# The weakness of finding descending sequences in ill-founded linear orders

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**Abstract.** We prove that the Weihrauch degree of the problem of finding a bad sequence in a non-well quasi order (BS) is strictly above that of finding a descending sequence in an ill-founded linear order (DS). This corrects our mistaken claim in [8], which stated that they are Weihrauch equivalent. We prove that König's lemma KL and the problem $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ of enumerating a given non-empty countable closed subset of $2^{\mathbb{N}}$ are not Weihrauch reducible to DS either, resolving two main open questions raised in [8].

**Keywords:** Weihrauch reducibility · well quasi-orders · first-order part.

## 1 Introduction and background

A quasi-order $(Q, \preceq)$ is called *well quasi-order* (abbreviated wqo) if, for every infinite sequence $(q_n)_{n\in\mathbb{N}}$ of elements of $Q$, there are $i, j$ with $i < j$ such that $q_i \preceq_Q q_j$. This can be restated by saying that a quasi-order is a wqo if it contains no infinite bad sequences, where a sequence $(q_n)_{n<\alpha}$ is called bad if $q_i \not\preceq_Q q_j$ for every $i < j < \alpha$. Equivalently, wqo's can be defined as quasi-orders that contain no infinite descending sequence and no infinite antichain. There is an extensive literature on the theory of wqo's. For an overview, we refer the reader to [10].

We study the difficulty of solving the following computational problems:

- given a countable ill-founded linear order, find an infinite Descending Sequence in it (DS), and
- given a countable non-well quasi-order, find a Bad Sequence in it (BS).

A suitable framework for this is the Weihrauch lattice (see [2] for a self-contained introduction). Several results on DS were proved in our previous paper [8]; however, [8, Prop. 4.5] falsely claims that DS and BS are Weihrauch equivalent. In Theorem 1 we refute our claim by proving that the first-order part of BS is not Weihrauch reducible to DS. On the other hand, the deterministic part and the $k$-finitary parts of BS are Weihrauch equivalent to the corresponding parts of DS (Theorem 3, Corollary 5).

We will also resolve (negatively) two main open questions raised in [8, Questions 6.1 and 6.2], namely whether $\mathsf{KL}$ and $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ are Weihrauch reducible to $\mathsf{DS}$ (Corollaries 2, 3). The core of our proof (Theorem 2) is

$$\mathsf{lim} \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f \mid \widehat{\mathsf{ACC}_{\mathbb{N}}} \times f \leq_{\mathrm{W}} \mathsf{DS}\}.$$

That is, even though $\widehat{\mathsf{ACC}_{\mathbb{N}}}$ is fairly weak (in particular it is below $\mathsf{lim}$, $\mathsf{KL}$ and $\mathsf{DS}$), $\mathsf{DS}$ cannot even compute $\widehat{\mathsf{ACC}_{\mathbb{N}}} \times f$ if $f \not\leq_{\mathrm{W}} \mathsf{lim}$. The existence of the maximum above provides an example of a "parallel quotient" [5, Remark 1].

In the rest of this section we briefly introduce relevant notions in Weihrauch complexity, followed with a note that $\mathsf{BS}$ is equivalent to its restriction to partial orders (Proposition 2).

A *represented space* $\mathbf{X} = (X, \delta_{\mathbf{X}})$ consists of a set $X$ and a (possibly partial) surjection $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. Many mathematical objects of interest can be represented in standard ways which we do not spell out here (see e.g. [2, Def. 2.5]), such as: $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$, $\mathbb{N}^{<\mathbb{N}}$, initial segments of $\mathbb{N}$, the set of binary relations on $\mathbb{N}$, the set of $\Gamma$-definable subsets of $\mathbb{N}$ where $\Gamma$ is a pointclass in the projective hierarchy, countable Cartesian products and countable disjoint unions of represented spaces.

A *problem* $f$ is a (possibly partial) multivalued function between represented spaces $\mathbf{X}$ and $\mathbf{Y}$, denoted $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$. For each $x \in X$, $f(x)$ denotes the set of possible outputs (i.e., *$f$-solutions*) corresponding to the input $x$. The *domain* $\mathrm{dom}(f)$ is the set of all $x \in X$ such that $f(x)$ is non-empty. Such $x$ is called an *$f$-instance*. If $f(x)$ is a singleton for all $x \in \mathrm{dom}(f)$, we say $f$ is *single-valued* and write $f :\subseteq \mathbf{X} \to \mathbf{Y}$. In this case, if $y$ is the $f$-solution to $x$, we write $f(x) = y$ instead of (the formally correct) $f(x) = \{y\}$. We say a problem is *computable* (resp. *continuous*) if there is some computable (resp. continuous) $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if $p$ is a name for some $x \in \mathrm{dom}(f)$, then $F(p)$ is a name for an $f$-solution to $x$.

A problem $f$ is *Weihrauch reducible* to a problem $g$, written $f \leq_{\mathrm{W}} g$, if there are computable maps $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if $p$ is a name for some $x \in \mathrm{dom}(f)$, then

1. $\Phi(p)$ is a name for some $y \in \mathrm{dom}(g)$, and
2. if $q$ is a name for some $g$-solution of $y$, then $\Psi(p, q)$ is a name for some $f$-solution of $x$.

If $\Phi$ and $\Psi$ satisfy the above, we say that $f \leq_{\mathrm{W}} g$ *via* $\Phi$, $\Psi$.

