Journal of Logic, Language and Information
https://doi.org/10.1007/s10849-024-09420-w

Logical Models of Mathematical Texts: The Case of Conventions for Division by Zero

Jan A. Bergstra[1] **· John V. Tucker**[2]

Accepted: 26 June 2024

Abstract

Arithmetical texts involving division are governed by conventions that avoid the risk of problems to do with division by zero (DbZ). A model for elementary arithmetic texts is given, and with the help of many examples and counter examples a partial description of what may be called *traditional conventions* on DbZ is explored. We introduce the informal notions of legal and illegal texts to analyse these conventions. First, we show that the legality of a text is algorithmically undecidable. As a consequence, we know that there is no simple sound and complete set of guidelines to determine unambiguously how DbZ is to be avoided. We argue that these observations call for further explorations of mathematical conventions. We propose a method using logics to progress the analysis of legality versus illegality: arithmetical texts in a model can be transformed into logical formulae over special total algebras that are able to approximate partiality but in a total world. The algebras we use are called *common meadows*. Our dive into informal mathematical practice using formal methods opens up questions about DbZ which we address in conclusion.

Keywords Division by zero · Arithmetic · Traditional conventions for writing mathematics · Legal texts · Illegal texts · Undecidability · Common meadows

1 Introduction

A partial function is a function $f : X \to Y$ that does not have values for all of its arguments: for at least one argument $x \in X$ there is no result $f(x)$. Division x/y in the arithmetic of numbers is a partial function. On the natural numbers $\mathbb{N}$ and the integers

✉ John V. Tucker
j.v.tucker@swansea.ac.uk

Jan A. Bergstra
j.a.bergstra@uva.nl; janaldertb@gmail.com

1 Informatics Institute, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands

2 Department of Computer Science, Swansea University, Bay Campus, Fabian Way, Swansea SA1 8EN, UK

Published online: 13 July 2024

$\mathbb{Z}$, there are infinitely many arguments y for which x/y has no value; on the rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$ there is just one: $y = 0$.

In writing about division, conventions guard against partiality by writing texts such as:

$$f(x) = 1/x \text{ for } x \neq 0 \text{ or } f(x) = 1/(x-5) \text{ for } x \neq 5.$$

Indeed, said differently, writing simply

$$f(x) = 1/x \text{ or } f(x) = 1/(x-5)$$

invites a teacher to reach for the red pencil to point out the omissions and deduct marks! The above texts can be said to be proper or improper, well-formed or ill-formed, or even correct or wrong; we will prefer the terms *legal* and *illegal*.

In practice, writing legal texts means adhering to 'traditional conventions' that try to avoid DbZ. Why do we make this strict distinction for DbZ? At work seems to be this convention about partiality:

Avoidance Principle *Do not write down mathematical expressions unless you know they denote something.*

In this paper we investigate the ways $1/0$ casts its shadow on the basic act of writing legal arithmetical texts, where legal takes its meaning from traditional conventions motivated by the Avoidance Principle. We wish to cast light on some current mathematical practices and conventions regarding partiality. For this purpose, writing down $1/0$ is sufficiently provocative to demonstrate that such traditional conventions are deeply established.[1]

But what are these traditional conventions? Our somewhat ambitious aims are: (i) to establish a problem worthy of research—the elucidation and specification of traditional conventions on partiality; and (ii) to propose a methodology and some mathematical tools to begin to undertake this investigation.

In writing mathematical texts, especially elementary texts, we make declarations, definitions, assumptions, assertions, deductions, exercises, etc. These are common elements of mathematical texts. To model arithmetical texts, we propose a simple language *VEAT* for *very elementary arithmetical texts* containing division. To explore formally the consequences of the traditional conventions as they apply to division by zero, we seek to isolate the rules that separate the legal texts from the illegal texts in the language *VEAT*.

By arithmetic, we have in mind the elementary arithmetic of numbers as found in school books around the world. By texts, we have in mind syntax: lists containing formalisations of variable declarations, assumptions, assertions, implications, and constructed using the operations of arithmetic in expressions—specifically, sums, differences, products, divisions. Texts need not be true, but they ought to be legal. In the course of our analysis, we prove this fact:

Theorem *For the mathematical text language VEAT, the legality/illegality of texts is algorithmically undecidable.*

[1] There are other partial functions in arithmetic to play with, such as subtraction on the naturals, but they cannot attract the same level of attention; indeed, the partiality of subtraction motivates the invention of the integers to dispense with the issue.

Thus, there does *not* exist a finite system of rules that can establish precisely which texts are legal and which are illegal. This theorem sets the scene for the techniques that make up the main body of the paper: we do not see this negative result as the end but as the beginning of a more complex forensic investigation of traditional conventions about DbZ. This leads to a number of new notions—e.g., *logical poles*—and four viable definitions of legality for the texts of *VEAT*.

We propose to turn to logical methods to try to formalise semantic conventions for legality in *VEAT*. Now logics are well understood and far easier to work with when applied to total algebras rather than to partial algebras. Our aim is to develop a formal model of legal texts using first order logical formulae over total arithmetical algebras.

The algebras we choose are *common meadows*. Meadows are fields with explicit division operators added (Bergstra & Tucker, 2007); common meadows are meadows in which division is made total by adding an absorptive value $\bot$ in order to define $1/0 = \bot$, (Bergstra & Ponse, 2016, 2021). Our standard common meadow is that of the rational numbers $\mathbb{Q}_\bot$. In summary:

Method *Our method chooses a logic L over a common meadow M. On transforming texts from the language VEAT into formulae in the logic L,*

$$T \in VEAT \to \phi_T \in L,$$

we examine the correspondence and classify the legal texts and illegal texts of VEAT by analysing the well-formed formulae in L representing the texts.

Thus, this method searches for old and new logics that yield more faithful formal approximations to legal texts. All are applied to common meadows and we expect that the well-formed formulae we characterise to be a superset of the (translations of the) legal texts. Thus, we seek to uncover and approximate formally the underlying logic of writing conventions.

Our ideas are general and can be applied to all partial functions. We focus on division because this work belongs to a (now extensive) programme on the semantics of computing with arithmetic structures (which we sketch later). In particular, division perfectly displays key semantic issues at the level of elementary mathematical education.

Structure of the paper. In Sect. 2, we discuss some of the background work that led to this investigation. In Sect. 3 we begin by analysing four kinds of divisions $\div$ and looking at partiality. In Sect. 4, we look at the conventions for dealing with DbZ by examining several examples of texts and classifying them as legal or illegal; we also define the tiny language *VEAT* of arithmetical texts. In Sect. 5, we prove the theorem above. In Sect. 6, we introduce the common meadows that have division but whose values are always total using a value $\bot$, i.e, $1 \div 0 = \bot$. We propose to effect a translation of texts in *VEAT* into logical formulae over these structures to model and clarify formally legality/illegality. In the last section we reflect on where we are in the quest and on technical issues to do with DbZ.

