Penrose limits of I-branes, twist-compactified D5-branes, and spin chains

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In this paper we consider the Penrose limit in the case of two gravity duals. One of them, consists of compactified I-branes [intersecting sets of D5-branes over $(1+1)$ dimensions]. The second consists of D5-branes compactified on a circle. Both compactifications preserve supersymmetry. We find a match of the oscillators and masses of string modes on the resulting pp wave against a spin chain in a $(2+1)$-dimensional field theory in the first case, and a spin chain in a $(1+1)$-dimensional field theory in the second.

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I. INTRODUCTION

In the AdS/CFT correspondence [1], and even more so in the gauge/gravity duality generalizations, often times the holographic dictionary and/or the exact matching of the two sides is not very clear. In order to gain additional information, through specializing to a useful subset, one tool at our disposal is the Penrose limit. The Penrose limit leads, on the gravity side, to a plane-parallel wave (pp wave), and on the field theory side, to a spin chain, as first seen in [2]. The Penrose limit has been used to make progress in less understood cases, for instance in [3–7].

A particular case of interest is that of confining field theories. In a confining field theory, like the $(3+1)$-dimensional case of [8] or the $(2+1)$-dimensional case of MNa [9], the Penrose limit leads to a theory of long cyclic hadronlike objects, dubbed "annulons" in [10]. But the resulting "spin chain" is not well-understood either, though in the MNa case progress was made in [5].

It would then be useful to study confining theories in different dimensions, like the (fibered) I-brane case, whose (singular) holographic dual was written in [11], and whose nonsingular supergravity background was found and studied in [12]. Also, in the related case of S^1 twisted-compactified D5-branes whose nonsingular background was found and studied in [13]. This is the subject of this paper. The I-brane theory is localized at the $(1+1)$-dimensional intersection of two sets of D5-branes, but it was found in [11] that one dimension appears dynamically, so the theory should be understood in $(2+1)$ dimensions. On the other hand, the theory of D5's compactified on S^1 is dual to a confining $(4+1)$-dimensional quantum field theory (QFT) with eight supercharges.

The paper is organized as follows. In Sec. II we discuss in detail the Penrose limit along two possible geodesics in the nonsingular geometry of I-branes [12]. The two geodesics lead to the same pp wave and, after some choice is made, we write it in the form of a parallelizable plane wave [14]. We then discuss the spectrum of the string and the associated spin chain, matching oscillations in QFT and in string theory.

Analogously, in Sec. III we discuss the Penrose limit in the background of S^1-twisted compactified D5 branes preserving eight supersymmetries (SUSYs) [13]. The pp wave is also parallelizable and preserves 24 SUSYs. A proposal for the associated spin chain is given. Section IV presents a summary and closing remarks, together with the proposal for further study in different backgrounds.

II. PENROSE LIMIT OF FIBERED I-BRANES AND DUAL SPIN CHAIN

The $(1+1)-$dimensional theory obtained when two stacks of D5-branes intersect along two space-time directions, is called I-brane theory [11]. As explained there, an extra world volume coordinate appears, leading to a $(2+1)$-dimensional theory. After a twisted compactification on a shrinking circle, the dual background was found to be *nonsingular* in [12]. This provided an "IR completion" to an otherwise singular gravity dual.[1]

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[1]See Ref. [15] for a Penrose limit on the singular background.

The supergravity background dual to the fibered I-branes, in the D5-brane string frame is relevant to the dual field theory. For simplicity, we write the S-dual NS5-brane theory, keeping in mind that for the field theory analysis we should work with D5-branes. The NS5 solution reads

$$\begin{aligned} ds_{st}^2 &= -dt^2 + dx^2 + 4Q^2 f(\rho)d\varphi^2 + \frac{d\rho^2}{f(\rho)} + \frac{N_B}{4}\left[\hat{\omega}_1^2 + \hat{\omega}_2^2 + (\hat{\omega}_3 - e_A A)^2\right] \\ &\quad + \frac{N_A}{4}\left[\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + (\tilde{\omega}_3^2 - e_B B)^2\right] \\ H_3 &= \frac{2}{e_A} d(\hat{\omega}_3 \wedge A) + \frac{2}{e_B} d(\tilde{\omega}_3 \wedge B) + 2N_B \mathrm{Vol}(S_A^3) + 2N_A \mathrm{Vol}(S_B^3). \\ \Phi &= -Q\rho, \end{aligned} \tag{2.1}$$

where $e_{A,B}^2 = 8/N_{B,A}$, N_A, N_B are integers, $\hat{\omega}^i$ and $\tilde{\omega}^i$ are the Maurer-Cartan forms for $su(2)$, given by

$$\begin{aligned} \hat{\omega}_1 &= \cos\psi_A d\theta_A + \sin\psi_A \sin\theta_A d\phi_A, \qquad \tilde{\omega}_1 = \cos\psi_B d\theta_B + \sin\psi_B \sin\theta_B d\phi_B, \\ \hat{\omega}_2 &= -\sin\psi_A d\theta_A + \cos\psi_A \sin\theta_A d\phi_A, \qquad \tilde{\omega}_2 = -\sin\psi_B d\theta_B + \cos\psi_B \sin\theta_B d\phi_B, \\ \hat{\omega}_3 &= d\psi_A + \cos\theta_A d\phi_A, \qquad \tilde{\omega}_3 = d\psi_B + \cos\theta_B d\phi_B. \end{aligned}$$

The 1-forms are

$$A = Q_A \zeta(\rho) d\varphi, \qquad B = Q_B \zeta(\rho) d\varphi, \tag{2.2}$$

where we have defined the functions

$$\begin{aligned} f(\rho) &= 1 - \tilde{m} e^{-2Q\rho} - \frac{(Q_A^2 + Q_B^2)}{2Q^2} e^{-4Q\rho}, \\ \zeta(\rho) &= e^{-2Q\rho} - e^{-2Q\rho_+}, \end{aligned} \tag{2.3}$$

with $Q = \sqrt{\frac{1}{N_A} + \frac{1}{N_B}}$ is a background charge in the string world sheet theory, and ρ_+ is the larger of the two solutions of $f(\rho) = 0$, $\rho_\pm$. We also defined $\tilde{m} = \frac{m}{4Q^2}$, with m the standard mass parameter.

We note that $\rho = \rho_+$ is the end of the space, corresponding to the IR of the field theory.

A. Penrose limit

Below we study the Penrose limit for the geometry in Eqs. (2.1)–(2.3). We find two possible geodesics about which to expand and write the associated pp wave. We then show the equivalence of the two plane waves obtained. After that we comment on the amount of SUSY preserved and quantize the string on this background.

