Submanifolds, Fibre Bundles, and Cofibrations in Noncommutative Differential Geometry

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Abstract

This is a thesis in noncommutative differential geometry. Equipping algebras with differential calculi, we propose noncommutative differential equivalents of some concepts from topology: submanifolds and fibre bundles. Further, we consider some ideas towards noncommutative versions of cofibrations and retracts. We also give a new diagrammatic calculus on Temperley-Lieb algebras.

Declarations and Statements

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed: James Edward Blake, **Date**: 18th January 2024

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s). Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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An earlier version of Part IV of the thesis is uploaded to the arXiv as [7], where the authors are listed in alphabetical order.

Contribution by the candidate James Blake: Writing the paper, calculations of theory and examples **(50%)**

Contribution by PhD supervisor Edwin Beggs: Initial idea of extending the theory of fibre bundles to use positive maps instead of algebra maps, ideas for the examples, regular discussion and checking of work **(50%)**

Candidate signature: James Edward Blake

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Part I

Introduction

1 Noncommutative Differential Geometry

In classical differential geometry, the objects of study are (infinitely) differentiable manifolds, and the vector fields and differential forms associated with them. On a differentiable manifold M, on each local coordinate patch (also called a chart) there are vector fields $\frac{\partial}{\partial x^i}$ which span the tangent bundle TM. Dual to these are 1-forms $\mathrm{d}x^i$ which span the cotangent bundle T^*M. Note that it only makes sense to talk about these on local coordinate patches. Denoting $C^\infty(M)$ for the algebra of smooth (i.e. infinitely differentiable) real-valued functions $M \to \mathbb{R}$, the duality is by the bimodule map $ev : TM \otimes T^*M \to C^\infty(M)$ given on local coordinate patches by $ev(\frac{\partial}{\partial x^i}, \mathrm{d}x^j) = \delta_{i,j}$. Globally we write $\Omega^1(M)$ for the space of 1-forms, which is regarded abstractly as the image of the linear map $\mathrm{d} : C^\infty(M) \to \Omega^1(M)$ satisfying $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i$ and the Leibniz rule $\mathrm{d}(fg) = f\mathrm{d}(g) + \mathrm{d}(f)g$. Higher differential forms $\Omega^n(M)$ can also be defined via the wedge product $\wedge : \Omega^n(M) \otimes \Omega^m(M) \to \Omega^{n+m}(M)$. For example, $\Omega^2(\mathbb{R}^2)$ contains the 2-form $\mathrm{d}x^1 \wedge \mathrm{d}x^2$. In particular the extension of the differential to higher calculi squares to zero, giving the following cochain complex, called the de Rham complex.

$$0 \longrightarrow C^\infty(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} \cdots$$

Associated to this cochain complex are the de Rham cohomology groups, defined as:

$$H^n(M) = \ker(\mathrm{d} : \Omega^n(M) \to \Omega^{n+1}(M))/\mathrm{im}(\mathrm{d} : \Omega^{n-1}(M) \to \Omega^n(M)).$$

However, in noncommutative differential geometry, it was observed that all of this theory, along with other objects such as metrics and connections on vector bundles, can also be defined when $C^\infty(M)$ is replaced by any algebra A, usually associative and unital, but which need not be commutative. The objective of the field is to find as many noncommutative analogues of classical constructions as possible, along with considering differential phenomena which occur only in a noncommutative setting. In the preliminaries section below this, we give an overview of the key definitions and results in noncommutative differential geometry.

The Gelfand-Naimark theorem [28] gives an interpretation of these noncommutative analogues, giving a one-to-one correspondence between compact Hausdorff spaces and commutative unital C*-algebras. Thus we interpret noncommutative unital C*-algebras as algebras of functions on hypothetical "noncommutative" compact Hausdorff spaces. The theory of C*-algebras was developed in the early 20th century by Von Neumann [44], and many others including Gelfand and Naimark [28].

Also in the early 20th century, it was observed that the algebra of observables in quantum theory is noncommutative. These were connected in the 1980s by Alain Connes [19], whose theory of spectral triples gave a noncommutative analogue of the Dirac operator from physics, and also an idea of a differential on noncommutative algebras as $\mathrm{d}a = [D, a]$. However, the notion of calculus that we work with in this paper is the one proposed by Woronowicz in 1989 [58] for quantum groups such as $\mathbb{C}_q[SU_2]$, which are noncommutative analogues of Lie groups.

This thesis contains no physics, but noncommutative differential geometry finds various applications in physics, including to noncommutative space time models by Majid such as in [32] and Chapter 9 of [10], to gauge theories and the standard model by Chamseddine and Connes in [18], and to geodesics in quantum mechanics by Beggs and Majid [9].

Within mathematics, there are applications to Hopf algebras as in chapter 2 of [10], and applications to Hopf algebroids by Majid and Simão [37], where the study of bimodule connections from algebroids gives rise to a theory of jet bundles. There are also applications to C*-algebras by considering differentiation of the KSGNS construction (see chapter 5 of [30] for details on the KSGNS construction), as was used in [6] to study noncommutative geodesics, and also as we use extensively in Parts V and VI of this thesis. There are also applications to noncommutative generalisations of grassmanians and Lie theory such as in [17] and [16]

In this thesis, we begin with a review of preliminaries from noncommutative differential geometry, closely following definitions and notation used in the book [10]. The overarching theme of the thesis is to introduce noncommutative analogues of various concepts from topology, and calculate a number of examples of each, with an emphasis on quantum group examples and discrete examples.

Section II contains the first original content, where we give a diagrammatic differential calculus on Temperley-Lieb algebras. This calculus is new to the extent of our knowledge, and the section also serves as an illustration of a number of techniques we use throughout the thesis.

In Section III we introduce a new possible definition for noncommutative submanifolds, and prove a number of results about the differential geometric properties of algebras satisfying this definition.

In Section IV we take an existing definition for noncommutative fibre bundles via algebra maps and generalise it to use bimodules and completely positive maps instead, showing we can still obtain a Leray-Serre spectral sequence. We also uploaded this to the arXiv as [7].

In Section V we look at ideas for noncommutative analogues of retracts and neighbourhood retracts. Since we only began work on this section quite late in the project, our investigations are still in the early stages and there is a lot of future work we could do.

Lastly, in Section VI we look at what Quillen's definition of cofibration might mean in a noncommutative differential context. We focus on the problem of lifting time-dependent states, which correspond to paths, since lifting paths is a necessary first step towards lifting homotopies. This section is just a few initial ideas and calculations towards something whose completed version, if it exists, would be much more complicated.

With the possible exception of the section on Temperley-Lieb algebras, all the subjects in this thesis — submanifolds, fibre bundles, retracts, and cofibrations — are related classically. Classical embedded submanifolds give neighbourhood retracts via tubular neighbourhoods. One definition of topological cofibrations is in terms of neighbourhood deformation retracts. Fibre bundles are examples of fibrations, and fibrations are dual to cofibrations.

2 Preliminaries

The key definitions and notation we use closely follow the setup of the book [10], but so that this thesis can be read as a self-contained document we also present them here.

The first thing we do with an algebra is to equip it with a calculus, after which we can consider structures like connections on modules over the algebra.

Definition 2.1. A vector space A over $\mathbb{C}$ is called an *algebra* if there is a multiplication operation which is distributive with respect to the vector space addition. Further, we say that A is associative if $a(bc) = (ab)c$ for all $a, b, c \in A$, and call A unital if there exists an element $1 \in A$ such that $1.a = a.1$ for all $a \in A$.

In this thesis, we will assume all algebras are associative and unital unless otherwise stated. The one non-unital algebra we consider is the algebra D in the retracts section.

Definition 2.2. ([10] Section 1.1) Given algebras A and B, we call vector space E:
(1) A *left A-module* if there is a left action $a \triangleright e \in E$ of A on E which respects algebra multiplication in the sense that $(a_1a_2) \triangleright e = a_1 \triangleright (a_2 \triangleright e)$. We write $E \in {}_A\mathcal{M}$.
(2) A *right B-module* if there is a right action $e \triangleleft b \in E$ of B on E which respects algebra multiplication in the sense that $e \triangleleft (b_1b_2) = (e \triangleleft b_1) \triangleleft b_2$. We write $E \in \mathcal{M}_B$.
(3) An *A-B bimodule* if E is both a left A-module and a right B-module and the actions commute in the sense that $(a \triangleright e) \triangleleft b = a \triangleright (e \triangleleft b)$. We write $E \in {}_A\mathcal{M}_B$. An A-A bimodule is called an A-bimodule for short.

Actions on modules, especially on calculi, may be also written as $a.e$ or simply ae. In the example $\mathbb{C}G$ for G a finite group, we actually see both the triangle and dot notations appearing, where the dot is the action on the calculus and the triangle is the action on a representation.
We can take the tensor product of an A-B bimodule E with a B-C bimodule F, and obtain an A-C bimodule $E \otimes_B F$. This has actions $a.(e \otimes f) = (a.e) \otimes f$ and $(e \otimes f).c = e \otimes (f.c)$. The characteristic property of $\otimes_B$ is that $e.b \otimes f = e \otimes b.f$ — i.e. that we can move elements of B across the tensor product.
Note that since every module over $\mathbb{C}$ can be regarded as a $\mathbb{C}$ module by multiplication, we can always take $B = \mathbb{C}$ and get a tensor product $E \otimes_{\mathbb{C}} F$, which it is customary to write as $E \otimes F$.

Definition 2.3. ([10] Definition 1.1) For an associative unital algebra A, we say that an A-bimodule Ω^1_A is a (differential) *calculus* on A if $\Omega^1_A = \mathrm{span}\{a'\mathrm{d}a \mid a, a' \in A\}$ for a linear map $\mathrm{d} : A \to \Omega^1_A$ satisfying $\mathrm{d}(ab) = a\mathrm{d}b + \mathrm{d}a.b$, which we call the exterior derivative.

The calculus is called connected if $\ker \mathrm{d} = \mathbb{K}.1$, where $\mathbb{K}$ is the field of scalars of A.

Here we always take $\mathbb{K} = \mathbb{C}$. If we drop the condition that Ω^1_A is spanned by elements of the form $a'\mathrm{d}a$ and allow elements not of this form, then Ω^1_A is called a generalised calculus. We consider generalised calculi in the Temperley-Lieb algebras section.
There are potentially many different calculi that any given algebra can be equipped with, and so we always need to specify which calculus we are using. If we talk about a calculus Ω^1_A on an algebra A without specifying which calculus it is, then Ω^1_A is arbitary.
Note that while in the calculus on a manifold we always have $\mathrm{d}b.a = a.\mathrm{d}b$, this non-commutative definition makes no such assumption, and this gives rise to a number of phenomena such as inner calculi which only appear in a noncommutative setting due to commutators not vanishing.

Definition 2.4. ([10] Definition 1.3) If there exists an element $\theta \in \Omega^1_A$ such that $\mathrm{d}a = [\theta, a]$ for all $a \in A$, where $[\theta, a] = \theta a - a\theta$ denotes the commutator, then we say that the calculus Ω^1_A is *inner* by θ.

Definition 2.5. ([10] Chapter 6) For an algebra A with calculus Ω^1_A, *right vector fields* on A are the set of right module maps

$$\mathfrak{X}^R_A = \mathrm{Hom}_A(\Omega^1_A, A),$$

i.e. which satisfy $X(\xi.a) = X(\xi).a$ for $\xi \in \Omega^1_A$, $a \in A$.
The *left vector fields* on A are the set of left module maps

$$\mathfrak{X}^L_A = {}_A\mathrm{Hom}(\Omega^1_A, A),$$

i.e. which satisfy $X(a.\xi) = aX(\xi)$ for $\xi \in \Omega^1_A$, $a \in A$.

The vector fields are an A-bimodule, with $X \in \mathfrak{X}^R_A$ having actions

$$(aX)(\xi) = a.X(\xi), \quad (Xa)(\xi) = X(a\xi),$$

and $X \in \mathfrak{X}^L_A$ having actions

$$(aX)(\xi) = X(\xi.a), \quad (Xa)(\xi) = X(\xi).a.$$

The actions on left vector fields look a little strange, but this comes from the requirement that the evaluation map $ev : \Omega^1_A \otimes_A \mathfrak{X}^L_A \to A$ be a bimodule map.
In the classical case where 1-forms commute with functions on a manifold, this definition is equivalent to derivations on $C^\infty(M)$, but in the noncommutative case these do not coincide, and we have to choose one definition over another. In Section III we go into more detail as to why we have chosen to take vector fields as dual to 1-forms instead of being derivations, but in short the main reason is because we can't take connections on it otherwise.

Definition 2.6. ([10] Definition 3.1) A right A-module F is said to be *right finitely generated projective* if there are a finite number of module elements $f_i \in F$ and right module maps $e^i \in \mathrm{Hom}_A(E, A)$ such that each $f \in F$ can be decomposed as $f = \sum_i f_i.e^i(f)$.

It follows that each $e \in \mathrm{Hom}_A(E, A)$ can be decomposed as $e = \sum_i e(f_i).e^i$, and that $\mathrm{Hom}_A(E, A)$ is a left A-module and is left finitely generated projective.
If the calculus Ω^1_A is right finitely generated projective, the vector fields $\mathfrak{X}^R_A$ are left finitely generated projective. Likewise, if Ω^1_A is left finitely generated projective, then $\mathfrak{X}^L_A$ is right finitely generated projective.
So far we have looked at first order calculi Ω^1_A, but similarly to the de Rham complex in classical differential geometry we can define higher order calculi Ω^n_A. This is given by equipping the algebra with a differential graded algebra structure as in Definition 1.30 of [10], where we extend the differential as

$$\mathrm{d}(\omega \wedge \rho) = (\mathrm{d}\omega) \wedge \rho + (-1)^n \omega \wedge \mathrm{d}\rho$$

for $\omega \in \Omega^n_A$, $\rho \in \Omega^n_A$, and following from the surjectivity condition of first order calculi, an element of Ω^n_A is the wedge product of n elements of Ω^1_A. This formula for the differential makes $\mathrm{d}^2 = 0$. So for example an element of Ω^2_A is of the form $\mathrm{d}a_1 \wedge \mathrm{d}a_2$ for some $a_1, a_2 \in A$, and satisfies $\mathrm{d}a_1.a_3 \wedge \mathrm{d}a_2 = \mathrm{d}a_1 \wedge a_3.\mathrm{d}a_2$ for $a_3 \in A$. As for what relations this wedge product satisfies, there are many possibilities, but one used very commonly and which has a minimal number of relations is the maximal prolongation calculus, discussed later in this section.
Next we look at connections on modules and bimodule connections on bimodules.

The idea of a bimodule connection was introduced in [24], [23] and [42] and used in [27], [31]. It was used to construct connections on tensor products in [15].

Definition 2.7. ([10] Definition 3.18) A *left connection* on a left A-module E is a linear map $\nabla_E : E \to \Omega^1_A \otimes_A E$ obeying the left Leibniz rule

$$\nabla_E(a.e) = \mathrm{d}a \otimes e + a.\nabla_E e, \quad e \in E, a \in A.$$

Its *curvature* $R_E : E \to \Omega^2_A \otimes_A E$ is defined by

$$R_E e = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\nabla_E e.$$

By Lemma 3.19 of [10], the curvature of a left connection is always a left A-module map. A connection is said to be flat if its curvature is zero.

This is related to the classical definition of connection by the formulae

$$\nabla s = \mathrm{d}x^i \otimes \nabla_i s, \qquad \nabla_i s = (ev \otimes \mathrm{id})(\frac{\partial}{\partial x^i} \otimes \nabla s),$$

where s is a section.

There is a similar definition for right connections on right modules.

Definition 2.8. A *right connection* on a right A-module E is a linear map $\tilde{\nabla}_E : E \to E \otimes_A \Omega^1_A$ obeying the right Leibniz rule

$$\tilde{\nabla}_E(e.a) = \tilde{\nabla}_E(e).a + e \otimes \mathrm{d}a,$$

and its *curvature* $\tilde{R}_E : E \to E \otimes_A \Omega^2_A$ is defined by

$$\tilde{R}_E(e) = (\mathrm{id} \otimes \mathrm{d} + \tilde{\nabla}_E \wedge \mathrm{id})\tilde{\nabla}_E e.$$

Similarly, the curvature of a right connection is always a right module map.

On a B-A bimodule, we can ask if a left connection satisfies a version of the right Leibniz rule as well, i.e. whether the left connection is compatible with the right module structure.

Definition 2.9. ([10] Generalisation of Definition 3.66) We say that a left connection $\nabla_E : E \to \Omega^1_B \otimes_B A$ on a B-A bimodule E is a *bimodule connection* if the map $\sigma_E : E \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B E$ given by

$$\sigma_E(e \otimes \mathrm{d}a) = \nabla(e.a) - \nabla_E(e).a, \quad e \in E, a \in A$$

is a bimodule map.

Since σ_E is defined in terms of ∇_E, it is not additional data. Rather, after specifying the right module structure, being a bimodule connection or not is a property of any given left connection.

The bimodule map σ_E lets us move first-order calculi from one side of the bimodule to another, but in order to move higher order calculi we require extendability.

Definition 2.10. ([10] Definition 4.10) A left bimodule connection ∇_E on a B-A bimodule E is called *extendable* if $\sigma_E : E \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B E$ extends for all $n \geq 1$ to $\sigma_E : E \otimes_A \Omega^n_A \to \Omega^n_B \otimes_B E$ such that for all $m \geq 1$

$$(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E)(\sigma_E \otimes \mathrm{id}) = \sigma_E(\mathrm{id} \otimes \wedge) : E \otimes_A \Omega^n_A \otimes_A \Omega^m_A \to \Omega^{n+m}_B \otimes_B E.$$

For a right bimodule connection extendability means that σ_E extends as $\sigma_E : \Omega^n_B \otimes_B E \to E \otimes_A \Omega^n_A$ by the formula $\sigma_E(\xi \wedge \eta \otimes e) = (\sigma_E \wedge \mathrm{id})(\xi \otimes \sigma_E(\eta \otimes e))$. This can also be written as $(\mathrm{id} \otimes \wedge)(\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E) = \sigma_E(\wedge \otimes \mathrm{id})$.

We use extendability in the fibre bundles section.

Definition 2.11. ([10] Lemma 1.32) Every first order calculus Ω^1 on A has a '*maximal prolongation*' Ω_{max} to an exterior algebra, where for every relation $\sum_i a_i.\mathrm{d}b_i = \sum_j \mathrm{d}r_j.s_j$ in Ω^1 for $a_i, b_i, r_j, s_j \in A$ we impose the relation $\sum_i \mathrm{d}a_i \wedge \mathrm{d}b_i = -\sum_j \mathrm{d}r_j \wedge \mathrm{d}s_j \in \Omega^2_{\max}$. This is extended to higher forms, but no new relations are added.

Lemma 2.12. *(Corollary 5.3 and 5.4 of [1]) If a B-A bimodule E has bimodule connection ∇_E, and the following two conditions hold:*

- *A is equipped with maximal prolongation calculi for its higher calculi,*
- *the curvature R_E of the connection is a bimodule map,*

then extendability of σ_E is automatic.

In the presence of a metric, we can ask if the metric is compatible with the metric (in this case we say it preserves the metric or is metric-preserving).

A right connection ∇_E is said to preserve an inner product $\langle , \rangle : \overline{E} \otimes_B E \to A$ on E if $\mathrm{d}\langle \overline{e_1}, e_2 \rangle = \langle \overline{e_1}, \nabla_E(e_2)_{(1)} \rangle \nabla_E(e_2)_{(2)} + \nabla_E(e_1)^*_{(2)} \langle \overline{\nabla_E(e_1)_{(1)}}, e_2 \rangle$ for all $e_1, e_2 \in E$, using a form of Sweedler notation $\nabla_E(e) = \sum \nabla_E(e)_{(1)} \otimes \nabla_E(e)_{(2)}$ for tensor products. (A version for left connections is given in Definition 8.33 of [10].) If there is a right connection ∇_E on E, then there is also a left connection $\nabla_{\overline{E}}$ on $\overline{E}$, given by $\nabla_{\overline{E}}(\overline{e}) = \nabla_E(e)^*_{(2)} \otimes \overline{\nabla_E(e)_{(1)}}$. We use this notation to write the metric preservation in string diagrams in Figure 1.

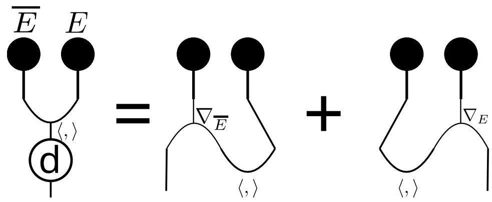


Figure 1: Preliminaries: Illustration of the metric preservation equation

3 List of Algebras

We give an overview of the algebras we use in this thesis and their standard calculi. A history of these calculi can be found in the notes at the end of Chapters 1-3 of [10], but we give citations here too.

Example 3.1. $\mathbf{C^\infty(M)}$ For a smooth manifold M, the set of smooth functions $M \to \mathbb{R}$ is a commutative algebra with unit the function sending all of M to 1. It has differential

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i,$$

and 1-forms commute with elements of the algebra.

We use this in Sections 13.1 and 13.2. $\diamond$

Example 3.2. $\mathbb{C}_\mathbf{q}[\mathbf{S^1}]$ ([10] Example 1.11. Calculus on quantum circle originally by Majid in [35].) The Algebraic Circle $\mathbb{C}_q[S^1] = \mathbb{C}[t, t^{-1}]$ is the algebra of polynomials in t and t^{-1}, and its standard calculus is given by

$$\Omega^1 = \mathbb{C}[t, t^{-1}].\mathrm{d}t, \quad \mathrm{d}t.f(t) = f(qt)\mathrm{d}t, \quad \mathrm{d}f = \frac{f(qt) - f(t)}{t(q-1)}\mathrm{d}t.$$

It has $*$-structure given by $t^* = t^{-1}$.

We use this in Sections 13.3 and 13.4 and 16.3 and 16.1 and 26 and 42.2. $\diamond$

Example 3.3. $\mathbb{C}_\mathbf{q}[\mathbf{D}]$ ([10] Example 3.40. Calculi on the quantum disk are originally by Vaksman in [55].) The quantum disk $\mathbb{C}_q[D]$ with $0 < q < 1$ is the q-deformation

of functions on the unit disk. It is generated by elements z and $\overline{z}$, with relation $z\overline{z} = q^{-2}\overline{z}z - q^2 + 1$ and $*$-structure $z^* = \overline{z}$. The algebra has a grading $|z| = 1$ and $|\overline{z}| = -1$. There is a grade zero element $w = 1 - \overline{z}z$, which satisfies $zw = q^{-2}wz$ and $\overline{z}w = q^2 w\overline{z}$.
It has calculus generated by $\mathrm{d}z$ and $\mathrm{d}\overline{z}$, which satisfy relations

$$\mathrm{d}z \wedge \mathrm{d}\overline{z} = -q^{-2}\mathrm{d}\overline{z} \wedge \mathrm{d}z, \quad z.\mathrm{d}z = q^{-2}\mathrm{d}z.z, \quad z.\mathrm{d}\overline{z} = q^{-2}\mathrm{d}\overline{z}.z$$
$$\mathrm{d}z \wedge \mathrm{d}z = \mathrm{d}\overline{z} \wedge \mathrm{d}\overline{z} = 0, \quad \overline{z}.\mathrm{d}z = q^2\mathrm{d}z.\overline{z}, \quad \overline{z}.\mathrm{d}\overline{z} = q^2\mathrm{d}\overline{z}.\overline{z}$$

A dual basis of the calculus is given by $e_z = \mathrm{d}_z$, $e_{\overline{z}} = \mathrm{d}\overline{z}$, and $e^z, e^{\overline{z}} \in \mathfrak{X}^R_{\mathbb{C}_q[D]}$ such that $e_z(e^z) = e_{\overline{z}}(e^{\overline{z}}) = 1$ and $e_z(e^{\overline{z}}) = e_{\overline{z}}(e^z) = 0$. These vector fields satisfy commutation relations

$$z.e_z = q^2 e_z.z, \quad \overline{z}.e_z = q^{-2}e_z.\overline{z}, \quad z.e_{\overline{z}} = q^2 e_{\overline{z}}.z, \quad \overline{z}.e_{\overline{z}} = q^{-2}e_{\overline{z}}.\overline{z}.$$

We use this in Sections 13.3 and 16.3. ◇

Example 3.4. $\mathbb{C}_{\mathbf{q}}[\mathbf{M_2}]$ ([10] Proposition 2.13 for algebra, [5] page 27 for calculus) The algebra $\mathbb{C}_q[M_2]$ has generators a, b, c, d and relations:

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \quad da - ad = (q - q^{-1})bc.$$

Note that in $\mathbb{C}_q[M_2]$, unlike in $\mathbb{C}_q[GL_2]$, $\mathbb{C}_q[SL_2]$ or $\mathbb{C}_q[SU_2]$, we make no assumption on the value or invertibility of the determinant $\det_q = ad - q^{-1}bc$.
It has a 1-parameter family of 4D calculi. Writing α for the free parameter and $\lambda = q - q^{-1}$, the calculus is freely generated by elements e_a, e_b, e_c, e_d and inner by $\theta = e_a + e_d$, which gives the differential. The commutation relations on the generators are:

$$e_a\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = q^{2\alpha}\begin{pmatrix} q^2a & b \\ q^2c & d \end{pmatrix}e_a, \quad [e_b, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)]_{q^{1+2\alpha}} = q^{1+2\alpha}\lambda\left(\begin{smallmatrix} 0 & a \\ 0 & c \end{smallmatrix}\right)e_a$$
$$[e_c, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)]_{q^{1+2\alpha}} = q^{1+2\alpha}\lambda\left(\begin{smallmatrix} b & 0 \\ d & 0 \end{smallmatrix}\right)e_a, \quad [e_d, \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)]_{q^{2\alpha}} = q^{2\alpha}\lambda\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)e_b$$
$$[e_d, \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)]_{q^{2\alpha}} = q^{2\alpha}\lambda\left(\begin{smallmatrix} ae_c+\lambda be_a \\ ce_c+\lambda de_a \end{smallmatrix}\right)$$

This notation is a shorthand, so for example one of the relations is

$$e_b.d - q^{1+2\alpha}de_b = q^{1+2\alpha}\lambda ce_a.$$

We use this in Section 13.7. ◇

Example 3.5. $\mathbb{C}_\mathbf{q}[\mathbf{SU_2}]$ ([10] Proposition 2.13 for algebra, Example 2.32 for calculus. Calculus originally by Woronowicz in [58].) The algebra $\mathbb{C}_q[SU_2]$ is a Hopf algebra, generated by elements a, b, c, d with the same relations as $\mathbb{C}_q[M_2]$, plus the additional relation that $\det_q = 1$, where $\det_q = ad - q^{-1}bc$. It has $*$-structure $q^* = q$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -q^{-1}c \\ -qb & a \end{smallmatrix}\right)$.
It has a standard vector space basis $\{b^m c^n\} \cup \{a^k b^m c^n\} \cup \{d^k b^m c^n\}$ for $m, n \geq 0$ and $k > 0$.
It has a freely generated left-covariant 3D star-calculus with basis

$$e^- = d\mathrm{d}b - qb\mathrm{d}d, \quad e^+ = q^{-1}a\mathrm{d}c - q^{-2}c\mathrm{d}a, \quad e^0 = d\mathrm{d}a - qb\mathrm{d}c.$$

Equipping the elements with grading $|a| = |c| = 1$, $|b| = |d| = -1$, the commutation relations are

$$e^\pm f = q^{|f|} f e^\pm, \quad e^0 f = q^{2|f|} f e^0.$$

The exterior derivative is

$$\mathrm{d}a = ae^0 + qbe^+, \quad \mathrm{d}b = ae^- - q^{-2}be^0, \quad \mathrm{d}c = ce^0 + qde^+\mathrm{d}d = ce^- - q^{-2}de^0.$$

We use this in Sections 13.4 and 16.1 and 42.4. ◇

Example 3.6. $\mathbb{C}_\mathbf{q}[\mathbf{S^2}]$ ([10] Lemma 2.34 for algebra, Proposition 2.35 for calculus. Calculi on quantum spheres originally by Podleś in [47].) The algebra $\mathbb{C}_q[S^2]$ is the subalgebra of $\mathbb{C}_q[SU_2]$ of elements of degree zero with respect to the $\mathbb{Z}$-grading $|a| = |c| = 1$, $|b| = |d| = -1$. It has generators $x = -q^{-1}bc$, $z = cd$, $z^* = -qab$, and relations

$$xz = q^2 xz, \quad z^* x = q^{-2} x z^*, \quad zz^* = q^2 x(1 - q^2 x), \quad z^* z = x(1 - x).$$

Alternatively, the degree zero elements of $\mathbb{C}_q[SU_2]$ are generated by ac^*, ca^* and cc^*.
It inherits a calculus from $\mathbb{C}_q[SU_2]$ by discarding the generator e^0, but this calculus is not left-covariant.
We use this in Section 42.4. ◇

Example 3.7. $\mathbb{C}_\mathbf{q}[\mathbb{C}^\mathbf{2}]$ ([10] Example 2.66 for algebra, Example 2.79 for calculus) The quantum plane $\mathbb{C}_q[\mathbb{C}^2]$ has generators x, y with relation $yx = qxy$. It has calculus generated by $\mathrm{d}x$ and $\mathrm{d}y$ with relations

$$\mathrm{d}x.x = q^2 x\mathrm{d}x, \quad \mathrm{d}x.y = qy\mathrm{d}x, \quad \mathrm{d}y.x = qx\mathrm{d}y + (q^2 - 1)y\mathrm{d}x, \quad \mathrm{d}y.y = q^2 y\mathrm{d}y.$$

We use this in Section 13.7. ◇

Example 3.8. $\mathbb{C}_\theta[\mathbb{T}^2]$ ([10] Example 1.36. Calculi on the noncommutative torus were first studied by Connes and Rieffel in [20].) The noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$ has generators u, v with the relation $vu = e^{i\theta}uv$ for a real parameter θ. It has star structure $u^* = u^{-1}$ and $v^* = v^{-1}$. It has calculus $\Omega^1 = \mathbb{C}_\theta[\mathbb{T}^2].\{\mathrm{d}u, \mathrm{d}v\}$, with left action given by multiplication and right action given by

$$\mathrm{d}u.u = u.\mathrm{d}u, \quad \mathrm{d}v.v = v.\mathrm{d}v, \quad \mathrm{d}v.u = e^{i\theta}u.\mathrm{d}v, \quad \mathrm{d}u.v = e^{-i\theta}v.\mathrm{d}u.$$

We use this in Section 26. ◇

Example 3.9. $\mathbf{C}(\mathbf{X})$ ([10] Proposition 1.24. Differentials on finite sets were first used by Connes in [19] then extensively studied by Majid in [34].) For a finite set X, the algebra $C(X)$ of complex-valued functions on X has basis δ_x for $x \in X$, which are defined as $\delta_x(y) = \delta_{x,y}$. Calculi $\Omega^1_{C(X)}$ on such algebras are given by finite graphs with vertices the elements of X, where in any given direction between two vertices there is at most one arrow, and there are no arrows from a vertex to itself. The basis of the calculus is $\omega_{x\to y}$ for each arrow $x \to y$ of the graph, and the left and right actions are $f.\omega_{x\to y} = f(x)\omega_{x\to y}$ and $\omega_{x\to y}.f = \omega_{x\to y}f(y)$ respectively. The calculus is connected if and only if the underlying (undirected) graph is connected. The calculus is inner by $\theta = \sum\limits_{x\to y} \omega_{x\to y}$, with exterior derivative

$$\mathrm{d}f = [\theta, f] = \sum_{x\to y} \big(f(y) - f(x)\big)\omega_{x\to y}.$$

This implies in particular that

$$\mathrm{d}\delta_z = \sum_{x\to y} \big(\delta_{z,y} - \delta_{z,x}\big)\omega_{x\to y} = \sum_{x\to z} \omega_{x\to z} - \sum_{z\to y} \omega_{z\to y}.$$

We use this in Sections 13.6 and 34.2. ◇

Example 3.10. $\mathbf{C}(\mathbf{G})$ ([10] Proposition 1.52) If G is a finite group, then the complex-valued functions $C(G)$ have differential calculi coming from Cayley graphs on G regarded as a finite set. These are determined by a subset $\mathcal{C} \subset G\backslash\{e\}$, and the arrows go from $x \to xa$ for $x \in G$ and $a \in \mathcal{C}$. Note that in this thesis, we use the symbol $\subset$ with the meaning $\subseteq$, so for sets A and B, the statement $A \subset B$ includes the possibility $A = B$.

Left covariant calculi are given by $\Omega^1 = \mathbb{C}(G).\Lambda^1$ as a free module over the vector space Λ^1, which has basis $e_a = \sum_{x\in G} \omega_{x\to xa}$. The differential satisfies $\omega_{x\to xa} = \delta_x \mathrm{d}\delta_{xa}$. We have:

$$e_a.f = R_a(f)e_a, \quad \mathrm{d}f = \sum_{a\in\mathcal{C}} (R_a(f) - f)e_a$$

for $f \in \mathbb{C}(G)$, where we denote $R_a(f)(g) = f(ga)$. The calculus is inner by $\theta = \sum_{a\in\mathcal{C}} e_a$, and is connected if and only if $\mathcal{C}$ is a generating set, and is right-covariant if and only if $\mathcal{C}$ is stable under conjugation. If $\mathcal{C}$ has inverses, then Ω^1 is a $*$-calculus by $e_a^* = -e_a^{-1}$.
We use this in Sections 13.5 and 16.2 and 42.3. ◇

Example 3.11. $\mathbb{C}$**G** ([10] Theorem 1.47) For a finite group G, the Hopf algebra $\mathbb{C}G$ is the linear extension of the group. It has star structure $x^* = x^{-1}$ for all $x \in G$. Its translation-invariant calculi are given by right $\mathbb{C}G$-modules Λ^1_G and maps $\zeta : G \to \Lambda^1$ satisfying

$$\zeta(xy) = \zeta(x) \triangleleft y + \zeta(y), \quad \forall x, y \in G$$

which are called cocycles. The calculus is then given as a free module $\Omega^1 = \mathbb{C}G.\Lambda^1_G$. Each pair (Λ^1_G, ζ) of a right $\mathbb{C}G$-module and a cocycle therefore gives a translation-invariant calculus. The right action on the calculus is $v.x = x(v \triangleleft x)$, and the differential is $\mathrm{d}x = x\zeta(x)$, for all $x \in G$ and $v \in \Lambda^1$. The calculus is connected if and only if $\zeta(x) \neq 0$ for all $x \in G\backslash\{e\}$.
We use this in Sections 24 and 42.1 and 34.1. ◇

Example 3.12. $\mathbf{M_2}(\mathbb{C})$ ([10] Example 1.8. Calculi on matrix algebras originally by Beggs and Majid in [11].) The algebra $M_2(\mathbb{C})$ of 2x2 complex-valued matrices has basis elements $E_{11}, E_{12}, E_{21}, E_{22}$ consisting of matrices with a 1 in the specified entry and all other entries zero. It has an inner calculus given by $\theta' = E_{12}s' + E_{21}t'$, where s' and t' are central.
We use this in Section 25. ◇

Example 3.13. $\mathbb{C}$**Hg** ([10] Example 4.62. The calculus on $\mathbb{C}Hg$ was extended in [10] from [36] and [38].) The Heisenberg group Hg is a multiplicative matrix group

$$Hg = \{ \left(\begin{smallmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{smallmatrix}\right) : n, m, k \in \mathbb{Z}\}$$

with generators $u = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. These satisfy relation $uv = wvu$, and the generator w is central. The group algebra $\mathbb{C}Hg$ has calculi of the same type as Example 3.11. We look at a specific calculus given by via the right representation Λ^1 with basis e^u, e^v, e^w and right actions

$$e^u \triangleleft u = e^u, \qquad e^v \triangleleft u = e^v - \frac{1}{2}e^w, \qquad e^u \triangleleft v = e^u + \frac{1}{2}e^w, \qquad e^v \triangleleft v = e^v,$$

with all actions on e^w leaving it invariant and $\triangleleft w = \mathrm{id}$ acting trivially. There is a cocycle $\zeta : \mathbb{C}Hg \to \Lambda^1$ given by $\zeta(x) = e^x$ for all $x = u, w, v$, and this gives a left-covariant calculus $\Omega^1(\mathbb{C}Hg) = \Lambda^1.\mathbb{C}Hg$ with differential $\mathrm{d}x = x\zeta(x)$.
We use this in Section 42.2. $\diamond$

Part II

Temperley-Lieb Algebras

Abstract

We give a diagrammatic differential calculus on Temperley-Lieb algebras which we believe to be new, and use this to give a bimodule connection on bimodules over the algebra.

4 Background

Temperley-Lieb algebras are a family of algebras defined by Neville Temperley and Elliott Lieb in [54], which are often drawn using diagrams. A vector space basis of $TL_n(\delta)$ for $n \geq 1$ and $\delta \in \mathbb{C}\backslash\{0\}$ is given by rectangular diagrams with n points on each of its left and right edges respectively, with each point connected to another on either side by a curve which does not cross any other curve or leave the rectangle. The product of diagrams is their horizontal concatenation, subject to the rule that a diagram containing a closed loop is equal to δ times the same diagram with that loop removed. The identity element is the diagram where each point is connected by a straight line to the point directly opposite. For example, the algebra $TL_3(\delta)$ has three generators $\{\mathrm{id}, e_1, e_2\}$, and basis $\{\mathrm{id}, e_1, e_2, e_1e_2, e_2e_1\}$, which is drawn in Figure 2.

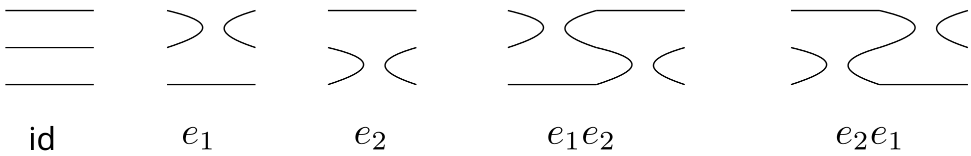


Figure 2: Temperley-Lieb: A basis of $TL_3(\delta)$

In the diagrams for $TL_n(\delta)$, we number the points at the top of the rectangle as the 1st and the points at the bottom as the nth. The diagram $e_k \in TL_n(\delta)$, where $1 \leq k \leq n-1$, has point k on each side connected to point $k+1$ on the same side, with all other points connected by a straight line to the one opposite. These satisfy the four Temperley-Lieb-Jones relations [26] **(1)** $e_i^2 = \delta e_i$ for all $1 \leq i \leq n-1$, **(2)** $e_ie_{i+1}e_i = e_i$ for all $1 \leq i \leq n-2$, **(3)** $e_ie_{i-1}e_i = e_i$ for all $2 \leq i \leq n-1$, **(4)** $e_ie_j = e_je_i$ for all $1 \leq i, j \leq n-1$ such that $|i-j| \neq 1$.

A general Temperley-Lieb algebra $TL_n(\delta)$ is generated by the identity diagram and diagrams e_i for $1 \leq k \leq n-1$.

5 Calculus

As far as we know, the following calculus on Temperley-Lieb algebras is new.

Definition 5.1. Define $\Xi_n^r(\delta)$ for $0 \leq r \leq n$ to be the vector space of diagrams in $TL_n(\delta)$ that have r dots placed on their n curves, subject to the following rules and relations.

1. Dots can be placed anywhere on a curve except the endpoints.
2. No dot can be placed directly above another.
3. Dots can be slid along curves, but when sliding one dot past another, the diagram is multiplied by a factor of -1.
4. A diagram with a dot on a closed loop is equal to zero.

We define a wedge product $\wedge : \Xi_n^r(\delta) \otimes \Xi_n^{r'}(\delta) \to \Xi_n^{r+r'}(\delta)$ by concatenating diagrams.

It follows that a diagram in $\Xi_n^r(\delta)$ with two dots on the same curve must be zero.

Definition 5.2. Define ξ_i in $\Xi_n^{r+1}(\delta)$ as the diagram ξ in $\Xi_n^r(\delta)$ with a dot added along a curve i according to the following rules.

1. If i is a left-to-left curve or a left-to-right curve, then the dot is added to the left of all other dots in ξ.
2. If i is a right-to-right curve, then the dot is added to the right of all other dots in ξ.
3. If i is a closed loop, then adding a dot makes the entire diagram equal to zero.

This allows us to define a differential map looking like Figure 3.

Proposition 5.3. *The map* $\mathrm{d} : \Xi_n^r(\delta) \to \Xi_n^{r+1}(\delta)$ *given by*

$$\mathrm{d}\xi = \sum_{\substack{i \in \text{ left-to-left} \\ \text{curves of } \xi}} \xi_i - \sum_{\substack{i \in \text{ right-to-right} \\ \text{curves of } \xi}} (-1)^{|\xi|} \xi_i \tag{1}$$

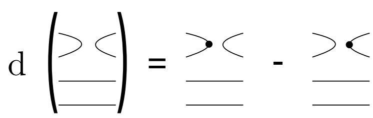


Figure 3: Temperley-Lieb: Illustration of $\mathrm{d}(e_1)$ for $e_1 \in TL_4$

satisfies $\mathrm{d}^2 = 0$ *and the graded Leibniz rule* $\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^{|\xi|}\xi \wedge \mathrm{d}\eta$ *and is thus a differential.*

Proof. **(1)** Firstly we show the Leibniz rule for $r = 0$. We assign signs to curves as follows. A left-to-left curve has sign $+1$, a right-to-right curve has sign -1, and left-to-right curves and closed loops have sign 0. The Leibniz rule says that the sign of each curve in the product MN is equal to the sum of the signs of its component curves in M and N, where each time a curve crosses the joining line of M and N it marks the start of a new component curve. Next, we consider the signs of each type of curve in MN.
A left-to-left curve is either contained entirely in M or meets the joining line between M and N an even number of times, and starts and ends with left-to-right curves. Consequently the sum of its component curve signs is $+1$.
Similarly, a right-to-right curve is either contained entirely in N or meets the joining line an even number of times, and hence the sum of its component curve signs is -1.
A left-to-right curve in MN meets the joining line an odd number of times, and both starts and ends with left-to-right curves, so the sum of its component curve signs is zero.
Lastly, every closed loop in MN is obtained by taking a left-to-left loop and replacing the first and last component curves with a single right-to-right curve in M. This decreases the degree in M by one, so its overall degree is zero.
(2) In Figure 4 we show the graded Leibniz rule on a product $\xi \wedge \eta$. In the diagram we draw only one curve, which we assume to not have any dots on it already.

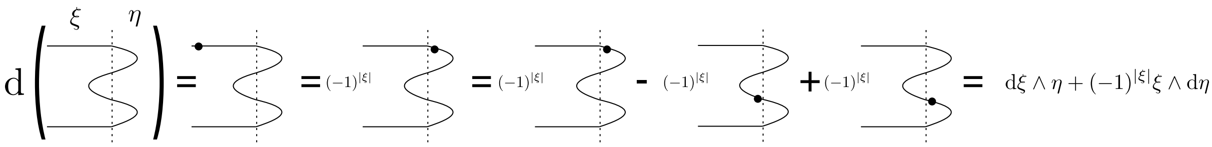


Figure 4: Temperley-Lieb: Part of the proof of the graded Leibniz rule for Temperley-Lieb algebras

(3) Next, we show that $\mathrm{d}^2 = 0$. In Figure 5, we see that $\mathrm{d}^2(e_i) = 0$ for $e_i \in TL_n(\delta)$.

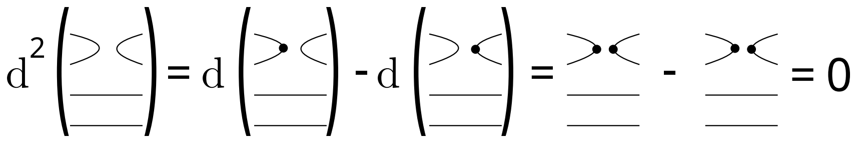


Figure 5: Temperley-Lieb: Illustration of how $\mathrm{d}^2(e_i) = 0$

Using the graded Leibniz rule, this implies that $\mathrm{d}^2 = 0$ for any element of $TL_n(\delta)$, seeing as the e_i generate Temperley Lieb algebras. Using the Leibniz rule on d^2 of a general 1-form then gives $\mathrm{d}^2(a\mathrm{d}e_i) = \mathrm{d}(\mathrm{d}a \wedge \mathrm{d}e_i + a \wedge \mathrm{d}^2 e_i) = \mathrm{d}(\mathrm{d}a \wedge \mathrm{d}e_i) = \mathrm{d}^2 a \wedge \mathrm{d}e_i = 0$. This shows that $\mathrm{d}^2 = 0$ on all 1-forms. But all n-forms are wedge products of 1-forms, and so the graded Leibniz rule applies and shows that $\mathrm{d}^2 = 0$ in general. □

Proposition 5.4. *The calculus $\Omega^1_{TL_n}$ given by the above differential* d *is spanned by the elements of Ξ^1_n except the identity with dots on it.*

Proof. The generator e_i with a dot on its left-to-left curve can be obtained as $\frac{1}{\delta}\mathrm{d}e_i.e_i$, while the generator e_i with a dot put its right-to-right curve can be obtained as $-\frac{1}{\delta}e_i.\mathrm{d}e_i$. Products of these generators with other generators also lie in the calculus, and since every left-to-left or right-to-right curve necessarily passes through the curved part of an e_i, dots can be put on those too. Likewise for left-to-right curves that start and finish at different heights, or which change height along the way.
For a straight horizontal line which is adjacent to a left-to-left or right-to-right curve, we can use the Temperley-Lieb-Jones relations to deform it and introduce a bend. This process can be recursively applied to all straight horizontal lines in a diagram, provided that the diagram contains least one e_i. Figure 6 illustrates this process.

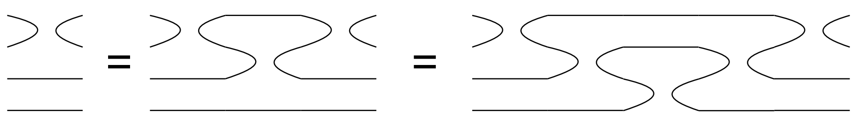


Figure 6: Temperley-Lieb: Illustration of recursively bending curves

However, the diagram consisting entirely of straight horizontal lines is the one exception, to which we have no way of adding dots. □

If we write id_i for the identity diagram with a dot on its ith curve, then the calculus would be inner by $\theta = \sum_i \mathrm{id}_i$, were it not for the fact that identity diagrams with a dot

— and hence θ — do not lie in the calculus. However, the generalised calculus obtained by dropping the requirement of being spanned by the image of d is inner by θ.

6 Some calculations of de Rham Cohomology

Next we do some calculations of de Rham cohomology.

Proposition 6.1. *The de Rham cohomology of $TL_n(\delta)$ satisfies the following.*

1. $H^m(TL_n) = 0$ *for* $m > n$*, since diagrams with two dots on a single line are zero.*

2. $H^0(TL_n) \cong \mathbb{C}$ *and* $H^n(TL_n) \cong 0$ *for all* n.

3. $H^0(TL_2) = \mathbb{C}$ *and* $H^1(TL_2) = 0$.

Proof. (1) The kernel of the differential $\mathrm{d} : TL_n \to \Omega^1_{TL_n}$ consists of multiples of the identity diagram and is hence 1-dimensional, so $H^0(TL_n) \cong \mathbb{C}$.
(2) Every diagram in $\Omega^n_{TL_n}$ has n dots and can be obtained (up to a complex factor) as the derivative of the same diagram but with $n-1$ dots (where we omit a dot on a left-to-left or right-to-right curve). This fact of $\mathrm{d} : \Omega^{n-1}_{TL_n} \to \Omega^k_{TL_n}$ being surjective implies that $H^n(TL_n) \cong 0$.
(3) For the specific case TL_2, we see that $H^1(TL_2) \cong 0$ since both the image of $\mathrm{d} : TL_2 \to \Omega^1_{TL_2}$ and the kernel of $\Omega^1_{TL_2} \to \Omega^2_{TL_2}$ are spanned by the derivative of $e_1 \in TL_2$. □

7 Bimodule Connections

Having defined a calculus on Temperley-Lieb algebras, there is a particularly nice class of bimodule connections that can be represented in diagram form.
We can define a vector space $E_{m,n}(\delta)$ as the set of noncrossing pairings of $m+n$ points on two opposite sides of a rectangle, with m points on the left and n points on the right. This is only well-defined in the case where $m-n$ is even, so a pairing for each point is possible. It has the structure of a $TL_m(\delta)$-$TL_n(\delta)$ bimodule by the left and right actions of composition with diagrams in $TL_m(\delta)$ and $TL_n(\delta)$ on the left and right respectively. In the case $m = n$, the bimodule $E_{n,n}(\delta)$ reduces to the algebra $TL_n(\delta)$.

Temperley-Lieb algebras have a star operation of flipping diagrams horizontally, which extends to bimodules as $*: E_{m,n} \to E_{n,m}$, and gives rise to an inner product $\langle , \rangle : \overline{E_{m,n}} \otimes E_{m,n} \to TL_n(\delta)$ on the bimodules by $\langle \overline{M}, N \rangle = M^* N$.

Proposition 7.1. *A zero-curvature metric-preserving extendable right bimodule connection on $E_{m,n}$ is given by the map $\nabla_R : E_{m,n} \to E_{m,n} \otimes_{TL_n} \Omega^1_{TL_n}$ shown in Figure 7 for the cases $E_{4,6}$ and $E_{2,6}$, but has a clear generalisation to other $E_{m,n}$. The bimodule map $\sigma : \Omega^1_{TL_m} \otimes_{TL_m} E_{m,n} \to E_{m,n} \otimes_{TL_n} \Omega^1_{TL_n}$ for the connection is given by $\sigma(\mathrm{d}a \otimes e) = \nabla_R(ae) - a\nabla_R(e)$. The map σ is drawn in Figure 8 for the case $E_{4,6}$, but is generalised in a similar manner to the diagrams for ∇_R.*

Figure 7: Temperley-Lieb: Illustration of right connections ∇_R on $E_{4,6}$ and $E_{2,6}$ respectively

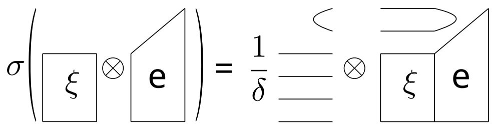
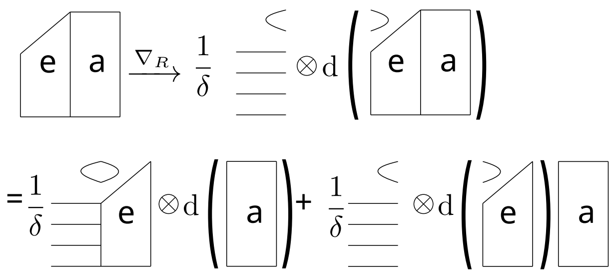


Figure 8: Temperley-Lieb: Illustration of σ on $E_{4,6}$

Proof. **(1)** The right Leibniz rule for ∇_R is shown diagrammatically in Figure 9.



$$= \ e \otimes \mathrm{d}a + \nabla_R(e).a$$

Figure 9: Temperley-Lieb: Proof of the right Leibniz rule for ∇_R

(2) Next we show that the curvature of ∇_R is zero. Recall that the curvature for a right connection ∇_R is given by $R_E = (\mathrm{id} \otimes \mathrm{d} + \nabla_R \wedge \mathrm{id})\nabla_R$. In our case, since $\mathrm{d}^2 = 0$ and our

∇_E contains an d, the curvature reduces to $R_E = (\nabla_R \wedge \mathrm{id})\nabla_R$. We simplify this further in Figure 10.

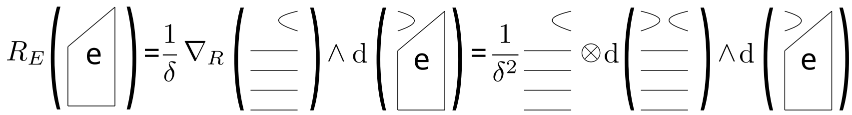


Figure 10: Temperley-Lieb: Calculation of curvature for a Temperley-Lieb bimodule.

The curvature calculated in the figure vanishes, because putting a dot on either of the bends in the middle term will form a dotted closed loop.

(3) Next we calculate the formula for σ. Firstly, we calculate $\nabla_R(ea)$ in Figure 11.

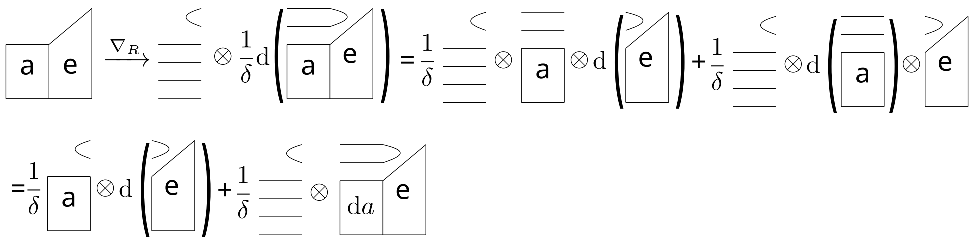


Figure 11: Temperley-Lieb: Calculation of $\nabla_R(ea)$

Then we calculate $a\nabla_R(e)$ in Figure 12.

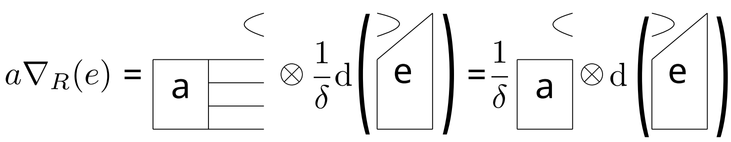


Figure 12: Temperley-Lieb: Calculation of $a\nabla_R(e)$

Subtracting the two gives the formula for σ when $\xi = \mathrm{d}a$, and the formula for σ on a general ξ follows from the result we showed in part (2).

(4) Next, we show that σ is a bimodule map. From the diagram by which σ is defined, we can see it is a right module map. We show in figure 13 that σ is also a left module map, where we use the fact that the tensor product is over TL_n, allowing us to move an element of the algebra from one side to the other.

(5) Next we show that the connection ∇_R preserves the metric. The metric preservation equation for the connection ∇_R is given in Figure 14, and can be seen to hold, since if we add a closed loop to the L.H.S. and multiply by $\frac{1}{\delta}$ to cancel it out, we can use the Leibniz rule to obtain the R.H.S.

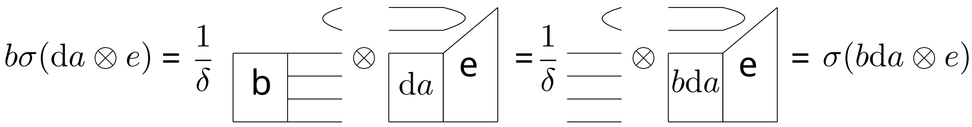


Figure 13: Temperley-Lieb: Proof that σ_E is a left module map

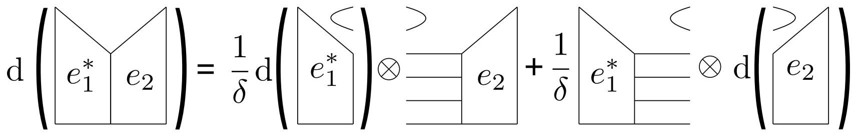


Figure 14: Temperley-Lieb: Proof of metric preservation by ∇_R

(6) Lastly we show that σ is extendable. We need to show that $\sigma(\xi \wedge \eta \otimes e) = (\sigma \wedge \mathrm{id})(\xi \otimes \sigma(\eta \otimes e))$. The proof is Figure 15, which uses the formula for σ.

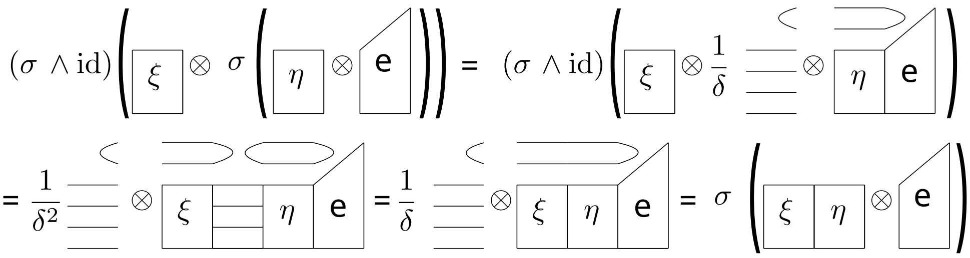


Figure 15: Temperley-Lieb: Proof of extendability of σ

□

We remark that for $n \leq m-2$, the map $\nabla_L : E_{n,m} \to \Omega^1_{TL_n} \otimes_{TL_n} TL_m$ shown in Figure 16 is a left connection, but here we only use the right connection.

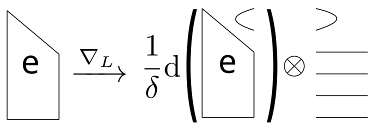


Figure 16: Temperley-Lieb: Illustration of the left connection ∇_L on $E_{6,4}$

We note that there are no nonzero $e \in E$ satisfying $\nabla_L(e) = 0$.

The connections we gave above are by no means the only connections on modules over Temperley-Lieb algebras, as seen in the following remark.

Remark 7.2. *(See [10] Example 3.22 for this type of connection) If we equip $TL_n(\delta)$ with inner generalised calculus, then every left $TL_n(\delta)$-module has a left connection ${}_\theta\nabla(e) =$*

$\theta \otimes e$ with curvature

$$_{\theta}R(e) = (\mathrm{d}\theta - \theta \wedge \theta) \otimes e = -\theta \wedge \theta \otimes e.$$

This connection is different to the one we gave above. The connection doesn't look to have any zeroes either, though its curvature may be non-zero.

8 Vector fields

Having defined calculi on Temperley-Lieb algebras, this allows us to define vector fields. The right vector fields $\mathfrak{X}^R_{TL_n}$ are linear maps $X : \Omega^1_{TL_n}(\delta) \to TL_n(\delta)$ which are right module maps, i.e. they satisfy $X(\xi e_i) = X(\xi)e_i$ for $\xi \in \Omega^1_{TL_n}(\delta)$ and e_i the generators of $TL_n(\delta)$.

8.1 Example: TL_2

First we look at the case $n = 2$. We note that we also calculated the $n = 2$ case by hand in order to verify that the Mathematica code was giving the correct result, so that we could have a higher level of trust in the Mathematica output for the $n = 3$ case which is too long to reasonably calculate by hand.

Definition 8.1. For the 2 basis elements of TL_2 write $y_1 = 1$ and $y_2 = e_1$. For the 2 basis elements of $\Omega^1_{TL_2}$ write ξ_1 and ξ_2 for e_1 with a dot on the left-to-left and right-to-right curves respectively.

Proposition 8.2. *A general right vector field $X \in \mathfrak{X}^R_{TL_2}$ on TL_2 is given by $X(\xi_k) = \sum_{j=1}^{2} x_{k,j}y_j$, where x is a 2×2 complex-valued matrix given as follows.*

$$x = \begin{pmatrix} 0 & a_{1,2} \\ a_{2,1} & -\frac{a_{2,1}}{\delta} \end{pmatrix}$$

The calculus $\Omega^1_{TL_2}$ is not finitely generated projective.

Proof. (1) Since the vector field X is a right module map, $X(\xi_k e_1) = \sum_{j=1}^{2} a_{k,j}y_je_1$.

The matrix a_1 gives the outcome of multiplying the 2 basis elements of TL_2 on the right by e_1. The matrix b_1 gives the outcome of multiplying the 2 (vector space) basis elements of $\Omega^1_{TL_2}$ on the right by e_1.