2 On Texts, Conventions and Division

We briefly sketch some background to this investigation into conventions, including technical work on the algebra of division.

2.1 Texts and Conventions

Mathematical texts share structures and notations that are common to many countries. They are organised by mathematical conventions that have evolved over centuries, even millennia—the structure created by the definitions, postulates, common notions, and propositions in Euclid's *13 Elements* is an ancient example. The emergence of arithmetic and algebra in their practical and theoretical forms expanded the nature of mathematical texts and developed their own conventions, notably in the search for clarity and precision of thought.

The sort of mathematical texts we have in mind are texts describing ideas, deductions and calculations with numbers; typical authors and readers of such texts are mathematicians, teachers, and school pupils who must 'show their workings'. Despite centuries of refinement and a concern for rigour, not all mathematical ideas are understood universally, even in arithmetical texts.

Elementary arithmetic as taught in schools and practiced in everyday life for centuries is the mathematics of addition, subtraction, multiplication and division. The mathematics has been largely settled since the 16th Century with 'modern' notation and vernacular textbooks Morgan (2024). Yet, the teaching and interpretations of elementary arithmetic are not always settled professionally: in the case of division, there are divergent positions on division by zero and the idea of fraction.

Thus, specifically motivated by problems of division by zero in computing, and by the role(s) of fractions in mathematics education, in Bergstra and Tucker (2023), we constructed a systematic informal description of a consensus on elementary arithmetic. Examining what we termed Raw Arithmetic, we made explicit what ideas and options were widely accepted, rejected or simply varied according to particular perspectives or personal tastes. Our account recorded many flexible precepts that could generate a range of informal options on behalf of arithmetical practitioners. This analysis of requirements we termed Naive Arithmetic. It identified a significant gap between working practices and the world of precisely defined formal calculi. In a sequel (Bergstra & Tucker, 2024), to bridge that gap, we introduced an informal axiomatisation called Synthetic Arithmetic designed to resolve ambiguities and prepare for more formal reasoning underpinned by logic.

The technical needs of division require special attention to fractions. The explorations of Raw Arithmetic reveal considerable conceptual difficulties with this term. One issue is the range of meanings implied by the use of the word 'fraction', which impacts any discussion of $\frac{1}{0}$.

2.2 Axiomatising and Partiality

The sequence of Raw, Naive and Synthetic stages constitute a systematic reflection on elementary arithmetical texts. By uncovering certain notions and conventions we are led to some idea of 'type-checking for conformance' against these conventions. In fact, we show in this paper that any attempt at type-checking against conventions in arithmetical texts, even in a narrowly designed language fragment, proves to be very complicated *viz.* theoretically undecidable and pragmatically very challenging.

The three stages also constitute a rather thorough informal preparation to accompany a formal axiomatisation of arithmetic. In our context, one role of an axiomatisation is to create a formalism that removes a need for 'type-checking' mathematical texts against the principles of (say) the informal axiomatisation of Synthetic Arithmetic.

We believe that the value of a systematic analysis to a teacher is to surface insights hidden in their practice and so can contribute to his or her mastery of the subject matter. As an analytical tool, one objective of formalisation is to detect, make explicit and settle technically subtle points and relations; this process leads to a ramification of options and, so, to a plurality of formal calculi. The four stages of Raw, Naive, Synthetic and Formal constitute a methodology that can be of use wherein a practice might benefit from analytical reflection (see Bergstra and Tucker (2024)).

Our early interest in formalisations of arithmetic are motivated by thinking about the algebra and logic of computer arithmetics, and our studies of computer arithmetics as abstract data types that need axiomatic specifications; this we summarise later in Sect. 7.3. Data types in computing are structures modelled by sets with operations (including boolean operations), i.e., many sorted algebras (Ehrich et al., 1997). Computer programs invoking these operations must not fail to return a value of some kind—when executed, they must be total operations. Moreover, these algebras are specified axiomatically by equations and conditional equations expressing properties of the operations (Ehrich et al., 1997). Partiality once present is not easy to control in computation or formal reasoning (cf. Andreka et al. (1988); Gavilanes-Franco and Lucio-Carasco (1990); Jones and Middelburg (1994); Robinson (1989)).

So, the key point is this: on employing division in a computer then $\frac{x}{0}$ must return data of *some* kind. We have studied some five basic equational calculi, primarily distinguished by semantic decisions about the meaning of $\frac{1}{0}$. Here, we meet some formal calculi with semantics that are currently recognisable in the schoolroom, based on two such decisions: the cases when

(i) $\frac{1}{0}$ does not exist—the expected school orthodoxy, and
(ii) $\frac{1}{0}$ is an error flag—the common semantics of school calculators.

We have studied this latter case in some detail in the build-up to the current state of our theory in Bergstra and Tucker (2023), Bergstra and Tucker (2024).

3 Division and Partiality

3.1 Syntax for Arithmetic

Notions of division can be defined in terms of multiplication in several ways.

Let Σ_r be a signature with 0, 1 as constants and function symbols for addition $x + y$, opposite $-x$, and multiplication $x \cdot y$. Σ_r is the signature of both rings and fields. We add a function symbol $\div$ for division, with the design decision that $x \div y$ is intentionally partial. This is a signature Σ_m for meadows (to be defined shortly) (Bergstra & Tucker, 2007).

This syntax allows us to introduce the notion of fracterm (Bergstra & Ponse, 2016):

Definition 3.1 A *fracterm* is a term over the meadow signature Σ_m whose leading function symbol is division.

To make $\div$ partial is a design decision and is consistent with multiplication as repeated addition and division as repeated subtraction. Thus, the decision does *not* suggest that it would be impossible, or even difficult, to specify useful values for $1 \div 0$; in fact, there is a long tradition of assigning values to $1 \div 0$, e.g., the value 0 or the symbol ∞: for more information we refer to these surveys (Bergstra, 2019) and Bergstra (2021) and subsect. 7.3.

3.2 What is Division?

The partial division operator we use is this:

Definition 3.2 Given a structure A for the signature Σ_r of rings. The possibly partial function $\div$ is defined on A as follows: for $a, b, c \in A$, $a \div b = c$ if, and only if, c is the unique element of A such that $b \cdot c = c \cdot b = a$. We refer to this as the *standard partial division operator*.

Definition 3.3 If $a \div b = c$ then c is called the *quotient* of a and b.

Proposition 3.1 *Let A be a structure with the signature Σ_r of rings. Suppose*

(i) $A \models (\forall x)[0 \cdot x = 0]$ *and*
(ii) A has more than one element.

Then a standard partial operator $\div$ is not a total function as $\frac{x}{0}$ is not defined.

Proof Let $a \in A$, $a \neq 0$ and suppose $a \div 0 = c$. Then, comparing Definition 3.2,

$$a \div 0 = c \iff 0 \cdot c = c \cdot 0 = a.$$

By (i), $a = c \cdot 0 = 0$, which contradicts the choice of a.