1. First limit: Geodesic on ψ_A, ψ_B

We consider a geodesic moving (besides the time t) in the directions ψ_A and ψ_B, and fixed at $x = \phi_A = \phi_B = 0$, (we could have replaced $x = 0$ with some arbitrary $x = x_0$ as well), $\theta_A = \theta_B = \pi/2$ and $\rho = \rho_+$.

Note that $\phi_A = \phi_B = 0$ and $\theta_A = \theta_B = \pi/2$ are necessary in order to have a solution of the geodesic equation $\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$. Moreover, as we are interested in understanding the IR of the theory, we must focus on the geodesics that sit at $\rho = \rho_+$.

The geodesics we are interested in, are further defined by a rotation of an angle α between the two spatial directions ψ_A and ψ_B. This means that we make the change of variables,

$$\begin{aligned} \psi_A &\to \frac{2}{\sqrt{N_B}}(\cos(\alpha)\psi_A + \sin(\alpha)\psi_B) \equiv \tilde{\psi}_A, \\ \psi_B &\to \frac{2}{\sqrt{N_A}}(-\sin(\alpha)\psi_A + \cos(\alpha)\psi_B) \equiv \tilde{\psi}_B, \end{aligned} \tag{2.4}$$

and take the geodesic to be on $\tilde{\psi}_B$. One defines light cone coordinates as usual,

$$\begin{aligned} t &= \frac{1}{\sqrt{2}}(u - v), \\ \tilde{\psi}_A &= \frac{1}{\sqrt{2}}(u + v). \end{aligned} \tag{2.5}$$

The coordinate ρ is kept always slightly off the special point $\rho = \rho_+$, where $f(\rho_+) = 0$, by the rescaling with L written below. The Penrose rescaling is

$$\begin{aligned} & v \to \frac{v}{L^2}, \qquad \theta_{A,B} \to \frac{\pi}{2} + \frac{2\theta_{A,B}}{L\sqrt{N_{B,A}}}, \qquad \phi_{A,B} \to \frac{2\phi_{A,B}}{L\sqrt{N_{B,A}}}, \\ & x \to \frac{x}{L}, \qquad \rho \to \rho_+ + \frac{\rho}{L^2}, \qquad \psi_B \to \frac{\psi_B}{L}, \\ & u \to u, \qquad \varphi \to \varphi. \end{aligned} \tag{2.6}$$

Note that the φ was not rescaled, since it will become an angular variable in the pp wave (so it is not a length variable, as to be rescaled by $1/L$), in the same way that it was done, for instance, in [5,7]. We also need to change coordinates according to[2]

$$\rho \to \frac{\sinh(Q(\rho_+ - \rho_-))}{e^{Q(\rho_+ - \rho_-)}}\rho^2, \qquad \varphi \to \frac{e^{Q(\rho_+ - \rho_-)}}{4Q^2 \sinh(Q(\rho_+ - \rho_-))}\varphi. \tag{2.7}$$

Then in the $L \to \infty$ limit, after rescaling the metric by L^2, we obtain the pp wave

$$\begin{aligned} L^2 ds^2 &= 2du(dv + A_\varphi d\varphi - A d\phi_A - B d\phi_B) + dx^2 + d\rho^2 \\ &\quad + \rho^2 d\varphi^2 + d\theta_A^2 + d\phi_A^2 + d\theta_B^2 + d\phi_B^2 + d\psi_B^2, \\ L^2 H_3 &= dA \wedge d\phi_A \wedge du + dB \wedge d\phi_B \wedge du + dA_\varphi \\ &\quad \wedge d\varphi \wedge du, \\ \Phi &= 0. \end{aligned} \tag{2.8}$$

We have defined

$$\begin{aligned} A_\varphi &= \frac{1}{2} e^{-2Q\rho_+} \rho^2 (Q_A \cos(\alpha) - Q_B \sin(\alpha)), \\ A &= \sqrt{\frac{2}{N_B}} \cos(\alpha)\theta_A, \qquad B = \sqrt{\frac{2}{N_A}} \sin(\alpha)\theta_B. \end{aligned} \tag{2.9}$$

We can write the B-field corresponding to the above field strength H_3,

$$L^2 B_2 = A d\phi_A \wedge du + B d\phi_B \wedge du + A_\varphi d\varphi \wedge du. \tag{2.10}$$

Finally, we go to Cartesian coordinates, from the (ρ, φ) space to (x_1, x_2), and define the corresponding rotated A_φ into (A_1, A_2),

$$\begin{aligned} A_1 &= -\frac{1}{2} x_2 e^{-2Q\rho_+} (Q_A \cos(\alpha) - Q_B \sin(\alpha)), \\ A_2 &= \frac{1}{2} x_1 e^{-2Q\rho_+} (Q_A \cos(\alpha) - Q_B \sin(\alpha)). \end{aligned} \tag{2.11}$$

Relabeling $(\theta_A, \phi_A, \theta_B, \phi_B, x, \psi_B) \to (x_3, x_4, x_5, x_6, x_7, x_8)$, the pp wave solution becomes

$$\begin{aligned} L^2 ds^2 &= 2du(dv + A_1 dx_1 + A_2 dx_2 - A dx_4 - B dx_6) \\ &\quad + \delta_{ij} dx^i dx^j, \\ L^2 B_2 &= -A_1 du \wedge dx_1 - A_2 du \wedge dx_2 - A du \wedge dx_4 \\ &\quad - B du \wedge dx_6, \\ \Phi &= 0, \quad i, j = 1, \ldots, 8. \end{aligned} \tag{2.12}$$

Let us now study a different geodesic and Penrose limit.

2. Second limit: Geodesic on $\psi_A, \psi_B, \phi_A, \phi_B$

Another possible Penrose limit, still describing excitations in the IR of the dual field theory, is for a null geodesic that moves on a combination of $\psi_A, \psi_B, \phi_A, \phi_B$, still at $x = 0$ (or $x = x_0$, in general), as well as $\rho = \rho_+$, but now at $\theta_A = \theta_B = 0$ (instead of $\pi/2$), and the three remaining coordinates (from which the combination of the null motion is taken) are also still fixed at 0.