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & \delta \end{pmatrix}, \quad b_1 = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{pmatrix}$$

We solve the following equation in Mathematica.

$$\text{Solve}[\{\text{Simplify}[x.\text{a1} - \text{b1}.x] == 0\}, \text{Flatten}[\text{x}]]$$

Set the output of the above as relations sub, then do Simplify[x//. sub], giving:

$$x = \begin{pmatrix} 0 & a(1,2) \\ a(2,1) & -\frac{a(2,1)}{\delta} \end{pmatrix}$$

(2) The matrix x is a function of two independent variables, so $\mathfrak{X}^R_{TL_2}$ has 2 basis elements given by

$$x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 1 & -\frac{1}{\delta} \end{pmatrix}$$

In order for $\Omega^1_{TL_2}$ to be finitely generated projective, we want to find $\tau_1, \tau_2 \in \Omega^1_{TL_2}$ such that any $\eta \in \Omega^1_{TL_2}$ can be decomposed as $\eta = \sum_{i=1}^{2} X_i(\eta).\tau_i$. This gives the following two equations:

$$\xi_1 = e_1.\tau_1, \quad \xi_2 = \tau_1 - \frac{1}{\delta} e_1.\tau_2$$

But by drawing the equation in diagrams, we see that the first equation has no solutions for τ_1 and τ_2, since no choice of τ_1 can put a dot on the left-to-left loop of $e_1.\tau_1$. This shows that there does not exist a dual basis for $\Omega^1_{TL_2}$, and hence it is not finitely generated projective. □

8.2 Example: TL_3

Next we look at the case $n = 3$.

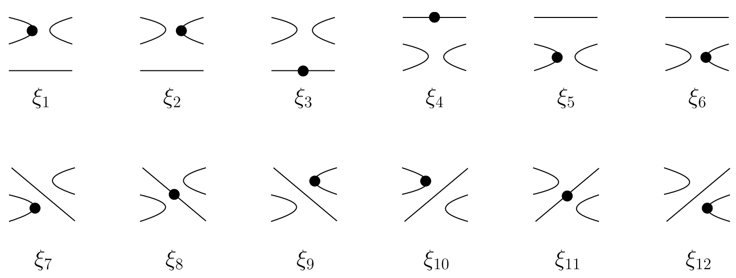


Figure 17: Temperley-Lieb: The 12 basis elements of $\Omega^1_{TL_3}$

Definition 8.3. For the 5 basis elements of TL_3 write $y_1 = 1$, $y_2 = e_1$, $y_3 = e_2$, $y_4 = x_1 = e_2e_1$, $y_5 = x_2 = e_1e_2$.
Write ξ_i for the 12 basis elements of $\Omega^1_{TL_n}$, each given by one of the four diagrams y_2, y_3, y_4, y_5 with a dot on one of their three lines, as drawn in Figure 17.

Proposition 8.4. *A general right vector field $X \in \mathfrak{X}^R_{TL_3}$ on TL_3 is given by $X(\xi_k) = \sum_{j=1}^{5} x_{k,j}y_j$, where x is a 12×5 matrix given as follows.*

$$x = \begin{pmatrix}
0 & a_{1,2} & 0 & a_{1,4} & 0 \\
-\delta a_{2,2}-a_{2,5} & a_{2,2} & a_{2,3} & -\frac{a_{2,3}}{\delta} & a_{2,5} \\
0 & \delta a_{2,5}+a_{2,2} & 0 & \frac{(\delta^2-1)a_{2,3}}{\delta}-\delta a_{2,2}-a_{2,5} & 0 \\
0 & 0 & \delta a_{6,4}+a_{6,3} & 0 & \frac{(\delta^2-1)a_{6,2}}{\delta}-\delta a_{6,3}-a_{6,4} \\
0 & 0 & a_{5,3} & 0 & a_{5,5} \\
-\delta a_{6,3}-a_{6,4} & a_{6,2} & a_{6,3} & a_{6,4} & -\frac{a_{6,2}}{\delta} \\
0 & a_{5,5} & 0 & a_{5,3} & 0 \\
0 & \frac{(\delta^2-1)a_{6,2}}{\delta}-\delta a_{6,3}-a_{6,4} & 0 & \delta a_{6,4}+a_{6,3} & 0 \\
(\frac{1}{\delta^2}-1)a_{6,2}+\frac{\delta^2(-a_{9,2})+\delta a_{6,3}+a_{6,4}+a_{9,2}}{\delta} & a_{9,2} & \frac{a_{6,2}}{\delta}+a_{6,4}+a_{9,2} & -\frac{\delta(a_{6,4}+a_{9,2})+a_{6,2}}{\delta^2} & \frac{(\delta^2-1)a_{6,2}-\delta(\delta a_{6,3}+a_{6,4}+a_{9,2})}{\delta^2} \\
0 & 0 & a_{1,4} & 0 & a_{1,2} \\
0 & 0 & \frac{(\delta^2-1)a_{2,3}}{\delta}-\delta a_{2,2}-a_{2,5} & 0 & \delta a_{2,5}+a_{2,2} \\
\delta(a_{2,5}-a_{12,2})+\frac{a_{12,2}}{\delta}+a_{2,2} & a_{12,2} & -\frac{a_{2,3}}{\delta}-a_{2,5}+a_{12,2} & -\frac{a_{12,2}}{\delta}-a_{2,2}+a_{2,3} & -\frac{a_{12,2}}{\delta}
\end{pmatrix}$$

The calculus $\Omega^1_{TL_3}$ is not finitely generated projective.

Proof. (1) Since the vector field X is a right module map, $X(\xi_k e_i) = \sum_j a_{k,j}y_je_i$.
The matrices a_1 and a_2 give the outcome of multiplying the 5 basis elements of TL_3 on the right by e_1 or e_2 respectively. The matrices b_1 and b_2 give the outcome of multiplying the 12 (vector space) basis elements of $\Omega^1_{TL_3}$ on the right by e_1 and e_2 respectively.

$$a_1 = \begin{pmatrix} 0&1&0&0&0 \\ 0&\delta&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&\delta&0 \\ 0&1&0&0&0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&\delta&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&\delta \end{pmatrix} \quad b_1 = \begin{pmatrix}
\delta&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&\delta&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&\delta&0&0&0&0&0 \\
0&0&0&0&0&0&0&\delta&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$b_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&\delta&0&0&0&0&0&0&0&0 \\ 0&0&0&0&\delta&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&\delta&0&0 \\ 0&0&0&0&0&0&0&0&0&0&\delta&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad x = \begin{pmatrix} a(1,1) & a(1,2) & a(1,3) & a(1,4) & a(1,5) \\ a(2,1) & a(2,2) & a(2,3) & a(2,4) & a(2,5) \\ a(3,1) & a(3,2) & a(3,3) & a(3,4) & a(3,5) \\ a(4,1) & a(4,2) & a(4,3) & a(4,4) & a(4,5) \\ a(5,1) & a(5,2) & a(5,3) & a(5,4) & a(5,5) \\ a(6,1) & a(6,2) & a(6,3) & a(6,4) & a(6,5) \\ a(7,1) & a(7,2) & a(7,3) & a(7,4) & a(7,5) \\ a(8,1) & a(8,2) & a(8,3) & a(8,4) & a(8,5) \\ a(9,1) & a(9,2) & a(9,3) & a(9,4) & a(9,5) \\ a(10,1) & a(10,2) & a(10,3) & a(10,4) & a(10,5) \\ a(11,1) & a(11,2) & a(11,3) & a(11,4) & a(11,5) \\ a(12,1) & a(12,2) & a(12,3) & a(12,4) & a(12,5) \end{pmatrix}$$

Then solve the following equation in Mathematica.

$$\text{Solve}[\{\text{Simplify}[x.\text{a1} - \text{b1}.x] == 0, \text{Simplify}[x.\text{a2} - \text{b2}.x] == 0\}, \text{Flatten}[\text{x}]]$$

Set the output of the above as relations sub. Then do Simplify$[x//.\,\text{sub}]$. Then:

$$x = \begin{pmatrix} 0 & a_{1,2} & 0 & a_{1,4} & 0 \\ -\delta a_{2,2}-a_{2,5} & a_{2,2} & a_{2,3} & -\frac{a_{2,3}}{\delta} & a_{2,5} \\ 0 & \delta a_{2,5}+a_{2,2} & 0 & \frac{(\delta^2-1)a_{2,3}}{\delta}-\delta a_{2,2}-a_{2,5} & 0 \\ 0 & 0 & \delta a_{6,4}+a_{6,3} & 0 & \frac{(\delta^2-1)a_{6,2}}{\delta}-\delta a_{6,3}-a_{6,4} \\ 0 & 0 & a_{5,3} & 0 & a_{5,5} \\ -\delta a_{6,3}-a_{6,4} & a_{6,2} & a_{6,3} & a_{6,4} & -\frac{a_{6,2}}{\delta} \\ 0 & a_{5,5} & 0 & a_{5,3} & 0 \\ 0 & \frac{(\delta^2-1)a_{6,2}}{\delta}-\delta a_{6,3}-a_{6,4} & 0 & \delta a_{6,4}+a_{6,3} & 0 \\ (\frac{1}{\delta^2}-1)a_{6,2}+\frac{\delta^2(-a_{9,2})+\delta a_{6,3}+a_{6,4}+a_{9,2}}{\delta} & a_{9,2} & \frac{a_{6,2}}{\delta}+a_{6,4}+a_{9,2} & -\frac{\delta(a_{6,4}+a_{9,2})+a_{6,2}}{\delta^2} & \frac{(\delta^2-1)a_{6,2}-\delta(\delta a_{6,3}+a_{6,4}+a_{9,2})}{\delta^2} \\ 0 & 0 & a_{1,4} & 0 & a_{1,2} \\ 0 & 0 & \frac{(\delta^2-1)a_{2,3}}{\delta}-\delta a_{2,2}-a_{2,5} & 0 & \delta a_{2,5}+a_{2,2} \\ \delta(a_{2,5}-a_{12,2})+\frac{a_{12,2}}{\delta}+a_{2,2} & a_{12,2} & -\frac{a_{2,3}}{\delta}-a_{2,5}+a_{12,2} & -\frac{a_{12,2}}{\delta}-a_{2,2}+a_{2,3} & -\frac{a_{12,2}}{\delta} \end{pmatrix}$$

as required.

(2) The matrix x is a function of 12 independent variables, and so $\mathfrak{X}^R_{TL_3}$ has 12 basis elements given as follows.

$$x_1 = \begin{pmatrix} 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0&0&0&0&0 \\ -\delta&1&0&0&0 \\ 0&1&0&-\delta&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&-\delta&0&1 \\ 1&0&0&-1&0 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&1&-\frac{1}{\delta}&0 \\ 0&0&0&\delta-\frac{1}{\delta}&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&\delta-\frac{1}{\delta}&0&0 \\ 0&0&-\frac{1}{\delta}&1&0 \end{pmatrix} \quad x_5 = \begin{pmatrix} 0&0&0&0&0 \\ -1&0&0&0&1 \\ 0&\delta&0&-1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&-1&0&\delta \\ \delta&0&-1&0&0 \end{pmatrix} \quad x_6 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$

$$x_7 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_8 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&\delta-\frac{1}{\delta} \\ 0&0&0&0&0 \\ 0&1&0&0&-\frac{1}{\delta} \\ 0&0&0&0&0 \\ 0&\delta-\frac{1}{\delta}&0&0&0 \\ \frac{1}{\delta^2}-1&0&\frac{1}{\delta}&-\frac{1}{\delta^2}&1-\frac{1}{\delta^2} \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_9 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&-\delta \\ 0&0&0&0&0 \\ -\delta&0&1&0&0 \\ 0&0&0&0&0 \\ 0&-\delta&0&1&0 \\ 1&0&0&0&-1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$

$$x_{10} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&\delta&0&-1 \\ 0&0&0&0&0 \\ -1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&-1&0&\delta&0 \\ \frac{1}{\delta}&0&1&-\frac{1}{\delta}&-\frac{1}{\delta} \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_{11} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ \frac{1}{\delta}-\delta&1&1&-\frac{1}{\delta}&-\frac{1}{\delta} \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad x_{12} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ \frac{1}{\delta}-\delta&1&1&-\frac{1}{\delta}&-\frac{1}{\delta} \end{pmatrix}$$

Can we find a set of 12 τ_i such that any $\eta \in \Omega^1_{TL_3}$ can be decomposed as $\eta = \sum_{i=1}^{12} X_i(\eta).\tau_i$? This gives the following conditions on the τ_i for a dual basis with the above X_i.

$$\xi_1 = e_1.\tau_1 + x_1.\tau_2$$
$$\xi_2 = (-\delta.1 + e_1).\tau_3 + (e_2 - \frac{1}{\delta}x_1).\tau_4 + (-1 + x_2).\tau_5$$
$$\xi_3 = (e_1 - \delta x_1).\tau_3 + (\delta - \frac{1}{\delta})x_1.\tau_4 + (\delta e_1 - x_1).\tau_5$$
$$\xi_4 = ((\delta - \frac{1}{\delta})x_2).\tau_8 + (e_2 - \delta x_2).\tau_9 + (\delta e_2 - x_2).\tau_{10}$$
$$\xi_5 = e_2.\tau_6 + x_2.\tau_7$$
$$\xi_6 = (e_2 - \frac{1}{\delta}x_2).\tau_8 + (-\delta.1 + e_2).\tau_9 + (-1 + x_1).\tau_{10}$$
$$\xi_7 = x_1.\tau_6 + e_1.\tau_7$$
$$\xi_8 = (\delta - \frac{1}{\delta})e_1.\tau_8 + (-\delta e_1 + x_1).\tau_9 + (-e_1 + \delta x_1).\tau_{10}$$
$$\begin{aligned} \xi_9 = {} & ((\frac{1}{\delta^2} - 1).1 + \frac{1}{\delta}e_2 - \frac{1}{\delta^2}x_1 + (1 - \frac{1}{\delta^2})x_2).\tau_8 + (1 - x_2).\tau_9 \\ & + (\frac{1}{\delta}.1 + e_2 - \frac{1}{\delta}x_1 - \frac{1}{\delta}x_2).\tau_{10} + ((\frac{1}{\delta} - \delta).1 + e_1 + e_2 - \frac{1}{\delta}x_1 - \frac{1}{\delta}x_2).\tau_{11} \end{aligned}$$
$$\xi_{10} = x_2.\tau_1 + e_2.\tau_2$$
$$\xi_{11} = (-\delta e_2 + x_2).\tau_3 + (\delta - \frac{1}{\delta})e_2.\tau_4 + (-e_2 + \delta x_2).\tau_5$$
$$\xi_{12} = (1 - x_1).\tau_3 + (-\frac{1}{\delta}e_2 + x_1).\tau_4 + (\delta.1 - e_2).\tau_5 + ((\frac{1}{\delta} - \delta).1 + e_1 + e_2 - \frac{1}{\delta}x_1 - \frac{1}{\delta}x_2).\tau_{12}$$

The following pair of equations when drawn diagrammatically

$$\begin{cases} \xi_5 & = e_2.\tau_6 + x_2.\tau_7 \\ \xi_7 & = x_1.\tau_6 + e_1.\tau_7 \end{cases}$$

can be seen to have no solutions, and hence $\Omega^1_{TL_3}$ is not finitely generated projective. $\square$

8.3 Example: TL_2 with Extended Calculus

When we looked at vector fields for the calculi on TL_2 with the standard differential calculi, they turned out to not be finitely generated projective. But what about when we add in the identity diagram with dots, to get the extended calculus $\hat{\Omega}^1_{TL_2}$?

Denote as before $y_1 = \text{id}$ and $y_2 = e_1$ for the basis elements of $TL_2(\delta)$, but now write ξ_1 for the identity diagram with a dot on its top line, ξ_2 for the identity diagram with a dot on its bottom line, ξ_3 for e_1 with a dot on its left-to-left curve, and ξ_4 for e_1 with a dot on its right-to-right curve. These four ξ_i give a vector space basis of the extended calculus. Denote as before a_1 for the 2x2 matrix of outcomes of multiplying the two basis elements of TL_2 on the right by e_1, and b_1 for the 4x4 matrix of outcomes of multiplying the four basis elements of $\hat{\Omega}^1_{TL_2}$ on the right by e_1.

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & \delta \end{pmatrix}, \qquad b_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{pmatrix}$$

We solve the following equation in Mathematica.

$$\text{Solve}[\{\text{Simplify}[x.\text{a1} - \text{b1}.x] == 0\}, \text{Flatten}[\text{x}]]$$

Set the output of the above as relations sub, then do $\text{Simplify}[x//.\,\text{sub}]$, giving:

$$x = \begin{pmatrix} a_{1,1} & \frac{-a_{1,1}+a_{3,2}}{\delta} \\ a_{2,1} & \frac{-a_{2,1}+a_{3,2}}{\delta} \\ 0 & a_{3,2} \\ a_{4,1} & \frac{-a_{4,1}}{\delta} \end{pmatrix}$$

The matrix x is a function of 4 independent variables $a_{1,1}$, $a_{2,1}$, $a_{3,2}$, $a_{4,1}$. Thus $\mathfrak{X}^R_{TL_2}$ has 4 basis elements given as follows.

$$x_1 = \begin{pmatrix} 1 & -\frac{1}{\delta} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 1 & -\frac{1}{\delta} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & \frac{1}{\delta} \\ 0 & \frac{1}{\delta} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & -\frac{1}{\delta} \end{pmatrix}$$

We want to find $\tau_1, \tau_2, \tau_3, \tau_4 \in \hat{\Omega}^1_{TL_2}$ that form a dual basis along with these x_i, so that any $\eta \in \hat{\Omega}^1_{TL_2}$ can be decomposed as $\eta = \sum_{i=1}^{4} X_i(\eta).\tau_i$. This gives the following equations.

$$\begin{aligned} \xi_1 &= \tau_1 - \frac{1}{\delta} e_1.\tau_1 + \frac{1}{\delta} e_1.\tau_3 \\ \xi_2 &= \tau_2 - \frac{1}{\delta} e_1.\tau_2 + \frac{1}{\delta} e_1.\tau_3 \\ \xi_3 &= \tau_3 \\ \xi_4 &= \tau_4 - \frac{1}{\delta} e_1.\tau_4 \end{aligned}$$

The last equation only has solution $\tau = 0$, so the extended calculus is not finitely generated projective either.

9 Future Ideas and Discussion

Temperley-Lieb algebras are related to projections onto subfactors of Von Neumann algebras, and so it might be interesting to investigate if these diagrammatic calculi also say anything about calculi on von Neumann algebras.

We might also consider whether our diagrammatic calculi on Temperley-Lieb diagrams have any generalisation to planar algebras as invented by Vaughan Jones. This would require a new definition of a calculus as something other than a bimodule.

Part III

Noncommutative Submanifolds

Abstract

We introduce a new approach to submanifolds in noncommutative differential geometry, characterised by the use of vector fields. Given an algebra B with differential calculus Ω^1_B, we define when a surjective algebra map $\pi : B \to A$ is a co-embedding, in which case we construct complementary tangent and normal bundles and a calculus on A making π differentiable, which are all finitely generated projective if Ω^1_B is. Every connection on Ω^1_B can be projected to a connection on the submanifold calculus Ω^1_A, satisfying a version of the Gauss-Codazzi equations for curvature, and which is compatible with a Hermitian metric if the original connection is. We calculate a number of commutative and noncommutative examples.

10 Introduction

In classical differential geometry, an inclusion $i : M \hookrightarrow N$ of an embedded submanifold induces a surjective algebra map $\pi : C^\infty(N) \to C^\infty(M)$ between the smooth real-valued functions on the manifolds, given by restriction. In noncommutative differential geometry, the algebra of functions on a manifold is replaced by a general associative algebra, and while we gain the ability to perform coordinate-free calculations and consider a number of quantum-only phenomena, it comes at the cost of no longer having an obvious notion of submanifold. There are three existing approaches at tackling this problem [40, 21, 3], but all three operate on the definition of vector fields as derivations, and use calculi based on that. We work with differential calculi on algebras and their dual the vector fields, as defined in the book [10], and use the idea from [21] of a noncommutative embedding (co-embedding) as a surjective algebra map $\pi : B \to A$. We define tangent and normal bundles associated with a surjective algebra map, and if the two are complementary then we call the map a co-embedding. Given a co-embedding we construct a calculus Ω^1_A on the submanifold algebra A, which we call the submanifold calculus. The submanifold calculus Ω^1_A has the properties that the co-embedding π used to construct it is differentiable, i.e. $\mathrm{d}\pi = \pi\mathrm{d}$, and that Ω^1_A is finitely generated projective if the larger calculus Ω^1_B on B is finitely generated projective. We also show that subject to a certain condition, the composition of co-embeddings is again a co-embedding. We calculate a number of examples of co-embeddings, both classical and non-classical, where the non-classical examples all use non-derivation calculi and are therefore are unique to our new definition. Given a co-embedding $\pi : B \to A$ and a covariant derivative on Ω^1_B, we construct a covariant derivative on the submanifold calculus Ω^1_A. Following an idea from [3], we show the curvature of this new connection is related to the curvature of the original connection by a non-commutative variant of the Gauss-Codazzi equations. We show that this connection on the submanifold calculus preserves a Hermitian metric if the original connection preserves a Hermitian metric, and calculate some noncommutative examples illustrating this. We conclude by considering what might be necessary to further generalise the idea of a co-embedding to be a positive map instead of an algebra map, seeing as certain parts of the theory seem to have a nice generalisation.

10.1 Existing Approaches

There are currently three existing approaches to extending the concept of submanifolds from classical geometry to a noncommutative setting. In the 1996 paper [40], Masson introduces a definition of when a quotient of an algebra by an ideal is a subalgebra. Masson's approach was further developed in the 2020 paper [21] by D'Andrea, via the notion of surjective algebra maps as co-embeddings, which was a major inspiration for our work here. D'Andrea proves that a surjective algebra map $\pi : A \to B$ with kernel J induces a map of derivations $\pi^* : \mathrm{Der}_\pi(\mathrm{A}) \to \mathrm{Der}(\mathrm{B})$, where

$$\mathrm{Der}_\pi(\mathrm{A}) := \{\mathrm{D} \in \mathrm{Der}(\mathrm{A}) \mid \mathrm{D}(\mathrm{A}) \in \mathrm{J} \quad \forall \mathrm{a} \in \mathrm{J}\}.$$

and says that if the induced map π^* is surjective then π is called a co-embedding, and B a submanifold algebra of A, in which case there is a surjective homomorphism $\Omega_{\mathrm{Der}}(A) \to \Omega_{\mathrm{Der}}(B)$. These two papers operate on the definition of derivations as noncommutative vector fields, and defines calculi correspondingly. See section 1.1 of [40] for details on these calculi, which were originally invented in [22].

We borrow D'Andrea's notation of a co-embedding as a surjective algebra map $\pi : B \to A$ satisfying certain properties, but instead of defining it as a co-embedding when it is differentiable, we ask for the existence of tangent and normal bundles and then construct a calculus on A with respect to which the co-embedding is differentiable. Our definition of horizontal vector fields Hor is an adaptation of D'Andrea's $\mathrm{Der}_\pi(\mathrm{A})$, and so the foundations of our approach owe a lot to the paper [21].

Another recent approach can also be found in the 2021 paper [3] by Arnlind and Norkvist. It uses another notion of calculus called pseudo-Riemannian calculi, but also takes vector fields to be derivations. Section 4 of [3] on minimal embeddings provided the inspiration for us to investigate whether our notion of submanifold also gives rise to a Gauss equation for curvature.

10.2 Why Not Take Vector Fields to be Derivations?

In classical geometry, vector fields can be defined either as derivations or as the dual to 1-forms, since both of these coincide. But in a noncommutative context they do not coincide, and we need to pick one to use as our definition.

Suppose we regard noncommutative vector fields on an algebra A as the derivations Der(A) — the maps $D : A \to A$ satisfying $D(aa') = aD(a') + D(a)a'$. Then in the classical case where A is the commutative algebra of smooth functions on a manifold, a vector field such as $\frac{\partial}{\partial x}$ is indeed a derivation. However on a general algebra which need not be commutative, we are not guaranteed to have a lot of derivations. Moreover, Der(A) is not a module over A when A is noncommutative, since the maps aD and Da fail to be derivations, because $(aD)(xy) = aD(x)y + xaD(y)$. It a module over the centre $Z(A)$, but this is liable to lose a lot of information for algebras with a small or trivial centre, and it creates a new problem of the vector fields being a module over a different algebra to the differential forms.

The other approach, which we use, is to regard noncommutative vector fields as the left or right dual of the 1-forms Ω^1_A, so there are right vector fields $\mathfrak{X}^R_A = \mathrm{Hom}_\mathrm{A}(\Omega^1_\mathrm{A}, \mathrm{A})$ and left vector fields $\mathfrak{X}^L_A = {}_A\mathrm{Hom}(\Omega^1_\mathrm{A}, \mathrm{A})$. These are A-bimodules, which means we can take left and right connections and bimodule connections on them. Moreover, as occurs classically, connections on the 1-forms give dual connections on the vector fields. We make extensive use of this duality in the sections later about connections.

11 Noncommutative Submanifolds

Suppose we have a surjective algebra map $\pi : B \to A$ with kernel $J := \ker(\pi)$. Since π is an algebra map, we have $\pi(jb) = \pi(j)\pi(b) = 0$ and $\pi(bj) = \pi(b)\pi(j) = 0$ for all $b \in B$, $j \in J$. Therefore $jb, bj \in J$, making J a two-sided ideal in B. By definition of kernel, the following sequence of algebras and algebra maps is short exact.

$$0 \longrightarrow J \hookrightarrow B \xrightarrow{\pi} A \longrightarrow 0 \tag{2}$$

If Ω^1_B is a differential calculus on B, then the right vector fields are the dual $\mathfrak{X}^R_B := \mathrm{Hom_B}(\Omega^1_\mathrm{B}, \mathrm{B})$, and by definition satisfy $X(\eta b) = X(\eta)b$. The left and right B-actions on $\mathfrak{X}^R_B$ are given for all $b \in B$, $X \in \mathfrak{X}^R_B$, $\eta \in \Omega^1_B$ as

$$(bX)(\eta) := bX(\eta), \qquad (Xb)(\eta) := X(b\eta).$$

If Ω^1_B is right finitely generated projective then $\mathfrak{X}^R_B$ is a left finitely generated projective B-bimodule, in which case there would be a finite dual basis $\sum_i X_i \otimes e^i \in \mathfrak{X}^R_B \otimes \Omega^1_B$ of $\mathfrak{X}^R_B$ which for all $\xi \in \Omega^1_B$ satisfies $\xi = \sum_i X_i(\xi).e^i$.

11.1 Restricting Vector Fields

Denote A_π for the algebra A regarded as a right B-module with action $a \triangleleft b = a\pi(b)$ for a surjective algebra map $\pi : B \to A$. We propose that restriction (via π) of a left B-module to A is given by the functor $\mathcal{R}_\pi : {}_B\mathcal{M} \to {}_A\mathcal{M}$ defined by

$$E \mapsto A_\pi \otimes_B E, \qquad T \mapsto \mathrm{id} \otimes T, \quad \forall T : E \to F \text{ left module maps.}$$

Proposition 11.1. *Let E be a left B-module, and suppose $\pi : B \to A$ is a surjective algebra map. Then the map $R_\pi : E \to A_\pi \otimes_B E$ given by $R_\pi(e) = 1 \otimes e$ is surjective and satisfies $R_\pi(b.e) = \pi(b)R_\pi(e)$. Note that composition of two restriction functors is another restriction functor, i.e. for surjective algebra maps $\psi : C \to B$ and $\phi : B \to A$, we have $R_{\phi\circ\psi} = R_\phi \circ R_\psi$.*

Proof. **(1)** First we show surjectivity of R_π. A general element of $A_\pi \otimes_B E$ is of the form $\sum_i a_i \otimes e_i$. But as π is surjective, there exist $b_i \in B$ such that $\pi(b_i) = a_i$ for each i. Therefore $\sum_i a_i \otimes e_i = 1 \otimes \sum_i b_i.e_i = R_\pi(\sum_i b_i.e_i)$, so R_π is surjective.

(2) The second property follows from $R_\pi(b.e) = 1 \otimes b.e = \pi(b) \otimes e = \pi(b).R_\pi(e)$.

(3) The fact that $R_{\phi\circ\psi} = R_\phi \circ R_\psi$ is since $A_\phi \otimes_B B_\psi \cong A_{\phi\circ\psi}$. □

Next we show that R_π preserves the property of being left finitely generated projective.

Proposition 11.2. *Suppose that $\pi : B \to A$ is a surjective algebra map, and that E is a left finitely generated projective left B-module — i.e. each $\xi \in E$ decomposes as $\xi = \sum_i e_i(\xi).e^i$ for some dual basis $e^i \in E$, $e_i \in {}_B\mathrm{Hom(E, B)}$. Then the left A-module $F := R(E) = A_\pi \otimes_B E$ is left finitely generated projective, with dual basis $f^i = 1 \otimes e^i \in F$ and*

$$f_i \in {}_A\mathrm{Hom}(F, A), \quad f_i(a \otimes e) = a\pi(e_i(e)),$$

satisfying $a \otimes e = \sum_i f_i(a \otimes e).f^i$ for all $a \otimes e \in F$.

Proof. (1) Firstly, we show that f_i is well-defined as a map over $\otimes_B$. To do this, we need to show $f_i(a \otimes be) = f_i(a\pi(b) \otimes e)$. The left hand side of this equation is

$$f_i(a \otimes be) = a\pi(e_i(be)) = a\pi(be_i(e)) = a\pi(b)\pi(e_i(e)).$$

The right hand side of the equation is

$$f_i(a\pi(b) \otimes e) = a\pi(b)\pi(e_i(e)).$$

These coincide, and so f_i is well-defined.

(2) Secondly, we show F is left finitely generated projective. Using that E is finitely generated projective, any $a \otimes \xi \in F$ decomposes as

$$a \otimes \xi = a \otimes \sum_i e_i(\xi).e^i = \sum_i a\pi(e_i(\xi)) \otimes e^i = \sum_i f_i(a \otimes \xi).f^i.$$

Hence F is left finitely generated projective by dual basis $\sum_i f_i \otimes f^i$. □

Taking $E = \mathfrak{X}^R_B$ gives an A-B bimodule $A_\pi \otimes_B \mathfrak{X}^R_B$, which is finitely generated projective if $\mathfrak{X}^R_B$ is finitely generated projective. We regard it as the restriction of $\mathfrak{X}^R_B$ to A.

11.2 Tangent Bundle

Suppose that $\pi : B \to A$ is a surjective algebra map, and that B has calculus Ω^1_B. We say that a vector field $X \in \mathfrak{X}^R_B$ is horizontal if it lies in the direction of A, which is defined as follows.

Proposition 11.3. *For a surjective algebra map $\pi : B \to A$, the vector space*

$$\mathrm{Hor} := \{X \in \mathfrak{X}^R_B \mid \pi X(\mathrm{d}j) = 0 \quad \forall j \in J\} \tag{3}$$

is a B-sub-bimodule of $\mathfrak{X}^R_B$. Note that this is equivalent to

$$\mathrm{Hor} := \{X \in \mathfrak{X}^R_B \mid X(\mathrm{d}j) \in J \quad \forall j \in J\}.$$

Proof. We show the following for all $b \in B$, $X \in \mathrm{Hor}$, and $j \in J$.

(1) Firstly, we show $bX \in \mathrm{Hor}$. Using the left action on vector fields and the fact that π is an algebra map, we have $\pi(bX)(\mathrm{d}j) = \pi(b.X(\mathrm{d}j)) = \pi(b).\pi X(\mathrm{d}j) = 0$.

(2) Secondly, we show $Xb \in \mathrm{Hor}$. Using the right action on vector fields, the Leibniz rule and the fact that right vector fields are right module maps, we have $\pi(Xb)(\mathrm{d}j) = \pi X(b.\mathrm{d}j) = \pi X(\mathrm{d}(bj) - \mathrm{d}b.j) = \pi X(\mathrm{d}(bj)) - \pi X(\mathrm{d}b.j) = \pi X(\mathrm{d}(bj)) - \pi X(\mathrm{d}b)\pi(j) = 0$, since both terms are zero. □

A justification of this definition can be seen later in Example 13.1.

Also, we apologise to the reader for the similarity between the notations Hom and Hor. Hom denotes a set of morphisms, while Hor denotes the horizontal vector fields.

As Hor is in particular a left B-module, applying the restriction construction from earlier gives an A-B bimodule $T_A := A_\pi \otimes_B \mathrm{Hor}$.

We will make extensive use of the following lemma in calculating Hor in examples.

Lemma 11.4. *Suppose $\pi : B \to A$ is a surjective algebra map and that B has right finitely generated projective calculus Ω^1_B, so that $\mathfrak{X}^R_B$ is left finitely generated projective with dual basis $e_i \in \Omega^1_B$ and $e^i \in \mathfrak{X}^R_B$. If the ideal $J = \ker(\pi) \subset B$ has a finite number of generators j_i, then $X \in \mathrm{Hor}$ if and only if both of the following two statements hold.*

***(1)** $X(\mathrm{d}j_i) \in J$ for all generators j_i of J.*

***(2)** $Xj_i \in J\mathfrak{X}^R_B$ for all generators j_i of J.*

Proof. The definition of Hor is the set of vector fields $X \in \mathfrak{X}^R_B$ such that $X(\mathrm{d}j) \in J$ for all $j \in J$. This in particular implies condition 1, that $X(\mathrm{d}j_i) \in J$ for all generators j_i of J. A general element of J is given by $\sum_i j_i\beta_i$ for $\beta_i \in B$. Evaluating a horizontal right vector field $X \in \mathrm{Hor}$ on this, and then using condition 1 we get:

$$X(\mathrm{d}(\sum_i j_i\beta_i)) = \sum_i \Big(X(\mathrm{d}j_i).\beta_i + X(j_i.\mathrm{d}\beta_i)\Big) = \sum_i X(\mathrm{d}(j_i.\mathrm{d}\beta_i) = \sum_i (Xj_i)(\mathrm{d}\beta_i).$$

This is in J precisely when $Xj_i \in J\mathfrak{X}^R_B$ for all generators j_i of J. This is condition 2. □

We note that if the j_i form a linear basis of J as a vector space, then we only need to check condition 1, as all β_i are in the field.

11.3 Normal Bundle

In the classical case where $A = C(N)$, $B = C(M)$ for M an embedded submanifold of a smooth manifold N, (where restriction is in the sense of restricting the domain as functions) there is a direct sum decomposition $\mathfrak{X}^R_B\big|_A = T_A \oplus N_A$ into tangent and normal bundles of M in N. In this section we define a candidate for the normal bundle N_A for the case with general algebras.

Definition 11.5. If B has calculus Ω^1_B and $\pi : B \to A$ is a surjective algebra map with kernel J, we define $\pi(\mathfrak{X}^R_B)\mid_{\mathrm{d}J.B}$ to be the subspace of $\mathrm{Hom_B(dJ.B, A)}$ consisting of all $\pi \circ X$ for $X \in \mathfrak{X}^R_B$.

Proposition 11.6. *The vector space $\pi(\mathfrak{X}^R_B)\mid_{\mathrm{d}J.B}$ is an A-B bimodule with left A-action $a \triangleright (\pi \circ X) = a(\pi \circ X)$ and right B-action $(\pi \circ X) \triangleleft b = \pi(Xb)$ for all $a \in A$, $b \in B$, $X \in \mathfrak{X}^R_B$.*

Proof. (1) Firstly we show that the left and right actions are well-defined. Suppose $(\pi \circ X)(\mathrm{d}j.b) = 0$ for all $b \in B$ and $j \in J$. Then $(a \triangleright \pi \circ X)(\mathrm{d}j.b) = a(\pi \circ X)(\mathrm{d}j.b) = 0$, so the left action is well-defined. Also, since J is a two-sided ideal, for all $b, b' \in B$ and $j \in J$ we have

$$\begin{aligned}(\pi \circ X \triangleleft b')(\mathrm{d}j.b) &= (\pi \circ X)(b'.\mathrm{d}j.b) = (\pi \circ X)(\mathrm{d}(b'j).b) - (\pi \circ X)(\mathrm{d}b'.jb)\\ &= (\pi \circ X)(\mathrm{d}(b'j)).\pi(b) - (\pi \circ X)(\mathrm{d}b').\pi(jb) = 0.\end{aligned}$$

Hence the right action is well-defined.

(2) Secondly we show that the actions commute. For all $b, b' \in B$ and $j \in J$ we have

$$(a \triangleright (\pi \circ X \triangleleft b'))(\mathrm{d}j.b) = (a \triangleright \pi \circ X)(b'.\mathrm{d}j.b) = a(\pi \circ X)(b'.\mathrm{d}j.b) = ((a \triangleright \pi \circ X) \triangleleft b)(\mathrm{d}j.b).$$

□

Definition 11.7. Suppose $\pi : B \to A$ is a surjective algebra map with kernel J. Then we define the linear map

$$T_\pi : A_\pi \otimes_B \mathfrak{X}_B^R \to \pi(\mathfrak{X}_B^R) \mid_{\mathrm{d}J.B}, \qquad T_\pi(a \otimes X) = a(\pi \circ X) \mid_{\mathrm{d}J.B} . \tag{4}$$

By the notation $a(\pi \circ X) \mid_{\mathrm{d}J.B}$ we mean the restriction of the vector field $a(\pi \circ X) : \Omega_B^1 \to A$ to the subset of elements of the form $\mathrm{d}j.b \in \Omega_B^1$ for $j \in J$, $b \in B$.

Proposition 11.8. *The map T_π is a surjective A-B bimodule map.*

Proof. **(1)** T_π is a left module map because $T_\pi(a'a \otimes X) = a'a(\pi \circ X) \mid_{\mathrm{d}J.B} = a'T_\pi(a \otimes X)$. It is a right module map because $T_\pi(a \otimes Xb) = a\pi(Xb) \mid_{\mathrm{d}J.B} = a\pi(Xb) \mid_{\mathrm{d}J.B} \triangleleft b = T_\pi(a \otimes X) \triangleleft b$. Since these actions commute, T_π is a bimodule map.

(2) T_π is surjective because every $\pi \circ X$ in $\pi(\mathfrak{X}_B^R) \mid_{\mathrm{d}J.B}$ is equal to $T_\pi(1 \otimes X)$. □

Theorem 11.9. *Suppose B has calculus Ω_B^1, and that $\pi : B \to A$ is a surjective algebra map with kernel J. Then for the following sequence of A-B modules and A-B module maps, the following numbered statements hold.*

$$0 \longrightarrow A_\pi \otimes_B \mathrm{Hor} \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} A_\pi \otimes_B \mathfrak{X}_B^R \xrightarrow{T_\pi} \pi(\mathfrak{X}_B^R)\big|_{\mathrm{d}J.B} \longrightarrow 0 \tag{5}$$

(1) *If the map* $\mathrm{id} \otimes \mathrm{inc}$ *is injective, the sequence is exact.*

(2) *There exists a direct sum decomposition $A_\pi \otimes_B \mathfrak{X}_B^R = (A_\pi \otimes_B \mathrm{Hor}) \oplus N$ for some A-B module N if and only if the sequence is split as A-B bimodules.*

(3) *If the sequence splits as A-B bimodules, the module N is isomorphic to $\pi(\mathfrak{X}_B^R)\big|_{\mathrm{d}J.B}$, and we call it the normal bundle.*

(4) *If the sequence splits and the vector fields $\mathfrak{X}_B^R$ are left finitely generated projective (or equivalently if the calculus Ω_B^1 is right finitely generated projective), then the three modules in the above exact sequence are also left finitely generated projective as A-modules.*

(5) *If $\pi(\mathfrak{X}_B^R)\big|_{\mathrm{d}J.B}$ is left finitely generated projective, then the exact sequence splits.*

Proof. **(1)** The map

$$T_\pi : A_\pi \otimes_B \mathfrak{X}^R_B \to \pi(\mathfrak{X}^R_B)\,|_{\mathrm{d}J.B}, \qquad T_\pi(a \otimes X) = a(\pi \circ X)\,|_{\mathrm{d}J.B}$$

has kernel $A_\pi \otimes_B \mathrm{Hor}$. By surjectivity of π, a general element of the domain $A_\pi \otimes_B \mathfrak{X}^R_B$ can be put in the form $1 \otimes X$ for some $X \in \mathfrak{X}^R_B$. The set of X with $T_\pi(1 \otimes X) = 0$ is given by

$$\{X \in \mathfrak{X}^R_B \mid \pi X(\mathrm{d}j.b) = 0, \quad \forall j \in J, b \in B\}.$$

Since π is surjective, this set is equal to Hor.

(2) This is a result from page 282 of [10].

(3) Suppose that $A_\pi \otimes_B \mathfrak{X}^R_B = (A_\pi \otimes_B \mathrm{Hor}) \oplus N$ for some A-B module N. We know from earlier that $\ker(T_\pi) = A_\pi \otimes_B \mathrm{Hor}$. Therefore no nonzero element of N is in $\ker(T_\pi)$, and thus the restriction of T_π to N is injective. Furthermore, as T_π is surjective, its restriction to the non-kernel elements N must also be surjective. Hence N is isomorphic to $\pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}J.B}$.

(4) The calculus Ω^1_B is right finitely generated projective if and only if the vector fields $\mathfrak{X}^R_B$ are left finitely generated projective. Proposition 11.2 says that $A_\pi \otimes_B \mathfrak{X}^R_B$ is left finitely generated projective if $\mathfrak{X}^R_B$ is, in which case any element can be decomposed as $a \otimes X = \sum_i a\pi(X_i) \otimes e_i$. As direct summands of a left finitely generated projective module are also left finitely generated projective, it follows that $A_\pi \otimes_B \mathrm{Hor}$ and N are left finitely generated projective. The isomorphism $N \cong \pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}J.B}$ implies that $\pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}J.B}$ is also left finitely generated projective.

(5) This is a result from page 282 of [10]. □

Remark 11.10. *The condition of requiring the map* $\mathrm{id} \otimes \mathrm{inc}$ *in the sequence to be injective means that if* $\sum_i 1 \otimes X_i \in A_\pi \otimes_B \mathrm{Hor}$ *is nonzero, then its inclusion* $\sum_i 1 \otimes X_i \in A_\pi \otimes_B \mathfrak{X}^R_B$ *is also nonzero. In all reasonable examples this would be very surprising if it did not hold, and perhaps with some module theory it could be shown to hold in general. However we leave the condition as part of the definition for now, to allow for the possibility that there might be some infinite dimensional example designed to make it fail.*

Definition 11.11. Given a surjective algebra map $\pi : B \to A$, we call π a co-embedding and say that A is a *noncommutative submanifold* of B if the following splits as a short exact sequence of A-B bimodules and A-B bimodule maps.

$$0 \longrightarrow A_\pi \otimes_B \mathrm{Hor} \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} A_\pi \otimes_B \mathfrak{X}^R_B \xrightarrow{T_\pi} \pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}J.B} \longrightarrow 0$$

If it only splits as a sequence of left A-modules and left A-module maps, we call π a *weak coembedding.*

Remark 11.12. *We will see later that a weak coembedding is sufficient to obtain a submanifold calculus, but not sufficient to project connections onto that submanifold calculus.*

To show that any given example satisfies the definition, we need to show $\mathrm{id} \otimes \mathrm{inc}$ is injective, and then either find a direct sum decomposition $A_\pi \otimes_B \mathfrak{X}^R_B = (A_\pi \otimes_B \mathrm{Hor}) \oplus N$ or to show that $\pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}J.B}$ is left finitely generated projective.

11.4 Submanifold Calculus

Lemma 11.13. $J\Omega^1_B \subset dJ.B$

Proof. Suppose $j \in J$ and $b \in B$. Then by the Leibniz rule, $j\mathrm{d}b = \mathrm{d}(jb) - \mathrm{d}j.b$. But as J is a right ideal, $jb \in J$, so both terms are in $\mathrm{d}J.B$. □

Proposition 11.14. *Suppose we have a surjective algebra map $\pi : B \to A$, where B has calculus Ω^1_B. Then the vector space $\mathfrak{X}^R_A := A_\pi \otimes_B \mathrm{Hor}_{\pi^{-1}}$ is an A-bimodule with left action multiplication and right action $(1 \otimes X) \triangleleft a = 1 \otimes X\pi^{-1}(a)$.*

Proof. Recall that $T_A = A_\pi \otimes_B \mathrm{Hor}$ is an A-B bimodule. We show that the right A-action is well-defined by showing that for all $X \in \mathrm{Hor}$ and $j \in J$, we have $1 \otimes X.j = 0$. But for all $\xi \in \Omega^1_B$, we have $1 \otimes (Xj)(\xi) = 1 \otimes X(j\xi) = \pi X(j\xi) \otimes 1$. Lemma 11.13 above says that $j\xi \in \mathrm{d}J.B$, and hence the facts that X is a right vector field and π is an algebra map imply that $\pi X(j\xi) = 0$. □

Definition 11.15. Suppose $\pi : B \to A$ is a surjective algebra map, and B has calculus Ω^1_B. We call the A-bimodule $\mathfrak{X}^R_A$ the *vector fields* on A. We call its dual, the A-bimodule

$$\tilde{\Omega}^1_A := {}_A\mathrm{Hom}(\mathfrak{X}^{\mathrm{R}}_{\mathrm{A}}, \mathrm{A}), \tag{6}$$

the extended *submanifold calculus* on A.

Proposition 11.16. *If Ω^1_B is right finitely generated projective then $\tilde{\Omega}^1_A$ is also right finitely generated projective.*

Proof. We showed earlier that if Ω^1_B is right finitely generated projective, the vector fields $\mathfrak{X}^R_B$ are left finitely generated projective, and the splitting of the sequence implies that $T_A = A_\pi \otimes_B \mathrm{Hor}$ is left finitely generated projective. Just changing the right module structure of T_A to get $\mathfrak{X}^R_A$ preserves the property of being left finitely generated projective, and hence its dual $\tilde{\Omega}^1_A$ is right finitely generated projective. □

Each element $\xi \in \tilde{\Omega}^1_A$ is specified by its evaluation on each $a \otimes X \in \mathfrak{X}^R_A$ to give an element of A. We use this to define an exterior derivative on A with values in $\tilde{\Omega}^1_A$.

Proposition 11.17. *The linear map $\mathrm{d}_A : A \to \tilde{\Omega}^1_A$ given by the following equation for all $a, a' \in A$ and $X \in \mathrm{Hor}$ satisfies the Leibniz rule and is thus an exterior derivative.*

$$ev((a \otimes X) \otimes \mathrm{d}_A a') = a.\pi X(\mathrm{d}(\pi^{-1} a')). \tag{7}$$

Proof. Choose $b', b'' \in B$ such that $\pi b' = a'$ and $\pi b'' = a''$. Then:

$$\begin{aligned}
ev(a \otimes X \otimes \mathrm{d}_A(a'a'')) &= a.\pi X(\mathrm{d}(\pi^{-1}(a'a''))) \\
&= a.\pi X(\mathrm{d}(b'b'')) \\
&= a.\pi X(\mathrm{d}b'.b'' + b'.\mathrm{d}b'') \\
&= a.\pi(X(\mathrm{d}b').b'') + a.\pi((Xb')(\mathrm{d}b'')) \\
&= a.\pi X(\mathrm{d}b')a'' + a.\pi((Xb')(\mathrm{d}b'')) \\
&= ev(a \otimes X \otimes \mathrm{d}_A a').a'' + ev(a \otimes Xb' \otimes \mathrm{d}_A a'') \\
&= ev(a \otimes X \otimes \mathrm{d}_A a'.a'') + ev(a \otimes X \otimes a'\mathrm{d}_A a'') \\
&= ev(a \otimes X \otimes (\mathrm{d}_A a'.a'' + a'.\mathrm{d}_A a'')),
\end{aligned}$$

meaning d_A obeys the Leibniz rule and is thus an exterior derivative. □

This almost meets the requirements to be a calculus, but we are not guaranteed the surjectivity axiom. Thus we define the subset $\Omega^1_A := \{a.\mathrm{d}_A a' \mid a, a' \in A\}_{\mathrm{span}}$ to be the largest subset of $\tilde{\Omega}^1_A$ satisfying the surjectivity condition, and call it the submanifold calculus. Later in examples we calculate whether $\tilde{\Omega}^1_A = \Omega^1_A$ or not, and find that there are instances of both cases.

Proposition 11.18. *If B has calculus Ω^1_B and A is equipped with submanifold calculus Ω^1_A coming from a co-embedding $\pi : B \to A$, then the co-embedding extends to a linear map $\pi^* : \Omega^1_B \to \Omega^1_A$ given by*

$$ev((a \otimes X) \otimes \pi^*(\xi)) = a.\pi(X(\xi)), \tag{8}$$

satisfying $\pi^*(\mathrm{d}b) = \mathrm{d}_A(\pi(b))$ and $\pi^*(\xi b) = \pi^*(\xi)\pi(b)$ and $\pi^*(b\xi) = \pi(b)\pi^*(\xi)$ for all $b \in B$, $a \in A$.

Proof. By definition of π^*, we have for all $a \in A$, $b \in B$, $\xi \in \Omega^1_B$, $X \in \mathrm{Hor}$

$$\pi^*(\mathrm{d}b) = a.\pi(X(\mathrm{d}b)) = a.\pi(X(\mathrm{d}(\pi^{-1}\pi b))) = \mathrm{d}_A(\pi(b)),$$

and

$$\begin{aligned} ev((a \otimes X) \otimes \pi^*(\xi.b)) &= a.\pi(X(\xi.b)) = a.\pi(X(\xi)).\pi(b) = ev((a \otimes X) \otimes \pi^*(\xi)).\pi(b) \\ &= ev((a \otimes X) \otimes \pi^*(\xi).\pi(b)). \end{aligned}$$

Using the Leibniz rule, we calculate:

$$\begin{aligned} \pi^*(b.\mathrm{d}b') &= \pi^*(\mathrm{d}(bb')) - \pi^*(\mathrm{d}b.b') = \mathrm{d}_A(\pi(b)\pi(b')) - \mathrm{d}_A(\pi(b))\pi(b') = \pi(b)\mathrm{d}_A(\pi(b')) \\ &= \pi(b)\pi^*(\mathrm{d}b'). \end{aligned}$$

Combined with the result that $\pi^*(b.\xi) = \pi(b).\pi^*(\xi)$, this implies $\pi^*(b.\xi) = \pi(b).\pi^*(\xi)$ for all $b \in B$, $\xi \in \Omega^1_B$. □

In summary, this means that our construction takes an algebra map $\pi : B \to A$ and constructs a calculus on A with respect to which the map π is differentiable.

11.5 Submanifolds of Submanifolds

Suppose that algebra maps $\psi : C \to B$ and $\pi : B \to A$ are co-embeddings. We show in the following that (subject to an injectivity condition) their composition $\pi \circ \psi : C \to A$ is also a co-embedding, similarly to how being a submanifold is a transitive property in classical geometry.

There is a B-bimodule

$$\mathrm{Hor}(\pi) = \{X \in \mathfrak{X}^R_B \mid \pi X(\mathrm{d}j) = 0, \quad \forall j \in \ker(\pi)\}$$

and C-bimodules

$$\mathrm{Hor}(\psi) = \{Y \in \mathfrak{X}^R_C \mid \psi Y(\mathrm{d}k) = 0, \quad \forall k \in \ker(\psi)\},$$

$$\mathrm{Hor}(\pi \circ \psi) = \{Y \in \mathfrak{X}^R_C \mid (\pi \circ \psi)Y(\mathrm{d}n) = 0, \quad \forall n \in \ker(\pi \circ \psi)\}.$$

Lemma 11.19. *If $\psi : C \to B$ and $\pi : B \to A$ are algebra maps, then, denoting ψ^{-1} for the pre-image set, we have*

$$\psi^{-1}(\ker \pi) = ker(\pi \circ \psi) : C \to A. \tag{9}$$

Proof. We prove equality by showing that each is a subset of the other.
(1) If $n \in \psi^{-1}(\ker \pi)$ then $\psi(n) \in \ker \pi$, so $(\pi \circ \psi)(n) = 0$. Hence $\psi^{-1}(\ker \pi) \subset \ker(\pi \circ \psi)$.
(2) If $n \in \ker(\pi \circ \psi)$, then $(\pi \circ \psi)(n) = 0$, i.e. $\psi(n) \in \ker \pi$, i.e. $n \in \psi^{-1}(\ker \pi)$. Hence $\ker(\pi \circ \psi) \subset \psi^{-1}(\ker \pi)$. □

Proposition 11.20. *Suppose $\psi : C \to B$ and $\pi : B \to A$ are co-embeddings, and that B is equipped with submanifold calculus from ψ. Then*

$$\mathrm{Hor}(\pi) \cong B_\psi \otimes_C \mathrm{Hor}(\pi \circ \psi)_{\psi^{-1}}. \tag{10}$$

Since $A_\pi \otimes_B B_\psi \cong A_{\pi\circ\psi}$, it follows that:

$$A_\pi \otimes_B \mathrm{Hor}(\pi)_{\pi^{-1}} \cong A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\pi \circ \psi)_{(\pi\circ\psi)^{-1}}, \tag{11}$$

meaning that if $\pi \circ \psi$ is also a co-embedding then the submanifold calculi on A from π and from $\pi \circ \psi$ are isomorphic.

Proof. Equipping B with submanifold calculus from ψ means its vector fields are $\mathfrak{X}^R_B = B_\psi \otimes_C \mathrm{Hor}(\psi)_{\psi^{-1}}$. By surjectivity of ψ a general element can be expressed as $1 \otimes Y$ for some $Y \in \mathrm{Hor}(\psi)$. In the following calculations, we first use the definition of $\mathrm{Hor}(\pi)$, then secondly a combination of the assumption that B is equipped with submanifold calculus and the definition of the differential on the submanifold calculus as in Equation 7, then thirdly we invoke Lemma 11.19, and then lastly recognise the definition of $\mathrm{Hor}(\pi \circ \psi)$.

$$\mathrm{Hor}(\pi) \cong \{X \in \mathfrak{X}^R_B \mid \pi X(\mathrm{d}j) = 0 \quad \forall j \in \ker \pi\}$$

$$= \{1 \otimes Y \in B_\psi \otimes_C \mathrm{Hor}(\psi)_{\psi^{-1}} \mid (\pi \circ \psi)Y(\mathrm{d}\pi^{-1}j) = 0 \quad \forall j \in \ker \psi\}$$
$$= \{1 \otimes Y \in B_\psi \otimes_C \mathrm{Hor}(\psi)_{\psi^{-1}} \mid (\pi \circ \psi)Y(\mathrm{d}n) = 0 \quad \forall n \in \ker(\pi \circ \psi)\}$$
$$= B_\psi \otimes_C \mathrm{Hor}(\pi \circ \psi)_{\psi^{-1}}.$$

□

Recall that as in Exercise 2.8 of [50], the Splitting Lemma says that a short exact sequence of left A-modules and left A-module maps

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

splits if and only if there exists a left A-module map $q : B \to A$ such that $q \circ i = \mathrm{id}_A$.

Proposition 11.21. *Suppose $\psi : C \to B$ and $\pi : B \to A$ are co-embeddings, where B is equipped with submanifold calculus, and that the map*

$$\mathrm{id} \otimes \mathrm{inc} : A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\pi \circ \psi) \to A_{\pi\circ\psi} \otimes_C \mathfrak{X}^R_C$$

is injective. Then the composition $\pi \circ \psi : C \to A$ is also a co-embedding.

Proof. For the co-embeddings π and ψ, there are split short exact sequences

$$0 \longrightarrow A_\pi \otimes_B \mathrm{Hor}(\pi) \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} A_\pi \otimes_B \mathfrak{X}^R_B \xrightarrow{T_\pi} \pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}\ker(\pi).B} \longrightarrow 0$$

$$0 \longrightarrow B_\psi \otimes_C \mathrm{Hor}(\psi) \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} B_\psi \otimes_C \mathfrak{X}^R_C \xrightarrow{T_\psi} \psi(\mathfrak{X}^R_C)\big|_{\mathrm{d}\ker(\psi).C} \longrightarrow 0$$

where $T_\pi(a \otimes X) = a(\pi \circ X) \mid_{\mathrm{d}(\ker(\pi)).B}$ and $T_\psi(b \otimes Y) = b(\psi \circ Y) \mid_{\mathrm{d}(\ker(\psi)).C}$.
These have splitting maps $u_\pi : A_\pi \otimes_B \mathfrak{X}^R_B \to A_\pi \otimes_B \mathrm{Hor}(\pi)$ and $u_\psi : B_\psi \otimes_C \mathfrak{X}^R_C \to B_\psi \otimes_C \mathrm{Hor}(\psi)$ whose compositions with their respective $\mathrm{id} \otimes \mathrm{inc}$ give the identity on the left hand side of the sequence.
The composition $\pi \circ \psi$ is a surjective algebra map, so to prove it is a co-embedding it remains to show that the following short exact sequence splits (exactness is by the assumption).

$$0 \longrightarrow A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\pi \circ \psi) \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} A_{\pi\circ\psi} \otimes_C \mathfrak{X}^R_C \xrightarrow{T_{\pi\circ\psi}} (\pi \circ \psi)(\mathfrak{X}^R_C)\big|_{\mathrm{d}\ker(\pi\circ\psi).C} \longrightarrow 0$$

where $T_{\pi\circ\psi}(a \otimes Y) = a((\pi \circ \psi) \circ X) \mid_{\mathrm{d}(\ker(\pi\circ\psi)).C}$.

But since B is equipped with the submanifold calculus, $\mathfrak{X}^R_B = B_\psi \otimes_C \mathrm{Hor}(\psi)$, so by the lemma above, the sequence for π is equivalent to

$$0 \longrightarrow A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\pi \circ \psi) \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\psi) \longrightarrow \pi(\mathfrak{X}^R_B)\big|_{\mathrm{d}\ker(\pi).B} \longrightarrow 0$$

with splitting map $\hat{u}_\pi : A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\psi) \to A_{\pi\circ\psi} \otimes_C \mathrm{Hor}(\pi \circ \psi)$.
The composition $\hat{u}_\pi \circ (\mathrm{id} \otimes u_\psi)$ gives the desired splitting, since

$$\hat{u}_\pi \circ (\mathrm{id} \otimes u_\psi)(1 \otimes X) = \hat{u}_\pi(1 \otimes X) = 1 \otimes X.$$

Hence $\pi \circ \psi$ is also a co-embedding. □

The injectivity condition would be automatic if for example A_π was flat as a right B-module.

12 Left-sided Theory

Here we provide a version of the theory for $\mathfrak{X}^L_B$, but omit to write the proofs since they entirely mirror the right-handed version. For an associative unital algebra B with calculus Ω^1_B, the vector fields $\mathfrak{X}^L_B = {}_B\mathrm{Hom}(\Omega^1_B, B)$ satisfy $X(b\xi) = bX(\xi)$ and are a B-bimodule, with left and right actions given for all $b \in B$, $\xi \in \Omega^1_B$ by

$$(bX)(\xi) = X(b\xi), \qquad (Xb)(\xi) = X(\xi).b.$$

If Ω^1_B is left finitely generated projective then $\mathfrak{X}^L_B$ is right finitely generated projective.
Given a surjective algebra map $\pi : B \to A$, the kernel $J = \ker(\pi)$ is a two-sided ideal of B. Write ${}_\pi A$ for the B-A bimodule given by the algebra A with right A-action given by multiplication, and left B-action given by $b \triangleright a = \pi(b)a$. The restriction via π of a right B-module E to A is given by $E \otimes_B {}_\pi A$, and this is right finitely generated projective if E is. The set

$$\mathrm{Hor}^L = \{X \in \mathfrak{X}^L_B \mid \pi X(\mathrm{d}j) = 0, \quad \forall j \in J\} \tag{12}$$

is a B-bimodule. We define a tangent bundle of A in B by $T_A = \mathrm{Hor}^L \otimes_B {}_\pi A$.
The subspace $\pi(\mathfrak{X}^L_B)\big|_{B.\mathrm{d}J}$ of ${}_B\mathrm{Hom}(\mathrm{d}J.B, A)$ consisting of all $\pi \circ X$ for $X \in \mathfrak{X}^L_B$ is a B-A bimodule with left B-action $b \triangleright (\pi \otimes X) = \pi(bX)$ and right A-action $(\pi \circ X) \triangleleft a = (\pi \circ X).a$.