With Definition 3.2, the partial function $\div$ is unambiguously defined. It is also possible that the interpretation of $\div$ is a proper partial function but is not standard; this will happen for instance if $0 \div 0$ is considered undefined and $x \div 0 = 0$ when $x \neq 0$.

This Definition 3.2 is sufficiently general as long as multiplication is commutative. A central algebraic idea is this, Bergstra and Tucker (2007):

Definition 3.4 A *meadow* is a field G expanded with a division operator $\div$, which we denote $G(\div)$.

In case of a non-commutative multiplication operator, two additional division operators arise, left division $\div_l$, and right division $\div_r$, which may or may not differ for a given A. In the above definition, for left division only $c \cdot_l b = a$ is required; and for right division only $b \cdot_r c = a$ is required. The relations between the following division operations are not obvious unless multiplication is commutative, in which case the four definitions for $\div$, $\div_l$, $\div_r$, $\div_{lr}$, respectively, determine the same functions.

Definition 3.5 Given a structure A for the signature Σ_r of rings. The possibly partial function $\div_l$ (called the standard partial left division operator) is defined as follows: for $a, b, c \in A$, $a \div_l b = c$ if, and only if, c is the unique element of A such that $c \cdot b = a$.

Definition 3.6 Given a structure A for the signature Σ_r of rings. The possibly partial function $\div_r$ (called the standard partial right division operator) is defined as follows: for $a, b, c \in A$, $a \div_r b = c$ if, and only if, c is the unique element of A such that $b \cdot c = a$.

Definition 3.7 Given a structure A for the signature Σ_r of rings. The possibly partial function $\div_{lr}$ (called the standard partial left/right division operator) is defined as follows: for $a, b, c \in A$, $a \div_{lr} b = c$ if, and only if, c is the unique element of A such that $c \cdot b = a$ or $b \cdot c = a$ (or both).

We will allow structures where division is defined on some arguments where standard partial division is not defined; in such cases, we will not say that a structure is equipped with a standard partial division operator.

3.3 On Defining Partial Functions

Defining partial functions in a generic manner is non-trivial. For instance, the square root function is partial on the real numbers and total on the complex numbers, if one adopts the convention to return principal square roots, such as $+\sqrt{-1}$. The notion of a principal square root is quite specific and does not generalise simply to arbitrary rings and fields. This raises the question if there is a general definition of a square root function. For instance:

Definition 3.8 Given a structure A for the signature Σ_r of rings, a partial or total unary function $f(-)$ is called a *square root function* if for all $a \in A$,

(i) $f(a)$ is defined if, and only if, for some $b \in A$, $b \cdot b = a$, and
(ii) if $f(a)$ is defined then $f(a) \cdot f(a) = a$.

Proposition 3.2 *There are infinitely many square root functions on the ring $\mathbb{Z}$ integers in the sense of Definition 3.8.*

It seems reasonable to say "let $\sqrt{x}$ be a square root function". In practice, however, that is far too liberal. By choosing the principal square root one finds a function which is holomorphic outside the non-positive reals (assuming one works in a field of complex numbers).

In the light of this digression, we see that the traditional convention to think in terms of a unique reasonable result—so evident in the case of division—breaks down already in the case of square roots. That a result is sought which *uniquely* satisfies a certain condition may be standard practice in the case of division, but it is not a paradigm which derives from how to deal with partial functions in general.

Proposal 3.1 *Disallowing say* $0 \div 0 = 0$ *as an element of the graph of* $\div$ *is a matter of design. Here, the design decision in the definition of division works against disambiguation in case more than one element* c *meets the multiplication criterion (here* $0 \cdot c = 0$*).*

One might call 0 the ‘principal quotient’ of 0 and 0 in which case it would become plausible to include $0 \div 0 = 0$ in the graph of $\div$. Doing so is not standard practice. However, there is no notion of “understanding division” which would stand in the way of adopting $0 \div 0 = 0$ just as there is no notion of “understanding square roots” which stands in the way of adopting $\sqrt{-1} = i$. The fact of the matter is that conventions have developed otherwise so that $0 \div 0$ is generally considered to be undefined.

4 What are Legal Texts Concerning DbZ?

We now begin our exploration of what are the legal texts according to traditional conventions operating on DbZ. We introduce informally a concrete syntax for a simple language *VEAT* and give a stock of examples of legal and illegal fragments of mathematical texts in *VEAT*. For our purposes, *VEAT* is sufficiently simple that it does not need a formal definition for either its syntax or semantics—indeed, that it is informal is appropriate for our investigation of informal conventions.

4.1 An Informal Model for Texts on Elementary Arithmetic

4.1.1 Arithmetic Data

The arithmetic nature of *VEAT* is established by *sorts* that name arithmetic types such as natural numbers, integers, rationals; typically these are nat, int, rat. To these are added *arithmetical properties* $R, S, \ldots$ of numbers based on arithmetical operations and tests, typically these are the operations

$$+, -, \cdot, \div, \text{ and } = .$$

Thus, inside the statements of *VEAT* are arithmetic formulae, which are written over the signature Σ_m of the meadow (cf. Sect. 3). Semantically, these arithmetical operations and tests will be interpreted in the integers and rationals. Of course, central to our investigation are the arithmetic terms which feature the operator $\div$, especially the fracterms of Definition 3.1.

4.1.2 Very Elementary Arithmetical Language

A *text* in the very elementary arithmetical language *VEAT* is a sequence of statements separated by semi-colons. The model will have different classes of statements and to start only these classes are considered:

1. Declarations naming global variables that we call *parameters*: given $x, y, \ldots : sort$
2. Additional assumptions about parameters, written in the form: moreover R. Here R is an equation $t = r$, or a negated equation $t \neq r$, where t and r are division-free; these are the atomic formulae of the language.
3. Assertions that are arithmetical properties: now R.
4. Assertions combined by the implication connective: now-if R then S, with R, S atomic formulae.

Here given, moreover and now are reserved words which indicate the role of a statement or part of a statement in the language.

The texts can be adapted to include more familiar means of expression. For example, we might write

$$\text{given } x, y : sort \text{ such that } R$$

as an alternate for

$$\text{given } x, y : sort; \text{ moreover } R.$$

4.1.3 An Informal Explanation of Meaning for *VEAT*

A *VEAT* text may be understood as a sequence of assertions. The idea is that *VEAT* texts are read from left to right as sequences (or top to bottom if displayed vertically). Variables may be declared only once in a text, and from the position of declaration onwards a variable keeps the same type and value. Variables may be used in assertions only after having been declared.

Given a valuation σ of variables into the rationals $\mathbb{Q}$, each text ϕ in *VEAT* can be true (T), false (F) or meaningless (M). Each assertion in ϕ can be evaluated: the result is M if an evaluation of a term involved yields undefined—shortly to be indicated by $\perp$; otherwise, the result is either T or F accordingly.

