution on Θ that results if a mediator tries to implement $\mu \in \Delta\Theta$ in Γ but player $i \in N$ deviates according to α_i. We set $\alpha \star \mu = (\alpha_i \star \mu)_{i \in N}$ for a vector of $i = 1, ..., n$, unilateral deviations of a distribution μ. We have the following definitions.
Let Γ be a N-person normal form game. Then Γ has a non-empty set of correlated equilibria $\mu \in CE(\Gamma) \neq \emptyset$ if and only if for every profile $\mu \in CE(\Gamma)$ in $CE(\Gamma)$ the **aggregate deviation function** Ψ_N of the game Γ is defined by:

$$\Psi_N(\cdot,\cdot) : \Delta\Theta \bigtimes \Delta\Theta \longrightarrow \mathbb{R},$$

verifies:

$$\Psi_N(\alpha \star \mu, \mu) := \sum_{j \in N} [U_j(\alpha_j \star \mu) - U_j(\mu)] \leqslant 0, \forall \alpha = (\alpha_j)_{j \in N}$$

with $\alpha_j \star \mu(\theta_{-i}, \hat{\theta}_i) = \sum_{\theta_i} \alpha_i(\hat{\theta}_i|\theta_i)\mu(\theta_{-i}, \theta_i)$ for every $\hat{\theta}_i \in \Theta_i$ and for all $\hat{\theta}_{-i} \in \hat{\Theta}_{-i}$.
Call **coherent** the pure strategies that are played with positive probability in at least one correlated equilibrium. The set of **jointly coherent strategies**, denoted $C(S)$, in the *game* Γ for a subset of players S is the set

$$C(S) := \{\theta_S \in \Theta_S | \exists p \in \mathtt{CE}(\Gamma), p(\theta_S \times \Theta_{-S}) > 0\}.$$

Our aim will be to characterize the set of pure best replies for the subset of players S of an arbitrary coalition structure $\mathscr{C}(N)$ in the partitioned game, $\Gamma_{\mathscr{C}(N)}$ for profiles of correlated $p_{\mathscr{C}(N)} = (p_S)_{S \in \mathscr{C}(N)}$ where each p_S is a correlated equilibrium distribution of a coalitional game $\Gamma_S(p_{-S})$ with the additional property that $p_{\mathscr{C}(N)}$ form a

mixed Nash equilibrium of $\Gamma_{\mathscr{C}(N)}$ i.e., $p_{\mathscr{C}(N)}$ is a Nash coalitional correlated equilibrium of the whole game Γ.
Next, we let define the set of jointly coherent strategies for S in each *coalitional game* $\Gamma_S(p_{-S})$ as:

$$C_S(p_{-S}) := \{\theta_S : \exists p_S \in \mathtt{CE}(\Gamma_S(p_{-S})), \mathtt{supp}(p_S) \ni \theta_S\}$$

where $\mathtt{CE}(\Gamma_S(p_{-S}))$ denotes the set of correlated equilibrium distributions of S into the coalitional game $\Gamma_S(p_{-S})$. It is well-known that the set of correlated equilibria is nonempty (see Hart and Schmeidler (1989)) and one can therefore deduce that the*best reply* correspondence of every coalition of players $S \in \mathscr{C}(N)$ –which represent the players— of the partitioned game is always *non-empty*.

Feasible deviation plans of a coalition under the class of NCCEs strategies. Now, fix a non-trivial coalition structure $\mathscr{C}(N)$. To each $-S = \{S' \neq S | S' \in \mathscr{C}(N)\}$, we denote p_{-S} as the *product measure* $p_{-S} = \prod_{S' \in -S} p_{S'}$ and we (ab)use of the notation $\Delta\Theta_{-S} \equiv \times_{S' \in -S} \Delta\Theta_{S'}$ for the set of probability measures p_S. If $p = p = (p_S, p_{-S})$ is a NCCE of Γ, we consider the induced $\mathscr{C}(N)$ -profile of coalitional equilibrium game $\Gamma_S(p) = (\Gamma_S(p_{-S}), \Gamma_S(p_{-S}))$.

The following result gives a characterization of the $\mathscr{C}(N)$ -'partitioned games'. It notably shows that the joint best

Proposition 2.2 (Characterization of the $\mathscr{C}(N)$-'partitioned games')**.** *The* $\mathscr{C}(N)$ *-partitioned game*

$$\Gamma_{\mathscr{C}(N)} = \langle \mathscr{C}(N), \hat{\Theta}_S, W_S : \Theta_S \times \Theta_{-S} \longrightarrow \mathbb{R} \rangle$$

of a game Γ *is given by the multi-linear payoff functions*

$$p_{-S} \longmapsto W_S(\theta_S; p_{-S}) = \sum_{i \in S} U_i(\theta_S; p_{-S}).$$

For the class of $\mathscr{C}(N)$*-NCCE, the joint best response set of a coalition is defined by the* p_{-S} ***-parametrized Ky-fan aggregate deviation function*** *of* S *in* Γ *:*

$$p_{-S} \longmapsto \Psi_S(p_S; \alpha_S; p_{-S}) = W_S(\alpha_S \star p_S; p_{-S}) - W_S(p_S; p_{-S})$$

relative to the vectors of feasible unilateral deviations $(\alpha_S^i \star p_S)_{i \in S}$ *with* $\alpha_S^i \star p_S \in D(p_S, i)$ *of players* $i \in S$ *from a recommendation* p_S.

Proof. Let $\Theta_S^{\text{IESDA}}(p_{-S})$ denote the set of actions that survive the *iterated elimination of strictly dominated actions* (IESDA) in the p_{-S}-coalitional subgame $\Gamma_S(p_{-S})$ played by the players in coalition $S \in \mathscr{C}(N)$. By definition of a NCCE, p, $p_S \in BR_S(p_{-S}) = CE_S(p_{-S})$ and the subset $\Theta_S^{\text{IESDA}}(p_{-S})$ contains the supports of all the correlated equilibrium distributions $CE_S(p_{-S})$ of $\Gamma_S(p_{-S})$. Hence, $supp(p_S) \subseteq \Theta_S^{\text{IESDA}}(p_{-S})$. To express a $\mathscr{C}(N)$- NCCE in terms of the IESDA of Γ, we first need to define and characterize the $\mathscr{C}(N)$-partitioned game played across the coalitions in $\mathscr{C}(N)$ defined by:

$$\Gamma_{\mathscr{C}(N)} = \langle \mathscr{C}(N), \Theta_S, W_S : \Theta_S \times \Theta_{-S} \longrightarrow \mathbb{R} \rangle.$$

We say that $\hat{\theta}_S$ (resp. $\hat{p}_S \in \Delta\Theta_S$) is a feasible pure deviation by coalition S from θ_S (resp. p_S) if there is a map $\alpha_S : \Theta_S \longrightarrow \Delta\Theta_S$ such that for all $\theta_S \in \Theta_S$, we have $\alpha_S(\hat{\theta}_S|\theta_S) = 1$ (resp. $\hat{p}_S = \alpha_S \star p_S = \sum_{\hat{\theta}_S}(\hat{\theta}_S|\theta_S) p_S(\theta_S)$. Let $D(p_S, S)$ denote the set of feasible deviations by coalition S from p_S and note that $D(p_S, S)$ is always non-empty since a coalition always has the trivial "deviation" consisting of each member of the coalition obeying his own recommendation i.e., $p_S \in D(p_S, S)$.
A joint mixed strategy strategy p_S for a coalition S is a **self-enforcing joint best response** in coalitional game $\Gamma_S(p_{-S})$ i.e., $p_S \in BR_S(p_{-S})$ only if p_S is a *correlated equilibrium* of $\Gamma_S(p_{-S})$. Consider a set of **deviation plans** for players $i \in S$ as a set of mappings $\eta_i : \Theta_S \longrightarrow \Delta(\Theta_i), i \in S$ (a transition probability). Denoting by $\eta_i(\hat{\theta}_i|\theta_i)$ the probability that player $i \in S$ will play $\hat{\theta}_i$ when recommended θ_i. Let $\eta_S(\hat{\theta}_S|\theta_S) = (\eta_i(\hat{\theta}_i|\theta_i))_{i \in S}$ be a profile of unilateral deviation plans for players i in coalition S. We have a best response $p_S \in BR_S(p_{-S})$ for coalition S in $\Gamma_S(p_{-S})$ when there is no $i \in S$ with an unilateral feasible deviation plan η_i inducing a distribution $\eta_i \star p_S = \hat{p}_S$:

$$\hat{p}_S(\hat{\theta}_S^i = (\hat{\theta}_i, \theta_{S_{-i}})) = \eta_i \star p_S(\hat{\theta}_S^i) = \sum_{\theta_S = (\theta_i) \in \Theta_i} \eta_i(\hat{\theta}_i|\theta_i) p_S(\theta_i, \theta_{S_{-i}})$$

such that $U_i(\hat{p}_S; p_{-S}) > U_i(p_S; p_{-S})$. Let denote the set of feasible deviation plans for $i \in S$ from p_S by $D(p_S, i)$. We write

$$\eta_i \star p_S^i(\hat{\theta}_i) = \hat{p}_S(\hat{\theta}_i, \Theta_{S_{-i}}) = \sum_{\theta_{S_{-i}} \in \Theta_{S_{-i}}} \eta_i \star p_S(\hat{\theta}_i, \theta_{S_{-i}})$$

for the *marginal probability* of i to play $\hat{\theta}_i$ under i's feasible deviation plan η_i.
A correlated equilibrium for S in $\Gamma_S(p_{-S})$ is a correlated strategy from which no individual has a feasible improving deviation. Hence, we consider the correlated equilibrium distributions of $\Gamma_S(p_{-S})$ as the subset

$$\mathrm{CE}_S(p_{-S}) := \mathrm{D}^*(p_S, S; p_{-S}) = \{p_S \in \Delta\Theta_S | \nexists \alpha^i_S \star p_S = \hat{p}_S \in \mathrm{D}(\mathrm{P}_S, i), \mathrm{U}_S(\hat{p}_S; p_{-S}) > \mathrm{U}_i(p_S; p_{-S})\}$$

and notes that it corresponds, by construction, to the joint best responses of coalition S in $\Gamma_S(p_{-S})$. The mixed best response correspondence of the $\mathscr{C}(\mathrm{N})$-partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ defined by

$$p_{-S} \longmapsto \mathrm{BR}_S(p_{-S}) = \mathrm{CE}_S(p_{-S}) = \{p_S \in \Delta\hat{\Theta}_S | \mathrm{W}_S(p_S; p_{-S}) \geqslant \mathrm{W}_S(\hat{p}_S; p_{-S}), \forall \hat{p}_S \in \Theta_S\}$$

Hence, it follows that the payoff functions W_S defining the $\mathscr{C}(\mathrm{N})$-partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ must be such that

$$\mathrm{BR}_S(p_{-S}) = \mathrm{CE}_S(p_{-S}) = \{p_S \in \Delta\hat{\Theta}_S | \mathrm{W}_S(p_S; p_{-S}) - \mathrm{W}_S(\hat{p}_S; p_{-S}) \geqslant 0, \forall \hat{p}_S = \alpha_S \star p_S \in \mathrm{D}(p_S, S)\}.$$

The above implies that

$$p_{-S} \longmapsto \mathrm{W}_S(\hat{p}_S; p_{-S}) = \sum_{i \in S} \mathrm{U}_i(\hat{p}_S; p_{-S}),$$

so that

$$\mathrm{BR}_S(p_{-S}) = \{p_S \in \Delta\hat{\Theta}_S | \forall i \in S, \Psi_S(p_S; \alpha_S; p_{-S}) \leqslant 0, \forall \alpha^i_S \star p_S = \hat{p}_S \in \mathrm{D}(\hat{p}_S, i)\}.$$

where each $p_{-S} \longmapsto \Psi_S(p_S; \alpha_S; p_{-S}) = \mathrm{W}_S(\alpha_S \star p_S; p_{-S}) - \mathrm{W}_S(p_S; p_{-S})$ defines the (p_{-S}-*parametrized Ky-fan aggregate deviation function* of S in Γ with respect to the vectors of feasible unilateral deviations $(\alpha^i_S \star p_S)_{i \in S}$ of players $i \in S$ from a recommendation p_S, $\alpha^i_S \star p_S \in \mathrm{D}(p_S, i)$ (Equivalently, $\Psi_S(\cdot, \cdot; p_{-S})$ is the Ky-fan aggregate deviation function of the coalitional game $\Gamma_S(p_{-S})$).

□

3. Existence and characterization of non-trivial Nash coalitional correlated equilibria in finite games

As remarked in Examples 2-3, there may exist some non-trivial NCCEs which are not forming a regular correlated equilibrium of the original game but for which not every coalition is a non-singleton subset of players using a proper correlated strategy.

In this section, we want to analyze the conditions for the existence of such $\mathscr{C}(\mathrm{N})$-NCCEs wherein we only require that there exists *at least one* non-singleton subset of players S that plays into a *properly mixed*–non-degenerate–correlated equilibrium distribution p^*_S that does *not* form a regular mixed Nash equilibrium of their induced coalitional game $\Gamma_S(p^*_{-S})$.
As formally discussed below, the identification of the class of non-trivial NCCEs, which do not form a regular mixed Nash equilibrium of the original game follows from the characterization of $\mathscr{C}(\mathrm{N})$ -partitioned games which possess some properly mixed Nash equilibria. The definitions below provide the formal elements to establish a first characterization of the games where the existence of non-trivial NCCEs is guaranteed, in terms of their $\mathscr{C}(\mathrm{N})$ -partitioned games.

3.1. Proper Nash coalitional correlated equilibria and 'nice partitioned games' in finite games.

Fix a non-trivial partition $\mathscr{C}(\mathrm{N})$ of the player set N of game Γ, S in $\mathscr{C}(\mathrm{N})$ and suppose that Γ admits a partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ (see the definition above). Let $\mathscr{C}(\mathrm{N})$ and $\mathscr{C}'(\mathrm{N})$ be two partitions of N in the set of partitions $\mathscr{C}$. We say that $\mathscr{C}'(\mathrm{N})$ is a *refinement* of $\mathscr{C}(\mathrm{N})$ and write $\mathscr{C}'(\mathrm{N}) \subset \mathscr{C}(\mathrm{N})$ if every $\mathrm{T} \in \mathscr{C}'(\mathrm{N})$, is such that $\mathrm{T} \subseteq \mathrm{S} \in \mathscr{C}(\mathrm{N})$ and there is at least one element $\mathrm{T} \in \mathscr{C}'(\mathrm{N})$, such that $\mathrm{T} \subset \mathrm{S}$ for some $\mathrm{S} \in \mathscr{C}(\mathrm{N})$. For every *non-singleton coalition* S, let $\Delta^*(\Theta_S)$ denote the set of **proper probability measures** over Θ_S, which is the subset of probability measures p_S in $\Delta(\Theta_S)$ where *all* the players' strategies in S get correlated so that measure p_S can*not* be further decomposed into a *finer* (finite) product measure i.e.,

$$\Delta^*(\Theta_S) = \left\{ p_S \in \Delta(\Theta_S) : \nexists p_S = (p_T)_{T \in \mathscr{C}(S)} \in \prod_{T \in \mathscr{C}(S)} \Delta(\Theta_T) \text{ s.t } p_S = \otimes_{T \in \mathscr{C}(S)} p_T \text{ for some } \mathscr{C}(S) \right\}.$$

With the above, we obtain the following refinement of NCCE.

Definition 3.1. The set of proper $\mathscr{C}(N)$-Nash coalitional correlated equilibria of a game Γ is the subset $NE^*(\mathscr{C}(N))$ of the properly mixed Nash equilibrium distributions $NE(\mathscr{C}(N))$ of the induced nice partitioned game $\Gamma^*_{\mathscr{C}(N)} = \langle \Delta(\Theta_S), U_S \rangle_{S \in \mathscr{C}(N)}$. This set is given by:

$$NE^*(\mathscr{C}(N)) = \{p = (p_{S'})_{S' \in \mathscr{C}(N)} | p \in NE(\mathscr{C}(N)) \text{ and } \exists S \in \mathscr{C}(N), |S| \geqslant 2, p_S \in \Delta^*(\Theta_S)\}.$$

Definition 3.2. Fix a *non-trivial partition* $\mathscr{C}(N)$, of the N players of a game Γ. The $\mathscr{C}(N)$ **nice partitioned game** of Γ is a partitioned game denoted,

$$\Gamma^*_{\mathscr{C}(N)} \equiv \left\langle \mathscr{C}(N), \{\Theta_S, U_S\}_{S \in \mathscr{C}(N)} \right\rangle$$

where:

(1) *The payoff function of each coalition* S *is defined by :*

$$U_S(\theta_S, \theta_{-S}) := \min_{\widehat{\theta}_S \in \Theta_S} \Psi_S(\widehat{\theta}_S, \theta_S; \theta_{-S})$$

and for each θ_{-S}, *the function,*

$$\Psi_S(\cdot, \cdot; \theta_{-S}) : \Theta_S \bigtimes \Theta_{-S} \longrightarrow \mathbb{R},$$

is the **aggregate deviation function** *of the coalitional game* $\Gamma_S(\theta_{-S})$:

$$\Psi_S(\theta'_S, \theta_S; \theta_{-S}) := \sum_{j \in S} \left[U_j(\theta'_j, \theta_{S \setminus j}; \theta_{-S}) - U_j(\theta_j, \theta_{S,i}; \theta_{-S}) \right].$$

(2) *The set of pure best replies of each coalition* S, $BR_S(p_{-S})$, *is non-empty for every profile* p_{-S} *in* Θ_{-S} *and coincides with the intersecting set of jointly coherent strategies and non-empty set of pure Nash equilibria* $\mathcal{C}_S(p_{-S}) \cap NE(\Gamma_S(p_{-S}))$ *of the subset of players* S *of the coalitional game* $\Gamma_S(p_{-S})$.

The hallmark of a nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$ is that a profile θ_S is a pure best response Nash equilibrium of coalition S in $\Gamma_S(p_{-S})$ to a tuple of mixed strategies p_{-S} of the other coalitions $-S$ if and only if θ_S is a *pure Nash equilibrium* of $\Gamma_S(p_{-S})$, which is equivalent to the condition:

$$\Psi_S(\theta'_S, \theta_S; p_{-S}) \leqslant 0, \forall \theta'_S \in \Theta_S.$$

Hence, a game Γ possesses a nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$ iff in every mixed Nash equilibrium $p = (p_S, p_{-S})$ the pure best replies $BR_S(p_{-S})$ of each coalition $S in \mathscr{C}(N)$ forms a *pure Nash equilibrium* of the coalitional game $\Gamma_S(p_{-S})$ induced by the mixed strategy profile p_{-S} of coalitions $-S \in \mathscr{C}(N) \setminus S$. This property will generally fail for a $\mathscr{C}(N)$-NCCEs $p = (p_S, p_{-S})$ of an arbitrary game since the support of a correlated equilibrium distribution p_S is only guaranteed to lie inside the set of *iterated non strictly dominated actions* (IESDA) of the induced coalitional game $\Gamma_S(p_{-S})$. Note that the property that every element in the support of a mixed Nash equilibrium $p = (p_S, p_{-S})$ of a nice partitioned game is a pure Nash equilibrium of the coalitional game $\Gamma_S(p_{-S})$ follows from the property of the aggregate deviation function of a normal form game introduced by Nikaido and Isoda (1955) and Fan (Fan, 1972). This function is notably used by Hart and Schmeidler (1989) to prove the existence of a correlated equilibrium.