There is a surjective B-A bimodule map given by

$$T_\pi : \mathfrak{X}^L_B \otimes_B {}_\pi A \to \pi(\mathfrak{X}^L_B)\big|_{B.\mathrm{d}J}, \qquad T_\pi(X \otimes a) = \Big(\pi \circ X\big|_{B.\mathrm{d}J}\Big).a. \tag{13}$$

The notation $\pi \circ X\big|_{B.\mathrm{d}J}$ denotes restriction of the map $\pi \circ X : \Omega^1_B \to A$ to the subset of the domain consisting of elements of the form $b.\mathrm{d}j$ for some $j \in J$, $b \in B$.

Theorem 12.1. *Suppose $\pi : B \to A$ is a surjective algebra map. Then for the following sequence of A-B modules and A-B module maps, the following numbered statements hold.*

$$0 \longrightarrow \mathrm{Hor}^L \otimes_B {}_\pi A \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} \mathfrak{X}^L_B \otimes_B {}_\pi A \xrightarrow{T_\pi} \pi(\mathfrak{X}^L_B)\big|_{B.\mathrm{d}J} \longrightarrow 0 \tag{14}$$

(1) *If $\mathrm{inc} \otimes \mathrm{id}$ is injective, then the sequence is exact.*
(2) *There exists a direct sum decomposition $\mathfrak{X}^L_B \otimes_B {}_\pi A = (\mathrm{Hor}^L \otimes_B {}_\pi A) \oplus N$ for some B-A bimodule N if and only if the sequence is split as B-A bimodules.*
(3) *If the sequence splits as B-A bimodules, the module N is isomorphic to $\pi(\mathfrak{X}^L_B)\big|_{B.\mathrm{d}J}$, and we call it the normal bundle.*
(4) *If the sequence splits and the vector fields $\mathfrak{X}^L_B$ are right finitely generated projective (or equivalently if the calculus Ω^1_B is left finitely generated projective), then the three modules in the above exact sequence are also right finitely generated projective.*
(5) *If $\pi(\mathfrak{X}^L_B)\big|_{B.\mathrm{d}J}$ is right finitely generated projective, then the exact sequence splits.*

If $\pi : B \to A$ is a surjective algebra map for which the above short exact sequence splits, then we say π is a (left) co-embedding. We define the vector fields on A as the A-bimodule

$$\mathfrak{X}^L_A := {}_{\pi^{-1}}\mathrm{Hor}^L \otimes_B {}_\pi A$$

which has left action $a \triangleright X \otimes 1 = \pi^{-1}(a).X \otimes 1$, which is well-defined by the properties of Hor^L. We define the extended submanifold calculus by

$$\tilde{\Omega}^1_A = \mathrm{Hom}_A(\mathfrak{X}^L_A, A).$$

This is left finitely generated projective if Ω^1_B is left finitely generated projective, and we denote Ω^1_A for the maximal subset spanned by the exterior derivative

$$\mathrm{d}_A : A \to \Omega^1_A, \qquad ev(\mathrm{d}_A a' \otimes X \otimes a) = \pi X(\mathrm{d}(\pi^{-1}a')).a. \tag{15}$$

As in the right-handed case we have the following proposition.

Proposition 12.2. *A left co-embedding $\pi : B \to A$ extends to a linear map*

$$\pi^* : \Omega^1_B \to \tilde{\Omega}^1_A, \qquad \mathrm{d}b \mapsto \mathrm{d}_A(\pi(b)) \tag{16}$$

satisfying $\pi^*(\xi b) = \pi^*(\xi)\pi(b)$ and $\pi^*(b\xi) = \pi(b)\pi^*(\xi)$ for all $b \in B$, $a \in A$.

The composition of two left co-embeddings is a left co-embedding, subject to the condition that the following map is injective.

$$\mathrm{inc} \otimes \mathrm{id} : \mathrm{Hor}(\pi \circ \psi) \otimes_{C\ \pi\circ\psi} A \to \mathfrak{X}^L_C \otimes_{C\ \pi\circ\psi} A$$

13 Submanifold Examples

In this section, we look at several examples, both classical and non-classical, which are tabulated below. The table contains one row per example, and records the algebras, the algebra map π between them, and the generators of the kernel J of π, and of the tangent and normal bundles. The last column is whether the submanifold calculus satisfies the surjectivity axiom of being spanned by the image of the exterior derivative d_A.

B	**A**	$\boldsymbol{\pi : B \to A}$	**J gen.**	$\boldsymbol{T_A}$ **gen.**	$\boldsymbol{N_A}$ **gen.**	$\boldsymbol{\tilde{\Omega}^1_A = \Omega^1_A}$**?**
$C^\infty(\mathbb{R}^2)$	$C^\infty(\mathbb{R}^2)\big\vert_{\mathbb{R}}$	Restriction	y	$1 \otimes \frac{\partial}{\partial x}$	$1 \otimes \frac{\partial}{\partial y}$	Y
$C^\infty(\mathbb{R})$	$C^\infty(\mathbb{R})\big\vert_{[0,1]}$	Restriction	f vanishing on $[0,1]$	$1 \otimes \frac{\partial}{\partial x}$	0	Y
$\mathbb{C}_q[D]$	$\mathbb{C}_{q^2}[S^1]$	$\pi(z) = t$, $\pi(\bar{z}) = t^{-1}$	$1 - \bar{z}z$	$q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}$	$1 \otimes e_z$	Y
$\mathbb{C}_q[SU_2]$	$\mathbb{C}_{q^2}[S^1]$	$\pi(a) = t$, $\pi(d) = t^{-1}$, $\pi(b) = \pi(c) = 0$	b, c	$1 \otimes e_0$	$1 \otimes e_+$, $1 \otimes e_-$	Y
$C(G)$	$C(H)$	$\pi(\delta_g) = \delta_g$ if $g \in H$, $\pi(\delta_g) = 0$ else	δ_g for all $g \in G \backslash H$	$1 \otimes e_a$ for $a \in \mathcal{C} \cap H$	$1 \otimes e_a$ for all $a \notin \mathcal{C} \cap H$	Y
$C(X)$ for X a finite set	$C(\{x_0\})$ for some $x_0 \in X$	Restriction	δ_x for all $x \in X \backslash \{x_0\}$	$1 \otimes f_{x_0 \leftarrow x}$ for arrows $x \to x_0$	$1 \otimes f_{y \leftarrow x}$ for arrows $x \to y$, $x \in X$, $y \in X \backslash \{x_0\}$	N
$\mathbb{C}_q[M_2]$	$\mathbb{C}_q[\mathbb{C}^2]$	$\pi(a) = x$, $\pi(c) = y$ $\pi(b) = \pi(d) = 0$	b, d	$1 \otimes e^b$, $1 \otimes e^d$	$1 \otimes e^a$, $1 \otimes e^c$	N

Table 1: Submanifolds Examples

We note that the last example $\pi : \mathbb{C}_q[M_2] \to \mathbb{C}_q[\mathbb{C}^2]$ is not a co-embedding but a weak co-embedding since the normal bundle fails to be a right B-module.

In each of these examples, we start by assuming a calculus Ω^1_B on the algebra B, which is of the type mentioned in square brackets at the start of each example, but we do not assume at first any calculus on A, since that is what we construct using the calculus on B using our theory of submanifolds.

13.1 Classical Example: x-axis in $\mathbb{R}^2$

[Algebras: See Example 3.1 for $C^\infty(M)$ and its calculus]

We start with a classical example: the inclusion of the x-axis into $\mathbb{R}^2$. The algebra $B = C^\infty(\mathbb{R}^2)$ is generated (up to completion) by functions x and y defined by $x(b_1, b_2) = b_1$, $y(b_1, b_2) = b_2$ respectively, and its calculus Ω^1_B is freely generated by $\mathrm{d}x$ and $\mathrm{d}y$. We take $A = C^\infty(\mathbb{R}^2)\big|_{\mathbb{R}} \cong C^\infty(\mathbb{R})$ as the restriction of B to the x-axis. The restriction map $\pi : B \to A$ is surjective with kernel J generated by y. A right vector field $X \in \mathfrak{X}^R_B$ takes the form $X = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y}$ for some $f_x, f_y \in B$, from which we calculate $X(\mathrm{d}x) = f_x$ and $X(\mathrm{d}y) = f_y$. The restricted vector fields are the B-A bimodule $A_\pi \otimes_B \Omega^1_B$, which has general element $g \otimes (f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y})$ and hence generators $1 \otimes \frac{\partial}{\partial x}$ and $1 \otimes \frac{\partial}{\partial y}$. A horizontal vector field satisfies $\pi X(\mathrm{d}j) = 0$ for all $j \in J$, i.e. $\pi(f_y) = 0$, and since Ω^1_B is commutative, condition 2 of Lemma 11.4 holds automatically. Hence Hor is generated by $\frac{\partial}{\partial x}$. The tangent bundle T_A is then generated by $1 \otimes \frac{\partial}{\partial x}$ and the normal bundle N_A by $1 \otimes \frac{\partial}{\partial y}$. The normal bundle N_A is an A-B bimodule, since multiplying $\frac{\partial}{\partial y}$ on the right by some $b \in C^\infty(\mathbb{R}^2)$ never produces multiples of $\frac{\partial}{\partial x}$. The map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$$

has trivial kernel, since if $a \otimes \frac{\partial}{\partial x}$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Hence $\pi : B \to A$ is a co-embedding. The A-bimodule $\mathfrak{X}^R_A = A_\pi \otimes_B \mathrm{Hor}_{\pi^{-1}}$ has right dual the left module maps $\tilde{\Omega}^1_{C^\infty(\mathbb{R})} = {}_A\mathrm{Hom}(\mathfrak{X}^\mathrm{R}_\mathrm{A}, \mathrm{A})$, which is a right finitely generated projective A-bimodule. The exterior derivative $\mathrm{d}_A : A \to \tilde{\Omega}^1_A$ is given by $ev((a \otimes X) \otimes \mathrm{d}_A a') = a.\pi X(\mathrm{d}(\pi^{-1} a'))$ and the submanifold calculus $\tilde{\Omega}^1_{C^\infty(\mathbb{R})}$ is spanned by $\mathrm{d}_A x$.

13.2 Classical Example: Closed interval in $\mathbb{R}$

[Algebras: See Example 3.1 for $C^\infty(M)$ and its calculi.]

Our next example is another classical one: The inclusion of the closed interval $[0,1]$ as the closure of a submanifold of $\mathbb{R}$. Let $B = C^\infty(\mathbb{R})$ and $A = C^\infty(\mathbb{R})\big|_{[0,1]}$, with right finitely generated projective calculus Ω^1_B. The restriction $\pi : B \to A$ is a surjective algebra map with kernel J the smooth functions vanishing on $[0,1]$. A vector field $\mathfrak{X}^R_B$ takes the form $x = g\frac{\partial}{\partial x}$ for some $g \in B$. The restricted vector fields $\mathfrak{X}^R_B\big|_A = A_\pi \otimes_B \mathfrak{X}^R_B$ have general element $k\otimes\frac{\partial}{\partial x}$ for some $k \in A$, and hence are generated as a left A-module by $1\otimes\frac{\partial}{\partial x}$. Next we calculate Hor. As in the previous example, since Ω^1_B is commutative, condition 2 of Lemma 11.4 holds automatically. Elements $X = g\frac{\partial}{\partial x}$ of Hor satisfy $X(j) = g\frac{\partial}{\partial x}(j) \in J$ for each smooth function j vanishing on $[0,1]$. But for a smooth function j vanishing on $[0,1]$, by continuity its derivatives must also vanish on $[0,1]$, so $X(j) \in J$. Hence $\mathrm{Hor} = \mathfrak{X}^R_B$, so $T_A = A_\pi \otimes_B \mathfrak{X}^R_A$ and $N_A = 0$, making π a co-embedding. The submanifold calculus $\tilde{\Omega}^1_A = {}_A\mathrm{Hom}(\mathfrak{X}^{\mathrm{R}}_{\mathrm{A}}, \mathrm{A})$ is spanned by $\mathrm{d}_A x$, the dual of $1 \otimes \frac{\partial}{\partial x}$. This example illustrates that under our definition, the closure of a manifold is indistinguishable from a manifold, since $[0,1]$ has the same tangent bundle, normal bundle and submanifold calculus as $(0,1)$.

13.3 Non-Classical Example: Algebraic Circle in Quantum Disk

[Algebras: See Example 3.3 for $\mathbb{C}_q[D]$ and its calculus, and Example 3.2 for $\mathbb{C}_q[S^1]$ and its calculus]

Next we look at the non-classical example of embedding the algebraic circle $A = \mathbb{C}[t, t^{-1}]$ into the quantum disk $B = \mathbb{C}_q[D]$, for $0 \le q \le 1$.

There is a surjective algebra map $\pi : B \to A$ given by $\pi(z) = t$, $\pi(\bar{z}) = t^{-1}$ with kernel J generated by $w := 1 - \bar{z}z$. This is known to be differentiable if A is equipped with the 1-dimensional calculus generated by $\mathrm{d}t$ with relation $\mathrm{d}t.t = q^2 t.\mathrm{d}t$, in which case the quantum circle algebra is denoted $\mathbb{C}_{q^2}[S^1]$. However, here we instead start by making no assumptions on the calculus of A and show that the submanifold calculus obtained from π as a co-embedding coincides with this.

As Ω^1_B is right finitely generated projective, there exist dual basis elements $e_z, e_{\bar{z}} \in \mathfrak{X}^R_B$ such that any vector field $X \in \mathfrak{X}^R_B$ decomposes as $X = \sum_{a\in\{z,\bar{z}\}} X_a e_a$ for some $X_a \in B$.

Then $e_a(\mathrm{d}b) = \delta_{a,b}$ for $a, b \in \{z, \bar{z}\}$, and $X(\mathrm{d}z) = X_z$ and $X(\mathrm{d}\bar{z}) = X_{\bar{z}}$.

Proposition 13.1. *The tangent bundle T_A is generated as a left A-module by $Y := q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}$.*

Proof. Vector fields $X \in \mathrm{Hor}$ are characterised by the property that $X(\mathrm{d}j) \in J$ for all $j \in J$, which implies $X(\mathrm{d}w) \in J$. Using the fact that $\mathrm{d}w = -q^2z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z = \mathrm{d}\bar{z}.z - q^2\mathrm{d}z.\bar{z}$, we calculate:

$$X(\mathrm{d}w) = X(\mathrm{d}\bar{z}.z - q^2\mathrm{d}z.\bar{z}) = X(\mathrm{d}\bar{z}).z - q^2X(\mathrm{d}z).\bar{z} = X_{\bar{z}}.z - q^2X_z.\bar{z}$$

Thus for all $X \in \mathrm{Hor}$ we have $0 = \pi X(\mathrm{d}w) = \pi(X_{\bar{z}}).t - q^2\pi(X_z).t^{-1}$, which is equivalent to $\pi(X_z) = q^2\pi(X_{\bar{z}}).t^2$. Next we check if there are any conditions on horizontal vector fields coming from condition 2 of Lemma 11.4. But by the commutation relations of the calculus, w commutes with the calculus, and so:

$$X_ze_zw + X_{\bar{z}}e_{\bar{z}}w = X_zwe_z + X_{\bar{z}}we_{\bar{z}} \in Je_z + Je_{\bar{z}}.$$

Hence this gives no additional conditions.

A general element of the tangent bundle $T_A = A_\pi \otimes_B \mathrm{Hor}$ takes the form $p(t) \otimes X$, for $p(t)$ a Laurent polynomial and $X \in \mathrm{Hor}$. We can expand:

$$\begin{aligned} p(t) \otimes X &= p(t) \otimes (X_ze_z + X_{\bar{z}}e_{\bar{z}}) \\ &= p(t)\pi(X_z) \otimes e_z + p(t)\pi(X_{\bar{z}}) \otimes e_{\bar{z}} \\ &= q^2p(t)\pi(X_{\bar{z}}).t^2 \otimes e_z + p(t)\pi(X_{\bar{z}}) \otimes e_{\bar{z}} \\ &= p(t)\pi(X_{\bar{z}}).(q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}) \end{aligned}$$

We didn't have any further restrictions on $X_{\bar{z}}$, so $\pi(X_{\bar{z}})$ is a general polynomial in A, meaning that $p(t)\pi(X_{\bar{z}})$ is a general polynomial too. Thus T_A as a left A-module is spanned by $Y := q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}$. □

A normal bundle N_A can then be given as the span of $1 \otimes e_z$, giving a direct sum decomposition $A_\pi \otimes_B \mathfrak{X}^R_B = T_A \oplus N_A$. From the commutation relations listed in Example 3.2, we can see that the normal bundle is closed under the right B-action, and thus a B-A module. The map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$$

has trivial kernel, since if $f(t).\big(q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}\big)$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Hence π is a co-embedding.

But in [10] there is already a differential structure on $\mathbb{C}_{q^2}[S^1]$ given by $\Omega^1_A = \mathbb{C}[t, t^{-1}].\mathrm{d}t$, $\mathrm{d}t.f(t) = f(q^2t).\mathrm{d}t$, $\mathrm{d}f = \frac{f(q^2t)-f(t)}{t(q^2-1)}\mathrm{d}t$. Does the submanifold calculus coincide with this? We start by calculating the commutation relations on $\mathfrak{X}^R_A$.

Proposition 13.2. *If we write $Y = q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}} \in \mathfrak{X}^R_A$, then the module $\mathfrak{X}^R_A$ has commutation relation $Y.t = q^{-2}tY$.*

Proof. We calculate $(e_z.z)(\mathrm{d}z) = e_z(z.\mathrm{d}z) = e_z(q^{-2}\mathrm{d}z.z) = q^{-2}e_z(\mathrm{d}z).z = q^{-2}z = q^{-2}ze_z(\mathrm{d}z) = q^{-2}(z.e_z)(\mathrm{d}z)$. Also $e_z(\mathrm{d}\bar{z}) = 0$, so $e_z.z = q^{-2}z.e_z$. Next, $(e_{\bar{z}}.z)(\mathrm{d}\bar{z}) = e_{\bar{z}}(z.\mathrm{d}\bar{z}) = e_{\bar{z}}(q^{-2}\mathrm{d}\bar{z}.z) = q^{-2}z = q^{-2}(z.e_{\bar{z}})(\mathrm{d}\bar{z})$. Also $e_{\bar{z}}(\mathrm{d}z) = 0$, so $e_{\bar{z}}.z = q^{-2}z.e_{\bar{z}}$. Therefore $Y.t = (q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}})z = q^2t^2 \otimes q^{-2}ze_z + 1 \otimes q^{-2}ze_{\bar{z}} = q^{-2}t(q^2t^2 \otimes e_z + 1 \otimes e_{\bar{z}}) = q^{-2}tY$. □

The submanifold calculus is $\tilde{\Omega}^1_A := {}_A\mathrm{Hom}(\mathfrak{X}^{\mathrm{R}}_{\mathrm{A}}, \mathrm{A})$, and it has basis element $q^2t^2\mathrm{d}_Az + \mathrm{d}_A\bar{z}$, which is the dual of Y.

Proposition 13.3. *The calculus $\tilde{\Omega}^1_A$ has commutation relation $\mathrm{d}t.t = q^2t.\mathrm{d}t$.*

Proof. The exterior derivative $\mathrm{d}_A : A \to \tilde{\Omega}^1_A$ is a linear map defined by the equation $ev(Y \otimes \mathrm{d}_At) = q^2t^2\pi e_z(\mathrm{d}(\pi^{-1}(t))) + \pi e_{\bar{z}}(\mathrm{d}(\pi^{-1}(t))) = q^2t^2\pi e_z(\mathrm{d}z) + \pi e_{\bar{z}}(\mathrm{d}z) = q^2t$. Thus, $ev(Y \otimes t.\mathrm{d}t) = ev(Yt \otimes \mathrm{d}t) = ev(q^{-2}tY \otimes \mathrm{d}t) = q^{-2}t.ev(Y \otimes \mathrm{d}t) = q^{-2}t.q^2t^2 = t^3$. Then, we have $ev(Y \otimes \mathrm{d}t.t) = ev(Y \otimes \mathrm{d}t)t = q^2t^3$. Therefore $\mathrm{d}t.t = q^2t.\mathrm{d}t$. □

This coincides with the usual relation on the calculus for $\mathbb{C}_{q^2}[S^1]$, so the calculi are isomorphic.

In retrospect, since the submanifold calculus is designed to always make the co-embedding differentiable, it should be no surprise that our construction gives the usual calculus on $\mathbb{C}_{q^2}[S^1]$. However, the generator of the tangent bundle we obtained along the way was not at all obvious. Our choice of normal bundle was sufficient for the purposes of being complementary, but was somewhat arbitrary.

We continue with this example in Section 16.3, where we look at Hermitian metrics and connections.

13.4 Non-Classical Example: Algebraic Circle in Quantum SU_2

[Algebras: See Example 3.5 for $\mathbb{C}_q[SU_2]$ and its 3D calculus, and Example 3.2 for $\mathbb{C}_q[S^1]$ and its calculus]

In this non-classical example we look at $A = \mathbb{C}[t, t^{-1}]$ as a noncommutative submanifold of $B = \mathbb{C}_q[SU_2]$.

Proposition 13.4. *The surjective algebra map $\pi : B \to A$ given by $\pi(a) = t$, $\pi(d) = t^{-1}$, $\pi(b) = \pi(c) = 0$ has kernel J generated by b and c, so $J = bB + cB$.*

Proof. As a vector space, $\mathbb{C}_q[SU_2]$ has a linear basis given by $a^n b^r c^s$ and $d^n b^r c^s$. Therefore $\mathbb{C}_q[SU_2]/ker(\pi)$ consists of elements a^n and d^n, for $n \geq 0$, as $\pi(a^n) = t^n$ and $\pi(d^n) = t^{-n}$. The complement of this, $\ker(\pi)$, then consists of the span of $a^n b^r c^s$ and $d^n b^r c^s$ for r, s not both zero, which is just the span of b and c. □

As Ω^1_B is right finitely generated projective, there exist $e_+, e_-, e_0 \in \mathfrak{X}^R_B$ such that any vector field $X \in \mathfrak{X}^R_B$ decomposes as $X = \sum_{i \in \{+,-,0\}} X_i e_i$ for some $X_i \in B$. We also have $e_i(e^j) = \delta_{i,j}$ for $i, j \in \{+, -, 0\}$, and $X(e^i) = X_i$. It can be calculated that the commutation relations for the vector fields e_i are as follows.

$$e_\pm f = q^{-|f|} f e_\pm, \quad e_0 f = q^{-2|f|} f e_0.$$

Proposition 13.5. *The tangent bundle T_A is generated by $Y := 1 \otimes e_0$, and the normal bundle N_A by $1 \otimes e_+$ and $1 \otimes e_-$.*

Proof. Each element X of Hor satisfies X satisfies $X(\mathrm{d}b), X(\mathrm{d}c) \in J$. We use the commutation relations $ae^- = q^{-1}e^-.a$, $be^0 = q^2 e^0.b$, $ce^0 = q^{-2}e^0.c$ and $de^+ = qe^+.d$ to calculate:

$$X(\mathrm{d}b) = X(ae^- - q^{-2}be^0) = X(q^{-1}e^-.a - e^0.b) = q^{-1}X_-.a - X_0.b,$$

$$X(\mathrm{d}c) = X(ce^0 + qde^+) = X(q^{-2}e^0.c + q^2e^+.d) = q^{-2}X_0.c + q^2X_+.d$$

Therefore for all $X \in$ Hor, we have $0 = \pi X(\mathrm{d}b) = q^{-1}\pi(X_-).t$ and $0 = \pi X(\mathrm{d}c) = q^2\pi(X_+).t^{-1}$, which implies that $\pi(X_+) = \pi(X_-) = 0$. We show that condition 2 of Lemma 11.4 is satisfied and presents no further conditions on elements of Hor, using the commutation relations on the calculus and then that J is a two-sided ideal:

$$X_+e_+j + X_-e_-j + X_0e_0j = q^{|j|}X_+je_+ + q^{|j|}X_-je_- + q^{2|j|}X_0je_0 \in Je_+ + Je_- + Je_0.$$

Consequently, a general element of $T_A = A_\pi \otimes_B \mathrm{Hor}$ takes the form

$$p(t) \otimes (X_+e_+ + X_-e_- + X_0e_0) = p(t)\pi(X_0) \otimes e_0,$$

where $p(t) \in A$ is a polynomial. The left hand side of the tensor product is free to take any value in A, and hence T_A is generated as a left A-module by $Y := 1 \otimes e_0$. By linear independence of the e_0, e_+, e_-, a complementary normal bundle N_A is generated by $1 \otimes e_+$ and $1 \otimes e_-$. □

The normal bundle is a right B-module because the right B-action $1 \otimes e_\pm$ never produces a $1 \otimes e_0$ term, and thus N_A is an A-B bimodule. The map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$$

has trivial kernel, since if $f(t) \otimes e_0$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Hence π is a co-embedding.

Next we look at the commutation relations on the vector fields and submanifold calculus.

Proposition 13.6. *The vector fields $\mathfrak{X}^R_A$ have commutation relation $Y.t = q^{-2}t.Y$, where $Y = 1 \otimes e_0$.*

Proof. Using the fact that $e^0.a = q^2a.e^0$, we calculate:

$$\begin{aligned} ev(e_0.a \otimes e^0) &= ev(e_0 \otimes a.e^0) = ev(e_0 \otimes q^{-2}e^0.a) = q^{-2}ev(e_0 \otimes e^0).a \\ &= q^{-2}a = q^{-2}a.ev(e_0 \otimes e^0) = q^{-2}ev(a.e_0 \otimes e^0). \end{aligned}$$

Thus $e_0.a = q^{-2}a.e_0$. Therefore:

$$Y.t = 1 \otimes e_0.a = 1 \otimes q^{-2}a.e_0 = q^{-2}t(1 \otimes e_0) = q^{-2}t.Y$$

□

Proposition 13.7. *The calculus $\tilde{\Omega}^1_A$ has commutation relation $\mathrm{d}t.t = q^2t.\mathrm{d}t$.*

Proof. The exterior derivative $\mathrm{d}_A : A \to \tilde{\Omega}^1_A$ is a linear map defined by the equation $ev(Y \otimes \mathrm{d}_At) = \pi e_0(\mathrm{d}(\pi^{-1}(t))) = \pi e_0(\mathrm{d}a) = \pi e_0(ae^0 + qbe^+) = \pi e_0(q^{-2}e^0.a + q^2e^+.b) = q^{-2}t$. Then, we calculate the right module structure of $\tilde{\Omega}^1_A$, by $ev(Y \otimes \mathrm{d}_At.t) = ev(Y \otimes \mathrm{d}_At.t) = q^{-2}t^2$, and

$$ev(Y \otimes t.\mathrm{d}_At) = ev(Yt \otimes \mathrm{d}_At) = ev(q^{-2}tY \otimes \mathrm{d}_At) = q^{-2}t(q^{-2}t) = q^{-4}t^2.$$

Hence $\mathrm{d}t.t = q^2t.\mathrm{d}t$. □

This coincides with the usual relation, so the submanifold calculus is the same as the calculus on $\mathbb{C}_{q^2}[S^1]$.
Since $ev(Y \otimes \mathrm{d}_A t) = q^{-2} ev(Y \otimes (e^0 \otimes 1)).t$, it follows that $\mathrm{d}_A t = q^{-2}(e^0 \otimes 1).t$.
We continue looking at this example in Section 16.1, where we look at Hermitian metrics and connections.

13.5 Non-Classical Example: Functions on a Finite Group

[Algebras: See Example 3.10 for $C(G)$ and its calculi]
In this example we look at the embedding of a subgroup H of a finite group G. There is a surjective algebra map $\pi : B \to A$ given on the basis elements of $C(G)$ by

$$\pi(\delta_g) = \begin{cases} \delta_g & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

with kernel J spanned by δ_g for $g \in G \backslash H$. Hence each $X \in \mathrm{Hor}$ satisfies $\pi X(\mathrm{d}\delta_g) = 0$ for all $g \in G\backslash H$. As Ω^1_B is right finitely generated projective, there exist dual basis elements $\{e_b\}_{b \in \mathcal{C}}$ in $\mathfrak{X}^R_B$ such that each $X \in \mathfrak{X}^R_B$ decomposes as $X = \sum_{b \in \mathcal{C}} X_b e_b$ for some $X_b \in B$. Then $e_a(e^b) = \delta_{a,b}$, and $X(\mathrm{d}\delta_c) = X_c$ for $a, b, c \in \mathcal{C}$.

Proposition 13.8. *Horizontal vector fields* $X \in \mathrm{Hor}$ *are characterised by the property* $\pi(X_a) = 0$ *for all* $a \in \mathcal{C}\backslash H$.

Proof. We begin by noting that since we have an actual linear basis of J, condition 2 of Lemma 11.4 holds automatically, and we need only check condition 1.
We know that for $f \in C(G)$ we have $\mathrm{d}f = \sum_{a \in \mathcal{C}} e^a.(f - R_{a^{-1}}(f))$. Thus $\mathrm{d}\delta_g = \sum_{a \in \mathcal{C}} e^a.(\delta_g - \delta_{ga})$. Applying $X \in \mathrm{Hor}$, we get:

$$X(\mathrm{d}\delta_g) = \sum_{a \in \mathcal{C}} X_a.(\delta_g - \delta_{ga}).$$

If $g \in G\backslash H$ then $X(\mathrm{d}\delta_g) \in J$, so $\pi \sum_{a \in \mathcal{C}} X_a.(\delta_g - \delta_{ga}) = 0$. Seeing as π is the map restricting to H, this means having $\sum_{a \in \mathcal{C}} X_a.(\delta_g - \delta_{ga})$ vanish on H. But the fact that $g \notin H$ means that $\delta_g(h)$ always vanishes, so the above is equivalent to having $-\sum_{a \in \mathcal{C}} X_a.\delta_{ga}$ vanish on H, which in turn is equivalent to having $\sum_{a \in \mathcal{C} : ga \in H} X_a.\delta_{ga}$ vanish on H.

Evaluating this at $h \in H$, all terms vanish except the one with $ga = h$ (if such a term exists), i.e. with $a = g^{-1}h$, so we get $X_{g^{-1}h}(h) = 0$ for all $h \in H$ and $g \in \mathcal{C}\backslash H$.
However, given any $a \in \mathcal{C}\backslash H$ and $h \in H$, there exists some $g \in G\backslash H$ such that $a = g^{-1}h$, i.e. that $h = ag$. Therefore the condition reduces to having $X_a(h) = 0$ for all $h \in H$ and $a \in \mathcal{C}\backslash H$, i.e. $\pi(X_a) = 0$ for $a \in \mathcal{C}\backslash H$. □

Proposition 13.9. *The tangent bundle T_A is generated as a left A-module by $Y_a := 1 \otimes e_a$ for all $a \in \mathcal{C} \cap H$, and the normal bundle by $1 \otimes e_a$ for all $a \notin \mathcal{C} \cap H$. Note that it is possible for the intersection $\mathcal{C} \cap H$ to be empty, based on our choices of $\mathcal{C}$ and H.*

Proof. A general element of $T_A = A_\pi \otimes_B \mathrm{Hor}$ can be expanded as

$$f \otimes X = f \otimes \sum_{a \in \mathcal{C}} X_a e_a = \sum_{a \in \mathcal{C} \cap H} f.\pi(X_a) \otimes e_a.$$

As there are no restrictions on X_a for $a \in \mathcal{C} \cap H$, the left hand side of the tensor product is a general element of $C(H)$, and hence T_A is generated by $Y_a := 1 \otimes e_a$ for $a \in \mathcal{C} \cap H$. By linear independence of the e_a, the tangent bundle N_a is generated by $1 \otimes e_a$ for $a \notin \mathcal{C} \cap H$. □

Since the right B-action on the vector fields e_g just multiplies them by functions, the normal bundle N_A is a B-A bimodule. The map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$$

has trivial kernel, since if $f \otimes \sum_{a \in \mathcal{C}} X_a e_a$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Thus $\pi : B \to A$ is a co-embedding.
As usual, we show that the submanifold calculus $\tilde{\Omega}^1_A$ coincides with the usual calculus on $C(H)$.

Proposition 13.10. *The exterior derivative is given by $ev(Y_a \otimes \mathrm{d}_A f) = f - R_{a^{-1}}(f)$.*

Proof. We calculate for all $h \in H$ and $a \in \mathcal{C} \cap H$

$$\begin{aligned} ev(Y_a \otimes \mathrm{d}_A f) &= \pi e_a(\mathrm{d}(\pi^{-1}(f))) = \pi e_a(\mathrm{d}f) = \pi e_a(\sum_{b \in \mathcal{C}} e^b.(f - R_{b^{-1}}(f))) = \pi(f - R_{a^{-1}}(f)) \\ &= f - R_{a^{-1}}(f) \end{aligned}$$

□

Proposition 13.11. *The commutation relations on* $\mathfrak{X}^R_A$ *are given by* $Y_a.f = R_{a^{-1}}(f).Y_a$ *for all* $f \in C(H)$ *and* $a \in \mathcal{C} \cap H$.

Proof. First we calculate the right module structure of $\mathfrak{X}^R_{C(H)}$.

$$(e_a.f)(e_a) = e_a(f.e^a) = e_a(e^a.R_{a^{-1}}(f)) = e_a(e^a).R_{a^{-1}}(f) = R_{a^{-1}}(f) = R_{a^{-1}}(f)e_a(e^a)$$
$$= (R_{a^{-1}}(f).e_a)(e^a).$$

Hence $e_a.f = R_{a^{-1}}(f).e_a$. Therefore:

$$Y_a.\delta_h = 1 \otimes e_a.f = 1 \otimes R_{a^{-1}}(f)e_a = R_{a^{-1}}(f).(1 \otimes e_a) = R_{a^{-1}}(f)Y_a.$$

So $Y_a.f = R_{a^{-1}}(f).Y_a$ for all $f \in C(H)$ and $a \in \mathcal{C} \cap H$. □

Proposition 13.12. *The commutation relations on* $\tilde{\Omega}^1_A$ *are given by* $\tilde{e}^a f = R_a(f)\tilde{e}^a$, *where* $\tilde{e}^a := \sum_{x\in G} \delta_x \mathrm{d}_A \delta_{xa}$ *for* $a \in \mathcal{C} \cap H$.

Proof. Define $\tilde{e}^a := \sum_{x\in G} \delta_x \mathrm{d}_A \delta_{xa}$ for $a \in \mathcal{C} \cap H$. The right module structure of $\tilde{\Omega}^1_A$ is:

$$ev(Y_a \otimes fe^a) = ev(Y_a\pi(f) \otimes e^a) = ev(Y_a f \otimes e^a) = ev(R_{a^{-1}}(f).Y_a \otimes e^a)$$
$$= R_{a^{-1}}(f)ev(Y_a \otimes e^a) = R_{a^{-1}}(f) = ev(Y_a \otimes e^a)R_{a^{-1}}(f) = ev(Y_a \otimes e^a.R_{a^{-1}}(f)),$$

for all $f \in C(H)$ and $a \in \mathcal{C} \cap H$. Hence $f\tilde{e}^a = \tilde{e}^a R_{a^{-1}}(f)$, i.e. $\tilde{e}^a f = R_a(f)\tilde{e}^a$. □

This coincides with the usual commutation relations on the calculus. Also, for this particular example, we do indeed have $\tilde{\Omega}^1_A = \Omega^1_A = \{\delta_h \mathrm{d}_A \delta_{h'} \mid h, h' \in H\}_{\text{span}}$, on account of the fact that the graph has no multiple edges or loops, and all calculi on finite sets are of this form.

We continue looking at this example in Section 16.2, where we look at Hermitian metrics and connections.

13.6 Non-classical Example: Point in a Finite Graph

[See Example 3.9 for $C(X)$ and its calculi]

Take $B = C(X)$ for some finite set X, and suppose we make a choice of a finite graph to determine its calculus Ω^1_B. Take $A = C(\{x_0\})$ for x_0 a fixed element of X, and let π be the map $\pi : C(X) \to C(\{x_0\})$ given by restriction of functions to x_0. This is a surjective algebra map with kernel:

$$J = \{f \in C(X) \mid f(x_0) = 0\} = \{f \in C(X) \mid f.\delta_{x_0} = 0\}. \tag{17}$$

A basis of J is then given by δ_x for all $x \in X \backslash \{x_0\}$. Next we calculate the horizontal vector fields. Since we have an actual linear basis of J, condition 2 of Lemma 11.4 holds automatically, and we need only check condition 1.

$$\mathrm{Hor} = \{X \in \mathfrak{X}^R_{C(X)} \mid \pi X(\mathrm{d}j) = 0 \quad \forall j \in J\} = \{X \in \mathfrak{X}^R_{C(X)} \mid X(\mathrm{d}\delta_z)\delta_{x_0} = 0 \quad \forall z \in X\backslash\{x_0\}\}.$$

Hence for each $z \neq x_0$, $X \in \mathrm{Hor}$, we have:

$$0 = \sum_{x\to z} X_{x\to z}\delta_{x,x_0} - \sum_{z\to y} X_{z\to y}\delta_{z,x_0} = X_{x_0\to z} - \sum_{x_0\to y} X_{x_0\to y}$$

But since this holds for all $z, y \in X$, it follows that $X_{x_0\to y} = 0$ for all $y \in X$, i.e. for any arrow out of x_0. Since a horizontal vector field has no $f_{z\leftarrow x_0}$ component for any z, the horizontal vector fields are spanned by $f_{y\leftarrow x}$ for all $y \in X$, $x \in X\backslash\{x_0\}$.

$$\mathrm{Hor} = \{X \in \mathfrak{X}^R_{C(X)} \mid X = \sum_{x\to y, x\neq x_0} X_{x\to y} f_{y\leftarrow x} \text{ for some } X_{x\to y} \in \mathbb{C}\} \tag{18}$$

The tangent bundle is $T_A = C(\{x_0\}) \otimes_\pi \mathrm{Hor}$, and a general element of T_A takes the form $\delta_{x_0} \otimes X$ for some $X \in \mathrm{Hor}$. Since $\pi(\delta_{x_0}) = \delta_{x_0}$ and $\delta_{x_0}.f_{y\leftarrow x} = \delta_{x_0,y} f_{y\leftarrow x}$, we calculate:

$$\delta_{x_0} \otimes \sum_{x\to y, x\neq x_0} X_{x\to y} f_{y\leftarrow x} = 1 \otimes \sum_{x\to y, x\neq x_0} X_{x\to y}\delta_{y,x_0} f_{y\leftarrow x} = 1 \otimes \sum_{x\to x_0} X_{x\to x_0} f_{x_0\leftarrow x}$$

Hence T_A is spanned by $Y_x := 1 \otimes f_{x_0\leftarrow x}$ for each arrow $x \to x_0$. The normal bundle is therefore spanned by $1 \otimes f_{y\leftarrow x}$ for each arrow $x \to y$ for $x \in X$, $y \in X\backslash\{x_0\}$. Since the normal bundle is closed under the right B-action, it is a B-A bimodule. The map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$$

has trivial kernel, since if $\sum_{x\to x_0} \lambda_x.Y_x$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Therefore π is a co-embedding.

Defining $\mathfrak{X}^R_{C(\{x_0\})} := C(\{x_0\})_\pi \otimes_{C(X)} \mathrm{Hor}_{\pi^{-1}}$, the submanifold calculus is

$$\tilde{\Omega}^1_{C(\{x_0\})} = {}_{C(\{x_0\})}\mathrm{Hom}(\mathfrak{X}^R_{C(\{x_0\})}, C(\{x_0\})),$$

which has exterior derivative $\mathrm{d}_A : C(\{x_0\}) \to \tilde{\Omega}^1_{C(\{x_0\})}$ satisfying for arrows $x \to x_0$:

$$ev(Y_x \otimes \mathrm{d}_A\delta_{x_0}) = \pi f_{x_0\leftarrow x}(d(\pi^{-1}(\delta_{x_0}))) = \pi f_{x_0\leftarrow x}(\mathrm{d}\delta_{x_0}) = \pi f_{x_0\leftarrow x}\big(\sum_{w\to x_0}\omega_{w\to x_0} - \sum_{x_0\to y}\omega_{x_0\to y}\big)$$
$$= \pi(\delta_x) = \delta_{x,x_0}\delta_{x_0} = 0.$$

Hence the exterior derivative d_A is the zero map, and $\Omega^1_A = 0$. But if there exist arrows $x \to x_0$, the tangent bundle has a non-trivial basis, and hence so does its dual $\tilde{\Omega}^1_A$. This case is an example where $\tilde{\Omega}^1_A$ is bigger than Ω^1_A, and the submanifold calculus does not satisfy the surjectivity axiom of everything being of the form $a.\mathrm{d}_A a'$.

13.7 Non-classical Example: $\mathbb{C}_q[\mathbb{C}^2]$ in $\mathbb{C}_q[M_2]$

[Algebras: See Example 3.7 for $\mathbb{C}_q[\mathbb{C}^2]$ and its calculus, and Example 3.4 for $\mathbb{C}_q[M_2]$ and its calculus]

Next, we show that the quantum plane $A = \mathbb{C}_q[\mathbb{C}^2]$ is a noncommutative submanifold of the q-deformed matrices $B = \mathbb{C}_q[M_2]$. To the extent of our knowledge, the algebra map used in this example is new.

Recall that $\mathbb{C}_q[\mathbb{C}^2]$ is the algebra with generators x, y and relation $yx = qxy$, while the algebra $\mathbb{C}_q[M_2]$ has generators a, b, c, d and relations:

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \quad da - ad = (q - q^{-1})bc.$$

Note that in $\mathbb{C}_q[M_2]$, unlike in $\mathbb{C}_q[GL_2]$, $\mathbb{C}_q[SL_2]$ or $\mathbb{C}_q[SU_2]$, we make no assumption on the value or invertibility of the determinant $\det_q = ad - q^{-1}bc$.

Page 27 of [5] gives a 1-parameter family of 4D calculi on $\mathbb{C}_q[GL_2]$. Writing α for the free parameter and $\lambda = q - q^{-1}$, the calculus is freely generated by elements e_a, e_b, e_c, e_d and inner by $\theta = e_a + e_d$, which gives the differential. The commutation relations on the generators are:

$$e_a\begin{pmatrix} a & b \\ c & d\end{pmatrix} = q^{2\alpha}\begin{pmatrix} q^2a & b \\ q^2c & d\end{pmatrix}e_a, \quad [e_b, \begin{pmatrix} a & b \\ c & d\end{pmatrix}]_{q^{1+2\alpha}} = q^{1+2\alpha}\lambda\begin{pmatrix} 0 & a \\ 0 & c\end{pmatrix}e_a$$
$$[e_c, \begin{pmatrix} a & b \\ c & d\end{pmatrix}]_{q^{1+2\alpha}} = q^{1+2\alpha}\lambda\begin{pmatrix} b & 0 \\ d & 0\end{pmatrix}e_a, \quad [e_d, \begin{pmatrix} a \\ c\end{pmatrix}]_{q^{2\alpha}} = q^{2\alpha}\lambda\begin{pmatrix} b \\ d\end{pmatrix}e_b$$
$$[e_d, \begin{pmatrix} b \\ d\end{pmatrix}]_{q^{2\alpha}} = q^{2\alpha}\lambda\begin{pmatrix} ae_c+\lambda be_a \\ ce_c+\lambda de_a\end{pmatrix}$$

This notation is a shorthand, so for example one of the relations is

$$e_b.d - q^{1+2\alpha}de_b = q^{1+2\alpha}\lambda ce_a.$$

Lemma 13.13. *The relations on Ω^1_B can be re-written as:*

$$ae_a = q^{-2-2\alpha}e_a.a \qquad be_a = q^{-2\alpha}e_a.b$$
$$ce_a = q^{-2-2\alpha}e_a.c \qquad de_a = q^{-2\alpha}e_a.d$$

$$ae_b = q^{-1-2\alpha}e_b.a \qquad be_b = q^{-1-2\alpha}e_b.b - \lambda q^{-2-2\alpha}e_a.a$$
$$ce_b = q^{-1-2\alpha}e_b.c \qquad de_b = q^{-1-2\alpha}e_b.d - \lambda q^{-2-2\alpha}e_a.c$$

$$ae_c = q^{-1-2\alpha}e_c.a - \lambda q^{-2\alpha}e_a.b \qquad be_c = q^{-1-2\alpha}e_c.b$$
$$ce_c = q^{-1-2\alpha}e_c.c - \lambda q^{-2\alpha}e_a.d \qquad de_c = q^{-1-2\alpha}e_c.d$$

$$ae_d = q^{-2\alpha}e_d.a - \lambda q^{-1-2\alpha}e_b.b + \lambda^2 q^{-2-2\alpha}e_a.a \qquad be_d = q^{-2\alpha}e_d.b - \lambda q^{-1-2\alpha}e_c.a$$
$$ce_d = q^{-2\alpha}e_d.c - \lambda q^{-1-2\alpha}e_b.d + \lambda^2 q^{-2-2\alpha}e_a.c \qquad de_d = q^{-2\alpha}e_d.d - \lambda q^{-1-2\alpha}e_c.c$$

Proof. For brevity we omit the full calculations, but we calculate in the order of e_a through to e_d and re-arrange and substitute until all algebra elements are on the right. □

As the calculus is freely generated, the vector fields are also freely generated, and hence in particular right finitely generated projective. This means there exist $e^a, e^b, e^c, e^d \in \mathfrak{X}^R_B$ such that $e^i(e_j) = \delta_{i,j}$ for $i, j \in \{a, b, c, d\}$, and every vector field $X \in \mathfrak{X}^R_B$ decomposes as $X = \sum_{i\in\{a,b,c,d\}} X_i e^i$ for some $X_i \in B$. This implies $X(e_i) = X_i$.

Lemma 13.14. *The vector fields $\mathfrak{X}^R_{\mathbb{C}_q[M_2]}$ have commutation relations:*

$$e^a.a = q^{-2-2\alpha}a.e^a - \lambda q^{-2\alpha}b.e^c + \lambda^2 q^{-2-2\alpha}a.e^d \qquad e^b.a = q^{-1-2\alpha}a.e^b - \lambda q^{-1-2\alpha}b.e^d$$
$$e^c.a = q^{-1-2\alpha}a.e^c \qquad e^d.a = q^{-2\alpha}a.e^d$$

$$e^a.b = q^{-2\alpha}b.e^a - \lambda q^{-2-2\alpha}a.e^b \qquad e^b.b = q^{-1-2\alpha}b.e^b$$
$$e^c.b = q^{-1-2\alpha}b.e^c - \lambda q^{-1-2\alpha}a.e^d \qquad e^d.b = q^{-2\alpha}b.e^d$$

$$e^a.c = q^{-2-2\alpha}c.e^a - \lambda q^{-2\alpha}d.e^c + \lambda^2 q^{-2-2\alpha}c.e^d \qquad e^b.c = q^{-1-2\alpha}c.e^b - \lambda q^{-1-2\alpha}d.e^d$$

$$e^c.c = q^{-1-2\alpha}c.e^c \qquad e^d.c = q^{-2\alpha}c.e^d$$

$$e^a.d = q^{-2\alpha}d.e^a - \lambda q^{-2-2\alpha}c.e^b \qquad e^b.d = q^{-1-2\alpha}d.e^b$$
$$e^c.d = q^{-1-2\alpha}d.e^c \qquad e^d.d = q^{-2\alpha}d.e^d$$

Proof. We calculate the commutation relations using the fact that $coev(1) = \sum_{i=a,b,c,d} e_i \otimes e^i$ is central, on account of $coev$ being a bimodule map.

(1) Firstly we calculate:

$$\begin{aligned}\sum_i ae_i \otimes e^i &= \Big(q^{-2-2\alpha}e_a.a\Big) \otimes e^a + \Big(q^{-1-2\alpha}e_b.a\Big) \otimes e^b + \Big(q^{-1-2\alpha}e_c.a - \lambda q^{-2\alpha}e_a.b\Big) \otimes e^c \\ &\quad + \Big(q^{-2\alpha}e_d.a - \lambda q^{-1-2\alpha}e_b.b + \lambda^2 q^{-2-2\alpha}e_a.a\Big) \otimes e^d \\ &= e_a \otimes \big(q^{-2-2\alpha}a.e^a - \lambda q^{-2\alpha}b.e^c + \lambda^2 q^{-2-2\alpha}a.e^d\big) + e_b \otimes \big(q^{-1-2\alpha}a.e^b - \lambda q^{-1-2\alpha}b.e^d\big) \\ &\quad + e_c \otimes \big(q^{-1-2\alpha}a.e^c\big) + e_d \otimes \big(q^{-2\alpha}a.e^d\big)\end{aligned}$$

Equating this with $\sum_i e_i \otimes e^i.a$ and comparing the RHS of the tensor product gives the equations above for moving a from right to left.

(2) Next we calculate:

$$\begin{aligned}\sum_i be_i \otimes e^i &= \Big(q^{-2\alpha}e_a.b\Big) \otimes e^a + \Big(q^{-1-2\alpha}e_b.b - \lambda q^{-2-2\alpha}e_a.a\Big) \otimes e^b + \Big(q^{-1-2\alpha}e_c.b\Big) \otimes e^c \\ &\quad + \Big(q^{-2\alpha}e_d.b - \lambda q^{-1-2\alpha}e_c.a\Big) \otimes e^d \\ &= e_a \otimes \big(q^{-2\alpha}b.e^a - \lambda q^{-2-2\alpha}a.e^b\big) + e_b \otimes \big(q^{-1-2\alpha}b.e^b\big) \\ &\quad + e_c \otimes \big(q^{-1-2\alpha}b.e^c - \lambda q^{-1-2\alpha}a.e^d\big) + e_d \otimes \big(q^{-2\alpha}b.e^d\big)\end{aligned}$$

Equating this with $\sum_i e_i \otimes e^i.b$ and comparing the RHS of the tensor product gives the equations above for moving b from right to left.

(3) Next we calculate:

$$\begin{aligned}\sum_i ce_i \otimes e^i &= \Big(q^{-2-2\alpha}e_a.c\Big) \otimes e^a + \Big(q^{-1-2\alpha}e_b.c\Big) \otimes e^b + \Big(q^{-1-2\alpha}e_c.c - \lambda q^{-2\alpha}e_a.d\Big) \otimes e^c \\ &\quad + \Big(q^{-2\alpha}e_d.c - \lambda q^{-1-2\alpha}e_b.d + \lambda^2 q^{-2-2\alpha}e_a.c\Big) \otimes e^d \\ &= e_a \otimes \big(q^{-2-2\alpha}c.e^a - \lambda q^{-2\alpha}d.e^c + \lambda^2 q^{-2-2\alpha}c.e^d\big) \\ &\quad + e_b \otimes \big(q^{-1-2\alpha}c.e^b - \lambda q^{-1-2\alpha}d.e^d\big)\end{aligned}$$

$$+ e_c \otimes \left(q^{-1-2\alpha} c.e^c\right) + e_d \otimes \left(q^{-2\alpha} c.e^d\right)$$

Equating this with $\sum_i e_i \otimes e^i.c$ and comparing the RHS of the tensor product gives the equations above for moving c from right to left.

(4) Lastly we calculate:

$$\sum_i de_i \otimes e^i = \left(q^{-2\alpha} e_a.d\right) \otimes e^a + \left(q^{-1-2\alpha} e_b.d - \lambda q^{-2-2\alpha} e_a.c\right) \otimes e^b + \left(q^{-1-2\alpha} e_c.d\right) \otimes e^c$$

$$+ \left(q^{-2\alpha} e_d.d - \lambda q^{-1-2\alpha} e_c.c\right) \otimes e^d$$

$$= e_a \otimes \left(q^{-2\alpha} d.e^a - \lambda q^{-2-2\alpha} c.e^b\right) + e_b \otimes \left(q^{-1-2\alpha} d.e^b\right)$$

$$+ e_c \otimes \left(q^{-1-2\alpha} d.e^c\right) + e_d \otimes \left(q^{-2\alpha} d.e^d\right)$$

Equating this with $\sum_i e_i \otimes e^i.d$ and comparing the RHS of the tensor product gives the equations above for moving d from right to left. □

Provided that the q-determinant $\det_q = ad - q^{-1}bc$ is not invertible (i.e. we are not in the GL_2 or SL_2 case), there is a surjective algebra map

$$\pi : \mathbb{C}_q[M_2] \to \mathbb{C}_q[\mathbb{C}^2], \qquad \pi(a) = x, \quad \pi(c) = y, \quad \pi(b) = \pi(d) = 0. \tag{19}$$

Since we can use the commutation relations to put b and d on the left of every term, the kernel J of π is $bB + dB$.

Proposition 13.15. *Horizontal vector fields $X \in \mathrm{Hor}$ are characterised by $\pi(X_a) = 0$ and $\pi(X_c) = 0$.*

Proof. In parts 1 and 2 of this proof we calculate the implications of condition 1 of Lemma 36, and in part 3 we calculate the implications of its condition 2.

(1) Firstly we calculate $\pi X(\mathrm{d}b)$. Using that the calculus is inner by $\theta = e_a + e_d$, we calculate $X(\mathrm{d}b)$ as:

$$X(\mathrm{d}b) = X([\theta, b]) = X(e_a.b + e_d.b - be_a - be_d)$$

$$= X_a.b + X_d.b - X(q^{-2\alpha} e_a.b + q^{-2\alpha} e_d.b - \lambda q^{-1-2\alpha} e_c.a)$$

$$= X_a.b + X_d.b - q^{-2\alpha} X_a.b - q^{-2\alpha} X_d.b + \lambda q^{-1-2\alpha} X_c.a$$

$$= (1 - q^{-2\alpha}) X_a.b + (1 - q^{-2\alpha}) X_d.b + \lambda q^{-1-2\alpha} X_c.a$$

The map π sends every term containing b to zero, and so $\pi X(\mathrm{d}b) = \lambda q^{-1-2\alpha}\pi(X_c).x$ For X to be horizontal, we therefore need $0 = \lambda q^{-1-2\alpha}\pi(X_c).x$. Hence $\pi(X_c) = 0$.

(2) Next we calculate $\pi X(\mathrm{d}d)$ as:

$$\begin{aligned} X(\mathrm{d}d) &= X([\theta, d]) = X(e_a.d + e_d.d - de_a - de_d) \\ &= X_a.d + X_d.d - X(q^{-2\alpha}e_a.d + q^{-2\alpha}e_d.d - \lambda q^{-1-2\alpha}e_c.c) \\ &= (1 - q^{-2\alpha})X_a.d + (1 - q^{-2\alpha})X_d.d + \lambda q^{-1-2\alpha}X_c.c \end{aligned}$$

The map π sends every term containing d to zero, so we get $\pi X(\mathrm{d}d) = \lambda q^{-1-2\alpha}\pi(X_c).y$, and for X to be horizontal we need this to vanish, so $\pi(X_c) = 0$.

(3) Lastly, we calculate the implications of condition 2 of Lemma 11.4. Using the commutation relations on the vector fields from Lemma 13.14, we first calculate:

$$\begin{aligned} Xb = \sum_i X_i e^i.b &= X_a\big(q^{-2\alpha}b.e^a - \lambda q^{-2-2\alpha}a.e^b\big) + X_b\big(q^{-1-2\alpha}b.e^b\big) \\ &\quad + X_c\big(q^{-1-2\alpha}b.e^c - \lambda q^{-1-2\alpha}a.e^d\big) + X_d\big(q^{-2\alpha}b.e^d\big) \\ &= \big(X_a q^{-2\alpha}b\big)e^a + \big(-X_a\lambda q^{-2-2\alpha}a + X_b q^{-1-2\alpha}b\big)e^b \\ &\quad + \big(X_c q^{-1-2\alpha}b\big)e^c + \big(-X_c\lambda q^{-1-2\alpha}a + X_d q^{-2\alpha}b\big)e^d. \end{aligned}$$

Every coefficient containing b or d is automatically in J since J is a two-sided ideal, and for the coefficients $X_a\lambda q^{-2-2\alpha}a$ and $X_c\lambda q^{-1-2\alpha}a$ to be in J, we require $X_a \in J$ and $X_c \in J$. Next we calculate:

$$\begin{aligned} Xd &= \sum_i X_i e^i.d \\ &= X_a\big(q^{-2\alpha}d.e^a - \lambda q^{-2-2\alpha}c.e^b\big) + X_b\big(q^{-1-2\alpha}d.e^b\big) + X_c\big(q^{-1-2\alpha}d.e^c\big) + X_d\big(q^{-2\alpha}d.e^d\big) \\ &= \big(X_a q^{-2\alpha}d\big)e^a + \big(-X_a\lambda q^{-2-2\alpha}c + X_b q^{-1-2\alpha}d\big)e^b + \big(X_c q^{-1-2\alpha}d\big)e^c + \big(X_d q^{-2\alpha}d\big)e^d \end{aligned}$$

For the coefficient $X_a\lambda q^{-2-2\alpha}c$ to be in J, we require $X_a \in J$. Thus the total set of conditions on horizontal vector fields is $X_a, X_c \in J$. □

It follows that the tangent bundle T_A as a left A-module has two generators $1 \otimes e^b$ and $1 \otimes e^d$. Since all right actions on the vector fields e^b and e^d produce only multiples of e^b and e^d (we can read this off the right column of Lemma 13.14), so T_A is closed under the right B-action, making it an A-B bimodule. A complement to this, and therefore a normal bundle N_A, is spanned by the two generators $1 \otimes e^a$ and $1 \otimes e^c$. This is a left A-module, but we can read off the left column of Lemma 13.14 that it is not closed under the right action, and so N_A is not a right B-module. However, we don't encounter any problems with injectivity of the map

$$\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B,$$

since if $f(x,y) \otimes e^b + g(x,y) \otimes e^d$ is nonzero in $A_\pi \otimes_B \mathrm{Hor}$ then it is also nonzero in $A_\pi \otimes_B \mathfrak{X}^R_B$. Therefore unlike previous examples, the map $\pi : B \to A$ here gives a weak coembedding. We remind the reader that a weak co-embedding is when the sequence splits only as a sequence of left A-modules, not as B-A bimodules. This is still sufficient to get a submanifold calculus, which we calculate in the remainder of this example, but it does mean that we do not know how to project connections to that submanifold calculus. The vector fields on $A = \mathbb{C}_q[\mathbb{C}^2]$ are defined as $\mathfrak{X}^R_A := A_\pi \otimes_B \mathrm{Hor}_{\pi^{-1}}$, and we calculate their commutation relations as follows:

Proposition 13.16. *The vector fields $\mathfrak{X}^R_A$ have commutation relations:*

$$(1 \otimes e^d).x = q^{-2\alpha} x.(1 \otimes e^d), \qquad (1 \otimes e^d).y = q^{-2\alpha} y.(1 \otimes e^d)$$
$$(1 \otimes e^b).x = q^{-1-2\alpha} x.(1 \otimes e^b), \qquad (1 \otimes e^b).y = q^{-1-2\alpha} y.(1 \otimes e^b)$$

Proof. Using the relations $e^d.a = q^{-2\alpha} a.e^d$ and $e^d.c = q^{-2\alpha} c.e^d$ and $e^b.a = q^{-1-2\alpha} a.e^b - \lambda q^{-1-2\alpha} b.e^d$ and $e^b.c = q^{-1-2\alpha} c.e^b - \lambda q^{-1-2\alpha} d.e^d$, we calculate:

$$(1 \otimes e^d).x = 1 \otimes e^d.\pi^{-1}(x) = 1 \otimes e^d.a = q^{-2\alpha} 1 \otimes a.e^d = q^{-2\alpha} 1.\pi(a) = q^{-2\alpha} x.(1 \otimes e^d),$$

and

$$(1 \otimes e^d).y = 1 \otimes e^d.c = 1 \otimes q^{-2\alpha} c.e^d = q^{-2\alpha} y.(1 \otimes e^d),$$

and

$$(1 \otimes e^b).x = 1 \otimes e^b.a = 1 \otimes \left(q^{-1-2\alpha} a.e^b - \lambda q^{-1-2\alpha} b.e^d\right) = q^{-1-2\alpha} x.(1 \otimes e^b) - 0,$$

and

$$(1 \otimes e^b).y = 1 \otimes e^b.c = 1 \otimes \left(q^{-1-2\alpha} c.e^b - \lambda q^{-1-2\alpha} d.e^d\right) = q^{-1-2\alpha} y.(1 \otimes e^b) - 0.$$

□

The submanifold calculus is given by $\tilde{\Omega}^1_A := {}_A\mathrm{Hom}(\mathfrak{X}^{\mathrm{R}}_{\mathrm{A}}, \mathrm{A})$, and we calculate the differential and commutation relations as follows.