We first do the coordinate change (replacement),

$$\begin{aligned} \psi_{A,B} &\to \hat{\psi}_{A,B} - \hat{\phi}_{A,B}, \qquad \theta_{A,B} \to 2\hat{\theta}_{A,B}, \\ \phi_{A,B} &\to \hat{\phi}_{A,B} + \hat{\psi}_{A,B}. \end{aligned} \tag{2.13}$$

Then, we consider the same rotation as in Eq. (2.4), but now for $\hat{\psi}_A$ and $\hat{\psi}_B$ (instead of ψ_A and ψ_B),

$$\begin{aligned} \hat{\psi}_A &\to \frac{1}{\sqrt{N_B}} (\cos(\alpha)\hat{\psi}_A + \sin(\alpha)\hat{\psi}_B), \\ \hat{\psi}_B &\to \frac{1}{\sqrt{N_A}} (-\sin(\alpha)\hat{\psi}_A + \cos(\alpha)\hat{\psi}_B), \end{aligned} \tag{2.14}$$

such that the geodesic motion is on the redefined $\hat{\psi}_A$. We use the same redefinition of light cone coordinates u, v in Eq. (2.5), for $(t, \hat{\psi}_A)$.

We perform the coordinate change indicated in Eq. (2.7), together with the Penrose rescaling,

$$\begin{aligned} &v \to \frac{v}{L^2}, \quad \hat{\theta}_{A,B} \to \frac{\hat{\theta}_{A,B}}{L\sqrt{N_{B,A}}}, \quad x \to \frac{x}{L}, \quad \rho \to \rho_+ + \frac{\rho}{L^2}, \\ &\hat{\psi}_B \to \frac{\hat{\psi}_B}{L} \quad u \to u, \quad \hat{\phi}_A \to \hat{\phi}_A, \quad \hat{\phi}_B \to \hat{\phi}_B. \end{aligned} \tag{2.15}$$

Multiplying the metric by L^2 and taking the $L \to \infty$ limit gives

[2]Note that it is only the redefined ρ that rescales by $1/L$ in the Penrose limit, as required by the Penrose theorem, since only this variable becomes the radial coordinate on the plane.

$$
\begin{aligned}
L^2 ds^2 &= -\frac{1}{2}\left(\frac{\hat\theta_A^2}{N_B}\cos^2(\alpha) + \frac{\hat\theta_B^2}{N_A}\sin^2(\alpha)\right)du^2 + 2du(dv + A_\varphi d\varphi)\\
&\quad + dx^2 + d\rho^2 + \rho^2 d\varphi^2 + d\hat\theta_A^2 + \hat\theta_A^2 d\hat\phi_A^2 + d\hat\theta_B^2 + \hat\theta_B^2 d\hat\phi_B^2 + d\hat\psi_B^2,\\
L^2 H_3 &= \cos(\alpha)\sqrt{\frac{2}{N_B}}\hat\theta_A du\wedge d\hat\theta_A \wedge d\hat\phi_A - \sin(\alpha)\sqrt{\frac{2}{N_A}}\hat\theta_B du \wedge d\hat\theta_B\wedge d\hat\phi_B + dA_\varphi \wedge d\varphi\wedge du\\
\Phi &= 0,
\end{aligned}
\tag{2.16}
$$

where A_φ is the same as in the geodesic in the previous case, Eq. (2.9). We can also find the 2-form potential that leads to $H_3 = dB_2$,

$$
L^2 B_2 = \frac{1}{\sqrt{2N_B}}\cos(\alpha)\hat\theta_A^2 du\wedge d\hat\phi_A - \frac{1}{\sqrt{2N_A}}\sin(\alpha)\hat\theta_B^2 du\wedge d\hat\phi_B - A_\varphi du\wedge d\varphi. \tag{2.17}
$$

In the above pp wave solution, we have three two-dimensional subspaces written in polar coordinates: (ρ,φ) and $(\hat\theta_{A,B},\hat\phi_{A,B})$. We switch to Cartesian coordinates (x_1,x_2), (x_3,x_4), and (x_5,x_6), respectively. Relabeling $(x,\hat\psi_B)\to(x_7,x_8)$ we find

$$
\begin{aligned}
L^2 ds^2 &= -\frac{1}{2}\left(\frac{1}{N_B}(x_3^2+x_4^2)\cos^2(\alpha) + \frac{1}{N_A}(x_5^2+x_6^2)\sin^2(\alpha)\right)du^2 + 2du(dv + A_1 dx_1 + A_2 dx_2) + \delta_{ij}dx^i dx^j,\\
L^2 B_2 &= -A_1 du\wedge dx_1 - A_2 du\wedge dx_2 + B_3 du\wedge dx_3 + B_4 du\wedge dx_4 + B_5 du\wedge dx_5 + B_6 du \wedge dx_6,\\
\Phi &= 0,
\end{aligned}
\tag{2.18}
$$

where A_1, A_2 are written in Eq. (2.11) and

$$
\begin{aligned}
B_3 &= \frac{1}{\sqrt{2N_B}}\cos(\alpha)x_4, \qquad B_4 = -\frac{1}{\sqrt{2N_B}}\cos(\alpha)x_3,\\
B_5 &= -\frac{1}{\sqrt{2N_A}}\sin(\alpha)x_6, \qquad B_6 = \frac{1}{\sqrt{2N_A}}\sin(\alpha)x_5.
\end{aligned}
\tag{2.19}
$$

This pp wave solution is called a "gyratonic" pp wave [16,17], being created by spinning objects moving at the speed of light. Let us discuss the equivalence of the plane waves obtained above and the preserved SUSY.

3. Equivalence of pp waves and supersymmetry

We now show that both pp waves in Eqs. (2.12) and (2.18) are the same, and moreover are of a general type called parallelizable in [14].

We start by denoting, in (2.12),

$$
2A_1 = -ax_2, \qquad 2A_2 = +ax_1, \qquad 2A = bx_3, \qquad 2B = cx_5, \tag{2.20}
$$

where a, b, c are given by

$$
\begin{aligned}
a &= e^{-2Q\rho_+}(Q_A\cos(\alpha) - Q_B\sin(\alpha)),\\
b &= 2\sqrt{\frac{1}{2N_B}}\cos(\alpha), \qquad c = 2\sqrt{\frac{1}{2N_A}}\sin(\alpha).
\end{aligned}
\tag{2.21}
$$

The pp wave solution in Eq. (2.12) becomes then

$$
\begin{aligned}
ds^2 &= 2dudv + du[-ax_2dx_1 + ax_1dx_2\\
&\quad - 2bx_3dx_4 - 2cx_5dx_6] + \sum_{i=1}^{8} dx_i dx_i,\\
B_2 &= \frac{a}{2}du\wedge(x_2dx_1 - x_1dx_2) - bx_3 du\wedge dx_4\\
&\quad - cx_5 du\wedge dx_6,\\
\Phi &= 0.
\end{aligned}
\tag{2.22}
$$