For each coalition S we then have:

$$\forall S \in \mathscr{C}(N), \mathtt{BR}_S(p_{-S}) = \arg \max_{\theta_{-S} \in \Theta_{-S}} U_S(\theta_S, p_{-S})$$

which gives the pure best reply correspondences, $\mathtt{BR}_S(\cdot)$, of the subset of players S in $\Gamma^*_{\mathscr{C}(N)}$.

Thereafter we say that a game Γ **admits a nice partitioned game** $\Gamma^*_{\mathscr{C}(N)}$ for a non-trivial partition $\mathscr{C}(N)$ if it is defined by (1) and (2). Below we want to find some sufficient conditions that guarantee that the set of *mixed Nash equilibria* of the partitioned game $\mathtt{NE}^*(\Gamma_{\mathscr{C}(N)})$ is non-empty.

Consider the (non-empty) set of jointly coherent strategies, $C^*(S)$, of a *non-singleton coalition* S of $\Gamma^*_{\mathscr{C}(N)}$ that are played by S under a correlated equilibrium distribution p_S when there is a profile of CEDs $p_{\mathscr{C}(N)} = (p_S, p_{-S})$ which forms a *mixed Nash equilibrium* of $\Gamma^*_{\mathscr{C}(N)}$ i.e.,

$$C^*(S) := \left\{ \theta_S : \exists p_{\mathscr{C}(N)} = (p_S : S \in \mathscr{C}(N)) \in \mathtt{NE}(\Gamma_{\mathscr{C}(N)}), \mathtt{supp}(p_S) \ni \theta_S \right\}.$$

We say that a (*non-singleton*) coalition S is a **meta-player** of $\Gamma^*_{\mathscr{C}(N)}$ if the optimal set of pure choices $\theta_S = (\theta_i : i \in S) \in \mathtt{BR}_S(p_{-S}) \subseteq C^*(S)$ of coalition S is not equal to the individual pure best replies of its components $\theta_i \in \mathtt{BR}^i_S(p_{-S}) \equiv \mathtt{BR}_i(p_{-S})$, for all $i \in S$, in every $\Gamma_S(p_{-S})$ i.e.,

$$\mathtt{BR}_S(p_{-S}) \subseteq C^*(S) \Longrightarrow \mathtt{BR}_S(p_{-S}) \neq \bigtimes_{i\in S} \mathtt{BR}^i_S(p_{-S}).$$

The property that a non-singleton subset of players S's pure best replies is not the sum of its parts is a key property to guarantee the existence of non-trivial NCCEs which does not boils down to the regular correlated equilibria of the original game. It is helpful to express this property to the following well-known *exchangeable* or *rectangularity property* of Nash equilibria in 'nice games' (see e.g., Moulin (1986)). In the simplest case of *bi-matrix games* Γ, a set of Nash equilibria $\mathtt{NE}(\Gamma)$ of a game Γ are **exchangeable** or the game has the **rectangular property** if

$$(p_1, p_2), (p_1^{'}, p_2^{'}) \in \mathtt{NE}(\Gamma) \Longrightarrow (p_1^{'}, p_2), (p_1, p_2^{'}) \in \mathtt{NE}(\Gamma).$$

Here, we need to extend this property to the set of *jointly coherent strategies* in N-player games. More specifically, take a subset of S players playing the coalitional games $\Gamma_S(p_{-S})$.

Joint coherent strategies in 'nice partitioned games'. The extension of the rectangular property to the set of *jointly coherent strategies* of S in the nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$ is that *every* set of jointly coherent strategies that can be played into one of its Nash equilibria, $C^*_S(p_{-S}) \subseteq C^*(S)$, writes as the Cartesian product of its components:

$$\forall C^*_S(p_{-S}) \subseteq C^*(S), C^*_S(p_{-S}) = \bigtimes_{i\in S} C^*_i(p_{-S}),$$

where $C^*_i(p_{-S})$ designates the set of pure strategies of $i \in S$ which lies into the support of a correlated equilibrium distribution p_S of players S in coalitional game $\Gamma_S(p_{-S})$, that induces a mixed Nash equilibrium of the nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$. As demonstrated in Theorem 2, in the case of convex potential games, the failure of the rectangular property will indeed concerns the set of *pure strategy Nash equilibria* of the coalitional games.

Proposition 3.3. *Consider a finite game* Γ *which admits a nice partitioned game* $\Gamma^*_{\mathscr{C}(N)}$ *for a non-trivial partition* $\mathscr{C}(N)$. *The game* $\Gamma^*_{\mathscr{C}(N)}$ *is played by a meta-player* $S \in \mathscr{C}(N)$ *if and only if there is a set* $C^*_S(p_{-S}) \subseteq C^*(S)$ *of jointly coherent strategies of* S *which fails the* ***rectangularity property*** *for some coalitional games* $\Gamma_S(p_{-S})$.

Proof. **See Appendix B.2**.

Nice partitioned games with the rectangularity property of their set of jointly coherent strategies that are played into some Nash equilibria are not played by meta-players. It is well-known that for two-player games, the rectangularity property holds if and only if the set of Nash equilibria is a convex set (see e.g., Moulin, 1986). So, this means that coalitions of partitioned games made-up of exactly two players cannot be viewed as meta-players if the game has a convex set of Nash equilibria. It is also useful to note that when S is a meta-player, then the set of jointly coherent strategies $\mathtt{PNE}_S((p_{-S}))$ can*not* be a singleton set forming a PSNE since

$$\{\theta^*_S\} = \mathtt{BR}_S(\theta^*_{-S}) = \bigtimes_{i\in S} \{\theta^*_i\} = \bigtimes_{i\in S} \mathtt{BR}_i(\theta^*_{-S}),$$

would contradict the definition of S as a meta-player. Hence any game $\Gamma_S(p_{-S})$ with a meta-player cannot have a unique PSNE $\theta^*_S = (\theta^*_i : i \in S)$. On the other hand, observe that if there exists a unique PSNE $\theta^*_S = (\theta^*_i : i \in S)$, then the rectangularity property is true (since there must have at least two PSNEs in $\mathtt{PNE}_S((p_{-S}))$ to have a failure of the rectangularity property).

A key argument behind the proof of existence of proper NCCEs then relies onto the observation that the failure of the rectangular property for a subset of players S (that belongs to a nontrivial partition) and the existence of a mixed Nash equilibrium in the partitioned game guarantees the existence of a correlated equilibrium distribution of the whole game Γ which can*not* coincide with a regular mixed Nash equilibrium. In the light of the above discussion, it thus follows that there exists a proper NCCE in Γ (which does not form a regular Nash equilibrium of Γ) whenever there exists at least one meta-player S in $\Gamma^*_{\mathscr{C}(N)}$ with the property that

$$\exists p_{\mathscr{C}(N)} = (p_S), C^*_S(p_{-S}) \neq \emptyset, \forall S \in \mathscr{C}(N).$$

A first method that ensures that the above condition holds for some profiles $p_{\mathscr{C}(\mathrm{N})}$ is to look directly at the correlated equilibrium polytope of the partitioned game and apply the results of Nau et al. (2004).
The existence of a non-trivial $\mathscr{C}(\mathrm{N})$-NCCE is guaranteed If a coalition structure $\mathscr{C}(\mathrm{N})$ with a cardinality $|\mathscr{C}(\mathrm{N})| \geqslant 3$ possesses a $\mathscr{C}(\mathrm{N})$ -partitioned game which admits a properly mixed Nash equilibrium $(p^*_{\mathrm{S}}, p^*_{-\mathrm{S}})$ with p^*_{S} a correlated equilibrium distribution in $\Delta^*(\Theta_{\mathrm{S}})$ for the non-singleton subset of players S. For coalition structures where there only exist two coalitions i.e., $|\mathscr{C}(\mathrm{N})| = 2$, it is also sufficient that $\mathscr{C}(\mathrm{N})$ has a non-singleton subset of players S and there exists a properly mixed Nash equilibrium $(p^*_{\mathrm{S}}, p^*_{-\mathrm{S}})$ of the partitioned game of Γ has a $\mathscr{C}(\mathrm{N})$ -partitioned game wherein both coalitions of players S and $-$S use a non-degenerate optimal mixed strategy p^*_{S} and $p^*_{-\mathrm{S}}$.

Proposition 3.4. *Consider a finite game* Γ *which admits a nice partitioned game* $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ *for a non-trivial partition* $\mathscr{C}(\mathrm{N})$. *Suppose* $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ *has at least one non-singleton subset of players* $\mathrm{S} \in \mathscr{C}(\mathrm{N})$ *that forms a meta-player and the existence of a Nash equilibrium in the relative interior of its correlated equilibrium polytope,* $\mathrm{CE}(\Gamma^*_{\mathscr{C}(\mathrm{N})})$.[9] *Then* Γ *has at least one proper* $\mathscr{C}'(\mathrm{N})$*-NCCE (not necessarily properly mixed) for a (non-trivial) partition,* $\mathscr{C}'(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N})$, *that is a refinement of* $\mathscr{C}(\mathrm{N})$.

Proof. See **Appendix B.3**.

Properly mixed Nash coalitional correlated equilibria through 'nice and tight partitioned games'. The above result has a limitation. It indeed deals with proper NCCEs which can be 'pathological', in the sense that some coalitions may play into a degenerate correlated equilibrium distribution, as in the case of the three-player game of Example 1. In this example, the trivial $\{\{1,2\},\{3\}\}$-NCCE $p = p_{12} \otimes \delta_{\mathbf{B}_3}$ corresponds to a correlated equilibrium distribution which is a product measure of the *NCCE* $p_{12} = \frac{1}{2}(\mathbf{H}_1, \mathbf{H}_2) \oplus \frac{1}{2}(\mathbf{T}_1, \mathbf{T}_2)$ of player 1 and 2 played into their *coalitional game* $\Gamma_{12}(\delta_{\mathbf{B}_3})$ and the *degenerate* distribution $\delta_{\mathbf{B}_3}$ for player 3.
The object of the following sections will be to give some sufficient conditions for the existence of proper NCCEs wherein *all* coalitions uses a *proper randomization.* More precisely, we carry our analysis by looking at the existence of those proper NCCEs wherein each coalition plays into some *finitely supported non-degenerate* correlated equilibrium distributions.
We say that a mixed Nash equilibrium $p^* = (p^*_i)$ of a game Γ is **properly mixed** (the term is taken from Echenique and Edlin, 2004) if the equilibrium that is not in pure strategies i.e., the *support*, $\mathbf{supp}(p^*_i)$ is *not* a singleton set for *all* i. The next definition extends this property to the case of NCCEs.

Definition 3.5. Consider a finite game Γ and fix a non-trivial partition $\mathscr{C}(\mathrm{N}) \notin \{\mathscr{C}^*(\mathrm{N}), \mathscr{C}^{**}(\mathrm{N})\}$. We say that a non-trivial NCCE $p^*_{\mathscr{C}(\mathrm{N})} = (p^*_{\mathrm{S}} : \mathrm{S} \in \mathscr{C}(\mathrm{N}))$ of a game Γ is a **properly mixed** $\mathscr{C}(\mathrm{N})$-**Nash coalitional correlated equilibrium** (for short **PNCCE**) of Γ if the correlated equilibrium distribution p^*_{S} of every subset of players S in coalitional game $\Gamma_{\mathrm{S}}(p^*_{-\mathrm{S}})$ is properly mixed.

The class of PNCCEs does capture the richness of what NCCEs can be used for in the modeling of the interactions occurring simultaneously within and between coalitions. An example of a PNCCE is the NCCE found in Example 1. In this game, there indeed exists a $\{\{1,2\},\{3\}\}$-NCCE in which every subset of players in the partition plays a non-degenerate mixed strategy, which corresponds to a correlated equilibrium distribution (CED) that does not induce a regular mixed Nash equilibrium for the two coalitions of players.
The identification of the existence of PNCCEs is based upon the following notion of *tight game* introduced by Nitzan (2005). (Nitzan, 2005) *A finite game* Γ *is* ***tight*** *if in every correlated equilibrium* $p^* \in \Delta(\Theta_{\mathrm{N}})$, *all incentives constraints are tight i.e., for each* $i \in \mathrm{N}$, *and for all* $\theta_i, \theta'_i \in \Theta_i$,

$$\sum_{\theta_{-i}} p^*(\theta_i, \theta_{-i}) \left[\mathrm{U}_i(\theta_i, \theta_{-i}) - \mathrm{U}_i(\theta'_i, \theta_{-i}) \right] = 0,$$

for all $\theta_{\mathrm{N}} = (\theta_i, \theta_{-i})$.

If the game is tight, then every pure strategy, hence also every mixed strategy of player i is a best-response to p_{-i}. The three-player matching pennies game of Example 1 is not tight. Indeed, the game has two pure Nash equilibrium strategy profiles: One in which the row and column players (players 1 and 2) play $(\mathbf{H}_1, \mathbf{H}_2)$ and the matrix player 3 plays $(\mathbf{T}_3)$ and another one in which the row and column players (players 1 and 2) play $(\mathbf{T}_1, \mathbf{T}_2)$ and

[9] The 'relative interior' of a convex set P in R^d is the interior of P in the affine hull of P.

the matrix player 3 plays $(\mathbf{H}_3)$. Since every (pure) Nash equilibrium is a correlated equilibrium, this shows that this game does not have the property to be tight.
We are now in a position to obtain some sufficient conditions for the existence of PNCCEs for finite games by imposing the tightness' property to the (finite) partitioned game.

Proposition 3.6. *Consider a finite game* Γ *which admits a nice partitioned game* $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ *for a non-trivial partition* $\mathscr{C}(\mathrm{N})$. *If the game* $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ *is a tight game with at least one meta-player* S, *then* Γ *has at least one* $\mathscr{C}'(\mathrm{N})$*-PNCCE with* $\mathscr{C}'(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N})$ *a partition that is a (possibly weak) refinement of* $\mathscr{C}(\mathrm{N})$.

Proof. See **Appendix B.4**.

4. Existence of non-trivial NCCEs in continuous games: The case of (exact) partition potentials

In this section we now formulate some results for games which possess non-trivial NCCEs by characterizing the continuous potential games subclass of infinite games where some non-singleton coalitions use some *independent non-degenerate correlated equilibria* resulting in a tuple of correlated strategies lying outside the set of the correlated equilibria of the original game. Combined with our last main theorem 3—characterizing the class of games where there is coincidence between coalitional equilibria of Ray and Vohra and the NCCEs—will provide the identification of a class of games where there exist some coalitional equilibria a la Ray and Vohra forming some non-trivial NCCEs.

4.1. **Existence of non-trivial NCCEs in mixed strategies in continuous convex games.** The above existence for arbitrary coalition structures, $\mathscr{C}(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N}), \mathscr{C}^{**}(\mathrm{N})$, of a $\mathscr{C}(\mathrm{N})$-Nash coalitional correlated equilibrium is formulated for the class of finite games. Hence, it cannot be directly applied to the class of games where players have compact and convex sets of pure strategies (in short, continuous games). In order to apply the 'tightness property' in the class of *convex and compact* games, we shall therefore introduce a method which consists in exploiting the combined properties of smooth partition potential games (see Monderer and Shapley (1996), Uno (2007, 2011)) and the property of the correlated equilibrium distributions in convex games (Neyman, 1997), by converting the analysis of the initial continuous game to the one of a tight finite game. More specifically, a key step in the proof is to analyze an *auxiliary finite partitioned game* that guarantees the existence of at least one proper NCCE where all coalitions play a non-degenerate CED. The next example below gives an illustration of how partitioned games arise in partition potential games (Uno, 2007, 2011) and motivates the focus onto the refined subset of non-trivial NCCEs wherein each coalition of players plays a non-degenerate correlated equilibrium distribution over a set of *pure Nash equilibria*.
The idea is to characterize the specific case where players inside each coalition of players S randomizes according to a correlated equilibrium distribution p^*_{S} that is a *mixture of the set of pure Nash equilibria* $\mathtt{PNE}_{\mathrm{S}}(p^*_{-\mathrm{S}})$, of their coalitional game $\Gamma_{\mathrm{S}}(p^*_{-\mathrm{S}})$.

Example 4 : Existence of $\mathscr{C}(\mathrm{N})$-canonical Nash correlated equilibria in (finite) potential games.

Consider the game Γ described by Table 4 below. This is a four-player game $\mathrm{N} = \{1,2,3,4\}$. We analyze the scenario where the players are split into two coalitions, $\{\{1,2\},\{3,4\}\}$. Let S_k denote a coalition of players $\mathrm{S}_k \in \{\mathrm{S}_1, \mathrm{S}_2\}$ with $\mathrm{S}_1 = \{1,2\}$ and $\mathrm{S}_2 = \{3,4\}$. Let Θ_{kl} be the set of pure strategies of player l of coalition S_k and θ_{kl} be one of the pure strategies available to player l of coalition S_k with a space of pure strategies, $\Theta_{kl} = \{\mathtt{A}_{kl}, \mathtt{B}_{kl}\}$, $k = 1,2$ and $l = 1,2,3,4$. The tables below describe the payoff matrices which are mutually induced for each coalition S_k when the other coalition S_{-k} plays its pure Nash equilibrium profiles $\mathtt{PNE}_{\mathrm{S}_{-k}}(\mathbf{A}_{\mathrm{S}_{-k}}) = \mathtt{PNE}_{\mathrm{S}_{-k}}(\mathbf{B}_{\mathrm{S}_{-k}}) = \{(\mathbf{A}_{\mathrm{S}_k}), (\mathbf{B}_{\mathrm{S}_k})\}$ where $\mathbf{A}_{\mathrm{S}_k} = (\mathrm{A}_{ki}, \mathrm{A}_{kj})$ and $\mathbf{B}_{\mathrm{S}_k} = (\mathrm{B}_{ki}, \mathrm{B}_{kj})$, $k = 1,2$.

$\Gamma_{\mathrm{S}_k}(\mathbf{A}_{\mathrm{S}_{-k}}), k=1,2$	A_{kj}	B_{kj}
A_{ki}	2,2	0,0
B_{ki}	0,0	1,1

$\Gamma_{\mathrm{S}_k}(\mathbf{B}_{\mathrm{S}_{-k}}), k=1,2$	A_{kj}	B_{kj}
A_{ki}	1,1	0,0
B_{ki}	0,0	2,2

In this example, there exists some $\mathscr{C}(\mathrm{N})$-mixed Nash equilibria wherein each coalition randomizes over the set of pure Nash equilibria of his coalition given the randomization of the other coalitions and the resulting tuple forms a mixed Nash equilibrium across the two coalitions. To see this, consider the table below representing the payoff values taken by the potential functions associated to the two coalitional games $\widehat{\Gamma}_{S_k}(\mathbf{A}_{S_{-k}})$ and $\widehat{\Gamma}_{S_{-k}}(\mathbf{B}_{S_{-k}})$.