Proposition 13.17. *The submanifold calculus has differential* $\mathrm{d}_A : A \to \Omega^1_A$ *given by* $\mathrm{d}_A x$ *and* $\mathrm{d}_A y$*, satisfying* $ev((1\otimes e^d)\otimes \mathrm{d}_A x) = (1-q^{-2\alpha})x$ *and* $ev((1\otimes e^d)\otimes \mathrm{d}_A y) = (1-q^{-2\alpha})y$*, and* $ev((1\otimes e^b)\otimes \mathrm{d}_A x) = ev((1\otimes e^b)\otimes \mathrm{d}_A y) = 0$*.*

Proof. Recall that the differential $\mathrm{d}_A : A \to \Omega^1_A$ is given for $X \in \mathrm{Hor}$ and $\alpha, \alpha' \in A$ by:

$$ev((\alpha\otimes X)\otimes \mathrm{d}_A\alpha') = \alpha.\pi X(\mathrm{d}(\pi^{-1}\alpha')).$$

(1) Using this formula, we calculate:

$$\begin{aligned} ev((1\otimes e^b)\otimes \mathrm{d}_A x) &= \pi e^b(\mathrm{d}\pi^{-1}(x)) = \pi e^b(\mathrm{d}a) = \pi e^b([\theta,a]) = \pi e^b\big((e_a+e_d)a - a(e_a+e_d)\big) \\ &= -\pi e^b(a.e_a + a.e_d) \\ &= -\pi e^b(q^{-2-2\alpha}e_a.a + q^{-2\alpha}e_d.a - \lambda q^{-1-2\alpha}e_b.b + \lambda^2 q^{-2-2\alpha}e_a.a) \\ &= \lambda q^{-1-2\alpha}e^b(e_b).\pi(b) = 0 \end{aligned}$$

(2) Likewise we calculate

$$\begin{aligned} ev((1\otimes e^b)\otimes \mathrm{d}_A y) &= \pi e^b([\theta,c]) = \pi e^b\big((e_a+e_d)c - c(e_a+e_d)\big) \\ &= -\pi e^b(c.e_a + c.e_d) \\ &= -\pi e^b(q^{-2-2\alpha}e_a.c + q^{-2\alpha}e_d.c - \lambda q^{-1-2\alpha}e_b.d + \lambda^2 q^{-2-2\alpha}e_a.c) \\ &= \lambda q^{-1-2\alpha}e^b(e_b)\pi(d) = 0 \end{aligned}$$

(3) Next we calculate:

$$\begin{aligned} ev((1\otimes e^d)\otimes \mathrm{d}_A x) &= \pi e^d([\theta,a]) = \pi e^d\big((e_a+e_d)a - a(e_a+e_d)\big) \\ &= \pi e^d(e_d).\pi(a) - \pi e^d(a.e_a + a.e_d) \\ &= x - \pi e^d(q^{-2-2\alpha}e_a.a + q^{-2\alpha}e_d.a - \lambda q^{-1-2\alpha}e_b.b + \lambda^2 q^{-2-2\alpha}e_a.a) \\ &= x - q^{-2\alpha}e^d(e_d).\pi(a) = (1-q^{-2\alpha})x \end{aligned}$$

(4) Lastly we calculate:

$$ev((1\otimes e^d)\otimes \mathrm{d}_A y) = \pi e^d([\theta,c]) = \pi e^d\big((e_a+e_d)c - c(e_a+e_d)\big)$$

$$= y - \pi e^d(q^{-2-2\alpha}e_a.c + q^{-2\alpha}e_d.c - \lambda q^{-1-2\alpha}e_b.d + \lambda^2 q^{-2-2\alpha}e_a.c)$$
$$= (1-q^{-2\alpha})y$$

□

Remark 13.18. *The paper [5] giving the calculus on $\mathbb{C}[M_2]$ states that in the case $\alpha = -\frac{1}{2}$ the calculus descends to the quotient $\det_q = 1$, giving the standard 4D calculus on $\mathbb{C}_q[SL_2]$. In our example, this would give $ev((1 \otimes e^d) \otimes \mathrm{d}_A x) = (1-q)x$ and $ev((1 \otimes e^d) \otimes \mathrm{d}_A y) = (1-q)y$. Thus in the case $\alpha = -\frac{1}{2}$ and $q = 1$, the submanifold calculus comes out with zero differential.*

One known example of a covariant calculus, as found in Example 2.79 of [10] on $\mathbb{C}_q[\mathbb{C}^2]$ is generated by $\mathrm{d}x$ and $\mathrm{d}y$ subject to the following commutation relations:

$$\mathrm{d}x.x = q^2 x\mathrm{d}x, \quad \mathrm{d}x.y = qy\mathrm{d}x, \quad \mathrm{d}y.x = qx\mathrm{d}y + (q^2-1)y\mathrm{d}x, \quad \mathrm{d}y.y = q^2 y\mathrm{d}y.$$

Proposition 13.19. *The submanifold calculus on $A = \mathbb{C}_q[\mathbb{C}^2]$ has commutation relations:*

$$\mathrm{d}_A x.x = q^{2\alpha} x\mathrm{d}_A x, \qquad \mathrm{d}_A x.y = q^{-1+2\alpha} y\mathrm{d}_A x$$
$$\mathrm{d}_A y.x = q^{1+2\alpha} x\mathrm{d}_A y, \qquad \mathrm{d}_A y.y = q^{2\alpha} y\mathrm{d}_A y$$
$$\mathrm{d}_A y.x = q\mathrm{d}_A x.y$$

In the case $\alpha = 1$, these imply those of the standard calculus.

Proof. **(1)** We use $(1 \otimes e^d).x = q^{-2\alpha}x.(1 \otimes e^d)$ and $ev((1 \otimes e^d) \otimes \mathrm{d}_A x) = (1-q^{-2\alpha})x$ to calculate:

$$\begin{aligned} ev((1 \otimes e^d) \otimes x.\mathrm{d}_A x) &= ev((1 \otimes e^d).x \otimes \mathrm{d}_A x) \\ &= q^{-2\alpha}x.ev((1 \otimes e^d) \otimes \mathrm{d}_A x) \\ &= q^{-2\alpha}x.((1-q^{-2\alpha})x) = ((1-q^{-2\alpha})x).q^{-2\alpha}x \\ &= ev((1 \otimes e^d) \otimes \mathrm{d}_A x).q^{-2\alpha}x \end{aligned}$$

Hence $x\mathrm{d}_A x = q^{-2\alpha}\mathrm{d}_A x.x$, which re-arranges to $\mathrm{d}_A x.x = q^{2\alpha}x\mathrm{d}_A x$

(2) We use $(1 \otimes e^d).y = q^{-2\alpha}y.(1 \otimes e^d)$ and $ev((1 \otimes e^d) \otimes \mathrm{d}_A x) = (1-q^{-2\alpha})x$ to calculate:

$$ev((1 \otimes e^d) \otimes y.\mathrm{d}_A x) = ev((1 \otimes e^d).y \otimes \mathrm{d}_A x)$$

$$
\begin{aligned}
&= q^{-2\alpha} y.ev((1 \otimes e^d) \otimes \mathrm{d}_A x) \\
&= q^{-2\alpha} y\big((1 - q^{-2\alpha})x\big) \\
&= \big((1 - q^{-2\alpha})x\big) q^{1-2\alpha} y \\
&= ev((1 \otimes e^d) \otimes \mathrm{d}_A x) q^{1-2\alpha} y
\end{aligned}
$$

Hence $y\mathrm{d}_A x = q^{1-2\alpha}\mathrm{d}_A x.y$, which re-arranges to $\mathrm{d}_A x.y = q^{-1+2\alpha} y \mathrm{d}_A x$.

(3) We use $(1 \otimes e^d).x = q^{-2\alpha} x.(1 \otimes e^d)$ and $ev((1 \otimes e^d) \otimes \mathrm{d}_A y) = (1 - q^{-2\alpha})y$ to calculate:

$$
\begin{aligned}
ev((1 \otimes e^d) \otimes x.\mathrm{d}_A y) &= ev((1 \otimes e^d).x \otimes \mathrm{d}_A y) \\
&= q^{-2\alpha} x.ev((1 \otimes e^d) \otimes \mathrm{d}_A y) \\
&= q^{-2\alpha} x.\big((1 - q^{-2\alpha})y\big) \\
&= \big((1 - q^{-2\alpha})y\big).q^{-1-2\alpha} y \\
&= ev((1 \otimes e^b) \otimes \mathrm{d}_A y).q^{-1-2\alpha} y
\end{aligned}
$$

Hence $x\mathrm{d}_A y = q^{-1-2\alpha}\mathrm{d}_A y.x$, which re-arranges to $\mathrm{d}_A y.x = q^{1+2\alpha} x \mathrm{d}_A y$.

(4) We use $(1 \otimes e^d).y = q^{-2\alpha} y.(1 \otimes e^d)$ and $ev((1 \otimes e^d) \otimes \mathrm{d}_A y) = (1 - q^{-2\alpha})y$ to calculate:

$$
\begin{aligned}
ev((1 \otimes e^d) \otimes y.\mathrm{d}_A y) &= ev((1 \otimes e^d)y. \otimes \mathrm{d}_A y) \\
&= q^{-2\alpha} y.ev((1 \otimes e^d) \otimes \mathrm{d}_A y) \\
&= q^{-2\alpha} y.\big((1 - q^{-2\alpha})y\big) \\
&= \big((1 - q^{-2\alpha})y\big).q^{-2\alpha} y \\
&= ev((1 \otimes e^b) \otimes \mathrm{d}_A y).q^{-2\alpha} y.
\end{aligned}
$$

Hence $y\mathrm{d}_A y = q^{-2\alpha}\mathrm{d}_A y.y$, which re-arranges to $\mathrm{d}_A y.y = q^{2\alpha} y \mathrm{d}_A y$.

(5) Next we show the relation $\mathrm{d}_A y.x = q\mathrm{d}_A x.y$, which has no corresponding version in the standard calculus on $\mathbb{C}_q[\mathbb{C}^2]$.

$$
\begin{aligned}
ev((1 \otimes e^d) \otimes (\mathrm{d}_A y.x - q\mathrm{d}_A x.y)) &= ev((1 \otimes e^d) \otimes \mathrm{d}_A y).x - q ev((1 \otimes e^d) \otimes \mathrm{d}_A x).y \\
&= (1 - q^{-2\alpha})(yx - qxy) = 0.
\end{aligned}
$$

(6) Lastly, we show that in the case $\alpha = 1$, these imply the relations of the standard calculus on $\mathbb{C}_q[\mathbb{C}^2]$. For the relations $\mathrm{d}_A x.x = q^{2\alpha} x \mathrm{d}_A x$ and $\mathrm{d}_A x.y = q^{-1+2\alpha} y \mathrm{d}_A x$ and $\mathrm{d}_A y.y = q^{2\alpha} y \mathrm{d}_A y$ this is easily seen, but the relation

$$
\mathrm{d}_A y.x = q^{1+2\alpha} x \mathrm{d}_A y
$$

is not so easily seen to imply $\mathrm{d}_A y.x = qx\mathrm{d}_A y + (q^2-1)y\mathrm{d}_A x$. But if we calculate the difference between them as:

$$qx\mathrm{d}_A y + (q^2-1)y\mathrm{d}_A x - q^3 x\mathrm{d}_A y = (q^2-1)\big(y\mathrm{d}_A x - qx\mathrm{d}_A y\big),$$

and then evaluate on $(1\otimes e^d)$, we get:

$$ev\Big((1\otimes e^d)\otimes(q^2-1)\big(y\mathrm{d}_A x - qx\mathrm{d}_A y\big)\Big) = (q^2-1)(1-q^{-2})(yx-qxy) = 0.$$

□

Thus we obtain a family of calculi with parameter α, where the case $\alpha = 1$ gives a quotient of the standard calculus on $\mathbb{C}_q[\mathbb{C}^2]$.

Having done lots of calculations to get here, we now check via the following proposition that this calculus does indeed preserve the relation $yx - qxy = 0$ on $\mathbb{C}_q[\mathbb{C}^2]$.

Proposition 13.20. *This submanifold calculus satisfies* $\mathrm{d}_A(yx - qxy) = 0$.

Proof. Using the Leibniz rule and then the commutation relations $y\mathrm{d}_A x = q^{1-2\alpha}\mathrm{d}_A x.y$ and $x\mathrm{d}_A y = q^{-1-2\alpha}\mathrm{d}_A y.x$ to move algebra elements to the right, we calculate:

$$\begin{aligned}\mathrm{d}_A(yx-qxy) &= \mathrm{d}_A y.x + y\mathrm{d}_A x - q\mathrm{d}_A x.y - qx\mathrm{d}_A y\\ &= \mathrm{d}_A y.x + q^{1-2\alpha}\mathrm{d}_A x.y - q\mathrm{d}_A x.y - q^{-2\alpha}\mathrm{d}_A y.x\end{aligned}$$

This is determined by its evaluation on e^d, so we substitute in $ev((1\otimes e^d)\otimes \mathrm{d}_A x) = (1-q^{-2\alpha})x$ and $ev((1\otimes e^d)\otimes \mathrm{d}_A y) = (1-q^{-2\alpha})y$ to get:

$$(1-q^{-2\alpha})\big(yx - q^{1-2\alpha}xy - qxy - q^{-2\alpha}yx\big) = (1-q^{-2\alpha})\big((yx-qxy)+q^{-2\alpha}(yx-qxy)\big) = 0$$

□

This is an example where $\tilde{\Omega}^1_A \neq \Omega^1_A$, since the evaluation of $\mathrm{d}_A x$ and $\mathrm{d}_A y$ on $(1\otimes e^b)$ is zero, so the span of d_A is smaller than the dual of $\mathfrak{X}^R_A$. This example is different to our other examples, in the sense that not only is it a weak co-embedding, but also that we chose quite a strange algebra map that is only an algebra map when the quantum determinant is not invertible, and ended up getting a quotient of the usual calculus on A as a result.

14 Non-examples

In this section we look at three cases where the definition is not met and what goes wrong. These are tabulated below. Note that the notation $C^\infty(\mathbb{R}^2)\big|_+$ denotes the restriction of $C^\infty(\mathbb{R}^2)$ to the union of the x and y axes (which is not a manifold).

B	**A**	**What goes wrong**
Certain algebras	$\mathbb{C}$	Not every algebra has a surjective algebra map to $\mathbb{C}$.
$C^\infty(\mathbb{R}^2)$	$C^\infty(\mathbb{R}^2)\big|_{\text{axes}}$	The normal bundle fails to be a left A-module.
$\mathbb{C}_q[S^1]$	$\mathbb{C}_q[S^2]$	There is no algebra map between these.

Table 2: Submanifolds Non-Examples

14.1 Classical Counterexample: A Point in Some Algebras

If we regard the algebra $\mathbb{C}$ as corresponding to a point, one might hope for it to have a co-embedding into any associative unital algebra. However, this would require a surjective algebra map $\pi : B \to \mathbb{C}$, which not every algebra has. If B is a Hopf algebra over the complex numbers then there is a counit $\epsilon : B \to \mathbb{C}$ which is an algebra map, but it may not be surjective, since there are some Hopf algebras with ϵ sending everything to zero.
If there does exist a surjective algebra map $\pi : B \to \mathbb{C}$, what we can say is that $\pi(1) = 1$ and the kernel doesn't contain any element of B with a multiplicative inverse.
The lack of surjective algebra maps $\pi : B \to \mathbb{C}$ in general indicates that it might be worth looking at more general types of maps than algebra maps, such as completely positive maps, since states on algebras are much more plentiful.

14.2 Classical Counterexample: Union of Axes in $\mathbb{R}^2$

We borrow this example from [21], which didn't satisfy D'Andrea's definition of a co-embedding, and show that things also go wrong under our definition. Take $B = \mathbb{R}^2$, and $A = C^\infty(\mathbb{R}^2)\big|_{\text{axes}}$, denoting the restriction of smooth functions on $\mathbb{R}^2$ to the union of the x and y axes. This set is not a smooth manifold, because of the point $(0, 0)$. The restriction map $\pi : B \to A$ is a surjective algebra map. Its kernel J consists of functions vanishing on both axes, and is therefore generated by xy.

Each vector field $X \in \mathfrak{X}_B^R$ satisfies $X(\mathrm{d}(xy)) = f_x \frac{\partial(xy)}{\partial x} + f_y \frac{\partial(xy)}{\partial y} = f_x y + f_y x$. Horizontal vector fields satisfy $X(\mathrm{d}(xy)) \in J$, meaning that $f_x y + f_y x$ must vanish on both axes, which in turn means $y f_x(0,y) = 0$ and $x f_y(x,0) = 0$ for all $x, y \in \mathbb{R}$. By continuity of f_x and f_y, it follows that $f_x(0,y) = 0$ and $f_y(x,0) = 0$ for all $x, y \in \mathbb{R}$.
A general element of $T_A = A_\pi \otimes_B \mathrm{Hor}$ then takes the form

$$g \otimes X = g \otimes (f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y}) = g\pi(f_x) \otimes \frac{\partial}{\partial x} + g\pi(f_y) \otimes \frac{\partial}{\partial y}.$$

We see that $g\pi(f_x)$ vanishes on the y-axis, and $g\pi(f_y)$ vanishes on the x-axis. This means that $g\pi(f_x)$ is divisible by x, and that $g\pi(f_y)$ by y. Thus T_A is generated as a left A-module by $x \otimes \frac{\partial}{\partial x}$ and $y \otimes \frac{\partial}{\partial y}$. However, we do not have a co-embedding, or even a weak co-embedding, because we cannot construct a normal bundle.

Proposition 14.1. *In this example, the normal bundle fails to be a left A-module.*

Proof. For A to be a noncommutative submanifold of B, we need the existence of an A-B module N_A such that $A_\pi \otimes_B \mathfrak{X}_B^R = T_A \oplus N_A$. We can see that $1 \otimes \frac{\partial}{\partial x}$ is not of the form $(A.x \otimes \frac{\partial}{\partial x} + A.y \otimes \frac{\partial}{\partial y})$, and thus it can be expressed as a sum $1 \otimes \frac{\partial}{\partial x} = h_x + n_x$, where $h_x \in A_\pi \otimes_B \mathrm{Hor}$, and $n_x \in N_A$. Multiplying on the left by x gives $x \otimes \frac{\partial}{\partial x} = x h_x + x n_x$, which lies in $A_\pi \otimes_B \mathrm{Hor}$, meaning that the component $x n_x \in N_A$ is equal to zero, using the fact that N_A is a left A-module. This implies that n_x vanishes on the x-axis minus zero, and therefore by continuity that it vanishes on the entire x-axis. However, we had $1 \otimes \frac{\partial}{\partial x} = h_x + n_x$, so if we restrict this to the x-axis then we get $h_x = 1 \otimes \frac{\partial}{\partial x}$. But this isn't of the form $(A.x \otimes \frac{\partial}{\partial x} + A.y \otimes \frac{\partial}{\partial y})$. This gives a contradiction with our assumption N_A was a left A-module. □

Although there is no normal bundle, we can still go ahead and calculate a submanifold calculus, but we find that it differs from the classical calculus on A.

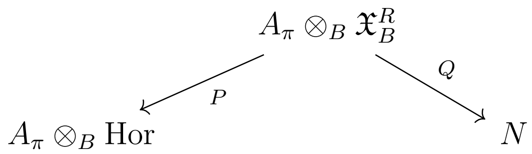
Proposition 14.2. *In this example, the submanifold calculus $\tilde{\Omega}_A^1$ contains elements not found in the classical case.*

Proof. For a general element $\xi \in \tilde{\Omega}_A^1$, we have $\xi(1 \otimes \frac{\partial}{\partial x}) = f$ for some $f \in A$. We then define $\mathrm{d}_A x$ dual to $1 \otimes \frac{\partial}{\partial x}$ by $\xi = f.\mathrm{d}_A x$. Then we have $\xi(x \otimes \frac{\partial}{\partial x}) = x\xi(1 \otimes \frac{\partial}{\partial x}) = xf$. We likewise have $\xi(y \otimes \frac{\partial}{\partial x}) =: yf$.

However, observe that $\eta = \frac{1}{x}.\mathrm{d}x$ gives a valid element of $\tilde{\Omega}^1_A$, seeing as $\eta(\pi(f_x) \otimes \frac{\partial}{\partial x}) = \frac{\pi(f_x)}{x} \in A$ for instance. This shows that the calculus $\tilde{\Omega}^1_A$ does not take the form $A.\mathrm{d}x$, and thus is not equal to the restriction of the classical case. $\square$

15 Restricting Connections - Gauss Equation for Curvature

Definition 15.1. If B has calculus Ω^1_B and a surjective algebra map $\pi : B \to A$ is a co-embedding — i.e. the map $\mathrm{id} \otimes \mathrm{inc} : A_\pi \otimes_B \mathrm{Hor} \to A_\pi \otimes_B \mathfrak{X}^R_B$ is injective and there is a splitting $A_\pi \otimes_B \mathfrak{X}^R_B = (A_\pi \otimes_B \mathrm{Hor}) \oplus N$ as A-B bimodules for some $N \subset A_\pi \otimes_B \mathfrak{X}^R_B$ complementary to the tangent bundle — then we denote the projections to the tangent and normal bundles as P and Q respectively.



By splitting of the exact sequence, these are A-B bimodule maps satisfying $P + Q = \mathrm{id}$.

Proposition 15.2. *Given a co-embedding $\pi : B \to A$ and a left connection $\nabla_B : \mathfrak{X}^R_B \to \Omega^1_B \otimes_B \mathfrak{X}^R_B$, there is a left connection $\nabla'_B : A_\pi \otimes_B \mathfrak{X}^R_B \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathfrak{X}^R_B)$ defined by*

$$\nabla'_B(a \otimes X) = \mathrm{d}a \otimes 1 \otimes X + a(\pi \otimes 1 \otimes \mathrm{id})\nabla_B X \tag{20}$$

or equivalently

$$\nabla'_B(1 \otimes X) = (\pi \otimes 1 \otimes \mathrm{id})\nabla_B X \tag{21}$$

for $a \in A$, $x \in \mathfrak{X}^R_B$. The curvature of ∇'_B is given by $R'_B(1 \otimes X) = (\pi \otimes 1 \otimes \mathrm{id})R_B(X)$.

Proof. **(1)** Firstly, we show ∇'_B is well-defined over $\otimes_B$. Since π is surjective, there exists $b \in B$ such that $\pi(b) = a$. We calculate

$$\begin{aligned}\nabla'_B(1 \otimes bX) &= (\pi \otimes 1 \otimes \mathrm{id})\nabla_B(bX) = (\pi \otimes 1 \otimes \mathrm{id})(\mathrm{d}b \otimes X + b.\nabla_B(X)) \\ &= \mathrm{d}a \otimes 1 \otimes X + a(\pi \otimes 1 \otimes \mathrm{id})\nabla_B X = \nabla'_B(a \otimes X)\end{aligned}$$

as required. This also shows the left Leibniz rule for connections.

(2) Next we show the formula for R'_B. Denoting $\nabla_B e = \xi \otimes Y$ and $\nabla_B f = \eta \otimes Z$, then:

$$\begin{aligned} R'_B(1 \otimes X) &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla'_B)\nabla'_B(1 \otimes X) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla'_B)(\pi(\xi) \otimes 1 \otimes Y) \\ &= \mathrm{d}\pi(\xi) \otimes 1 \otimes Y - \pi(\xi) \wedge \nabla'_B(1 \otimes Y) = \mathrm{d}\pi(\xi) \otimes 1 \otimes Y - \pi(\xi) \wedge \pi(\eta) \otimes 1 \otimes Z \\ &= (\pi \otimes 1 \otimes \mathrm{id})(\mathrm{d}\xi \otimes Y - \mathrm{id} \wedge \nabla_B)\nabla_B(X) = (\pi \otimes 1 \otimes \mathrm{id})R_B(X). \end{aligned}$$

□

To clarify this notation, if $\nabla_B X = \xi \otimes f$, then $\nabla'_B(1 \otimes X) = \pi(\xi) \otimes 1 \otimes f$.
Similarly to classical geometry, the restriction of a connection to the tangent bundle of a submanifold splits into a sum of a connection on the submanifold plus the second fundamental form.

Proposition 15.3. *If we restrict the domain of ∇'_B to the tangent bundle, then for each $e \in A_\pi \otimes_B \mathrm{Hor}$ there is a splitting*

$$\nabla'_B(e) = \nabla_A(e) + \alpha(e) \tag{22}$$

for the left connection

$$\nabla_A = (\mathrm{id} \otimes P)\nabla'_B : A_\pi \otimes_B \mathrm{Hor} \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathrm{Hor}) \tag{23}$$

and the left A-module map

$$\alpha = (\mathrm{id} \otimes Q)\nabla'_B : A_\pi \otimes_B \mathrm{Hor} \to \Omega^1_A \otimes_A N. \tag{24}$$

Proof. The splitting is obvious from $P + Q = \mathrm{id}$ and expressing the connection as

$$\nabla'_B(1 \otimes X) = (\pi^* \otimes (P + Q)(1 \otimes -))\nabla_B(X),$$

but we need to show that ∇_A is a left connection and α a left A-module map. Using that ∇'_B is a connection, we show these as

$$\nabla_A(ae) = (\mathrm{id} \otimes P)(\mathrm{d}a \otimes e + a.\nabla'_B(e)) = \mathrm{d}a \otimes e + a.\nabla_A(e)$$

and

$$\alpha(ae) = (\mathrm{id} \otimes Q)(\mathrm{d}a \otimes e + a.\nabla'_B(e)) = a.\alpha(e),$$

as required. □

In classical differential geometry the second fundamental form is a tensor, and in a non-commutative context this corresponds to α being a module map.
The following theorem is the equivalent of the Gauss equation for curvature in classical differential geometry.

Theorem 15.4. ***(Gauss Equation)*** *The left connection* $\nabla_A : A_\pi \otimes_B \mathrm{Hor} \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathrm{Hor})$ *given by* $\nabla_A = (\mathrm{id} \otimes P)\nabla'_B$ *has curvature* $R_A : A_\pi \otimes_B \mathrm{Hor} \to \Omega^2_A \otimes_A (A_\pi \otimes_B \mathrm{Hor})$ *given by*

$$R_A(f) = (\mathrm{id} \otimes P)R'_B(f) + (\mathrm{id} \wedge \beta)\alpha(f) \tag{25}$$

where $\beta : N \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathrm{Hor})$ *is a left module map given by*

$$\beta = (\mathrm{id} \otimes P)\nabla'_B. \tag{26}$$

Proof. If $\nabla'_B(e) = \xi \otimes h \in \Omega^1_A \otimes_A (A_\pi \otimes_B \mathfrak{X}^R_B)$, then $\nabla_A(e) = \xi \otimes Ph$ and $\alpha(e) = \xi \otimes Qh$. On one hand we calculate

$$R_A(e) = \mathrm{d}\xi \otimes Ph - \xi \wedge \nabla_A(Ph).$$

On the other hand we calculate

$$(\mathrm{id} \otimes P)R'_B(e) = \mathrm{d}\xi \otimes Ph - \xi \wedge (\mathrm{id} \otimes P)\nabla'_B h.$$

The difference between the two is

$$\begin{aligned} R_A(e) - (\mathrm{id} \otimes P)R'_B(e) &= \xi \wedge (\mathrm{id} \otimes P)\nabla'_B h - \xi \wedge \nabla_A(Ph) \\ &= \xi \wedge (\mathrm{id} \otimes P)\nabla'_B(Ph) + \xi \wedge (\mathrm{id} \otimes P)\nabla'_B(Qh) - \xi \wedge \nabla_A(Ph) \\ &= \xi \wedge (\mathrm{id} \otimes P)\nabla'_B(Qh) \\ &= (\mathrm{id} \otimes (\mathrm{id} \otimes P)\nabla'_B)\alpha(e) \\ &= (\mathrm{id} \wedge \beta)\alpha(e), \end{aligned}$$

which shows the Gauss equation. Lastly, we see that β is a left module map because for all $e \in N$ and $a \in A$,

$$\beta(ae) = (\mathrm{id} \otimes P)\nabla'_B(ae) = (\mathrm{id} \otimes P)(\mathrm{d}a \otimes e + a.\nabla'_B(e)) = a.\beta(e),$$

as required. $\square$

The above theory is for left connections on the right vector fields $\mathfrak{X}_B^R$, but a similar version exists for right connections on the left vector fields $\mathfrak{X}_B^L$. If instead $\pi : B \to A$ is a left-handed co-embedding, i.e. $\mathfrak{X}_B^L \otimes_{B\,\pi} A = (\mathrm{Hor}^L \otimes_{B\,\pi} A) \oplus N$ for some B-A bimodule $N \subset \mathfrak{X}_B^L \otimes_{B\,\pi} A$ complementary to the tangent bundle, then there are projections P and Q which sum to the identity given as follows:

$$\begin{array}{ccc} & \mathfrak{X}_L^R \otimes_{B\,\pi} A & \\ \swarrow^{P} & & \searrow^{Q} \\ \mathrm{Hor}^L \otimes_{B\,\pi} A & & N \end{array}$$

Theorem 15.5. *Given a left-handed co-embedding $\pi : B \to A$ and a right connection $\tilde{\nabla}_B : \mathfrak{X}_B^L \to \mathfrak{X}_B^L \otimes_B \otimes_B \Omega_B^1$ with curvature $\tilde{R}_B : \mathfrak{X}_B^L \to \mathfrak{X}_B^L \otimes \Omega_B^2$, there is a right connection $\tilde{\nabla}'_B : \mathfrak{X}_B^L \otimes_{B\,\pi} A \to (\mathfrak{X}_B^L \otimes_{B\,\pi} A) \otimes_A \Omega_A^1$ given by*

$$\tilde{\nabla}'_B(X \otimes 1) = (\mathrm{id} \otimes 1 \otimes \pi)\tilde{\nabla}_B(X) \tag{27}$$

with curvature $\tilde{R}'_B(1 \otimes X) = (\mathrm{id} \otimes 1 \otimes \pi)\tilde{R}_B(X)$. There is a right connection $\tilde{\nabla}_A : \mathrm{Hor}^L \otimes_{B\,\pi} A \to (\mathrm{Hor}^L \otimes_{B\,\pi} A) \otimes_A \Omega_A^1$ given by

$$\tilde{\nabla}_A = (P \otimes \mathrm{id})\tilde{\nabla}'_B \tag{28}$$

and a right A-module map $\tilde{\alpha} : \mathrm{Hor}^L \otimes_{B\,\pi} A \to N \otimes_A \Omega_A^1$ given by

$$\tilde{\alpha} = (Q \otimes \mathrm{id})\tilde{\nabla}'_B, \tag{29}$$

and a right module map $\tilde{\beta} : N \to (\mathrm{Hor}^L \otimes_{B\,\pi} A) \otimes_A \Omega_A^1$ given by

$$\tilde{\beta} = (P \otimes \mathrm{id})\tilde{\nabla}'_B. \tag{30}$$

The curvature $\tilde{R}_A : \mathrm{Hor}^L \otimes_{B\,\pi} A \to (\mathrm{Hor}^L \otimes_{B\,\pi} A) \otimes_A \Omega_A^2$ of $\tilde{\nabla}_A$ is given by

$$\tilde{R}_A(f) = (P \otimes \mathrm{id})\tilde{R}'_B(f) - (\tilde{\beta} \wedge \mathrm{id})\tilde{\alpha}(f). \tag{31}$$

Proof. The only part of this not clear by symmetry is the flipped Gauss equation 31. If $\tilde{\nabla}'_B(e) = h \otimes \xi \in (\mathfrak{X}_B^L \otimes_{B\,\pi} A) \otimes_A \Omega_A^1$, then $\tilde{\nabla}_A(e) = Ph \otimes \xi$, and $\tilde{\alpha}(e) = Qh \otimes \xi$. On one hand we calculate

$$\tilde{R}_A(e) = Ph \otimes \mathrm{d}\xi + \tilde{\nabla}_A(Ph) \wedge \xi.$$

On the other hand we calculate

$$(P \otimes \mathrm{id})\tilde{R}'_B(e) = Ph \otimes \mathrm{d}\xi + (P \otimes \mathrm{id})\tilde{\nabla}'_B h \wedge \xi.$$

The difference between the two is

$$\begin{aligned}
\tilde{R}(e) - (P \otimes \mathrm{id})\tilde{R}'_B(e) &= \tilde{\nabla}_A(Ph) \wedge \xi - (P \otimes \mathrm{id})\tilde{\nabla}'_B h \wedge \xi \\
&= \tilde{\nabla}_A(Ph) \wedge \xi - (P \otimes \mathrm{id})\tilde{\nabla}'_B(Ph) \wedge \xi - (P \otimes \mathrm{id})\tilde{\nabla}'_B(Qh) \wedge \xi \\
&= -(P \otimes \mathrm{id})\tilde{\nabla}'_B(Qh) \wedge \xi \\
&= -((P \otimes \mathrm{id})\tilde{\nabla}'_B \otimes \mathrm{id})\tilde{\alpha}(e) \\
&= -(\tilde{\beta} \wedge \mathrm{id})\tilde{\alpha}(e),
\end{aligned}$$

as required. □

16 Hermitian Metrics

(See chapter 8.4 of [10] for the notation we use in this section on Hermitian metrics — especially the G notation.)

If B is a $*$-algebra and E a left B-module, then a Hermitian metric on E is a B-bimodule map $\langle , \rangle_E : E \otimes \overline{E} \to B$ satisfying $\langle e, \overline{f} \rangle^* = \langle f, \overline{e} \rangle$ for all $e, f \in E$. Each nondegenerate Hermitian metric is specified by an invertible map $G : \overline{E} \to E^\circ = {}_B\mathrm{Hom}(E, B)$, as in the following diagram.

$$\begin{array}{ccc}
E \otimes \overline{E} & \xrightarrow{\langle , \rangle_E} & B \\
\big\downarrow{\scriptstyle \mathrm{id} \otimes G} & \nearrow{\scriptstyle ev} & \\
E \otimes E^\circ & &
\end{array}$$

If E is left finitely generated projective with dual basis $e^i \in E$, $e_i \in E^\circ$, then the inner product is described by matrices $g^{ij} = \langle e^i, \overline{e^j} \rangle_E$. The condition of being Hermitian is equivalent to $(g^{ij})^* = g^{ji}$. There is a corresponding notion of Hermitian metric on a right module. The key thing about Hermitian metrics is that unlike Riemannian metrics they only require a one-sided module, and are defined over the tensor product $\otimes_{\mathbb{C}}$.

Recall that there is a standard isomorphism $bb : E \to \overline{\overline{E}}$ adding a double bar, with inverse bb^{-1} removing a double bar.

Given a Hermitian metric $\langle,\rangle_E : E\otimes\overline{E}\to B$ on a left B-module E, there is a Hermitian metric with the bar on the other side on the left vector fields $E^\circ = \mathfrak{X}^L_B$ given by

$$\langle,\rangle_{E^\circ} = ev\circ(bb^{-1}\overline{G^{-1}}\otimes\mathrm{id}) : \overline{E^\circ}\otimes E^\circ\to B. \tag{32}$$

We can see this works by breaking it down into the following sequence of maps:

$$\overline{E^\circ}\otimes E^\circ \xrightarrow{\overline{G^{-1}}\otimes\mathrm{id}} \overline{\overline{E}}\otimes E^\circ \xrightarrow{bb^{-1}\otimes\mathrm{id}} E\otimes E^\circ \xrightarrow{\quad ev\quad} B\ .$$

In the definition of a co-embedding it suffices for the normal bundle to be complementary to the tangent bundle, but in examples where a Hermitian metric is known on Ω^1_B, we can use this to find a normal bundle which is really orthogonal to the tangent bundle. We calculate several examples of this at the end of the section.

Next, we look at metric preservation. Recall that given a left B-module E and a left connection $\nabla_E : E\to\Omega^1_B\otimes_B E$, denoting $\nabla_E(e)=\xi\otimes f$ for some $e\in E$, there is a right connection $\nabla_{\overline{E}} : \overline{E}\to\overline{E}\otimes_B\Omega^1_B$ given by $\nabla_{\overline{E}}(\overline{e}) = \overline{f}\otimes\xi^*$.

The following lemma will be useful in showing that various connections in this section are metric preserving.

Lemma 16.1. *If E is a left B-module equipped with a Hermitian metric $\langle,\rangle_E : E\otimes\overline{E}\to B$, then a left connection $\nabla_E : E\to\Omega^1_B\otimes_B E$ preserves the metric if and only if $\mathbb{W}(G) = (G\otimes\mathrm{id})\nabla_{\overline{E}} - \nabla_{E^\circ}\circ G : \overline{E}\to E^\circ\otimes_B\Omega^1_B$ vanishes.*

Proof. Using the assumption $\mathbb{W}(G)=0$ and the definition $\langle,\rangle = ev(\mathrm{id}\otimes G)$, we calculate

$$\begin{aligned}
\mathrm{d}\langle,\rangle &= \mathrm{d}(ev(\mathrm{id}\otimes G))\\
&= (\mathrm{id}\otimes ev)(\nabla_E\otimes G) + (ev\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_{E^\circ}G)\\
&= (\mathrm{id}\otimes ev)(\nabla_E\otimes G) + (ev\otimes\mathrm{id})(G\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_{\overline{E}})\\
&= (\mathrm{id}\otimes\langle,\rangle)(\nabla_E\otimes\mathrm{id}) + (\langle,\rangle\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_{\overline{E}})
\end{aligned}$$

as required. □

Note that by symmetry the above Lemma also works for a right connection.

Proposition 16.2. *(Proposition 3.32 of [10], and Equation (3.17) on page 229 of [10])* *If Ω^1_B is left finitely generated projective with left connection ∇_E, there exists a unique right connection on $(\Omega^1_B)^\circ = \mathfrak{X}^L_B$ given by*

$$\nabla_{E^\circ} : E^\circ \to E^\circ \otimes_A \Omega^1_A, \qquad \nabla_{E^\circ}(\gamma) = e_i \otimes \mathrm{d}(ev(e^i \otimes \gamma)) - e_i \otimes (\mathrm{id} \otimes ev)(\nabla_E e^i \otimes \gamma)$$

satisfying

$$(\mathrm{id} \otimes ev)(\nabla_E \otimes \mathrm{id}) + (ev \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{\nabla}_{E^\circ}) : E \otimes E^\circ \to \Omega^1$$

Conversely, if Ω^1_B is right finitely generated and has a right connection $\nabla_E : \Omega^1_B \to \Omega^1_B \otimes_B \Omega^1_B$, then on the right vector fields $E^\sharp = \mathfrak{X}^R_B$ we have a left connection $\nabla_{E^\sharp} : \mathfrak{X}^R_B \to \Omega^1_B \otimes_B \mathfrak{X}^R_B$ given by

$$\nabla_{E^\sharp}(f) = \mathrm{d}(ev(f \otimes e^i)) \otimes e_i - (ev \otimes \mathrm{id})(f \otimes \nabla_E(e^i)) \otimes e_i. \tag{33}$$

In the case where we have a dual basis, the following corollary is useful for calculating curvatures of flipped connections.

Corollary 16.2.1. *It follows that*

$$R_E(e^i) = -\sum_j (ev \otimes \mathrm{id})(e^i \otimes \tilde{R}_{e^\circ}(e_j)) \otimes e^j$$

and

$$\tilde{R}_{E^\circ}(e_j) = -\sum_i e_i \otimes (\mathrm{id} \otimes ev)(R_E(e^i) \otimes e_j)$$

Proof. Expanding R_E and $\tilde{R}_{E^\circ}$ using the dual bases of E and E° respectively and substituting into each the equation $(\mathrm{id} \otimes ev)(\nabla_E \otimes \mathrm{id}) + (ev \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{\nabla}_{E^\circ})$, we get:

$$\begin{aligned}\tilde{R}_{E^\circ}(e_j) &= \sum_i e_i \otimes (ev \otimes \mathrm{id})(e^i \otimes \tilde{R}_{E^\circ}(e_j)) \\ &= -\sum_i e_i \otimes (\mathrm{id} \otimes ev)(R_E(e^i) \otimes e_j),\end{aligned}$$

and

$$\begin{aligned}\tilde{R}_{E^\circ} &= \sum_j (\mathrm{id} \otimes ev)(R_e(e^i) \otimes e_j) \otimes e^j \\ &= -\sum_i e_i \otimes (\mathrm{id} \otimes ev)(R_E(e^i) \otimes e_j),\end{aligned}$$

as required. □

Next we prove a central result of this section, that a connection on Ω^1_B preserves a Hermitian metric if and only if its dual connection on $\mathfrak{X}^L_B$ preserves the dual Hermitian metric.

Theorem 16.3. *Denoting $E = \Omega^1_B$ and $E^\circ = \mathfrak{X}^R_B$, a left connection $\nabla_E : E \to \Omega^1_B \otimes_B E$ preserves the Hermitian metric $\langle , \rangle_E = ev \circ (\mathrm{id} \otimes G) : E \otimes \overline{E} \to B$ if and only if the right connection $\nabla_{E^\circ} : E^\circ \to E^\circ \otimes_B \Omega^1_B$ given by*

$$\nabla_{E^\circ}(\gamma) = e_i \otimes \mathrm{d}(ev(e^i \otimes \gamma)) - e_i \otimes (\mathrm{id} \otimes ev)(\nabla_E e^i \otimes \gamma) \tag{34}$$

preserves the Hermitian metric

$$\langle , \rangle'_E = ev \circ (bb^{-1}\overline{G^{-1}} \otimes \mathrm{id}) : \overline{E^\circ} \otimes E^\circ \to B. \tag{35}$$

Proof. In the first part of the proof we assume ∇_E is metric preserving and show that ∇_{E° is metric preserving, then in the second part we show the implication the other way around. **(1)** For ∇_E to be metric preserving means $\mathbb{W}(G) = 0$. The metric $\langle , \rangle' : \overline{E^\circ} \otimes E^\circ \to B$ uses map $bb^{-1}\overline{G^{-1}} : \overline{E^\circ} \to E$, so we need to show $\mathbb{W}(bb^{-1}\overline{G^{-1}}) = 0$, i.e.

$$\nabla_E bb^{-1}\overline{G^{-1}} = (\mathrm{id} \otimes bb^{-1}\overline{G^{-1}})\nabla_{\overline{E^\circ}}. \tag{36}$$

Since G is an isomorphism, every $\alpha \in E^\circ$ is equal to $G(\overline{e})$ for some $e \in E$. Thus we calculate the left hand side of (36) as:

$$\nabla_E bb^{-1}\overline{G^{-1}}(\overline{\alpha}) = \nabla_E bb^{-1}\overline{(G^{-1} \circ G)(\overline{e})} = \nabla_E(e)$$

Using the assumption that ∇_B is metric preserving, then writing $\nabla_E(e) = k \otimes f$ we calculate

$$\nabla_{E^\circ} \circ G(\overline{e}) = (G \otimes \mathrm{id})\nabla_{\overline{E}}(\overline{e}) = (G \otimes \mathrm{id})(\overline{f} \otimes k^*) = G(\overline{f}) \otimes k^*,$$

which implies $\nabla_{\overline{E^\circ}} \circ \overline{G(\overline{e})} = k \otimes \overline{G(\overline{f})}$. Using this, we calculate the right hand side of (36) as:

$$\begin{aligned}(\mathrm{id} \otimes bb^{-1}\overline{G^{-1}})\nabla_{\overline{E^\circ}}(\overline{\alpha}) &= (\mathrm{id} \otimes bb^{-1}\overline{G^{-1}})\nabla_{\overline{E^\circ}}\overline{G(\overline{e})} \\ &= (\mathrm{id} \otimes bb^{-1}\overline{G^{-1}})(k \otimes \overline{G(\overline{f})}) \\ &= k \otimes bb^{-1}\overline{G^{-1}G(\overline{f})}\end{aligned}$$

$$= k \otimes bb^{-1}\overline{\overline{f}}$$

$$= k \otimes f = \nabla_E(e),$$

and this shows the result.

(2) Writing $\nabla_{E^\circ} = \gamma \otimes \eta$, the assumption that ∇_{E° is metric preserving means

$$\nabla_E bb^{-1}\overline{G^{-1}}(\overline{\alpha}) = (\mathrm{id} \otimes bb^{-1}\overline{G^{-1}})\nabla_{\overline{E^\circ}}(\overline{\alpha})$$
$$= \eta^* \otimes bb^{-1}\overline{G^{-1}(\gamma)}$$

This implies

$$\nabla_{\overline{E}} G^{-1}(\alpha) = G^{-1}(\gamma) \otimes \eta. \tag{37}$$

We want to show for all $e \in E$ that

$$\nabla_{E^\circ} \circ G(\overline{e}) = (G \otimes \mathrm{id})\nabla_{\overline{E}}(\overline{e}). \tag{38}$$

Since G is an isomorphism, for each $e \in E$ we can write $\overline{e} = G^{-1}(\alpha)$ for some $\alpha \in E^\circ$. Also, denote $\nabla_{E^\circ}(\alpha) = \gamma \otimes \eta$. Making this substitution, we can re-write the left hand side of (38) as

$$\nabla_{E^\circ} \circ G(\overline{e}) = \nabla_{E^\circ}(\alpha) = \gamma \otimes \eta.$$

Using the substitution and equation (37), we calculate the right hand side of (38) as

$$(G \otimes \mathrm{id})\nabla_{\overline{E}}(\overline{e}) = (G \otimes \mathrm{id})\nabla_{\overline{E}} G^{-1}(\alpha)$$
$$= (G \otimes \mathrm{id})(G^{-1}(\gamma) \otimes \eta)$$
$$= \gamma \otimes \eta = \nabla_{E^\circ}(\alpha),$$

as required. □

Denote $F = \tilde{\Omega}^1_A$, so $\mathfrak{X}^L_A = F^\circ$. These are paired by the evaluation

$$ev(\mathrm{d}_A a' \otimes (X \otimes a)) = \pi X(\mathrm{d}(\pi^{-1}a')).a.$$

The restriction of the right connection ∇_{E° to a right connection ∇_{F° on the tangent bundle is given by:

$$\nabla_{F^\circ}(X \otimes 1) = (P(- \otimes 1) \otimes \pi^*)\nabla_{E^\circ} X. \tag{39}$$

A Hermitian metric on $\mathfrak{X}^L_A = {}_\pi\mathfrak{X}^L_B \otimes {}_\pi A$ is given by

$$\langle,\rangle'_F : \overline{\mathfrak{X}^L_A} \otimes \mathfrak{X}^L_A \to A, \qquad \langle\overline{Y\otimes a}, X\otimes a'\rangle'_F = a^*.\pi\langle\overline{Y}, X\rangle'.a' \tag{40}$$

We want to show that ∇_{F° preserves this Hermitian metric. We show the following intermediate step.

Lemma 16.4. *Denote the right module $C = \mathfrak{X}^L_B \otimes_B {}_\pi A$, and $\nabla_{E^\circ}X = Y\otimes\xi$. Then the right connection $\nabla_C : C \to C\otimes_A \Omega^1_A$ given by $\nabla_C(X\otimes a) = Y\otimes\pi^*(\xi).a + X\otimes \mathrm{d}a$ preserves the Hermitian metric $\langle,\rangle_C : \overline{C}\otimes C \to A$ given by $\langle\overline{X'\otimes a'}, X\otimes a\rangle_C = a'^*.\pi(\langle\overline{X'}, X\rangle_{E^\circ}).a$.*

Proof. Differentiating the inner product $\langle,\rangle_C$ and using the assumption that $\langle,\rangle_{E^\circ}$ is metric preserving, and denoting $\nabla_{E^\circ}X' = Y'\otimes\xi'$, we calculate:

$$\begin{aligned}\mathrm{d}\langle\overline{X'\otimes a'}, X\otimes a\rangle_C &= \mathrm{d}(a')^*\pi\langle\overline{X'}, X\rangle_{E^\circ}.a + a'^*\pi\langle\overline{X'}, X\rangle_{E^\circ}.\mathrm{d}a + a'^*\mathrm{d}\pi\langle\overline{X'}, X\rangle_{E^\circ}.a\\ &= \mathrm{d}(a')^*\pi\langle\overline{X'}, X\rangle_{E^\circ}.a + a'^*\pi\langle\overline{X'}, X\rangle_{E^\circ}.\mathrm{d}a + \pi\Big(\langle\overline{X'}, Y\rangle_{E^\circ}.\xi + \xi'\langle\overline{Y'}, X\rangle_{E^\circ}\Big).a\\ &(\langle,\rangle_C.\mathrm{id})(\overline{X'\otimes a'}\otimes\nabla_C(X\otimes a)) + (\mathrm{id}.\langle,\rangle_C)(\nabla_{\overline{C}}(\overline{X'\otimes a'})\otimes X\otimes a)\end{aligned}$$

But this is precisely the metric preservation equation for ∇_C. □

The only difference between $\langle,\rangle_C$ and $\langle,\rangle_{F^\circ}$ is restricting the domain of the vector fields, and similarly the only difference between ∇_C and ∇_{F° is restricting the domain and composing with a projection. In the case where the tangent and normal bundles are orthogonal with respect to the inner product, we have $\langle\overline{c'}, c\rangle_C = \langle\overline{c'}, Pc\rangle_C$ for all $c, c' \in C$. Hence the metric preservation equation also holds for ∇_{F°.

Then lastly we want to flip one more time using Theorem 16.3 to end up in the submanifold calculus. If the connection on the tangent bundle is metric-preserving then the one on the submanifold calculus is automatically metric preserving too, by the theorem earlier.

Equation 3.17 on page 229 of [10] says that if F is a right finitely generated projective module with dual basis $f_i\otimes f^i \in F\otimes F^\sharp$ with right connection $\tilde{\nabla}_F$, then there is a unique left connection $\nabla_{F^\sharp} : F^\sharp \to \Omega^1\otimes_A F^\sharp$ given by

$$\nabla_{F^\sharp}(\beta) = \mathrm{d}(ev(\beta\otimes f_i))\otimes f^i - (ev\otimes\mathrm{id})(\beta\otimes\tilde{\nabla}_F f_i)\otimes f^i \tag{41}$$

Hence we flip ∇_A to get a left connection on the submanifold calculus, which is metric preserving. So the overall result of the process is that we started with a Hermitian metric-preserving connection on Ω^1_B, and obtained a Hermitian metric preserving connection on the submanifold calculus Ω^1_A.

16.1 Example: Algebraic Circle in Quantum SU_2

We look again at the co-embedding of $A = \mathbb{C}_q[S^1]$ into $B = \mathbb{C}_q[SU_2]$ via the surjective algebra map $\pi : B \to A$, $\pi(a) = t$, $\pi(d) = t^{-1}$, $\pi(b) = \pi(c) = 0$, as we calculated in Section 13.4.

As detailed in section 7 of [12], the algebra $\mathbb{C}_q[SU_2]$ doesn't have a Riemannian metric on its 3D calculus, but there is a Hermitian metric given by g diagonal, with nonzero real entries g^{++}, g^{--}, g^{00}. The normal bundle we calculated earlier is orthogonal with respect to this. The paper [12] gives a family of left connections $\nabla_B : \Omega^1_B \to \Omega^1_B \otimes_B \Omega^1_B$ on the 3D calculus on $\mathbb{C}_q[SU_2]$ invariant under the right $\mathbb{C}\mathbb{Z}$-coaction and preserving the Hermitian metric, for parameters $n_\pm, r \in \mathbb{R}$, $\nu, m_+ \in \mathbb{C}$ by the following values on invariant elements.

$$
\begin{aligned}
\nabla^L_B(e^0) &= re^0 \otimes e^0 + \nu e^+ \otimes e^- + q^{-1} g_{++} g_{--}^{-1} m_+^* e^- \otimes e^+ \\
\nabla^L_B(e^+) &= n_+ e^0 \otimes e^+ + m_+ e^+ \otimes e^0 \\
\nabla^L_B(e^-) &= n_- e^0 \otimes e^- + q^{-1} g_{00} g_{--}^{-1} \nu^* e^- \otimes e^0
\end{aligned}
$$

In the following proposition we calculate out in full the curvature of this connection.

Dualising to the (left) vector fields, there is a right connection for $i \in \{+, -, 0\}$ given with summation implicit as

$$
\tilde{\nabla}_B : \mathfrak{X}^L_B \to \mathfrak{X}^L_B \otimes_B \Omega^1_B, \qquad \tilde{\nabla}_B(X) = e_i \otimes \mathrm{d}(ev(e^i \otimes X)) - e_i \otimes (\mathrm{id} \otimes ev)(\nabla_B e^i \otimes X),
$$

which preserves the Hermitian metric on the vector fields. We calculate:

$$
\begin{aligned}
\tilde{\nabla}_B(e_0) &= -r(e_0 \otimes e^0) - \nu(e_+ \otimes e^+) - q^{-1} g_{00} g_{--}^{-1} \nu^* (e_- \otimes e^+) \\
\tilde{\nabla}_B(e_+) &= -n_+(e_0 \otimes e^+) - m_+(e_+ \otimes e^0) \\
\tilde{\nabla}_B(e_-) &= -n_-(e_0 \otimes e^-) - q^{-1} g_{00} g_{--}^{-1} \nu^* (e_- \otimes e^0).
\end{aligned}
$$

Using the fact that $\pi(e^0) = t^{-1}\mathrm{d}t$ and $\pi(e^{\pm}) = 0$, we calculate $\tilde{\nabla}'_B : \mathfrak{X}^L_B \otimes_B {}_\pi A \to (\mathfrak{X}^L_B \otimes_B {}_\pi A) \otimes_A \Omega^1_A$ given by $\tilde{\nabla}'_B(1 \otimes X) = (\mathrm{id} \otimes 1 \otimes \pi)\tilde{\nabla}_B(X)$ as

$$\tilde{\nabla}'_B(e_0) = -re_0 \otimes 1 \otimes t^{-1}\mathrm{d}t$$
$$\tilde{\nabla}'_B(e_+) = -me_+ \otimes 1 \otimes t^{-1}\mathrm{d}t$$
$$\tilde{\nabla}'_B(e_-) = -q^{-1}g_{00}g_{--}^{-1}\nu^* e_- \otimes 1 \otimes t^{-1}\mathrm{d}t.$$

Recall that there is a projection map $P : \mathfrak{X}^L_B \otimes_B {}_\pi A \to \mathrm{Hor}^L \otimes_B {}_\pi A$, and along with the algebra map π this defines a projection of the right connection on $\mathfrak{X}^L_B$ to a right connection on the tangent bundle.

$$\nabla_A : \mathrm{Hor}^L \otimes_B {}_\pi A \to (\mathrm{Hor}^L \otimes_B {}_\pi A) \otimes_A \Omega^1_A, \qquad \nabla_A(X \otimes 1) = (P \otimes \pi)\tilde{\nabla}'_B(1 \otimes X),$$

On the generator $e_0 \otimes 1$ of the tangent bundle this is:

$$\nabla_A(e_0 \otimes 1) = -rP(e_0 \otimes 1) \otimes t^{-1}\mathrm{d}t = -re_0 \otimes 1 \otimes t^{-1}\mathrm{d}t$$

This preserves the Hermitian metric $\langle \overline{Y \otimes a}, X \otimes a' \rangle = a^*(\pi\langle \overline{Y}, X\rangle')a'$.
Lastly we flip the connection one more time using the formula from Proposition 16.2 to get a left connection $\tilde{\nabla}_A : \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ on the submanifold calculus given by

$$\begin{aligned}\tilde{\nabla}_A(e^0 \otimes 1) &= -(ev \otimes \mathrm{id})(e^0 \otimes \nabla_A e_0) \otimes (1 \otimes e^0)\\ &= rt^{-1}\mathrm{d}t \otimes \mathrm{d}t\end{aligned}$$

Recall that $\mathrm{d}_A t = q^{-2}(e^0 \otimes 1).t$, so

$$\begin{aligned}\tilde{\nabla}_A(\mathrm{d}_A t) &= q^{-2}\tilde{\nabla}_A(e^0 \otimes t) = q^{-2}\tilde{\nabla}_A(e^0.a \otimes 1) = \tilde{\nabla}_A(t.e^0 \otimes 1) = \mathrm{d}t \otimes (e^0 \otimes 1) + t.\tilde{\nabla}(e^0 \otimes 1)\\ &= q^2\mathrm{d}t \otimes \mathrm{d}t.t + r\mathrm{d}t \otimes \mathrm{d}t = (q^6 t + r)\mathrm{d}t \otimes \mathrm{d}t.\end{aligned}$$

Next we look at curvature and the Gauss equation.