An implication $\phi =$ now-if R then S, is evaluated in the "short-circuit manner": if R evaluates to F then ϕ evaluates to T. If R evaluates to T then ϕ evaluates to the evaluation result of S. If R evaluates to M then so does ϕ.

For a text the following qualitative aspects may be distinguished:

- *Syntactic correctness*. For instance,

$$\text{given } x : \mathbb{Q}; \text{ given } x : \mathbb{Q}; \text{ now } x = x$$

  is not syntactically correct as x is declared twice; it is assumed that syntactic correctness is independent of properties of consistency, validity, and legality. So, the use of say $\frac{1}{0}$ will not be taken for a syntactic problem.
- *Consistency*. The combination of all assumptions that occur in the text is consistent.

- *Validity*. For each valuation that meets all declarations and assumptions it is the case that all assertions are true.
- *Legality*. Division by zero is avoided in an appropriate manner.

4.1.4 Legality: Positive and Negative Examples, and Undecided Cases

Here are some alternately legal and illegal texts in $VEAT$.

A legal text: given $x : \mathbb{Q}$; moreover $x \neq 0$; now $\frac{x}{x} = 1$.

The argument is simple: given that x is rational and non-zero, it is known thereafter (i.e., in later assertions) that x is non-zero; in this case, it can be used to show that the denominator of the only fracterm contained in the text is non-zero.

A non-legal text: given $x : \mathbb{Q}$; moreover $x = 0$; now $\frac{x}{x} = 1$.

Consider the valuation where $x = 0$, then the declaration and the first assumption succeeds, but calculating the fracterm fails on division by zero.

Legal: given $x : \mathbb{Q}$; moreover $x = 0$; moreover $x \neq 0$; now $x = 1$.

In the absence of any facterms legality boils down to syntactic correctness, which is satisfied in this case.

Non-legal: given $x : \mathbb{Q}$; now $\frac{x}{x} = 1$.

Consider the valuation where $x = 0$, then the declaration succeeds, and subsequently, i.e., while reading from left to right, calculating the fracterm fails on division by zero.

Legal: given $x : \mathbb{Q}$; moreover $x \neq 0$; moreover $x = 0$; now $x = 0$.

In the absence of any facterms, syntactic correctness suffices.

Non-legal given $x : \mathbb{Q}$; moreover $x \neq 1$; now $\frac{x}{x} = 1$.

On the basis of the first and only assumption, it can be shown that the fracterm that comes later can have a zero denominator.

Legal: given $x : \mathbb{Q}$; now-if $x \neq 0$ then $\frac{x}{x} = 1$.

The argument for legality is non-trivial in this case. Two features combine for their legality: first of all, for whatever valuation one may choose, when reading the text from left to right and evaluating only the expressions that need evaluation, no division by zero is encountered. Here it matters that now-if R then S is supposedly evaluated in a short-circuit fashion. Secondly, one finds that the denominator of the (unique) fracterm at hand is not demonstrably equal to zero, on the basis of preceding declarations. Importantly, we won't count the condition $x \neq 0$ of the conditional assertion among the preceding assumptions.

Non-legal: given $x : \mathbb{Q}$; now-if $0 \neq 0$ then $\frac{0}{0} = 1$.

Now the first argument still holds: for any valuation, evaluating the declarations and assertions in a left to right order, and stopping when an evaluation yields false, will not run into division by zero. But the text contains a fracterm which under all circumstances (i.e., for all valuations) will give rise to division by zero. Such fracterms are non-legal.

This last example also illustrates why the condition (in this case $0 \neq 0$) of the conditional assertion is not considered to be an assumption. Indeed, from an assumption $0 \neq 0$ it would possible to prove that whatever denominator is non-zero, thus arguing that there is no possible issue with division by zero. However, the latter argument is unconvincing in the presence of a non-legal term.

In general, a term may be considered *non-legal* if its value always equals $\perp$. A legal text may not contain a non-legal term.

Definition 4.1 A denominator of a fracterm in a text which is demonstrably equal to zero on the basis of preceding assumptions in the text will be called a *logical pole*. Absence of logical poles is a requirement for legality.

Legal: $\mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ x \neq 0;\ \mathsf{now\text{-}if}\ 0 \neq 0\ \mathsf{then}\ \frac{1}{x} = 1.$
In this case the only denominator around can be proven to be non-zero on the basis of the conjunction of preceding declarations and assumptions: no logical poles.

Non-legal: $\mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ x = 0;\ \mathsf{now\text{-}if}\ 0 \neq 0\ \mathsf{then}\ \frac{1}{x} = 1.$
Now the only fracterm involved is not non-legal by itself, but it is non-legal in this particular context where the assumption $x = 0$ may be used: a logical pole

Legal: $\mathsf{given}\ x : \mathbb{Q};\ \mathsf{now\text{-}if}\ 0 \neq 0\ \mathsf{then}\ \frac{1}{x} = 1.$ Absence of logical poles.

Non-legal: $\mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ x = 0;\ \mathsf{now\text{-}if}\ x \neq 0\ \mathsf{then}\ \frac{x}{x} = 1.$
Presence of a logical pole.

4.1.5 Legality, a Problematic Concept?

The purpose of a set of conventions is to specify legal versus illegal texts. Here, however, are two examples of texts for which it seems not to be the case that there is any clear *intuition* about legality. The combination of assumptions is inconsistent, which by itself does not stand in the way of legality.

$\phi \equiv \mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ x = 0;\ \mathsf{moreover}\ x \neq 0;\ \mathsf{now}\ \frac{x}{x} = 1.$

$\psi \equiv \mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ x \neq 0;\ \mathsf{moreover}\ x = 0;\ \mathsf{now}\ \frac{x}{x} = 1.$

We introduce safety as a property of texts, and use it in four different notions of legality, each of which has some value.

Definition 4.2 A text is *safe* if for every valuation of its variables, each of the assertions (i.e., declarations, assumptions and conclusions) evaluates to either true or false when evaluating these linearly from left to right and terminating evaluation at the first assertion that takes value false.

Example of a safe text: $\mathsf{given}\ x : \mathbb{Q};\ \mathsf{moreover}\ 1 = 0;\ \mathsf{now}\ \frac{1}{0} = 0.$

Example of a non-safe text: $\mathsf{given}\ x, y : \mathbb{Q};\ \mathsf{moreover}\ 1 \neq x;\ \mathsf{now}\ \frac{1}{x} = y.$

Here are the notions of legality.

Definition 4.3 1. A text is $legal_A$ if for each occurrence of a fracterm F in it, it is the case that:

(a) the whole of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^w(F)$, and
(b) from $C^w(F)$ it can be shown that the denominator of F is non-zero.