Weihrauch reducibility forms a preorder on problems. We say $f$ and $g$ are *Weihrauch equivalent*, written $f \equiv_{\mathrm{W}} g$, if $f \leq_{\mathrm{W}} g$ and $g \leq_{\mathrm{W}} f$. The $\equiv_{\mathrm{W}}$-equivalence classes (*Weihrauch degrees*) are partially ordered by $\leq_{\mathrm{W}}$. Among the numerous algebraic operations in the Weihrauch degrees, we consider:

– for problems $f_i :\subseteq \mathbf{X}_i \rightrightarrows \mathbf{Y}_i$, the *parallel product*

$$f_0 \times f_1 :\subseteq \mathbf{X}_0 \times \mathbf{X}_1 \rightrightarrows \mathbf{Y}_0 \times \mathbf{Y}_1 \text{ defined by } (x_0, x_1) \mapsto f_0(x_0) \times f_1(x_1),$$

i.e., given an $f_0$-instance and an $f_1$-instance, solve both,

– for a problem $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, the *(infinite) parallelization*

$$\widehat{f} :\subseteq \mathbf{X}^{\mathbb{N}} \rightrightarrows \mathbf{Y}^{\mathbb{N}} \text{ defined by } (x_i)_i \mapsto \prod_i f(x_i),$$

i.e., given a countable sequence of $f$-instances, solve all of them.

These operations are defined on problems, but they all lift to the Weihrauch degrees. Parallelization even forms a closure operator, i.e., $f \leq_{\mathrm{W}} \widehat{f}$, $f \leq_{\mathrm{W}} g$ implies $\widehat{f} \leq_{\mathrm{W}} \widehat{g}$, and $\widehat{\widehat{f}} \equiv_{\mathrm{W}} \widehat{f}$.

The Weihrauch degrees also support a number of interior operators, which have been used to separate degrees of interest (see e.g. [13, §3.1]). For any problem $f$ and any represented space $\mathbf{X}$,

$$\mathrm{Det}_{\mathbf{X}}(f) := \max_{\leq_{\mathrm{W}}}\{g \leq_{\mathrm{W}} f \mid g \text{ has codomain } \mathbf{X} \text{ and is single-valued}\}$$

exists [8, Thm. 3.2]. We call $\mathrm{Det}_{\mathbb{N}^{\mathbb{N}}}(f)$ the *deterministic part of* $f$ and denote it by $\mathrm{Det}(f)$ for short. Observe that [8, Prop. 3.6] can be generalized slightly:

**Proposition 1.** $\mathrm{Det}(f) \leq_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbf{2}}(f)}$.

*Proof.* Suppose $g$ is single-valued, has codomain $\mathbb{N}^{\mathbb{N}}$, and $g \leq_{\mathrm{W}} f$. Define a single-valued problem $h$ as follows: Given $n, m \in \mathbb{N}$ and a $g$-instance $x$, produce 1 if $g(x)(n) \geq m$, otherwise produce 0. It is easy to see that $g \leq_{\mathrm{W}} \widehat{h}$ and $h \leq_{\mathrm{W}} f$. The latter implies $h \leq_{\mathrm{W}} \mathrm{Det}_{\mathbf{2}}(f)$ and so $g \leq_{\mathrm{W}} \widehat{h} \leq_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbf{2}}(f)}$.

For any problem $f$ and $\mathbf{X} = \mathbb{N}$ or $\mathbf{k}$, it is also known that

$$\max_{\leq_{\mathrm{W}}}\{g \leq_{\mathrm{W}} f \mid g \text{ has codomain } \mathbf{X}\}$$

exists. For $\mathbf{X} = \mathbb{N}$ we call it the *first-order part of* $f$ [6, Thm. 2.2], denoted by ${}^1f$, while for $\mathbf{X} = \mathbf{k}$ we call it the $\mathbf{k}$*-finitary part of* $f$ [4, Prop. 2.9], denoted by $\mathrm{Fin}_{\mathbf{k}}(f)$. We have $\mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} {}^1f$ and $\mathrm{Det}_{\mathbf{k}}(f) \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(f) \leq_{\mathrm{W}} {}^1f$.

To study the problems $\mathsf{DS}$ and $\mathsf{BS}$ from the point of view of Weihrauch reducibility, we need to introduce the represented spaces of linear orders and quasi-orders. We only work with countable linear orders/quasi-orders with domain contained in $\mathbb{N}$. We represent a linear order $(L, \leq_L)$ with the characteristic function of the set $\{\langle n, m\rangle : n \leq_L m\}$. Likewise, we represent a quasi-order $(Q, \preceq_Q)$ with the characteristic function of the set $\{\langle n, m\rangle : n \preceq_Q m\}$.

We conclude this section observing a fact about $\mathsf{BS}$ which was implicit in [8].

**Proposition 2.** $\mathsf{BS}$ *is Weihrauch equivalent to its restriction to partial orders.*

*Proof.* Given a non-well quasi-order $(Q, \preceq_Q)$ where $Q \subseteq \mathbb{N}$, compute the set $S = \{a \in Q \mid (\forall b <_{\mathbb{N}} a)(a \not\preceq_Q b \text{ or } b \not\preceq_Q a)\}$. The restriction $(S, \preceq_Q)$ is a non-well partial order because it is isomorphic to the partial order of $\preceq_Q$-equivalence classes.

Henceforth we will use Proposition 2 without mention.

## 2 Separating BS and DS

We shall separate BS and DS by separating their first-order parts.