Proposition 16.5. *When the algebra $B = \mathbb{C}_q[SU_2]$ is equipped with maximal prolongation calculi, the connection ∇_E has curvature $R_E : \Omega^1_B \to \Omega^2_B \otimes_B \Omega^1_B$ given by*

$$\begin{aligned}R_E(e^0) = &\Big(rq^3 - |\nu|^2 q^{-1}g_{00}g_{--}^{-1} + q|m_+|^2 g_{++}g_{--}^{-1}\Big)e^+ \wedge e^- \otimes e^0\\ &+ \Big(-\nu q^2[2]_{q^{-2}} + q^{-4}r\nu - \nu n_-\Big)e^+ \wedge e^0 \otimes e^-\end{aligned}$$

$$+\Big(q^{-1}g_{++}g_{--}^{-1}m_+^*q^{-2}[2]_{q^{-2}}+q^3rg_{++}g_{--}^{-1}m_+^*\Big)e^-\wedge e^0\otimes e^+$$

$$\begin{aligned}R_E(e^+)=&\Big(n_+q^3-q^{-1}g_{++}g_{--}^{-1}|m_+|^2\Big)e^+\wedge e^-\otimes e^+\\&+\Big(-m_+q^2[2]_{q^{-2}}+q^4n_+m_+-m_+r\Big)e^+\wedge e^0\otimes e^0\end{aligned}$$

$$\begin{aligned}R_E(e^-)=&\Big(n_-q^3+q|\nu|^2g_{00}g_{--}^{-1}\Big)e^+\wedge e^-\otimes e^-\\&+\Big(q^{-3}g_{00}g_{--}^{-1}\nu^*[2]_{q^{-2}}+q^{-5}n_-g_{00}g_{--}^{-1}\nu^*-rq^{-1}g_{00}g_{--}^{-1}\nu^*\Big)e^-\wedge e^0\otimes e^0\end{aligned}$$

Proof. Recall from Example 2.32 of [10] that when $\mathbb{C}_q[SU_2]$ is equipped with maximal prolongation calculus, then denoting $[n]_q=(1-q^n)/(1-q)$ for a q-integer:

$$\begin{aligned}&\mathrm{d}e^0=q^3e^+\wedge e^-, &&\mathrm{d}e^+=-q^2[2]_{q^{-2}}e^+\wedge e^0, &&\mathrm{d}e^-=q^{-2}[2]_{q^{-2}}e^-\wedge e^0,\\&e^-\wedge e^+=-q^2e^+\wedge e^-, &&e^0\wedge e^+=-q^{+4}e^+\wedge e^0, &&e^0\wedge e^-=-q^{-4}e^-\wedge e^0,\\&e^\pm\wedge e^\pm=e^0\wedge e^0=0\end{aligned}$$

Hence we calculate

$$\begin{aligned}R_E(e^0)=&(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_E)\nabla_E(e^0)\\=&r\mathrm{d}e^0\otimes e^0+\nu\mathrm{d}e^+\otimes e^-+q^{-1}g_{++}g_{--}^{-1}m_+^*\mathrm{d}e^-\otimes e^+\\&-re^0\wedge\nabla_E(e^0)-\nu e^+\wedge\nabla_E(e^-)-q^{-1}g_{++}g_{--}^{-1}m_+^*e^-\wedge\nabla_E(e^+)\\=&rq^3e^+\wedge e^-\otimes e^0-\nu q^2[2]_{q^{-2}}e^+\wedge e^0\otimes e^-+q^{-1}g_{++}g_{--}^{-1}m_+^*q^{-2}[2]_{q^{-2}}e^-\wedge e^0\otimes e^+\\&-re^0\wedge\Big(re^0\otimes e^0+\nu e^+\otimes e^-+q^{-1}g_{++}g_{--}^{-1}m_+^*e^-\otimes e^+\Big)\\&-\nu e^+\wedge\Big(n_-e^0\otimes e^-+q^{-1}g_{00}g_{--}^{-1}\nu^*e^-\otimes e^0\Big)\\&-q^{-1}g_{++}g_{--}^{-1}m_+^*e^-\wedge\Big(n_+e^0\otimes e^++m_+e^+\otimes e^0\Big)\\=&\Big(rq^3-|\nu|^2q^{-1}g_{00}g_{--}^{-1}+q|m_+|^2g_{++}g_{--}^{-1}\Big)e^+\wedge e^-\otimes e^0\\&+\Big(-\nu q^2[2]_{q^{-2}}+q^{-4}r\nu-\nu n_-\Big)e^+\wedge e^0\otimes e^-\\&+\Big(q^{-1}g_{++}g_{--}^{-1}m_+^*q^{-2}[2]_{q^{-2}}+q^3rg_{++}g_{--}^{-1}m_+^*\Big)e^-\wedge e^0\otimes e^+\end{aligned}$$

and

$$R_E(e^+)=(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_E)\nabla_E(e^+)$$

$$
\begin{aligned}
&= n_+ \mathrm{d}e^0 \otimes e^+ + m_+ \mathrm{d}e^+ \otimes e^0 - n_+ e^0 \wedge \nabla_E(e^+) - m_+ e^+ \wedge \nabla_E(e^0) \\
&= n_+ q^3 e^+ \wedge e^- \otimes e^+ - m_+ q^2 [2]_{q^{-2}} e^+ \wedge e^0 \otimes e^0 - n_+ e^0 \wedge \Big(n_+ e^0 \otimes e^+ + m_+ e^+ \otimes e^0 \Big) \\
&\quad - m_+ e^+ \wedge \Big(r e^0 \otimes e^0 + \nu e^+ \otimes e^- + q^{-1} g_{++} g_{--}^{-1} m_+^* e^- \otimes e^+ \Big) \\
&= \Big(n_+ q^3 - q^{-1} g_{++} g_{--}^{-1} |m_+|^2 \Big) e^+ \wedge e^- \otimes e^+ \\
&+ \quad \Big(- m_+ q^2 [2]_{q^{-2}} + q^4 n_+ m_+ - m_+ r \Big) e^+ \wedge e^0 \otimes e^0
\end{aligned}
$$

and

$$
\begin{aligned}
R_E(e^-) &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E) \nabla_E(e^-) \\
&= n_- \mathrm{d}e^0 \otimes e^- + q^{-1} g_{00} g_{--}^{-1} \nu^* \mathrm{d}e^- \otimes e^0 - n_- e^0 \wedge \nabla_E(e^-) - q^{-1} g_{00} g_{--}^{-1} \nu^* e^- \wedge \nabla_E(e^0) \\
&= n_- q^3 e^+ \wedge e^- \otimes e^- + q^{-1} g_{00} g_{--}^{-1} \nu^* q^{-2} [2]_{q^{-2}} e^- \wedge e^0 \otimes e^0 \\
&\quad - n_- e^0 \wedge \Big(n_- e^0 \otimes e^- + q^{-1} g_{00} g_{--}^{-1} \nu^* e^- \otimes e^0 \Big) \\
&\quad - q^{-1} g_{00} g_{--}^{-1} \nu^* e^- \wedge \Big(r e^0 \otimes e^0 + \nu e^+ \otimes e^- + q^{-1} g_{++} g_{--}^{-1} m_+^* e^- \otimes e^+ \Big) \\
&= \Big(n_- q^3 + q |\nu|^2 g_{00} g_{--}^{-1} \Big) e^+ \wedge e^- \otimes e^- \\
&\quad + \Big(q^{-3} g_{00} g_{--}^{-1} \nu^* [2]_{q^{-2}} + q^{-5} n_- g_{00} g_{--}^{-1} \nu^* - r q^{-1} g_{00} g_{--}^{-1} \nu^* \Big) e^- \wedge e^0 \otimes e^0
\end{aligned}
$$

□

Since $\Omega^2_A = 0$, we know on dimensional grounds that the curvature $\tilde{R}_A$ of $\tilde{\nabla}_A$ vanishes. But even though we already know the result, in order to illustrate the (left-sided) Gauss equation 31, we calculate the terms on the other side of the equation anyway and show they vanish. Recall that $\pi(e^0) = t^{-1}\mathrm{d}t$ and $\pi(e^\pm) = 0$.

Firstly we show that $(P \otimes \mathrm{id})\tilde{R}'_B$ vanishes. For brevity, denote $R_E(e^i) = \sum_k \omega_{i,k} \otimes e^k$. Then

$$
\begin{aligned}
\tilde{R}_E(e_j) &= -\sum_i e_i \otimes (\mathrm{id} \otimes ev)(R_E(e^i) \otimes e_j) = -\sum_{i,k} e_i \otimes (\mathrm{id} \otimes ev)(\omega_{i,k} \otimes e^k \otimes e_j) \\
&= -\sum_i e_i \otimes \omega_{i,j}.
\end{aligned}
$$

Hence

$$
\tilde{R}'_B(e_j) = -\sum_i e_i \otimes 1 \otimes \pi(\omega_{i,j}).
$$

We calculate

$$(P\otimes \mathrm{id})\tilde{R}'_B(e_0) = -\sum_i P(e_i\otimes 1)\otimes \pi(\omega_{i,0}) = -e_0\otimes 1\otimes \pi(\omega_{0,0})$$
$$= -e_0\otimes 1\otimes \left(rq^3 - |\nu|^2 q^{-1} g_{00} g_{--}^{-1} + q|m_+|^2 g_{++} g_{--}^{-1}\right)\pi(e^+\wedge e^-).$$

Again, since $\Omega^2_A = 0$, this vanishes. Next, we calculate $\tilde{\alpha} = (Q\otimes \mathrm{id})\nabla'_B : A_\pi \otimes_B \mathrm{Hor} \to \Omega^1_A \otimes_A N$. Using that $Q(e_0\otimes 1) = 0$ and $Q(e_\pm \otimes 1) = e_{\pm\otimes 1}$, we calculate:

$$\tilde{\alpha}(e_0\otimes 1) = -rQ(e_0\otimes 1)\otimes \pi(e^0) - \nu Q(e_+\otimes 1)\otimes \pi(e^-) - q^{-1} g_{++} g_{--}^{-1} m_+^* Q(e_-\otimes 1)\otimes \pi(e^+)$$
$$= 0.$$

Hence the term $-(\tilde{\beta}\wedge \mathrm{id})\tilde{\alpha}$ vanishes, so the Gauss equation 31 says $\tilde{R}_A = 0$.

16.2 Example: Functions on Finite Groups

As we calculated in Section 13.5, if H is a subset of a finite group G, then there is a co-embedding $\pi : C(G) \to C(H)$.

As in section 6 of [12], the algebra $B = C(G)$ for G a finite group has left covariant calculus $\Omega^1_{C(G)}$ specified by a subset $\mathcal{C} \subset G\backslash\{e\}$. A left invariant Hermitian structure which is a right module map can be written as

$$G : \overline{\Lambda^1 C(G)} \to (\Lambda^1 C(G))^\circ, \qquad G(\overline{e^a}) = e_a . g^{a,a},$$

where $g^{a,a}$ is real. It is a right comodule map if for all $a \in \mathcal{C}$ and $x \in G$ it satisfies $g^{xax^{-1},xax^{-1}} = g^{a,a}$. Since the metric is diagonal, the normal bundle corresponding to a subgroup as we calculated earlier is orthogonal. In the following we assume that all entries of g are equal. The paper [12] gives a left invariant connection on the calculus of $C(G)$ by

$$\nabla^L_B(e^a) = -\hat{\Gamma}^a_{b,c} e^b \otimes e^c,$$

which preserves the metric if $\hat{\Gamma}^a_{d,c} = (\hat{\Gamma}^c_{d^{-1},a})^*$.

Example 16.6. For the specific example $G = S_3$ with calculus specified by

$$\mathcal{C} = \{(12), (23), (31)\},$$

a left invariant connection on the calculus is specified for $a, c, d \in \mathbb{R}$ and $b \in \mathbb{C}$ by

$$\hat{\Gamma}^x_{x,x} = a - 1, \quad \hat{\Gamma}^x_{y,z} = c, \quad \hat{\Gamma}^x_{y,x} = d - 1, \quad \hat{\Gamma}^x_{x,y} = b^*, \quad \hat{\Gamma}^y_{x,x} = b.$$

for arbitrary but distinct $x, y, z \in \mathcal{C}$. So for example $\hat{\Gamma}^x_{x,x}$ denotes when all three indices of the Christoffel symbols are the same, while $\hat{\Gamma}^x_{y,z}$ denotes when they are all different. Next we calculate the dual of this connection to the left vector fields. Recall there is a right connection for $i \in \mathcal{C}$ given as (summation implicit)

$$\tilde{\nabla}_B : \mathfrak{X}^L_B \to \mathfrak{X}^L_B \otimes_B \Omega^1_B, \qquad \tilde{\nabla}_B(X) = e_i \otimes \mathrm{d}(ev(e^i \otimes X)) - e_i \otimes (\mathrm{id} \otimes ev)(\nabla_B e^i \otimes X),$$

We calculate this for $X = e_j$ for some fixed $j \in \mathcal{C}$.

$$\begin{aligned}\tilde{\nabla}_B(e_j) = e_{(12)} &\otimes \left(e^{(12)}\hat{\Gamma}^{(12)}_{(12),j} + e^{(23)}\hat{\Gamma}^{(12)}_{(23),j} + e^{(31)}\hat{\Gamma}^{(12)}_{(31),j}\right) \\ +e_{(23)} &\otimes \left(e^{(12)}\hat{\Gamma}^{(23)}_{(12),j} + e^{(23)}\hat{\Gamma}^{(23)}_{(23),j} + e^{(31)}\hat{\Gamma}^{(23)}_{(31),j}\right) \\ +e_{(31)} &\otimes \left(e^{(12)}\hat{\Gamma}^{(31)}_{(12),j} + e^{(23)}\hat{\Gamma}^{(31)}_{(23),j} + e^{(31)}\hat{\Gamma}^{(31)}_{(31),j}\right)\end{aligned}$$

For the specific case $j = (12)$, using the formulae above for the Christoffel symbols, this becomes:

$$\begin{aligned}\tilde{\nabla}_B(e_{(12)}) = e_{(12)} &\otimes \left(e^{(12)}(a-1) + e^{(23)}(d-1) + e^{(31)}(d-1)\right) \\ +e_{(23)} &\otimes \left(e^{(12)}b + e^{(23)}b^* + e^{(31)}c\right) \\ +e_{(31)} &\otimes \left(e^{(12)}b + e^{(23)}c + e^{(31)}b^*\right)\end{aligned}$$

For every subgroup H of S_3, the algebra $C(H)$ is a submanifold algebra of $C(S_3)$, with submanifold calculus given in the standard way by the set $\mathcal{C} \cap H$. For example, the subgroup of S_3 generated by the 3-cycles would have the zero calculus, as none of the 2-cycles in $\mathcal{C}$ lie in that subgroup. So instead, we look at the more interesting subgroup $H = \{e, (12)\}$, which has submanifold calculus generated by $e^{(12)}$.

Using a projection P as in the previous example, along with the algebra map π, the connection $\tilde{\nabla}_B$ restricts to the tangent bundle $\mathrm{Hor}^L \otimes_{B\,\pi} A$ as a right connection:

$$\nabla_A : \mathrm{Hor}^L \otimes_{B\,\pi} A \to (\mathrm{Hor}^L \otimes_{B\,\pi} A) \otimes_A \Omega^1_A, \qquad \nabla_A(X \otimes 1) = (P(- \otimes 1) \otimes \pi^*)\tilde{\nabla}_B X,$$

which we calculate on the generator as

$$\nabla_A(e_{(12)} \otimes 1) = e_{(12)} \otimes 1 \otimes e^{(12)}(a-1).$$

Lastly we flip the connection one more time using the formula from Proposition 16.2 to get a left connection $\tilde{\nabla}_A : \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ on the submanifold calculus given by

$$\tilde{\nabla}_A(e^{(12)} \otimes 1) = -(ev \otimes \mathrm{id})(e^{(12)} \otimes \nabla_A(e_{(12)})) \otimes e^{(12)} = -(a-1)e^{(12)} \otimes e^{(12)}.$$

◇

Next, we look at curvature and the (left-handed) Gauss equation 31, but for $B = C(G)$ and $A = C(H)$ for G an arbitrary finite group with subgroup H. If Ω^1_B is freely generated by e^a for $a \in \mathcal{C} \subset G\backslash\{e\}$, then Ω^1_A is freely generated by e^a for $a \in \mathcal{C}_H := \mathcal{C} \cap H$. As before, a general left covariant connection on Ω^1_B is given by

$$\nabla^L_B(e^a) = -\hat{\Gamma}^a_{b,c} e^b \otimes e^c, \quad a \in \mathcal{C}.$$

The corresponding right connection on $\mathfrak{X}^L_B$ is

$$\tilde{\nabla}_B(e_j) = \sum_{i,k \in \mathcal{C}} e_i \otimes e^k \hat{\Gamma}^i_{kj}, \quad j \in \mathcal{C}$$

and so

$$\tilde{\nabla}'_B(1 \otimes e_j) = \sum_{i \in \mathcal{C}, k \in \mathcal{C}_H} e_i \otimes 1 \otimes e^k \hat{\Gamma}^i_{kj}, \quad j \in \mathcal{C}$$

We also calculate

$$\tilde{\alpha}(e_j \otimes 1) = \sum_{i \in \mathcal{C}\backslash\mathcal{C}_H, k \in \mathcal{C}_H} e_i \otimes 1 \otimes e^k \hat{\Gamma}^i_{kj}, \quad j \in \mathcal{C}_H$$

and

$$\tilde{\beta}(e_j \otimes 1) = \sum_{i,k \in \mathcal{C}_H} e_i \otimes 1 \otimes e^k \hat{\Gamma}^i_{kj}, \quad j \in \mathcal{C}\backslash\mathcal{C}_H.$$

We check the left-handed Gauss equation 31. For $j \in \mathcal{C}_H$, we have

$$\begin{aligned}(\tilde{\beta} \wedge \mathrm{id})\tilde{\alpha}(e_j \otimes 1) &= \sum_{k \in \mathcal{C}_H, i \in \mathcal{C}\backslash\mathcal{C}_H} \tilde{\beta}(e_i \otimes 1) \wedge e^k \hat{\Gamma}^i_{kj} \\ &= \sum_{k \in \mathcal{C}_H, i \in \mathcal{C}\backslash\mathcal{C}_H} \sum_{k', i' \in \mathcal{C}_H} e_{i'} \hat{\Gamma}^{i'}_{k'i} \otimes 1 \otimes e^{k'} \wedge e^k \hat{\Gamma}^i_{kj}.\end{aligned}$$

We want this to be equal to $(P \otimes \mathrm{id})\tilde{R}'_B(e_j) - \tilde{R}_A(e_j)$. Firstly,

$$\tilde{R}'_B(e_j \otimes 1) = \sum_{i \in \mathcal{C}, k \in \mathcal{C}_H} \big(e_i \otimes 1 \otimes \mathrm{d}e^k \hat{\Gamma}^i_{kj} + \tilde{\nabla}'_B(e_i \otimes 1) \wedge e^k \hat{\Gamma}^i_{kj} \big), \quad j \in \mathcal{C}_H.$$

Also

$$\tilde{\nabla}_A(e_j \otimes 1) = \sum_{i,k \in \mathcal{C}_H} e_i \otimes 1 \otimes e^k \hat{\Gamma}^i_{kj}, \quad j \in \mathcal{C}_H,$$

from which we calculate

$$\tilde{R}_A(e_j \otimes 1) = \sum_{i,k \in \mathcal{C}_H} \hat{\Gamma}^i_{kj}\big(e_i \otimes 1 \otimes \mathrm{d}e^k - \tilde{\nabla}_A(e_i \otimes 1) \wedge e^k\big), \quad j \in \mathcal{C}_H.$$

Hence

$$\begin{aligned}(P \otimes \mathrm{id})\tilde{R}'_B(e_j) - \tilde{R}_A(e_j) &= \sum_{k,i \in \mathcal{C}_H} \big(e_i \otimes 1 \otimes \mathrm{d}e^k \hat{\Gamma}^i_{kj} + \tilde{\nabla}'_B(e_i \otimes 1) \wedge e^k \hat{\Gamma}^i_{kj}\big) \\ &\quad - \sum_{i,k \in \mathcal{C}_H} \hat{\Gamma}^i_{kj}\big(e_i \otimes 1 \otimes \mathrm{d}e^k - \tilde{\nabla}_A(e_i \otimes 1) \wedge e^k\big) \\ &= \sum_{i \in \mathcal{C} \setminus \mathcal{C}_H, k \in \mathcal{C}_H} \tilde{\nabla}'_B(e_i \otimes 1) \wedge e^k \hat{\Gamma}^i_{kj} \\ &= \sum_{i \in \mathcal{C} \setminus \mathcal{C}_H, k \in \mathcal{C}_H} \sum_{i',k \in \mathcal{C}_H} e_{i'} \otimes e^{k'} \hat{\Gamma}^{i'}_{k'j} \wedge e^k \hat{\Gamma}^i_{kj}.\end{aligned}$$

This is precisely what we calculated $(\tilde{\beta} \wedge \mathrm{id})\tilde{\alpha}(e_j \otimes 1)$ to be, and thus we can see that the Gauss equation is satisfied. Note that there was never any uncertainty as to whether it would be true - we just calculated out all the terms to illustrate that it holds in an example.

16.3 Example: Quantum Circle in Quantum Disk

As we calculated in Section 13.3 (with right vector fields), the algebra $B = \mathbb{C}_q[D]$ has noncommutative submanifold $A = \mathbb{C}_{q^2}[S^1]$ via the algebra map $\pi : B \to A$, $\pi(z) = t$, $\pi(\overline{z}) = t^{-1}$, which has kernel generated by $w = 1 - \overline{z}z$. In Exercise 8.7 of [10], after adding an additional element w^{-1} to the quantum disk, a Hermitian metric on its calculus $\Omega^1_{\mathbb{C}_q[D]}$ is given (specifically, derived from a Riemannian metric). However, the addition of the inverse element w^{-1} would cause π to fail to be an algebra map, since if it was an algebra map then $1 = \pi(ww^{-1}) = \pi(w)\pi(w^{-1}) = 0$ which is a contradiction. This makes sense geometrically, since if z was a complex number, then w would be $1 - |z|^2$, and to make that invertible would mean excluding the unit circle, which is precisely the submanifold we want to look at. Thus we have to find a different Hermitian metric on $\Omega^1_{\mathbb{C}_q[D]}$ which doesn't require w to be invertible.

A Hermitian metric $\langle , \rangle : \overline{\Omega^1_{\mathbb{C}_q[D]}} \otimes \Omega^1_{\mathbb{C}_q[D]} \to \mathbb{C}_q[D]$ is given for $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ by $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, i.e.

$$\langle \overline{\mathrm{d}z}, \mathrm{d}z \rangle = \alpha, \quad \langle \overline{\mathrm{d}\overline{z}}, \mathrm{d}\overline{z} \rangle = \beta, \quad \langle \overline{\mathrm{d}z}, \mathrm{d}\overline{z} \rangle = \langle \overline{\mathrm{d}\overline{z}}, \mathrm{d}z \rangle = 0.$$

We note that while the metric in Exercise 8.7 of [10] which comes from a quantum metric and requires w to be invertible is over $\otimes_{\mathbb{C}_q[D]}$, this metric is over $\otimes_{\mathbb{C}}$.

Remark 16.7. *This metric is rotation invariant, in the sense of being invariant under the differentiable action $S^1 \otimes \mathbb{C}_q[D] \to \mathbb{C}_q[D]$ given by*

$$e^{i\theta} \triangleright z = e^{i\theta} z, \quad e^{i\theta} \triangleright \overline{z} = e^{-i\theta} \overline{z}, \quad e^{i\theta} \triangleright \mathrm{d}z = e^{i\theta} \mathrm{d}z, \quad e^{i\theta} \triangleright \mathrm{d}\overline{z} = e^{-i\theta} \mathrm{d}\overline{z},$$

since

$$\langle \overline{e^{i\theta} \triangleright \mathrm{d}z}, e^{i\theta} \triangleright \mathrm{d}z \rangle = \langle \overline{e^{i\theta}.\mathrm{d}z}, e^{i\theta}.\mathrm{d}z \rangle = e^{i\theta} e^{-i\theta} \langle \overline{\mathrm{d}z}, \mathrm{d}z \rangle = \langle \overline{\mathrm{d}z}, \mathrm{d}z \rangle$$

et cetera for all four combinations.

When we calculated the tangent bundle earlier with generator $Y = q^2 t^2 \otimes e_z + 1 \otimes e_{\overline{z}}$, we took a normal bundle as being generated by $1 \otimes e_z$. It was complementary to the tangent bundle, which was sufficient for the purposes of calculating a submanifold, but in order to get a connection which preserves the corresponding Hermitian metric when we project to the tangent bundle, we need the tangent and normal bundles to be orthogonal with respect to the metric.

Proposition 16.8. *When $\Omega^1_{C_q[D]}$ is equipped with a Hermitian metric of the form $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ for $\alpha, \beta \in \mathbb{C}\backslash\{0\}$, an orthogonal normal bundle is generated by*

$$t^2 \otimes e_z - (q^2)^* \frac{\alpha}{\beta} \otimes e_{\overline{z}}. \tag{42}$$

Proof. To work out out which elements of $A_\pi \otimes_B \mathfrak{X}^R_B$ are orthogonal to $Y = q^2 t^2 \otimes e_z + 1 \otimes e_{\overline{z}}$, we calculate

$$\langle \overline{\mu t^2 \otimes e_z + \lambda \otimes e_{\overline{z}}}, q^2 t^2 \otimes e_z + 1 \otimes e_z \rangle = \mu^* q^2 \alpha + \beta \lambda^* = 0.$$

Setting this equal to zero gives $\lambda^* = -\mu^* q^* \frac{\alpha}{\beta}$, and starring this gives $\lambda = -\mu (q^*)^2 \frac{\alpha}{\beta}$. Since we just need a generator, we are free to set $\mu = 1$, which gives the result. □

Proposition 16.9. *With tangent bundle generated by $Y = q^2t^2 \otimes e_z + 1 \otimes e_{\overline{z}}$ and normal bundle generated by $t^2 \otimes e_z - (q^2)^* \frac{\alpha}{\beta} \otimes e_{\overline{z}}$, the projection map $P : A_\pi \otimes_B \mathfrak{X}^R_B \to A_\pi \otimes_B \mathrm{Hor}$ is given by*

$$P(1 \otimes e_z) = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\big((qq^*)^2\frac{\alpha}{\beta} \otimes e_z + q^2t^2 \otimes e_{\overline{z}}\big)$$
$$P(1 \otimes e_{\overline{z}}) = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\big((q^*)^2\frac{\alpha}{\beta}t^{-2} \otimes e_z + 1 \otimes e_{\overline{z}}\big).$$

Proof. The projection map is rank 1, and is thus of the form

$$P\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} q^2t^2 \\ 1 \end{smallmatrix}\right).\left(\begin{smallmatrix} a & b \end{smallmatrix}\right).\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} q^2t^2.a & q^2t^2.b \\ a & b \end{smallmatrix}\right).\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right),$$

where the elements a, b are determined by equations

$$\left(\begin{smallmatrix} a & b \end{smallmatrix}\right).\left(\begin{smallmatrix} q^2t^2 \\ 1 \end{smallmatrix}\right) = 1, \quad \left(\begin{smallmatrix} a & b \end{smallmatrix}\right).\left(\begin{smallmatrix} t^2 \\ -(q^*)^2\frac{\alpha}{\beta} \end{smallmatrix}\right) = 0.$$

Thus

$$q^2t^2a + b = 1, \quad t^2a - (q^*)^2\frac{\alpha}{\beta}b = 0.$$

Hence $a = (q^*)^2\frac{\alpha}{\beta}t^{-2}b$, which implies $(qq^*)^2\frac{\alpha}{\beta}b + b = 1$, so

$$a = \frac{(q^*)^2\frac{\alpha}{\beta}t^{-2}}{1+(qq^*)^2\frac{\alpha}{\beta}}, \quad b = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}$$

Hence

$$P\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\left(\begin{smallmatrix} (qq^*)^2\frac{\alpha}{\beta} & q^2t^2 \\ (q^*)^2\frac{\alpha}{\beta}t^{-2} & 1 \end{smallmatrix}\right).\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right).$$

Setting (v, w) as $(1, 0)$ and $(0, 1)$ respectively gives

$$P(1 \otimes e_z) = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\big((qq^*)^2\frac{\alpha}{\beta} \otimes e_z + q^2t^2 \otimes e_{\overline{z}}\big)$$
$$P(1 \otimes e_{\overline{z}}) = \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\big((q^*)^2\frac{\alpha}{\beta}t^{-2} \otimes e_z + 1 \otimes e_{\overline{z}}\big).$$

□

Next we look at connections. Denote $e^1 = \mathrm{d}z$ and $e^2 = \mathrm{d}\overline{z}$, and e_1, e_2 their respective duals in $\mathfrak{X}^R_B$. The extension of π to the calculi is $\pi(e^1) = \pi(\mathrm{d}z) = \mathrm{d}(\pi(z)) = \mathrm{d}t$, and $\pi(e^2) = \mathrm{d}(t^{-1}) = -t^{-1}.\mathrm{d}t.t^{-1}$.

Proposition 16.10. *The right connection* $\nabla_E : \Omega^1_B \to \Omega^1_B \otimes_B \Omega^1_B$ *given by* $\nabla_E(e^i) = -e^j \otimes \Gamma_j{}^i$ *preserves an inner product* $\langle , \rangle_E : \overline{\Omega^1_B} \otimes_B \Omega^1_B \to B$ *if for* $i, j \in \{1, 2\}$

$$0 = \Gamma_1{}^1 + \Gamma_1{}^{1*}, \quad 0 = \Gamma_2{}^2 + \Gamma_2{}^{2*}$$
$$0 = \Gamma_2{}^1 + \frac{\alpha}{\beta}\Gamma_1{}^{2*}, \quad 0 = \Gamma_1{}^2 + \frac{\alpha}{\beta}\Gamma_2{}^{1*}.$$

It is torsion free if $0 = e^j \wedge \Gamma_j{}^i$

Proof. (1) Firstly we show the condition for metric preservation. Substituting $\overline{e^k} \otimes e^i$ into the metric preservation equation, we get

$$\mathrm{d}g^{ki} = -\langle \overline{e^k} \otimes e^j \rangle_E \otimes \Gamma^i_j - (\Gamma^k_p)^* \langle \overline{e^p}, e^i \rangle_E = -g^{kj}\Gamma^i_j - (\Gamma^k_p)^* g^{pi}.$$

This can be written as $-g\Gamma - \Gamma^* g = \mathrm{d}g$, where

$$\Gamma = \begin{pmatrix} \Gamma_1{}^1 & \Gamma_1{}^2 \\ \Gamma_2{}^1 & \Gamma_2{}^2 \end{pmatrix}, \qquad \Gamma^* = \begin{pmatrix} \Gamma_1{}^{1*} & \Gamma_2{}^{1*} \\ \Gamma_1{}^{2*} & \Gamma_2{}^{2*} \end{pmatrix}.$$

Substituting in the specific values of our metric g whose coefficients are constants, the equation becomes:

$$0 = g\Gamma + \Gamma^* g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \Gamma_1{}^1 & \Gamma_1{}^2 \\ \Gamma_2{}^1 & \Gamma_2{}^2 \end{pmatrix} + \begin{pmatrix} \Gamma_1{}^{1*} & \Gamma_2{}^{1*} \\ \Gamma_1{}^{2*} & \Gamma_2{}^{2*} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$
$$= \begin{pmatrix} \alpha(\Gamma_1{}^1 + \Gamma_1{}^{1*}) & \alpha\Gamma_1{}^2 + \beta\Gamma_2{}^{1*} \\ \beta\Gamma_2{}^1 + \alpha\Gamma_1{}^{2*} & \beta(\Gamma^2{}_2 + \Gamma^2{}_2{}^*). \end{pmatrix}$$

Re-arranging gives the equations.

(2) Secondly, we show the condition for torsion free. The torsion of a right connection is given by $T_\nabla = \wedge\nabla + \mathrm{d} : \Omega^1 \to \Omega^2$, and for this example ∇_B is torsion free if

$$\mathrm{d}e^i = \wedge\nabla_B(e^i) = -e^j \wedge \Gamma_j{}^i.$$

But since $\mathrm{d}e^i = 0$, this implies $0 = e^j \wedge \Gamma_j{}^i$. □

This condition for metric preservation is quite a mild one, since it only affects the star-structure.

We use Proposition 16.2 to calculate the left connection $\nabla_B : \mathfrak{X}^R_B \to \Omega^1_B \otimes_B \mathfrak{X}^R_B$ given by

$$\nabla_B(f) = \mathrm{d}(ev(f \otimes e^i)) \otimes e_i - (ev \otimes \mathrm{id})(f \otimes \nabla_E(e^i)) \otimes e_i.$$

On the generators, this is

$$\nabla_B(e_1) = \Gamma^i_{\ 1} \otimes e_i, \quad \nabla_B(e_2) = \Gamma^i_{\ 2} \otimes e_i.$$

Hence the left connection $\nabla'_B : A_\pi \otimes_B \mathfrak{X}^R_B \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathfrak{X}^R_B)$ defined as $\nabla'_B(1 \otimes X) = (\pi \otimes 1 \otimes \mathrm{id})\nabla_B(X)$ is given on the generators of $A_\pi \otimes_B \mathfrak{X}^R_B$ by

$$\nabla'_B(1 \otimes e_1) = \pi(\Gamma^i_{\ 1}) \otimes 1 \otimes e_i, \quad \nabla'_B(1 \otimes e_2) = \pi(\Gamma^i_{\ 2}) \otimes 1 \otimes e_i.$$

Using this, the left Leibniz rule, and the fact that $\mathrm{d}(t^2) = t\mathrm{d}t + \mathrm{d}t.t = (1+q)t.\mathrm{d}t$, we calculate ∇'_B on the generator $Y = q^2t^2 \otimes e_z + 1 \otimes e_{\overline{z}}$ of the tangent bundle $A_\pi \otimes_B \mathrm{Hor}^R$ as

$$\begin{aligned}\nabla'_B(Y) &= q^2\nabla'_B(t^2.1 \otimes e_1) + \nabla'_B(1 \otimes e_2)\\ &= q^2t^2.\nabla'_B(1 \otimes e_1) + q^2\mathrm{d}(t^2) \otimes 1 \otimes e_1 + \nabla'_B(1 \otimes e_2)\\ &= q^2t^2\pi(\Gamma^i_{\ 1}) \otimes 1 \otimes e_i + q^2(1+q)t.\mathrm{d}t \otimes 1 \otimes e_1 + \pi(\Gamma^i_{\ 2}) \otimes 1 \otimes e_i.\end{aligned}$$

The projection $P : A_\pi \otimes_B \mathfrak{X}^R_B \to A_\pi \otimes_B \mathrm{Hor}$ is the identity on multiples of $Y = q^2t^2 \otimes e_z + 1 \otimes e_{\overline{z}}$, and sends elements orthogonal to Y to zero.
Using the projection P to the tangent bundle, along with $\mathrm{d}t.t = q^2t.\mathrm{d}t$ we calculate the left connection $\nabla_A : A_\pi \otimes_B \mathrm{Hor} \to \Omega^1_A \otimes_A (A_\pi \otimes_B \mathrm{Hor})$ given by $\nabla_A = (\mathrm{id} \otimes P)\nabla'_B$ as

$$\nabla_A(Y) = q^2t^2\pi(\Gamma^i_{\ 1}) \otimes P(1 \otimes e_i) + q^2(1+q^2)t.\mathrm{d}t \otimes P(1 \otimes e_1) + \pi(\Gamma^i_{\ 2}) \otimes P(1 \otimes e_i)$$

Seeing as this is quite messy, we omit to fully expand the evaluations of the projection P in general. Certain choices of Christoffel symbols could simplify this a lot. For example, in the case where the Christoffel symbols of ∇_B are zero (which is metric preserving and torsion free), we get

$$\nabla_A(Y) = q^2(1+q^2)t.\mathrm{d}t \otimes \frac{1}{1+(qq^*)^2\frac{\alpha}{\beta}}\big((qq^*)^2\frac{\alpha}{\beta} \otimes e_z + q^2t^2 \otimes e_{\overline{z}}\big)$$

Denote this as

$$\nabla_A(Y) = \lambda_1 t.\mathrm{d}t \otimes 1 \otimes e_z + \lambda_2 t^3\mathrm{d}t \otimes 1 \otimes e_{\overline{z}}$$

for the appropriate constants $\lambda_1, \lambda_2 \in \mathbb{C}$.

We can dualise the connection one last time to get a right connection on the submanifold calculus $\nabla_{\rm sub} : \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ given by

$$\nabla_{\rm sub}(\xi) = Y' \otimes {\rm d}(ev(Y \otimes \xi)) - Y' \otimes ({\rm id} \otimes ev)(\nabla_A Y \otimes \xi),$$

where we write Y' for the dual of Y. Since the calculus is 1-dimensional, we have $Y' = {\rm d}t.f(t)$ for some polynomial $f(t) \in \mathbb{C}_{q^2}(t)$. For each $g(t)$, we want a dual basis decomposition ${\rm d}t.g(t) = Y'.Y({\rm d}t.g(t))$. But since Y is a vector field, this implies $Y' = {\rm d}t.Y({\rm d}t)$. We calculate

$$Y({\rm d}t) = q^2t^2\pi e_z({\rm d}\pi^{-1}x) + \pi e_{\overline{z}}({\rm d}\pi^{-1}x) = q^2t^2\pi e_z(e^z) + \pi e_{\overline{z}}(e^z) = q^2t.$$

Hence $Y' = q^2{\rm d}t.t$.

For the case where we take the Christoffel symbols of ∇_B as zero, the dual connection on the submanifold calculus is given by

$$\begin{aligned}\nabla_{\rm sub}({\rm d}t) &= q^2{\rm d}t.t \otimes {\rm d}(ev(Y \otimes {\rm d}t)) - q^2{\rm d}t.t \otimes ({\rm id} \otimes ev)(\nabla_A Y \otimes {\rm d}t)\\ &= q^4{\rm d}t.t \otimes {\rm d}t - q^2{\rm d}t.t \otimes ({\rm id} \otimes ev)(\nabla_A Y \otimes {\rm d}t)\\ &= q^4{\rm d}t.t \otimes {\rm d}t - q^2{\rm d}t.t \otimes \lambda_1 t.{\rm d}t.q^2t^2\\ &= \big(q^6t - \lambda_1 q^{16}t^4\big){\rm d}t \otimes {\rm d}t\\ &= \Big(q^6t - (1+q^2).\frac{(qq^*)^2\frac{\alpha}{\beta}}{1+(qq^*)^2\frac{\alpha}{\beta}}q^{18}t^4\Big){\rm d}t \otimes {\rm d}t.\end{aligned}$$

The coefficient here is not nice at all, even though the Christoffel symbols of the original connection on Ω^1_B were zero but it makes sense that a connection on Ω^1_A maps to a multiple of ${\rm d}t \otimes {\rm d}t$.

17 Can We Define Submanifolds via Positive Maps?

Since algebra maps and two sided ideals are not always readily available for any given algebra, it might be of interest to consider how much of the theory still works in the context of positive maps and left ideals. For example, while the algebraic circle is a noncommutative submanifold of $\mathbb{C}_q[SU_2]$, the restriction of that co-embedding to the quantum sphere does not give a co-embedding of the algebraic circle. This is because the restriction of π to the quantum sphere (degree 0 elements of quantum SU_2) is no longer surjective. However, from a geometrical perspective we would hope to somehow be able to embed the quantum circle into the quantum sphere, in the same way a classical sphere has an equator.

Later in Section V we look at a positive maps approach to retracts (which are related but not equivalent to submanifolds), but here let's look at how much of the theory of submanifolds can be done when we replace algebra maps by positive maps.

In the case where A and B are C*-algebras, and $\phi : B \to A$ a completely positive surjective map, it is possible to make generalisations of J and Hor which reduce to the original definitions when ϕ is an algebra map.

Proposition 17.1. *If A and B are C*-algebras, and $\phi : B \to A$ a completely positive surjective map, the subset*

$$J = \{j \in B \mid \phi(j^* j) = 0\} \tag{43}$$

is a left ideal of B, and a subset of $\ker(\phi)$. *An equivalent definition is*

$$J = \{j \in B \mid je_0 = 0\}$$

for the $e_0 \in E$ giving ϕ as $\phi(b) = \langle \overline{e_0}, be_0 \rangle$ via the KSGNS construction.

Proof. **(1)** Firstly we show that J is a left ideal of B, i.e. that for all $b \in B$, $j \in J$ we have $bj \in J$. Every $j \in J$ satisfies $0 = \phi(j^*j) = \langle \overline{je_0}, je_0 \rangle$, which implies by definition of inner product that $je_0 = 0$. Hence $\phi((bj)^*(bj)) = \langle \overline{bje_0}, bje_0 \rangle = \langle \overline{0}, 0 \rangle = 0$, so $bj \in J$.

(2) Next we show the equivalence of the two definitions. Seeing as $je_0 = 0$, this means $\phi(j) = \langle \overline{e_0}, je_0 \rangle = 0$, so $J \subset \ker(\phi)$.

An equivalent definition of J is therefore $J = \{j \in B \mid je_0 = 0\}$. □

In the case where ϕ is a *-algebra map, we get $\phi(j^*j) = \phi(j)^*\phi(j) \geq 0$, with equality precisely when $\phi(j) = 0$. Therefore in the *-algebra map case, $J = \ker(\phi)$ as before.

Using this, we also generalise the concept of horizontal vector fields to completely positive maps as follows.

Proposition 17.2. *For a completely positive map $\phi : B \to A$, the following subset* $\mathrm{Hor}(\phi) \subset \mathfrak{X}^R_B$ *given as follows is a B-bimodule.*

$$\mathrm{Hor}(\phi) := \{X \in \mathfrak{X}^R_B \mid X(\mathrm{d}j) \in J, \quad \forall j \in J\} \tag{44}$$

Proof. Let $X \in \mathrm{Hor}(\phi)$ and $b \in B$.

(1) Firstly, if $X(\mathrm{d}j) \in J$ then $(bX)(\mathrm{d}j) = b.X(\mathrm{d}j)$. But J is a left ideal, so $b.X(\mathrm{d}j) \in J$. Thus $\mathrm{Hor}(\phi)$ is a left B-module.

(2) Next, $(Xb)(\mathrm{d}j) = X(b.\mathrm{d}j) = X(\mathrm{d}(bj) - \mathrm{d}b.j) = X(\mathrm{d}(bj)) - X(\mathrm{d}b.j) = X(\mathrm{d}(bj)) - X(\mathrm{d}b).j$. But $bj \in J$, as J is a left ideal, so $X(\mathrm{d}(bj)) \in J$. Also we can see that $X(\mathrm{d}b) \in B$, so $X(\mathrm{d}b).j \in J$. Hence $\mathrm{Hor}(\phi)$ is a right B-module.

Since $((bX)b')(\xi) = (b(Xb'))(\xi)$, the actions commute, making $\mathrm{Hor}(\phi)$ is a B-bimodule.

□

However, without ϕ being an algebra map, it isn't clear how to obtain a tangent bundle from this, since the right action on A_ϕ is not well-defined otherwise. To proceed any further would require a new definition of restriction of vector fields.

18 Future Ideas and Discussion

Question: Kernel of π^*

For a co-embedding $\pi : B \to A$, if we denote by K the kernel of its extension π^* to the calculi where A is equipped with submanifold calculus, then there is a short exact sequence of algebras and algebra maps:

$$0 \longrightarrow K \xrightarrow{inc} \Omega^1_B \xrightarrow{\pi^*} \Omega^1_A \longrightarrow 0 \tag{45}$$

But what is K?

Question: Star Structure

If B and A are star algebras and π a star-algebra map, and if B is equipped with a star-calculus, is the submanifold calculus on A then a star-calculus, and do we have $*\mathrm{d} = \mathrm{d}*$? This is a question to which the author is very much interested in knowing the answer. In the examples we calculated, the submanifold calculus often came out as the standard calculus on A which has a known $*$-structure, but whether it holds in general is currently unknown.

Question: Left-right Symmetry

Are the submanifold calculi obtained from left and right vector fields isomorphic? It seems likely that they would be, as long as $\mathfrak{X}_B^R \cong \mathfrak{X}_B^L$. Also, this is likely the key to getting the star structure to work, since the star of a left vector field is a right vector field and vice versa.

Question: Restricting Bimodule Connections

Does our procedure for restricting connections to the submanifold calculi send a bimodule connection on $\mathfrak{X}_B^R$ to a bimodule connection on $\mathfrak{X}_A^R$? We know the formula for the connection, so we just have to check that the associated σ is a bimodule map.

Interpreting Projection of Connections

Given a co-embedding $\pi : B \to A$, we have a procedure which takes a connection on Ω^1_B which preserves a Hermitian metric, and produces a connection on the submanifold calculus Ω^1_A which preserves a corresponding Hermitian metric. But the coefficients we get in examples are often complicated. It would be interesting to know why, since this might give some insight into exactly what kind of embedding of submanifolds is occurring. One example we could begin by looking at is the co-embedding $\pi : C^\infty(M) \to C^\infty(N)$ for N an embedded smooth submanifold of a smooth manifold M, and then projecting connections.

Part IV

Differential Fibre Bundles via Bimodules

Abstract

We construct a Leray-Serre spectral sequence for fibre bundles for de Rham cohomology on noncommutative algebras, generalising an existing definition which uses algebra maps as morphisms to now use bimodules as morphisms. The fibre bundles are bimodules with zero-curvature extendable bimodule connections satisfying an additional condition. By the KSGNS construction, completely positive maps between C*-algebras correspond to Hilbert C*-bimodules. We give three examples of fibre bundles, involving group algebras, matrix algebras, and the quantum torus

19 Introduction

Fibre bundles are an object in classical topology, finding applications in fields such as gauge bundles in physics. A fibre bundle is defined as a map $\pi : E \to B$ from the total space to the base space, satisfying the property that there exists a third space F called the fibre which can be associated in a continuous manner with the pre-image $\pi^{-1}\{b\}$ for each $b \in B$. Associated to each fibre bundle is a Leray-Serre spectral sequence, which allows calculation via homological algebra of the cohomology of the total space with coefficients in a group. The reader may wish to refer to chapters 2 and 9 of the topology textbook [52] for a detailed reference on fibre bundles and spectral sequences respectively.

Our objective is to generalise and calculate examples of fibre bundles and their associated Leray-Serre spectral sequences in the context of noncommutative geometry, where spaces are replaced by algebras. To extend the concept of fibre bundles to a noncommutative setting, where spaces are replaced by algebras with differential calculi, we take B to be an algebra of functions on a hypothetical base space of a fibre bundle, and A as the algebra corresponding to the total space. Since switching from spaces to algebras reverses the direction of functions, a noncommutative fibre bundle now goes from B to A.

A previous work in this direction is the 2008 paper [25], which proposes a definition of noncommutative fibre bundles giving rise to a Leray-Serre spectral sequence, but has the drawback of being limited to the case where the base space algebra is the algebra of functions on a locally compact Hausdorff space.

In another approach, the 2005 paper [8] and its extension in [14] defines a noncommutative fibre bundle as an algebra map between general algebras, and constructs a Leray-Serre spectral sequence converging to the de Rham sheaf cohomology of the total space with coefficients in a bimodule. This has a number of examples such as the noncommutative Hopf fibre bundle, which can be found in chapter 4 of the book [10].

In this paper, we reformulate this second definition [14] of noncommutative fibre bundles so as to no longer require an algebra map but a bimodule. This allows for the calculation of sheaf cohomology via spectral sequences for new examples which were not possible under existing approaches. Specifically, for noncommutative fibre bundles not coming from algebra maps, such as the noncommutative fibre bundle of the quantum circle in the quantum torus, which we calculate at the end.

With the additional data of an inner product on the bimodule, our examples can be regarded as equivalent to fibre bundles being completely positive maps (which generalise algebra maps), via the KSGNS construction giving a correspondence between C^* Hilbert-bimodules and positive maps, which we review in more detail later.

We begin with a review of spectral sequences and the necessary background from noncommutative differential geometry, before presenting our new definition, in which a bimodule differential fibre bundle is a bimodule equipped with a zero-curvature extendable bimodule connection, to which we show there is an associated Leray-Serre spectral sequence. We conclude by calculating two finite-dimensional examples of bimodule differential fibre bundles, followed by one infinite-dimensional example.

Our first example of a bimodule differential fibre bundle is between group algebras, with base algebra $\mathbb{C}G$ and total algebra $\mathbb{C}X$, for a subgroup $G \subset X$. This happens to come from a differentiable algebra map, and so could also have been calculated using existing theory, but since it can be nicely calculated in full it serves to illustrate our theory.

Our second example is between complex-valued matrix algebras, with base space algebra $M_2(\mathbb{C})$ and total space algebra $M_3(\mathbb{C})$. The bimodule from this example gives a differentiable map which is completely positive but not an algebra map, and so requires our new definition.

Our third example has base space algebra the quantum circle $\mathbb{C}_q[S^1]$ and total space algebra the quantum torus $\mathbb{C}_\theta[\mathbb{T}^2]$. This is infinite dimensional, and also does not come from an algebra map.

20 Background

20.1 Spectral Sequences

A spectral sequence (E_r, d_r) is a series of two-dimensional lattices called pages, denoted E_r, with each page r having entry $E_r^{p,q}$ in position $(p,q) \in \mathbb{Z}^2$ and differentials $\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ satisfying $\mathrm{d}_r^2 = 0$. By convention, the p-axis is horizontal and the q-axis is vertical. In the case we consider, only the top-right quadrant $(p, q \geq 0)$ of page 0 has nonzero entries. The differentials d_r on the rth page move right by r entries and down by $r-1$. Seeing as the differentials square to zero, we can take their cohomology. The $(r+1)$th page is defined as the cohomology of the rth page. Figure 18 illustrates what a spectral sequence looks like on pages $r = 0, 1, 2, 3$.

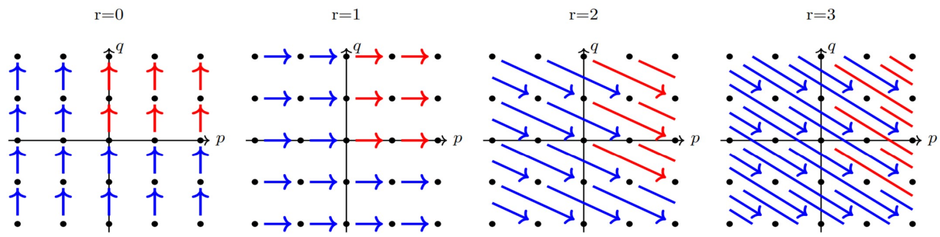


Figure 18: Fibre Bundles: Illustration of successive pages of a spectral sequence [45]

A spectral sequence converges if there is a fixed page after which all subsequent pages are the same. Once taking cohomology no longer changes a spectral sequence, it is said to have stabilised, and position (p,q) on the stable pages is denoted as $E_\infty^{p,q}$.
The spectral sequence we use is a variant of the Leray-Serre spectral sequence, which arises from a filtration.

Definition 20.1. Given a cochain complex C^n of vector spaces with linear differential $\mathrm{d} : C^n \to C^{n+1}$ satisfying $\mathrm{d}^2 = 0$, we say that a sequence of subspaces $F^m C \subset C$ for $m \geq 0$ is a *decreasing filtration* of C if the following three conditions are satisfied.

1. $\mathrm{d}F^mC \subset F^mC$ for all $m \geq 0$.
2. $F^{m+1}C \subset F^mC$ for all $m \geq 0$.
3. $F^0C = C$ and $F^mC^n := F^mC \cap C^n = \{0\}$ for all $m > n$.

Given such a filtration, the spectral sequence with first page $E_1^{p,q} = H^{p+q}(\frac{F^pC}{F^{p+1}C})$ converges to $H^*(C, \mathrm{d})$ in the sense that $H^k(C, \mathrm{d}) = \bigoplus_{p+q=k} E_\infty^{p,q}$. This can be read off the stabilised sequence as the direct sum along the north-west to south-east diagonals.

20.2 Bimodules and Connections

We looked at bimodule connections in the preliminaries, but now we need to look in more depth at extendability results.

Proposition 20.2. *(See Corollary 5.4 of [1]) If the algebra A has maximal prolongation calculi for its higher calculi, and if the curvature R_E is also a left module map, then extendability of σ_E is automatic.*

Next, there are a few results which are proven in [10] for left connections but which we need to prove for right connections, since the right handed versions are not necessarily just mirror images of the left handed versions, as we see in Lemma 20.5.

Lemma 20.3. *Let E be a B-A bimodule with extendable right bimodule connection (∇_E, σ_E). The connection ∇_E extends to higher calculi as*

$$\nabla_E^{[n]} = \mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id} : E \otimes_A \Omega_A^n \to E \otimes_A \Omega_A^{n+1} \tag{46}$$

Proof. We need to show that $\nabla_E^{[n]}$ is well-defined over the tensor-product $\otimes_A$, and so we check that $\nabla_E^{[n]}(ea \otimes \eta) = \nabla_E^{[n]}(e \otimes a\eta)$. The two sides of this equation can be expanded as

$$(\mathrm{id} \otimes \mathrm{d})(ea \otimes \eta) + (\nabla_E \wedge \mathrm{id})(ea \otimes \eta) = ea \otimes \mathrm{d}\eta + \nabla_E(e)a \wedge \eta + e \otimes \mathrm{d}a \wedge \eta$$

and

$$(\mathrm{id} \otimes \mathrm{d})(e \otimes a\eta) + (\nabla_E \wedge \mathrm{id})(e \otimes a\eta) = e \otimes \mathrm{d}a \wedge \eta + e \otimes a \wedge \mathrm{d}\eta + \nabla_E(e) \wedge a\eta.$$

Since $ea \otimes \mathrm{d}\eta = e \otimes a \wedge \mathrm{d}\eta$ and $\nabla_E(e) \wedge a\eta = \nabla_E(e)a \wedge \eta$, the above equations imply the desired result that $\nabla_E^{[n]}(ea \otimes \eta) = \nabla_E^{[n]}(e \otimes a\eta)$. □

Lemma 20.4. *Let E be a B-A bimodule with extendable right bimodule connection (∇_E, σ_E). Then $\nabla_E^{[n+1]} \circ \nabla_E^{[n]} = R_E \wedge \mathrm{id} : E \otimes_A \Omega_A^n \to E \otimes_A \Omega_A^{n+2}$, where $R_E = (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})\nabla_E : E \to E \otimes_A \Omega_A^2$ is the curvature of ∇_E.*

Proof. Writing out $\nabla_E^{[n+1]} \circ \nabla_E^{[n]}$ in string diagrams and then expanding $\mathrm{d}\wedge$ and using associativity of $\wedge$ gives Figure 19, which gives the desired result. □

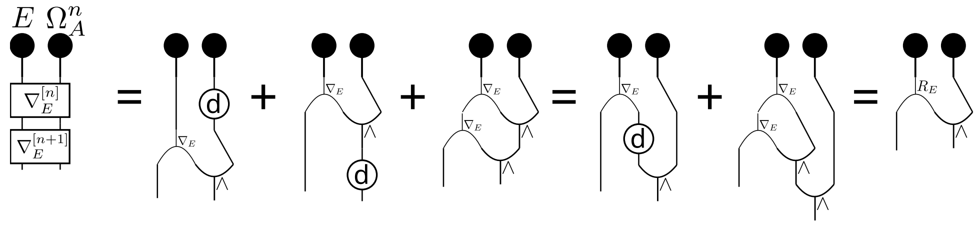


Figure 19: Fibre Bundles: Diagrammatic proof that $\nabla_E^{[n+1]} \circ \nabla_E^{[n]} = R_E \wedge \mathrm{id}$

The following lemma almost mirrors the one on page 304 of [10], although we see that by switching sides a power of -1 is introduced, so the construction is not symmetric.

Lemma 20.5. *Let E be a B-A bimodule with extendable right bimodule connection (∇_E, σ_E) whose curvature R_E is a left module map. Then for all $n \geq 1$ we have the following equation:*

$$\nabla_E^{[n]} \circ \sigma_E = \sigma_E(\mathrm{d} \otimes \mathrm{id}) + (-1)^n(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E) : \Omega_B^n \otimes_B E \to E \otimes_A \Omega_A^{n+1}. \quad (47)$$

Proof. Recall that the curvature $R_E = \nabla_E^{[1]} \circ \nabla_E = (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})\nabla_E$ is always a right module map.

(1) First we show the $n = 1$ case. Let $b \in B$, $e \in E$, and write $\nabla_E(e) = f \otimes \xi$. Then we calculate:

$$\begin{aligned}
R_E(be) &= (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})\nabla_E(be) = (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})(bf \otimes \xi + \sigma_E(\mathrm{d}b \otimes e)) \\
&= \nabla_E(bf) \wedge \xi + bf \otimes \mathrm{d}\xi + \nabla^{[1]}\sigma_E(\mathrm{d}b \otimes e) \\
&= b.\nabla_E(f) \wedge \xi + \sigma_E(\mathrm{d}b \otimes f) \wedge \xi + bf \otimes \mathrm{d}\xi + \nabla^{[1]}\sigma_E(\mathrm{d}b \otimes e) \\
&= b.R_E(e) + \sigma_E(\mathrm{d}b \otimes f) \wedge \xi + \nabla^{[1]}\sigma_E(\mathrm{d}b \otimes e)
\end{aligned}$$

However, since R_E is a left module map, it satisfies $R_E(be) = b.R_E(e)$, and so we have:

$$0 = \sigma_E(\mathrm{d}b \otimes f) \wedge \xi + \nabla^{[1]}\sigma_E(\mathrm{d}b \otimes e)$$

Using this result, for a general 1-form $\eta = c\mathrm{d}b$ (summation omitted) we calculate:

$$\begin{aligned}
&\sigma_E(c\mathrm{d}b \otimes f) \wedge \xi + \nabla^{[1]}\sigma_E(c\mathrm{d}b \otimes e) \wedge \xi = c\sigma_E(\mathrm{d}b \otimes f) + \nabla^{[1]}(c\sigma_E(\mathrm{d}b \otimes e)) \\
&= c\big(\sigma_E(\mathrm{d}b \otimes f) \wedge \xi + \nabla_E^{[1]}\sigma_E(\mathrm{d}b \otimes e)\big) + (\sigma_E \wedge \mathrm{id})(\mathrm{d}c \otimes \sigma_E(\mathrm{d}b \otimes e)) \\
&= 0 + (\sigma_E \wedge \mathrm{id})(\mathrm{d}c \otimes \sigma_E(\mathrm{d}b \otimes e)) = \sigma_E(\mathrm{d}c \wedge \mathrm{d}b \otimes e) = \sigma_E(\mathrm{d}(c\mathrm{d}b) \otimes e)
\end{aligned}$$

where we have used that $0 = \sigma_E(\mathrm{d}b \otimes f) + \nabla^{[1]}\sigma_E(\mathrm{d}b \otimes e)$ and then the extendability of σ_E. Re-arranging this, we get:

$$\begin{aligned}
\nabla^{[1]}\sigma_E(\eta \otimes e) &= \sigma_E(\mathrm{d}\eta \otimes e) - \sigma_E(\eta \otimes f) \wedge \xi \\
&= \sigma_E(\mathrm{d} \otimes \mathrm{id})(\eta \otimes e) + (-1)^1(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E)(\eta \otimes e)
\end{aligned}$$

This shows the $\nabla_E^{[1]}\sigma_E$ case.

(2) Next we suppose the formula holds for $\nabla_E^{[n]}\sigma$ and use induction to show it for $n+1$. Suppose $\eta, \xi \in \Omega^1_B$ and $e \in E$. Expressing $\nabla^{[n+1]}\sigma_E(\eta \wedge \xi \otimes e)$ in string diagrams in Figure 20, we use extendability of σ_E, then the formula for $\nabla_E^{[n+1]}$, then the Leibniz rule on $\wedge$, then recognise the formula for $\nabla_E^{[n]}$, then use the induction assumption, then use associativity of $\wedge$, then recognise the formula for $\nabla_E^{[n]}$, then use the induction assumption again, then finally we re-arrange using the Leibniz rule for $\wedge$ and associativity of $\wedge$ and extendability of σ. Hence $\nabla_E^{[n+1]} \circ \sigma_E = \sigma_E(\mathrm{d} \otimes \mathrm{id}) + (-1)^n(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E)$. □

In particular, if $R_E = 0$ then the composition $\nabla_E^{[n+1]} \circ \nabla_E^{[n]} = R_E \wedge \mathrm{id}$ vanishes, making the flat connection ∇_E a cochain differential. We use this later to give a filtration.

21 Theory: Fibre Bundles (Right-handed Version)

The paper [14] gives a definition of differential fibre bundles, in which given an algebra map $\pi : B \to A$ which extends to a map π^* of differential graded algebras, where differential forms of degree p in the base and q in the fibre are given by the quotient $\frac{\pi^*\Omega_B^p \wedge \Omega_A^q}{\pi^*\Omega_B^{p+1} \wedge \Omega_A^{q-1}}$.

In this paper we take a similar approach, but instead represent these forms by a quotient that doesn't require an algebra map. This comes at the cost of now needing a bimodule instead of just a module, and a bimodule connection instead of just a one-sided connection. But since Hilbert C*-bimodules with inner products correspond via the KSGNS

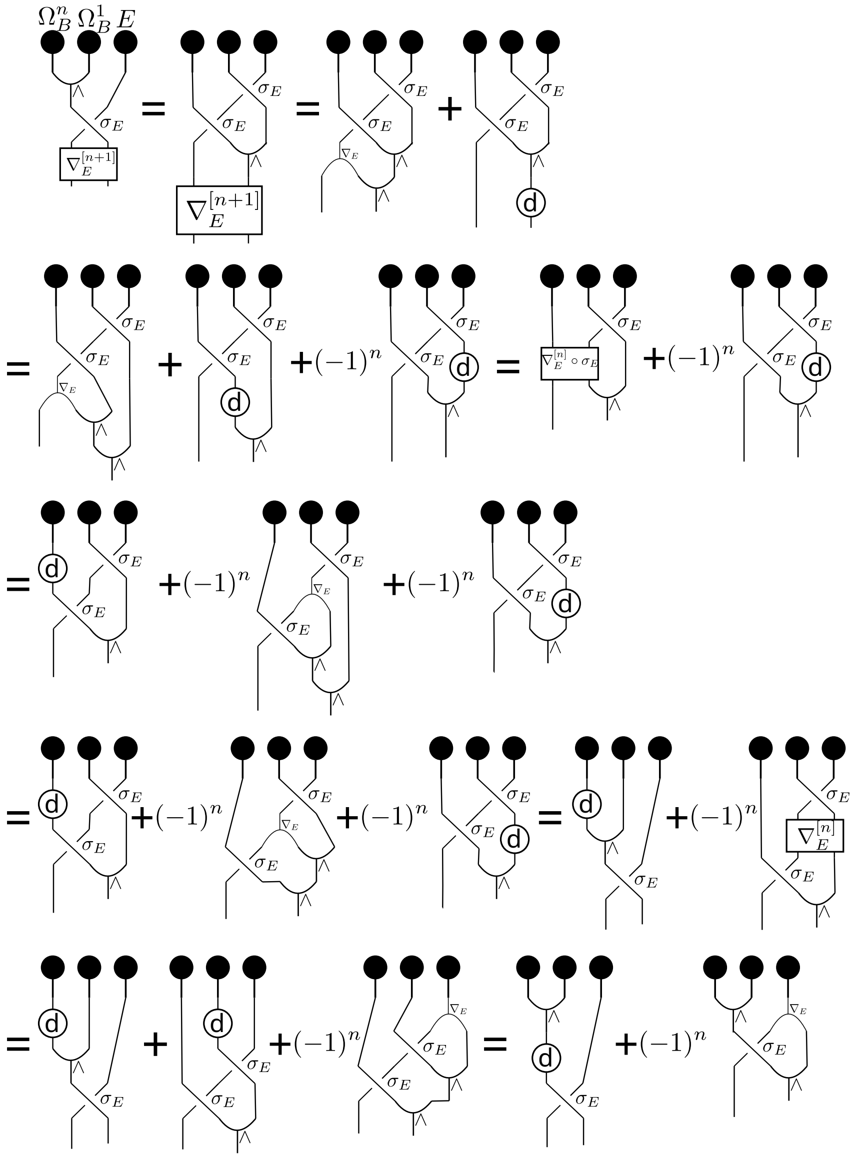


Figure 20: Fibre Bundles: Induction step in proof that $\nabla_E^{[n+1]} \circ \sigma_E = \sigma_E(\mathrm{d} \otimes \mathrm{id}) + (-1)^n(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E)$

construction to completely positive maps, and every *-algebra map is also a completely positive map, this constitutes a generalisation. We start by defining the filtration.