2. A text is *legal*$_B$ if it is safe and in addition for each occurrence of a fracterm F in it, it is the case that:

(a) the whole of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^w(F)$, and
(b) from $C^w(F)$ it cannot be shown that the denominator of F equals 0.

3. A text is $legal^{forward}$ if for each occurrence of a fracterm F in it, it is the case that:

(a) some initial segment of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^{forward}(F)$, and
(b) from $C^{forward}(F)$ it can be shown that the denominator of F is non-zero.

4. A text is $legal^{backward}$ if for each occurrence of a fracterm F in it, it is the case that:

(a) some final segment of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^{backward}(F)$, and
(b) from $C^{backward}(F)$ it can be shown that the denominator of F is non-zero.

Consider some examples.

Proposition 4.1 *Both ϕ and ψ above are neither legal$_A$ nor legal$_B$. However,*

(a) ϕ is $legal^{backward}$ but not $legal^{forward}$.
(b) ψ is $legal^{forward}$ but not $legal^{backward}$.

Next, let $\chi \equiv$ given $x : \mathbb{Q}$; now-if $x \neq 0$ then $\frac{x}{x} = 1$.
We find that χ is safe, that χ is legal$_B$, while χ is not legal$_A$, not legal^{forward} and not legal^{backward}.

Let $\chi_0 \equiv$ given $x : \mathbb{Q}$; now-if $0 = 1$ then $\frac{x}{x} = 1$.
We find that χ_0 is safe, and that χ_0 is not legal in any of the four senses.

Let $\chi_1 \equiv$ given $x : \mathbb{Q}$; now-if $0 \neq 1$ then $\frac{x}{x} = 1$.
We find that χ_1 is safe, that χ_1 is legal$_B$, while χ_1 is not legal$_A$, not legal^{forward} and not legal^{backward}.

At present, we do not have clarity on the question whether or not one of the four definitions of legality of arithmetical texts (for the limited notation of *VEAT*) should be preferred as being closer to arithmetical practice, or if any other definition of legality may be put forward which might enjoy more credibility. However, we do suggest that

(i) legal$_A$ is to be preferred over legal$_B$ as it offers certainty; and

(ii) legal^{backward} is to be preferred over legal^{forward} as it is local.

Different criteria for non-legality can be found in a similar manner.

Definition 4.4 1. A text is *illegal$_A$* if for some occurrence of a fracterm F in it, it is the case that:

   (a) the whole of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^w(F)$, and
   (b) from $C^w(F)$ it can be shown that the denominator of F is zero.

2. A text is *illegal$_B$* if it is safe and in addition for some occurrence of a fracterm F in it, it is the case that:

   (a) the whole of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^w(F)$, and
   (b) from $C^w(F)$ it cannot be shown that the denominator of F is non-zero.

3. A text is *illegal^{forward}* if for each occurrence of a fracterm F in it, it is the case that:

   (a) some initial segment of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^{forward}(F)$, and
   (b) from $C^{forward}(F)$ it can be shown that the denominator of F is zero.

4. A text is *illegal^{backward}* if for each occurrence of a fracterm F in it, it is the case that:

   (a) some final segment of the assumptions (= declarations and assertions with the keyword moreover) preceding F taken together are consistent with conjunction $C^{backward}(F)$, and
   (b) from $C^{backward}(F)$ it can be shown that the denominator of F is zero.

5 Undecidability of Textual Legality

We will first formulate two generalisations of some of the above examples.

5.1 Polynomials and Undecidability

Proposition 5.1 *(Polynomial legality proposition.) Let $p(x_1, \ldots, x_k)$ be a polynomial (i.e., an expression over Σ_r). Suppose that for all integer values $n_1, \ldots, n_k$, $\mathbb{Q} \models p(n_1, \ldots, n_k) \neq 0$. Then the text:*
   given $x_1, \ldots, x_k : \mathbb{Z}$; now $1 \div p(x_1, \ldots, x_k) \neq 0$
is legal.

Proposition 5.2 *(Polynomial non-legality proposition.) Let $p(x_1, \ldots, x_k)$ be a polynomial (i.e., an expression over Σ_r). Suppose that for some tuple of integer values $n_1, \ldots, n_k$, $\mathbb{Q} \models p(n_1, \ldots, n_k) = 0$. Then the text:*

given $x_1, \ldots, x_k : \mathbb{Z}$; now $1 \div p(x_1, \ldots, x_k) \neq 0$

is not legal.

Solvability of Diophantine equations over the integers is undecidable; this is a fundamental theorem of Matijasevitch, who proved that computably enumerable sets are diophantine, which finally solved Hilbert's 10-th problem in the negative. See, e.g., the discussion in Manin (1977) and Matiyasevich (1993).

Let V be be a set of natural numbers which is computably enumerable but not computable. As a consequence of Matijasevitch's Diophantine Theorem, there is a number k and a $k + 1$ variable polynomial $p_V(x_1, \ldots, x_k, y)$ such that $n \in V$ if, and only if, for some $n_1, \ldots, n_k \in \mathbb{Z}$ it is the case that $\mathbb{Q} \models p(n_1, \ldots, n_k, n) = 0$.

Theorem 5.1 *Consider the family T_n of texts parametrised by $n \in \mathbb{N}$:*

$$T_n \equiv \textit{given } x_1, \ldots, x_k : \mathbb{Z}; \text{ now } 1 \div p_V(x_1, \ldots, x_k, n) \neq 0.$$

Then the legality of text T_n is an undecidable property of n.

Proof The result follows from the definition of p_V in combination with both above propositions: T_n is not legal if, and only if, $n \in V$.

Here is a slightly stronger fact.

Corollary 5.2 *Safety is not decidable.*

Proof In the proof above legality and safety coincide.

5.2 Some Conclusions Concerning Traditional Conventions on DbZ Legality

From Theorem 5.1 we infer the following conclusions:

1. The rules behind the notion of legality in the practice of school arithmetic cannot be simple, sound and complete at the same time. Here simple means finite in number with computably enumerable consequences; sound means the rules generate only legal texts; and complete means that the rules generate all legal texts.
2. In as far as attempts are made to write down rules for text legality in relation to DbZ, divergence between the results of different attempts is to be expected.
3. Text legality with respect to DbZ differs for different underlying fields of numbers.
4. In arithmetics where the general polynomial legality or non-legality propositions 5.1 and 5.2 fail or are unknown, the undecidability result may disappear. Instead the question will arise for which polynomials can the polynomial legality proposition be maintained; and, similarly, for the polynomial non-legality proposition.
5. Extracting rule sets for text legality in connection with DbZ is a valid research topic.

In the teaching of arithmetic, the idea that understanding expression or text legality comes or should come in advance of "learning arithmetic" is not plausible. It is remarkable that the idea that DbZ can be avoided if only "one properly understands numbers" is so wide-spread. The problem is a hard example of the complexity of type checking.