We can do a similar coordinate transformation as was done, for instance, in [3], and first define

$$
z_1 \equiv x_1 + ix_2, \tag{2.23}
$$

such that

$$
x_1dx_2 - x_2dx_1 = -\frac{i}{2}(\bar z dz - zd\bar z), \qquad dx_1^2 + dx_2^2 = dzd\bar z, \tag{2.24}
$$

and then, redefining

$$z_1 = e^{-iau/2} w_1, \qquad \bar{z}_1 = e^{+iau/2} \bar{w}_1, \tag{2.25}$$

we find

$$dz_1 d\bar{z}_1 - \frac{i}{2} a du (\bar{z}_1 dz_1 - z_1 d\bar{z}_1) = dw_1 d\bar{w}_1 - \frac{a^2}{4} du^2 |w_1|^2, \tag{2.26}$$

so the metric becomes

$$ds^2 = 2dudv + du\left[-\frac{a^2}{4}|w_1|^2 - 2bx_3 dx_4 - 2cx_5 dx_6\right] + dw_1 d\bar{w}_1 + \sum_{i=3}^{8} dx_i dx_i. \tag{2.27}$$

Then, the shift

$$v \to v + \frac{b}{2} x_3 x_4 + \frac{c}{2} x_5 x_6 \tag{2.28}$$

takes the metric to

$$ds^2 = 2dudv + du\left[-\frac{a^2}{4}|w_1|^2 du - b(x_3 dx_4 - x_4 dx_3) - c(x_5 dx_6 - x_6 dx_5)\right] + dw_1 d\bar{w}_1 + \sum_{i=3}^{8} dx_i dx_i. \tag{2.29}$$

Finally, we repeat for the pairs (x_3, x_4) and (x_5, x_6) the same steps as for (x_1, x_2), and get

$$ds^2 = 2dudv - du^2\left[\frac{a^2}{4}|w_1|^2 + \frac{b^2}{4}|w_2|^2 + \frac{c^2}{4}|w_3|^2\right] + \sum_{a=1}^{3} dw_a d\bar{w}_a + dx_7^2 + dx_8^2. \tag{2.30}$$

We go back to real coordinates by $w = x_1' + ix_2'$ and drop the primes, to find the metric

$$ds^2 = 2dudv - \frac{du^2}{4}[a^2(x_1^2 + x_2^2) + b^2(x_3^2 + x_4^2) + c^2(x_5^2 + x_6^2)] + \sum_{i=1}^{8} dx_i^2. \tag{2.31}$$

The field strength of the B field is, originally,

$$H_3 = dB_2 = du \wedge [adx_1 \wedge dx_2 + bdx_3 \wedge dx_4 + cdx_5 \wedge dx_6], \tag{2.32}$$

but then neither the transformation (2.25), nor the shift (2.28) changes it, so it has the same form in the final variables. We can, moreover, choose a gauge in which

$$B_2 = \frac{1}{2} du \wedge [a(x_2 dx_1 - x_1 dx_2) + b(x_4 dx_3 - x_3 dx_4) + c(x_6 dx_5 - x_5 dx_6)]. \tag{2.33}$$

As before, $\Phi = 0$. For the pp wave on the second geodesic, see Eq. (2.18), using the same definitions in (2.21), we can put it in the form,

$$\begin{aligned} ds^2 &= 2dudv - du[x_2 dx_1 - x_1 dx_2] - \frac{du^2}{4}[b^2(x_3^2 + x_4^2) + c^2(x_5^2 + x_6^2)] + \sum_{i=1}^{8} dx_i dx_i, \\ B_2 &= \frac{du}{2}[a(x_2 dx_1 - x_1 dx_2) + b(x_4 dx_3 - x_3 dx_4) + c(x_6 dx_5 - x_5 dx_6)], \\ \Phi &= 0. \end{aligned} \tag{2.34}$$

We see that, with respect to the previous case, we have an intermediate case, where the transformations were done for the (x_3, x_4) and (x_5, x_6) pairs, but it remains to do for the (x_1, x_2) pair. Once that is done, the same solution is obtained.

This common solution is also of the type called "parallelizable" in [14], namely a solution of the type

$$\begin{aligned} ds^2 &= 2dudv - (du)^2[a_1^2(x_1^2 + x_2^2) + a_2^2(x_3^2 + x_4^2) + a_3^2(x_5^2 + x_6^2) + a_4^2(x_7^2 + x_8^2)] + \sum_{i=1}^{8} dx_i^2 \\ H &= du \wedge (2a_1 dx_1 \wedge dx_2 + 2a_2 dx_3 \wedge dx_4 + 2a_3 dx_5 \wedge dx_6 + 2a_4 dx_7 \wedge dx_8) \Rightarrow \\ B &= du \wedge [a_1(x_1 dx_2 - x_2 dx_1) + a_2(x_3 dx_4 - x_4 dx_3) + a_3(x_5 dx_6 - x_6 dx_5) + a_4(x_7 dx_8 - x_8 dx_7)]. \end{aligned} \tag{2.35}$$

From the table on page 16 in the paper [14], we see that the solution (since it is of the generic type) has only 16 supercharges (a generic pp wave has 16 supercharges or 1/2 SUSY, but depending on the solution, it can have perhaps more).

On the other hand, we remember the definitions (2.21), as well as the fact, shown in [12], that the background is supersymmetric if

$$e_A Q_B = \pm e_B Q_A \Rightarrow Q_B \sqrt{N_A} = \pm Q_A \sqrt{N_B}. \quad (2.36)$$

Then, we *choose* the rotation parameter α (which was free until now) such that

$$\cos\alpha \propto \sqrt{N_A}, \qquad \sin\alpha \propto \sqrt{N_B} \Rightarrow \tan\alpha = \sqrt{\frac{N_B}{N_A}}. \quad (2.37)$$

Then we obtain $b = c$, and moreover $a = 0$, for the supersymmetric background. This makes sense, since from Table 2 on page 16 of [14], we see that in this case we have 24 supercharges; the 16 generic ones, plus 8 additional ones. In this case we have $U(2) \times O(4)$ symmetry (R-symmetry in the field theory).

We summarize (up to this point). We study the nonsingular background corresponding to I-branes compactified on a circle and preserving SUSY in Eq. (2.1) in the NS-frame. We consider its Penrose limit along two possible geodesics, showing that the result for both geodesics is the same. A parallelizable background in Eq. (2.35) for $a_1 = a_4 = a = 0$ and $a_2 = a_3 = b = c$, preserving twenty-four SUSYs. Let us now study the quantization of a string on this plane wave background.