Proposition 21.1. *Let E be a B-A bimodule with extendable zero-curvature right bimodule connection (∇_E, σ_E). For $m \leq n$, the cochain complex $C^n = E \otimes_A \Omega^n_A$ with differential $\mathrm{d}_C := \nabla_E^{[n]} : C^n \to C^{n+1}$ gives the following filtration.*

$$F^m C^n = im\Big(\sigma_E \wedge \mathrm{id} : \Omega^m_B \otimes_B E \otimes_A \Omega^{n-m}_A \to E \otimes_A \Omega^n_A\Big) \tag{48}$$

Proof. (1) The first property we need for a filtration is $\mathrm{d}_C F^m C \subset F^m C$ for all $m \geq 0$.

This means showing $\nabla_E^{[n]} F^m(E \otimes_A \Omega_A^n) \subset \bigoplus_{n' \geq 0} F^m(E \otimes_A \Omega_A^{n'})$.

In the calculations in Figure 21 we start with $\nabla_E^{[n]}(\sigma_E \wedge \mathrm{id})$, then use the fact that $\nabla_E^{[n]} = \mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id}$, then use associativity of $\wedge$ and expand $\mathrm{d}\wedge$, then recognise the formula for $\nabla_E^{[m]}$, then use the formula $\nabla_E^{[m]} \circ \sigma_E = \sigma_E(\mathrm{d} \otimes \mathrm{id}) + (-1)^m(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E)$ we showed earlier, then use associativity of $\wedge$, then recognise the formula for $\nabla_E^{[n-m]}$.

This is in $F^{m+1}C^{n+1} + F^m C^{n+1}$. However, as we will show in the next step, the filtration is decreasing, so as required, it is contained in $F^m C$.

(2) The second property we need for a filtration is $F^{m+1}C \subset F^m C$ for all $m \geq 0$.

In a differential calculus (as opposed to a more general differential graded algebra), elements of the higher calculi can all be decomposed into wedge products of elements of Ω^1, and so $\Omega_B^{m+1} = \Omega_B^m \wedge \Omega_B^1$. Let $\xi \in \Omega_B^m$, $\eta \in \Omega_B^1$, $e \in E$, $\kappa \in \Omega_A^{n-m-1}$. Then $\xi \wedge \eta \otimes e \otimes \kappa \in \Omega_B^{m+1} \otimes_B E \otimes \Omega_A^{n-m-1}$, so the map $\sigma_E \wedge id$ takes it to $E \otimes_A \Omega_A^n$, and the image of all such things is $F^{m+1}C^n$. We have the string diagram Figure 22 for $(\mathrm{id} \otimes \wedge)(\sigma \otimes \mathrm{id})(\wedge \otimes \mathrm{id} \otimes \mathrm{id})(\xi \otimes \eta \otimes e \otimes \kappa)$, where we use that σ_E is extendable and that $\wedge$ is associative. This shows that $F^{m+1}C^n$ lies in $\mathrm{im}(\sigma_E \wedge \mathrm{id}) : \Omega_B^m \otimes_B E \otimes_A \Omega_A^{n-m} \to E \otimes_A \Omega_A^n$, i.e. in $F^m C^n$, and hence that the filtration is decreasing in m.

(3) The third property we need is $F^0 C = C$.

$$F^0 C^n = \mathrm{im}(\sigma_E \wedge \mathrm{id}) : B \otimes_B E \otimes_A \Omega_A^n \to E \otimes_A \Omega_A^n$$

Recalling that $\sigma_E(1 \otimes e) = e \otimes 1$ when $m = 0$, the set $F^0 C^n$ consists of elements $b.e \otimes \xi$, which gives all of C^n.

(4) The final property we need is $F^m C^n := F^m C \cap C^n = \{0\}$ for all $m > n$. This holds because for $m > n$, we have $\Omega^{n-m} = 0$, giving $F^m C^n = im(\sigma_E \wedge \mathrm{id}) : 0 \to C^n$, which has zero intersection with C^n. $\square$

Definition 21.2. Given a filtration as above, we define *differential forms with coefficients* in E of degree p in the fibre and q in the base as the quotient

$$M_{p,q} := \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\sigma_E(\Omega_B^p \otimes_B E) \wedge \Omega_A^q}{\sigma_E(\Omega_B^{p+1} \otimes_B E) \wedge \Omega_A^{q-1}}, \tag{49}$$

and from these we denote forms with coefficients in E of degree q in the fibre only as:

$$N_q := M_{0,q} = \frac{C^q}{F^1 C^q} = \frac{E \otimes_A \Omega_A^q}{\sigma_E(\Omega_B^1 \otimes_B E) \wedge \Omega_A^{q-1}}. \tag{50}$$

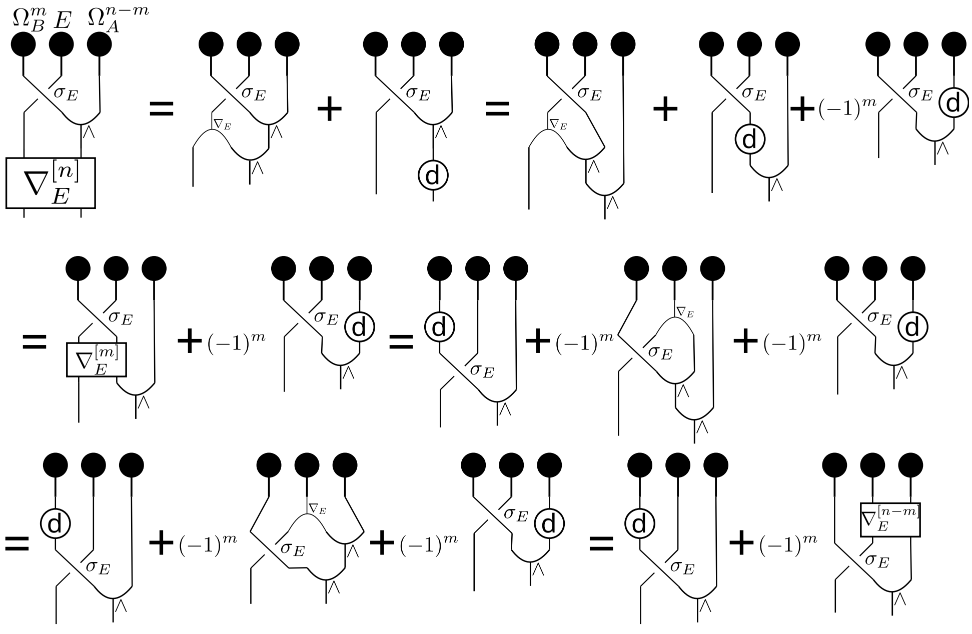


Figure 21: Fibre Bundles: Proof that $\mathrm{d}_C(F^mC) \subset F^mC$

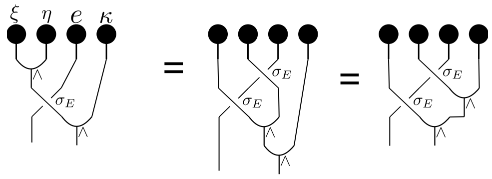


Figure 22: Fibre Bundles: Proof the filtration is decreasing

Proposition 21.3. *Let E be a B-A bimodule with extendable zero-curvature right bimodule connection (∇_E, σ_E). Then there is a well-defined surjective linear map:*

$$g : \Omega_B^p \otimes_B N_q \to M_{p,q}, \qquad g(\xi \otimes [e \otimes \eta]) = [(\sigma_E \wedge \mathrm{id})(\xi \otimes e \otimes \eta)]. \tag{51}$$

Proof. Surjectivity follows from the definition of the map, so we only need to show that g is well-defined on equivalence classes, i.e. that if $[e \otimes \eta] = 0$ then we also have $[(\sigma_E \wedge \mathrm{id})(\xi \otimes e \otimes \eta)] = 0$. By definition, we have $[e \otimes \eta] = 0 \in N_q$ if and only if $e \otimes \eta = (\sigma_E \wedge \mathrm{id})(\xi' \otimes f \otimes \eta')$ for some $\xi' \in \Omega_B^1$, $f \in E$, $\eta' \in \Omega_A^{q-1}$ (summation implicit). Thus, using associativity of $\wedge$ and then extendability of σ, we can re-write $g(\xi \otimes [e \otimes \eta])$ as in Figure 23, which we can see is in the image of $\sigma_E \wedge \mathrm{id} : \Omega_B^{p+1} \otimes_B E \otimes_A \Omega_A^{q-1} \to E \otimes_A \Omega_A^{p+q}$, and hence has equivalence class zero in $M_{p,q}$. □

In a classical fibre bundle, the differential forms on the total space would split into a direct sum of forms in the direction of the base space and forms in the direction of the

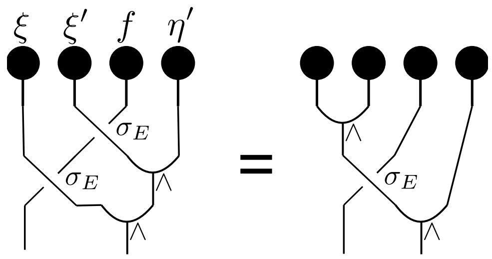


Figure 23: Fibre Bundles: Re-writing $g(\xi \otimes [e \otimes \eta])$

fibre, but in a noncommutative context there is no obvious algebra that can be called the fibre. Consequently, in the following definition of a bimodule noncommutative fibre bundle (note that the algebra maps approach employs a similar idea), we take the quotient of forms on the total space by forms on the base space as a stand-in for forms on the fibre. In the classical case where there is a direct sum, this quotient reduces to the usual differential forms on the fibre.

Definition 21.4. For algebras A and B, we call a B-A bimodule E a *(bimodule) differential fibre bundle* if it satisfies the following three properties:

(1) There is an extendable zero-curvature right bimodule connection (∇_E, σ_E) on E.

(2) For all $p \geq 1$ the calculi Ω^p_B are flat as right modules

(3) For all $p, q \geq 0$ the map g is an isomorphism.

Remark 21.5. *Recall that flatness of Ω^p_B as a right B-module means that if*

$$0 \longrightarrow E_1 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} E_1 \longrightarrow 0$$

is a short exact sequence of left B-modules and left B-module maps then the following sequence of left B-modules and left B-module maps is also short exact.

$$0 \longrightarrow \Omega^p_B \otimes_B E_1 \xrightarrow{\mathrm{id}\otimes\phi_1} \Omega^p_B \otimes_B E_2 \xrightarrow{\mathrm{id}\otimes\phi_2} \Omega^p_B \otimes_B E_3 \longrightarrow 0.$$

If Ω^p_B is finitely generated projective, then flatness is automatic.

In the remainder of this section, we show that for each bimodule differential fibre bundle, we can construct a Leray-Serre spectral sequence. The following Lemma corresponds to Lemma 4.64 of [10], but with a number of differences to adapt it from algebra maps fibre bundles to bimodule fibre bundles. One difference is that here we no longer need to make any assumption about the flatness of E as a module, since E is now built into g. This allows us to calculate sheaf cohomology with coefficients in a wider range of bimodules.

Lemma 21.6. *For a bimodule differential fibre bundle (E, ∇_E), there is a cochain complex*

$$\cdots \xrightarrow{d} M_{p,q-1} \xrightarrow{d} M_{p,q} \xrightarrow{d} M_{p,q+1} \xrightarrow{d} \cdots ,$$

whose differential on $M_{p,q}$ is $[\nabla_E^{[p+q]}]$, and whose cohomology we denote as $\hat{H}^q(M_{p,q})$. Then by equation (50), $\hat{H}^q(N) := \hat{H}^q(M_{0,q})$. The isomorphism g is a (graded) cochain map, and extends to the following isomorphism of cohomology:

$$\hat{g} : \Omega^p_B \otimes_B \hat{H}^q(M_{0,q}) \to \hat{H}^q(M_{p,q}), \qquad \xi \otimes [[e \otimes \eta]] \mapsto [[\sigma_E(\xi \otimes e) \wedge \eta]]. \tag{52}$$

Proof. (1) The differential is well-defined by Proposition 21.1, and satisfies $\mathrm{d}^2 = 0$ by flatness of ∇_E.

(2) Recall that (E, ∇_E) being a bimodule differential fibre bundle implies that for all $p, q \geq 0$ there is an isomorphism $g : \Omega^p_B \otimes_B N_q \to M_{p,q}$. We need to show that the differential $[\nabla_E^{[p+q]}]$ commutes with g, i.e. that the following diagram commutes:

$$\begin{array}{ccc} \Omega^p_B \otimes_B N_q & \xrightarrow{(-1)^p \mathrm{id} \otimes [\nabla_E^{[q]}]} & \Omega^p_B \otimes_B N_{q+1} \\ \downarrow g & & \downarrow g \\ M_{p,q} & \xrightarrow{[\nabla_E^{[p+q]}]} & M_{p,q+1} \end{array}$$

In the proof that $(F^m C, \nabla_E^{[n]})$ is a filtration, we calculated (diagrammatically) that

$$\nabla_E^{[p+q]}(\sigma_E \wedge \mathrm{id}) = \sigma_E(\mathrm{d} \otimes \mathrm{id}) \wedge \mathrm{id} + (-1)^p(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E^{[q]}).$$

However, we know that the term $\sigma_E(\mathrm{d} \otimes \mathrm{id}) \wedge \mathrm{id}$ has equivalence class zero in $M_{p,q+1}$. Taking equivalence classes therefore gives

$$[\nabla_E^{[p+q]} \circ g] = (-1)^p[(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E^{[q]})].$$

Going the other way around the diagram, we get

$$g((-1)^p \mathrm{id} \otimes [\nabla_E^{[q]}]) = (-1)^p[(\sigma_E \wedge \mathrm{id})(\mathrm{id} \otimes \nabla_E^{[q]})].$$

These coincide, so the diagram commutes.

(3) Secondly, we need to show that g extends to cohomology, i.e. that the map

$$\Omega^p_B \otimes_B \hat{H}^q(N_q) \to \hat{H}^q(M_{p,q}), \qquad \xi \otimes [[e \otimes \eta]] \mapsto [[\sigma_E(\xi \otimes e) \wedge \eta]]$$

is an isomorphism. Make the following two definitions.

$$Z_{p,q} := \mathrm{im(d)} : \mathrm{M_{p,q-1}} \to \mathrm{M_{p,q}}, \qquad \mathrm{K_{p,q}} := \ker(\mathrm{d}) : \mathrm{M_{p,q}} \to \mathrm{M_{p,q+1}}$$

Therefore $\hat{H}^{p+q}(M_{p,q}) = \frac{K_{p,q}}{Z_{p,q}}$.

Next we show that the differential $\mathrm{d} : M_{0,q} \to M_{0,q+1}$ is a left B-module map. We take $[\nabla^{[q]}_E(b.e \otimes \eta)]$ and apply the definition of $\nabla^{[q]}$, then use the Leibniz rule, then use the fact that $[\sigma_E(\mathrm{d}b \otimes e) \wedge \eta] = 0$ to calculate:

$$\begin{aligned}
[\nabla^{[q]}_E(b.e \otimes \eta)] &= [b.e \otimes \mathrm{d}\eta] + [\nabla_E(b.e) \wedge \eta] \\
&= [b.e \otimes \mathrm{d}\eta] + [\sigma_E(\mathrm{d}b \otimes e) \wedge \eta] + [b\nabla_E(e) \wedge \eta] \\
&= [b(\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})(e \otimes \eta)] \\
&= [b\nabla^{[q]}_E(e \otimes \eta)].
\end{aligned}$$

Hence there is an exact sequence of left B-modules and left B-module maps:

$$0 \longrightarrow K_{0,q} \xrightarrow{\mathrm{inc}} M_{0,q} \xrightarrow{\mathrm{d}} Z_{0,q+1} \longrightarrow 0$$

Taking the tensor product with the flat right module Ω^p_B gives another exact sequence:

$$0 \longrightarrow \Omega^p_B \otimes_B K_{0,q} \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} \Omega^p_B \otimes_B M_{0,q} \xrightarrow{\mathrm{id}\otimes\mathrm{d}} \Omega^p_B \otimes_B Z_{0,q+1} \longrightarrow 0$$

Applying g to the elements of this sequence, the first part of the proof tells us that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^p_B \otimes_B K_{0,q} & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & \Omega^p_B \otimes_B M_{0,q} & \xrightarrow{\mathrm{id}\otimes\mathrm{d}} & \Omega^p_B \otimes_B Z_{0,q+1} & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow g & & \downarrow g & & \\
 & & M_{p,q} & \xrightarrow{\mathrm{id}} & M_{p,q} & \xrightarrow{(-1)^p\mathrm{d}} & M_{p,q+1} & &
\end{array}$$

Note that the middle instance of g is an isomorphism, while the first and third are merely injective. This diagram gives the following two isomorphisms.

$$Z_{p,q+1} \cong \Omega^p_B \otimes_B Z_{0,q+1}, \qquad K_{p,q+1} \cong \Omega^p_B \otimes_B K_{0,q+1},$$

and consequently

$$Z_{p,q} \cong \Omega^p_B \otimes_B Z_{0,q}, \qquad K_{p,q} \cong \Omega^p_B \otimes_B K_{0,q}.$$

By definition of $\hat{H}^q(N) = \frac{K_{0,q}}{Z_{0,q}}$, we have another short exact sequence:

$$0 \longrightarrow Z_{0,q} \xrightarrow{\text{inc}} K_{0,q} \longrightarrow \hat{H}^q(N) \longrightarrow 0$$

Taking the tensor product with the flat right module Ω^p_B gives the exact sequence:

$$0 \longrightarrow \Omega^p_B \otimes_B Z_{0,q} \xrightarrow{\text{id}\otimes\text{inc}} \Omega^p_B \otimes_B K_{0,q} \longrightarrow \Omega^p_B \otimes_B \hat{H}^q(N) \longrightarrow 0$$

Therefore

$$\Omega^p_B \otimes_B \hat{H}^q(N) \cong \frac{\Omega^p_B \otimes_B K_{0,q}}{\Omega^p_B \otimes_B Z_{0,q}} \cong \frac{K_{p,q}}{Z_{p,q}} = \hat{H}^{p+q}(M_{p,q}).$$

This is the isomorphism we wanted to show. □

Proposition 21.7. *If for $e \in E$, $\xi \in \Omega^q_A$ we denote (with summation implicit)*

$$\nabla^{[q]}_E(e \otimes \xi) = \sigma_E(\eta \otimes f) \wedge \kappa \in \sigma_E(\Omega^1_B \otimes E) \wedge \Omega^q_A,$$

then the map

$$\nabla_q : \hat{H}^q(N) \to \Omega^1_B \otimes_B \hat{H}^q(N), \qquad \nabla_q([[e \otimes \xi]]) = \eta \otimes [[f \otimes \kappa]] \tag{53}$$

defines a zero-curvature left connection on the cohomology of the fibre.

Proof. (1) Firstly, we show that the map ∇_q is well-defined. Since $\hat{H}^q(N) = \frac{\ker[\nabla^{[q]}_E]}{\mathrm{im}[\nabla^{[q-1]}_E]}$, it follows that for all $e \in E$, $\xi \in \Omega^q_A$ such that $[[e \otimes \xi]] \in \hat{H}^q(N)$, the equivalence class $[\nabla^{[n]}_E(e \otimes \xi)]$ vanishes in N_{q+1}. But for $\nabla^{[n]}_E(e \otimes \xi)$ to lie in the denominator of $N_{q+1} = \frac{E \otimes_A \Omega^{q+1}_A}{\sigma_E(\Omega^1_B \otimes_B E) \wedge \Omega^q_A}$ means that there exist $\eta \in \Omega^1_B$, $f \in E$, $\kappa \in \Omega^q_A$ such that

$$\nabla^{[n]}_E(e \otimes \xi) = \sigma_E(\eta \otimes f) \wedge \kappa \in \sigma_E(\Omega^1_B \otimes E) \wedge \Omega^q_A.$$

Applying the isomorphism $\hat{g}^{-1}$ to $[[\nabla^{[n]}_E(e \otimes \xi)]]$ then gives $\eta \otimes [[f \otimes \kappa]]$.

(2) Next, we show for all $b \in B$ that ∇_q satisfies the left Leibniz rule. We take $\nabla^{[q]}_E(be \otimes \xi)$ and use the definition of $\nabla^{[q]}_E$ then the braided Leibniz rule for ∇_E and then once again the definition of $\nabla^{[q]}_E$ to calculate:

$$\nabla^{[q]}_E(be \otimes \xi) = (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})(be \otimes \xi)$$

$$= be \otimes \mathrm{d}\xi + \nabla_E(be) \wedge \xi$$

$$= be \otimes \mathrm{d}\xi + \sigma_E(\mathrm{d}b \otimes e) \wedge \xi + b\nabla_E(e) \wedge \xi$$

$$= \sigma_E(\mathrm{d}b \otimes e) \wedge \xi + b\nabla_E^{[q]}(e \otimes \xi)$$

Taking equivalence classes and using the isomorphism $\hat{g}$ gives the desired result that

$$\nabla_q(b.[[e \otimes \xi]]) = \mathrm{d}b \otimes [[e \otimes \xi]] + b\nabla_q([[e \otimes \xi]]).$$

(3) Lastly, we show that the curvature, $R_q = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_q)\nabla_q$ vanishes.
Denoting $\nabla_E^{[q]}(f \otimes \kappa) = \sigma_E(\eta' \otimes f') \wedge \kappa'$, we have:

$$R_q([[e \otimes \xi]]) = \mathrm{d}\eta \otimes [[f \otimes \kappa]] + \eta \wedge \nabla_q([[f \otimes \kappa]])$$

$$= \mathrm{d}\eta \otimes [[f \otimes \kappa]] + \eta \wedge \eta' \otimes [[f' \otimes \kappa']]$$

To show this vanishes, we want to show $\mathrm{d}\eta \otimes [f \otimes \kappa] + \eta \wedge \eta' \otimes [f' \otimes \kappa'] = 0$.
As the curvature R_E of ∇_E vanishes, we have:

$$0 = \nabla_E^{[q+1]} \circ \nabla_E^{[q]}(e \otimes \xi) = \nabla_E^{[q+1]}(\sigma_E(\eta \otimes f) \wedge \kappa)$$

$$= (\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \wedge \nabla_E)(\sigma_E(\eta \otimes f) \wedge \kappa)$$

Taking equivalence classes and using the isomorphism g, we get

$$0 = (\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \wedge \nabla_E)(\eta \otimes [f \otimes \kappa])$$

$$= \mathrm{d}\eta \otimes [f \otimes \kappa] + \eta \wedge \nabla_E([f \otimes \kappa])$$

$$= \mathrm{d}\eta \otimes [f \otimes \kappa] + \eta \wedge \eta' \otimes [f' \otimes \kappa']$$

as required. Hence $R_q = 0$. □

Recall that the sheaf cohomology group $H^p(B, \hat{H}^q(N), \nabla_q)$ is defined as the cohomology at $\Omega^p_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^q(N)$ in the following sequence (which is not necessarily exact).

$$0 \longrightarrow \hat{H}^q(N) \xrightarrow{\nabla_q} \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^q(N) \xrightarrow{\nabla_q^{[1]}} \Omega^2_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^q(N) \xrightarrow{\nabla_q^{[2]}} \cdots$$

In the sense of [8], equipping $\hat{H}^q(N)$ with a zero-curvature connection makes it a sheaf, i.e. we can do sheaf cohomology with coefficients in $\hat{H}^q(N)$.
There is a spectral sequence for the filtration, which has first page

$$E_1^{p,q} = H^{p+q}(M_{p,q}) \cong H^{p+q}(\Omega^p_B \otimes_B N_q) = \Omega^p_B \otimes_B \hat{H}^q(N),$$

and second page position (p, q) given by $H^p(B, \hat{H}^q(N), \nabla_q)$, and which converges to $H(A, E, \nabla_E)$ in the sense described in the background section.

22 Theory: Fibre Bundles (Left-handed Version)

By symmetry of modules, this construction can be mirrored to use an A-B bimodule E with an extendable zero-curvature left bimodule connection (∇_E, σ_E), where $\nabla_E : E \to \Omega^1_A \otimes_A E$ and $\sigma_E : E \otimes_B \Omega^1_B \to \Omega^1_A \otimes_A E$. In this case, zero curvature means $R_E = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\nabla_E = 0$. The bimodule connection satisfies

$$\nabla^{[n]}_E \sigma_E = (\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}) + \sigma_E(\mathrm{id} \otimes \mathrm{id}) : E \otimes_B \Omega^n_B \to \Omega^{n+1}_A \otimes_A E.$$

The cochain complex $C^n = \Omega^n_A \otimes_A E$ with differential $C^n \to C^{n+1}$ given by

$$\mathrm{d}_C = \nabla^{[n]}_E = \mathrm{id} \otimes \mathrm{d} + (-1)^n \nabla_E \wedge \mathrm{id}$$

has a filtration

$$F^m C^n = \mathrm{im}(\mathrm{id} \wedge \sigma_{\mathrm{E}}) : \Omega^{\mathrm{n-m}}_{\mathrm{A}} \otimes_{\mathrm{A}} \mathrm{E} \otimes_{\mathrm{B}} \Omega^{\mathrm{m}}_{\mathrm{B}} \to \Omega^{\mathrm{n}}_{\mathrm{A}} \otimes_{\mathrm{A}} \mathrm{E}.$$

The quotients for the fibre are given as follows.

$$M_{p,q} := \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\Omega^q_A \wedge \sigma_E(E \otimes_B, \Omega^p_B)}{\Omega^{q-1}_A \wedge \sigma_E(E \otimes_B \Omega^{p+1}_B)}$$
$$N_q := M_{0,q} = \frac{C^q}{F^1 C^q} = \frac{\Omega^q_A \otimes_A E}{\Omega^{q-1}_A \wedge \sigma_E(E \otimes_B \Omega^1_B)},$$

There is then a well-defined map

$$g : N_q \otimes_B \Omega^p_B \to M_{p,q}, \qquad [\eta \otimes e] \otimes \xi \mapsto [(\mathrm{id} \wedge \sigma)(\eta \otimes e \otimes \xi)]$$

which extends to cohomology.

We say that E is a differential fibre bundle if g is an isomorphism for all $p, q \geq 0$ and if the calculi Ω^p_B are flat as left modules for all $p \geq 0$

On the cohomology we have the following a zero-curvature right connection.

$$\nabla_q : \hat{H}^q(N) \to \hat{H}^q(N) \otimes_B \Omega^1_B, \qquad \nabla_q([[\xi \otimes e]]) = [[\kappa \otimes f]] \otimes \eta,$$

where $\nabla^{[q]}_E(\xi \otimes e) = \kappa \wedge \sigma_E(f \otimes \eta) \in \Omega^q_A \wedge \sigma_E(E \otimes_B \Omega^1_B) \subset \Omega^{q+1}_A \otimes_A E$ with summation implicit. Assuming that we have a differential fibre bundle, there is then a spectral sequence converging to $H(A, E, \nabla_E)$ with first page position (p, q) given by $E^{p,q}_1 = \hat{H}^q(N) \otimes_B \Omega^p_B$ and second page position (p, q) given by $H^p(B, \hat{H}^q(N), \nabla_q)$.

23 Positive Maps and the KSGNS Construction

In this section we discuss the KSGNS construction and its relation to our definition of bimodule fibre bundles.

In a C*-algebra A, a positive element is one that can be written in the form a^*a for some $a \in A$. A linear map $\phi : B \to A$ between C*-algebras is called positive if it maps positive elements to positive elements, and completely positive if for all $n \geq 2$ the map

$$\phi : M_n(B) \to M_n(A), \qquad \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \to \begin{pmatrix} \phi(b_1) & \phi(b_2) \\ \phi(b_3) & \phi(b_4) \end{pmatrix}$$

is also positive.

Every *-algebra map is completely positive, and so are all positive linear functions $B \to \mathbb{C}$. The KSGNS theorem (see [30] for reference) gives a correspondence between bimodules and completely positive maps. Suppose A and B are C*-algebras, and E a Hilbert B-A bimodule with an inner product $\langle , \rangle : \overline{E} \otimes_B E \to A$. We use $\overline{E}$ to denote the conjugate module of E as defined as in [13], which has elements $\overline{e}$ is an element of $\overline{E}$ for each $e \in E$, and satisfies $\lambda\overline{e} = \overline{\lambda^* e}$ for scalars $\lambda \in \mathbb{C}$, and has A-B bimodule structure given by $a\overline{e} = \overline{ea^*}$ and $\overline{e}b = \overline{b^* e}$ for $a \in A$, $b \in B$, $e \in E$.

According to the KSGNS theorem, if a map $\phi : B \to A$ can be written in the form $\phi(b) = \langle \overline{e}, be \rangle$ for some $e \in E$, then ϕ is completely positive. Conversely, if we have a completely positive map $\phi : B \to A$ between unital C*-algebras, then we can construct a Hilbert B-A bimodule E and find an element $e_0 \in E$ such that $\phi(b) = \langle \overline{e_0}, be_0 \rangle$. The process to construct this bimodule is to first take the B-A bimodule $B \otimes A$ with actions given by multiplication, equipped with inner product $\langle , \rangle : \overline{B \otimes A} \otimes_B B \otimes A \to A$ given by $\langle \overline{b \otimes a}, b' \otimes a' \rangle = a^* \phi(b^* b') a'$. Next we quotient the bimodule by all zero-length elements with respect to this inner product. Lastly we take the completion with respect to the inner product, and obtain the bimodule E.

Proposition 23.1. *(Proposition 4.86 of [10]) Suppose that A is a unital dense *-subalgebra of a C*-algebra, (E, ∇_E, σ_E) a right B-A bimodule connection which is extendable with curvature R_E a bimodule map and $\langle , \rangle : \overline{E} \otimes_B E \to A$ a semi-inner product A-module structure preserved by ∇_E. If $e \in E$ obeys $\nabla_E(e) = 0$ then $\phi : \Omega_B \to \Omega_A$, $\phi(\xi) = (\langle , \rangle \otimes \mathrm{id})(\overline{E} \otimes \sigma_E(\xi \otimes e))$ is a cochain map, i.e. $\mathrm{d} \circ \phi = \phi \circ \mathrm{d}$.*

24 Example: Group Algebras

[Algebras: See Example 3.11 for $\mathbb{C}G$ and its calculi]

In this first example, we look at a fibre bundle which could also be handled by existing theory, but which is nicely calculable and thus verifies that the bimodules approach to fibre bundles gives the right kind of results. In the two examples after this, we look at one finite and one infinite-dimensional example for which we do not have algebra maps and so the algebra maps approach would not work, but where by the bimodule approach we are able to get Leray-Serre spectral sequences.

Associated to each finite group X is its group algebra $\mathbb{C}X$, whose basis is given by the elements of X. A general element of $\mathbb{C}X$ takes the form $\sum_{x\in X} \lambda_x x$, where λ_x are complex numbers. Note that in general, $\mathbb{C}X$ is not commutative unless X is commutative.

For a right representation V of X, a surjective map $\omega : \mathbb{C}X \to V$ satisfying $\omega(xy) = \omega(x) \triangleleft y + \omega(y)$ for $x, y \in X$ is called a cocycle. This property allows the calculation of ω on any element of X as a product of generators. It follows that $\omega(x^{-1}) = -\omega(x) \triangleleft x^{-1}$ and that $\omega(1) = 0$. By results in [33], left covariant calculi on $\mathbb{C}X$ are classified by cocycles. These calculi are given by $\Omega^1_{\mathbb{C}X} = \Lambda^1_{\mathbb{C}X} \otimes \mathbb{C}X$ with exterior derivative $\mathrm{d}x = x\omega(x)$, right action $(v \otimes x).y = v \otimes xy$ and left action $x.(v \otimes y) = v \triangleleft x^{-1} \otimes xy$. We abbreviate the calculus as $\Omega^1_{\mathbb{C}X} = \Lambda^1_{\mathbb{C}X}.\mathbb{C}X$. The calculus is connected if and only if for all $x \in X\backslash\{0\}$ we have $\omega(x) \neq 0$. For a connected calculus, we have $H_{dR}(\mathbb{C}X) = \Lambda_{\mathbb{C}X}$.

Lemma 24.1. *Let X be a finite group with calculus given by a right representation V and a cocycle $\omega : \mathbb{C}X \to V$, and which has a subgroup G. Then subspace W of V spanned by $\omega(g)$ for all $g \in G$ is a right representation of G, and has complement $W^\perp$ which is also right representation of G.*

Proof. The cocycle condition $\omega(x) \triangleleft y = \omega(xy) - \omega(y)$ defines a right action on W, which gives a calculus on $\mathbb{C}G$. Since G is a finite group, the representation V has an invariant inner product $\overline{V} \otimes V \to C$ (invariant meaning $\langle \overline{v \triangleleft g}, v \triangleleft g\rangle = \langle \overline{v}, v\rangle$), from which it follows that $V = W \oplus W^\perp$, where $W^\perp$ is the perpendicular complement of W. The vector space $W^\perp$ is then also a representation of G. □

The restriction of ω to a cocycle $\mathbb{C}G \to W$ gives a calculus on the subgroup G.

Proposition 24.2. *If for the higher calculi on $\mathbb{C}X$ we assume that $\mathrm{d}(V)=0$, then the wedge product $\wedge$ is antisymmetric on invariant elements $\Lambda_{\mathbb{C}X}$.*

Proof. Since $v \triangleleft x = x^{-1}vx$, it follows that $x(v\triangleleft x)=vx$. Applying d to this and using the assumption that $\mathrm{d}(V)=0$, we obtain $\mathrm{d}x\wedge(v\triangleleft x)=-v\wedge \mathrm{d}x$.
Using this, we calculate the following.

$$v\wedge\omega(x)=v\wedge(x^{-1}\mathrm{d}x)=(vx^{-1})\wedge\mathrm{d}x=x^{-1}(v\triangleleft x^{-1})\wedge\mathrm{d}x=-x^{-1}\mathrm{d}x\wedge v=-\omega(x)\wedge v.$$

Since the images $\omega(x)$ span V, this proves that $\wedge$ is antisymmetric on $\Lambda_{\mathbb{C}X}$. □

Now we look at fibre bundles. Suppose G is a finite subgroup of a group X, and take $A=\mathbb{C}X$, $B=\mathbb{C}G$ as in the discussion of fibre bundles earlier. Equip $\mathbb{C}X$ with calculus as above for $\Lambda^1_{\mathbb{C}X}=V$ and some cocycle $\omega:\mathbb{C}X\to V$ for some right representation V of $\mathbb{C}X$. For the higher calculi on $\mathbb{C}X$ take maximal prolongation plus the assumption $\mathrm{d}(V)=0$. For the calculus on $\mathbb{C}G$ take $\Lambda^1_{\mathbb{C}G}=W=\omega(\mathbb{C}G)$ with cocycle the restriction of ω to $\mathbb{C}G$, and maximal prolongation for the higher calculi.

Proposition 24.3. *A $\mathbb{C}G$-$\mathbb{C}X$ bimodule is given by $E=\mathbb{C}X$ with left and right actions given by multiplication. When the algebras are equipped with the calculi above there is a zero-curvature extendable right bimodule connection on E given by (∇_E,σ_E), where*

$$\nabla_E:\mathbb{C}X\to\mathbb{C}X\otimes_{\mathbb{C}X}\Omega^1_{\mathbb{C}X},\qquad x\mapsto 1\otimes\mathrm{d}x,$$

$$\sigma_E:\Omega^1_{\mathbb{C}G}\otimes_{\mathbb{C}G}\mathbb{C}X\to\mathbb{C}X\otimes_{\mathbb{C}X}\Omega^1_{\mathbb{C}X},\qquad \mathrm{d}g\otimes x\mapsto 1\otimes\mathrm{d}g.x.$$

Proof. The connection satisfies the condition $\nabla_E(gx)=\sigma_E(\mathrm{d}g\otimes x)+g\nabla_E(x)$ required to be a bimodule connection, since $\sigma_E(\mathrm{d}g\otimes x)=1\otimes(\mathrm{d}(gx)-g\mathrm{d}x)=1\otimes\mathrm{d}g.x$. The curvature is zero because d has zero curvature. The connection is extendable as $\sigma_E(\xi\otimes x)=1\otimes\xi.x$ for all $\xi\in\Omega^n_{\mathbb{C}G}$. □

Proposition 24.4. *Equip $A=\mathbb{C}X$, $B=\mathbb{C}G$ with calculi as above, the B-A bimodule $E=\mathbb{C}X$ with actions given by multiplication. The right bimodule connection (∇_E,σ_E) as above, given by $\nabla_E(x)=1\otimes\mathrm{d}x$ and $\sigma_E(\mathrm{d}g\otimes x)=1\otimes\mathrm{d}g.x$, is a differential fibre bundle. The fibres are $N_q\cong(W^\perp)^{\wedge q}.\mathbb{C}X$, on which a differential $\mathrm{d}:N_q\to N_{q+1}$ is given by*

$$\mathrm{d}(\xi.x)=(-1)^{|\xi|}\xi\wedge\pi^\perp(\omega(x)\triangleleft x^{-1}).x \tag{54}$$

for $\xi \in (W^\perp)^{\wedge q}$ and $x \in X$, and where we write $\pi^\perp$ for the projection $V \to W^\perp$ which has kernel W. The differential $\nabla_q : \hat{H}^q(N) \to \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^q(N)$ is given by

$$\nabla_q([\xi.x]) = \pi(\omega(x) \triangleleft x^{-1}) \otimes [\xi.x]. \tag{55}$$

The fibre bundle E gives rise to a spectral sequence converging to $H(\mathbb{C}X, E, \nabla_E) \cong H_{\mathrm{dR}}(\mathbb{C}X)$ with second page position (p,q) given by $H^p(\mathbb{C}G, \hat{H}^q(N), \nabla_q)$

Proof. **(1)** Firstly we show that E is a differential fibre bundle. The calculi $\Omega^p_B = \Omega^p_{\mathbb{C}G}$ are finitely generated projective for all $p \geq 0$ and therefore flat as modules, and the bimodule connection has zero curvature and is extendable. Lastly we need to show that the map $g : \Omega^p_{\mathbb{C}G} \otimes_{\mathbb{C}G} M_{0,q} \to M_{p,q}$ given by $g(\xi \otimes [e \otimes \eta]) = [(\sigma_E \wedge \mathrm{id})(\xi \otimes e \otimes \eta)] = [e \otimes \xi \wedge \eta]$ is an isomorphism. Using the fact that $x\xi = x\xi x^{-1}x = (\xi \triangleleft x^{-1})x$ to move all elements of the group to the right, and then the fact that $V = W \oplus W^\perp$, we calculate:

$$M_{p,q} = \frac{\sigma_E(\Omega^p_{\mathbb{C}X} \otimes_{\mathbb{C}X} E) \wedge \Omega^q_{\mathbb{C}G}}{\sigma_E(\Omega^{p+1}_{\mathbb{C}X} \otimes_{\mathbb{C}X} E) \wedge \Omega^{q-1}_{\mathbb{C}G}} = \frac{W^{\wedge p} \wedge V^{\wedge q}}{W^{\wedge p+1} \wedge V^{\wedge q-1}}.\mathbb{C}X \cong W^{\wedge p} \otimes (W^\perp)^{\wedge q}.\mathbb{C}X.$$

The above isomorphism sends $[w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge v_{j_1} \wedge \cdots \wedge v_{j_q}] \to w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge v_{j_1} \wedge \cdots \wedge v_{j_q}$ where the w_{i_k} are basis elements of W and the v_{i_k} are basis elements of V. The map g sending $W^{\wedge p} \otimes (W^\perp)^{\wedge q} \in \Omega^p_{\mathbb{C}G} \otimes_{\mathbb{C}X} M_{0,q}$ to $W^{\wedge p} \otimes (W^\perp)^{\wedge q} \in M_{p,q}$ then is an isomorphism.
(2) The fibres are $N_q \cong \frac{\Omega^q_{\mathbb{C}X}}{\Omega^1_{\mathbb{C}G} \wedge \Omega^{q-1}_{\mathbb{C}X}} \cong (W^\perp)^{\wedge q}.\mathbb{C}X$. The differential $\mathrm{d} : N_q \to N_{q+1}$ is given by $\mathrm{d}(\xi.x) = (-1)^q[\xi \wedge \mathrm{d}x]$ for $\xi \in (W^\perp)^{\wedge q}$ and $x \in X$, but we can use the fact that $\mathrm{d}x = (\omega(x) \triangleleft x^{-1}).x$ to write the differential on N_q as $\mathrm{d}(\xi.x) = (-1)^{|\xi|}\xi \wedge \pi^\perp(\omega(x) \triangleleft x^{-1}).x$. The cohomology of the fibre is then $\hat{H}^q(N) = \frac{\ker \mathrm{d}: N^{q-1} \to N^q}{\mathrm{im} \mathrm{d}: N^q \to N^{q+1}}$, using this differential. The differential $\nabla_q : \hat{H}^q(N) \to \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^q(N)$ on the cohomology groups is given by

$$\nabla_q([\xi \otimes x]) = g^{-1}([\pi(\omega(x) \triangleleft x^{-1}) \wedge \xi.x]) = \pi(\omega(x) \triangleleft x^{-1}) \otimes [\xi.x].$$

□

Group algebras of finite groups are C*-algebras, with a $*$-map given by $(\lambda_x x)^* = \lambda_x^* x^{-1}$ and extended linearly. The bimodule E has an inner product $\langle , \rangle : \overline{E} \otimes_{\mathbb{C}G} E \to \mathbb{C}X$ given by $\langle \overline{x}, y \rangle = x^* y = x^{-1}y$, and the Leibniz rule shows that ∇_E preserves this inner product. On a C*-algebra with an inner product we can use the KSGNS construction to obtain positive maps. The kernel of ∇_E consists of $\mathbb{C}.e$, and so the positive map we get via the KSGNS construction is $\langle \overline{e}, ge \rangle = g$. This is just the inclusion map, which is an algebra map.

24.1 Example: S_3

Next, we do a full calculation of the spectral sequence for the example of S_3 and its subgroup generated by the cycle $u = (1, 2)$.

Example 24.5. Let $X = S_3$, denote transpositions as $u = (12)$ and $v = (23)$, and then define a subgroup $G = \{e, u\} \subset S_3$. An example of a right representation of X is given by $V = \mathbb{C}^2$ with right action $(v_1, v_2) \triangleleft x = (v_1, v_2)\rho(x)$ for the homomorphism $\rho : S_3 \to End(V)$ given by $\rho(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$and $\rho(v) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. To define a calculus on $X = \mathbb{C}S_3$ (and therefore by restriction a calculus on G) we need a cocycle $\omega : S_3 \to \mathbb{C}^2$ satisfying $\omega(xy) = \omega(x)\rho(y) + \omega(y)$. For the cocycle to be a well-defined linear map, we need to be able to apply ω to the three relations of S_3, which are $u^2 = e$, $v^2 = e$, and $uvu = vuv$. If we write $\omega(v) = (a, b)$ and $\omega(u) = (c, d)$, we have the following.

(1) Recalling that $\omega(e) = 0$, the relation $u^2 = e$ gives:

$$0 = \omega(u^2) = \omega(u)\rho(u) + \omega(u) = (c, d)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + (c, d) = (c, -d) + (c, d) = (2c, 0).$$

Hence $c = 0$. We can normalise to get $d = 1$ so that $\omega(u) = (0, 1)$.

(2) The relation $v^2 = e$ gives:

$$\begin{aligned} 0 &= \omega(v^2) = \omega(v)\rho(v) + \omega(v) = \omega(v)(\rho(v) + I_2) = (a, b)\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \\ &= \frac{1}{2}(a + \sqrt{3}b, \sqrt{3}a + 3b) \end{aligned}$$

Both equations arising from this give that $a = -\sqrt{3}b$. We already normalised when defining $\omega(u)$, so we simply have b as a free parameter, giving $\omega(v) = (-\sqrt{3}b, b)$.

(3) Finally we have the relation $uvu = vuv$. We calculate:

$$\begin{aligned} \omega(uvu) &= \omega(u) + \omega(uv)\rho(u) = \omega(u) + \omega(v)\rho(u) + \omega(u)\rho(v)\rho(u) \\ &= (0, 1) + (-\sqrt{3}b, b)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + (0, 1)\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \\ &= (0, 1) + (-\sqrt{3}b, -b) + (\frac{1}{2}\sqrt{3}, -\frac{1}{2}) = (\frac{1}{2}\sqrt{3} - \sqrt{3}b, \frac{1}{2} - b). \end{aligned}$$

Flipping u and v in the above, we calculate:

$$\begin{aligned} \omega(vuv) &= \omega(v) + \omega(vu)\rho(v) = \omega(v) + \omega(u)\rho(v) + \omega(v)\rho(u)\rho(v) \\ &= (-\sqrt{3}b, b) + (0, 1)\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} + (-\sqrt{3}b, b)\frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \end{aligned}$$

$$= (-\sqrt{3}b, b) + (\frac{1}{2}\sqrt{3}, \frac{1}{2}) + (0, -2b) = (\frac{1}{2}\sqrt{3} - \sqrt{3}b, \frac{1}{2} - b).$$

This shows that the equality $\omega(uvu) = \omega(vuv)$ follows automatically once we assume that $0 = \omega(u^2) = \omega(v^2)$, and hence we get no new restrictions on b as a result of this relation. This gives a 1-parameter family of 2D calculi on $\mathbb{C}X$, with Λ^1 generated by $e_u = \omega(u) = (0,1)$ and $e_v = \omega(v) = b(-\sqrt{3}, 1)$, where $b \in \mathbb{C}$ is a free parameter. Take the calculus on $\mathbb{C}G$ to be the vector space W generated by e_u, so $\Omega^0_{\mathbb{C}G} = \mathbb{C}\{1, u\}$ and $\Omega^1_{\mathbb{C}G} = \omega(u).\mathbb{C}\{1, u\}$.

We now calculate the de Rham cohomology. As long as $b \neq \frac{1}{2}$, this ω doesn't send any elements of X other than e to zero, the calculus on $\mathbb{C}X$ is connected, and hence has de Rham cohomology $H_{dR}(\mathbb{C}X) = \Lambda_{\mathbb{C}X}$. This gives $H^0_{dR}(\mathbb{C}X) \cong \mathbb{C}$ and $H^1_{dR}(\mathbb{C}X) \cong \mathbb{C} \oplus \mathbb{C}$, while $H^2_{dR}(\mathbb{C}X)$ is a quotient of $\mathbb{C}^2 \wedge \mathbb{C}^2$. Since the wedge product is antisymmetric under the assumption $\mathrm{d}V = 0$, a basis of this is given by $\omega(u) \wedge \omega(v)$, and $H^2_{dR}(\mathbb{C}X) \cong \mathbb{C}$.

We now calculate the Leray-Serre spectral sequence explicitly for this example, where $E = A = \mathbb{C}X$, $B = \mathbb{C}G$ for $X = S_3$ and G the subgroup generated by the cycle $u = (12)$. As $\omega(u) = (0,1)$ is a basis of W, it follows that $(1,0)$ is a basis of $W^\perp$, and hence $\pi^\perp(x,y) = (x, 0)$.

From the formula above that $N_q \cong (W^\perp)^{\wedge q}.\mathbb{C}X$, we have $N_0 \cong \mathbb{C}X$ and $N_1 \cong (1,0).\mathbb{C}X$. All the other N_q are zero, since $W^{\wedge 2}$ and $(W^\perp)^{\wedge 2}$ are zero, seeing as W and $W^\perp$ are 1-dimensional and the wedge product is antisymmetric on V.

The one non-trivial differential is therefore $\mathrm{d} : N_0 \to N_1$, given by $\mathrm{d}x = \pi^\perp(\omega(x) \triangleleft x^{-1}).x$. The kernel of $\mathrm{d} : N_0 \to N_1$ is two-dimensional with basis elements e and u. The reason that the identity element e lies in the kernel is because $\omega(e) = 0$, while u is in the kernel because $\omega(u) \triangleleft u^{-1} = \omega(u)\rho^{-1}(u) = (0,1)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{-1} = (0, -1)$, which is sent by $\pi^\perp$ to zero. The image of $\mathrm{d} : N_0 \to N_1$ is four-dimensional with basis elements $(1,0).v$, $(1,0).uv$, $(1,0).vu$, $(1,0).uvu$.

Hence $H^0(N)$ is two-dimensional with basis elements $[e]$ and $[u]$, while $H^1(N)$ is two-dimensional with basis $[(1,0).e]$ and $[(1,0).u]$.

The differential $\nabla_0 : \hat{H}^0(N) \to \Omega^1_B \otimes_B \hat{H}^0(N)$ is given on basis elements by $\nabla_0(e) = \pi(\omega(e) \triangleleft e^{-1}) \otimes [e] = 0$ and $\nabla_0(u) = \pi(\omega(u) \triangleleft u^{-1}) \otimes [u] = (0, -1) \otimes [u]$.

The differential $\nabla_0 : \hat{H}^1(N) \to \Omega^1_B \otimes_B \hat{H}^1(N)$ is given on basis elements by $\nabla_1([(1,0).e]) =$

$\pi(\omega(e) \triangleleft e^{-1}) \otimes [(0,1).e] = 0$ and $\nabla_1([(0,1).u]) = \pi(\omega(u) \triangleleft u^{-1}) \otimes [(0,1).u] = (0,-1) \otimes [(0,1).u]$.

Hence ∇_0 has kernel spanned by $[e]$ and image spanned by $(0,1).[u]$, while ∇_1 has kernel spanned by $[(0,1).e]$ and image spanned by $(0,1) \otimes [(0,1).u]$.

Seeing as $\Omega^p_{\mathbb{C}X} = 0$ for $p \geq 2$ and $\hat{H}^q(N) = 0$ for $q \geq 2$, the sequences for the cohomology are the following two.

$$0 \longrightarrow \hat{H}^0(N) \xrightarrow{\nabla_0} \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^0(N) \longrightarrow 0$$

$$0 \longrightarrow \hat{H}^1(N) \xrightarrow{\nabla_1} \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^1(N) \longrightarrow 0$$

$H^0(B, \hat{H}^0(N), \nabla_0)$ is the cohomology at $\hat{H}^0(N)$, which is $\frac{\ker(\nabla_0)}{\mathrm{im}(0)} \cong \langle [e] \rangle_{\mathrm{span}} \cong \mathbb{C}$.

$H^1(B, \hat{H}^0(N), \nabla_0)$ is the cohomology at $\Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^0(N)$, which is $\frac{\Omega^1_{\mathbb{C}G} \otimes \hat{H}^1(N)}{\mathrm{im}(\nabla_0)} \cong \langle (0,1) \otimes [e] \rangle_{\mathrm{span}} \cong \mathbb{C}$.

$H^0(B, \hat{H}^1(N), \nabla_1)$ is the cohomology at $\hat{H}^1(N)$, which is $\frac{\ker(\nabla_1)}{\mathrm{im}(0)} \cong \langle [(0,1).e] \rangle_{\mathrm{span}} \cong \mathbb{C}$.

$H^1(B, \hat{H}^1(N), \nabla_1)$ is the cohomology at $\Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} \hat{H}^1(N)$, which is $\frac{\Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G}}{\mathrm{im}\nabla_1} \cong \langle (1,0) \otimes [(1,0).e] \rangle_{\mathrm{span}} \cong \mathbb{C}$.

Page 2 of the Leray-Serre spectral sequence has entries $E_2^{p,q} = H^p(\mathbb{C}G, \hat{H}^q(N), \nabla_q)$, with $E_2^{0,0}, E_2^{0,1}, E_2^{1,0}, E_2^{1,1}$ as its nonvanishing entries. This is stable already, and hence the nontrivial cohomology groups are the following direct sums along diagonals.

$$H^0(\mathbb{C}S_3, E, \nabla_E) \cong H^0(B, \hat{H}^0(N), \nabla_0) \cong \mathbb{C}$$
$$H^1(\mathbb{C}S_3, E, \nabla_E) \cong H^1(B, \hat{H}^0(N), \nabla_0) \oplus H^0(B, \hat{H}^1(N), \nabla_1) \cong \mathbb{C} \oplus \mathbb{C}$$
$$H^2(\mathbb{C}S_3, E, \nabla_E) \cong H^1(B, \hat{H}^1(N), \nabla_1) \cong \mathbb{C}$$

This is the same as the de Rham cohomology $H_{dR}(\mathbb{C}X)$ that we calculated earlier. ◇

Note that in [39], a different calculus on S_3 is obtained by using the same right action ρ but on the representation $V = M_2(\mathbb{C})$ instead of $V = \mathbb{C}^2$.

25 Example: Matrices

[Algebras: See Example 3.12 for $M_2(\mathbb{C})$ and its calculus]

In [10] an inner calculus on the matrix algebra $M_2(\mathbb{C})$ is given by $\mathrm{d}b = [\theta', b] = \theta' b - b\theta'$ for $b \in M_2(\mathbb{C})$ and inner element $\theta' = E_{12}s' + E_{21}t'$, where s' and t' are central (i.e. they commute with any algebra element). The maximal prolongation calculus has the relation $s' \wedge t' = t' \wedge s'$.

We extend this idea to $M_3(\mathbb{C})$, giving it an inner calculus by $\theta = E_{12}s + E_{21}t + E_{33}u$ for central elements s, t, u. The differential $\mathrm{d} : M_3(\mathbb{C}) \to \Omega^1_{M_3(\mathbb{C})}$ is then given by $\mathrm{d}a = [\theta, a] = [E_{12}, a]s + [E_{21}, a]t + [E_{33}, a]u$, which on a general matrix in $M_3(\mathbb{C})$ is the following.

$$\mathrm{d}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e-a & f \\ 0 & -d & 0 \\ 0 & -g & 0 \end{pmatrix} s + \begin{pmatrix} -b & 0 & 0 \\ a-e & b & c \\ -h & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & -c \\ 0 & 0 & -f \\ g & h & 0 \end{pmatrix} u \tag{56}$$

From this we can see that $\mathrm{d}E_{33} = 0$, which means the calculus is not connected, since a connected calculus needs $\ker \mathrm{d} = \mathbb{C}.I_3$.

For a higher order inner calculus, the differential is given by $\mathrm{d}\xi = \theta \wedge \xi - (-1)^{|\xi|}\xi \wedge \theta$ for the inner element θ. For example, since $|u| = 1$, we have $\mathrm{d}u = \theta \wedge u + u \wedge \theta$, and similarly for s and t.

Proposition 25.1. *Equipping $M_3(\mathbb{C})$ with higher order inner calculus for the inner element $\theta = E_{12}s + E_{21}t + E_{33}u$ necessitates that $s \wedge t = t \wedge s = u \wedge u$.*

Proof. As the calculus is inner, the differential is given by $\mathrm{d}a = \theta a - a\theta$. If we apply the differential twice to an element $a \in M_3(\mathbb{C})$, we get $\mathrm{d}^2 a = \theta \wedge (\theta a - a\theta) - (\theta a - a\theta) \wedge \theta = \theta \wedge \theta a - \theta \wedge a\theta + \theta \wedge a\theta - a\theta \wedge \theta = \theta \wedge \theta a - a\theta \wedge \theta = [\theta \wedge \theta, a]$. For d to be well-defined as a differential we need $\mathrm{d}^2 a$ to vanish, so $\theta \wedge \theta$ needs to be central so that its commutator with anything vanishes. We calculate $\theta \wedge \theta = (E_{12}s + E_{21}t + E_{33}u) \wedge (E_{12}s + E_{21}t + E_{33}u) = E_{11}s \wedge t + E_{22}t \wedge s + E_{33}u \wedge u$. The only central elements of $M_3(\mathbb{C})$ are multiples of I_3, and hence for $\theta \wedge \theta$ to be central we require $s \wedge t = t \wedge s = u \wedge u$. □

Although the additional assumptions that $u \wedge t = t \wedge u$ and $u \wedge s = s \wedge u$ are not mandatory, we make these as well so that all the generators of the calculi commute. Based on a private communication [29], these extra assumptions bring the growth of the calculi down from exponential to polynomial. With these additional assumptions, the derivatives of the calculi's basis elements are $\mathrm{d}s = 2s \wedge \theta$, $\mathrm{d}t = 2t \wedge \theta$ and $\mathrm{d}u = 2u \wedge \theta$.

For $A = M_3(\mathbb{C})$ and $B = M_2(\mathbb{C})$, an example of a B-A bimodule is given by $E = M_{2,3}(\mathbb{C})$.

Proposition 25.2. *Suppose we equip A with inner calculus as above given by inner element $\theta = E_{12}s + E_{21}t + E_{33}u$. Then a right zero-curvature connection $\nabla_E : E \to$*

$E \otimes_A \Omega^1_A$ *satisfying* $\nabla_E(e_0) = 0$ *for* $e_0 = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{smallmatrix}\right)$ *is well-defined and takes the form* $\nabla_E(e_0 a) = e_0 \otimes \mathrm{d}a$. *This connection becomes an extendable bimodule connection by the bimodule map* $\sigma_E : \Omega^1_B \otimes_B E \to E \otimes_A \Omega^1_A$ *given by* $\sigma_E(\mathrm{d}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes e_0) = e_0 \otimes \mathrm{d}\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, *which satisfies* $\sigma_E(s' \otimes e_0) = e_0 \otimes s$ *and* $\sigma_E(t' \otimes e_0) = e_0 \otimes t$.

Proof. **(1)** First we show well-definedness of ∇_E. Observing that $e_0.\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in E$, the image of this under the linear map ∇_E must be zero, meaning that the differential must satisfy $e_0 \otimes \mathrm{d}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & i \end{smallmatrix}\right) = 0 \in E \otimes_A \Omega^1_A$. Note that this wouldn't be true in the universal calculus. We calculate using the differential above that $e_0 \otimes \mathrm{d}E_{3i} = e_0 \otimes \Big(0 + 0 + [E_{33}, E_{3i}]u\Big) = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{smallmatrix}\right) \otimes (E_{3i} - \delta_{i,3}E_{3,3})u = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{smallmatrix}\right)(E_{3i} - \delta_{i,3}E_{3,3}) \otimes u = 0$, seeing as nonzero entries of $(E_{3i} - \delta_{i,3}E_{3,3})$ can only lie in the third row, and thus ∇_E is well-defined.
(2) Secondly, we calculate ∇_E. We can see that every element of E is of the form $e_0.a$, since $e_0.M_3(\mathbb{C}) = M_{2,3}(\mathbb{C}) = E$. Therefore, using the Leibniz rule and the assumption $\nabla_E(e_0) = 0$, we calculate the connection as $\nabla_E(e_0 a) = \nabla_E(e_0).a + e_0 \otimes \mathrm{d}a = e_0 \otimes \mathrm{d}a$.
(3) Thirdly, the map σ_E satisfies $\sigma_E(\mathrm{d}b \otimes e_0) = \nabla_E(be_0) - b\nabla_E(e_0)$. But $\nabla_E(e_0) = 0$, so $\sigma_E(\mathrm{d}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes e_0) = \nabla_E(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)e_0) = \nabla_E(e_0\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)) = e_0 \otimes \mathrm{d}\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ as required.
(4) Next, we show $\sigma_E(s' \otimes e_0) = e_0 \otimes s$ and $\sigma_E(t' \otimes e_0) = e_0 \otimes t$. In the calculus on B, we have $\mathrm{d}E_{21} = [E_{12}, E_{21}]s' + [E_{21}, E_{21}]t' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)s'$, and likewise on the calculus on A. Therefore, using the fact that σ_E is a bimodule map and that s' is central and also the formula above for σ_E,

$$\begin{aligned} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\sigma_E(s' \otimes e_0) &= \sigma_E(s'\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes e_0) = \sigma_E(\mathrm{d}E_{21} \otimes e_0) \\ &= e_0 \otimes \mathrm{d}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) = e_0 \otimes \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)e_0 \otimes s. \end{aligned}$$

However, as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is invertible, this implies $\sigma_E(s' \otimes e_0) = e_0 \otimes s$. The result $\sigma_E(t' \otimes e_0) = e_0 \otimes t$ follows similarly by considering $\mathrm{d}E_{12} = [E_{12}, E_{12}]s' + [E_{21}, E_{12}]t' = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)t'$.
(5) Lastly, we show extendability. Since $B = M_2(\mathbb{C})$ is equipped with maximal prolongation calculus, Corollary 5.3 of [1] says that every zero-curvature bimodule connection is extendable. $\square$

Next we show that with this bimodule and connection we do indeed get a fibre bundle.