**Theorem 1.** *$^1\mathsf{BS} \not\leq_\mathrm{W} {^1\mathsf{DS}}$ and so $\mathsf{DS} <_\mathrm{W} \mathsf{BS}$.*

Recall from [8, Thm. 4.10] that $^1\mathsf{DS} \equiv_\mathrm{W} \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$, which is the problem of producing an upper bound for a finite subset of $\mathbb{N}$ (given via a $\Pi^1_1$ code). Observe that $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$ is *upwards closed*, i.e., if $n \in g(x)$ then $m \in g(x)$ for all $m > n$.

**Lemma 1.** *Let $f$ be a problem with codomain $\mathbb{N}$. The following are equivalent:*

1. *there exists an upwards closed problem $g$ with codomain $\mathbb{N}$ such that $f \leq_\mathrm{W} g$;*
2. *there is a computable procedure which takes as input any $x \in \mathrm{dom}(f)$ and produces a sequence $p_x \in \mathbb{N}^\mathbb{N}$ of guesses for $f$-solutions to $x$ which is correct cofinitely often.*

*Proof.* For 1. $\Rightarrow$ 2., let $g$ be upwards closed and assume $f \leq_\mathrm{W} g$ via $\Phi$ and $\Psi$. Given $x \in \mathrm{dom}(f)$, run the computations $(\Psi(x, m))_{m\in\mathbb{N}}$ in parallel. Once some $\Psi(x, m)$ halts, we output its result and cancel $\Psi(x, n)$ for all $n < m$. This produces a sequence of numbers. The fact that $g$ is upwards closed guarantees that cofinitely many elements of this sequence are elements of $f(x)$.

For the converse direction, for every $x \in \mathrm{dom}(f)$, let $p_x \in \mathbb{N}^\mathbb{N}$ be as in the hypothesis. Define $M_x := \max\{m \mid p_x(m) \notin f(x)\}$ and let $g(x) := \{n \mid n > M_x\}$. Clearly $g$ is upwards closed. The fact that $f \leq_\mathrm{W} g$ follows from the fact that $x \mapsto p_x$ is computable.

Given a non-well quasi-order $(Q, \preceq_Q)$, we say that a finite sequence $\sigma$ is extendible to an infinite $\preceq_Q$-bad sequence (or, more compactly, $\sigma$ is $\preceq_Q$-extendible) if there is a $\preceq_Q$-bad sequence $(q_n)_{n\in\mathbb{N}}$ such that $(\forall i < |\sigma|)(\sigma(i) = q_i)$. We omit the order whenever there is no ambiguity.

Observe that $^1\mathsf{BS}$ can compute the problem "given a non-well partial order $(P, \leq_P)$, produce an element of $P$ that is extendible to an infinite bad sequence". In light of Lemma 1, to prove Theorem 1 it suffices to show that one cannot computably "guess" solutions for BS. In other words, given a computable procedure which tries to guess extendible elements in a non-wqo, we want to construct a non-wqo $P$ on which the procedure outputs a non-extendible element infinitely often. This would imply that $^1\mathsf{BS} \not\leq_\mathrm{W} \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$. The non-wqos $P$ we construct will be "tree-like" in the following sense:

**Definition 1.** *A* tree decomposition *of a partial order $(P, \leq_P)$ consists of a tree $T \subseteq 2^{<\mathbb{N}}$ and a function $\iota\colon T \to P$ such that:*

1. *If $w_1, w_2 \in T$ and $w_1$ is a proper prefix of $w_2$ (written $w_1 \sqsubset w_2$), then $\iota(w_1) <_P \iota(w_2)$.*

2. *$P$ is partitioned into finite $P$-intervals, where each interval has the form*

$$(wb] = \{v \in P \mid \iota(w) <_P v \leq_P \iota(wb)\}$$

*for some vertex $wb \in T$ (with final entry $b$), or $(\varepsilon] = \{\iota(\varepsilon)\}$ (where $\varepsilon$ denotes the root of $2^{<\mathbb{N}}$). For $v \in P$ let $\lceil v \rceil \in T$ be uniquely defined by $v \in (\lceil v \rceil]$.*
3. *If $w_1, w_2 \in T$ are incompatible, so are $\iota(w_1)$ and $\iota(w_2)$ (i.e. they have no common upper bound in $P$).*

The following lemma is straightforward.

**Lemma 2.** *If $\iota\colon T \to P$ is a tree decomposition, then $P$ has no infinite descending sequences. Moreover, $T$ is wqo (i.e. it has finite width) if and only if $P$ is wqo. In other words, $T$ has an infinite antichain iff so does $P$.*

*Proof.* The fact that every partial order that admits a tree decomposition does not have an infinite descending sequence follows from the fact that if $(v_n)_{n\in\mathbb{N}}$ is an infinite descending sequence in $P$, then since every interval $(\lceil v_n \rceil]$ is finite, up to removing duplicates, the sequence $(\lceil v_n \rceil)_{n\in\mathbb{N}}$ would be an infinite descending sequence in $T$.

If $(w_n)_{n\in\mathbb{N}}$ is an infinite antichain in $T$ then, by definition of tree decomposition, $(\iota(w_n))_{n\in\mathbb{N}}$ is an infinite antichain in $P$. Conversely, if $(v_n)_{n\in\mathbb{N}}$ is an infinite antichain in $P$, then for every $n$, for all but finitely $m$, $\lceil v_n \rceil$ is $\sqsubseteq$-incomparable with $\lceil v_m \rceil$. In particular, we can obtain an infinite antichain in $T$ by choosing a subsequence $(v_{n_i})_{i\in\mathbb{N}}$ such that, for every $i \neq j$, $\lceil v_{n_i} \rceil$ and $\lceil v_{n_j} \rceil$ are $\sqsubseteq$-incomparable.