4. Quantization of string modes

We start by writing the equations of motion for a string in generic parallelizable plane wave in Eq. (2.35). For constant values of a_i with $i = 1, 2, 3, 4$, and choosing the light cone gauge $u = \alpha' p^+ \tau$ we find for the first pair of coordinates (X^1, X^2),

$$\begin{aligned} &\Box X^1 - (a_1 \alpha' p^+)^2 X^1 + 2(a_1 \alpha' p^+)\partial_\sigma X^2 = 0, \\ &\Box X^2 - (a_1 \alpha' p^+)^2 X^2 - 2(a_1 \alpha' p^+)\partial_\sigma X^1 = 0. \end{aligned} \quad (2.38)$$

There are analog equations for the pairs (X^3, X^4) with the constant a_2. For (X^5, X^6) with constant a_3 and for (X^7, X^8) with constant a_4. The equation for the u-coordinate is implied by the Virasoro constraint and the equation for the v-coordinate is automatically satisfied.

In the case of our plane wave with 24 SUSYs ($a = 0$, $b = c$), the string spectrum on the pp wave is easily obtained. In the light cone gauge $\partial_\tau u = \alpha' p^+$, and with the usual ansatz,

$$X^1 = X_0^1 e^{-i\omega t + in\sigma}, \qquad X^2 = X_0^2 e^{-i\omega t + in\sigma}, \quad (2.39)$$

the equations of motion for the (decoupled from the rest) (x_1, x_2) system become

$$\begin{aligned} &(\omega^2 - n^2 - (a\alpha' p^+)^2) X_0^1 + 2in(a\alpha' p^+) X_0^2 = 0, \\ &(\omega^2 - n^2 - (a\alpha' p^+)^2) X_0^2 - 2in(a\alpha' p^+) X_0^1 = 0, \end{aligned} \quad (2.40)$$

so that

$$\begin{aligned} &\frac{X_0^2}{X_0^1} = \frac{+2in(a\alpha' p^+)}{\omega^2 - n^2 - (a\alpha' p^+)^2} = \frac{\omega^2 - n^2 - (a\alpha' p^+)^2}{-\text{i}n(a\alpha' p^+)} \Rightarrow \\ &\omega = \sqrt{n^2 + (a\alpha' p^+)^2 \pm 2n(a\alpha' p^+)} = |n \pm a\alpha' p^+|, \end{aligned} \quad (2.41)$$

but, with the usual rescaling of σ, this gives

$$\omega = \left| a \pm \frac{n}{\alpha' p^+} \right|. \quad (2.42)$$

The same for the coordinates (x_3, x_4) and (x_5, x_6) and constants b, c. In the SUSY case ($a = 0$, $b = c$), we have

$$\begin{aligned} &\omega_\pm = \left| b \pm \frac{n}{\alpha' p^+} \right|, \quad i = 3, 4, 5, 6, \\ &\omega = \frac{n}{\alpha' p^+}, \quad i = 1, 2, 7, 8, \end{aligned} \quad (2.43)$$

so two complex modes of the same mass b, and two complex massless modes since, coming from the (x_1, x_2) and (x_7, x_8) pairs.

B. Dual spin chain

Now we would like to try to reproduce the string spectrum on the pp wave derived above from the dual quantum field theory, originally defined in $1 + 1$ dimensions. Here, we only concentrate on the $n = 0$ modes, corresponding to the Bogomoln'yi-Prasad-Sommerfeld operators in the supersymmetric field theory case. The $n/(\alpha' p^+) \propto 1/J$ terms are left for future work.

1. Field theory

As explained in [11,12], the $(1 + 1)$-dimensional theory, with coordinates x_0 and x_1, comes from the intersection of two sets of D5-branes in type-IIB string theory. The corresponding low-energy field theory is the theory with gauge groups $SU(N_A)_{N_B} \times SU(N_B)_{N_A}$, with both Yang-Mills (YM) and Chern-Simons (CS) terms in $(2 + 1)$ dimensions, and with fermion bifundamentals under the gauge groups, reduced to $(1 + 1)$ dimensions. The theory also has an $SO(4) \times SO(4)$ R-symmetry, corresponding to the two three-spheres S^3's in the gravity dual, see Eq. (2.1).

This is manifest in field theory as the rotation of the two sets of four coordinates parallel to each D5-brane, but transverse to the $(1+1)$-dimensional intersection.

The $(2+1)$-dimensional theory (before the reduction) has some similarity with the theory for the $(2+1)$-dimensional Guarino-Jafferis-Varela (GJV) model [18] (which is also of YM + CS type, though only for one gauge group), whose spin chain coming from a Penrose limit was described in [3]. It is therefore conceivable that one could write the action, in a similar way with the GJV case, then reduce on the S^1. Parts of the theory were written down in [11]. The authors of that paper considered only the gauge fields $A_{(1)}$ and $A_{(2)}$, coming from the two sets of D5-branes, and their interaction with the bifundamental fermions. One should have the scalars of the D5-branes, $\phi^I_{(1)}$, with $I=2$, 3, 4, 5 (transverse to the first set of D5-branes) and $\phi^J_{(2)}$ (transverse to the second set of D5-branes), with $J=6$, 7, 8, 9, on which the $SO(4)\times SO(4)$ R-symmetry should act.

In the Penrose limit, the eight scalars will pair up into four complex ones, $Z, \bar{Z}$ (corresponding to the u and v directions in the gravity dual), and $W_1 = X_1 + iX_2$, $W_2 = X_3 + iX_4$, $W_3 = X_5 + iX_6$ (we renamed the scalars, without explaining the notation, since we do not yet know how they generically pair up into these). Then the spin chain will be, as usual, with insertions of $W_i, \bar{W}_i$ (six oscillators) and D_i (two oscillators) inside the trace of the operator corresponding to the vacuum, $\mathrm{Tr}[Z^J]$.

It is not clear how to get the case of frequencies of string oscillators with arbitrary a, b, c's from field theory, though perhaps that is because of having less supersymmetries, and so there will be more arbitrary renormalizations when going from weak coupling (SYM) to strong coupling (AdS).

However, the supersymmetric case is easier to understand. In this case, we construct the complex scalars,

$$Z = \phi^2_{(1)} + i\phi^6_{(2)}, \qquad W_1 = \phi^3_{(1)} + i\phi^7_{(2)},$$
$$W_2 = \phi^4_{(1)} + i\phi^5_{(1)}, \qquad W_3 = \phi^8_{(2)} + i\phi^9_{(2)}. \tag{2.44}$$

Then, from the $U(2)\times O(4)$ R-symmetry derived from the supersymmetric pp wave previously found, the $SO(4)$ acts on W_2, W_3 (that have the same value for $\Delta - J$, corresponding to the same mass, b, in the supersymmetric pp wave), and the $U(2)$ acts on (Z, W_1) (both corresponding to massless fields in the supersymmetric pp wave). There are also the modes corresponding to insertions of D_i, $i=0$, 1, but those correspond to the directions parallel to the field theory, so they are special.