Proposition 25.3. *Suppose* $B = M_2(\mathbb{C})$ *and* $A = M_3(\mathbb{C})$ *are equipped with the above calculi. Then the* B-A *bimodule* $E = M_{2,3}(\mathbb{C})$ *with the bimodule connection* (∇_E, σ_E)

from earlier gives a differential fibre bundle, and thus a spectral sequence converging to $H(A, E, \nabla_E) = H(M_3(\mathbb{C}), M_{2,3}(\mathbb{C}), \nabla_E)$.

Proof. For all $p \geq 0$ the calculi $\Omega^p_B = \Omega^p_{M_2(\mathbb{C})}$ are finitely generated projective and hence flat as modules. The bimodule connection (∇_E, σ_E) satisfies the requirements of having zero curvature and being extendable. The last property we need to show is therefore that the map $g : \Omega^p_B \otimes_B M_{0,q} \to M_{p,q}$ given by $g(\xi \otimes [e \otimes \eta]) = [(\sigma_E \wedge \mathrm{id})(\xi \otimes e \otimes \eta)]$ is an isomorphism.

Since $E = e_0.M_3(\mathbb{C})$, the forms on $M_3(\mathbb{C})$ of degree p in the fibre and q in the base are given by the quotient

$$M_{p,q} = \frac{\sigma_E(\Omega^p_B \otimes_B E) \wedge \Omega^q_A}{\sigma_E(\Omega^{p+1}_B \otimes_B E) \wedge \Omega^{q-1}_A} \cong \frac{\sigma_E(\Omega^p_{M_2(\mathbb{C})} \otimes_{M_2(\mathbb{C})} e_0) \wedge \Omega^q_{M_3(\mathbb{C})}}{\sigma_E(\Omega^{p+1}_{M_2(\mathbb{C})} \otimes_{M_2(\mathbb{C})} e_0) \wedge \Omega^{q-1}_{M_3(\mathbb{C})}}.$$

Everything in the numerator is of the form $(s \text{ or } t)^{\wedge(p+q-k)} \wedge u^{\wedge k}.M_3(\mathbb{C})$ for some $0 \leq k \leq q$, while everything in the denominator is of the form $(s \text{ or } t)^{\wedge(p+q-k+1)} \wedge u^{\wedge(k-1)}.M_3(\mathbb{C})$ for $0 \leq k \leq q$. Since $u \wedge u = s \wedge t$, it follows that if an element of the numerator has $k \geq 2$ then it lies in the denominator. But if an element of the numerator has $k < q$ then it has to lie in the denominator. Therefore $M_{p,q} = 0$ for $q \geq 2$, and (omitting to write the equivalence classes) a basis of $M_{p,0}$ is given by $e_0 \otimes s^{\wedge r} \wedge t^{\wedge(p-r)}$ for some $0 \leq r \leq p$, while a basis of $M_{p,1}$ is given by $e_0 \otimes s^{\wedge r} \wedge t^{\wedge(p-r)} \wedge u$.

In the case $q = 0$, the map g is given on basis elements as

$$(s')^{\wedge r} \wedge (t')^{\wedge(p-r)} \otimes e_0 \longmapsto e_0 \otimes s^{\wedge r} \wedge t^{\wedge(p-r)}.$$

The map g here is an isomorphism, since it just re-arranges the order of the tensor product and and re-labels s' and t', which introduces no new relations.

Similarly in the case $q = 1$, the map g is given on basis elements as

$$(s')^{\wedge r} \wedge (t')^{\wedge(p-r)} \otimes e_0 \otimes u \longmapsto e_0 \otimes s^{\wedge r} \wedge t^{\wedge(p-r)} \wedge u,$$

which is an isomorphism. □

Next we calculate this limit of the spectral sequence.

Proposition 25.4. *The nonzero cohomology groups of A with coefficients in E can be calculated via the Leray-Serre spectral sequence as* $H^0(A, E, \nabla_E) \cong \mathbb{C}$, $H^1(A, E, \nabla_E) \cong \mathbb{C}^6$, $H^2(A, E, \nabla_E) \cong \mathbb{C}^5$.

Proof. (1) The space N_0 is isomorphic to $e_0.A$, which has six-dimensional vector space basis $e_0.E_{ij}$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$ (i.e. excluding the bottom row). The space N_1 is isomorphic to $e_0.A \otimes u$, which has six-dimensional vector space basis $e_0.E_{ij} \otimes u$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Since we showed earlier that $M_{p,q} = 0$ for $q \geq 2$, this means that all the $N_i = M_{0,i} = 0$ for $i \geq 2$.
(2) The differential $\mathrm{d} : N_0 \to N_1$ and is given by $\mathrm{d}([e_0.E_{ij}]) = [\nabla_E(e_0.E_{ij})] = [e_0 \otimes \mathrm{d}E_{ij}] = [e_0.[E_{33}, E_{ij}] \otimes u]$. The kernel has four-dimensional basis $[e_0.E_{11}]$, $[e_0.E_{12}]$, $[e_0.E_{21}]$, $[e_0.E_{22}]$. The image has two-dimensional basis $[e_0.E_{13} \otimes u]$ and $[e_0.E_{23} \otimes u]$.
(3) Consequently $\hat{H}^0(N)$ is four-dimensional with basis elements $[[e_0.E_{11}]]$, $[[e_0.E_{12}]]$, $[[e_0.E_{21}]]$, $[[e_0.E_{22}]]$. Also $\hat{H}^1(N)$ is four-dimensional with basis elements $[[e_0.E_{11} \otimes u]]$, $[[e_0.E_{12} \otimes u]]$, $[[e_0.E_{21} \otimes u]]$, $[[e_0.E_{22} \otimes u]]$.
(4) Next we calculate $\nabla_0 : \hat{H}^0(N) \to \Omega^1_B \otimes_B \hat{H}^0(N)$ on the basis elements of $\hat{H}^0(N)$. For $1 \leq i, j \leq 2$ we have $\nabla_0([[e_0.E_{ij}]]) = g^{-1}([[e_0 \otimes \mathrm{d}E_{ij}]])$. We calculate

$$\begin{aligned}
\nabla_0(e_0.E_{12}) &= g^{-1}([[e_0 \otimes (E_{22} - E_{11})t]]) = t' \otimes [[e_0.(E_{22} - E_{11})]], \\
\nabla_0(e_0.E_{21}) &= s' \otimes [[e_0.(E_{11} - E_{22})]], \\
\nabla_0(e_0.E_{11}) &= -s' \otimes [[e_0.E_{12}]] + t' \otimes [[e_0.E_{21}]] = -\nabla_0(e_0.E_{22}).
\end{aligned}$$

Hence ∇_0 has one-dimensional kernel with basis $[[e_0.(E_{11} + E_{22})]]$, and three-dimensional image with basis elements $t' \otimes [[e_0.(E_{22} - E_{11})]]$, $s' \otimes [[e_0.(E_{11} - E_{22})]]$, $t' \otimes [[e_0.E_{21}]] - s' \otimes [[e_0.E_{12}]]$.
(5) Next we calculate $\nabla_1 : \hat{H}^1(N) \to \Omega^1_B \otimes_B \hat{H}^1(N)$ on the basis elements of $\hat{H}^1(N)$. For $1 \leq i, j \leq 2$, we have

$$\begin{aligned}
\nabla^{[1]}_E(e_0.E_{ij} \otimes u) &= \nabla_E(e_0.E_{ij}) \wedge u + e_0.E_{ij} \otimes \mathrm{d}u = e_0 \otimes [\theta, E_{ij}] \wedge u + 2e_0.E_{ij} \otimes \theta \wedge u \\
&= e_0 \otimes \Big((E_{12}E_{ij} + E_{ij}E_{12})s + (E_{21}E_{ij} + E_{ij}E_{21})t\Big) \wedge u \\
&= \sigma_E(s' \otimes e_0.(E_{12}E_{ij} + E_{ij}E_{12})) \wedge u + \sigma_E(t' \otimes e_0.(E_{21}E_{ij} + E_{ij}E_{21})) \wedge u.
\end{aligned}$$

Consequently,

$$\nabla_1([[e_0.E_{ij} \otimes u]]) = s' \otimes [[e_0.(E_{12}E_{ij} + E_{ij}E_{12}) \otimes u]] + t' \otimes [[e_0.(E_{21}E_{ij} + E_{ij}E_{21}) \otimes u]].$$

Using this, we calculate $\nabla_1([[e_0.E_{12} \otimes u]]) = t' \otimes [[e_0 \otimes u]]$ and $\nabla_1([[e_0.E_{21} \otimes u]]) = s' \otimes [[e_0 \otimes u]]$ and $\nabla_1([[e_0.E_{11} \otimes u]]) = s' \otimes [[e_0.E_{12} \otimes u]] + t' \otimes [[e_0.E_{21} \otimes u]] = \nabla_1([[e_0.E_{22} \otimes u]])$.

Hence the kernel of ∇_1 has one-dimensional basis $[[e_0.(E_{11} - E_{22}) \otimes u]]$, while the image has three-dimensional basis $t' \otimes [[e_0 \otimes u]]$, $s' \otimes [[e_0 \otimes u]]$, $s' \otimes [[e_0.E_{12} \otimes u]] + t' \otimes [[e_0.E_{21} \otimes u]]$.

(6) Next we work out the quotients for cohomology.

(i) Firstly, $H^0(B, \hat{H}^0(N), \nabla_0) \cong \frac{\ker(\nabla_0)}{\mathrm{im}(0)} \cong \mathbb{C}$.

(ii) Secondly, $H^1(B, \hat{H}^0(N), \nabla_0) \cong \frac{\Omega^1_B \otimes \hat{H}^0(N)}{\mathrm{im}(\nabla_0)}$. Seeing as Ω^1_B is a free module with two basis elements and $\hat{H}^0(N)$ is four-dimensional, the vector space $\Omega^1_B \otimes_B \hat{H}^0(N)$ is eight-dimensional. The quotient is therefore five dimensional, and an example of a basis of $\frac{\Omega^1_B \otimes \hat{H}^0(N)}{\mathrm{im}(\nabla_0)}$ is given by $[s' \otimes [[e_0.E_{11}]]]$, $[s' \otimes [[e_0.E_{12}]]]$, $[s' \otimes [[e_0.E_{21}]]]$, $[t' \otimes [[e_0.E_{11}]]]$, $[t' \otimes [[e_0.E_{12}]]]$. Hence $H^1(B, \hat{H}^0(N), \nabla_0) \cong \mathbb{C}^5$.

(iii) Thirdly, $H^0(B, \hat{H}^1(N), \nabla_1) \cong \frac{\ker(\nabla_1)}{\mathrm{im}(0)} \cong \mathbb{C}$.

(iv) Lastly, $H^1(B, \hat{H}^1(N), \nabla_1) \cong \frac{\Omega^1_B \otimes_B \hat{H}^1(N)}{\mathrm{im}(\nabla_1)}$. Seeing as Ω^1_B is a free module with two basis elements and $\hat{H}^1(N)$ is four-dimensional, the vector space $\Omega^1_B \otimes_B \hat{H}^1(N)$ is eight-dimensional. Taking the quotient by the three-dimensional $\mathrm{im}(\nabla_1)$ gives a five-dimensional vector space. Hence $H^1(B, \hat{H}^1(N), \nabla_1) \cong \mathbb{C}^5$.

(7) Page 2 of the Leray-Serre spectral sequence has entries $E_2^{p,q} = H^p(B, \hat{H}^q(N), \nabla_q)$, with $E_2^{0,0}, E_2^{0,1}, E_2^{1,0}, E_2^{1,1}$ as its nonvanishing entries. This is stable already, and hence the nontrivial cohomology groups are the following direct sums along diagonals.

$$H^0(A, E, \nabla_E) \cong H^0(B, \hat{H}^0(N), \nabla_0) \cong \mathbb{C}$$
$$H^1(A, E, \nabla_E) \cong H^1(B, \hat{H}^0(N), \nabla_0) \oplus H^0(B, \hat{H}^1(N), \nabla_1) \cong \mathbb{C}^5 \oplus \mathbb{C} \cong \mathbb{C}^6$$
$$H^2(A, E, \nabla_E) \cong H^1(B, \hat{H}^1(N), \nabla_1) \cong \mathbb{C}^5.$$

□

The bimodule E has inner product $\langle , \rangle : \overline{E} \otimes_B E \to A$ given by $\langle \overline{x}, y \rangle = x^* y$, where $*$ is the conjugate transpose map. As matrix algebras are C*-algebras, the KSGNS construction says that the map $\phi : B \to A$ given by $\phi(b) = \langle \overline{e_0}, b e_0 \rangle$ is completely positive. For $e_0 = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{smallmatrix}\right)$ then ϕ is not an algebra map, seeing as $\phi(I_2) = 4I_3 \neq I_3$, and algebra maps have to send the identity to the identity.

Moreover, $\nabla_E(e_0) = 0$, so for ϕ to be a cochain map we just need metric preservation, which holds because of the following. Recall that for the right connection ∇_E on E, we have $\nabla_E(e_0 a) = e_0 \otimes \mathrm{d}a$, which gives a corresponding left connection $\nabla_{\overline{E}}$ on $\overline{E}$ given by

$\nabla_{\overline{E}}(\overline{e_0 a}) = \mathrm{d}a^* \otimes \overline{e_0}$, and that the inner product on E is given by $\langle \overline{x}, y\rangle = x^{-1}y$. Then:

$$\mathrm{d}a_1{}^*\langle \overline{e_0}, e_0 a_2\rangle + \langle \overline{e_0 a_1}, e_0\rangle \mathrm{d}a_2 = \mathrm{d}a_1{}^*.e_0^* e_0 a_2 + a_1^* e_0^* e_0 \mathrm{d}a_2 = 4\mathrm{d}a_1{}^*.a_2 + 4a_1^*\mathrm{d}a_2$$
$$= 4\mathrm{d}a_1^* a_2 = \mathrm{d}a_1^* e_0^* e_0 a_2 = \mathrm{d}\langle \overline{e_0 a_1}, e_0 a_2\rangle.$$

Thus by Proposition 23.1, ϕ is a completely positive cochain map, but not an algebra map.

26 Example: Quantum Circle in the Quantum Torus

[Algebras: See Example 3.8 for $\mathbb{C}_\theta[\mathbb{T}^2]$ and its calculus, and Example 3.2 for $\mathbb{C}_q[S^1]$ and its calculus]

Next, we look at an infinite dimensional example, with total space algebra the quantum torus $A = \mathbb{C}_\theta[\mathbb{T}^2]$ and base space algebra the quantum circle $B = \mathbb{C}_q[S^1]$. Note that the quantum torus $\mathbb{C}_\theta[\mathbb{T}^2]$ is sometimes also denoted $\mathbb{T}^2_\theta$. As in Example 1.36 of [10], the noncommutative torus is generated as a complex algebra by two invertible generators u, v with the relation $vu = e^{i\theta}uv$ for a real parameter θ. It has calculus $\Omega^1_{\mathbb{C}_\theta[\mathbb{T}^2]} = \mathbb{C}_\theta[\mathbb{T}^2].\{\mathrm{d}u, \mathrm{d}v\}$, with right module structure is given by the following relations.

$$\mathrm{d}u.u = u.\mathrm{d}u, \quad \mathrm{d}v.v = v.\mathrm{d}v, \quad \mathrm{d}v.u = e^{i\theta}u.\mathrm{d}v, \quad \mathrm{d}u.v = e^{-i\theta}v.\mathrm{d}u$$

We take maximal prolongation calculi for the higher calculi on A, giving relations $\mathrm{d}u \wedge \mathrm{d}u = 0 = \mathrm{d}v \wedge \mathrm{d}v$ and $\mathrm{d}v \wedge \mathrm{d}u = -e^{i\theta}\mathrm{d}u \wedge \mathrm{d}v$. These relations imply $\Omega^2_A = \mathrm{d}u \wedge \mathrm{d}v.A$ and $\Omega^q_A = 0$ for $q \geq 3$, with every nonzero element of Ω^2_A a multiple of $\mathrm{d}u \wedge \mathrm{d}v$. The calculus on the quantum circle satisfies $\Omega^n_B = 0$ for $n \geq 2$.

There is a B-A bimodule E given by $E = \mathbb{C}_\theta[\mathbb{T}^2] \oplus \mathbb{C}_\theta[\mathbb{T}^2]$, with left B-action and right A-action given respectively by

$$t \triangleright (f \oplus g) = uf \oplus vg, \quad (f \oplus g) \triangleleft g' = fg' \oplus gg'.$$

Proposition 26.1. *If $B = \mathbb{C}_q[S^1]$ is equipped with classical calculus ($q = 1$), then there is a zero curvature right bimodule connection (∇_E, σ_E) on E, for the right connection $\nabla_E : E \to E \otimes_A \Omega^1_A$ given by*

$$\nabla_E(f \oplus 0) = (1 \oplus 0) \otimes \mathrm{d}f, \quad \nabla_E(0 \oplus g) = (0 \oplus 1) \otimes \mathrm{d}g$$

and the bimodule map $\sigma_E : \Omega^1_B \otimes_B E \to E \otimes_A \Omega^1_A$ *given by*

$$\sigma_E(\mathrm{d}t \otimes (f \oplus 0)) = (1 \oplus 0) \otimes \mathrm{d}u.f, \quad \sigma_E(\mathrm{d}t \otimes (0 \oplus g)) = (0 \oplus 1) \otimes \mathrm{d}v.g.$$

Since higher calculi are zero, this connection is automatically extendable.

Proof. In this proof, we start by using the q-deformed calculus on $\mathbb{C}_q[S^1]$ and then show that for σ_E to be a left module map we need $q = 1$.

(1) Firstly, we calculate the formula for σ_E on the generators.

$$\begin{aligned}\sigma_E(\mathrm{d}t \otimes (f \oplus 0)) &= \nabla_E(t.(f \oplus 0)) - t.\nabla_E(f \oplus 0)\\ &= \nabla_E(uf \oplus 0) - t.(1 \oplus 0) \otimes \mathrm{d}f\\ &= (1 \oplus 0) \otimes \mathrm{d}(uf) - (u \oplus 0) \otimes \mathrm{d}f\\ &= (1 \oplus 0) \otimes \mathrm{d}u.f + (1 \oplus 0) \otimes u.\mathrm{d}f - (u \oplus 0) \otimes \mathrm{d}f\\ &= (1 \oplus 0) \otimes \mathrm{d}u.f\end{aligned}$$

Here we used the standard formula for a bimodule connection that $\sigma(\mathrm{d}a \otimes e) = \nabla(a.e) - a.\nabla(e)$.

$$\begin{aligned}\sigma_E(\mathrm{d}t \otimes (0 \oplus g)) &= \nabla_E(t.(0 \oplus g)) - t.\nabla_E(0 \oplus g)\\ &= \nabla_E(0 \oplus vg) - (0 \oplus v) \otimes \mathrm{d}g\\ &= (0 \oplus 1) \otimes \mathrm{d}(vg) - (0 \oplus v) \otimes \mathrm{d}g\\ &= (0 \oplus 1) \otimes v.\mathrm{d}g + (0 \oplus 1)\mathrm{d}v.g - (0 \oplus v) \otimes \mathrm{d}g\\ &= (0 \oplus 1) \otimes \mathrm{d}v.g\end{aligned}$$

(2) Next we show this is a bimodule map. The calculations in part (1) show that σ_E is a right module map, so we just need to show it is a left module map. Using $t.\mathrm{d}t = q^{-1}\mathrm{d}t.t$ and $\mathrm{d}u.u = u.\mathrm{d}u$, we calculate:

$$\begin{aligned}\sigma_E(t.\mathrm{d}t \otimes (1 \oplus 0)) &= q^{-1}\sigma_E(\mathrm{d}t \otimes t.(1 \oplus 0))\\ &= q^{-1}\sigma_E(\mathrm{d}t \otimes (u \oplus 0))\\ &= q^{-1}(1 \oplus 0) \otimes \mathrm{d}u.u\\ &= q^{-1}(1 \oplus 0) \otimes u.\mathrm{d}u\\ &= q^{-1}(u \oplus 0) \otimes \mathrm{d}u\end{aligned}$$

$$= q^{-1} t \triangleright (1 \oplus 0) \otimes \mathrm{d}u$$

Hence σ_E is a left module map precisely when $q = 1$, i.e. when $\mathbb{C}_q[S^1]$ is equipped with its classical calculus. □

Having a zero curvature extendable bimodule connection, we show the last ingredient required for a bimodule fibre bundle.

Proposition 26.2. *The map* $g : \Omega^p_B \otimes_B M_{0,q} \to M_{p,q}$ *given by*

$$g(\xi \otimes [e \otimes \eta]) = [(\sigma_E \wedge \mathrm{id})(\xi \otimes e \otimes \eta)] \tag{57}$$

is an isomorphism, where forms of degree p in the fibre and q in the base are given by the formula:

$$M_{p,q} = \frac{\sigma_E(\Omega^p_B \otimes_B E) \wedge \Omega^q_A}{\sigma_E(\Omega^{p+1}_B \otimes_B E) \wedge \Omega^{q-1}_A} \tag{58}$$

Proof. **(1)** Putting algebra elements on the right, we calculate $M_{1,0}$, the 1-forms in just the fibre as:

$$\begin{aligned} M_{1,0} = \sigma_E(\Omega^1_B \otimes_B E) &\cong \sigma_E\Big(\mathrm{d}t \otimes \big((1 \oplus 0).A + (0 \oplus 1).A\big)\Big) \\ &\cong (1 \oplus 0) \otimes \mathrm{d}u.A + (0 \oplus 1) \otimes \mathrm{d}v.A. \end{aligned}$$

Noting that $M_{0,0} = E$, the map $g : \Omega^1_B \otimes_B M_{0,0} \to M_{1,0}$ sending

$$g(\mathrm{d}t \otimes (1 \oplus 0)) = (1 \oplus 0) \otimes \mathrm{d}u, \quad g(\mathrm{d}t \otimes (0 \oplus 1)) = (0 \oplus 1) \otimes \mathrm{d}v$$

is an isomorphism.

(2) We calculate $M_{0,1}$, the 1-forms in just the base as:

$$\begin{aligned} M_{0,1} = \frac{E \otimes \Omega^1_A}{\sigma_E(\Omega^1_B \otimes_B E)} &\cong \frac{(1 \oplus 0) \otimes (\mathrm{d}u.A + \mathrm{d}v.A) + (0 \oplus 1) \otimes (\mathrm{d}u.A + \mathrm{d}v.A)}{(0 \oplus 1) \otimes \mathrm{d}v.A + (1 \oplus 0) \otimes \mathrm{d}u.A} \\ &\cong (1 \oplus 0) \otimes \mathrm{d}v.A + (0 \oplus 1) \otimes \mathrm{d}u.A, \end{aligned}$$

but this is already trivially isomorphic to $\Omega^0_B \otimes_B M_{0,1}$.

(3) Using that $\Omega^2_B = 0$, we calculate $M_{1,1}$, the 2-forms of degree 1 in the fibre and 1 in the base as:

$$M_{1,1} = \sigma_E(\Omega^1_B \otimes_B E) \wedge \Omega^1_A \cong (1 \oplus 0) \otimes \mathrm{d}u \wedge \mathrm{d}v.A + (0 \oplus 1) \otimes \mathrm{d}v \wedge \mathrm{d}u.A$$

$$\cong (1 \oplus 0) \otimes \mathrm{d}u \wedge \mathrm{d}v.A + (0 \oplus 1) \otimes \mathrm{d}u \wedge \mathrm{d}v.A$$

Using the formula for $M_{0,1}$ above, we see that the map $g : \Omega^1_B \otimes_B M_{0,1} \to M_{1,1}$ sending

$$g(\mathrm{d}t \otimes (1 \oplus 0) \otimes \mathrm{d}v.A) = (1 \oplus 0) \otimes \mathrm{d}u \wedge \mathrm{d}v.A$$

and

$$g(\mathrm{d}t \otimes (0 \oplus 1) \otimes \mathrm{d}u.A) = (0 \oplus 1) \otimes \mathrm{d}v \wedge \mathrm{d}u.A \cong (0 \oplus 1) \otimes \mathrm{d}u \wedge \mathrm{d}v.A$$

is an isomorphism. □

Having shown that g is an isomorphism, we have a bimodule differential fibre bundle, so there exists a Leray-Serre spectral sequence converging to the sheaf cohomology of the quantum torus with coefficients in the bimodule E.

Equip E with inner product $\langle , \rangle : \overline{E} \otimes_B E \to A$ given by

$$\langle \overline{f_1 \oplus g_1}, f_2 \oplus g_2 \rangle = f_1^* f_2 + g_1^* g_2$$

take $e_0 = f_0 \oplus g_0$. Define $\phi : B \to A$ by

$$\phi(t^n) = \langle \overline{e_0}, b e_0 \rangle = f_0^* u^n f_0 + g_0^* v^n g_0$$

This is not an algebra map, even when $e_0 = 1 \oplus 1$. In the case $e_0 = 1 \oplus 1$, we have $\nabla_E(e_0) = 0$. Hence this example requires our new definition in terms of completely positive maps.

We could calculate the Leray-Serre spectral sequence, but for brevity we omit to do so.

27 Future Ideas and Discussion

One thing which is missing, both from the bimodules approach and the algebra maps approach to fibre bundles, is the idea of a trivial fibre bundle. In the case of a trivial principal bundle in Chapter 5 of [10] there are strong and weak definitions, but both are specific to that case with no clear generalisation.

Also, we might ask how much of the theory of principal bundles can also be done using bimodule-based fibre bundles. Algebra maps seem quite hard to remove from the theory of principal bundles.

Part V

Noncommutative Retracts and Neighbourhood Retracts

28 Noncommutative Retracts

Abstract

We consider some ideas of what a noncommutative retract between unital C*-algebras might be, in terms of completely positive maps and using the KSGNS construction. This generalises an existing definition by Lance. Associated to a completely positive map, there is an intermediate $*$-algebra, and we investigate how this is related to noncommutative neighbourhood retracts.

[51] In topology, given a space N with embedded subspace M, a retract is a map $r : N \to M$ such that the inclusion $\iota : M \to N$ satisfies $r \circ \iota = \mathrm{id}_M$. This implies r is surjective. If the spaces are smooth manifolds, the retract r induces an injective algebra map $r_* : C^\infty(M) \to C^\infty(N)$ by composition on the right with r.

Example 28.1. Denoting S^1 for the unit circle, the map $r : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ given by $r(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$ is a retract by the inclusion $inc : S^1 \to \mathbb{R}^2\setminus\{(0,0)\}$. It also gives a deformation retract by $r_t = (1-t)(x,y) + \frac{t}{\sqrt{x^2+y^2}}(x,y)$. $\diamond$

Example 28.2. Writing D for the unit disk, there is no retract associated to the inclusion $inc : S^1 \to D$, since there is no continuous map from the disk to the circle which fixes the boundary. However, there is a neighbourhood retract, as long as we exclude $(0,0)$ from the neighbourhood around S^1 in D. $\diamond$

In topology, deformation retracts are related to cofibrations as follows.

Proposition 28.3. *([48] Satz 1. See also [57] for English) The inclusion of a closed subspace A in a space X is a cofibration if and only if A is a neighborhood deformation retract of X.*

Correspondingly, a good notion of noncommutative deformation retracts might potentially give some insight into noncommutative cofibrations. Here we don't answer the question of what is a noncommutative deformation retract, but we consider ideas for versions of retracts and neighbourhood retracts. In our definitions, we make the choice of using completely positive maps instead of algebra maps, since there are not a lot of algebra maps, and the KSGNS construction gives a nice way of obtaining completely positive maps. For a reminder on completely positive maps and the KSGNS construction, refer to Section 23.

We mention that on page 55 of [30], there is a definition of when a completely positive map is called a retraction. Lance's definition of a retraction generalises conditional expectations, and in the case where the big algebra is unital, they reduce to conditional expectations. Our definition here is separate to this, although we do find later that conditional expectations of group algebras give an example under our definition of a retract.

Definition 28.4. For unital C*-algebras A and B and unital completely positive maps $\psi : A \to B$ and $\phi : B \to A$, we say that ψ is a *retract* by ϕ if $\phi \circ \psi = \mathrm{id}_A$.

Note that since $(\phi \circ \psi)(a) = a$, a retract ψ must always be injective, and ϕ always surjective.

By complete positivity, the KSGNS construction gives bimodules $E \in {}_B\mathcal{M}_A$ and $F \in {}_A\mathcal{M}_B$ with inner products $\langle , \rangle_E : \overline{E} \otimes_B E \to A$ and $\langle , \rangle_F : \overline{F} \otimes_A F \to B$ such that for all $a \in A$, $b \in B$, the maps ϕ and ψ are specified as $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E$ and $\psi(a) = \langle \overline{f_0}, af_0 \rangle_F$ for some fixed $e_0 \in E$, $f_0 \in F$. In particular, as ϕ and ψ are unital maps, we have $\langle \overline{e_0}, e_0 \rangle_E = 1$ and $\langle \overline{f_0}, f_0 \rangle_F = 1$.

Denote the bimodule $C = F \otimes_B E \in {}_A\mathcal{M}_A$ and the element $c = f_0 \otimes e_0 \in C$. There is an inner product:

$$\langle , \rangle_C : \overline{C} \otimes_A C \to A, \quad \langle \overline{f' \otimes e'}, f \otimes e \rangle_C = \langle \overline{e'}, \langle \overline{f'}, f \rangle_F e \rangle_E.$$

Hence $a = (\phi \circ \psi)(a) = \langle \overline{c}, ac \rangle_C$. We show the following proposition in generality before specialising to $G = C$ and $g = c$.

Proposition 28.5. *Suppose an A-bimodule G has a (possibly degenerate) inner product $\langle , \rangle_G : \overline{G} \otimes_A G \to A$ and $g \in G$, satisfying $\langle \overline{g}, ag \rangle_G = a$ for all $a \in A$. Then:*

1. *The element $ag - ga$ is length zero for all $a \in A$, i.e. that its inner product with everything is zero. Hence if the inner product $\langle , \rangle_G$ is nondegenerate, the element g is central.*

2. *The map* $Q : G \to A$ *given by* $Q(x) = \langle \overline{g}, x\rangle_G$ *is a surjective bimodule map.*

3. G *splits as a right module as* $G = \ker Q \oplus Ag$*, or as a left module by* $a \mapsto ag$ *as* $G = \ker Q \oplus gA$*. If* g *is central then* G *splits as a bimodule.*

Further, if the inner product $\langle , \rangle_G$ *is nondegenerate, then* g *is central, and* G *splits as a bimodule by the bimodule map* $P : G \to G$ *given by* $P(x) = Q(x).g$*.*

Proof. (1) We start with the assumption have $a = \langle \overline{g}, ag\rangle_G$. But since inner products are right module maps, we also have $1.a = \langle \overline{g}, g\rangle_G.a = \langle \overline{g}, ga\rangle_G$. Subtracting equations gives:

$$0 = \langle \overline{g}, ag - ga\rangle_G. \tag{59}$$

Using Equation 59, and then by adding $\langle \overline{g}, a^*g - ga^*\rangle_G.a$ which is the right hand side of Equation 59 with a replaced by a^* and multiplied on the right by a, and is hence still zero.

$$\begin{aligned}\langle \overline{ag - ga}, ag - ga\rangle_G &= \langle \overline{g}, a^*ag - a^*ga\rangle_G - a^*\langle \overline{g}, ag - ga\rangle_G = \langle \overline{g}, a^*ag - a^*ga\rangle_G \\ &= \langle \overline{g}, a^*ag - a^*ga + a^*ga - ga^*a\rangle_G = \langle \overline{g}, (a^*a)g - g(a^*a)\rangle_G\end{aligned}$$

But this is just Equation 59 with a replaced by a^*a and is hence zero, so $ag - ga$ has length zero.

(2) It is clear that Q is a right module map because inner products are bimodule maps. Using that $\langle \overline{g}, ag\rangle_G = a$ for all $a \in A$ (which implies surjectivity), we calculate:

$$Q(ax) = \langle \overline{g}, ax\rangle_G = \langle \overline{a^*g}, x\rangle_G = \langle \overline{a^*g - ga^*}, x\rangle_G + \langle \overline{ga^*}, x\rangle_G = 0 + aQ(x).$$

Hence Q is also a left module map.

(3) We show that P is a bimodule map. It is clear that P, being defined by $P(x) = Q(x)g$ is a left module map if Q is. But $P(xa) - P(x)a = Q(x).(ag - ga)$, which is length zero and vanishes precisely when g is central, i.e. when the inner product is nondegenerate. □

Corollary 28.5.1. *Given a retract* (ψ, F) *by* (ϕ, E) *as in Definition 28.4, and denoting* $c = f_0 \otimes e_0 \in C = F \otimes_B E$*, there is a surjective bimodule map* $Q : C \to A$ *given by* $Q(x) = \langle \overline{c}, x\rangle_C$*, and* $ac - ca$ *is length zero for all* $a \in A$*. Also* C *splits as* $G = \ker Q \oplus cA$ *or* $G = \ker Q \oplus Ac$*. If* $\langle , \rangle_C$ *is nondegenerate, then* c *is central and there exists a bimodule map* $P : C \to C$ *given by* $P(x) = Q(x).c$*, and* C *splits as a bimodule.*

Proof. Since $\phi \circ \psi = \mathrm{id}_A$, we have $a = \langle \overline{c}, ac \rangle_C$ for all $a \in A$. The rest follows by the proposition above. $\square$

We remark that it is not obvious exactly when $\langle , \rangle_C$ is nondegenerate, even if both $\langle , \rangle_E$ and $\langle , \rangle_F$ are.

Proposition 28.6. *Given a retract (ψ, F) by (ϕ, E) as in Definition 28.4, there is a right module map $\omega : F \to \overline{E}$ given by $\omega(f) = \overline{e_0}.\langle \overline{f_0}, f \rangle_F$, which makes the following diagram commute.*

$$\begin{array}{ccc} F \otimes_B E & \xrightarrow{Q} & A \\ \downarrow{\scriptstyle \omega \otimes \mathrm{id}} & \nearrow{\scriptstyle \langle , \rangle_E} & \\ \overline{E} \otimes_B E & & \end{array}$$

If the inner product $\langle , \rangle_E$ is nondegenerate, then:

(1) *ω is a bimodule map.*

(2) *$Be_0 A = Be_0$.*

(3) *$e_0 a = \psi(a) e_0$ for all $a \in A$.*

Proof. (1) We can see that ω is a right module map because inner products are bimodule maps, but we need to show that it is a left module map. We show that $\omega(af) - a\omega(f)$ has length zero, i.e. that its inner product with everything is zero. Using that Q and $\langle , \rangle_E$ are bimodule maps, we calculate for all $e \in E$:

$$\langle a\omega(f), e \rangle_E = a \langle \omega(f), e \rangle_E = aQ(f \otimes e) = Q(af \otimes e) = \langle \omega(af), e \rangle_E.$$

Hence by nondegeneracy of $\langle , \rangle_E$, we have $\omega(af) - a\omega(f) = 0$, making ω a left module map.

(2) By definition, $Be_0 \subset Be_0 A$. But we can re-write ω as $\omega(f) = \overline{\langle \overline{f}, f_0 \rangle_F e_0}$, so ω has image in $\overline{Be_0}$. But because $\omega(f_0) = \overline{e_0}$ and ω is a bimodule map, its image contains $\overline{Be_0 A}$, and hence $Be_0 A \subset Be_0$. Thus $Be_0 A = Be_0$.

(3) Since ω is a bimodule map, we have

$$\omega(af_0) = a\omega(f_0) = a\overline{e_0} = \overline{e_0 a^*}.$$

Because ψ is a completely positive map coming from a Hermitian inner product, it satisfies $\psi(a)^* = \psi(a^*)$. Using this, along with the definition of ω, we calculate:

$$\omega(af_0) = \overline{e_0} \langle \overline{f_0}, af \rangle_F = \overline{e_0} \psi(a) = \overline{\psi(a)^* e_0} = \overline{\psi(a^*) e_0}.$$

This gives that $e_0 a^* = \psi(a^*)e_0$ for all $a \in A$. Swapping a^* for a gives the result. $\square$

29 A Subalgebra Making J a Two-Sided Ideal

Next, we look at certain quotient associated with a retract. However, it isn't clear yet exactly how to interpret it.

Recall from the end of the submanifolds section that there is a bimodule $J = \{b \in B \mid be_0 = 0\}$ which is a left ideal in B. We have $Be_0 \cong B/J$ by the isomorphism $be_0 \mapsto [b]$ of left B-modules.

Proposition 29.1. *Given a retract (ψ, F) by (ϕ, E) as in Definition 28.4, plus the assumption that $\langle , \rangle_E$ is nondegenerate, it follows that the image of ψ lies in the algebra*

$$\mathcal{S} = \{b \in B \mid jb \subset J, \quad \forall j \in J\}, \tag{60}$$

which is a two sided ideal in S, and that the quotiented map $\psi/J : A \to \mathcal{S}/J$ is an algebra map.

Proof. For all $j \in J$ and $a \in A$, we have $0 = je_0 a = j\psi(a)e_0$, which means $J\psi(A) \subset J$. But also $\psi(a)je_0 = \psi(a).0 = 0$, so $\psi(A)J \subset J$. Hence ψ restricts to $\psi : A \to \mathcal{S}$.

Since J is a two-sided ideal of $\mathcal{S}$, there is an algebra map $\pi : \mathcal{S} \to \mathcal{S}/J$ sending each element to its equivalence class, and composing ψ with this we get $\pi \circ \psi : A \to \mathcal{S}/J$. Note that S/J is isomorphic as a B-A bimodule to $\mathcal{S}e_0$ (in fact this defines the right A-action on S/J). We calculate:

$$\psi(aa')e_0 = e_0.aa' = (\psi(a)e_0)a' = \psi(a)\psi(a')e_0,$$

and thus since π is an algebra map, we have

$$\pi(\psi(aa')) = \pi(\psi(a)\psi(a')) = (\pi \circ \psi)(a).(\pi \circ \psi)(a').$$

$\square$

Corollary 29.1.1. *Since $J \subset ker(\phi)$, it follows that if ϕ is a bijection and thereby that $J = 0$, then ψ would be an algebra map.*

The map $\phi : B \to A$ restricts to a map $\phi/J : \mathcal{S}/J \to A$, since $\phi(j) = 0$ for all $j \in J$. Also, this restriction is surjective, since $(\phi \circ \psi)(a) = a$.

We note that the algebra $\mathcal{S}$ is likely not a $*$-algebra, since there is no particular reason why having $jbe_0 = 0$ for all $j \in J$ should imply $jb^*e_0 = 0$ for all $j \in J$.

Since ψ is a completely positive map, $\psi(a^*) = \psi(a)^* \in \mathcal{S}^*$, meaning $\mathrm{im}(\psi) \subset \mathcal{S} \cap \mathcal{S}^*$. Since J is a two-sided ideal of $\mathcal{S}$, it follows that J^* is a two-sided ideal of $\mathcal{S}^*$, so $J \cup J^*$ is a two-sided ideal of $\mathcal{S} \cap \mathcal{S}^*$.

30 An Intermediate Algebra

Given a Hilbert C*-bimodule $E \in {}_B\mathcal{M}_A$, there is a B-bimodule $D = E \otimes_A \overline{E}$. In this section we show that D can be endowed with the structure of an algebra, which is associative and can be made unital under certain assumptions on E, and that there are various interesting maps associated to it. In a certain sense it behaves like an intermediate algebra between A and B.

Proposition 30.1. *If E is a B-A Hilbert C*-bimodule with inner product $\langle , \rangle : \overline{E} \otimes_B E \to A$, then $D = E \otimes_A \overline{E}$ is an associative $*$-algebra when equipped with multiplication*

$$(e_1 \otimes \overline{e_2})(e_3 \otimes \overline{e_4}) = e_1 \langle \overline{e_2}, e_3 \rangle \otimes e_4 \tag{61}$$

and star operation $(e_1 \otimes \overline{e_2})^* = e_2 \otimes \overline{e_1}$.

Proof. On one hand we have

$$\big((e_1 \otimes \overline{e_2})(e_3 \otimes \overline{e_4})\big)(e_5 \otimes \overline{e_6}) = \big(e_1 \otimes \langle \overline{e_2}, e_3 \rangle_E \overline{e_4}\big)\big)(e_5 \otimes \overline{e_6}) = e_1 \otimes \langle \overline{e_2}, e_3 \rangle_E \langle \overline{e_4}, e_5 \rangle_E \overline{e_6},$$

and on the other hand we have

$$(e_1 \otimes \overline{e_2})\big((e_3 \otimes \overline{e_4})(e_5 \otimes \overline{e_6})\big) = (e_1 \otimes \overline{e_2})\big((e_3 \otimes \langle \overline{e_4}, e_5 \rangle_E \overline{e_6})\big) = e_1 \otimes \langle \overline{e_2}, e_3 \rangle_E \langle \overline{e_4}, e_5 \rangle_E \overline{e_6}.$$

Hence D is an associative algebra. □

Note that that with the above set of assumptions D is not unital yet, but later with some extra conditions we add an identity.

Lemma 30.2. *If $E \in {}_B\mathcal{M}_A$ is a Hilbert C^*-bimodule with inner product $\langle , \rangle_E : \overline{E} \otimes_B E \to A$, then E is also a D-A bimodule with left D-action*

$$(e_1 \otimes \overline{e_2}) \triangleright e_3 = e_1 \langle \overline{e_2}, e_3 \rangle_E, \tag{62}$$

and the inner product descends from $\otimes_B$ to $\otimes_D$ as $\langle , \rangle'_E : \overline{E} \otimes_D E \to A$.

Proof. Positivity of $\langle , \rangle$ follows automatically, since it uses the same formula as $\langle , \rangle_E$. **(1)** Firstly, we show that the left D-action on E is well-defined. On one hand:

$$\begin{aligned}(e_1 \otimes \overline{e_2}) \triangleright \big((e_3 \otimes \overline{e_4}) \triangleright e_5\big) &= (e_1 \otimes \overline{e_2}) \triangleright \big(e_3 \langle \overline{e_4}, e_5 \rangle_E\big) \\ &= e_1 \langle \overline{e_2}, e_3 \rangle_E \langle \overline{e_4}, e_5 \rangle_E,\end{aligned}$$

while on the other hand:

$$\begin{aligned}\big((e_1 \otimes \overline{e_2}).(e_3 \otimes \overline{e_4})\big) \triangleright e_5 &= (e_1 \langle \overline{e_2}, e_3 \rangle_E \otimes e_4) \triangleright e_5 \\ &= e_1 \langle \overline{e_2}, e_3 \rangle_E \langle \overline{e_4}, e_5 \rangle_E.\end{aligned}$$

These coincide, so the left D-action is well-defined.
(2) Next, we show that the inner product descends to $\otimes_D$. On one hand

$$\begin{aligned}\langle \overline{e_1}, (e_2 \otimes \overline{e_3} \triangleright e_4) &= \langle \overline{e_1} \otimes e_2 \langle \overline{e_3}, e_4 \rangle_E \rangle'_E \\ &= \langle \overline{e_1}, e_2 \rangle'_E \langle \overline{e_3}, e_4 \rangle_E,\end{aligned}$$

while on the other hand

$$\begin{aligned}\langle \overline{e_1}, (e_2 \otimes \overline{e_3} \triangleright e_4) &= \langle \overline{(e_3 \otimes \overline{e_2}) \triangleright e_1}, e_4 \rangle'_E \\ &= \langle \overline{e_3 \langle \overline{e_2}, e_1 \rangle_E}, e_4 \rangle'_E \\ &= \langle \langle \overline{e_2}, e_1 \rangle^*_E . \overline{e_3}, e_4 \rangle'_E \\ &= \langle \langle \overline{e_1}, e_2 \rangle_E . \overline{e_3}, e_4 \rangle'_E \\ &= \langle \overline{e_1}, e_2 \rangle_E . \langle \overline{e_3}, e_4 \rangle'_E.\end{aligned}$$

These coincide, so the the inner product descends to $\otimes_D$. $\square$

Note that in the following we are not assuming that we have a retract or the existence of maps $A \to B$, merely a unital completely positive map $\phi : B \to A$, which need not necessarily be surjective.

Proposition 30.3. *Suppose we have a unital completely positive map $\phi : B \to A$, given by $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E$ for some element $e_0 \in E$ of a Hilbert C*-bimodule $E \in {}_B\mathcal{M}_A$ with inner product $\langle , \rangle_E : \overline{E} \otimes_B E \to A$. Then there is a map $\chi : D \to A$, given by*

$$\chi(e_1 \otimes \overline{e_2}) = \langle \overline{e_0}, e_1 \rangle_E \langle \overline{e_2}, e_0 \rangle_E, \tag{63}$$

satisfying $\chi(d^*) = \chi(d)^*$, and which is "positive" in the sense that $\chi(dd^*) \geq 0$ in A. Further, if $\langle \overline{e_0}, e_0 \rangle_E = 1$ for some $e_0 \in E$, then $\chi(e_0 \otimes \overline{e_0}) = 1$.
There is also a $*$-algebra map $\theta : A \to D$, given by

$$\theta(a) = e_0 a \otimes \overline{e_0}, \tag{64}$$

satisfying $\chi \circ \theta = \mathrm{id}_A$. This implies that χ is surjective, and that θ is injective.

Proof. **(1)** Firstly we calculate the properties of χ as

$$\chi(e_0 \otimes \overline{e_0}) = \langle \overline{e_0}, e_0 \rangle_E \langle \overline{e_0}, e_0 \rangle_E, = 1,$$

and

$$\chi(e_1 \otimes \overline{e_2})^* = \big(\langle \overline{e_0}, e_1 \rangle_E \langle \overline{e_2}, e_0 \rangle_E\big)^* = \langle \overline{e_0}, e_2 \rangle_E \langle \overline{e_1}, e_0 \rangle_E = \chi(e_2 \otimes \overline{e_1}) = \chi((e_1 \otimes \overline{e_2})^*),$$

as required.
(2) The map θ is an algebra map because:

$$\theta(a)\theta(a') = (e_0 a \otimes \overline{e_0}).(e_0 a' \otimes \overline{e_0}) = e_0 a \langle \overline{e_0}, e_0 \rangle_E a' \otimes \overline{e_0} = \theta(aa'),$$

and a $*$-algebra map because:

$$\theta(a)^* = (e_0 a \otimes \overline{e_0})^* = e_0 \otimes \overline{e_0 a} = e_0 a^* \otimes \overline{e_0} = \theta(a^*).$$

(3) We have the composition $\chi \circ \theta = \mathrm{id}_A$ because:

$$(\chi \circ \theta)(a) = \chi(e_0 a \otimes \overline{e_0}) = \langle \overline{e_0}, e_0 a \rangle_C \langle \overline{e_0}, e_0 \rangle_E = a.$$

□

The reason "positive" is in quotation marks is because although D is a $*$-algebra and χ obeys the equation for positivity, we have not yet shown D to be a C*-algebra.
In the following theorem, we add a couple more assumptions on E in order to give D an identity element, in which case we get a certain $*$-algebra map.

Theorem 30.4. *Suppose that $\phi : B \to A$ is a unital completely positive map given by $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E$ for a Hilbert C^*-bimodule $E \in {}_B\mathcal{M}_A$ with nondegenerate inner product $\langle , \rangle_E : \overline{E} \otimes_B E \to A$, plus the following assumptions:*

(i) *The bimodule E is right finitely generated projective.*

(ii) *The bimodule map $G : \overline{E} \to E' = \mathrm{Hom_A(E, A)}$ given by $G(\overline{e})(e') = \langle \overline{e}, e' \rangle_E$ is an isomorphism of bimodules. Note that this implies $\langle , \rangle_E$ is nondegenerate.*

Then denoting e^i and e_i for the dual basis of E, the algebra D has unit element given by

$$1_D := \sum_i e^i \otimes G^{-1}(e_i). \tag{65}$$

There exists a $*$-algebra map $\gamma : B \to D$ given by

$$\gamma(b) = b.1_D, \tag{66}$$

such that for the map $\chi : D \to A$ given by $\chi(e_1 \otimes \overline{e_2}) = \langle \overline{e_0}, e_1 \rangle_E \langle \overline{e_2}, e_0 \rangle_E$, we have

$$\phi = \chi \circ \gamma. \tag{67}$$

This implies χ is unital, in the sense of $\chi(1_D) = 1$.

Proof. **(1)** First we show that G is a bimodule map. First we calculate

$$G(\overline{e}.a)(f) = \langle \overline{e}.a, f \rangle = \langle \overline{e}, af \rangle = G(\overline{e})(af) = (G(\overline{e}.a))(f).$$

Hence G is a right module map. Next we calculate

$$G(a\overline{e}) = \langle a\overline{e}, f \rangle = a\langle \overline{e}, f \rangle = (aG(\overline{e}))(f).$$

Hence G is a left module map.

(2) We show that 1_D really is the unit of D, i.e. for all $(f \otimes \overline{g}) \in D$ that $(f \otimes \overline{g}).1_D = 1_D.(f \otimes \overline{g}) = (f \otimes \overline{g})$. Recall that $\langle , \rangle_E = ev \circ (G \otimes \mathrm{id})$ as in the following diagram:

$$\begin{array}{ccc} \overline{E} \otimes_B E & \xrightarrow{\langle , \rangle_E} & A \\ \downarrow{\scriptstyle G\otimes \mathrm{id}} & \nearrow{\scriptstyle ev} & \\ E' \otimes_B E & & \end{array}$$

Then since G is invertible it follows that $\langle , \rangle_E \circ (G^{-1} \otimes \mathrm{id}) = ev$, i.e. $\langle G^{-1}(e_i), f \rangle_E = e_i(f)$. Using this, we calculate:

$$\Big(\sum_i e^i \otimes G^{-1}(e_i)\Big)(f \otimes \overline{g}) = \sum_i e^i.\langle G^{-1}(e_i), f \rangle \otimes \overline{g} = \sum_i e^i.e_i(f) \otimes \overline{g} = f \otimes \overline{g}.$$

Hence $1_D.(f \otimes \overline{g}) = (f \otimes \overline{g})$. Next we calculate:

$$(f \otimes \overline{g})\Big(\sum_i e^i \otimes G^{-1}(e_i)\Big) = \sum_i f \otimes \langle \overline{g}, e^i \rangle . G^{-1}(e_i) = f \otimes \sum_i G^{-1}(\langle \overline{g}, e^i \rangle e_i).$$

We show as follows that $\overline{g} - \sum_i G^{-1}(\langle \overline{g}, e^i \rangle e_i)$ has zero length. For all $k \in E$ we have

$$\sum_i \langle G^{-1}(\langle \overline{g}, e^i \rangle e_i), k \rangle = \langle \overline{g}, e^i \rangle . e_i(k) = \langle \overline{g}, e^i . e_i(k) \rangle = \langle \overline{g}, k \rangle.$$

Hence by nondegeneracy of $\langle , \rangle_E$, it follows that $\overline{g} = \sum_i G^{-1}(\langle \overline{g}, e^i \rangle e_i)$, from which it follows that $(f \otimes \overline{g}).1_D = (f \otimes \overline{g})$.

(3) We show that $1_D^* = 1_D$. Using that 1_D is a two-sided identity, $1_D^* = 1_D^*.1_D$. Starring this equation gives $(1_D^*)^* = 1_D^*.(1_D^*)^*$, i.e. $1_D = 1_D^*.1_D = 1_D^*$, as required.

(4) We show that γ is a $*$-algebra map. Using that 1_D is the identity of D we calculate

$$\gamma(b)\gamma(b') = (b.1_D)(b'.1_D) = (b.1_D.b').1_D = (b.b').1_D.1_D = (b.b').1_D = \gamma(bb'),$$

which shows that γ is an algebra map. Moreover, it is a $*$-algebra map because

$$\gamma(b^*) = b^*.1_D = (b.1_D)^* = \gamma(b)^*.$$

(5) We show that $\phi = \chi \circ \gamma$.

$$\begin{aligned}(\chi \circ \gamma)(b) &= \chi(be_i \otimes G^{-1}(e^i)) = \langle \overline{e_0}, be_i \rangle_E \langle G^{-1}(e^i), e_0 \rangle_E = \langle \overline{e_0}, be_i \rangle_E \langle \overline{e_i}, e_0 \rangle_E \delta_{i,0} \\ &= \phi(b).\langle \overline{e_0}, e_0 \rangle_E = \phi(b).\end{aligned}$$

□

The maps (θ, χ) very nearly satisfy the definition of a retract, apart from the fact that θ isn't necessarily unital and we haven't shown D to be a C*-algebra. One way of interpreting the fact that θ is not unital, is that rather than D being a retract of A, the retract is given by the corner algebra (see [46]) $D' = pDp$ for the idempotent element $p = e_0 \otimes \overline{e_0}$. If D' is a C*-algebra, then $\phi : B \to A$ would decompose into the composition of a $*$-algebra map $\gamma : B \to D$ and a map $\chi : D \to A$ which restricts to a retract. In future work it would be interesting to consider more widely when we can make D and D' C*-algebras.

31 Interpretation of D and Neighbourhood Retracts

In classical geometry, the sphere S^2 has equator S^1. Around the equator, there is an open neighbourhood which we call T — the tropics, like the tropics of Capricorn and Cancer on the globe. We have an inclusion $S^1 \to S^2$, but no retract $S^2 \to S^1$ due to a lack of continuous maps. However, there is a retract $T \to S^1$ of the tropics onto the sphere by the inclusion $S^1 \to T$, and the tropics themselves have inclusion $T \to S^2$ into the sphere. Thus S^1 is not a retract, but a neighbourhood retract of S^2.
Dualising these maps to smooth functions on the manifolds, we get

- A surjective map $C^\infty(S^2) \to C^\infty(S^1)$ coming from the inclusion of the equator, which is like $\phi : B \to A$, apart from the fact that we didn't have to assume ϕ was surjective.
- No maps $C^\infty(S^1) \to C^\infty(S^2)$, and therefore no noncommutative retract of A in B unless we have extra information.
- An injective map $C^\infty(S^1) \to C^\infty(T)$ coming from the retract $T \to S^1$, which is like $\theta : A \to D$.
- A surjective map $C^\infty(T) \to C^\infty(S^1)$ coming from the inclusion of S^1 in T, which is like $\chi : D \to A$.
- A map $C^\infty(S^2) \to C^\infty(T)$, coming from inclusion of T in S^2, which is like $\gamma : B \to D$. It need not be surjective since T is an open set, so $C^\infty(T)$ can contain unbounded functions.

The algebra D and its associated maps match up with the classical setup of a neighbourhood retract. This is thanks to the conditions of finitely generated projectiveness of E and the invertibility of G, which in less nice examples are not guaranteed to hold. As a first idea of a definition of noncommutative deformation retract (but which very well may need refining), we could take the conditions of Theorem 30.4.

32 Classifying Bimodule Connections on E for a Retract

Next we classify bimodule connections on the bimodule E associated with a retract. In the content so far we have only looked at algebras, but to define connections we require calculi, and so suppose that A and B are equipped with calculi Ω^1_A and Ω^1_B respectively. Note that we make no assumptions on what these calculi are.

Suppose a unital completely positive map $\phi : B \to A$ given by $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E$ is a retract by a unital completely positive map $\psi : A \to B$ given by $\psi(a) = \langle \overline{f_0}, af_0 \rangle_F$, for Hilbert C*-bimodules $E \in {}_B\mathcal{M}_A$ and $F \in {}_A\mathcal{M}_B$ with nondegenerate inner products $\langle , \rangle_E : \overline{E} \otimes_B E \to A$ and $\langle , \rangle_F : \overline{F} \otimes_A F \to B$. Then, recalling that $e_0 a = \psi(a)e_0$, a right bimodule connection $\nabla_E : E \to E \otimes_A \Omega^1_A$ on E with bimodule map $\sigma_E : \Omega^1_B \otimes_B E \to E \otimes_A \Omega^1_A$ satisfies:

$$\begin{aligned} e_0 \otimes \mathrm{d}a &= \nabla_E(e_0 a) - \nabla_E(e_0).a \\ &= \nabla_E(\psi(a)e_0) - \nabla_E(e_0).a \\ &= \sigma_E(\mathrm{d}\psi(a) \otimes e_0) + \psi(a)\nabla_E(e_0) - \nabla_E(e_0).a \end{aligned}$$

If $\nabla_E(e_0) = 0$, which is one of the conditions needed for ϕ to be a cochain map, then we get:

$$\sigma_E(\mathrm{d}\psi(a) \otimes e_0) = e_0 \otimes \mathrm{d}a. \tag{68}$$

We also calculate the bimodule map associated to a right connection on E regarded as a D-A bimodule instead of a B-A bimodule.

Proposition 32.1. *If we take the same right connection ∇_E, but now regard E as a D-A bimodule by the left action $(e_1 \otimes \overline{e_2}) \triangleright e_3 = e_1 \langle \overline{e_2}, e_3 \rangle_E$, then the condition for ∇_E to be a bimodule connection is the existence of a D-A bimodule map $\tilde{\sigma}_E : \Omega^1_D \otimes_D E \to E \otimes_A \Omega^1_A$, satisfying:*

$$\tilde{\sigma}_E(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3) = \nabla_E(e_1).\langle \overline{e_2}, e_3 \rangle + e_1 \otimes \mathrm{d}\langle \overline{e_2}, e_3 \rangle - (e_1 \otimes \overline{e_2}) \triangleright \nabla_E(e_3). \tag{69}$$

If the connection ∇_E preserves the metric $\langle , \rangle_E$, then this becomes

$$\tilde{\sigma}_E(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3) = (\mathrm{id} \otimes \langle , \rangle_E)\big(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3\big). \tag{70}$$

Proof. On one hand we have:

$$\nabla_E((e_1 \otimes \overline{e_2}) \triangleright e_3) = \nabla_E(e_1).\langle \overline{e_2}, e_3 \rangle + e_1 \otimes \mathrm{d}\langle \overline{e_2}, e_3 \rangle.$$

But if $\tilde{\sigma}_E$ exists, then we also have:

$$\nabla_E((e_1 \otimes \overline{e_2}) \triangleright e_3) = (e_1 \otimes \overline{e_2}) \triangleright \nabla_E(e_3) + \tilde{\sigma}_E(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3).$$

Therefore $\tilde{\sigma}_E$ satisfies:

$$\tilde{\sigma}_E(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3) = \nabla_E(e_1).\langle \overline{e_2}, e_3 \rangle + e_1 \otimes \mathrm{d}\langle \overline{e_2}, e_3 \rangle - (e_1 \otimes \overline{e_2}) \triangleright \nabla_E(e_3).$$

Using the additional assumption of metric preservation and then the definition of the differential on D as $\mathrm{d}_D(e_1 \otimes \overline{e_2}) = \nabla_E(e_1) \otimes \overline{e_2} + e_1 \otimes \nabla_{\overline{E}}(\overline{e_2})$, we get

$$\begin{aligned}
&\tilde{\sigma}_E(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3) = \nabla_E(e_1).\langle \overline{e_2}, e_3 \rangle + e_1 \otimes (\langle , \rangle \otimes \mathrm{id})(\overline{e_2} \otimes \nabla_E(e_3)) \\
&\quad + e_1 \otimes (\mathrm{id} \otimes \langle , \rangle)(\nabla_{\overline{E}}(e_2) \otimes e_3) - (e_1 \otimes \overline{e_2}) \triangleright \nabla_E(e_3) \\
&= \nabla_E(e_1).\langle \overline{e_2}, e_3 \rangle + e_1 \otimes (\mathrm{id} \otimes \langle , \rangle)(\nabla_{\overline{E}}(e_2) \otimes e_3) \\
&= (\mathrm{id} \otimes \langle , \rangle_E)\big(\mathrm{d}_D(e_1 \otimes \overline{e_2}) \otimes e_3\big),
\end{aligned}$$

as required. □

33 A Differential Graded Algebra and a Cochain Map

Next we look at a differential graded algebra D_n, satisfying $D_0 = D$, and whose differential is defined in terms of a right connection on E. In the case when it is a bimodule connection, we show that there is an extension of the $*$-algebra map $\gamma : B \to D$ as a cochain map $\gamma_n : \Omega^n_B \to D_n$. Throughout this section, we always assume the conditions of Theorem 30.4, which we are regarding as the setup for a noncommutative neighbourhood retract. Since we are only assuming a neighbourhood retract, we are not assuming that we have a retract.