**Lemma 3.** *There is no computable procedure that, given in input a partial order which admits a tree decomposition, outputs an infinite sequence of elements of that partial order such that if the input is not wqo, then cofinitely many elements in the output are extendible to a bad sequence.*

We point out a subtle yet important aspect regarding Lemma 3: The procedure only has access to the partial order, not to a tree decomposition of it.

*Proof.* Fix a computable "guessing" procedure $g$ that receives as input a partial order (admitting a tree decomposition) and outputs an infinite sequence of elements in that partial order. We shall build a partial order $P$ together with a tree decomposition $\iota\colon T \to P$ in stages such that, infinitely often, $g$ outputs an element of $P$ that does not extend to an infinite bad sequence.

Start with $T_0 = \{\varepsilon\}$ and $P_0$ having a single element $v_\varepsilon$, with $\iota_0(\varepsilon) = v_\varepsilon$. In stage $s$, we have built a finite tree decomposition $\iota_s\colon T_s \to P_s$ and wish to extend it to some $\iota_{s+1}\colon T_{s+1} \to P_{s+1}$. The tree $T_{s+1}$ will always be obtained by giving each leaf in $T_s$ a single successor, and then adding two successors to exactly one of the new leaves. To decide which leaf gets two successors, say a finite extension $Q$ of $P_s$ is *suitable* for $\iota_s\colon T_s \to P_s$ if for every $v \in Q \setminus P_s$, there is exactly one leaf $w \in T_s$ such that $\iota_s(w) <_Q v$. Pick the left-most leaf $\sigma$ of $T_s$ with the following property:

> There is some suitable extension $Q$ of $P_s$ such that, when given $Q$, the guessing procedure $g$ would guess an element of $Q$ which is comparable with $\iota_s(\sigma)$.

To see that such $\sigma$ must exist, consider extending $P_s$ by adding an "infinite comb" (i.e. a copy of $\{0^n1^i \mid n \in \mathbb{N}, i \in \{0,1\}\}$) above the $\iota_s$-image of a single leaf in $T_s$. The resulting partial order $Q$ is non-wqo, admits a tree decomposition (obtained by extending $T_s$ and $\iota_s$ in the obvious way), and its finite approximations (extending $P_s$) are suitable for $\iota_s$. Hence, by hypothesis, $g$ eventually guesses some element, which must be comparable with $\iota_s(\sigma)$, for some leaf $\sigma \in T_s$ (because all elements of $Q$ are).

Having identified $\sigma$, we fix any corresponding suitable extension $Q$ of $P_s$. In order to extend $\iota_s$, we further extend $Q$ to $Q'$ by adding a new maximal element $v_w$ to $Q$ for each leaf $w \in T_s$ as follows: $v_w$ lies above all $v \in Q \setminus P_s$ such that $\iota_s(w) <_P v$, and is incomparable with all other elements (including the other new maximal elements $v_{w'}$). To extend $T_s$, we add a new leaf $\tau^\frown 0$ to $T_s$ for each leaf $\tau$, obtaining a tree $T'$. We extend $\iota_s$ to yield a tree decomposition $\iota' : T' \to Q'$ in the obvious way.

Finally, we add two successors to $\sigma^\frown 0$ in $T'$, i.e., define $T_{s+1} = T' \cup \{\sigma^\frown 00, \sigma^\frown 01\}$. We also add two successors $v_1$, $v_2$ to $\iota'(\sigma^\frown 0)$ in $Q'$ to obtain $P_{s+1}$, and extend $\iota'$ to $\iota_{s+1}$ by setting $\iota_{s+1}(\sigma^\frown 0i) = v_i$. This concludes stage $s$.

It is clear from the construction that $\iota : T \to P$ is a tree decomposition. Let us discuss the shape of the tree $T$. In stage $s$, we introduced a bifurcation above a leaf $\sigma_s$ of $T_s$. These are the only bifurcations in $T$. Observe that, whenever $s' < s$, $\sigma_s$ is either above or to the right of $\sigma_{s'}$, because every suitable extension of $P_s$ is also a suitable extension of $P_{s'}$ and, at stage $s'$, the chosen leaf was the left-most. Therefore $T$ has a unique non-isolated infinite path $p = \lim_s \sigma_s$, and a vertex $w$ in $T$ is extendible to an infinite antichain in $T$ if and only if it does not belong to $p$.

We may now apply Lemma 2 to analyze $P$. First, since $T$ is not wqo, neither is $P$. Second, we claim that if $v <_P \iota(\sigma)$ for some vertex $\sigma$ on $p$, then $v$ is not extendible to an infinite bad sequence. To prove this, suppose $v$ is extendible. Then so is $\iota(\sigma)$. The proof of Lemma 2 implies that $\lceil \iota(\sigma) \rceil = \sigma$ is extendible to an infinite antichain in $T$. So $\sigma$ cannot lie on $p$, proving our claim.

To complete the proof, observe that our construction of $\iota$ ensures that for each $s$, $g(P)$ eventually outputs a guess which is below $\iota(\sigma_s{}^\frown 0)$. Whenever $\sigma_s{}^\frown 0$ lies along $p$ (which holds for infinitely many $s$), this guess is wrong by the above claim.