$P_{S_k}(\cdot,\mathbf{A}_{S_{-k}}), k=1,2$	A_{kj}	B_{kj}
A_{ki}	2	0
B_{ki}	0	1

$P_{S_k}(\cdot,\mathbf{B}_{S_{-k}}), k=1,2$	A_{kj}	B_{kj}
A_{ki}	1	0
B_{ki}	0	2

As can be read out from the table above in this particular example, the set of maximizers of the parametrized potential function, $P_{S_k}(\cdot,\theta_{S_{-k}})$, of each coalitional game, $\Gamma_{S_k}(\cdot)$ is such that:

$$\arg\max_{\theta_{S_k}} P_{S_k}(\theta_{S_k},\theta_{S_{-k}}) = \begin{cases} \{\mathbf{A}_{S_k}\} \text{ if } \theta_{S_{-k}} = \mathbf{A}_{S_{-k}}, k=1,2; \\ \{\mathbf{B}_{S_k}\} \text{ if } \theta_{S_{-k}} = \mathbf{B}_{S_{-k}}, k=1,2. \end{cases}$$

To find out the set of $\{\{1,2\},\{3,4\}\}$-mixed Nash correlated equilibria we need to look for the set of mixed Nash equilibria of the *partitioned game* induced between the coalitions when they randomize over their respective sets of pure Nash equilibria in their partitioned game. This game played between the two coalitions is given in the table below.

$\Gamma_{\mathscr{C}(N)}$	$\mathbf{A}_{S_2}$	$\mathbf{B}_{S_2}$
$\mathbf{A}_{S_1}$	2,2	1,1
$\mathbf{B}_{S_1}$	1,1	2,2

The reading of the payoff matrix representing the partitioned game $\Gamma_{\mathscr{C}(N)}$ shows that $(\mathbf{A}_{S_1},\mathbf{A}_{S_2})$ and $(\mathbf{B}_{S_1},\mathbf{B}_{S_2})$ are two $\{\{1,2\},\{3,4\}\}$- Nash correlated equilibria of the whole game in pure strategies. Moreover, the set of mixed Nash equilibria of the two-player symmetric game played across the coalitions on in the *partitioned game*—gives the set of Nash correlated equilibria of the whole game. Here, we can check that the tuple of probability measures,

$$(p_{12},p_{34}) = (\frac{1}{2}\mathbf{A}_{S_1},\frac{1}{2}\mathbf{B}_{S_1}),(\frac{1}{2}\mathbf{A}_{S_2},\frac{1}{2}\mathbf{B}_{S_2}))$$

is the unique mixed Nash equilibrium of the *partitioned game*. In addition, note that this equilibrium has the property to be a properly mixed $\{\{1,2\},\{3,4\}\}$-NCCE, since, in equilibrium,each coalition $k=1,2$ properly randomizes over the set of pure Nash equilibria of their respective coalitional games. In the above Nash correlated equilibrium (p^*_{12},C), the mediator of coalition of players 1 and 2 properly randomizes over the set of pure Nash equilibria $\texttt{PNE}(\Gamma_{12}(C)) = \{\texttt{TA},\texttt{SB}\}$ of the induced coalitional game $\Gamma_{12}(C)$ of players 1 and 2. Actually, in this game, every profile (p^*_{12},C) such that p^*_{12} lies in the convex hull of $\texttt{PNE}(\Gamma_{12}(C))$ with, $0 < p^*_{12}(\texttt{TA}) \leqslant \frac{2}{3}$ and $0 < p^*_{12}(\texttt{SB}) \leqslant \frac{2}{3}$, is a $\mathscr{C}(N) = \{\{1,2\},\{3\}\}$-NCCE with this property. Clearly, this class of NCCEs requires 'nice games' wherein the collection of coalitional games played by the mediators of each coalition S in *equilibrium* possess a *non-empty set of pure Nash equilibria*. Thus, we cannot obtain the existence of NCCE with this property for arbitrary games.
Given a topological set Θ_i , let $\Delta(\Theta_i)$ denotes the set of regular probability measures over the Borel σ- algebra on Θ_i . A N-player *compact game* Γ is given by a compact set of strategies Θ_i for each player i and by a continuous payoff function $U = (U_i)_{i\in N}$ from Θ to $\mathbb{R}^N$. The set of mixed strategies for player i is $\tilde{\Theta}_i = \Delta\Theta_i$ and U is extended to $\tilde{\Theta} = \Delta(\prod_{i\in N}\Theta_i)$ by $U(p) = E_p U(\theta)$ with 'E' the expectation operator. Let $p^*_{-S} \equiv (p^*_{S'} : S' \neq S, S' \in \mathscr{C}(N))$.

Definition 4.1. Given a N-player *compact game* Γ, say that a $\mathscr{C}(N)$-Nash coalitional correlated equilibrium $p^*_{\mathscr{C}(N)} = (p^*_S)_{S\in\mathscr{C}(N)}$ of Γ is a $\mathscr{C}(N)$–**canonical (mixed) Nash coalitional correlated equilibrium** if each coalition of players $S \in \mathscr{C}(N)$ randomizes in a correlated equilibrium distribution over a subset of the pure Nash equilibria $\texttt{PNE}_S(p^*_{-S})$ of the coalitional game $\Gamma_S(p^*_{-S})$.

The first requirement for the existence of a $\mathscr{C}(N)$- canonical Nash coalitional correlated equilibrium is the existence of a non-empty set of pure Nash equilibria $\texttt{PNE}_S(p^*_{-S})$ of $\Gamma_S(p^*_{-S})$, for each coalition S. In a $\mathscr{C}(N)$-canonical Nash coalitional correlated equilibrium, every (non-degenerate) correlated equilibrium distribution p^*_S is a (proper) mixture of (certain) pure strategy Nash equilibria of the coalitional game $\Gamma_S(p^*_{-S})$. As stated below, the existence of such particular $\mathscr{C}(N)$-NCCE is guaranteed by weakening the notion of exact potential (Monderer

and Shapley, 1996), as follows.
Fix a partition $\mathscr{C}(\mathrm{N})$ of the N set of players. A exact potential for a coalitional game $\Gamma_S(\theta_{-S})$ is a function $P^{\theta_{-s}}$: $\Theta_S \longrightarrow \mathbb{R}$ such that for all $i \in S$ the condition,

$$U_i(\theta_i', \theta_{S\setminus i}; \theta_{-S}) - U_i(\theta_i, \theta_{S\setminus i}; \theta_{-S}) = P^{\theta_{-s}}(\theta_i', \theta_{S\setminus i}) - P^{\theta_{-s}}(\theta_i, \theta_{S\setminus i}),$$

holds for all $\theta_i', \theta_{S\setminus i}$ and θ_i. A game is **smooth** if for each $i \in N$, $U_i(\theta_N)$ has continuous partial derivatives with respect to each variable θ_i. We say that Γ has a **partitioned** $\mathscr{C}(\mathrm{N})$-**smooth** (or C^1)-**exact potential** if and only if *each* game $\Gamma_S(\theta_{-S})$ of the *family of games* $\Gamma_S(\cdot) \equiv \{\Gamma_S(\theta_{-S}) : \theta_{-S} \in \Theta_{-S}\}$ is a exact potential game. When there is a $\mathscr{C}(\mathrm{N})$-partitioned potential function, the partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$, can be directly defined with space of all pure strategies of of each coalition S by the $|S|$-fold Cartesian product $\Theta_S = \prod_{i\in S}\Theta_i$ of the original spaces of pure strategies in Γ. When Γ has a partitioned $\mathscr{C}(\mathrm{N})$-smooth (or C^1)-**exact potential**, this induces a **smooth partitioned game**,

$$\Gamma_{\mathscr{C}(\mathrm{N})} \equiv \left\langle \mathscr{C}(\mathrm{N}), \{\Theta_S, U_S\}_{S\in\mathscr{C}(\mathrm{N})} \right\rangle$$

where the 'coalitions' payoff functions $\{U_S\}_{S\in\mathscr{C}(\mathrm{N})}$ are given by the potential functions:

$$U_S : \Theta_S \bigtimes \Theta_{-S} \longrightarrow \mathbb{R}; U_S(\theta_S, \theta_{-S}) := P_S^{\theta_{-s}}(\theta_S).$$

The next example illustrates how the existence of a $\mathscr{C}(\mathrm{N})$-mixed Nash equilibria arises in a four-player potential game when there are two proper subsets of players and each mediator of each coalition S randomizes over the set of pure Nash equilibria. This provides the intuition behind the proof of our next result, which guarantees the existence of a $\mathscr{C}(\mathrm{N})$-mixed coalitional correlated equilibrium for the class of continuous games.

Theorem 1. *Consider a* N*-player strategic game with compact and convex strategy spaces and bounded and continuous payoffs* $\Gamma = \langle N, (\Theta_i, U_i)_{i\in N}\rangle$*. Fix a partition* $\mathscr{C}(\mathrm{N})$ *and assume that* Γ *has a partitioned* $\mathscr{C}(\mathrm{N})$*-*C^1*-concave exact potential. Then,* Γ *has at least one non-trivial* $\mathscr{C}(\mathrm{N})$*-NCCE and this equilibrium is necessarily a canonical* $\mathscr{C}(\mathrm{N})$*-mixed Nash coalitional correlated equilibrium of* Γ.
Proof. Exact potential games with continuous payoff functions have continuous exact potential functions and continuous functions on a compact set achieve a maximum. Hence, since every coalitional game $\Gamma_S(\theta_{-S})$ is a continuous exact potential game with compact strategy sets, the Lemma of Monderer and Shapley, (1996, Lemma 4.3) ensures the existence of (at least) a pure Nash equilibrium to each game $\Gamma_S(\theta_{-S})$. Moreover, when each subgame $\Gamma_S(\theta_{-S})$ is an exact C^1-concave, potential game, Neyman's Corollary (1997) implies that the set of (necessarily non-empty) pure Nash equilibria $\mathtt{PNE}_S(\theta_{-S})$ of $\Gamma_S(\theta_{-S})$ is a *convex subset* of Θ_S given by the set of maximizers of the potential function $P^{\theta_{-s}}$ i.e.,

$$\mathtt{PNE}_S(\theta_{-S}) = \prod_{i\in S} \arg\max_{\theta_i\in\Theta_i} U_i(\theta_i, \theta_{S\setminus i}; \theta_{-i}) = \arg\max_{\theta_S\in\Theta_S} P^{\theta_{-s}}(\theta_S).$$

Consider the coalitional game $\Gamma_S(p_{-S})$. The potential function of coalition S in game $\Gamma_S(\theta_{-S})$ defines the best response correspondence $\mathtt{BR}_S$ of S i.e.,

$$\theta_{-S} \longmapsto \mathtt{BR}_S(\theta_{-S}) \equiv \arg\max_{\theta_S\in\Theta_S} P^{\theta_{-s}}(\theta_S).$$

The collection of potential functions, $\theta_{-S} \longmapsto P^{\theta_{-s}}(\theta_S)$ of each coalition S induces a payoff function,

$$(\theta_S, \theta_{-S}) \longmapsto U_S(\theta_S, \theta_{-S})$$

where

$$U_S(\theta_S, \theta_{-S}) \equiv P_S(\theta_S, \theta_{-S})$$

for S in the partitioned game,

$$\Gamma_{\mathscr{C}(\mathrm{N})} \equiv \left\langle \mathscr{C}(\mathrm{N}), \{\Theta_S, U_S\}_{S\in\mathscr{C}(\mathrm{N})} \right\rangle.$$

So, it is w.l.o.g to consider the partitioned game,

$$\Gamma_{\mathscr{C}(\mathrm{N})} = \langle \mathscr{C}(\mathrm{N}), (\Theta_S, P_S)\rangle.$$

Since each coalitional game is an *exact* smooth potential game with convex strategy sets and bounded payoffs, Neyman (1997) entails that the set of pure NE, $\Theta_S^{\mathtt{PNE}}$, of each coalition S is a *convex* set and any correlated equilibrium of $\Gamma_S(\Theta_{-S})$ is a mixture in $\Theta_S^{\mathtt{PNE}}$.

Hence, one can define the (restricted) partitioned game,

$$\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*) \equiv \left\langle \mathscr{C}(\mathrm{N}), (\Theta_{\mathrm{S}}^{\mathtt{PNE}}, \mathrm{P_S}) \right\rangle,$$

wherein each set of strategy profiles of each coalition S is *restricted* to their (convex) set of *pure Nash equilibria* $\Theta_{\mathrm{S}}^{\mathtt{PNE}}$. Thus, taking the *mixed extension* of $\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*)$, the compactness and convexity of strategy sets of $\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*)$, means that one can apply Kakutani fixed point theorem to the game $\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*)$ to guaranty the existence of at least one mixed Nash equilibrium in the partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*)$ and hence the existence of at least one $\mathscr{C}(\mathrm{N})$-PNCCE in Γ wherein each mediator of S recommends to his coalition members to play a correlated equilibrium $\mathrm{P}_{\mathrm{S}}^*$ by randomizing over their set of *pure Nash equilibria* $\Theta_{\mathrm{S}}^{\mathtt{PNE}}$.

The second part of the proof consists in checking that the resulting $\mathscr{C}(\mathrm{N})$-PNCCE is also forming a $\mathscr{C}(\mathrm{N})$-partitioned mixed Nash equilibrium of Γ. In order to see this, note that when the other coalitions' profile of probability distributions is $p_{-\mathrm{S}}$, the mediator of coalition S has a best reply $p_{\mathrm{S}}^* \in \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}})$ given by the maximizers of the potential function $\mathrm{P}^{p_{-\mathrm{S}}}(\theta_{\mathrm{S}})$ of game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$. That is,

$$p_{-\mathrm{S}} \longmapsto \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}}) \equiv \arg \max_{p_{\mathrm{S}} \in \Delta(\Theta_{\mathrm{S}})} \mathrm{P}^{p_{-\mathrm{S}}}(p_{\mathrm{S}}).$$

The set of pure Nash equilibria of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$ are the set of pure strategies contained in $\mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}})$. This follows since in a NCCE where each coalition properly randomizes the induced non-degenerate correlated distributions $p_{-\mathrm{S}}^* = \otimes_{\mathrm{S}' \neq \mathrm{S}} p_{\mathrm{S}'}^*$ must render each coalition S indifferent between any profile θ_{S}^* in the (finite) support of p_{S}^*, since in a NCCE, the tuple (p_{S}^*) must form a regular mixed Nash equilibrium of the partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}(\Theta^*)$. From this, it follows that, $\forall \mathrm{S} \in \mathscr{C}(\mathrm{N})$, we have

$$\theta_{\mathrm{S}}^*(p_{-\mathrm{S}}^*), \theta_{\mathrm{S}}^{*'}(p_{-\mathrm{S}}^*) \in \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}}^*), \text{ iff } \theta_{-\mathrm{S}}^*, \theta_{\mathrm{S}}^{*'} \in \mathbf{supp}(p_{\mathrm{S}}^*).$$

By construction, when the potentials are *exact*, every pure Nash equilibria $\theta_{\mathrm{S}}^*(p_{-\mathrm{S}}^*)$ of the coalitional game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$ is a local maximum of the potential function $\mathrm{P}^{p^*{-\mathrm{S}}}(\theta_{\mathrm{S}}^*)$ i.e.,

$$\mathrm{P}^{p_{-\mathrm{S}}^*}(\theta_{\mathrm{S}}) \equiv \sum_{\theta_{-\mathrm{S}} \in \mathtt{supp}(p^*{}_{-\mathrm{S}})} \mathrm{P}^{\theta_{-\mathrm{S}}}(\theta_{\mathrm{S}}) p_{-\mathrm{S}}^*(\theta_{-\mathrm{S}}).$$

Since each profile $\theta_{\mathrm{S}}^*(p_{-\mathrm{S}}^*)$ must be a pure best reply of coalition S to $p_{-\mathrm{S}}^*$, each such profile of pure strategies is a maximizer of the induced potential function $\mathrm{P}^{p_{-\mathrm{S}}^*}$ of $\theta_{\mathrm{S}}^*(p_{-\mathrm{S}}^*)$ i.e.,

$$\mathbf{supp}(p_{\mathrm{S}}^*) \subset \arg \max_{\theta_{\mathrm{S}} \in \Theta_{\mathrm{S}}} \mathrm{P}^{p^*{-\mathrm{S}}}(\theta_{\mathrm{S}}).$$

Now note that when the collection of potentials of the family of games $\Gamma_{\mathrm{S}}(\cdot)$ are *exact potentials*, then the support $\mathbf{supp}(p_{\mathrm{S}}^*)$ of the correlated equilibrium distribution $p^*{}_{\mathrm{S}}$ of S must necessarily be contained in the set of pure Nash equilibria $\mathtt{PNE}(p^*{}_{-\mathrm{S}})$ of game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$. This follows from the application of Theorem 1 of Neyman (1997) to each game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$: *Every* correlated equilibrium p_{S}^* of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$ is a *mixture of pure Nash equilibria* of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$. It remains to prove that the set of pure Nash equilibria $\mathtt{PNE}_{\mathrm{S}}(p_{-\mathrm{S}}^*)$ of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}}^*)$ is given by the set:

$$\mathtt{PNE}_{\mathrm{S}}(p_{-\mathrm{S}}^*) = \bigcup_{\theta_{-\mathrm{S}} \in \mathtt{supp}(p_{-\mathrm{S}}^*)} \mathtt{PNE}_{\mathrm{S}}(\theta_{-\mathrm{S}}).$$

To see this, consider the convex hull

$$\mathtt{conv}(\Theta_{\mathrm{S}}^{\mathtt{PNE}}) \equiv \left\{ \widetilde{\theta}_{\mathrm{S}}^* : \widetilde{\theta}_{\mathrm{S}}^*(p_{-\mathrm{S}}) \equiv \sum_{\theta_{-\mathrm{S}} \in \Theta_{-\mathrm{S}}} \theta_{\mathrm{S}}^*(\theta_{-\mathrm{S}}) p_{-\mathrm{S}}(\theta_{-\mathrm{S}}), \theta_{\mathrm{S}}^*(\theta_{-\mathrm{S}}) \in \mathtt{PNE}_{\mathrm{S}}(\theta_{-\mathrm{S}}), p_{-\mathrm{S}} \in \Delta(\theta_{-\mathrm{S}}) \right\}$$

of the set of pure Nash equilibria $\Theta_{\mathrm{S}}^{\mathtt{PNE}} \equiv \{\mathtt{PNE}_{\mathrm{S}}(\theta_{-\mathrm{S}}) : \theta_{-\mathrm{S}} \in \Theta_{-\mathrm{S}}\}$ of the family of games $\Gamma_{\mathrm{S}}(\theta_{-\mathrm{S}})$. By construction, when the family of games $\Gamma_{\mathrm{S}}(\cdot)$ has an *exact* C^1-concave potential game, we can use, once more time, the fact have that every pure Nash equilibrium $\theta_{\mathrm{S}}^*(\theta_{-\mathrm{S}})$ of $\Gamma_{\mathrm{S}}(\theta_{-\mathrm{S}})$ is a maximizer of the potential,

$$\theta_{\mathrm{S}}^*(\theta_{-\mathrm{S}}) \in \arg \max_{\theta_{\mathrm{S}} \in \Theta_{\mathrm{S}}} \mathrm{P}^{\theta_{-\mathrm{S}}}(\theta_{\mathrm{S}}).$$