6 Common Meadows and Logics

We now turn to logics to model legality under traditional conventions. Our method for progressing legality/illegality using logics needs semantics that is clear, precise and very well understood. Logics defined over signatures have clear precise syntax; if satisfaction is defined by total algebras then they have clear precise semantics. Partial algebras cause considerable semantic complications even for equational logics. Thus, we choose total algebras that can mimic or simulate partiality to enable us to apply logics with easy and reliable semantics.

6.1 Common Meadows

The algebras we choose are enlargements $G(\div)$ of fields G with division to which are added the new element $\bot$, which is an *absorbtive element*, i.e., for all $x \in G$,

$$x + \bot = \bot, x \cdot \bot = \bot, -\bot = \bot, \text{etc.}$$

If $x \div y$ is not defined in G then we define $x \div y = \bot$.

This technique of adding $\bot$ to totalise the partial operations of an algebra A is quite general and has been studied in Bergstra and Tucker (2022a), where the process was specified by a general operator $\mathsf{Enl}_\bot(A)$ on algebras A. In addition, in Bergstra and Tucker (2022a), the operator $\mathsf{Enl}_\bot$ has a left inverse $\mathsf{Pdt}_\bot$ so that $\mathsf{Pdt}_\bot(\mathsf{Enl}_\bot(A)) = A$. The idea is that $\mathsf{Enl}_\bot$ totalises the partial operations of algebras by adding $\bot$, and $\mathsf{Pdt}_\bot$ recovers partial algebras by removing $\bot$.

The total algebras we will use for our investigation are these (Bergstra & Ponse, 2021, 2016):

Definition 6.1 For G a field,

$$\mathsf{Enl}_\bot(G(\div))$$

is a *common meadow*. In common meadows, for all x,

$$x \div 0 = \bot.$$

The design decision that is the focus of this paper is the use of a partial division operator on G; this will not be changed by our total methods. Now, notice that the legality of terms and formulae over a total algebra $\mathsf{Enl}_\bot(G(\div))$ is not an issue as they are defined by standard rules of first order logic, i.e., all well-formed first order formulae are legal in working with $\mathsf{Enl}_\bot(G(\div))$.

Secondly, note the important role of equality. Now, the underlying field G and the enlargement $\mathsf{Enl}_\bot(G(\div))$ are total algebras and therefore both come with a native notion of equality, written = in both cases. Whilst equality in the partial meadow $G(\div)$ is complicated, as there are several options as what to do when terms are undefined, equality in the common meadow $\mathsf{Enl}_\perp(G(\div))$ is an unambiguously defined standard notion. Indeed, to some extent equality in the common meadow approximates the use of $=$ in elementary arithmetic with $\div$ under traditional conventions. However, that approximation is a rough one: for instance, consider this well-formed and valid equation over $\mathsf{Enl}_\perp(G(\div))$:

$$\mathsf{Enl}_\perp(G(\div)) \models 1 \div 0 = (-1) \div 0.$$

Although legal in the first order logical language, this equation is to be rejected by mathematicians working with traditional conventions because of its illegal fracterm.

6.2 Logic of Traditional Conventions for Handling DbZ

Our catalogue of examples of texts and our formulations of legality and illegality reflect arithmetic practices in which

(i) division by zero must be avoided;
(ii) notations involving division by zero lie outside the realm of legal notations, which are to be used in school and perhaps even in mathematics generally;
(iii) there is conceptual evidence of the *non-existence* of (say) $1/0$, which suggests that avoiding division by zero is a practice which follows from the very understanding of arithmetic as a cognitive asset.

Clearly, the idea of legality for elementary arithmetic texts is part of a well-established social practice.

Proposal 6.1 *Legality conventions for elementary arithmetic constitute a well-established, seemingly quite uniform, social practice, abundantly present in elementary teaching. At the same time, no published research on these legality conventions seems to exist.*

What are these conventions? How to specify them? Are the legality conventions unique, or are there different traditions on the matter?

Proposal 6.2 *Traditional elementary arithmetic comes with conventional rules and guidelines, which may be implicit, regarding the legality of expressions for use in adequate texts. These legality conventions are more restrictive than the conventions that come with* $\mathsf{Enl}_\perp(\mathbb{Q}(\div))$*, which is evidenced by the expression* $1 \div 0$ *which is legal for* $\mathsf{Enl}_\perp(\mathbb{Q}(\div))$ *and which is not considered legal in traditional/conventional arithmetic.*

Having introduced the elementary arithmetic language *VEAT* and the total structure $\mathsf{Enl}_\perp(\mathbb{Q}(\div))$, the method comes into focus.

The method is to choose, adapt and design logics L over the common meadow $\mathsf{Enl}_\perp(\mathbb{Q}(\div))$, and transform texts from the language *VEAT* into formulae in the logic

L. In the well behaved world of logics over $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$, we can attempt to classify the well-formed formulae of L that represent the legal texts and illegal texts of *VEAT*.

Because of the simplicity of *VEAT*, the translation

$$T \in VEAT \rightarrow \phi_T \in L$$

will be straight-forward and easy to see:

(i) the separator ; of *VEAT* becomes a conjunction in L; and
(ii) conditionals become implications in L.

In fact, as we build up momentum, we might talk about traditional legality directly in L.

The logics we need depart from classical logics as their semantics must accommodate 'short circuit' reading of formulae to cope with our examples and definitions.

Starting with $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$ as a formal tool, the details of legality conventions for, say, school arithmetic are a topic for logical research. More explicitly:

Proposal 6.3 *Having formulated the problem, we conjecture that* $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$ *can serve as a point of departure for developing arithmetic with conventional notational constraints about DbZ.*

7 Concluding Remarks

We comment on various perspectives on division by zero and on some directions for further research.

7.1 Traditional Conventions and Formal Texts

Antipathy to division by zero is evident in the traditional conventions on writing arithmetical texts, wherein formulae that might allow division by zero are strictly avoided. We have posed the novel problem of establishing a set of precise rules or guidelines for writing arithmetical texts that capture these traditional conventions. We have shown, using computabilty theory, that such guidelines cannot be simple, sound and complete. But we have also introduced a method to explore the scope and limits of traditional conventions using formal logics applied to special total algebraic structures called common meadows. The logics can dissect partiality as it is manifested by division in arithmetical texts, using technical concepts that allow us to classify, in an approximate and rigorous way, the texts that are legal or illegal according to traditional conventions. Of course, as the approximations become more complex and the examples more subtle the question arises:

Does there exist a set of practical and precise rules or guidelines for writing arithmetical texts that capture conventions in traditional practice? Is there at least a kernel of precise rules or guidelines that can be applied soundly in informal practices?

At this stage, it is clear that traditional conventions do exist but it is not clear that they are unique or that there is a kernel that captures a stable social consensus, as simple examples persistently escape or contradict seemingly reasonable formal conditions. This paper has uncovered some barriers to be taken down. First, there are technical issues of formulating properties in logics. Second, there is the role of undecidability in making judgements in theory and practice. Third, there are the effects of inconsistency arising from false assumptions.