We may now complete the proof of Theorem 1.

*Proof (Theorem 1).* Suppose towards a contradiction that ${}^1\mathsf{BS} \leq_{\mathrm{W}} \boldsymbol{\Pi}^1_1\mathsf{-Bound}$. Since the problem of finding an element in a non-wqo which extends to an infinite bad sequence is first-order, it is Weihrauch reducible to $\boldsymbol{\Pi}^1_1\mathsf{-Bound}$ as well. Now $\boldsymbol{\Pi}^1_1\mathsf{-Bound}$ is upwards closed, so there is a computable guessing procedure for this problem (Lemma 1). However such a procedure cannot exist, even for partial orders which admit a tree decomposition (Lemma 3).

## 3 Separating KL and DS

Recall the following problems [2, §6, Def. 7.4(7), Thm. 8.10]:

$\mathsf{ACC}_{\mathbb{N}}$: Given an enumeration of a set $A \subseteq \mathbb{N}$ of size at most 1, find a number not in $A$.
$\mathsf{lim}$: Given a convergent sequence in $\mathbb{N}^{\mathbb{N}}$, find its limit.
$\mathsf{KL}$: Given an infinite finitely branching subtree of $\mathbb{N}^{<\mathbb{N}}$, find an infinite path through it.

It is known that $\widehat{\mathsf{ACC}_{\mathbb{N}}} <_{\mathrm{W}} \mathsf{lim} <_{\mathrm{W}} \mathsf{KL}$ (see [2]). For our separation of $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{DS}$ in Corollary 2, all we need to know about $\mathsf{KL}$ are the facts stated before the corollary. The core of our proof is the following.

**Theorem 2.** *Let $f$ be a problem. The following are equivalent:*

1. $\widehat{\mathsf{ACC}_{\mathbb{N}}} \times f \leq_{\mathrm{W}} \mathsf{DS}$
2. $f \leq_{\mathrm{W}} \mathsf{lim}$.

*Proof.* The implication from 2. to 1. follows from $\mathsf{lim} \leq_{\mathrm{W}} \mathsf{DS}$ (as shown in [8, Thm. 4.16]), as $\mathsf{lim}$ is closed under parallel product. For the other direction, we consider a name $x$ for an input to $f$ together with witnesses $\Phi, \Psi$ for the reduction. We show that, from them, we can uniformly compute an input $q$ to $\widehat{\mathsf{ACC}_{\mathbb{N}}}$ together with an enumeration of a set $W$ such that $W$ is the well-founded part of the (ill-founded) linear order $L$ built by $\Phi$ on $(q, x)$. We can then use $\mathsf{lim}$ to obtain the characteristic function of $W$. Having access to this lets us find an infinite descending sequence in $L$ greedily by avoiding ever choosing an element of $W$. From such a descending sequence $\Psi$ then computes a solution to $f$ for $x$.

It remains to construct $q = \langle q_0, q_1, \ldots \rangle$ and $W$ to achieve the above. At the beginning, $W$ is empty, and we extend each $q_i$ in a way that removes no solution from its $\mathsf{ACC}_{\mathbb{N}}$-instance. As we do so, for each $i \notin W$ (in parallel), we monitor whether the following condition has occurred:

> $L$ (as computed by the finite prefix of $(q, x)$ built/observed thus far) contains $i$ and some (finite) descending sequence $\ell$ such that
> 1. $\ell$ is $L$-above $i$ (i.e. $i <_L \min_L \ell$);
> 2. the functional $\Psi$, upon reading the current prefix of $(q, x)$ and $\ell$, produces some output $m$ for the $i$-th $\mathsf{ACC}_{\mathbb{N}}$-instance.

Once the above occurs for $i$ (if ever), we remove $m$ as a valid solution to $q_i$. This means that $\ell$ cannot be extendible to an infinite descending sequence in $L$, so $i$ must be in the well-founded part of $L$. Hence we shall enumerate $i$ into $W$. This completes our action for $i$, after which we return to monitoring the above condition for numbers not in $W$. This completes the construction.

It is clear that each $q_i$ is an $\mathsf{ACC}_{\mathbb{N}}$-instance (with solution set $\mathbb{N}$ if the condition is never triggered, otherwise with solution set $\mathbb{N} \setminus \{m\}$). Hence $L = \Phi(q, x)$ is an ill-founded linear order. As argued above, $W$ is contained in the well-founded part of $L$. Conversely, suppose $i$ lies in the well-founded part of $L$. Fix an infinite descending sequence $r$ which lies above $i$. Then $\Psi$ has to produce all $\mathsf{ACC}_{\mathbb{N}}$ answers upon receiving $(q,x)$ and $r$, including an answer to $q_i$. This answer is determined by finite prefixes only, and after having constructed a sufficiently long prefix of $q$, some finite prefix $\ell$ of $r$ will trigger the condition for $i$ (unless something else triggered it previously), which ensures that $i$ gets placed into $W$. This shows that $W$ is exactly the well-founded part of $L$, thereby concluding the proof.

**Corollary 1.** *If $f$ is a parallelizable problem (i.e., $f \equiv_{\mathrm{W}} \widehat{f}$) with $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$, then $f \leq_{\mathrm{W}} \mathsf{lim}$.*

*Proof.* Since $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$ and $f$ is parallelizable, we have $\widehat{\mathsf{ACC}_{\mathbb{N}}} \times f \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$. By the previous theorem, $f \leq_{\mathrm{W}} \mathsf{lim}$.