On the other hand, by construction, every distribution,

$$p^*_{\mathrm{S}}(p_{-\mathrm{S}}) = \widetilde{\theta}^*_{\mathrm{S}}(p_{-\mathrm{S}}) = \sum_{\theta_{-\mathrm{S}} \in \mathtt{supp}(p_{-\mathrm{S}})} \theta^*_{\mathrm{S}}(\theta_{-\mathrm{S}})p_{-\mathrm{S}}(\theta_{-\mathrm{S}}) \in \mathtt{conv}(\Theta^{\mathtt{PNE}}_{\mathrm{S}}),$$

forms a correlated equilibrium distribution p^*_{S} of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$. As a result, every distribution, $p^*_{\mathrm{S}}(p_{-\mathrm{S}})$ lies in the best response correspondence $\mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}})$ of S. Hence, in a $\mathscr{C}(\mathrm{N})$-NCCE the set of pure Nash equilibria $\mathtt{PNE}_{\mathrm{S}}(p^*_{-\mathrm{S}})$ of $\Gamma_{\mathrm{S}}(p^*_{-\mathrm{S}})$ is the set

$$\mathtt{PNE}_{\mathrm{S}}(p^*_{-\mathrm{S}}) = \bigcup_{\theta_{-\mathrm{S}} \in \mathtt{supp}(p^*_{-\mathrm{S}})} \mathtt{PNE}_{\mathrm{S}}(\theta_{-\mathrm{S}}) \subseteq \mathtt{BR}_{\mathrm{S}}(p^*_{-\mathrm{S}}), \forall \mathrm{S} \in \mathscr{C}(\mathrm{N}).$$

From this we have that *every* such correlated equilibrium distribution p^*_{S} lies in the convex hull $\mathtt{conv}(\Theta^{\mathtt{PNE}}_{\mathrm{S}})$ of the set of pure Nash equilibria of the family of games $\Gamma_{\mathrm{S}}(\cdot)$. From this it follows that every $\mathscr{C}(\mathrm{N})$-NCCE is a tuple $p^*_{\mathscr{C}(\mathrm{N})} = (p^*_{\mathrm{S}} : \mathrm{S} \in \mathscr{C}(\mathrm{N}))$ which verifies that

$$p^*_{\mathscr{C}(\mathrm{N})} \in \prod_{\mathrm{S} \in \mathscr{C}(\mathrm{N})} \mathtt{BR}_{\mathrm{S}}(p^*_{-\mathrm{S}}) \subseteq \prod_{\mathrm{S} \in \mathscr{C}(\mathrm{N})} \widetilde{\Theta}^{\mathtt{NE}}_{\mathrm{S}}.$$

□

EXISTENCE OF NON-TRIVIAL NASH COALITIONAL CORRELATED EQUILIBRIA IN INFINITE GAMES: THE CASE WHERE COALITIONS ARE 'META-PLAYERS'

4.2. **Spans of convex nice partitioned games.** To formulate our result concerning the existence of non-trivial Nash coalitional correlated equilibria (NCCEs) in games with infinite space of actions with continuous payoffs, we shall study the property put onto the structure of the constraints in the correlated equilibrium strategies used by the coalitions. To do so, we shall make use of the above notion of *tight game* we used in our characterization of finite games, by introducing the notion of a '*span of a convex (finite) game*'. The imposition of the 'tightness' condition to this finite game will then be one of the key properties to guarantee the existence of a set of the use of *finitely supported* correlated equilibrium distributions for each coalition that form NCCEs in the class of infinite games which are *continuous convex smooth partition potential games*.

Definition 4.2. Consider a $\mathscr{C}(\mathrm{N})$-player nice partitioned game $\Gamma^*_{\mathscr{C}(\mathrm{N})} = \langle \Theta_{\mathrm{S}}, \mathrm{U}_{\mathrm{S}} \rangle$ with compact and convex strategy spaces and bounded and continuous payoffs. We say that

$$\widehat{\Gamma}^*_{\mathscr{C}(\mathrm{N})} = \left\langle \widehat{\Theta}_{\mathrm{S}}, \widehat{\mathrm{U}}_{\mathrm{S}} \right\rangle$$

is a **span** of $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ induced by a finitely supported mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(\mathrm{N})} = (\widehat{p}_{\mathrm{S}})$ of Γ denoted $\widehat{\Gamma}_{\mathscr{C}(\mathrm{N})} \equiv$ **Span**$(\Gamma_{\mathscr{C}(\mathrm{N})})$, if the space of the pure strategy profiles of each subset of players S, $\widehat{\Theta}_{\mathrm{S}}$, is a finite subset of the pure best replies (of the restricted game $\widehat{\Gamma}_{\mathscr{C}(\mathrm{N})}$) , $\widehat{\mathtt{BR}}_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}})$ of S in $\Gamma_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}})$ i.e., $\widehat{\Theta}_{\mathrm{S}} \subseteq \mathtt{BR}_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}})$.

The span of a convex game requires a non-empty set of pure Nash equilibria (PSNEs). In the class of *nice partitioned games*, the set of PSNEs is non-empty and it coincides with the set of best replies of S to each profile of correlated (equilibrium) distributions $p_{-\mathrm{S}}$ of players $-\mathrm{S}$ induced by a mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(\mathrm{N})}$. The existence of nice partitioned games is guaranteed in continuous convex potential games by applying the theorem of Neyman (1997, Theorem 1) which establishes that *every* correlated equilibrium distribution is a mixture of the (convex) set of *pure strategy Nash equilibria* to each $\Gamma_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}})$. As in the previous section, an essential property to ensure the existence of a non-trivial NCCE is that the aggregate optimal behavior of a coalition of players S does *not* coincide with a tuple of the independent optimal choices of its members. When these properties are met, we obtain that the set of jointly coherent strategies is only made-up of *pure strategy Nash equilibria*. The upshot is then that the partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ of Γ is *nice* and the set of jointly coherent strategies for S, $\mathrm{C}^*(\mathrm{S})$, that are play into a Nash equilibrium of the span of the *nice partitioned game* $\Gamma_{\mathscr{C}(\mathrm{N})}$ coincides with the set of PSNEs of the induced collection of coalitional games of S. From this, we then conclude that the existence of a meta-player S in a nice partitioned game will *only require the failure of the rectangular property for the set of PSNEs* of the coalitional games. Formally, let

$$\widehat{\mathtt{PNE}}_i(p_{-\mathrm{S}}) := \left\{\theta_i \in \widehat{\Theta}_i : \exists \theta_{\mathrm{S}\setminus i} \in \widehat{\Theta}_{\mathrm{S}\setminus i}, (\theta_i, \theta_{\mathrm{S}\setminus i}) \in \mathtt{PNE}(p_{-\mathrm{S}})\right\}$$

be the set of pure strategies for player i in the span of a game Γ for which there exists a profile for players $\mathrm{S}\setminus i$ in the span of the game whose resulting profile $(\theta_i, \theta_{\mathrm{S}\setminus i})$ forms a best reply, hence a PSNE in $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$. As formally stated in Appendix B.5 and C., with the application of Neyman Theorem to the partition potential nice partitioned game, $\Gamma^*_{\mathscr{C}(\mathrm{N})}$, we obtain the equality of the set of PSNEs and the set of jointly coherent strategies for subsets of players S which are played into a Nash equilibrium of the partitioned game i.e.,

$$\forall \mathrm{S} \in \mathscr{C}(\mathrm{N}), \widehat{\mathtt{PNE}}_{\mathrm{S}}(p_{-\mathrm{S}}) = \widehat{\mathrm{C}}^*_{\mathrm{S}}(p_{-\mathrm{S}})$$

Let $\widehat{\mathtt{BR}}_i(p_{-\mathrm{S}})$ be the best response set of player i in the game $\widehat{\Gamma}_{\mathrm{S}}(p_{-\mathrm{S}})$. It follows that the best response of a player i corresponds (by construction) to the set of i-components that induce a PSNE in game $\widehat{\Gamma}_{\mathrm{S}}(p_{-\mathrm{S}})$ i.e.,

$$\widehat{\mathtt{BR}}_i(p_{-\mathrm{S}}) \stackrel{\mathtt{def}}{=} \widehat{\mathtt{PNE}}_i(p_{-\mathrm{S}}), i \in \mathrm{S}.$$

So similar to the previous sections, we have that a span $\widehat{\Gamma}^*_{\mathscr{C}(\mathrm{N})}$ of the nice partitioned game $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ has a meta-player S if and only if it fails the **rectangularity property** in the set of jointly coherent strategies $\widehat{\mathtt{PNE}}_{\mathrm{S}}(p_{-\mathrm{S}}) = \widehat{\mathrm{C}}^*_{\mathrm{S}}(p_{-\mathrm{S}})$ for S, so that either, $\widehat{\Theta}_{\mathrm{S}} = \widehat{\mathtt{BR}}_{\mathrm{S}}(p_{-\mathrm{S}}) \neq \bigtimes_{i\in\mathrm{S}} \widehat{\mathtt{BR}}_i(p_{-\mathrm{S}})$ is a finite set of best replies, or $\widehat{\Theta}_{\mathrm{S}} \subset \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}})$ is a finite subset of PSNEs of $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$ with $\widehat{\Theta}_{\mathrm{S}} \neq \bigtimes_{i\in\mathrm{S}} \widehat{\mathtt{BR}}_i(p_{-\mathrm{S}})$ and $\bigtimes_{i\in\mathrm{S}} \widehat{\mathtt{BR}}_i(p_{-\mathrm{S}}) \subseteq \bigtimes_{i\in\mathrm{S}} \mathtt{BR}_i(p_{-\mathrm{S}}), i \in \mathrm{S}$. When this holds for every subset of players S, we have by construction a span of the partitioned game that is played only by a set of meta-players. With the induced finite span of the partitioned game, we can directly study the analysis of the initial continuous convex game in terms of the analysis of its correlated equilibrium polytope. This results in the theorem below whose claim can be summarized as follows: Take a partition potential (convex and compact) game and fix a non-trivial partition of the player set N. Then, the existence of a proper NCE (wherein at least one coalition of players plays into a non-degenerate correlated equilibrium over their set of PSNEs) is guaranteed when there exists a partitioned game played by at least one meta-player $\mathrm{S} \subset \mathrm{N}$ whose span is *tight*. Hence, the final statement which asserts that the existence of a PNCE is guaranteed when the partitioned game is only played by meta-players.

Theorem 2. *Consider a* N*-player strategic game* $\Gamma = \langle \mathrm{N}, (\Theta_i, \mathrm{U}_i)_{i\in\mathrm{N}}\rangle$*. Fix a non-trivial partition* $\mathscr{C}(\mathrm{N})$*. If* Γ *has compact and convex strategy spaces and bounded and continuous payoffs with a nice partitioned* C^1*-concave* $\mathscr{C}(\mathrm{N})$*-exact potential function. Then the following two equivalent properties guarantee the existence of a non-trivial Nash coalitional correlated equilibrium* $\widehat{p}_{\mathscr{C}'(\mathrm{N})} = (\widehat{p}_{\mathrm{S}})$ in Γ:

(1) *The nice partitioned game has at least one meta-player* S *and a finitely supported mixed Nash equilibrium* $\widehat{p}_{\mathscr{C}'(\mathrm{N})} = (\widehat{p}_{\mathrm{S}})$ *for* $\mathscr{C}'(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N})$ *a (possibly weak) refinement of* $\mathscr{C}(\mathrm{N})$;

(2) *There exists a span of the nice partitioned game,*

$$\widehat{\Gamma}^*_{\mathscr{C}(\mathrm{N})} = \left\langle \widehat{\Theta}^*_{\mathrm{S}}, \widehat{\mathrm{U}}_{\mathrm{S}} \right\rangle,$$

played by at least one meta-player S i.e., $\widehat{\Theta}^*_{\mathrm{S}} \neq \bigtimes_{i\in\mathrm{S}} \widehat{\Theta}^*_i$ *where* $\widehat{\Theta}^*_i \subset BR_i(\widehat{p}_{-\mathrm{S}}), i \in \mathrm{S}$ *and* $\widehat{\Gamma}^*_{\mathscr{C}(\mathrm{N})}$ *is a tight game with* $\widehat{\Theta}^*_{\mathrm{S}} \subset \mathtt{BR}_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}}) \equiv \widehat{\mathrm{C}}^*_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}})$;

(3) *When in addition of (2), each player* T *in* $\mathscr{C}(\mathrm{N})$ *is a meta-player, then there exists a non-trivial NCCE* $\widehat{p}_{\mathscr{C}'(\mathrm{N})} = (\widehat{p}_{\mathrm{S}})$ *(with* $\mathscr{C}'(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N})$*), wherein each component* $\widehat{p}_{\mathrm{S}}$ *is a proper (joint) probability measure in some space,* $\Delta^*(\widehat{\Theta}^*_{\mathscr{C}(\mathrm{S})}) = \bigtimes_{\mathrm{V}\in\mathscr{C}(\mathrm{S})} \Delta^*(\widehat{\Theta}^*_{\mathrm{S}})$*, such that* $\mathscr{C}(\mathrm{S}) \neq \mathscr{C}^*(\mathrm{S})$.

Proof. **See Appendix C**

The stronger statement (3) in Theorem 2 is obtained by adding the extra requirement that the partitioned game is only made of meta-players. This allows to obtain the existence of a proper PNE wherein *each coalition* of players properly randomizes over the set of PSNEs of their (induced) coalitional game. In such PNCCEs $p_{\mathscr{C}(\mathrm{N})} = (p_{\mathrm{S}})$, the CED p_{S} of each coalition S is a joint probability measure which never forms a regular mixed Nash equilibrium of their coalitional game. However, notice that it is still possible that some of such CEDs are not proper: There may exist some CED p_{S} which do not belong to the set of **proper probability measures** over Θ_{S} $p_{\mathrm{S}} \notin \Delta^*(\Theta_{\mathrm{S}})$. Thus, as stated, claim (3) of theorem 2 ensures the existence of a PNCCE but does not allow to identify the (non-trivial) partition for which there exists a PNCCE. The characterization of the PNCCE can be obtained by imposing the additional requirement that every meta-player $\mathrm{S} \in \mathscr{C}(\mathrm{N})$ cannot be decomposed into a subset of meta-players $\mathscr{C}'(\mathrm{N}) \subsetneq \mathscr{C}(\mathrm{N})$ such that for all the *possible refinements* of the partition of the subset of players in S,$\mathscr{C}(\mathrm{S})$, the **rectangularity property** fails:

$$\mathtt{supp}(\widehat{p}_{\mathrm{S}}) \neq \bigtimes_{\mathrm{T}\in\mathscr{C}(\mathrm{S})} \text{ with } \widehat{\Theta}^*_{\mathrm{T}} \subset \mathrm{PNE}_{\mathrm{S}}(\widehat{p}_{-\mathrm{S}}), \widehat{\Theta}^*_{\mathrm{T}} \subset \mathtt{PNE}_{\mathrm{T}}(\widehat{p}_{-\mathrm{S}}) = \widehat{\mathrm{C}}^*_{\mathrm{T}}(\widehat{p}_{-\mathrm{S}}).$$

When this condition holds, one can*not* indeed find *any* refinement $\mathscr{C}'(\mathrm{N})$ of $\mathscr{C}(\mathrm{N})$ such that there is a properly mixed Nash equilibrium of $\Gamma^*_{\mathscr{C}}(\mathrm{N})$ forming the proper PNCCE of Γ since *every* component $\widehat{p}_\mathrm{S}$ of the properly mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(\mathrm{N})} = (\widehat{p}_\mathrm{S})$ (whose non-degenerate measure is asserted by the tight property of the partitioned game) cannot be decomposed into a product probability measure

$$\widehat{p}_{\mathscr{C}(\mathrm{S})} = \widehat{p}_\mathrm{S} = \otimes_{\mathrm{V}\in\mathscr{C}(\mathrm{S})}\widehat{p}_i, \forall \mathscr{C}(\mathrm{S}) \subsetneq \mathscr{C}^{**}(\mathrm{S})$$

and it follows that the partition $\mathscr{C}(\mathrm{N})$ must necessarily be the one *inducing the PNCCE* $\widehat{p}_{\mathscr{C}(\mathrm{N})} = (\widehat{p}_\mathrm{S})$ of Γ.