Perhaps no single logic can do. It is perfectly reasonable to explore *simultaneously* the use of a number of *different* logics to analyse components of legal and illegal texts. Perhaps a patchwork of different logics applied to common meadows can serve to define a kernel.

In this paper we have only scratched the surface: enough to establish the problem and its difficulty and to see the need, or at least the plausibility, of using common meadows and suitable logics. Some candidates worth investigating are 3-valued or 4-valued 'short circuit' logics (Bergstra et al., 1995; Ponse & Staudt, 2018).

The notation *VEAT* for texts on elementary arithmetic here comprises only a part of what is needed to capture at least the spirit of teaching notes or a simple textbook on arithmetic. Apart from primitives for modularisation of text (sections, chapters etc.), one needs at least: (i) function definitions, (ii) statements about function definitions (e.g., x is in the domain of function f), (iii) examples (importing a module with parameters and assumptions), (iv) links with figures and diagrams, (v) exercises, and (vi) exams.

For instance, exercises may be taken to have the following forms (with e an expression, and A an assertion):

"Is the expression e in simplified form?",

"Simplify the expression e?",

"Is the assertion A valid?",

"What is the value of the expression e?".

Detailing a format for modelling texts about elementary arithmetic is a project in its own right and may have unexpected uses.

It is easy to recognise the Avoidance Principle at work in mathematical writings. It assumes that text and meaning—syntax and semantics—are intimately connected and seeks to preserve verisimilitude for texts by playing safe with partiality. This is consistent with the practice of mathematicians who fuse syntax and semantics to the extent that syntax is *not acknowledged explicitly and disappears*. However, another attitude to partiality is possible, namely don't bother about legality at all and assume that in each case the semantics of the text will be evident, or at least sort itself out. This seems to amount to embracing the independence of syntax and making it primary in some sense. This attitude can be expressed as a Principle and studied too. Such directions connect with positions in the philosophy of mathematics, of course.

7.2 Other Directions

As to the form of traditional conventions, a more radical approach could be to embrace the evidence of practice directly and assemble a corpus of elementary mathematical texts and apply some form of machine learning to them. This approach has worked well in a not unrelated context: programmers' coding conventions (Allamanis et al., 2014). In many software development projects there is a kernel of precise rules or guidelines for the programmers to follow. The conventions are necessary to promote understanding and collaboration between coders on a development project and for later maintenance of their product. But programmers do *not* follow the rules despite being certain about the importance of their existence. One can say that the coding conventions fail to capture the practice of coding. As is established in Allamanis et al. (2014), machine learning can explore practice with the aim of improving the usefulness and value of the coding conventions.

The phenomena of association *v.* disassociation as highlighted in Nicaud et al. (2001) are important for acquiring an understanding of the psychology of reading texts on arithmetic—and hence the nature of traditional conventions. At a very high level of association, if something is incomprehensible it triggers disassociation and changes to a very low level of association where each detail matters a lot. In practice, the levels of association as maintained by individual persons differ in time, changing in an oscillating pattern as needed for understanding, and differ between persons.

7.3 Options for Total Algebras to Define DbZ

Adopting $1/0 = \bot$ as in common meadows is an option that provides a useful tool for investigating several technical details of conventional arithmetic, including its algebra and logic, and its semantic role in computing. Common meadows are fully motivated by such applications; indeed, common meadows constitute an unavoidable ingredient for the investigation and formalisation of elementary arithmetic as it occurs in practice.

Can other algebras with total division be used in our methods in place of common meadows?

Another well-known option for totalisation is to adopt $1/0 = 0$. Theoretical work in that direction has started with Komori (1975) and Ono (1983). In Bergstra and Tucker (2007), $1/0 = 0$ was adopted for the purpose of algebraic specification of an arithmetical data type of rationals. Since Bergstra and Tucker (2007), quite some theory has been developed for these total algebras, which are called *involutive meadows* because $(x^{-1})^{-1} = x$ for all x. In Bergstra and Middelburg (2015), the structure theory of fracterms is investigated under the assumption that $1/0 = 0$. An experiment with using $1/0 = 0$ as an assumption for an application in forensic reasoning can be found in Bergstra (2019). For calculations involving probability, having $1/0 = 0$ may be advantageous (Bergstra, 2019)—much more work will be needed, however, to confirm that suggestion. Adopting $1/0 = 0$, as a matter of convenience, not as a matter of principle, has become common practice for proof checking systems, of which there are many.

As stated above, adopting $1/0 = 0$ is rather uninformative regarding the practice of elementary arithmetic, for instance as present in teaching. In Michiwaki et al. (2016); Okumura and Saitoh (2018) a line of work is promoted where adopting $1/0 = 0$ (or rather a generalisation of that identity phrased in terms of evaluating a Laurent series in its pole) is claimed to be useful for analysis, geometry, and trigonometry. A book-length study of this direction is Saitoh (2021).

In this case, let $\mathbb{Q}_0(\div)$ denote the involutive meadow of rationals totalising division in $\mathbb{Q}(\div)$. Now considering closed terms in $\mathbb{Q}_0(\div)$, forces one to view a closed term such as $1 \div ((5-6)+1)$ as illegal, thereby confusing matters of legality and equality; this stands in the way of the separation of concerns which is a benefit of working with $\mathsf{Enl}_\perp(\mathbb{Q}(\div))$.

Yet another strand of DbZ theory comes from computer programming where adopting $1 \div 0 = +\infty$ has become a common practice, in part thanks to floating point conventions. Theoretical work in that direction, including the design of arithmetical data types such as the transrational numbers, starts with Anderson et al. (2007). In dos Reis et al. (2016); dos Reis (2019) and one finds an approach to real analysis based on the adoption of $1 \div 0 = +\infty$. An algebraic specification of the abstract data type of transrational numbers is given in Bergstra and Tucker (2020). A refinement of these types of models that integrate the approach of common meadows is Bergstra and Tucker (2022b).

Adopting $1 \div 0 = \infty$ (unsigned infinity) has been proposed in Setzer (1997) and analysed in significant detail in Carlström (2004, 2005) and Bergstra and Tucker (2021), in the theory of wheels. Although this option is not common in computational practice, though it suits models of exact real arithmetic, its significance undeniably rests on the observation that the wheels of rationals, reals, and complex numbers arise naturally and unavoidably when contemplating arithmetical data types which capture the topological characteristics of the Riemann sphere. It is fair to say that of the four mentioned options, wheels are most clearly rooted in classical mathematics.

References

Allamanis, M., Barr, E.T., Bird, C., & Sutton, C. (2014). Learning natural coding conventions. *FSE 2014: Proceedings of the 22nd ACM SIGSOFT International Symposium on Foundations of Software Engineering*, ACM.