Since $\mathsf{KL}$ is parallelizable, $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{KL}$, yet $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{lim}$, we obtain a negative answer to [8, Question 6.1]:

**Corollary 2.** $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{DS}$.

Similarly, consider the problem $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ of enumerating all elements (possibly with repetition) of a given non-empty countable closed subset of $2^{\mathbb{N}}$. Since $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ is parallelizable, $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{lim} \leq_{\mathrm{W}} \mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$, yet $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega} \not\leq_{\mathrm{W}} \mathsf{lim}$ [9, Prop. 6.12, 6.13, Cor. 6.16], we obtain a negative answer to [8, Question 6.2]:

**Corollary 3.** $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega} \not\leq_{\mathrm{W}} \mathsf{DS}$.

Note that $\mathsf{KL}$ is a parallelization of a first-order problem (such as $\mathsf{RT}^1_2$). Using a recent result of Pauly and Soldà [12], we can characterize (up to *continuous* Weihrauch reducibility $\leq^*_{\mathrm{W}}$) the parallelizations of first-order problems which are reducible to $\mathsf{DS}$.

**Corollary 4.** *If $\widehat{f} \leq^*_{\mathrm{W}} \mathsf{DS}$, then ${}^1f \leq^*_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. Therefore, for any first-order $f$,*

$$\widehat{f} \leq^*_{\mathrm{W}} \mathsf{DS} \quad \text{if and only if} \quad f \leq^*_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}.$$

*Proof.* If ${}^1f$ is continuous, the conclusion of the first statement is satisfied. Otherwise, $\mathsf{ACC}_{\mathbb{N}} \leq^*_{\mathrm{W}} f$ by [12, Thm. 1]. The relativization of Theorem 2 then implies $\widehat{f} \leq^*_{\mathrm{W}} \mathsf{lim}$. We conclude ${}^1f \leq^*_{\mathrm{W}} {}^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. The second statement then follows from $\widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{lim} \leq_{\mathrm{W}} \mathsf{DS}$.

## 4 The finitary part and deterministic part of BS

In this section, we show that $\mathsf{BS}$ and $\mathsf{DS}$ cannot be separated by looking at their respective finitary or deterministic parts. Recall from [8, Thms. 4.16, 4.31] that $\mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$ and $\mathrm{Fin}_{\mathbf{k}}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{RT}^1_k$. Since both the deterministic and the finitary parts are monotone, this implies that $\mathsf{lim} \leq_{\mathrm{W}} \mathrm{Det}(\mathsf{BS})$ and $\mathsf{RT}^1_k \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(\mathsf{BS})$, so we only need to show that the converse reductions hold.

To this end, we first introduce a technical lemma. For a fixed partial order $(P, \leq_P)$, we can define the following quasi-order on the (finite or infinite) $\leq_P$-bad sequences:

$$\alpha \trianglelefteq^P \beta :\iff \alpha = \beta \text{ or } (\exists i < |\alpha|)(\forall j < |\beta|)(\alpha(i) \leq_P \beta(j)).$$

We just write $\trianglelefteq$ when the partial order is clear from the context.

**Lemma 4.** *Let $(P, \leq_P)$ be a non-well partial order and let $\alpha, \beta$ be finite $\leq_P$-bad sequences. If $\alpha \trianglelefteq \beta$ and $\alpha$ is extendible to an infinite $\leq_P$-bad sequence, then so is $\beta$. If $\alpha$ is not extendible then there is an infinite $\leq_P$-bad sequence $B \in \mathbb{N}^\mathbb{N}$ such that $\alpha \trianglelefteq B$. (Hence $\alpha \trianglelefteq \beta$ for every initial segment $\beta$ of $B$.)*

*Proof.* To prove the first part of the theorem, fix $\alpha \trianglelefteq \beta$ and let $A \in \mathbb{N}^\mathbb{N}$ be an infinite $\leq_P$-bad sequence extending $\alpha$. Let also $i < |\alpha|$ be a witness for $\alpha \trianglelefteq \beta$. For every $j > i$ and every $k < |\beta|$, $\beta(k) \not\leq_P A(j)$ (otherwise $A(i) = \alpha(i) \leq_P \beta(k) \leq_P A(j)$ would contradict the fact that $A$ is a $\leq_P$-bad sequence), which implies that $\beta$ is extendible.

Assume now that $\alpha$ is non-extendible and let $F \in \mathbb{N}^\mathbb{N}$ be a $\leq_P$-bad sequence. We show that there is $i < |\alpha|$ and infinitely many $k$ such that $\alpha(i) <_P F(k)$. This is enough to conclude the proof, as we could take $B$ as any subsequence of $F$ with $\alpha(i) <_P B(k)$ for every $k$ (i.e. $\alpha \trianglelefteq B$).

Assume that, for every $i < |\alpha|$ there is $k_i$ such that for every $k \geq k_i$, $\alpha(i) \not\leq_P F(k)$ (since $P$ is a partial order, there can be at most one $k$ such that $\alpha(i) = F(k)$). Since $\alpha$ is finite, we can take $k := \max_{i<|\alpha|} k_i$ and consider the sequence $\alpha^\frown(F(k+1), F(k+2), \ldots)$. We have now reached a contradiction as this is an infinite $\leq_P$-bad sequence extending $\alpha$.