We finally note that in the class of games identified by Theorem 2, PNCCEs are no longer guaranteed when each meta-player is made-up of a subset of players equal to two. This claim follows directly from the observation that in bi-matrix tight games, Nash equilibria are necessarily exchangeable. [10]

Class of games where the Nash coalitional correlated equilibria intersect the coalitional equilibria of Ray and Vohra

We are now in a position to give some (tight) sufficient conditions that single out a class of games where NCCEs and coalitional equilibria coincide. In their definition of a coalitional equilibrium , Ray and Vohra, define the join best response of a coalition $\mathrm{S} \in \mathscr{C}(\mathrm{N})$ as a subset of undominated (or not strictly dominated) Pareto action profiles

$$\beta_\mathrm{S}(p_{-\mathrm{S}}) = \{\theta_\mathrm{S} \in \Theta_\mathrm{S} | \nexists \hat{\theta}_\mathrm{S}, \mathrm{U}_\mathrm{S}(\hat{\theta}_\mathrm{S}, p_{-\mathrm{S}}) >> \mathrm{U}_\mathrm{S}(\theta_\mathrm{S}, p_{-\mathrm{S}})\}.$$

where $\mathrm{U}_\mathrm{S}(\hat{\theta}_\mathrm{S}, p_{-\mathrm{S}}) = (\mathrm{U}_i(\hat{\theta}_\mathrm{S}, p_{-\mathrm{S}}))_{i\in\mathrm{S}} \in \mathbb{R}^{|\mathrm{S}|}$ denotes the payoff vector of coalition S when they play profile $\hat{\theta}_\mathrm{S}$ in coalitional game $\Gamma_\mathrm{S}(p_{-\mathrm{S}})$. On the other hand, the joint best response in every $\mathscr{C}(\mathrm{N})$-NCCE $p = (p_\mathrm{S}, p_{-\mathrm{S}})$ of a game Γ must be robust to the set of unilateral deviations of players inside each coalition $\mathrm{S} \in \mathscr{C}(\mathrm{N})$ in the induced $p_{-\mathrm{S}}$-coalitional game $\Gamma_\mathrm{S}(p_{-\mathrm{S}})$. The proof below find the conditions to obtain a non-empty convex and compact intersection between each coalition's best responses $\beta_\mathrm{S}(p_{-\mathrm{S}})$ (relative to a $\mathscr{C}(\mathrm{N})$-coalition structure) in the construction of the Ray and Vohra's *coalitional equilibrium*, and the best response subset of correlated equilibrium distributions $\mathrm{BR}_\mathrm{S}(p_{-\mathrm{S}})$ from which the $\mathscr{C}(\mathrm{N})$- NCCEs of Γ are defined. To do so will require to consider the counterpart of the set $\Theta^{\mathrm{IESDA}}_\mathrm{S}(p_{-\mathrm{S}})$ by defining the set of actions that survive the *iterated elimination of strictly dominated Pareto actions* (IESDPA) in subgame the $p_{-\mathrm{S}}$-coalitional subgame $\Gamma_\mathrm{S}(p_{-\mathrm{S}})$ played by the players in coalition $\mathrm{S} \in \mathscr{C}(\mathrm{N})$. When the sets $\Theta^{\mathrm{IESDA}}_\mathrm{S}(p_{-\mathrm{S}}) = \Theta^{\mathrm{IESDPA}}_\mathrm{S}(p_{-\mathrm{S}})$ for each coalition $\mathrm{S} \in \mathscr{C}(\mathrm{N})$, we then obtain the identity between the class of Ray and Vohra's EBAs and the extension of Aumann correlated equilibrium to multiple random devices across a disjoint subsets of players.
The Theorem below is stated for identifying the class of finite games where *all* the $\mathscr{C}(\mathrm{N})$- coalitional equilibria of a game Γ coincide with *all* the $\mathscr{C}(\mathrm{N})$-NCCEs of the game.

Theorem 3. *Consider a* N*-player game* $\Gamma = \langle \mathrm{N}, (\Theta_i, \mathrm{U}_i)_{i\in\mathrm{N}}\rangle$ *which satisfies the properties of Theorem 1 or 2 with finite action spaces* $\Theta_i, i = 1,.., n$ *and continuous payoff functions* $\mathrm{U}_i, i = 1,..., n$. *Fix any non-trivial coalition structure* $\mathscr{C}(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N}), \mathscr{C}^{**}(\mathrm{N})$. *Assume there exists a non-empty set of (non-trivial)* $\mathscr{C}(\mathrm{N})$*-NCCE* $p = (p_\mathrm{S}, p_{-\mathrm{S}})$ *of* Γ *(as e.g. per Theorem 1 or 2). Then, a sufficient condition for having this set of NCCEs to coincide with all the* $\mathscr{C}(\mathrm{N})$*-coalitional equilibria of* Γ *is that the set of* $\mathscr{C}(\mathrm{N})$*- iterated elimination of strictly dominated Pareto actions in* Γ *is (weakly) contained in the set of actions that survive the* $\mathscr{C}(\mathrm{N})$*- iterated elimination of strictly dominated actions i.e.,* $\Theta^{\mathrm{IESDA}}_{\mathscr{C}(\mathrm{N})}(p) \subseteq \Theta^{\mathrm{IESDPA}}_{\mathscr{C}(\mathrm{N})}(p)$ *in* Γ.
One can obviously weaken the requirement of a weakly inclusion of the $\mathscr{C}(\mathrm{N})$- Pareto undominated actions being contained into the $\mathscr{C}(\mathrm{N})$-IESDA set and still obtain for certain games that some of the $\mathscr{C}(\mathrm{N})$- coalitional equilibria form some of the $\mathscr{C}(\mathrm{N})$-NCCEs. Also notice that the result does *not* make any claim about the (im)possibility of games where *all* the $\mathscr{C}(\mathrm{N})$- coalitional equilibria coincide with the $\mathscr{C}(\mathrm{N})$-NCCEs. There may exist some games where this may happen even if the inclusion property fails. However, the theorem is tight in the sense that we are not guaranteed that some of the $\mathscr{C}(\mathrm{N})$- coalitional equilibria will indeed all form a $\mathscr{C}(\mathrm{N})$-NCCE (and conversely) for arbitrary games where the inclusion property does not hold. The (quasi-) concavity of the vectorial payoff functions $\mathrm{U}_\mathrm{S}(\cdot; p_{-\mathrm{S}}) = (\mathrm{U}_i(\cdot; p_{-\mathrm{S}})_{i\in\mathrm{S}} \in \mathbb{R}^{|\mathrm{S}|}$ automatically holds since every subset of players S is allowed to use correlated strategies over Θ_S (see Haeringer, 2004).[12]
Proof. See Appendix D.

[10] As discussed in Viossat (2003), this property however fails for tight games with more than two players.[11].

[12] This property in turns guarantees that the coalitions' best response correspondences are convex-valued.

Remark 1: For finite games, Moreno and Wooders (1996) show that if the collection of correlated strategies with support in the IESDA set has a Pareto-best element i.e., one that simultaneously maximizes the payoff of every player over that set of correlated strategies, then that strategy is a coalition-proof correlated equilibrium. Hence, an immediate corollary of Theorem 3 is that if $\Theta^{\mathrm{IESDA}}_{\mathscr{C}(\mathrm{N})}(p) \subseteq \Theta^{\mathrm{IESDPA}}_{\mathscr{C}(\mathrm{N})}(p)$ with p a (non-trivial) $\mathscr{C}(\mathrm{N})$-NCCE $p = (p_{\mathrm{S}}, p_{-\mathrm{S}})$ where each p_{S} is a correlated equilibrium distribution containing a Pareto-best element, then the coalitional equilibrium p is a NCCE with the additional property that p is made-up of *coalition-proof correlated equilibria* (one for each coalition of players).

Remark 2: It is well-known that infinite games may not possess undominated strategies for some players or that some strategies are dominated only by other dominated strategies. Hence, the additional qualifications of payoff continuity and compact space of actions is necessary to obtain the counterpart formulation of Theorem 0 for the class of infinite games (see e.g., Milgrom and Roberts, 1996).

5. Concluding Remarks

In this paper we have pointed out to a natural extension of Aumann notion of correlated equilibrium. In this extension, different disjoint coalitions of players are characterized by some independent sources generating the random private signals observed by each player of a coalition. Such a tuple of 'private roulettes' across the coalitions induces each player to play in a correlated equilibrium relative to the subgame played by the players of his coalition, given the correlated strategies used by the other coalitions. Using different tools (potential techniques and results on various properties of correlated equilibria), we have identified some games where these profiles of correlated equilibrium strategies are representing the mixed Nash equilibria of an induced non cooperative game played by the coalitions themselves. The bulk of the paper has identified the class of games where the resulting mixed Nash equilibria played by the coalitions (or the 'mediators') are neither the regular mixed Nash equilibria nor correlated equilibria of the original game, but coincide with some (of the non-trivial) *coalitional equilibria* of Ray and Vohra (1997).

Our main results show that the equilibrium played between the coalitions in Ray and Vohra can be seen as a generalization of the class of Aumann correlated equilibria when (disjoint) coalitions of players have access to different correlation devices (one correlation device per coalition). This generalization of the Aumann's original correlated equilibrium notion is not innocuous. It notably implies that the building block of Ray and Vohra's EBA is for certain classes of games characterized by the correlated equilibria of the game played within each coalition of players. Hence, when they exist, the pure strategies Nash equilibria of the game played by the players inside the coalition must be part of the pure best responses of the coalition. However, every time the game played inside a (non-singleton) coalition has a pure strategy Nash equilibrium profile that is Pareto dominated by another strategy profile, the original definition of a best response in Ray and Vorha (1997) exclude these strategies i.e., the pure strategies Nash equilibrium (hence degenerate correlated equilibria of the game played inside a coalition of players) cannot belong to the set of the joint best response set of the coalition.[13]

Our analysis opens several questions. Even in games where the set of CE of a coalition is larger than the set of NE and some CE outcomes may strictly improve upon the NE outcome for a coalition (given the correlated play of the other coalitions), the efficient outcome maximizing the total welfare for the coalition may not be attainable by the class of Aumann correlated strategies. Hence, while we have identified a class of games where the set of NCCEs and the set of Ray and Vohra's 'coalitional equilibria' (Ray and Vohra, 1997) coincide, there may have a class of games wherein the coalitions need to use some extended class of correlated strategies which might involve other forms of 'binding agreements' than an agreed correlation device as la Aumann. This will typically happen if the players of a coalition are playing a game of the class of the Prisoners' Dilemma game (given the correlated strategies of the other coalitions). In this class of games players may need to resort to an extended class of correlated strategies like e.g. soft correlated equilibrium (SCE) of Forgó (2005, 2010). Hence, the forms of correlated strategies–via 'binding agreements'– that will need to be used by the players in a EBA depend on the class of games this concept is being applied to. Other induced games played inside the coalitions in a ABE may be compatible with the use of another form of binding agreements like the class of "coarse correlated equilibrium" (CCE) strategies introduced by Moulin and Vial (1978) in order improve upon a completely mixed NE (see Ray and Gupta, 2013 and Moulin et

[13]As an example, take a 3 -player game where the best replies of a third player induces a prisoner dilemma for the two players forming a coalition. While the coalition must use some correlated strategies for their mixed strategies in general, the (Pareto dominated) pure Nash strategy profile of the game played by the coalition is not part of the pure best response set of the coalition in this case.

al, 2014). This might notably be the case if the game played by a coalition belongs to the class of strategically zero-sum games (where CEs cannot improve upon NE). Finally, we have only started to explore some of the ramification between the NCCEs of a game and the various refinement solution concepts of the correlated equilibrium such as strong correlated equilibria coalition-proof correlated equilibria with and without communication (see Moreno and Wooders, 1996 and Milgrom and Roberts, 1996). We leave this to some future research.

Appendix

Appendix B.2

Proposition 2 *A nice partitioned game* $\Gamma^*_{\mathscr{C}(\mathrm{N})}$ *is played by a meta-player* $\mathrm{S} \in \mathscr{C}(\mathrm{N})$ *if and only if the set* $\mathit{PNE}_{\mathrm{S}}((p_{-\mathrm{S}}))$ *of PSNEs of* S *fails to have the* ***rectangularity property*** *in some coalitional game* $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$.

Proof. Suppose

$$\mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}}) \neq \bigtimes_{i \in \mathrm{S}} \mathtt{BR}_i(p_{-\mathrm{S}}).$$

Then, this implies that

$$\mathtt{C}^*(p_{-\mathrm{S}}) \neq \bigtimes_{i \in \mathrm{S}} \mathtt{C}^*_i(p_{-\mathrm{S}})$$

where

$$\mathtt{C}^*_i(p_{-\mathrm{S}}) := \left\{\theta_i \in \Theta_i : \exists \theta_{\mathrm{S}\setminus i} \in \Theta_{\mathrm{S}\setminus i}, (\theta_i, \theta_{\mathrm{S}\setminus i}) \in \mathtt{C}^*(p_{-\mathrm{S}})\right\}.$$

This proves that S is a meta-player *only if* there is a failure of the rectangularity property for a subset of players S in the coalitional game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$. The converse direction is established symmetrically.

□

Appendix B.3

Proposition 3 *Consider a finite game* Γ *which admits a partitioned game* $\Gamma_{\mathscr{C}(\mathrm{N})}$ *for a non-trivial partition* $\mathscr{C}(\mathrm{N})$. *Suppose the partitioned game,* $\Gamma_{\mathscr{C}(\mathrm{N})}$ *has at least one subset of players* S *that forms a meta-player and the existence of a Nash equilibrium in the relative interior of its correlated equilibrium polytope,* $\mathrm{CE}(\Gamma_{\mathscr{C}(\mathrm{N})})$. *Then* Γ *has at least one proper* $\mathscr{C}'(\mathrm{N})$*-NCCE (not necessarily properly mixed) for a (non-trivial) partition,* $\mathscr{C}'(\mathrm{N}) \neq \mathscr{C}^*(\mathrm{N})$, *that is a refinement of* $\mathscr{C}(\mathrm{N})$.

Proof. Fix a non-trivial partition $\mathscr{C}(\mathrm{N})$. When the finite partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ has a mixed Nash equilibrium, $p_{\mathscr{C}(\mathrm{N})} = (p_{\mathrm{S}})$, the set of best reply profiles of each S is non-empty and must satisfy the indifference condition at the level of groups, as in any regular mixed Nash equilibrium. Hence, in a mixed Nash equilibrium, $p_{\mathscr{C}(\mathrm{N})}$, we must have

$$\mathtt{supp}(p_{\mathrm{S}}) \subseteq \ \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}}), \forall \mathrm{S} \in \mathscr{C}(\mathrm{N}),$$

where as remarked in the main text, each profile θ_{S} in $\mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}})$ is a jointly coherent strategy for S in the game $\Gamma_{\mathrm{S}}(p_{-\mathrm{S}})$. Note that the equality holds whenever $p_{\mathscr{C}(\mathrm{N})}$ is a quasi-strict mixed Nash equilibrium of the partitioned game. Now, let

$$\mathtt{NE}_{\mathrm{S}}(\Theta_{-\mathrm{S}}) := \left\{ p_{-\mathrm{S}} \in \bigtimes_{\mathrm{T} \in \mathscr{C}(\mathrm{N}):\mathrm{T} \neq \mathrm{S}} \Delta(\Theta_{\mathrm{T}}) : \exists p_{\mathscr{C}(\mathrm{N})} = (p_{\mathrm{S}}, p_{-\mathrm{S}}) \in \mathrm{NE}(\Gamma_{\mathscr{C}(\mathrm{N})}) \right\}$$

denote the set of CEDs profiles in $\Delta_{\mathrm{S}}(\Theta_{\mathscr{C}(\mathrm{N})}) := \bigtimes_{\mathrm{T} \in \mathscr{C}(\mathrm{N}):\mathrm{T} \neq \mathrm{S}} \Delta(\Theta_{\mathrm{T}})$ which correspond to the set of Nash equilibrium components for $-\mathrm{S}$ in the *nice partitioned game* $\Gamma_{\mathscr{C}(\mathrm{N})}$. We then obtain the set of all the best response strategy profiles for the subset of players S, denoted $\mathtt{BR}_{\mathrm{S}}(\mathtt{NE}_{\mathrm{S}}(\Theta_{-\mathrm{S}}))$, (in the collection coalitional games $\Gamma_{\mathrm{S}}(\cdot)$) which are in the supports of some mixed Nash equilibria $p_{\mathscr{C}(\mathrm{N})} = (p_{\mathrm{S}}, p_{-\mathrm{S}})$ of $\Gamma_{\mathscr{C}(\mathrm{N})}$. This set writes as:

$$\mathtt{BR}_{\mathrm{S}}(\mathtt{NE}_{\mathrm{S}}(\Theta_{-\mathrm{S}})) = \bigcup_{p_{-\mathrm{S}} \in \mathtt{NE}_{\mathrm{S}}(\Theta_{-\mathrm{S}})} \mathtt{BR}_{\mathrm{S}}(p_{-\mathrm{S}}).$$

Recall that the set of **jointly coherent strategies**, (denoted $\mathrm{C}^*(\mathrm{S})$ in the main text), for S in the *game* Γ is the set

$$\mathrm{C}^*(\mathrm{S}) := \left\{\theta_{\mathrm{S}} : \exists \mu_{\mathscr{C}(\mathrm{N})} = (p_{\mathrm{S}}) \in \mathtt{NE}(\Gamma_{\mathscr{C}(\mathrm{N})}), \mathtt{supp}(p_{\mathrm{S}}) \ni \theta_{\mathrm{S}}\right\}.$$

From the above, it thus follows–using the definitions given in the main text–that the set of coherent strategies for players in subset S is the set $C^*(S)$ which contains the non-empty subsets of all the best replies of players S i.e.,

$$\mathtt{BR}_S(\mathtt{NE}_S(\Theta_{-S})) = C^*(S). \tag{5.1}$$

Notice the implication of Eq.(1) for the subset of players S to be a meta-player. Indeed, when Eq. (1) holds, the set of jointly coherent coherent strategy profiles $C^*(S)$ of the subset of players S fails to have the **rectangular property** if and only if S is not a meta-player:

$$\mathtt{BR}_S(\mathtt{NE}_S(\Theta_{-S})) \neq \bigtimes_{i\in S} \mathtt{BR}_i(\mathtt{NE}_S(\Theta_{-S})) \subseteq C^*(S) \Longleftrightarrow C^*(S) \neq \bigtimes_{i\in S} C_i^*(S), \tag{5.2}$$

where $C_i^*(S)$ is the set:

$$C^*(i) := \left\{\theta_i : \exists \mathscr{C}(N), \exists \mu_{\mathscr{C}(N)} = \otimes_{S\in\mathscr{C}(N)} p_S \in \mathtt{CE}(\Gamma), \mathtt{supp}(p_S^i) \ni \theta_i\right\},$$

of **coherent** strategies for i in the *game* Γ with $p_S^i := \mathtt{marg}_{\Theta_i} p_S$ the marginal probability of i under the CED p_S. In addition, as mentioned in Remark 1, when S is a meta-player, then the correlated equilibrium polytope of $\Gamma_{\mathscr{C}(N)}$ can*not* be made-up of a singleton i.e.,

$$\mathtt{BR}_{\mathscr{C}(N)}(\mathtt{NE}_{\mathscr{C}(N)}) \neq \bigtimes_{S\in\mathscr{C}(N)} \mathtt{BR}_S(\mathtt{NE}_S(\Theta_{-S})) = \{\theta_S\}$$

for all $\mathscr{C}(N)$.

The claim of the proposition is finally obtained by applying the next result of Nau et al. (2004) to the partitioned game.