Anderson, J.A., Völker, N., & Adams, A.A. (2007). Perspecx machine VIII, axioms of transreal arithmetic. In J. Latecki, D. M. Mount and A. Y. Wu (eds.), *Vision Geometry XV*, 649902.

Andreka, H., Craig, W., & Nemeti, I. (1988). A system of logic for partial functions under existence-dependent Kleene equality. *Journal of Symbolic Logic, 53*(3), 834–839.

Bergstra, J. A. (2019). Adams conditioning and likelihood ratio transfer mediated inference. *Scientific Annals of Computer Science, 29*(1), 1–58.

Bergstra, J. A. (2019). Division by zero, a survey of options. *Transmathematica*. https://doi.org/10.36285/tm.v0i0.17

Bergstra, J. A. (2021). Division by zero in logic and computing. HAL Archives Ouvertes, https://hal.archives-ouvertes.fr/hal-03184956.

Bergstra, J. A., Bethke, I., & Rodenburg, P. H. (1995). A propositional logic with 4 values: true, false, divergent and meaningless. *Journal of Applied Non-Classical Logics, 5*(2), 199–217.

Bergstra, J. A., & Middelburg, C. A. (2015). Transformation of fractions into simple fractions in divisive meadows. *Journal of Applied Logic, 16*, 92–110.

Bergstra, J.A., & Ponse, A. (2015). Division by zero in common meadows. In R. de Nicola and R. Hennicker (eds.) *Software, Services, and Systems (Wirsing Festschrift)* Lecture Notes in Computer Science 8950, (pp.46–61). Springer. Also, in improved form: arXiv:1406.6878v4 [math.RA]. (2021)

Bergstra, J. A., & Ponse, A. (2016). Fracpairs and fractions over a reduced commutative ring. *Indigationes Mathematicae, 27*, 727–748.

Bergstra, J. A., & Tucker, J. V. (2007). The rational numbers as an abstract data type. *Journal of the ACM, 54*(2), 7.

Bergstra, J. A., & Tucker, J. V. (2020). The transrational numbers as an abstract data type. *Transmathematica*. https://doi.org/10.36285/tm.47

Bergstra, J. A., & Tucker, J. V. (2021). The wheel of rational numbers as an abstract data type. In P. James & M. Roggenbach (Eds.), *Workshop on Algebraic Development Techniques*, Lecture Notes in Computer Science 12669, (pp. 1–18).

Bergstra, J. A., & Tucker, J. V. (2022). Totalising partial algebras: teams and splinters. *Transmathematica*. https://doi.org/10.36285/tm.57

Bergstra, J.A., & Tucker, J.V. (2022) Symmetric transrationals: The data type and the algorithmic degree of its equational theory. In N. Jansen et al. (eds.) *A Journey From Process Algebra via Timed Automata to Model Learning - A Festschrift Dedicated to Frits Vaandrager on the Occasion of His 60th Birthday*, Lecture Notes in Computer Science 13560, (pp. 63–80). Springer.

Bergstra, J. A., & Tucker, J. V. (2023). On the axioms of common meadows: Fracterm calculus, flattening and incompleteness. *The Computer Journal, 66*(7), 1565–1572.

Bergstra, J.A., & Tucker, J.V. A complete finite axiomatisation of the equational theory of common meadows. https://doi.org/10.48550/arXiv.2307.04270

Bergstra, J. A., & Tucker, J. V. (2023). Naive fracterm calculus. *Journal of Universal Computer Science, 29*(9), 961–987. https://doi.org/10.3897/jucs.87563

Bergstra, J. A., & Tucker, J. V. (2024). Synthetic fracterm calculus. *Journal of Universal Computer Science, 30*(3), 289–308. https://doi.org/10.3897/jucs.107082

Carlström, J. (2004). Wheels - on division by zero. *Mathematical Structures in Computer Science, 14*(1), 143–184.

Carlström, J. (2005). *Partiality and choice, foundational contributions*. PhD. Thesis, Stockholm University.

dos Reis, T. S., Gomide, W., & Anderson, J. A. D. W. (2016). Construction of the transreal numbers and algebraic transfields. *IAENG International Journal of Applied Mathematics, 46*(1), 11–23.

dos Reis, T. S. (2019). Transreal integral. *Transmathematica*. https://doi.org/10.36285/tm.v0i0.13

Ehrich, H.D., Wolf, M., & Loeckx, J. (1997). *Specification of Abstract Data Types*. Vieweg + Teubner.

Gavilanes-Franco, A., & Lucio-Carasco, F. (1990). A first order logic for partial functions. *Theoretical Computer Science, 74*, 37–69.

Jones, C. B., & Middelburg, C. A. (1994). A typed logic of partial functions, reconstructed classically. *Acta Informatica, 31*, 399–430.

Komori, Y. (1975). Free algebras over all fields and pseudo-fields. Report 10, 9-15, Faculty of Science, Shizuoka University.

Matiyasevich, Y. V. (1993). *Hilbert's Tenth Problem*. MIT Press.

Manin, Y. (1977). *A Course in Mathematical Logic* (2nd ed.). Springer, 2010.

McDonnel, E.E. (1976). Zero divided by zero. *Proceedings APL 1976*, ACM, pp. 295–296. https://doi.org/10.1145/800114.803689.

Michiwaki, H., Saitoh, S., & Yamada, N. (2016). Reality of the division by zero $z/0 = 0$. *International Journal of Applied Physics and Mathematics*. https://doi.org/10.17706/ijapm.2016.6.1.1-8

Morgan, J. (2024). *Resourceaholic. Ideas and Resources for Teaching Secondary School Mathematics: Online Historical Maths Textbooks*. https://www.resourceaholic.com/p/digitised-antique-maths-textbooks.html.

Nicaud, J-.F., Bouhineau, D., & Gelis, J-.M. (2001). Syntax and semantics in algebra. *Proceedings 12th ICMI Study Conference*, The University of Melbourne, 2001. HAL archives-ouvertes https://hal.archives-ouvertes.fr/hal-00962023/document.

Ono, H. (1983). Equational theories and universal theories of fields. *Journal of the Mathematical Society of Japan, 35*(2), 289–306.

Okumura, H., & Saitoh, S. (2018). Applications of the division by zero calculus to Wasan geometry. *Global Journal of Advanced Research on Classical and Modern Geometries, 7*(2), 44–49.

Ponse, A., & Staudt, D. J. C. (2018). An independent axiomatisation for free short-circuit logic. *Journal of Applied Non-Classical Logics, 28*(1), 35–71.

Robinson, A. (1989). Equational logic of partial functions under Kleene equality: a complete and an incomplete set of rules. *Journal of Symbolic Logic, 54*(2), 354–362.

Saitoh, S. (2021). *Introduction to the Division by Zero Calculus*. Scientific Research Publishing Inc.

Setzer, A. (1997). *Wheels (draft)*. http://www.cs.swan.ac.uk/csetzer/articles/wheel.pdf.