2. Matching of oscillator modes and masses: Naive try

The correspondence between the pp wave (and gravity dual) coordinates and scalar fields in field theory is then as follows:

$$(x_1, x_2) = (\rho, \varphi) \to D_i, \qquad (x_7, x_8) = (x, \psi_B) \to W_1,$$
$$(x_3, x_4) = (\theta_A, \phi_A) \to W_2, \qquad (x_5, x_6) = (\theta_B, \phi_B) \to W_3. \tag{2.45}$$

The first, *naive*, try for the oscillator spectrum is as follows (we are in $1+1$ dimensions, so a scalar has mass dimension 0).

Field	Z	W_1	$\bar{Z}$	$\bar{W}_1$	$W_2, \bar{W}_2$	$W_3, \bar{W}_3$	D_{x_i}
Δ	0	0	0	0	0	0	1
J	−1	−1	1	1	0	0	0
$\Delta - J$	1	1	−1	−1	0	0	1
$H/\mu = \Delta - J - E_0$	0	0	−2	−2	−1	−1	0
Oscillator	⋯	x_8	⋯	⋯	x_3, x_4	x_5, x_6	x_1, x_2

Then the Hamiltonian has the correct value, if one multiplies it by $\mu = -1$. $\bar{Z}$ and $\bar{W}_1$ would get infinite masses by renormalization, and thus do not correspond to oscillators, as usual. Note that W_1 is massless, because of having the same J as Z, since ψ_A (for Z) and ψ_B (for W_1) are rotated into each other.

We see that we are missing x_7 (one massless oscillator), and the problem is traced to the fact that $\bar{W}_1$ is not an oscillator, just like $\bar{Z}$, so it does not enter the counting. How to fix this?

3. Matching of oscillators modes and masses: Correct version

The problem is resolved if we think better of the construction in [11]. The field theory is actually $(2+1)$-dimensional, where the extra spatial coordinate for the field theory was obtained by joining at $\bar{u} = \bar{v} = 0$ the radial coordinates $\bar{u} \equiv \sqrt{\sum_i \phi^i_{(1)}\phi^i_{(1)}}$ and $\bar{v} = \sqrt{\sum_j \phi^j_{(2)}\phi^j_{(2)}}$ (the radial coordinates transverse to each D5-brane). Let us call this direction w. Then we must add x_7 (corresponding to x) as the D_w oscillator in the $(2+1)$-dimensional field theory.

Then the correct table is [note that in $(2+1)$ dimensions, a scalar has dimension $1/2$, so we must redefine the unit of J to be $1/2$ as well].

Field	Z	W_1	$\bar{Z}$	$\bar{W}_1$	$W_2, \bar{W}_2$	$W_3, \bar{W}_3$	D_{x_i}	D_w
Δ	1/2	1/2	1/2	1/2	1/2	1/2	1	1
J	−1/2	−1/2	1/2	1/2	0	0	0	0
$\Delta - J$	1	1	0	0	1/2	1/2	1	1
$H/\mu = \Delta - J - E_0$	0	0	−1	−1	−1/2	−1/2	0	0
Oscillator	⋯	x_8	⋯	⋯	x_3, x_4	x_5, x_6	x_1, x_2	x_7

Now the Hamiltonian is the correct one if we multiply by $\mu = -2$. Of course, there seems to be a sort of overcounting the dimensions (which now sum up to 11, instead of the 10 of type IIB), but that is due to the fact that the dimension w is half of one dimension, and half of another, so does not truly exist independently: any point with $(\bar{u} \neq 0, \bar{v} \neq 0)$, does not belong to the world volume of the QFT, and the fibered I-brane gravity dual we consider only deals with these points.

In any case, as usual, the vacuum is given by

$$|0;p^+\rangle = \frac{1}{\sqrt{J}N^{J/2}}\mathrm{Tr}[Z^J], \tag{2.46}$$

and we insert the oscillators from the table above inside the trace, in order to obtain the oscillator states.

Now, we perform a similar study for the background dual to a $(4+1)$-dimensional confining QFT [13].

III. PENROSE LIMIT OF TWISTED D5-BRANES AND DUAL SPIN CHAIN

In [13], a gravity dual was considered for a single set of twisted compactified D5-branes on S^1_φ, with a world volume of $(t,\vec{x}_4)$, plus a fibration over the coordinate φ; the S^3 transverse to the single set of D5-branes is fibered over φ, obtaining a cigarlike geometry. One has a Wilson line holonomy inserted, and the theory preserves eight supercharges (1/4 SUSY).

The gravity dual background, in the NS5-brane frame, is given by

$$\begin{aligned} ds^2_{st} &= dx^2_{1,4} + f_s(\rho)d\varphi^2 + \frac{d\rho^2}{f_s(\rho)} \\ &\quad + \frac{N}{4}\left(\omega_1^2+\omega_2^2+\left(\omega_3-\sqrt{\frac{8}{N}}Q\zeta(\rho)d\varphi\right)^2\right), \\ H_3 &= 2N\mathrm{Vol}(S^3) + 2\sqrt{\frac{N}{8}}Qd(\zeta(\rho)\omega_3\wedge d\varphi), \\ \Phi &= -\frac{\rho}{\sqrt{N}}, \end{aligned} \tag{3.1}$$

where[3]

$$\begin{aligned} f_s(\rho) &= 1 - me^{-2\rho/\sqrt{N}} - 2Q^2e^{-4\rho/\sqrt{N}} \\ &= e^{-4\rho/\sqrt{N}}\left(e^{2\rho/\sqrt{N}}-e^{2\rho_+/\sqrt{N}}\right)\left(e^{2\rho/\sqrt{N}}-e^{2\rho_-/\sqrt{N}}\right), \\ \zeta(\rho) &= e^{-2\rho/\sqrt{N}} - e^{-2\rho_+/\sqrt{N}}, \end{aligned} \tag{3.2}$$

and, as in the I-brane case, the metric on the transverse S^3 is written in terms of the Maurer-Cartan forms of $su(2)$,

$$\begin{aligned} \omega_1 &= \cos(\psi)d\theta + \sin(\psi)\sin(\theta)d\phi, \\ \omega_2 &= -\sin(\psi)d\theta) + \cos(\psi)\sin(\theta)d\phi, \\ \omega_3 &= d\psi + \cos(\theta_A)d\phi. \end{aligned} \tag{3.3}$$

The $\rho_\pm$ are the two solutions of $f_s(\rho)=0$; namely,

$$e^{2\frac{\rho_\pm}{\sqrt{N}}} = \frac{m\pm\sqrt{m^2+8Q^2}}{2}. \tag{3.4}$$

The supersymmetric case corresponds to

$$e^{\frac{2\rho_+}{\sqrt{N}}} = \sqrt{2}Q, \qquad m=0. \tag{3.5}$$

As in the I-brane case, the end of the cigar, $\rho=\rho_+$, corresponds to the IR of the field theory.