Nau et al. (2004) [proposition 2] *If there is a Nash equilibrium in the relative interior of the correlated equilibrium polytope, then the Nash equilibrium assigns positive probability to every coherent strategy of every player.*

It follows that if there is a Nash equilibrium in the relative interior of the *non-singleton* correlated equilibrium polytope of the *partitioned game*, $\mathtt{CE}(\Gamma_{\mathscr{C}(N)})$, then, $\Gamma_{\mathscr{C}(N)}$, has a (not necessarily properly mixed) Nash equilibrium, $p_{\mathscr{C}(N)} = (p_S)$ whose component are the correlated equilibrium distributions, p_S, which assign positive probabilities to *every coherent strategy* in $C^*(S)$ i.e., $p_S(C^*(S)) = 1$. From the above it thus follows that $p_{\mathscr{C}(N)}$ cannot be a regular mixed Nash equilibrium of Γ since there must exist at least one subset of players S in the partitioned game $\Gamma_{\mathscr{C}(N)}$ with *at least two coherent strategy profiles,* which corresponds to the set of pure best replies of meta-player S (otherwise the polytope of $\Gamma_{\mathscr{C}(N)}$ would be a singleton and hence violates the property that S is a meta-player). The correlated equilibrium distribution p_S for S is thus a non-degenerate mixture of the coherent strategy profiles $C^*(S)$. The existence of a non-degenerate CED p_S for at least one S of the non-trivial partition $\mathscr{C}(N)$, implies that there is indifference condition for every pair of coherent profiles $\theta_S \in \mathtt{supp}(p_S) = C^*(S)$. Finally, the resulting CED p_S cannot be a regular mixed Nash equilibrium of $\Gamma_S(p_{-S})$. To this, it suffices to note that when the set of jointly coherent strategy profiles $C^*(S)$ of group S fails to have the **rectangularity property**, i.e.,

$$p_{\mathscr{C}(N)} = (p_S) \in \mathrm{NE}(\Gamma_{\mathscr{C}}(N)), \mathtt{supp}(p_S) = C^*(S) \text{ s.t. } \mathtt{supp}(p_S) \neq \bigtimes_{i\in S} \mathtt{supp}(p_S^i),$$

then this implies that the support of p_S can*not* be written as a Cartesian product and hence it follows that

$$p_S = \sum_{\theta_S\in C^*(S)} p_S(\theta_S)\delta_{\theta_S}$$

can*not* induce a *finite product measure* $p_S = \otimes_{i\in S} p_S^i$. It thus follows that p_S can*not* form a regular mixed Nash equilibrium of $\Gamma_S(p_{-S})$. From the above series of observations, we have therefore obtained the existence of a mixed Nash equilibrium $p_{\mathscr{C}(N)} = (p_S)$ of the partitioned game whose at least *one component* p_S cannot be a regular mixed Nash equilibrium of $\Gamma_S(p_{-S})$. This implies that the resulting non-degenerate PCE $p_{\mathscr{C}(N)} = (p_S)$ of Γ is a *proper* PCE, which does not form a regular mixed Nash equilibrium of Γ and such that there only exists a possible non-trivial refinement $\mathscr{C}'(N) \subseteq \mathscr{C}(N)$ that makes $p_{\mathscr{C}'(N)}$ a proper-PCE.

□

Appendix B.4

Proposition 4 *Consider a finite game* Γ *which admits a nice partitioned game* $\Gamma^*_{\mathscr{C}(N)}$ *for a non-trivial partition* $\mathscr{C}(N)$. *If the partitioned game* $\Gamma^*_{\mathscr{C}(N)}$ *is a tight game with at least one meta-player* S, *then* Γ *has at least one* $\mathscr{C}'(N)$*-PPCE with* $\mathscr{C}'(N) \neq \mathscr{C}^*(N)$ *a partition that is a (possibly weak) refinement of* $\mathscr{C}(N)$.
Proof. We start with the following lemmas.

Recall that a mixed Nash equilibrium p of a game Γ is completely mixed (or interior) if it assigns positive probabilities to all the player's pure strategies i.e., there is *full support*, $\mathbf{supp}(p) = \Theta_i$ for all i.

Viossat (2003, 2010): *Every finite tight game has a completely mixed Nash equilibrium.*
Proof. See Viossat Proposition 4 (2010).

□

From this result, we deduce the following.

Lemma 5.3. *Take any finite game* Γ. *If there exists a partitioned game* $\Gamma_{\mathscr{C}(N)}$ *for a non-trivial partition and if* $\Gamma^*_{\mathscr{C}(N)}$ *is a tight game, then it has a completely mixed Nash equilibrium* $p_{\mathscr{C}(N)} = (p_S)_{S \in \mathscr{C}(N)}$ *which necessarily induces the existence of a* $\mathscr{C}'(N)$*- PPCE of* Γ *(possibly, for a refinement* $\mathscr{C}'(N) \neq \mathscr{C}^*(N)$ *of* $\mathscr{C}(N)$*) if there is at least one meta-player* S *playing in* $\Gamma_S(p_{-S})$.

Proof. The existence of a completely mixed Nash equilibrium follows directly from the application of the following result of Viossat (2003, 2010) to the partitioned game.
We first note that when the partitioned game has a completely mixed Nash equilibrium (which is the case if it is a tight game), then the resulting tuple of correlated distributions $p_{\mathscr{C}(N)}$ necessarily induces the existence of a proper PCE in Γ. For a proper PCE to exist, we must by definition check that there exists a $\mathscr{C}'(N)$-PCE for a *non-trivial partition.* Observe that the claim is *not* that $p_{\mathscr{C}(N)}$ is necessarily the proper PCE, but that there at least exists a (non-trivial) refinement $\mathscr{C}'(N) \subseteq \mathscr{C}(N)$ that makes $p_{\mathscr{C}'(N)} = (p_T : T \in \mathscr{C}'(N))$ a proper PCE. When $p_{\mathscr{C}(N)}$ is completely mixed, this means that each component p_S of the mixed Nash equilibrium of $\Gamma^*_{\mathscr{C}(N)}$ is necessarily a *non-degenerate* CED. Since this tuple of CEDs forms a Nash equilibrium of the partitioned game, it follows immediately that the indifference condition holds for every S. Moreover, as noted in the previous proof of Proposition 3, when S is a meta player in game $\Gamma_S(p_{-S})$, then the distribution p_S cannot form a regular mixed Nash equilibrium of $\Gamma_S(p_{-S})$ since p_S cannot be written as a product measure, $p_S = \otimes_{i \in S} p^i_S$, and hence cannot form a regular mixed Nash equilibrium in $\Gamma_S(p_{-S})$.[14] From this, it follows that there cannot exist the trivial refinement $\mathscr{C}^*(N)$ that would make the Nash equilibrium $p_{\mathscr{C}^*(N)} = p_{\mathscr{C}(N)} = (p_S)$ of $\Gamma_{\mathscr{C}(N)}$ a regular mixed Nash equilibrium of Γ. Hence, since any possible existing refinement $\mathscr{C}'(N)$ is necessarily non-trivial, this proves that there exists a non-trivial partition $\mathscr{C}'(N) \neq \mathscr{C}^*(N)$ $p_{\mathscr{C}'(N)} = (p_S)$ which forms a proper PCE $p_{\mathscr{C}'(N)}$ of Γ. The fact that it is also completely mixed for each $T \neq S$ of $\mathscr{C}(N)$ shows that the resulting CED must indeed induce a PPCE in Γ.

□

APPENDIX C: PROOF THEOREM 2

Theorem 2 *Consider a* N*-player strategic game* $\Gamma = \langle N, (\Theta_i, U_i)_{i \in N} \rangle$. *Fix a non-trivial partition* $\mathscr{C}(N)$. *If* Γ *has compact and convex strategy spaces and bounded and continuous payoffs with a partitioned* C^1*-concave* $\mathscr{C}(N)$*-exact potential function. Then the following two equivalent properties guarantee the existence of a proper PNCCE* $\widehat{p}_{\mathscr{C}'(N)} = (\widehat{p}_S)$ *in* Γ*:*

(1) *There exists a* $\mathscr{C}(N)$*-nice partitioned game with at least one meta-player* S *and a finitely supported mixed Nash equilibrium* $\widehat{p}_{\mathscr{C}'(N)} = (\widehat{p}_S)$ *for* $\mathscr{C}'(N) \neq \mathscr{C}^*(N)$ *a (possibly weak) refinement of* $\mathscr{C}(N)$;

(2) *There exists a span of the nice partitioned game,*

$$\widehat{\Gamma}_{\mathscr{C}(N)} = \left\langle \widehat{\Theta}^*_S, \widehat{U}_S \right\rangle,$$

[14] Again, this follows because the condition to have a meta-player S is equivalent to requiring that one component p_S of the mixed Nash equilibrium $p_{\mathscr{C}(N)} = (p_S)$ of $\Gamma_{\mathscr{C}(N)}$, has the property that: $\mathtt{supp}(p_S) \neq \bigtimes_{i \in S} \mathtt{supp}(p^i_S)$ where $p^i_S = \mathtt{marg}_{\Theta_i} p_S$ denotes the marginal probability distribution of i in S.

played by at least one meta-player S *i.e.,* $\widehat{\Theta}_S^* \neq \bigtimes_{i\in S} \widehat{\Theta}_i^*$ *where* $\widehat{\Theta}_i^* \subset BR_i(\widehat{p}_{-S}), i \in S$ *and* $\widehat{\Gamma}_{\mathscr{C}(N)}$ *is a tight game with* $\widehat{\Theta}_S^* \subset \mathtt{BR}_S(\widehat{p}_{-S})$;

(3) *When in addition of (2), each player* T *in* $\mathscr{C}(N)$ *is a meta-player, then there exists a PPCE* $\widehat{p}_{\mathscr{C}'(N)} = (\widehat{p}_S)$ *wherein each component* p_S *is a proper (joint) probability measure in some space,* $\Delta^*(\widehat{\Theta}^*_{\mathscr{C}(S)}) = \bigtimes_{V\in\mathscr{C}(S)} \Delta^*(\widehat{\Theta}_S^*)$, *with* $\mathscr{C}(S) \neq \mathscr{C}^*(S)$.

Proof.
The proof of the theorem relies on the following series of lemmas:

Say that a nice partitioned game

$$\widehat{\Gamma}^*_{\mathscr{C}(N)} = \langle \widehat{\Theta}_S^*, \widehat{U}_S \rangle.$$

is **non-trivial** whenever the set of pure strategies is a non-singleton set for each player i.e., $|\widehat{\Theta}_S^*| \geqslant 2$ for every S.

Lemma 5.4. *The partitioned potential game has a properly mixed Nash equilibrium* $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ *wherein each player randomizes over its set of PSNEs of* $\Gamma_S(\widehat{p}_{-S})$ *if and only if there exists a span of the partitioned game that is a non-trivial tight game.*

Proof. From proposition 5, recall that Neyman theorem (Neyman, 1997) implies that the set of jointly coherent strategies for S $C^*(S)$ that are played into a Nash equilibrium of the nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$ are necessarily a set of PSNEs and every CED p_S is a mixture of these PSNEs. Moreover, as argued in proposition 5, with the use of Carathéodory theorem (see Aliprantis and Border, 2003), to each profile (p_S, p_{-S}), forming a mixed Nash equilibrium of the partitioned game, one can associate a *finite* subset of PSNEs, $\widehat{\Theta}_S^* \subset \mathtt{PNE}_S(p_{-S})$, which allows to write every correlated equilibrium distribution p_S as a *finitely supported probability distribution* of at most $m_S + 1$ points which forms a CED of game $\Gamma_S(p_{-S})$.[15] The CED p_{-S} is contained into the convex hull of maximizers of the potential of game $\Gamma_S(p_{-S})$:

$$\widehat{\Theta}_S^* \subset \mathrm{PNE}_S(p_{-S}) = \arg\max_{\theta_S} \mathrm{P}_S(\theta_S; p_{-S}).$$

The existence of a *finitely supported probability distribution* of at most $m_S + 1$ points which forms a CED of game $\Gamma_S(p_{-S})$ for every subset of players S is ensured by the existence of a finite tight game which defines a span of the partitioned game. To see this, consider a non-trivial tight span game of the partitioned game defined by,

$$\widehat{\Gamma}^*_{\mathscr{C}(N)} = \langle \widehat{\Theta}_S^*, \widehat{U}_S \rangle.$$

This game is finite because every $\widehat{\Theta}_S^*$ is a finite subset of the $|\widehat{\Theta}_S^*| \geqslant 2$ PSNEs of players S in $\Gamma_S(p_{-S})$. When $\widehat{\Gamma}^*_{\mathscr{C}(N)}$ is a tight game, the use of Viossat (2010) applied to the span of the game shows that this is equivalent to the existence of a totally mixed Nash equilibrium in the span $\widehat{\Gamma}^*_{\mathscr{C}(N)}$ of the nice partitioned game. Hence, the tightness of $\widehat{\Gamma}^*_{\mathscr{C}(N)}$ implies the existence of a finitely supported mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ in the partitioned game $\Gamma_{\mathscr{C}(N)}$. □

Lemma 5.5. *The nice partitioned game* $\Gamma^*_{\mathscr{C}(N)}$ *has a properly mixed Nash equilibrium* $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ *if and only if the probability distribution* $\widehat{\mu}_{\mathscr{C}(N)} = \otimes_{S\in\mathscr{C}(N)} \widehat{p}_S$ *induced by* $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ *is a non-degenerate CED of game* Γ.

Proof. From Lemma 2, for every fixed profile of correlated equilibrium distributions, $\widehat{p}_{-S}$ we have by construction of the span of the partitioned game that the correlated equilibrium polytope (parametrized by $\widehat{p}_{-S}$) of the span of the partitioned game, equals the correlated equilibrium polytope of the smooth partition potential game Γ_S i.e.,

$$\mathtt{conv}(\widehat{\Theta}_S^*) \subseteq \mathtt{CE}(\widehat{\Gamma}_S(\widehat{p}_{-S})) = \mathtt{CE}(\Gamma_S(\widehat{p}_{-S})) \subseteq \Delta(\widehat{\Theta}_S).$$

This is true for every subset of players S in $\mathscr{C}(N)$. Hence, if there exists a mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ which is properly mixed into the correlated equilibrium polytope of the nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$, we have that the profile $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ induces a correlated equilibrium distribution $\widehat{\mu}_{\mathscr{C}(N)} = \otimes_{S\in\mathscr{C}(N)} \widehat{p}_S$, which lies into

$$\mathtt{CE}(\Gamma_{\mathscr{C}(N)}) \subseteq \Delta(\widehat{\Theta}_{\mathscr{C}(N)}).$$

[15] The existence of the finite set of pure strategy profiles $\widehat{\Theta}_S^*$ will in general be dependent onto the mixed Nash equilibrium $(\widehat{p}_S)$ under consideration: What Caratheorodory theorem states is just that for each distribution, there exists a finite set with at most $d+1$ points, not that there exists the same set for every distribution.

□

Lemma 5.6. *When there exists a (non-trivial) span of the nice partitioned game that is tight and there is also a meta-player* S *playing in this game , then there exists a properly mixed Nash equilibrium in the polytope of the nice partitioned game* $\Gamma^*_{\mathscr{C}(N)}$ *which does not form a regular mixed Nash equilibrium of* Γ *i.e.,*

$$\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S), \textit{ such that } \widehat{p}_S \neq \otimes_{i\in S} p_i$$

for at least one S *in* $\mathscr{C}(N)$.

Proof. We first apply the property of tight games of Nitzan (2005) to the span of the partitioned game to ensure the existence of a non-degenerate mixed Nash equilibrium in the partitioned game. If the span of the partitioned game is tight, then it has a totally mixed Nash equilibrium $\widehat{p}^*_{\mathscr{C}(N)} = (\widehat{p}_S)$ where each $\widehat{p}_S$ is a correlated equilibrium distribution of $\widehat{\Gamma}_S(\widehat{p}_{-S})$ with a support into a subset of the PSNEs of the original partitioned game $\Gamma_{\mathscr{C}(N)}$ i.e.,

$$\texttt{supp}(\widehat{p}_S) = \widehat{\Theta}^*_S \subset \text{PNE}_S(\widehat{p}_{-S})$$

When the span of the partitioned game fails to have the **rectangularity property** this means that there exists a subset $\widehat{\Theta}^*_S$ of PSNEs of players S in $\Gamma_S(p_{-S})$ which cannot be written as a Cartesian product of $\widehat{\Theta}^*_i, i \in S$. On the other hand, proving that the corresponding tuple of probability distributions $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ forms a *proper* PCE of Γ is equivalent to showing that there is a probability measure $\widehat{p}_S$ of a coalition S which forms a correlated equilibrium distribution of $\Gamma_S(\widehat{p}_{-S})$ that is *not* a product probability measure in the polytope of $\Gamma_S(\widehat{p}_{-S})$. So consider the nice $\mathscr{C}(N)-$ partitioned game

$$\widehat{\Gamma}^*_{\mathscr{C}(N)} = \left\langle \widehat{\Theta}^*_S, \widehat{U}_S \right\rangle,$$

which is defined with a space of pure strategies for *at least one* subset of players S with the property that the set $\widehat{\Theta}^*_S \subset \texttt{BR}_S(p_{-S})$ fails the **rectangularity property** i.e., the set $\widehat{\Theta}^*_S$ cannot be written as the Cartesian product of the players' strategy spaces:

$$\widehat{\Theta}^*_S \neq \bigtimes_{i\in S} \widehat{\Theta}^*_i, \widehat{\Theta}^*_i \subset \texttt{BR}_i(p_{-S}),$$

then is cannot be that a probability distribution $\widehat{P}_S$ with full support , *supp*$(\widehat{P}_S) = \widehat{\Theta}^*_S$, can be decomposed as a product probability measure i.e.,

$$supp(\widehat{p}_S) = \widehat{\Theta}^*_S \text{ s.t. } \widehat{\Theta}^*_S \neq \bigtimes_{i\in S} \widehat{\Theta}^*_i \Longrightarrow \widehat{p}_S \neq \otimes_{i\in S} \widehat{P}_i.$$

This follows immediately by noting that there exists a regular mixed Nash equilibrium $p_{\mathscr{C}*(S)} = (p_i : i \in S)$ in $\Gamma_S(p_{-S})$ that generates the CED p_S of coalition S in a properly mixed Nash equilibrium $p_{\mathscr{C}(N)} = (p_S)$ of the nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$, *only if* the CED p_S can be written as the product probability measure

$$p_{\mathscr{C}*(S)} = p_S = \otimes_{i\in S} p_i.$$

There is existence of such a product probability measure, $p_{\mathscr{C}*(S)} = p_S$, if and only if there is independence of the sigma-algebras $\{\mathscr{F}_i : i \in S\}$ (see e.g., Billingsley, 1995). In the partitioned game $\Gamma_{\mathscr{C}(N)}$ this property of independence can therefore be satisfied *only if* we have the **rectangularity property**:

$$\widehat{\Theta}^*_S = \bigtimes_{i\in S} \widehat{\Theta}^*_i \subset \texttt{PNE}_S(\Gamma_S(p_{-S})), \widehat{\Theta}^*_i \subset \texttt{PNE}_i(\Gamma_S(p_{-S})).$$

Lemma 4 allows to conclude that there necessarily exists a Nash equilibrium $p_{\mathscr{C}'(N)}$ of a nice partitioned game $\Gamma^*_{\mathscr{C}(N)}$ that is proper PNCCE for a possible refinement $\mathscr{C}'(N) \neq \mathscr{C}^*(N)$ of $\mathscr{C}(N)$ if the rectangularity property fails in a collection of games $\Gamma_S(p_{-S})$ for at least one coalition of players S.
From the above series of arguments, we conclude that there exists a span of the partitioned game with at least one space of pure strategies for a subset of players S that fails to have the **rectangularity property** in game $\Gamma_S(\widehat{p}_{-S})$, then the resulting correlated equilibrium distribution $\widehat{p}_S$ cannot form a regular mixed Nash equilibrium of $\Gamma_S(\widehat{p}_{-S})$. It is necessarily a joint probability measure that forms a (real) correlated equilibrium distribution of $\Gamma_S(\widehat{p}_{-S})$. So, if $\widehat{\Gamma}_S(\widehat{p}_{-S})$ fails the **rectangularity property**—as assumed above—, then there exists at least one player S for which

cannot be written as a trivial Cartesian product i.e., $\widehat{\Theta}^*_S \neq \bigtimes_{i\in S} \widehat{\Theta}^*_i$. From this it follows that there is at least one component of the mixed Nash equilibrium $\widehat{p}^*_{\mathscr{C}(N)} = (\widehat{p}_S)$ of the span of the partitioned game that induces a correlated probability distribution which cannot be a product probability measure: $\widehat{p}_S \neq \otimes_{i\in S}\widehat{p}_i$.