Let $(P, \leq_P)$ be a partial order. We call a $A \subseteq P$ *dense* if for every $w \in P$ there is some $u \geq_P w$ with $u \in A$. We call it *upwards-closed*, if $w \in A$ and $w \leq_P u$ implies $u \in A$. By $\boldsymbol{\Sigma}^1_1\text{–}\mathsf{DUCC}$ we denote the following problem: Given a partial order $P$ and a dense upwards-closed subset $A \subseteq P$ (given via a $\Sigma^1_1$ code), find some element of $A$. We can think of a $\Sigma^1_1$ code for $(P, \leq_P)$ as a sequence $(T_n)_{n\in\mathbb{N}}$ of subtrees of $\mathbb{N}^{<\mathbb{N}}$ such that, for every $n, m \in \mathbb{N}$, $n \leq_P m$ iff $T_{\langle n,m\rangle}$ is ill-founded (and $n \in P$ iff $n \leq_P n$). We refer to [8] for a more detailed discussion on various presentations of orders.

**Proposition 3.** ${}^1\mathsf{BS} \leq_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{–}\mathsf{DUCC}$.

*Proof.* Let $f$ be a problem with codomain $\mathbb{N}$ and assume $f \leq_{\mathrm{W}} \mathsf{BS}$ via $\Phi, \Psi$. Fix $x \in \mathrm{dom}(f)$. Let $(P, \leq_P)$ denote the non-well partial order defined by $\Phi(x)$. We say that a finite $\leq_P$-bad sequence $\beta$ is *sufficiently long* if $\Psi(x, \beta)$ returns a natural number in at most $|\beta|$ steps.

To show that $f \leq_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{–}\mathsf{DUCC}$, it is enough to notice that Lemma 4 implies that the set of sufficiently long finite extendible bad sequences is $\Sigma^{1,x}_1$, non-empty, dense, and upwards-closed with respect to $\trianglelefteq^P$.

**Lemma 5.** $\boldsymbol{\Sigma}^1_1\text{–}\mathsf{DUCC} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{–}\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$*, where the latter denotes the restriction of the former to $2^{<\mathbb{N}}$ with the prefix ordering $\sqsubseteq$.*

*Proof.* Clearly, we only need to show that $\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC} \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$. Let the input be $((P, \leq_P), A)$ where $A$ is a $\boldsymbol{\Sigma}_1^1$, non-empty, dense, and upwards-closed subset of the partial order $(P, \leq_P)$. We uniformly define a computable labelling $\lambda\colon 2^{<\mathbb{N}} \to P$ such that $\lambda^{-1}(A)$ is non-empty, dense, and upwards-closed. This suffices to prove the claimed reduction, as the preimage of $A$ via $\lambda$ is (uniformly) $\boldsymbol{\Sigma}_1^1$ and, given $\sigma \in \lambda^{-1}(A)$, we can simply compute $\lambda(\sigma) \in A$.

Let $(x_n)_{n\in\mathbb{N}}$ be an enumeration of $P$. We define an auxiliary computable function $\overline{\lambda}\colon P \times 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ as follows: for every $i$, $\overline{\lambda}(x, 0^i) := \overline{\lambda}(x, 0^i 1) := x$. To define $\overline{\lambda}(x, 0^i 1 b^\frown\sigma)$ we distinguish two cases:

$$\overline{\lambda}(x, 0^i 1 b^\frown \sigma) := \begin{cases} \overline{\lambda}(x, \sigma) & \text{if } b = 0 \text{ or } x \not\leq_P x_i \\ \overline{\lambda}(x_i, \sigma) & \text{if } b = 1 \text{ and } x \leq_P x_i. \end{cases}$$

We then define $\lambda(\sigma) := \overline{\lambda}(x_0, \sigma)$. It is clear that $\lambda$ is computable and total. Let us show that $\lambda^{-1}(A)$ is a valid input for $\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$. Observe first that, for every $\sigma \sqsubseteq \tau$, $\lambda(\sigma) \leq_P \lambda(\tau)$, which implies that $\lambda^{-1}(A)$ is upwards-closed.

To prove the density, fix $\sigma \in 2^{<\mathbb{N}}$ and assume $\lambda(\sigma) \notin A$. Since $A$ is dense, there is $i$ such that $\lambda(\sigma) \leq_P x_i \in A$. Let $\tau \sqsupseteq \sigma$ be such that, for every $\rho$, $\lambda(\tau^\frown\rho) = \lambda(\rho)$. Notice that such $\tau$ always exists: indeed, if $\overline{\sigma}$ is the longest tail of $\sigma$ of the form $0^j$ or $0^j 1$ for some $j$, then $\tau := \sigma^\frown d0$, where $d = 1$ if $\overline{\sigma} = 0^j$ and $d = \varepsilon$ otherwise, satisfies the above requirement. In particular, $\lambda(\tau^\frown 0^i 11) = x_i$. This proves that $\lambda^{-1}(A)$ is dense and therefore concludes the proof.

**Theorem 3.** $\mathrm{Fin}_{\mathbf{k}}(\mathsf{BS}) \equiv_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}\big) \equiv_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{RT}_k^1$.

*Proof.* We have $\mathsf{RT}_k^1 \equiv_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(\mathsf{DS}) \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(\mathsf{BS}) \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}\big)$ by [8, Thm. 4.31] and Proposition 3. It remains to show that $\mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}\big) \leq_{\mathrm{W}} \mathsf{RT}_k^1$. By Lemma 5, it is enough to show that $\mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)\big) \leq_{\mathrm{W}} \mathsf{RT}_k^1$.