A. Penrose limit

In order to explore the IR of the field theory, we consider a geodesic at $\rho=\rho_+$, like in the I-brane case of Sec. II. Moreover, we now consider a geodesic moving in t and ψ (the coordinate on an equator of S^3) and at $\theta=\pi/2, \phi=0$, as well as $x^i=0$, $i=1,\ldots,4$ (or any constant x^i_0), and φ arbitrary.

We define the usual lightcone coordinates, now for (t,ψ),

$$\begin{aligned} t &= \frac{1}{\sqrt{2}}(u-v), \\ \psi &= \frac{1}{\sqrt{2}}(u+v), \end{aligned} \tag{3.6}$$

and then the standard rescalings in the Penrose theorem,

$$\begin{aligned} &v\to\frac{v}{L^2}, \qquad \theta\to\frac{\pi}{2}+\frac{2\theta}{L\sqrt{N}}, \qquad \phi\to\frac{2\phi_{A,B}}{L\sqrt{N}}, \\ &x^i\to\frac{x^i}{L}, \qquad \rho\to\rho_++\frac{\rho}{L^2}, \qquad u\to u, \qquad \varphi\to\varphi, \end{aligned} \tag{3.7}$$

where the only nontrivial case is that of φ, which does not rescale, since it is an angular coordinate (and stays so).

Then we take the Penrose limit, by multiplying the metric with L^2 and taking the limit $L\to\infty$, while also using the redefinition

$$\begin{aligned} \rho &\to \frac{1}{\sqrt{N}}\frac{\sinh\left(\frac{1}{\sqrt{N}}(\rho_+-\rho_-)\right)}{e^{\frac{1}{\sqrt{N}}(\rho_+-\rho_-)}}\rho^2, \\ \varphi &\to \frac{\sqrt{N}}{2}\frac{e^{\frac{1}{\sqrt{N}}(\rho_+-\rho_-)}}{4\sinh\left(\frac{1}{\sqrt{N}}(\rho_+-\rho_-)\right)}\varphi, \end{aligned} \tag{3.8}$$

to obtain the pp wave solution

[3]Note that here $e^{2\rho_-/\sqrt{N}}<0$, due to the fact that Q^2 has the opposite sign to the one in a black solution. That is so, since the solution is related to the black membrane through a double Wick rotation, that requires also $Q\to -iQ$ in order to keep the solution real. But the negative sign is no problem—the double Wick rotation means that $\rho_\pm$ are not interpreted as horizons—only ρ_+ is physical, and it is just the tip of a cigar geometry, locally the origin of a tangent plane. ρ_- on the other hand is just a mathematical construct.

$$L^2 ds^2 = 2du\left(dv - \sqrt{\frac{2}{N}}\theta d\phi + \frac{Q}{\sqrt{N}}e^{-2\rho_+/\sqrt{N}}\rho^2 d\varphi\right)^2 + d\rho^2 + \rho^2 d\varphi^2 + d\theta^2 + d\phi^2 + d\vec{x}^2,$$
$$L^2 H_3 = \sqrt{\frac{2}{N}}du\wedge d\theta\wedge d\phi + \frac{2Q}{\sqrt{N}}e^{-2\rho_+/\sqrt{N}}\rho du\wedge d\rho\wedge d\phi,$$
$$\Phi = 0. \tag{3.9}$$

We now move to Cartesian coordinates for (ρ, φ), namely (x^5, x^6), and relabel the rescaled (θ, ϕ) as (x^7, x^8), to obtain

$$ds^2_{PL} = 2du\left(dv - \sqrt{\frac{2}{N}}x_7 dx_8 + \frac{Q}{\sqrt{N}}e^{-2\rho_+/\sqrt{N}}(-x_6 dx_5 + x_5 dx_6)\right)^2 + d\vec{x}^2,$$
$$H_3 = \sqrt{\frac{2}{N}}du\wedge dx_7\wedge dx_8 + \frac{2Q}{\sqrt{N}}e^{-2\rho_+/\sqrt{N}}du\wedge dx_5\wedge dx_6,$$
$$\Phi = 0. \tag{3.10}$$

Defining

$$a = \frac{Q}{\sqrt{N}}e^{-2\rho_+/\sqrt{N}}, \qquad b = \frac{1}{2}\sqrt{\frac{2}{N}}, \tag{3.11}$$

we obtain the solution in the form

$$ds^2_{PL} = 2du[dv + a(-x_6 dx_5 + x_5 dx_6) - 2b x_7 dx_8] + d\vec{x}^2,$$
$$H_3 = 2a du\wedge dx_5\wedge dx_6 + 2b du\wedge dx_7\wedge dx_8,$$
$$\Phi = 0. \tag{3.12}$$

1. Coordinate change to parallelizable pp wave, and supersymmetry

We make a coordinate change to a parallelizable pp wave. Indeed, defining the complex coordinate,

$$z = x_5 + i x_6, \tag{3.13}$$

we get

$$-x_6 dx_5 + x_6 dx_5 = -\frac{i}{2}(\bar{z}dz - z d\bar{z}). \tag{3.14}$$

Then, the coordinate change

$$z = e^{-iau/2}w, \qquad \bar{z} = e^{iau/2}\bar{w} \tag{3.15}$$

means that

$$dz d\bar{z} + 2du\left(-\frac{ia}{2}(\bar{z}dz - z d\bar{z})\right) = dw d\bar{w} - a^2 w\bar{w} du^2. \tag{3.16}$$

Going back to Cartesian coordinates, we get the pp wave solution in the form,

$$ds^2 = -a^2(x_5^2 + x_6^2)du^2 + 2du(dv - 2b x_7 dx_8) + d\vec{x}^2,$$
$$H_3 = 2a du\wedge dx_5\wedge dx_6 + 2b du\wedge dx_7\wedge dx_8,$$
$$\Phi = 0. \tag{3.17}$$

Under these coordinate changes, H_3 did not change, as in the I-brane case. Finally, we shift v as

$$v \to v + b x_7 x_8, \qquad dv \to dv + b x_8 dx_7 + b x_7 dx_8, \tag{3.18}$$

which means that we put the term proportional to b in the same form as the term proportional to a was, so we can follow the previous steps again for this term.