Moreover, by construction, using lemma 3, we have that the correlated equilibrium polytope of the span of the partitioned game is a subpolytope $\mathtt{CE}(\widehat{\Gamma}^*_{\mathscr{C}(N)})$ of the correlated equilibrium polytope of game $\Gamma^*_{\mathscr{C}(N)}$:

$$\mathtt{conv}(\widehat{\Theta}^*_{\mathscr{C}(N)}) = \mathrm{CE}(\widehat{\Gamma}_{\mathscr{C}(N)}) \subsetneq \mathtt{CE}(\Gamma^*_{\mathscr{C}(N)}) \subseteq \Delta(\Theta_S).$$

□

The following Remark completes the proof of the equivalence of claims (1) and (2). When the span of the partitioned game is tight and one of the player S is a meta-player, then Remark 1 implies that at least one CED $\widehat{p}_S$ has a non-degenerate support. When the span of the game is tight and only played by meta-players, then the span of the partitioned game is a non-trivial game. In this case, it has a totally mixed Nash equilibrium $\widehat{p}_{\mathscr{C}(N)} = (\widehat{p}_S)$ with full support. Hence, $\widehat{p}_{\mathscr{C}(N)}$ is a properly mixed Nash equilibrium whose components p_S are probability measures in $\Delta(\widehat{\Theta}^*_S)$ with a finite full support $\mathtt{supp}(\widehat{p}_S) = \widehat{\Theta}^*_S$. The proof of claim (3) of theorem 2 then follows.

Corollary 5.7. *When in addition of property (2) in Theorem 2, each player* T *in* $\mathscr{C}(N)$ *is a meta-player, then there exists a PPCE* $\widehat{p}_{\mathscr{C}'(N)} = (\widehat{p}_T)$ *wherein each component* p_S *is a proper (joint) probability measure in some space,* $\Delta^*(\widehat{\Theta}^*_{\mathscr{C}'(T)}) = \bigtimes_{V\in\mathscr{C}'(T)} \Delta^*(\widehat{\Theta}^*_T)$*, with* $\mathscr{C}'(T) \neq \mathscr{C}^*(T)$ *for all* T *in* $\mathscr{C}'(N)$.

Proof. It suffices to apply the above series of lemmas to every sub-coalition of players T in $\mathscr{C}(N)$.

□

APPENDIX D: PROOF THEOREM 3

Proof Theorem 3: Strictly dominated actions of a coalition under NCCEs. A pure joint action $\theta^i_S \in \Theta_i$ is strictly dominated for i in coalition S in $\Gamma_S(p_{-S})$ if player $i \in S$ can unilaterally deviate from his (deterministic) recommendation to play θ_i and play according to a distribution $\eta_i \star_i p^i_S(\cdot) = p_S(\cdot, \Theta_{S_{-i}})$ over Θ_i which is the marginal of a correlated equilibrium distribution $p_S \in \mathrm{CE}_S(p_{-S})$ (while the other players S_{-i} follow their recommendations and play $\theta_{S_{-i}}$ with probability one i.e., $\eta_{S_{-i}}(\theta_{S_{-i}})|\theta_{S_{-i}}) = 1$.

Definition 5.8. A player $i \in S \in \mathscr{C}(N)$ has a $\mathscr{C}(N)$-**strictly dominated action** $\theta^i_S \in \Theta_i$ in a game Γ if there exists a $p_i \in \Delta\Theta_i$, and a $p_{-S} \in \Delta\Theta_{-S}$ such that

$$U_i(p_i, \theta_{S_{-i}}; p_{-S}) \geqslant U_i(\theta_S = (\theta^i_S, \theta_{S_{-i}}); p_{-S}), \forall \theta_{S_{-i}} \in \times_{j\in S\setminus\{i\}} \Theta_j.$$

In other words, an action θ^i_S is a $\mathscr{C}(N)$-strictly dominated action for $i \in S$ if there exists a coalitional game $\Gamma_S(p_{-S})$ where θ^i_S is a strictly dominated action.

Definition 5.9. A pure action θ^i_S of player $i \in S \in \mathscr{C}(N)$ is $\mathscr{C}(N)$-**undominated** in Γ if there exists a conjecture profile $(\mu^i_S, p_{-S}) \in \Delta\Theta_{S_{-i}} \times \Delta\Theta_{-S}$ such that $\theta^i_S \in \mathrm{BR}_i(\mu^i_S, p_{-S})$.

Note that the above definition of a strictly dominated action for a player deviates from the classical standard definition in several ways. We have the following properties:
(1)The strictly dominating action $\hat{p}^i_S$ of the deviating player is the *marginal of a correlated strategy* $p_S \in \Delta\Theta_S$. Hence, note that the condition of the definition does *not* entail the possibility to retrieve the classical equivalence between the undominated set of actions $\Theta^*_{S_i}$ of i in the original game Γ and and the set of i's pure best replies $\mathrm{BR}_{S_i}(\cdot)$ in Γ. Instead, the above definition says that the set of strictly dominated actions of a player $i \in S$ might be larger than what it would be in the standard definition relative to the game. This follows because under the definition of strictly dominated actions for a player i in a *non-singleton* coalition S, the best reply of i is only relative to the subgames $\Gamma_S(p_{-S})$. Hence, this eliminates the possibility for i in a coalition S to regard some actions θ^i_S which are only the best replies (in the original game Γ) to some *correlated beliefs* $p_{-i} \in \Delta(\Theta_{-i})$ which cannot be written as a product distribution i.e., $p_{-i} \neq p_{-S} \times p^{-i}_S \in \Delta(\Theta_{-S}) \times \Delta(\Theta_{S_{-i}})$. Hence, for a *non-singleton coalition* S, the set of correlated beliefs to which an action θ^i_S can be a pure best reply is only given by the subset of beliefs $\Delta(\Theta_{-S}) \times \Delta(\Theta_{S_{-i}}) \subset \Delta(\Theta_{-i})$. (2) We allow the strictly dominating action $\hat{p}^i_S$ of the deviating player to be a mixed action. So, in spite

of (1), this allows to retrieve the classical equivalence between the undominated set of actions $\Theta^{*}_{S_i}(p_{-S})$ of player $i \in S$ and the set of i's pure best replies $BR_{S_i}(\cdot; p_{-S})$ *relative to the subgame* $\Gamma_S(p_{-S})$. That is, the subset of $\mathscr{C}(N)$-undominated actions Θ^{*i}_{S} for $i \in S$ is undominated relative to the existence of a set of i's conjectures lying into the *product space* of probability measures $\Delta\Theta_{S_{-i}} \times \Delta\Theta_{-S}$. That is, an action of player i $\theta^i_S \in \Theta^i_S$ is a best reply $BR_{S_i}(p_{S_{-i}}; p_{-S})$ to some belief $p_{S_{-i}} \in \Delta\Theta_{S_{-i}}$ with $p_{S_{-i}} = p_S(\cdot; \Theta_{S_i})$ (the S_{-i}-marginal of p_S) in $\Gamma_S(p_{-S})$ if and only if θ^i_S is not strictly dominated by a $p_{S_i} = p_S(\cdot; \Theta_{S_{-i}})$ (the S_i-marginal of p_S).
(3) An immediate consequence of (1-2) is that the subset of $\mathscr{C}(N)$-undominated actions Θ^{*i}_{S} will generally form a *strict* subset of the N-undominated actions Θ^{*}_{i} of Γ since this classical dominance relation in Γ involves the space of conjectures strictly contained in $\Delta(\Theta_{S_{-i}} \times \Theta_{-S})$.

Strictly dominated action of a player in a (non-singleton) coalition. Let $C_{-S}(p_{-S}) = supp(p_{-S})$ denote the support of correlated strategy p_{-S}. To formally define the $\mathscr{C}(N)$-IESDA process of a game Γ, we first need to define the notion of ' strictly dominated action of a coalition' to arbitrary coalitional games $\Gamma_S(p_{-S})$. Given a fixed $\Gamma_S(p_{-S})$, define

$$D^{*}_{i}(p_S, i) = \{\hat{p}_S(\cdot, \Theta_{S_{-i}}) | \exists \hat{p}_S = \eta_i \star p^i_S \in D(\hat{p}_S, S) \cap CE_S(p_{-S})\}$$

as the set of marginals $p_S(\cdot, \Theta_{S_{-i}})$ of player $i \in S$ derived from the set of feasible deviations of coalition S $\hat{p}_S \in D(p_S, S)$ forming a CED $\hat{p}_S$ of $\Gamma_S(p_{-S})$. By construction, we must verify that under the play of coalition correlated equilibrium p_S in game $\Gamma_S(p_{-S})$, every θ^i_S in the support of the marginal of p_S lies in the best response set BR^i_S of i i.e., for all $\theta^i_S \in supp(p^i_S), \theta^i_S \in BR^i_S(p_S(|\theta^i_S), p_{-S})$.

Definition 5.10 (**strictly dominated action of a coalition**)**.** Fix a joint action $\theta_S = (\theta_{S_i}, \theta_{S_{-i}}) \in \Theta_S$ for coalition S. Say that θ_S is a $\mathscr{C}(N)$ **strictly dominated for coalition** $S \in \mathscr{C}(N)$ in Γ by a (potentially mixed) action (correlated strategy) $\hat{p}_S \in \Delta(\Theta_S)$ if there is a $p_{-S} \in \Delta(\Theta_{-S})$ inducing a p_{-S}- coalitional game $\Gamma_S(p_{-S})$ and a $i \in S$ with a *feasible unilateral deviation* $\eta_i \star \hat{p}_S \in D^{*}(\delta_{\theta_S}, i)$ such that $\theta^i_S = \theta_{S_i}$ is strictly dominated for player i in coalitional game $\Gamma_S(p_{-S})$.

$\mathscr{C}(N)$-Iterated Elimination strictly dominated actions (IESDA).

Definition 5.11. A finite $\mathscr{C}(N)$-sequence $\{(\Theta^{\ell}_{S})_{S \in \mathscr{C}(N)})^{L}_{\ell=0}$ is a process of $\mathscr{C}(N)$- **iterated elimination of strictly dominated actions** ($\mathscr{C}(N)$-IESDA) of $\{\Gamma_S(p_{-S})\}_{S \in \mathscr{C}(N)}$ for and all $p = (p_S, p_{-S}) \in \Delta\Theta_S \times \Delta\Theta_{-S}$ if for all $S \in \mathscr{C}(N)$ we have

i $\Theta^0_S = \Theta_S$ if $\Theta^0_i(p^0_{-S}) = \Theta_i$ for all $i \in S$; and for all $S \in \mathscr{C}(N)$ and $\ell = 0, 1, ..., L-1$:

ii $\Theta^{\ell+1}_S(p^{\ell+1}_{-S}) \subseteq \Theta^{\ell}_S(p^{\ell}_{-S})$ if $\Theta^{\ell+1}_{S_i}(p^{\ell+1}_{-S}) \subseteq \Theta^{\ell}_{S_i}(p^{\ell}_{-S})$ for all $i \in S$;

iii $\theta_S = (\theta^i_S)_{i \in S} \in \Theta^{\ell}_S(p^{\ell}_{-S}) \backslash \Theta^{\ell+1}_S(p^{\ell+1}_{-S})$ if for all $i \in S$, $\theta^i_S \in \Theta^{\ell}_{S_i}(p^{\ell}_{-S}) \backslash \Theta^{\ell+1}_{S_i}(p^{\ell+1}_{-S})$ only if θ^i_S is strictly dominated in a $p^{\ell}_{-S}-$ reduced coalitional strategic game $\Gamma_S(p^{\ell}_{-S})$:

$$\Gamma_S(\Theta^{\ell}_S; p^{\ell}_{-S}) = \langle S, (\Theta^{\ell}_j, U^{\ell}_j(\cdot; p^{\ell}_{-S}) : \prod_{S \in \mathscr{C}(N)} \Theta^{\ell}_S \longrightarrow \mathbb{R})_{j \in S} \rangle$$

where p^{ℓ}_{-S} is a distribution in $\prod_{S' \in \mathscr{C}(N) | S' \neq S} \Delta(\Theta^{\ell}_{S'})$

iv $\theta_S = (\theta^i_S)_{i \in S} \in \Theta^{L}_{S_i}$ if for *each* $\theta^i_S \in \Theta^{L}_{S_i}$ entails that θ^i_S is not strictly dominated for $i \in S$ in a p^{L}_{-S}- strategic game of coalition S :

$$\Gamma_S(\Theta^{L}; p^{L}_{-S}) = \langle S, (\Theta^{L}_j, U^{L}_j(\cdot; p^{L}_{-S}) : \prod_{S \in \mathscr{C}(N)} \Theta^{L}_S \longrightarrow \mathbb{R})_{j \in S} \rangle.$$

where p^{L}_{-S} is a distribution in $\prod_{S' \in \mathscr{C}(N) | S' \neq S} \Delta(\Theta^{L}_{S'})$.

The set $\Theta^{L}_S = \prod_{i \in S} \Theta^{L}_{S_i}$ denote the set of action profiles of coalition $S \in \mathscr{C}(N)$ that survive the $\mathscr{C}(N)$ -IESDA process in Γ and $\Theta^{L}_{\mathscr{C}(N)} = \prod_{S' \in \mathscr{C}(N)} \Theta^{L}_{S'}$ denotes the set of all the action profiles $(\theta^{'}_S)_{S' \in \mathscr{C}(N)}$ of coalition structure $\mathscr{C}(N)$ that survive the $\mathscr{C}(N)$- IESDA process in Γ.
The following corollary obtains immediately.

Corollary 5.12. *Suppose that $p = (p_S, p_{-S})$ is a tuple of correlated equilibria forming a $\mathscr{C}(N)$- NCCE of Γ. Then, we have that*

$$C_{\mathscr{C}(N)}(p) = \prod_{S' \in \mathscr{C}(N)} supp(p_{S'}) \subseteq \Theta^L_{\mathscr{C}(N)}.$$

Proof. Follows from the standard result that every action profile in the support of a correlated equilibrium distribution cannot be strictly dominated.

□

The above corollary entails that in a $\mathscr{C}(N)$- NCCE $p = (p_S, p_{-S})$ of a Γ, we have the property that for each coalition $S \in \mathscr{C}(N)$, there exists a sub collection of θ_{-S}-coalitional games $\{\Gamma_S(\theta_{-S}) | \theta_{-S} \in C_{-S}(p_S) \subseteq \Theta^L_{-S'}\}$ induced by the set of *strategy profiles* $\theta_{-S} = (\theta_{S'})_{S' \in \mathscr{C}(N)}$ in contained in the set of *joint coherent strategy profiles* $C_{-S}(p_S) \subseteq C_{-S}(\Delta_S)$ of the correlated equilibrium distributions of coalitions $-S \in \mathscr{C}(N)$.
A tuple $(\theta'_S)_{S' \in \mathscr{C}(N)}$ survives the $\mathscr{C}(N)$- IESDA process in Γ if the family of $p^L = (p^L_S, p^L_{-S})$-coalition games $\Gamma_{\mathscr{C}(N)}(p^L) = (\Gamma_S(p^L_{-S}), \Gamma_{-S}(p^L_S))$ of Γ is such that each joint action θ'_S of coalition S' survives the (standard) IESDA process in $\Gamma_{S'}(p^L_{-S'})$ with the property that $p^L_{-S'}$ is a correlated strategy profiles of coalitions $-S'$ over the set of actions $\Theta^L_{-S'}$ that survives IESDA process in games $\Gamma_{-S'}(p^L_{S'})$. This implies that each joint action $\theta_{S'}$ of coalition S' survives the (standard) IESDA process in $\Gamma_{S'}(p_{-S'})$ only if $\Gamma_{S'}(p_{-S'})$ is a coalitional game induced by a tuple of correlated strategies $p_{-S'}$ whose supports survive IESDA i.e., $supp(p_{-S'}) \subseteq \Theta^L_{-S'} = \prod_{\{S \neq S' | S \in \mathscr{C}(N)\}} \Theta^L_S$.
We verify that a trivial $\mathscr{C}(N)$-NCCE (θ_S, θ_{-S}) in pure strategies of Γ is a pure Nash equilibrium where each θ_S is a undominated strictly dominated action in coalitional game $\Gamma_S(\theta_{-S})$ for every coalition S. This property becomes a particular case when the game Γ is strict-dominance solvable. Say that a game is $\mathscr{C}(N)$-**strict-dominance solvable** if iterated deletion of strictly dominated strategies for each coalitions $S \in \mathscr{C}(N)$results in a unique strategy profile. Hence,a game Γ is $\mathscr{C}(N)$-strict-dominance solvable iff each $\Gamma_S^{p_{-S}}$ is strict-dominance solvable. For this class of games it is then obvious that there exists a unique $\mathscr{C}(N)$- CCE $\theta_N = (\theta_S)_{S \in \mathscr{C}(N)}$ which is trivial since it must form a pure strategy Nash equilibrium of Γ. Hence we have the following property:

All the games Γ which are $\mathscr{C}(N)$-strict-dominance solvable have an empty set of non-trivial $\mathscr{C}(N)$-NCCEs.

Existence of NCCEs. The existence of a $\mathscr{C}(N)$-NCCE is trivially guaranteed for every coalition structure $\mathscr{C}(N)$ when one does *not* require $\mathscr{C}(N)$-NCCEs to form a *properly mixed Nash equilibrium* $p = (p_S, p_{-S})$ of the *partitioned game* (with at least one component p_S required to be proper (non-degenerate) correlated equilibrium distribution of the coalitional game $\Gamma_S(p_{-S})$ for at least one non-singleton coalition of players S). Indeed, if for each coalition S, each p_S is a product probability measure in $\prod_{i \in S} \Delta\Theta_i$, then, the resulting product measure $p = p_S \otimes p_{-S} = prod_{i \in N} \Delta\Theta_i$ boils down to a regular mixed Nash equilibrium of Γ.