Let $f$ be a problem with codomain $k$ and assume $f \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1\text{–}\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$ via $\Phi, \Psi$. Observe that every $x \in \mathrm{dom}(f)$ induces a coloring $c\colon 2^{<\mathbb{N}} \to k$ as follows: run $\Psi(x, \sigma)$ in parallel on every $\sigma \in 2^{<\mathbb{N}}$. Whenever we see that $\Psi(x, \sigma)$ returns a number less than $k$, we define $c(\tau) := \Psi(x, \sigma)$ for every $\tau \sqsubseteq \sigma$ such that $c(\tau)$ is not defined yet. By density of $\Phi(x)$, $c$ is total.

By the Chubb-Hirst-McNicholl tree theorem [3], there is some $\sigma \in 2^{<\mathbb{N}}$ and some color $i < k$ such that $i$ appears densely above $\sigma$. We claim that if $i$ appears densely above $\sigma$ then $i \in f(x)$. To prove this, recall $\Phi(x)$ codes a set which is dense and upwards-closed. By density of both this set and the color $i$, we may fix some $\tau \sqsupseteq \sigma$ which lies in the set coded by $\Phi(x)$ and has color $i$. Then fix $\rho \sqsupseteq \tau$ such that $c(\tau)$ was defined to be $\Psi(x, \rho)$. Now $\rho$ lies in the set coded by $\Phi(x)$ as well, so $i = c(\tau) = \Psi(x, \rho)$ lies in $f(x)$.

Since a $k$-coloring of $2^{<\mathbb{N}}$ can be naturally turned into a $k$-coloring of $\mathbb{Q}$ (using a canonical computable order isomorphism between them), the problem "given a $k$-coloring of $2^{<\mathbb{N}}$, find $\sigma$ and $i$ such that $i$ appears densely above $\sigma$" can be solved by $\mathsf{RT}_k^1$, as shown in [11, Cor. 42].

It is immediate from the previous theorem that the finitary parts of $\mathsf{BS}$ and $\mathsf{DS}$ in the sense of Cipriani and Pauly [4, Definition 2.10] agree as well. Finally, we shall prove that the deterministic parts of $\mathsf{BS}$ and $\mathsf{DS}$ agree.

**Lemma 6.** *If* $\mathrm{Fin}_{\mathbf{2}}(f) \le_{\mathrm{W}} \mathsf{RT}^1_2$, *then* $\mathrm{Det}(f) \le_{\mathrm{W}} \mathsf{lim}$.

*Proof.* By algebraic properties of Det and $\mathrm{Fin}_{\mathbf{k}}$, we have

$$\mathrm{Det}(f) \le_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbf{2}}(f)} \le_{\mathrm{W}} \widehat{\mathrm{Fin}_{\mathbf{2}}(f)} \le_{\mathrm{W}} \widehat{\mathsf{RT}^1_2} \equiv_{\mathrm{W}} \mathsf{KL}.$$

So $\mathrm{Det}(f) \le_{\mathrm{W}} \mathrm{Det}(\mathsf{KL}) \equiv_{\mathrm{W}} \mathsf{lim}$: Use the fact $\mathsf{KL} \equiv_{\mathrm{W}} \mathsf{WKL} * \mathsf{lim}$ and the choice elimination principle [2, 11.7.25] (see also [8, Thm. 3.9]).

Since $\mathrm{Det}(\mathsf{BS}) \ge_{\mathrm{W}} \mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$ [8, Thm. 4.16], we conclude that:

**Corollary 5.** $\mathrm{Det}(\mathsf{BS}) \equiv_{\mathrm{W}} \mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$.

**Corollary 6.** $\mathrm{Det}_{\mathbb{N}}(\mathsf{BS}) \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

*Proof.* Since $\mathbb{N}$ computably embeds in $\mathbb{N}^{\mathbb{N}}$, for every problem $f$ we have $\mathrm{Det}_{\mathbb{N}}(f) \le_{\mathrm{W}} \mathrm{Det}(f)$. In particular, by Corollary 5, $\mathrm{Det}_{\mathbb{N}}(\mathsf{BS}) \le_{\mathrm{W}} \mathrm{Det}(\mathsf{BS}) \equiv_{\mathrm{W}} \mathsf{lim}$. Since $^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ ([1, Prop. 13.10], see also [13, Thm. 7.2]), this implies $\mathrm{Det}_{\mathbb{N}}(\mathsf{BS}) \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. The converse reduction follows from the fact that $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(\mathsf{DS})$ [8, Prop. 4.14].

We remark that for establishing $\mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}^1_1\text{−}\mathsf{DUCC}\big) \le_{\mathrm{W}} \mathsf{RT}^1_k$ in Theorem 3 it was immaterial that the set of correct solutions was provided as a $\Sigma^1_1$-set. If we consider any other represented point class $\boldsymbol{\Gamma}$ which is effectively closed under taking preimages under computable functions, and define $\boldsymbol{\Gamma}\text{−}\mathsf{DUCC}$ in the obvious way, we can obtain:

**Corollary 7.** $\mathrm{Fin}_{\mathbf{k}}(\boldsymbol{\Gamma}\text{−}\mathsf{DUCC}) \le_{\mathrm{W}} \mathsf{RT}^1_k$.

This observation could be useful e.g. for exploring the Weihrauch degree of finding bad arrays in non-better-quasi-orders (cf. [7]).

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