Finally, the result is that the pp wave is put in the parallelizable form,

$$ds^2 = 2du dv - [a^2(x_5^2 + x_6^2) + b^2(x_7^2 + x_8^2)]du^2 + d\vec{x}^2,$$
$$H_3 = 2a du\wedge dx_5\wedge dx_6 + 2b du\wedge dx_7\wedge dx_8,$$
$$\Phi = 0, \tag{3.19}$$

with a, b defined in Eq. (3.11). We also observe that the supersymmetric case, which is now also the extremal case, with $a = b$,

$$e^{\frac{2\rho_+}{\sqrt{N}}} = \sqrt{2}Q, \quad m = 0, \tag{3.20}$$

has the same form as the supersymmetric pp wave obtained from the fibered I-branes, which in the classification in [14], has 24 supercharges; the 16 generic ones, plus additional eight ones. The R-symmetry is also the same as before, $U(2)\times O(4)$.

B. Dual spin chain

The natural field theory dual would be the theory on the world volume of the D5-branes, with $(t, \vec{x}_4)$ and the compact φ, on which we can compactify. As we said, the theory has implicitly a Wilson-loop holonomy, breaking a quarter of the SUSY, preserving eight supercharges.

As in the I-brane case, we will only be able to describe the supersymmetric case, corresponding in the gravity dual to $a = b$. The generic case is unclear, but we expect that quantum corrections would spoil that analysis in any case.

We will see that, in fact, it is better to consider the same $(1+1)$-dimensional field theory as for the I-brane case (also since the supersymmetric pp wave is the same), but now understood as coming from the compactification of the $(4+1)$-dimensional theory in $(t, \vec{x}_4)$ on the missing S^3 (which was transverse in the I-brane case, but is now parallel in this fibered D5-brane case).

Then the field theory scalars are $4X$'s transverse to the D5-branes, let us call them X^6, X^7, X^8, X^9, plus 4 for the compactification on $S^3 \times S^1$, let's call them A_2, A_3, A_4, A_5, as they arise from gauge fields in compact directions that become scalars after compactification.

Then, if ψ rotates X^6, X^7, we take the complex scalar field $Z = X^6 + iX^7$ to be charged under J. As in the I-brane case, this has $\Delta = 0, J = -1$, so gives $\Delta - J = 1$, just like D_i, $i = 0, 1$, which now has $\Delta = 1, J = 0$, and after subtracting the ground state energy $E_0 = 1$, gives two of the massless oscillators. Then X^8, X^9 have $\Delta = 0, J = 0$, $\Delta - J = 0$, so after subtracting $E_0 = 1$ gives -1, so we obtain two of the massive oscillators; $(x_7, x_8) = (\theta, \phi)$ in the gravity dual.

However, in order to obtain the correct number of massless and massive oscillators, we now need to split A_a, where $a = 2, 3, 4, 5$ (for the four coordinates on $S^3 \times S^1$) into two massless oscillators and two massive oscillators [for $(x_5, x_6) = (\rho, \varphi)$ in the gravity dual].

To do so, we can write $A_5 \to X_\varphi$, so the field in the φ direction is thought of as a scalar; by multiplication with $1/g^2_{\rm YM}$, it has dimension 0 instead of dimension 1. Moreover, we can do the same for the overall field on the S^3,

$$\frac{1}{g^2_{\rm YM}} \sqrt{\sum_{a=2}^{4} A_a A_a} \to X_\rho, \tag{3.21}$$

with dimension 0, like a scalar. Then X_φ, X_ρ have $\Delta - J = 1$, and also correspond to massless excitations, specifically x_5, x_6 (for φ, ρ). This leaves A_2, A_3 as dimension 1 fields, giving the remaining two massless modes.

This procedure is consistent, but somewhat puzzling, yet is the only one possible that gives the same result as in the I-brane case, which was necessary, since the supersymmetric pp wave was the same.

As for the other fields, we shift the mass by the ground-state energy $E_0 = 1$, and multiply it by $\mu = -1$.

Thus, the final table is.

Field	Z	$\bar{Z}$	X^3, X^4	X_φ	X_ρ	$A_{a'}, a'=2, 3$	$D_{x_i}, i=0, 1$
Δ	0	0	0	0	0	1	1
J	-1	1	0	0	0	0	0
$\Delta - J$	1	-1	0	0	0	1	1
$H/\mu = \Delta - J - E_0$	0	-2	-1	-1	-1	0	0
Oscillator	$\cdots$	$\cdots$	x_7, x_8	x_6	x_5	x_3, x_4	x_1, x_2

The R-symmetry of $U(2) \times O(4) \simeq U(1) \times SO(3) \times SO(4)$ (modulo some discrete symmetries) in the supersymmetric case is understood in the field theory as the symmetry of S^1_φ, $S^2 \subset S^3$ (since S^3 was understood as an S^2 fibration, allowing for the split of A_2, A_3, A_4 into two massive and one massless excitation), and of $X^6, \ldots, X^9$, respectively.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have studied the Penrose limits of the gravity duals of fibered I-branes and D5-branes [12,13], in order to understand better the duality.

The I-branes gave a $(1+1)$-dimensional theory, dynamically extended to $(2+1)$ dimensions. We have found that in order to match the pp wave analysis obtained in the Penrose limit against a field theory, we must consider the theory in $(2+1)$ dimensions, in which case we can match the oscillators and their masses.

Perhaps more surprisingly, we have found that, in order to match the oscillators and masses for the case of the Penrose limit of the twisted S^1-compactified D5-branes against field theory, we need to consider the same $(1+1)$-dimensional theory of the original I-brane world volume, now obtained by reduction on the S^1 fibration circle, as well as the world volume S^3. Moreover, in the S^3, one of the directions becomes special also, although our analysis still is made from the $(1+1)$-dimensional point of view, and not a $(2+1)$-dimensional one.

In both fibered I-branes and D5-branes cases, we find that the *supersymmetric* pp wave, the only case for which we can find matching, is the same one, the generic parallelizable case with 24 supercharges (the standard 16 SUSYs, plus 8 more) from the classification in [14]. The nonsupersymmetric pp waves are different in the two cases, but it is not clear if or how to match, as in that case, there will presumably be uncontrollable corrections to the mass. We also found that, before a coordinate transformation, one of the generic Penrose limits on the I-brane case (with parameters $a \neq b \neq c$) gave us a so-called gyratonic pp wave [16,17].

There are many issues left for further work, among them notably the match in the generic, nonsupersymmetric case, as well as the match of the terms with $n/(\alpha' p^+)$ in the string oscillator frequencies, which correspond to $\propto 1/J$ terms, coming from gauge interactions, in the field theory side. Apart from this, backgrounds similar to the ones studied here, see for example [19,20,21,22,23] could be analyzed following our formalism. We expect results with similar qualitative features.

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