Lemma 5.13. *Fix a (in particular finite) game Γ with continuous payoff functions. Then, a $\mathscr{C}(N)$-NCCE exists for the coalition structure $\mathscr{C}(N)$.*

Proof. Using the property of the joint coalitional best responses of each coalition, this follows by a simple application of the Kakutani fixed point theorem assuring the existence of a Nash equilibrium in the game played across the coalitions. The details below are given for finite games.
Given a coalition structure $\mathscr{C}(N)$, let $CE(\Gamma_S(p_{-S}))$ denote the set of all the - **correlated equilibria** in the p_{-S}-coalitional game $\Gamma_S(p_{-S})$ induced by the correlated strategy profile p_{-S} of the other coalitions $-S$. If they exist, $CE(\Gamma_S(p_{-S}))$ contains the pure strategy Nash equilibria of $\Gamma_S(p_{-S})$. By construction, given a correlated strategy profile, p_{-S}, $CE(\Gamma_S(p_{-S}))$ coincides with the joint best response $BR_S(p_{-S})$ of coalition of players S. Thus, using the well-known fact that $CE(\Gamma_S(p_{-S}))$ is a convex and compact set (polytope), it follows using the concavity of the payoff functions $U_i(\cdot; p_{-S})$, that the map BR_S:

$$p_{-S} \longmapsto BR_S(p_{-S}) = CE(\Gamma_S(p_{-S})) \subseteq \Delta(\Theta_S)$$

is a non-empty, convex -valued and compact correspondence which has a closed graph. These properties apply to each coalition $-S \in \mathscr{C}(N)$. Given a coalition structure $\mathscr{C}(N)$, we simply denote a profile of correlated strategies by $p = (p_S)_{S \in \mathscr{C}(N)}$ and define for each such p the tuple of induced p_{-S}-coalitional games at p as the collection $\Gamma_{\mathscr{C}(N)}(p) \equiv \{\Gamma_S(p_S) | S \in \mathscr{C}(N)\}$ that is induced at profile p. We then consider the $|\mathscr{C}(N)|$-Cartesian product of coalitional best responses of each coalition S at p. It is given by the Cartesian product of correlated equilibria of each coalition :

$$\mathrm{CE}(\Gamma_{\mathscr{C}(\mathrm{N})}(p)) \equiv \prod_{\mathrm{S}\in\mathscr{C}(\mathrm{N})} \mathrm{CE}(\Gamma_{\mathrm{S}}(p_{\mathrm{S}})).$$

Hence, the combined best response correspondence (which inherit the properties of each $\mathrm{BR_S}$) defines a well-behaved correspondence which satisfies the usual conditions to apply the Kakutani's FPT and it follows that $\mathrm{BR}_{\mathscr{C}(\mathrm{N})}$: $\mathrm{CE}(\Gamma_{\mathscr{C}(\mathrm{N})}(p)) \twoheadrightarrow \mathrm{CE}(\Gamma_{\mathscr{C}(\mathrm{N})}(p))$ has a fixed point which is by construction a mixed strategy equilibrium p of the finite game $\Gamma_{\mathscr{C}(\mathrm{N})}(p)$ played across the coalitions in $\mathscr{C}(\mathrm{N})$. This proves the existence of a $\mathscr{C}(\mathrm{N})$-NCCE of finite game Γ for each coalition structure $\mathscr{C}(\mathrm{N})$.

□

Proof Theorem 3: Mixed Nash equilibria of the partitioned game under a NCCE. We first establish the necessary and sufficient conditions that must hold in the partitioned game to have a *non-trivial* $\mathscr{C}(\mathrm{N})$- NCCE $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ of a game Γ wherein each coalition of players are using *non-degenerate* correlated equilibrium distributions as their mixed Nash equilibrium strategies of the partitioned game.

Lemma 5.14. *Given a coalition structure* $\mathrm{S} \in \mathscr{C}(\mathrm{N})$*, each profile of coalition mixed strategies* $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ *of the* $\mathscr{C}(\mathrm{N})$*-partitioned game* $\Gamma_{\mathscr{C}(\mathrm{N})}$ *generates a mixed Nash equilibrium of* $\Gamma_{\mathscr{C}(\mathrm{N})}$ *forming a NCCE if and only if each* $p_{\mathrm{S}} \in \mathrm{BR_S}(p_{-\mathrm{S}}), \forall \mathrm{S} \in \mathscr{C}(\mathrm{N})$ *verifies the following properties:For all* $\mathrm{S} \in \mathscr{C}(\mathrm{N})$:

i *[coalitions' indifference condition] :*

$$\mathrm{W_S}(\theta_{\mathrm{S}}; p_{-\mathrm{S}}) = \mathrm{W_S}(\theta'_{\mathrm{S}}; p_{-\mathrm{S}}), \forall \theta'_{\mathrm{S}}, \theta_{\mathrm{S}} \in supp(p_{\mathrm{S}});$$

ii *[coalitions optimality]*

$$\mathrm{W_S}(\theta_{\mathrm{S}}; p_{-\mathrm{S}}) = \mathrm{W_S}(p_{\mathrm{S}}; p_{-\mathrm{S}}) \geqslant \mathrm{W_S}(\hat{p}_{\mathrm{S}}; p_{-\mathrm{S}}), \forall \hat{p}_{\mathrm{S}} = \alpha^i_{\mathrm{S}} \star p_{\mathrm{S}} \in \mathrm{D}(p_{\mathrm{S}}, i), \forall i \in \mathrm{S}.$$

Proof. This follows by applying the standard necessary and sufficient conditions for a mixed Nash equilibrium in a normal form game to the partitioned game $\Gamma_{\mathscr{C}(\mathrm{N})}$ with the characterization of the best response correspondences $p_{-\mathrm{S}} \longmapsto \mathrm{BR_S}(p_{-\mathrm{S}})$ (see the section '*Joint best response of a coalition and 'partitioned game*').

□

Corollary 5.15. *A coalitional equilibrium of* Γ $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ *forms a* $\mathscr{C}(\mathrm{N})$*- NCCE of a game* Γ *iff for each coalition* $\mathrm{S} \in \mathscr{C}(\mathrm{N})$*, the optimal coalitions' indifference conditions (i-ii) implies the individual indifference condition: For all* $i \in \mathrm{S}$ *and every pair of joint coherent strategies,* $\theta_{\mathrm{S}} = (\theta^i_{\mathrm{S}})$ *and* $\hat{\theta}_{\mathrm{S}} = (\hat{\theta}^i_{\mathrm{S}})$ *in* $supp(p_{\mathrm{S}})$*:*

$$\mathrm{W_S}(\theta_{\mathrm{S}}, p_{-\mathrm{S}}) = \mathrm{W_S}(\hat{\theta}_{\mathrm{S}}, p_{-\mathrm{S}}) \Longrightarrow \mathrm{U}_i(\theta_{\mathrm{S}}, p_{-\mathrm{S}}) = \mathrm{U}_i(\hat{\theta}_{\mathrm{S}}, p_{-\mathrm{S}}).$$

Proof. If $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ is a NCCE that is also a coalitional equilibrium of Γ, then each $p_{\mathrm{S}} \in \beta_{\mathrm{S}}(p_{-\mathrm{S}})$ must be a Pareto-best from among those strategies with support in $\Theta^{\mathrm{IESDA}}_{\mathrm{S}}(p_{-\mathrm{S}})$. This implies that all the pure strategies $\theta_{\mathrm{S}} = (\theta^i_{\mathrm{S}})$ and $\hat{\theta}_{\mathrm{S}} = (\hat{\theta}^i_{\mathrm{S}})$ in $supp(p_{\mathrm{S}})$ must be payoff equivalent for all $i \in \mathrm{S}$; otherwise, we could construct another correlated strategy that yielded at least one player $i \in \mathrm{S}$ a higher payoff, which would contradict that p_{S} also lies in $\beta_{\mathrm{S}}(p_{-\mathrm{S}})$.

□

Coalitions' best response correspondences. The next statement says that when profile of coalition mixed strategies $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ forms a $\mathscr{C}(\mathrm{N})$- NCCE there must exist a feasible deviation $\hat{\theta}_{\mathrm{S}} = \alpha_{\mathrm{S}} \star \theta_{\mathrm{S}}$ such that for every $\theta_{\mathrm{S}} = (\theta^i_{\mathrm{S}})$ in $supp(p_{\mathrm{S}})$:

$$\mathrm{W_S}(\theta_{\mathrm{S}}, \theta'_{-\mathrm{S}}) \geqslant \mathrm{W}(\hat{\theta}_{\mathrm{S}}, \theta'_{-\mathrm{S}})$$

for every $p_{-\mathrm{S}}(\theta'_{-\mathrm{S}}) > 0$.

Lemma 5.16. *A profile of coalition mixed strategies* $p = (p_{\mathrm{S}})_{\mathrm{S}\in\mathscr{C}(\mathrm{N})}$ *forms a* $\mathscr{C}(\mathrm{N})$*- NCCE of a game* Γ *only if every* $\theta_{\mathrm{S}} = (\theta^i_{\mathrm{S}})$ *in* $supp(p_{\mathrm{S}})$ *is an undominated action for coalition* S *such that for every* $i \in \mathrm{S}$, θ^i_{S} *is an undominated action in every coalitional game* $\Gamma_{\mathrm{S}}(\theta_{-\mathrm{S}})$ *for every* $p_{-\mathrm{S}}(\theta_{-\mathrm{S}}) > 0$;

Proof. By definition, $p = (p_S)_{S\in\mathscr{C}(N)}$ forms a NCCE of Γ *only if* each p_S is a correlated equilibrium distribution of $\Gamma_S(p_{-S})$. Hence, the following property must hold for every $S \in \mathscr{C}(N)$:

$$\theta_S = (\theta_S^i)_{i\in S} \in BR_S, p_{-S}) \Longleftrightarrow \theta_S^i \in BR_S^i(p_{S_{-i}}(\cdot|\theta_S^i), p_{-S}), \forall i \in S.$$

The above says that every profile θ_S in the support of p_S is a coalition best response of S in the sense that θ_S dominates every feasible deviation $\hat{\theta}_S = \alpha_S \star \theta_S$ iff every individual component θ_S^i is an individual best response for every $i \in S$ to a product belief $p_{S_{-i}}(\cdot|\theta_S^i) \times p_{-S} \in \Delta\Theta_{S_{-i}} \times \Delta\Theta_{-S}$. Hence, it follows that every component θ_S^i must be an undominated action of Γ. This proves that every profile θ_S in the support of a component p_S derived from a $\mathscr{C}(N)$-NCCE $p = (p_S)_{S\in\mathscr{C}(N)}$ of a game Γ has the property that for every component θ_S^i, there must exist some coalitional game $\Gamma_S(\theta_{-S})$ for every $p_{-S}(\theta_{-S}) > 0$ and profile $\hat{\theta}_S = (\hat{\theta}_S^i, \theta_{S_{-i}})$ such that

$$U_i(\theta_S^i, \theta_{S_{-i}}, \theta_{-S}) \geqslant U_i(\hat{\theta}_S^i, \theta_{S_{-i}}, \theta_{-S}), p_{-S}(\theta_{-S}) > 0.$$

Hence every p_S is a correlated equilibrium distribution of $\Gamma_S(p_{-S})$ in a NCCE of Γ only if θ_S^i is an undominated action every coalitional game $\Gamma_S(\theta_{-S})$ for $p_{-S}(\theta_{-S}) > 0$.

□

By construction the *pure best response* correspondence of a coalition S in the $\mathscr{C}(N)$-partitioned game $\Gamma_{\mathscr{C}(N)}$ are given by the subset of the *coherent* pure strategies (see Nau and McCardle (1990)) that are played with positive probability in at least one correlated equilibrium of some p_{-S}-coalitional game $\Gamma_S(p_{-S})$. Given a coalition structure $\mathscr{C}(N)$, we define the set of $\mathscr{C}(N)$-**jointly coherent strategies**, denoted $C_{\mathscr{C}(N)}(S)$, in the *game* Γ for a subset of players S as the subset of pure actions:

$$C_{\mathscr{C}(N)}(S) := \{\theta_S \in \Theta_S | \exists p_{-S} \in \Delta\Theta_{-S}, p_S \in CE_S(\Gamma_S(p_{-S})) \Longrightarrow supp(p_S) \ni \theta_S\}.$$

For each correlated strategy profile p_{-S}, we have the pure best response set of coalition $S \in \mathscr{C}(N)$ given by the subset of jointly coherent strategies of players $i \in S$:

$$p_{-S} \longmapsto BR_S(p_{-S}) := \{\hat{\theta}_S \in C_{\mathscr{C}(N)}(S) | W_S(\hat{\theta}_S; p_{-S}) \geqslant W_S(\theta_S; p_{-S}), \forall \theta_S \in \Theta_S\} \equiv C_S(\Theta_S)(p_{-S}).$$

The next result records this fact.

Lemma 5.17. *The pure best responses* $p_{-S} \longmapsto BR_S(p_{-S}) \subset \Theta_S$ *of a coalition* S *in the* $\mathscr{C}(N)$*-partitioned game* $\Gamma_{\mathscr{C}(N)}$ *are given by the subsets of coalition* S *coherent joint pure strategies* $C_S(\Theta_S)(p_{-S})$ *of* Γ.

Proof. Let $p_S \in CE_S(p_{-S})$ be a non-degenerated correlated equilibrium distribution of $\Gamma_S(p_{-S})$. For every vector of feasible deviation $\alpha_S \star p_S = (\alpha_S^i \star p_S)_{i\in S}$, we have:

$$W_S(\alpha_S \star p_S; p_{-S}) \leqslant W_S(p_S; p_{-S}) \Longleftrightarrow \Psi_S(p_S; \alpha_S; p_{-S}) \leqslant 0 \text{for all } \alpha_S \star p_S = \delta_{\theta_S}.$$

In particular

$$\Psi_S(p_S; \alpha_S; p_{-S}) = 0, \text{ for all } \alpha_S \star p_S = \delta_{\theta_S} \Longleftrightarrow$$
$$\sum_{i\in S}[U_i(\theta_S^i, p_{S_{-i}}(\cdot|\theta_S^i); p_{-S})) - U_i(p_S; p_{-S})] = 0,$$

for all vectors of feasible deviations $\delta_{\theta_S} = (\alpha_S^i \star p_S)_{i\in S}$ induced by joint coherent action profiles $\theta_S = (\theta_S^i)_{i\in S} \in C_S(\Theta_S)(p_{-S})$. Hence,

$$p_{-S} \longmapsto BR_S(p_{-S}) := \{\hat{\theta}_S \in C_{\mathscr{C}(N)}(S) | \Psi_S(p_S; \alpha_S; p_{-S}) = W_S(\alpha_S \star p_S; p_{-S}) - W_S(p_S; p_{-S}) \leqslant 0, \text{ for all } \alpha_S \star p_S = \delta_{\theta_S}\}$$

follows by definition.

□

Proof Theorem 3: Intersection of Ray and Vohra's '*best response property*and correlated equilibria. Consider the joint best response correspondence β_S of a coalition S under an arbitrary coalition structure $\mathscr{C}(N)$. A generalized definition of the Ray and Vohra's '*best response property* (relative to the $\mathscr{C}(N)$-partitioned game $\Gamma_{\mathscr{C}(N)}$) where each coalition $S \in \mathscr{C}(N)$ has access to correlated strategies in $\Delta\Theta_S$ is introduced in Haeringer (2004) and defined by:

$$\beta_S(p_{-S}) = \{\hat{p}_S \in \Delta\hat{\Theta}_S | \nexists p_S \in D(p_S, S), U_S(p_S; p_{-S}) = (U_i(p_S; p_{-S})_{i\in S}) >> U_S(\hat{p}_S; p_{-S}) = (U_i(\hat{p}_S; p_{-S})_{i\in S})\}.$$

For any coalition structure $\mathscr{C}(N)$, the set

$$\beta_{\mathscr{C}(N)} = \{(p = (p_S, p_{-S}) \in \Delta\Theta_S \times \Delta\Theta_{-S} | \beta_S(p_{-S}) \times \beta_{-S}(p_S) \neq \varnothing\}$$

is *non-empty* if utility functions are continuous and quasi- concave with respect to the (non-empty compact and convex) strategy set $\Delta\Theta_S \times \Delta\Theta_{-S}$ (see Ray and Vohra (1996), Proposition 2.2). Now we suppose that for each coalition $S \in \mathscr{C}(N)$, there exists a non empty compact and convex intersection $\Theta_S^L \cap \Theta_S^{LPareto}$ between the subset of action profiles of coalition $S \in \mathscr{C}(N)$ that survive the $\mathscr{C}(N)$ -IESDA process in Γ and the set $\Theta_{S^{Pareto}}$ of all the undominated Pareto action profiles of coalition $S \in \mathscr{C}(N)$ in Γ. Hence, by construction, the Pareto undominated strategy profiles of each coalition S is given by the (non-empty) projection set, $\Theta_S^{Pareto} \equiv proj_{\Theta_S}\beta_{\mathscr{C}(N)}$, of Pareto undominated strategy profiles in $\beta_{\mathscr{C}(N)}$ onto Θ_S i.e.,

$$\Theta_S^{Pareto} = \{\beta_S(p_{-S}) \subseteq \Delta\Theta_S | p_{-S} \in \Delta\Theta_{-S}\} \equiv \beta_S.$$

Thus, for each coalition $S \in \mathscr{C}(N)$, we have the property

$$\Theta_S^L \cap \Theta_S^{LPareto} \neq \emptyset \Longleftrightarrow \beta_S \cap BR_S \neq \emptyset$$

and

$$\beta_S \cap BR_S \neq \emptyset \Longleftrightarrow \exists p_{-S} \in \Delta\Theta_{-S}^L \text{ such that } \beta_S(p_{-S}) \cap BR_S(p_{-S}) \neq \emptyset.$$

For each coalition S, we obtain a non-empty intersection

$$BR_S^* \equiv \beta_S \cap BR_S = \{p_S \in \Delta\Theta_S^L | \exists p_{-S} \in \Delta\Theta_{-S}^L, p_S \in \beta_S(p_{-S}) \cap BR_S(p_{-S})\}$$

which is convex and compact if $\Theta_S^L \cap \Theta_S^{LPareto}$ is convex and compact. With the above, we deduce the existence of a non-empty and compact and convex valued correspondence

$$p_{-S} \longmapsto BR_S^*(p_{-S}) \equiv \beta_S(p_{-S}) \cap BR_S(p_{-S})$$

for each coalition $S \in \mathscr{C}(N)$. Ray and Vohra's result states that $\beta_{\mathscr{C}(N)}$ is non-empty if continuous utility functions are quasi- concave with respect to whether each β_S is defined over Θ_S^L or $\Delta\Theta_S^L$. The proof is then completed by applying the remark of Haeringer (2004):When β_S is defined over $\Delta\Theta_S^L$, the players' utility functions get automatically quasi-concave. Hence, the result that $BR_{\mathscr{C}(N)}^* = \prod_{S\in\mathscr{C}(N)} BR_S^*$ is non-empty for the class of best response correspondences BR_S^* of each coalition S defined over $\Delta\Theta_S^L$.

□

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