Proposition 33.1. *Given the conditions of Theorem 30.4, plus a right connection* $\nabla_E : E \to E \otimes_A \Omega^1_A$ *with curvature* R_E*, then the graded* B*-bimodules* $D_n = E \otimes_A \Omega^n_A \otimes_A \overline{E}$ *have derivation* $\mathrm{d} : D_n \to D_{n+1}$ *given by*

$$\mathrm{d}(e \otimes \xi \otimes \overline{f}) = \nabla_E(e) \wedge \xi \otimes \overline{f} + e \otimes \mathrm{d}\xi \otimes \overline{f} + (-1)^{|\xi|} e \otimes \xi \wedge \nabla_{\overline{E}}(\overline{f}), \qquad (71)$$

satisfying $\mathrm{d}^2 = R_E \wedge \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \wedge R_{\overline{E}}$. *In the case* $R_E = 0$, (D_n, d) *forms a cochain complex, and if* ∇_E *is also metric-preserving then* (D_n, d) *becomes a differential graded algebra with wedge product* $\wedge : D_n \otimes D_m \to D_{n+m}$ *given by*

$$(e \otimes \xi \otimes \overline{f}) \wedge (e' \otimes \xi' \otimes \overline{f'}) = e \otimes \xi \langle \overline{f}, e' \rangle \wedge \xi' \otimes \overline{f'}. \tag{72}$$

Proof. This differential was defined to satisfy the Leibniz rule, since it differentiates each of the components in the tensor product with the appropriate sign, but we still need to check it is well-defined over $\otimes_A$, and to calculate d^2, and calculate its interaction with the wedge $\wedge$.

(1) Firstly, we show this preserves the $\otimes_A$. On one hand

$$\begin{aligned}
&\mathrm{d}(ea \otimes \xi \otimes \overline{f}) = \nabla_E(ea) \wedge \xi \otimes \overline{f} + ea \otimes \mathrm{d}\xi \otimes \overline{f} + (-1)^{|\xi|} ea \otimes \xi \wedge \nabla_{\overline{E}}(\overline{f}) \\
&= \nabla_E(e)a \wedge \xi \otimes \overline{f} + e \otimes \mathrm{d}a \wedge \xi \otimes \overline{f} + e \otimes a\mathrm{d}\xi \otimes \overline{f} + (-1)^{|\xi|} e \otimes a\xi \wedge \nabla_{\overline{E}}(\overline{f}) \\
&= \nabla_E(e) \wedge (a\xi) \otimes \overline{f} + e \otimes \mathrm{d}(a\xi) \otimes \overline{f} + (-1)^{|\xi|} e \otimes (a\xi) \wedge \nabla_{\overline{E}}(\overline{f}) \\
&= \mathrm{d}(e \otimes a\xi \otimes \overline{f}).
\end{aligned}$$

But on the other hand

$$\begin{aligned}
&\mathrm{d}(e \otimes \xi \otimes a\overline{f}) = \mathrm{d}(e \otimes \xi \otimes \overline{fa^*}) = \nabla_E(e) \wedge \xi \otimes \overline{fa^*} + e \otimes \mathrm{d}\xi \otimes \overline{fa^*} + (-1)^{|\xi|} e \otimes \xi \wedge \nabla_{\overline{E}}(\overline{fa^*}) \\
&= \nabla_E(e) \wedge \xi a \otimes \overline{f} + e \otimes \mathrm{d}\xi.a \otimes \overline{f} + (-1)^{|\xi|} e \otimes \xi \otimes a\nabla_{\overline{E}}(\overline{f}) + (-1)^{|\xi|} e \otimes \xi \wedge \mathrm{d}a \otimes \overline{f} \\
&= \nabla_E(e) \wedge (\xi a) \otimes \overline{f} + e \otimes \mathrm{d}(\xi a) \otimes \overline{f} + (-1)^{|\xi|} e \otimes (\xi a) \otimes \nabla_{\overline{E}}(\overline{f}) \\
&= \mathrm{d}(e \otimes \xi a \otimes \overline{f}).
\end{aligned}$$

These coincide, so the operation d is well-defined over the tensor product $\otimes_A$.

(2) Secondly, we calculate d^2. Recall that we have $R_E = (\mathrm{id} \otimes \mathrm{d} + \nabla_E \wedge \mathrm{id})\nabla_E$ and $R_{\overline{E}} = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_{\overline{E}})\nabla_{\overline{E}}$.

Hence by Figure 24 we have $\mathrm{d}^2 = R_E \wedge \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \wedge R_{\overline{E}}$. If $R_E = 0$, then $R_{\overline{E}} = 0$ and $\mathrm{d}^2 = 0$.

(3) Thirdly, we show that $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}\beta$, where $(e \otimes \xi \otimes \overline{f}) \wedge (e' \otimes \xi' \otimes \overline{f'}) = e \otimes \xi \langle \overline{f}, e' \rangle \wedge \xi' \otimes \overline{f'}$. Firstly we calculate $\mathrm{d}(\alpha \wedge \beta)$ in Figure 25.

Then, we use metric preservation to calculate $\mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}\beta$ in Figure 26. We can see that the two diagrams are equal. □

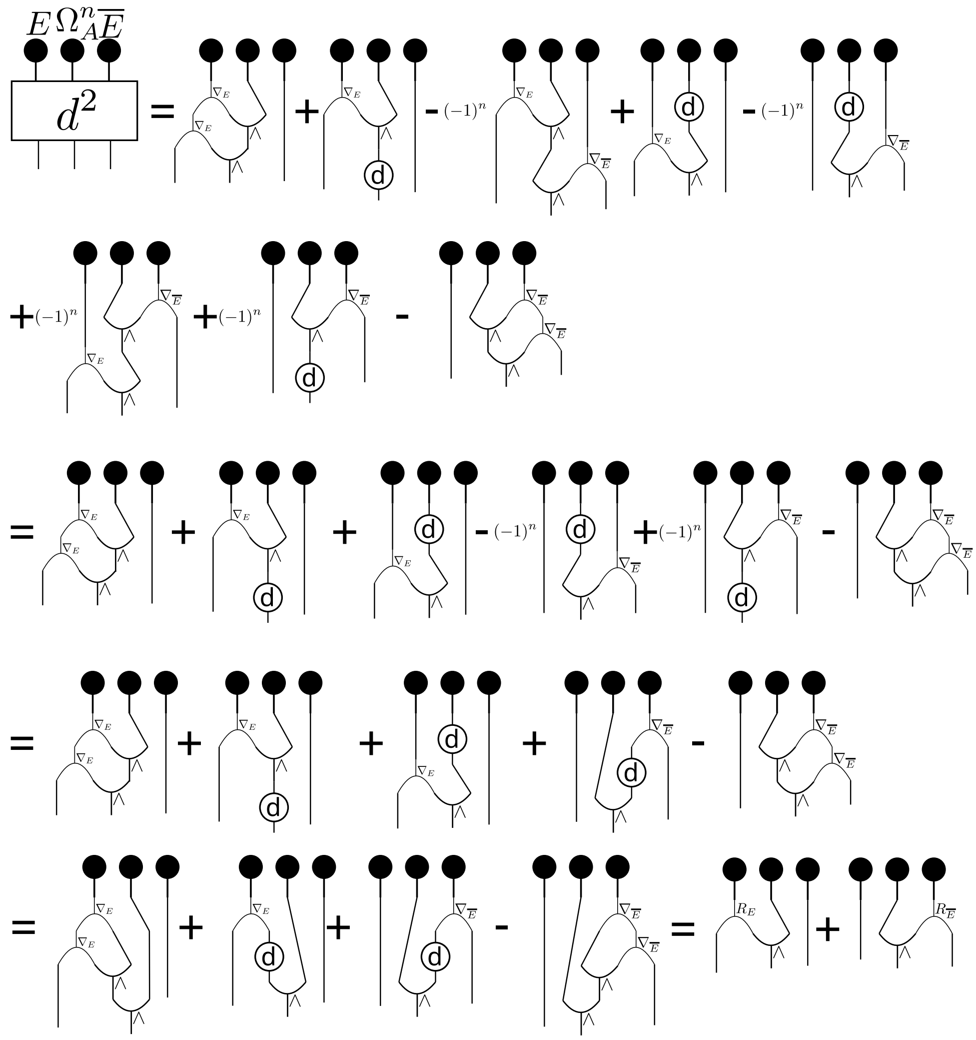


Figure 24: Retracts: Calculation of d^2

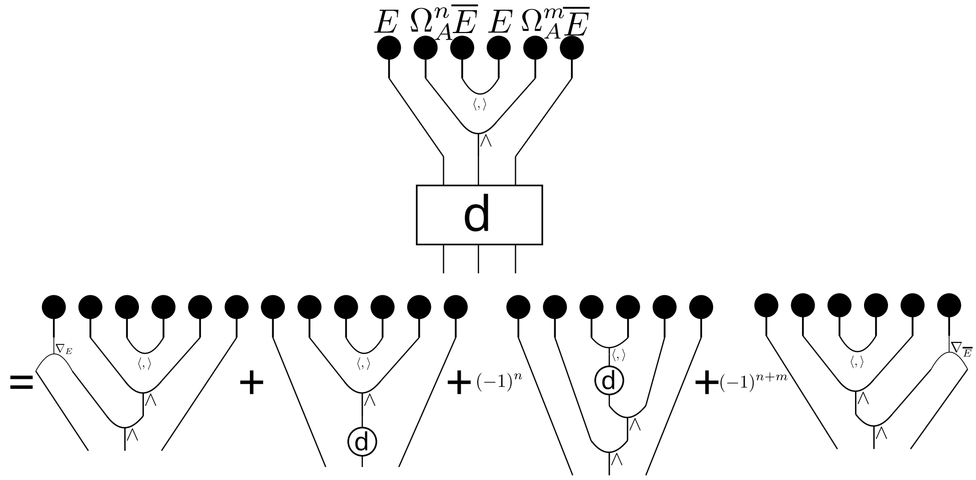


Figure 25: Retracts: Calculation of $\mathrm{d}(\alpha \wedge \beta)$

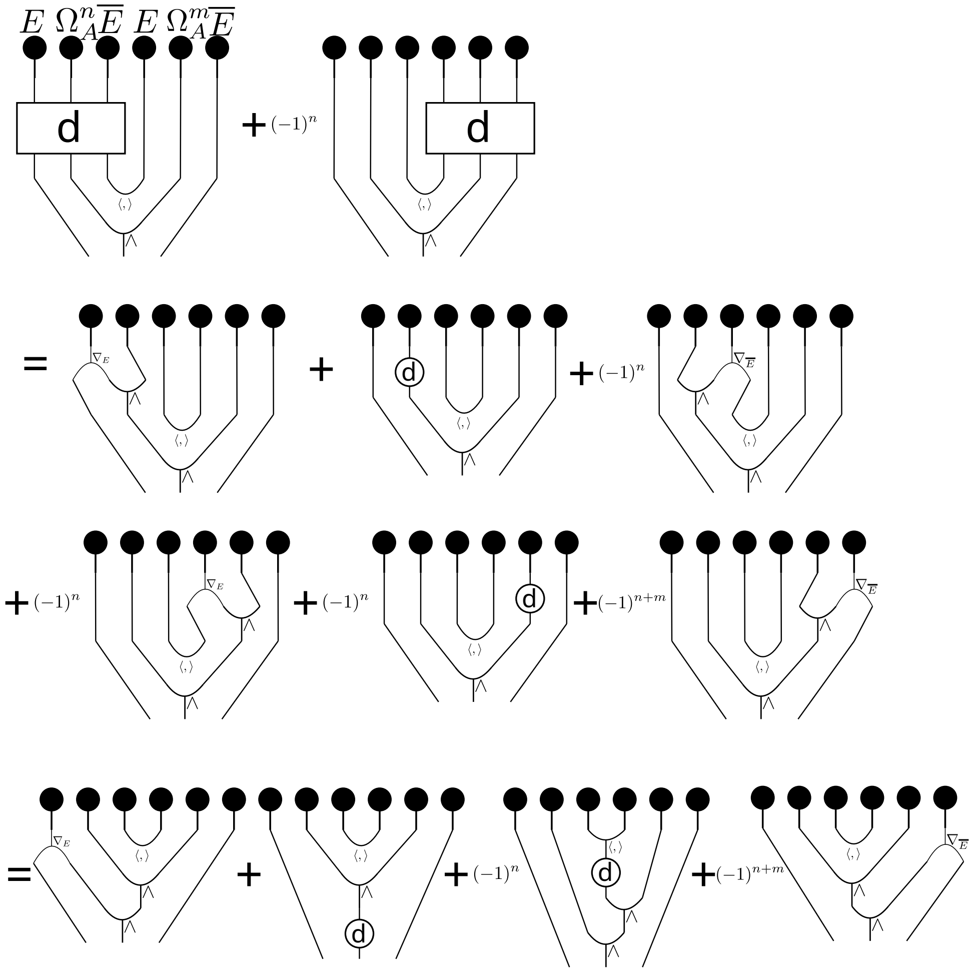


Figure 26: Retracts: Calculation of $\mathrm{d}\alpha\wedge\beta+(-1)^{|\alpha|}\alpha\wedge\mathrm{d}\beta$

Remark 33.2. *We note that although D_n is a B-bimodule, it can also be regarded as a D-bimodule. In future work, we would be interested in figuring out how to do this.*

Next we look at a map of differential graded algebras, defined in terms of a bimodule connection. Since we are only assuming a neighbourhood retract and not necessarily a retract, the formulae from the previous section do not necessarily apply to this σ_E.

Proposition 33.3. *Under the conditions of Proposition 33.1 for D_n to be a differential graded algebra, but now with ∇_E a zero curvature metric preserving extendable B-A bimodule connection on E, we can extend the $*$-algebra map $\gamma : B \to D$, $\gamma(b) = b.1_D$ to a map of differential graded algebras $\gamma_n : \Omega^n_B \to E\otimes_A \Omega^n_A \otimes_A \overline{E}$ as*

$$\gamma_n(\xi) = (\sigma_E \otimes \mathrm{id})(\omega\otimes 1_D). \tag{73}$$

By a map of differential graded algebras, we mean it commutes with d *and* $\wedge$.

Proof. By the assumptions that E is finitely generated projective and has G invertible, there exists an element $\langle,\rangle^{-1} \in E \otimes_A \overline{E}$, which satisfies

$$(\mathrm{id} \otimes \langle,\rangle)(\langle,\rangle^{-1} \otimes \mathrm{id}) = \mathrm{id}_E = (\langle,\rangle \otimes \mathrm{id})(\mathrm{id} \otimes \langle,\rangle^{-1}).$$

By these identities, we can see that $\langle,\rangle^{-1} = 1_D$. In the following diagrams, we use the notation $\langle,\rangle^{-1}$ instead of 1_D, and apologise to the reader for the confusing notation.

(1) Firstly in Figure 27 we show $\mathrm{d}\gamma_n = \gamma_{n+1}\mathrm{d}$, using the fact that metric preservation implies $(\nabla_E \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\overline{E}})\langle,\rangle^{-1} = 0$.

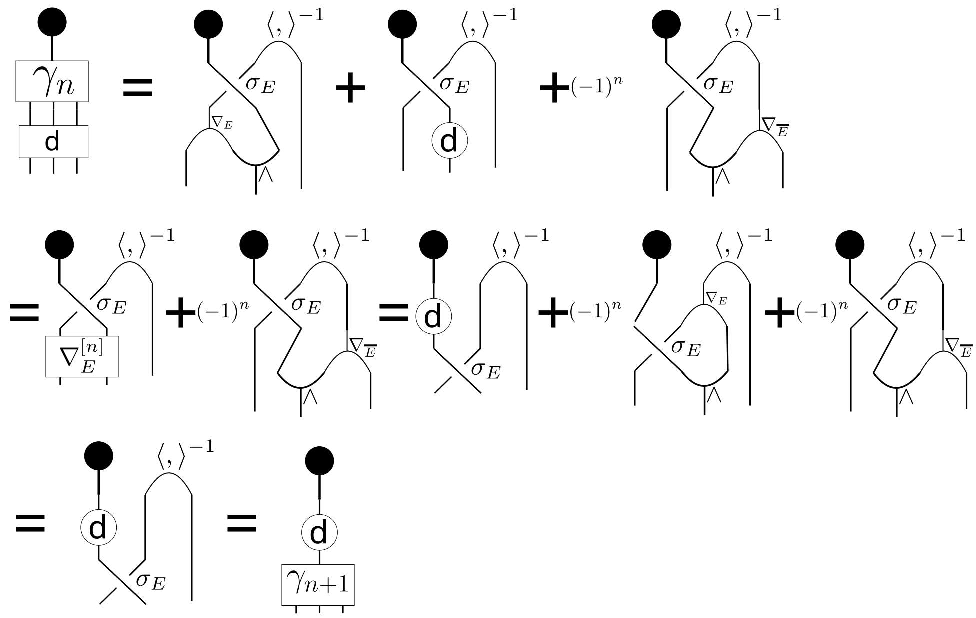


Figure 27: Retracts: Proof γ is a cochain map.

(2) Secondly in Figure 28 we show $\gamma_n(\omega) \wedge \gamma_m(\xi) = \gamma_{n+m}(\omega \wedge \xi)$, using the definition of $\langle,\rangle^{-1}$ and then extendability of σ_E.

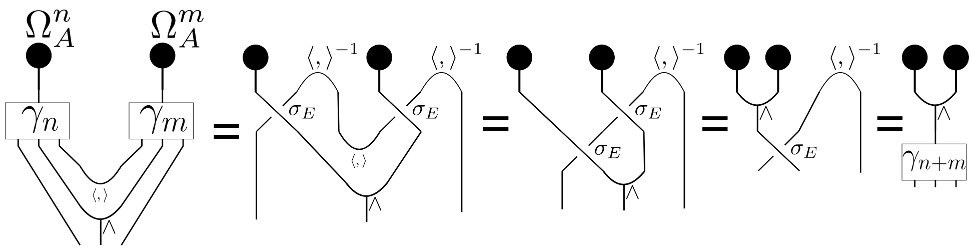


Figure 28: Retracts: Proof γ commutes with $\wedge$

□

When γ_n is a map of differential graded algebras, then $\ker(\gamma_n)$ is a differential graded algebra (and in fact when $n = 0$ a two-sided ideal), since if $\gamma_n(\xi) = 0$, then $\gamma_{n+1}(\mathrm{d}\xi) = \mathrm{d}\gamma_n(\xi) = \mathrm{d}0 = 0$, so $\mathrm{d}\xi \in \ker(\gamma_{n+1})$, and $\gamma_{n+m}(\xi \wedge \eta) = \gamma_n(\xi) \wedge \gamma_m(\eta) = 0 \wedge \gamma_m(\eta) = 0$. Hence the following is a short exact sequence of cochain complexes and cochain maps.

$$0 \longrightarrow \ker(\gamma_n) \longrightarrow \Omega^n_B \xrightarrow{\gamma_n} E \otimes_A \Omega^n_A \otimes_A \overline{E} \longrightarrow 0$$

We could then look at the relative cohomology of this sequence, as defined in Chapter 4.6 of [10].

34 Retracts Examples

34.1 Conditional Expectations on $\mathbb{C}X$

[Algebras: See Example 3.11 for $\mathbb{C}X$ and its calculi, for X a finite group]

Recall the following definition of a noncommutative conditional expectation.

Definition 34.1. [56] Let A and B be unital C^*-algebras, with $A \subset B$. Then a linear map $\mathbb{E} : B \to A$ is called a *conditional expectation* if:

1. $\mathbb{E}(1_B) = 1_A$
2. $\mathbb{E}$ is an A-bimodule map
3. $\mathbb{E}$ is positive, meaning that if $b \geq 0$ then $\mathbb{E}(b) \geq 0$.

Proposition 34.2. *Suppose X is a finite group with a subgroup G, so $\mathbb{C}G \subset \mathbb{C}X$, a conditional expectation is given by the linear extension $\mathbb{E} : \mathbb{C}X \to \mathbb{C}G$ of the map defined on X as*

$$\mathbb{E}(x) = \begin{cases} x & , x \in G \\ 0 & , x \notin G \end{cases}.$$

Proof. **(1)** Firstly, as subgroups contain the identity element of the bigger group, we have that $\mathbb{E}(e) = e$.

(2) Secondly, for $g \in G$, then $x \in G$ if and only if $gx \in G$, so $\mathbb{E}(gx) = g\mathbb{E}(x)$ and $\mathbb{E}(xg) = \mathbb{E}(x)g$, making $\mathbb{E}$ a $\mathbb{C}G$-bimodule map.

(3) Finally, we show that $\mathbb{E}$ is positive. Write x_i for the coset representatives of $F = X/G$. Because cosets are disjoint, we have $x_i \notin Gx_j$, so $x_i x_j^{-1} \notin G$, which means $\mathbb{E}(x_i x_j^{-1}) = 0$. Consequently we have an inner product $\langle , \rangle_F : F \otimes \overline{F} \to G$ given by $\langle x_i, \overline{x_j} \rangle = \delta_{i,j} e$. We also have an inner product $\langle , \rangle_G : G \otimes \overline{G} \to G$ given by $\langle g, h \rangle = gh^{-1}$.
Any element of $\mathbb{C}X$ can be written as a linear combination of terms gx_i. Thus $\langle gx_i, \overline{hx_j} \rangle = \overline{E}(gx_i x_j^{-1} h^{-1}) = g\mathbb{E}(x_i x_j^{-1})h^{-1} = \langle x_i, \overline{x_j} \rangle_F \langle g, \overline{h} \rangle_G$. But then $\langle gx_i, \overline{gx_i} \rangle = gg^* \geq 0$. □

We can regard the algebra $E = \mathbb{C}X$ as a $\mathbb{C}X$-$\mathbb{C}G$ bimodule, with actions given by multiplication. This is well-defined because G is a subgroup of X. The module E has inner product $\langle , \rangle_E : \overline{E} \otimes_{\mathbb{C}X} E \to \mathbb{C}G$ given by $\langle \overline{x}, y \rangle = \mathbb{E}(x^* y)$. Set $e_0 = 1$. This inner product gives rise to the positive map $\phi : \mathbb{C}X \to \mathbb{C}G$ as $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E = \mathbb{E}(b)$.
Alternatively we can regard $F = \mathbb{C}X$ as a $\mathbb{C}G$-$\mathbb{C}X$ bimodule, with actions also given by multiplication, and with inner product $\langle , \rangle_F : \overline{F} \otimes_{\mathbb{C}G} F \to \mathbb{C}X$ given by $\langle \overline{x}, y \rangle_F = x^* y$. Set $f_0 = 1 \in F$. This inner product gives rise to a map $\psi : \mathbb{C}G \to \mathbb{C}X$ via $\psi(g) = \langle \overline{f_0}, gf_0 \rangle_F = g \in \mathbb{C}X$. We have $\phi \circ \psi = \mathrm{id}$, and hence a retract.

Proposition 34.3. *The $\mathbb{C}X$-$\mathbb{C}G$ bimodule $E = \mathbb{C}X$ is right finitely generated projective, with dual basis $\sum_i e^i \otimes e_i$ and decomposition $\tilde{e} = \sum_i e^i . e_i(\tilde{e})$ for all $\tilde{e} \in E$, where the e^i are left coset representatives of G in X, so $\bigcup_i e^i G = X$. The $e_i \in E' = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}G)$ are defined by*

$$e_i(x) = \begin{cases} (e^i)^{-1} x, & \text{if } x \in e^i G \\ 0, & \text{else} \end{cases}.$$

Proof. We can see the decomposition holds for all $x \in E$ because:

$$\sum_i e^i . e_i(x) = e^i (e^i)^{-1} x = x.$$

□

Proposition 34.4. *The bimodule map $G : \overline{E} \to \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}G)$ given by $G(\overline{e})(f) = \langle \overline{e}, f \rangle_E$ is an isomorphism, and in fact is $G(\overline{e^i}) = e_i$.*

Proof. **(1)** Firstly we show that G is an isomorphism. An element $f \in \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}G)$ is determined by $f(e^i)$ for all i, since $f(e^i g) = f(e^i) g$. This corresponds to $|X|/|G| \times |G| =$

$|X|$ elements, so both domain and codomain of G are dimension $|X|$. Thus to show G is an isomorphism, we need only show it is injective. On a general element, we have

$$G(\sum_{x\in X}\lambda_x\overline{x})(y) = \sum_{x:x^{-1}y\in G}\lambda_x x^{-1}y.$$

But each term of this is linearly independent, since if $x_1^{-1}y = x_2^{-1}y$ then the invertibility of group elements implies $x_1 = x_2$. Thus $G(\sum_{x\in X}\lambda_x\overline{x})(y)$ vanishes only when $\lambda_x = 0$ for all $x \in E$, meaning that G is injective. **(2)** Next, we show that $G(\overline{e^i}) = e_i$. We want to find coefficients λ_x for which $G(\sum_x \lambda_x\overline{x}) = e_i$. Evaluating this on some $y \in X$, and then using the definition of e^i as coset representatives, we get

$$(e^i)^{-1}y = \sum_{x:x^{-1}y\in G}\lambda_x x^{-1}y = \sum_{x\in y^{-1}G}\lambda_x x^{-1}y = \sum_{x\in e^iG}\lambda_x x^{-1}y$$

Multiplying on the right by y^{-1} and then applying the $*$-operation, we get for all $y \in X$:

$$(e^i) = \sum_{x\in e^iG}\lambda_x x$$

For this to be true, we require $\lambda_x = \delta_{x,e^i}$, i.e. $G(\overline{e^i}) = e_i$. □

We therefore have an associative $*$-algebra $D = \mathbb{C}X \otimes_{\mathbb{C}G} \overline{\mathbb{C}X}$ with multiplication

$$(x_1 \otimes \overline{y_1})(x_2 \otimes \overline{y_2}) = x_1\mathcal{E}(y_1x_2^*) \otimes \overline{y_2}$$

and unit $1_D = \sum_i e^i \otimes \overline{e^i}$. Since E is a tensor product of finite dimensional C*-algebras, D should actually be a C*-algebra. There is a $*$-algebra map $\gamma : B \to D$ given by

$$\gamma(x) = x.1_D = x.\sum_i e^i \otimes \overline{e^i}.$$

There is a unital positive map $\chi : D \to A$ given by

$$\chi(e_1 \otimes \overline{e_2}) = \langle\overline{1}, e_1\rangle_E\langle\overline{e_2}, 1\rangle_E = \mathbb{E}(e_1).\big(\mathbb{E}(e_2)\big)^{-1}$$

which satisfies $\phi = \chi \circ \gamma$. Since D is a C*-algebra, $\theta : A \to D$ is a retract by $\chi : D \to A$. Thus not only is A a retract of B, but it is also a neighbourhood retract.

Next we look at calculi. Recall that $\Omega^1(\mathbb{C}X) = \mathbb{C}X.V$ for V a right representation of $\mathbb{C}X$ with $\mathrm{d}x = x.\zeta(x)$ for a cocycle $\zeta : X \to V$, i.e. satisfying $\zeta(xy) = \zeta(x) \triangleleft y + \zeta(y)$. Further, $\Omega^1(\mathbb{C}G) = \mathbb{C}G.W$ for $W \subset V$ the smallest right representation of G containing the image $\zeta(G)$.

Proposition 34.5. *In this example, a right connection* $\nabla_E : E \to E \otimes_{\mathbb{C}G} \Omega^1_{\mathbb{C}G}$ *given by* $\nabla_E(e^i) = \sum_j e^j \otimes \Gamma^i_j$ *preserves the metric* $\langle , \rangle_E : \overline{E} \otimes_{\mathbb{C}X} E \to \mathbb{C}G$ *given by* $\langle \overline{x}, y \rangle = \mathbb{E}(x^{-1}y)$ *when*

$$0 = \Gamma^i_j + (\Gamma^j_i)^*. \tag{74}$$

Proof. The metric preservation equation is

$$\mathrm{d}\langle \overline{e^j}, e^i \rangle_E = \langle \overline{e^j}, e^k \rangle_E \Gamma^i_k + (\Gamma^j_k)^* \langle \overline{e^k}, e^i \rangle_E.$$

But $\langle \overline{e^j}, e^i \rangle_E = \mathbb{E}((e^j)^{-1} e^i) = \delta_{j,i}.e$, its differential is zero, and the metric preservation equation becomes

$$0 = \Gamma^i_j + (\Gamma^j_i)^*$$

as required. $\square$

A linear basis of D is given by $e^i g \otimes \overline{e^j}$, so a general element of Ω^1_D is given by

$$\sum_{i,g,j} \mathrm{d}_D(e^i g \otimes \overline{e^j}) \lambda_{i,g,j} = \sum_{i,g,j} \Big(e^k \otimes \Gamma^i_k.g \otimes \overline{e^j} + e^i \otimes \mathrm{d}g \otimes \overline{e^j} + e^i \otimes g(\Gamma^j_k)^* \otimes \overline{e^k} \Big).$$

Taking the Christoffel symbols as zero and $g = 1$ gives $\mathrm{d}(\sum_{i,j} e^i \otimes e^j) = 0$, so Ω^1_D is not always connected.

34.2 Functions on Graphs

[Algebras: See Example 3.9 for $C(X)$ and its calculi]

Take $B = \mathbb{C}^4$ and $A = \mathbb{C}^3$, regarded as the functions on graphs of four and three disjoint points respectively, so that A is a noncommutative submanifold of B. The multiplication and star operations are elementwise on the vectors, and the units are the vectors with 1 in all entries.

There is a B-A bimodule $E = \mathbb{C}^4$ with actions and inner product $\langle , \rangle_E : \overline{E} \otimes_B E \to A$ given as follows.

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \triangleright \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} aw \\ bx \\ cy \\ dz \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \triangleleft \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \\ dz \end{pmatrix}, \quad \langle \overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \rangle_E = \begin{pmatrix} x_1^* y_1 \\ x_2^* y_2 \\ \frac{x_3^* y_3 + x_4^* y_4}{2} \end{pmatrix}.$$

Taking $e_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, the inner product gives rise to a completely positive map $\phi : \mathbb{C}^4 \to \mathbb{C}^3$ given by $\phi(b) = \langle \overline{e_0}, be_0 \rangle_E$, which on a specific element is $\phi(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) = \begin{pmatrix} a \\ b \\ \frac{c+d}{2} \end{pmatrix}$. This is unital.

There is also an A-B bimodule $F = \mathbb{C}^4$ with mirrored actions from E and inner product $\langle , \rangle_F : \overline{F} \otimes_A F \to B$ given as follows.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \triangleright \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} xa \\ yb \\ zc \\ zd \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \triangleleft \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} aw \\ bx \\ cy \\ dz \end{pmatrix}, \quad \langle \overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \rangle_F = \begin{pmatrix} x_1^* y_1 \\ x_2^* y_2 \\ x_3^* y_3 \\ x_4^* y_4 \end{pmatrix}.$$

Taking $f_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, this inner product gives rise to a completely positive map $\psi : A \to B$ given by $\psi(a) = \langle \overline{f_0}, af_0 \rangle_F$, which on a specific element is $\psi(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = \begin{pmatrix} a \\ b \\ c \\ c \end{pmatrix}$. This is unital. Because $\phi \circ \psi = \mathrm{id}_A$, we have a retract. In this example we also have that ψ is an algebra map. We note that while the metric $\langle , \rangle_E$ is nondegenerate, E here is likely not finitely generated projective due to the dimensions of the algebras.

35 Future Ideas and Discussion

Possible Example: Equator of Quantum Sphere

A neighbourhood retract is a weaker condition than a retract and means having a subspace which gives a retract, and we had an idea for what a noncommutative version of this might be. One example we would very much hope to be able to find a neighbourhood retract for is the quantum circle $\mathbb{C}_q[S^1]$ in the quantum sphere $\mathbb{C}_q[S^2]$, as a q-deformed version of the equator being a neighbourhood retract of the sphere. This is a question the author is particularly interested in, since in Section III the algebra $\mathbb{C}_q[S^1]$ turned out to not be a submanifold of $\mathbb{C}_q[S^2]$ via our definition due to a lack of algebra maps.

Missing Definition: Deformation Retract

Deformation retract is a stronger notion than retract, and is defined as a homotopy between the identity map and a retract. A literal interpretation of this in a noncommutative setting would be a time-dependent map $\psi_t : A \to B$ which at $t = 0$ is a retract

$\psi_0 : A \to B$ by some $\phi : B \to A$, and at time $t = 1$ is the identity $\psi_1 = \mathrm{id}_B : B \to B$. This is of course nonsensical unless A was a subalgbera of B, which is more restrictive of a condition than we would like, and so we currently have no satisfactory idea.

If we were able to define a deformation retract, we could investigate if it gives rise to a notion of neighbourhood deformation retract, in the same way as retracts gave an idea of neighbourhood retracts.

Is the Intermediate Algebra a Submanifold?

Another question we might ask is whether the surjective $*$-algebra map $\gamma : B \to D$, $\gamma(b) = b.1_D$ ever gives a co-embedding in the sense of Chapter III, and if so, what submanifold calculus does D inherit from Ω^1_B? The first step would be to understand the kernel of γ, which is determined by the kernel of the left action on E. It seems like a reasonable thing to hope that D might be a noncommutative submanifold of B under a nice enough set of conditions, in the same way that the tropics are a submanifold of the sphere.

Part VI

First Ideas Towards Noncommutative Differential Cofibrations

Abstract

We look at how Quillen's definition of cofibration might be interpreted in a noncommutative differential context. We use methods of noncommutative geodesics to solve a state path lifting problem and show for a number of examples that the natural idea of trivial cofibration satisfies this definition. We also speculate how this might relate to existing theory of noncommutative fibre bundles.

36 Introduction

An open question in noncommutative differential geometry, as posed at the end of [10] Chapter 4.6, is whether there is a good notion of differential cofibration. In combination with appropriate definitions of noncommutative fibrations and weak equivalences, the hope is that this would form a model category, allowing us to do noncommutative homotopy theory. To answer this question in full would go far beyond the scope of this PhD project, but we investigate a first few ideas in this direction, with a focus on trivial cofibrations. We note that there has also been recent work via other noncommutative differential approaches to construct analogues of homotopy groups, such as via reconstruction results in [53].

There is already a notion of a noncommutative fibre bundle as in [14], which we generalised from algebra maps to completely positive maps in Section IV of this thesis, although here in this very early stage of investigation into cofibrations we only consider the algebra maps version. However, this definition of a fibre bundle was defined to give rise to a Leray-Serre spectral sequence for cohomology, and so it would not be surprising if we needed additional assumptions to make it satisfy more fibration-like properties.

A cofibration is defined as a map satisfying a homotopy extension property. Daniel

Quillen [49] generalised this to more general categories, saying that a diagonal map h exists to make the following diagram commute when δ is a cofibration and ι is a fibration making the outside of the diagram commute, plus the condition that one of δ or ι is a weak equivalence.

$$\begin{array}{ccc} X & \longrightarrow & E \\ \Big\downarrow{\scriptstyle \delta} & \overset{h}{\nearrow} & \Big\downarrow{\scriptstyle \iota} \\ Y & \longrightarrow & B \end{array}$$

However, this diagram gives an equivalent definition of cofibrations in terms of fibrations and vice versa, so for example δ is a cofibration if and only if for every fibration ι making the outside of the diagram commute and is also a weak equivalence there is a diagonal map h which commutes with the diagram. Likewise, ι is a fibration if and only if for every trivial cofibration δ making the outside of the diagram commute and is also a weak equivalence there is an h making the diagram commute.

Re-phrasing Quillen's definition in terms of algebras, where the arrows go the other way around, gives the following.

Definition 36.1. In a category of algebras with morphisms maps between them (which are at the very least linear), and with certain maps designated as fibrations and cofibrations and weak equivalences, then for algebras A, B, X, Y, we call a morphism $\delta : Y \to X$ a *cofibration* if whenever $\iota : B \to A$ is a fibration which is a weak equivalence there always exists a morphism h that makes all diagrams of the following form commute.

$$\begin{array}{ccc} X & \xleftarrow[h_0]{} & A \\ {\scriptstyle \delta}\Big\uparrow & \overset{h}{\swarrow} & {\scriptstyle \iota}\Big\uparrow \\ Y & \xleftarrow[\alpha]{} & B \end{array}$$

Unlike in topology where we can easily find continuous maps between spaces, their noncommutative equivalent of algebra maps between algebras are not so readily available. Consequently, in our noncommutative differentiable context we look at differentiable completely positive maps. For the time being we restrict to looking at the case where the fibration ι is a differentiable injective algebra map. Under this set of assumptions, we see that given an ι satisfying these conditions and making the diagram commute (in the case of a trivial cofibration), it is actually possible to recover certain properties of the differential fibre bundles defined in [14], which were designed to give Leray-Serre spectral sequences.

In future work we could consider more generally Quillen's definition with ι a differentiable injective completely positive map, in which case we might hope for links to our bimodule definition of fibre bundles in Section IV.

37 Trivial Cofibrations

A particularly tractable class of examples is given by maps corresponding to classical cofibrations.

Definition 37.1. ([43] Definition 1.2) A continuous function $X \to Y$ between topological spaces is a classical *Serre fibration* if for every commuting square of the following form there exists for each n an $h : D^n \times I \to X$ to make the diagram commute.

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ \big\downarrow{\scriptstyle (\mathrm{id},0)} & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & Y \end{array}$$

Note that D^n denotes the closed n-disk.

The map $(\mathrm{id}, 0)$ in the above diagram gives the simplest example of a cofibration. In our noncommutative case, we take this differential cofibration to be δ_0, the evaluation map at $t = 0$ in the following diagram.

$$\begin{array}{ccc} D & \xleftarrow{\quad h_0 \quad} & A \\ {\scriptstyle \delta_0}\big\uparrow & \overset{h}{\swarrow} & \big\uparrow{\scriptstyle \iota} \\ C^\infty(\mathbb{R}) \otimes D & \xleftarrow[\alpha]{} & B \end{array}$$

Under a certain assumption on α the problem has the following particularly nice solution.

Proposition 37.2. *Suppose that ι is a differential fibration and that α and h_0 are completely positive maps such that the outside of the following diagram commutes.*

$$\begin{array}{ccc} D & \xleftarrow{\quad h_0 \quad} & A \\ {\scriptstyle \delta_0}\big\uparrow & \overset{h}{\swarrow} & \big\uparrow{\scriptstyle \iota} \\ C^\infty(\mathbb{R}) \otimes D & \xleftarrow[\alpha]{} & B \end{array}$$

If we make the extra assumption that there are $b_t \in B$ for $t \in \mathbb{R}$ such that $\alpha_t(b) = \alpha_0(b_t b b_t^*)$, then the dotted line $h_t(a) = h_0(\iota(b_t) a \iota(b_t)^*)$ makes the diagram commute.

Proof. The fact that $\delta_0 \circ h = h_0$ holds trivially, while we see that $h \circ \iota = \alpha$ by substituting $a = \iota(b)$ into the formula for $h_t(a)$. □

However, this is too restrictive of a case, since in general we cannot assume α to be of this form. For example when $D = \mathbb{C}$, the KSGNS construction says that for C*-algebras A and B, completely positive maps $\alpha : B \to C^\infty(\mathbb{R}) \otimes \mathbb{C}$ are each given by a $C^\infty(\mathbb{R})$-B Hilbert C*-bimodule E and an element $e \in E$ by $\alpha(b) = \langle eb, \overline{e} \rangle_E$.

From here onwards, we consider only the toy example of $D = \mathbb{C}$, which corresponds to the D^0 diagram in the classical definition of Serre fibrations. We call this the state path lifting problem for a fibration, since we are lifting time-dependent states. States correspond to points (or convex combinations thereof) in classical geometry, and so time-dependent states correspond to paths. Although the problem we solve is relatively small, lifting paths is a necessary first step towards lifting homotopies, and we hope that this can give some insight into the more general problem.

38 Classical Interpretation

In the classical lifting problem for a fibre bundle $\pi : M \to N$, we start with a path $p(t)$ in the base space N and a starting point $q(0)$ in the total space M with $\pi q(0) = p(0)$, and we want to extend $p(t)$ to a path $q(t)$ in the total space satisfying $\pi q(t) = p(t)$. In our differentiable context, we assume that $\dot{p}(t) = X_t$ for a time-dependent vector field X_t on the base space. We ask for this condition because we want the path to be differentiable, with its velocity vector described by a vector field. Similarly, we would like to define $q(t)$ as a time-dependent vector field Y_t on the total space, satisfying a condition corresponding to $\pi(q(t)) = p(t)$. We look now at this vector field lifting problem.

The idea of a connection of functions on the total space as a module over functions on the base space gives a way to lift vector fields, as we show in the noncommutative context, but it is possible to see classically too why this makes sense. A vector field X gives a directional derivative of a function as $D_x f(x) = X^i \frac{\partial f}{\partial X^i}$. Suppose the locally-defined vector fields $\frac{\partial}{\partial X^i}$ on the base space extend to Y_i on the total space. Then we define

$$\nabla_{X^i \frac{\partial}{\partial X^i}} f = X_i D_{Y_i} f$$

for functions f on the base space. Further, if $f = g \circ \pi$ for some function g on the base space, then:

$$\nabla_{X^i \frac{\partial}{\partial X^i}} (g \circ \pi) = (D_x g) \circ \pi,$$

as the Y_i extend $\frac{\partial}{\partial X^i}$. In particular $\nabla_X 1 = 0$. Recall that $\nabla_X e = (ev \otimes \mathrm{id})(X \otimes \nabla e)$. We next look at the noncommutative version.

39 Geodesics Preliminaries

The following definitions and propositions are from [6], and are key to constructing h in the case $D = \mathbb{C}$. Note that the calculus on A is arbitrary.

Definition 39.1. ([6] Example 4.3) Define the following inner product on $M = C^\infty(R) \otimes A$:

$$\langle f(t) \otimes a, \overline{g(t) \otimes b} \rangle_M = f(t) g(t)^* \langle a, \bar{b} \rangle_F \in C^\infty(\mathbb{R})$$

where $\langle , \rangle_F$ is a fixed (time independent) inner product on A.

Proposition 39.2. *([6] Proposition 5.1) For a unital algebra A with calculus Ω_A and $C^\infty(\mathbb{R})$ with its usual calculus $\Omega(\mathbb{R})$ we set $M = C^\infty(\mathbb{R}) \otimes A$ regarded as a $C^\infty(\mathbb{R})$-A bimodule. Then a general left bimodule connection on M is of the form, for $c \in C^\infty(\mathbb{R}) \otimes A$ and $\xi \in \Omega^1_A$,*

$$\nabla_M(c) = \mathrm{d}t \otimes (bc + \frac{\partial c}{\partial t} + K(\mathrm{d}c)), \qquad \sigma_M(1 \otimes \xi) = \mathrm{d}t \otimes K(\xi) \tag{75}$$

for some $b \in C^\infty(\mathbb{R}) \otimes A$ and $K \in C^\infty(\mathbb{R}) \otimes \mathfrak{X}^R_A$.

In our case we take the vector field as time-independent.

Proposition 39.3. *([6] Proposition 6.4) This connection on $M = C^\infty(\mathbb{R}) \otimes A$ preserves the inner product on M if for all $a \in A$ and $\xi \in \Omega^1_A$ the following equation holds:*

$$\langle (ba + K(\mathrm{d}a) + ab^*), \bar{1} \rangle = 0 = \langle K(\xi^*) - K(\xi)^*, \bar{1} \rangle$$

The first condition is called the divergence condition, and the second the reality condition.

Proposition 39.4. *([6] Proposition 6.5) If ∇_M preserves the inner product as defined above, and if $\nabla_M(m) = 0$ then the positive map $\phi(a) = \langle ma, \overline{m}\rangle$ satisfies*

$$\mathrm{d}t.\frac{\mathrm{d}}{\mathrm{d}t}\phi(a) = (\mathrm{id} \otimes \langle , \rangle)(\sigma_M \otimes \mathrm{id})(m \otimes \mathrm{d}a \otimes \overline{m}).$$

In particular $\frac{\mathrm{d}}{\mathrm{d}t}\phi(1) = 0$, so if we begin at $t = 0$ with a state on A (normalised to be 1 at $1 \in A$) then we have a state for all time.

In summary, the paper [6] says that metric preserving left connections on $M = C^\infty(R) \otimes A$ can be described in terms of vector fields, and for each zero m of such an equation we get a differential equation for time-dependent states on A.
The paper goes further and characterises in terms of time-dependent vector fields when these time-dependent states satisfy an equation of geodesic motion $\mathbb{\nabla}(\sigma_M) = 0$, but in our context we consider the more general case where their time evolution doesn't have to be geodesic.

40 State Path Lifting Problem for a Fibration

In order to construct the diagonal map h required for compatibility with cofibrations, we propose that the definition of a differential fibration should include the additional data of a certain extendable bimodule connection on the total space as a module over the base space. In the following, we construct a projection on the calculi, and a lifting of vector fields from the base space to the total space. The following proposition gives a lifting of a vector field from the base space to the total space.

Proposition 40.1. *Suppose we have a differentiable injective algebra map $i : B \to A$ and a left bimodule connection $\nabla_A : {}_iA \to \Omega^1_B \otimes_B {}_iA$ satisfying the property $\nabla_A(1) = 0$. If $X \in \mathfrak{X}^R_B$ is a right vector field on B, then there is a right vector field on A which we denote $Y : \Omega^1_A \to A$, specified by*

$$ev(Y \otimes \xi) = (ev \otimes \mathrm{id})(X \otimes \sigma_A(1 \otimes \xi)) \tag{76}$$

for any $\xi \in \Omega^1_A$. The vector field Y is a lifting of X, in the sense that it makes the *following diagram commute.*

$$\begin{array}{ccc} \Omega^1_A & \xrightarrow{Y} & A \\ {\scriptstyle\iota}\uparrow & & {\scriptstyle\iota}\uparrow \\ \Omega^1_B & \xrightarrow{\iota} & B \end{array}$$

Proof. **(1)** Firstly we show that Y is a right module map. But since σ_A is a bimodule map,

$$ev(Y\otimes\xi.a)=(ev\otimes\mathrm{id})(X\otimes\sigma_A(1\otimes\xi.a))=(ev\otimes\mathrm{id})(X\otimes\sigma_A(1\otimes\xi)).a=ev(Y\otimes\xi).a$$

for all $a\in A$ and $\xi\in\Omega^1_A$.
(2) Next we show that the diagram commutes. We calculate

$$\begin{aligned} ev(Y\otimes\iota(\eta)) &= (ev\otimes\mathrm{id})(X\otimes\sigma_A(1\otimes\iota(\eta)))=(ev\otimes\mathrm{id})(X\otimes\eta\otimes 1)=ev(X\otimes\eta)\triangleright 1 \\ &= \iota(ev(X\otimes\eta)) \end{aligned}$$

But since evaluation is a bimodule map, having $ev(Y\otimes\iota(\eta))=\iota(ev(X\otimes\eta))$ implies the diagram commutes. □

Supposing we have a connection on $C^\infty(\mathbb{R})\otimes B$, then the above result on geodesics gives a right vector field X on B. Because we have a fibration i, this lifts to a right vector field Y on A, and thus a connection on $C^\infty(\mathbb{R})\otimes A$ given by

$$\nabla_N(c)=\mathrm{d}t\otimes(i(b)c+\frac{\partial c}{\partial t}+Y(\mathrm{d}C)),\qquad \sigma_M(1\otimes\xi)=\mathrm{d}t\otimes Y(\xi) \tag{77}$$

For h to be a differentiable positive map, we want it to come from the zeroes of this connection.
Also, these connections are classified by the following.

Theorem 40.2. *Suppose $i:B\to A$ is a differentiable injective algebra map. This gives a B-A bimodule ${}_iA$ with left action $b.a=i(b)a$ and right action $a.a'=aa'$. Then:*
(1) *The following two statements are equivalent.*

(i) There is a left bimodule connection $\nabla_A:{}_iA\to\Omega^1_B\otimes_B{}_iA$ with $\nabla_A(1)=0$.

(ii) There is a B-A bimodule map $\sigma_A(1\otimes-):{}_i\Omega^1_A\to\Omega^1_B\otimes_B{}_iA$ with $\sigma_A(1\otimes i(\xi).a)=\xi\otimes a$ for all $\xi\in\Omega^1_B$ and $a\in A$.

***(2)** If the statements from part (1) are true, the B-A bimodule map $P = (i.\mathrm{id})\sigma_A(1\otimes -) : {}_i\Omega^1_A \to {}_i\Omega^1_A$ is a projection with image $i(\Omega^1_B).A$.*

Proof. (1.(i) $\Longrightarrow$ (ii)) Firstly, we show that (i) implies (ii). Suppose we have a left bimodule connection $\nabla_A : {}_iA \to \Omega^1_B \otimes_B {}_iA$ with the property $\nabla_A(1) = 0$. For any $a \in A$ and $b \in B$, we have $\nabla_A(b.a) = b.\nabla_A(a) + \mathrm{d}b \otimes a$. If we choose $a = 1$ then we can use the fact that $b.a = i(b)a$ to get $\nabla_A(i(b)) = \mathrm{d}b \otimes 1$. It is clear from the definition that $\sigma_A(1\otimes -)$ is a B-A bimodule map, so we just need show the last property. For all $b, b' \in B$ and $\xi \in \Omega^1_B$ we have $\sigma_A(1 \otimes i(b'\mathrm{d}b)) = \sigma_A(1 \otimes i(b')i(\mathrm{d}b)) = b'\sigma_A(1 \otimes i(\mathrm{d}b)) = b'\mathrm{d}b \otimes 1$. This shows for all $a \in A$ and $\xi \in \Omega^1_B$ the desired property that $\sigma_A(1 \otimes i(\xi).a) = \xi \otimes a$.
(1.(i) $\Longleftarrow$ (ii)) Secondly, we show that (ii) implies (i). Suppose we have a B-A bimodule map $\sigma_A(1 \otimes -) : {}_i\Omega^1_A \to \Omega^1_B \otimes_B {}_iA$ with $\sigma_A(1 \otimes i(\xi).a) = \xi \otimes a$ for all $\xi \in \Omega^1_B$ and $a \in A$. Define a map $\nabla_A : {}_iA \to \Omega^1_B \otimes_B {}_iA$ by $\nabla_A(a) = \sigma_A(1 \otimes \mathrm{d}a)$. Because $\mathrm{d}1 = 0$ this satisfies $\nabla_A(1) = 0$. We need to show that ∇_A is a left connection, meaning we need to prove that $\nabla_A(b.a) = \mathrm{d}b \otimes a + b.\nabla_A(a)$. This follows as $\nabla_A(b.a) = \nabla_A(i(b)a) = \sigma_A(1 \otimes \mathrm{d}(i(b)a)) = \sigma_A(1 \otimes \mathrm{d}(i(b)).a) + \sigma_A(1 \otimes i(b).\mathrm{d}a) = \sigma_A(1 \otimes i(b)).a + b.\sigma(1 \otimes \mathrm{d}a) = \mathrm{d}b \otimes a + b.\nabla_A(a)$. Note that here we have used injectivity of i to get $\sigma_A(1 \otimes i(b).\mathrm{d}a) = b.\sigma(1 \otimes \mathrm{d}a)$.
(2) Firstly, observe that the left hand side of the B-A bimodule map $\sigma_A : {}_iA \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B {}_iA$ is isomorphic to ${}_i\Omega^1_A$. Since $\nabla_A(1) = 0$, we have $\sigma_A(1\otimes \mathrm{d}a) = \nabla_A(a)$. Therefore, for all $b \in B$,

$$P(i(\mathrm{d}b)) = P(\mathrm{d}(i(b))) = (i.\mathrm{id})\nabla_A(i(b)) = (i.\mathrm{id})\nabla_A(b \triangleleft 1) = (i.\mathrm{id})(\mathrm{d}b \otimes 1) = i(\mathrm{d}b).$$

But since P is a right A-module map, it follows that $P(i(\mathrm{d}b).a) = i(\mathrm{d}b).a$ for all $b \in B$, $a \in A$. Hence $P = P^2$ so P is a projection. □

Given these results, it might make sense to include the existence of such a connection (or perhaps some other condition implying its existence) as part of the data of a noncommutative fibration.

41 How Does This Relate to Fibre Bundles?

The algebra maps definition of a fibre bundle, designed to give a Leray-Serre spectral sequence for cohomology, is the following.

Definition 41.1. ([14] Definition 4.1) For a differential graded algebra map $\iota : B \to A$, the forms of degree p on the base and degree q on the fibre are denoted

$$N_{p,q} = \frac{\iota\Omega_B^p \wedge \Omega_A^q}{\iota\Omega_B^{p+1} \wedge \Omega_A^{q-1}}, \qquad N_{p,0} = \iota\Omega_B^p \otimes_A A,$$

and ι is called a *differential fibre bundle* if the maps $g : \Omega_B^p \otimes_B N_{0,q} \to N_{p,q}$, i.e.

$$g : \Omega_B^p \otimes_B \frac{{}_\iota\Omega_A^q}{\iota\Omega_B^1 \wedge \Omega_A^{q-1}} \to \frac{\iota\Omega_B^p \wedge \Omega_A^q}{\iota\Omega_B^{p+1} \wedge \Omega_A^{q-1}}$$

given by $\xi \otimes [\eta] \mapsto [\iota(\xi) \wedge \eta]$ are isomorphisms for all $p, q \geq 0$.

In the previous section, we only assumed that ι is a differentiable injective algebra map, and not necessarily a fibre bundle. But the projection P had image $i(\Omega_B^1).A$, and therefore its kernel can be identified with the quotient $\Omega_A^1/i(\Omega_B^1).A$, which we observe appears in the theory of noncommutative fibre bundles as the 1-forms in the fibre only.
A question we can therefore pose: Given an ι and ∇_A making Quillen's diagram commute, can we equip A with higher calculi such that g becomes an isomorphism? This would make ι a noncommutative fibre bundle.
Answering this question would likely use the following definition:

Definition 41.2. If $\iota : B \to A$ is a differentiable injective algebra map, then define:

$$K^0 = {}_\iota A, \quad K^1 = {}_\iota \ker P, \quad K^n = \frac{{}_\iota\Omega_A^n}{\iota(\Omega_B^1) \wedge \Omega_A^{n-1}}, \quad \forall n \geq 2.$$

We end this section with a guess as to what might be a possible definition of ι being a weak equivalence in the case where it gives a noncommutative fibre bundle.
We could call a differential fibre bundle $\iota : B \to A$ a weak equivalence if the fibre has cohomology $H^q(N) = \delta_{q,0}\mathbb{C}$, where the cochain complex N has qth entry $N_{0,q}$ and the differential is a quotient of the ordinary differential. In this case, the second page of the Leray-Serre Spectral sequence has only one non-vanishing row, with entry $(p, 0)$ given by $H^p(B, \mathbb{C})$. This sequence is already stabilised, and so $H^p(B, \mathbb{C}) \cong H^p(A, E, \nabla_E)$.

42 Examples

In this section we give examples of fibrations where the lifting problem implies the existence of a diagonal map h, and for some of them we solve the differential equation for h explicitly.

42.1 Example: Group Algebras

[Algebras: See Example 3.11 for $\mathbb{C}G$ and its calculi]

Take $A = \mathbb{C}X$ and $B = \mathbb{C}G$ for a subgroup $G \subset X$. Equip $\mathbb{C}X$ with left covariant calculus $\Omega^1_{\mathbb{C}X} = \Lambda^1_{\mathbb{C}X}.\mathbb{C}X$ for a right representation $\Lambda^1_{\mathbb{C}X}$ spanned by a linear map $\omega : \mathbb{C}X \to \Lambda^1_{\mathbb{C}X}$ satisfying $\omega(xy) = \omega(x) \triangleleft y + \omega(y)$ for $x, y \in X$ called a cocycle. The differential is $\mathrm{d}x = x\omega(x)$. Similarly equip $\mathbb{C}G$ with $\Omega^1_{\mathbb{C}G} = \Lambda^1_{\mathbb{C}G}.\mathbb{C}G$ for $\Lambda^1_{\mathbb{C}G}$ the vector space spanned by the restriction $\omega|_{\mathbb{C}G}$.

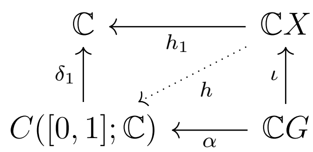
Take $i : \mathbb{C}G \to \mathbb{C}X$ to be the inclusion map. We want to explicitly find a bimodule map $\sigma_A(1 \otimes -) : {}_i\Omega^1_A \to \Omega^1_B \otimes_B {}_iA$ satisfying $\sigma_A(1 \otimes i^*(\xi).a) = \xi \otimes a$ for all $\xi \in \Lambda^1_B$ and $a \in A$. For $x, y \in X$, define

$$\sigma_A(1 \otimes \mathrm{d}x.y) = \begin{cases} \mathrm{d}x \otimes y, & \text{if } x \in G \\ 0, & \text{else} \end{cases}.$$

Then $\sigma_A(1 \otimes i^*(\xi).x) = \xi \otimes x$ for all $\xi \in \Omega^1_{\mathbb{C}G}$ and $x \in X$ as required. This gives a left bimodule connection $\nabla_{\mathbb{C}X} : {}_i\mathbb{C}X \to \Omega^1_{\mathbb{C}G} \otimes_{\mathbb{C}G} {}_i\mathbb{C}X$ with $\nabla_{\mathbb{C}X}(1) = 0$.

Seeing as $\Omega^1_{\mathbb{C}G} \subset \Omega^1_{\mathbb{C}X}$, it follows that $\mathfrak{X}^R_{\mathbb{C}G} \subset \mathfrak{X}^R_{\mathbb{C}X}$, and so the lifting of a vector field is simply its inclusion.

Example 42.1. We give an example of a choice of h_1, α and h to fill in the commutative diagram, in order to illustrate what kind of maps go there. We note that this of course does not constitute a proof, and is just an illustration.



This could have maps

$$h_1(x) = \begin{cases} 0, & \text{if } x \neq e \\ 1, & \text{if } x = e \end{cases}, \quad \alpha_t(g) = \begin{cases} 0, & \text{if } g \neq e \\ t, & \text{if } g = e \end{cases}, \quad h_t(x) = \begin{cases} 0, & \text{if } x \neq e \\ t, & \text{if } x = e \end{cases}$$

and where δ_1 means evaluation at 1. ◇

Remark 42.2. *One interesting thing about this example is that there is a conditional expectation* $\mathbb{E} : \mathbb{C}X \to \mathbb{C}G$ *— the one from Section 34.1. If we had* $h_1 = \alpha_1 \circ \mathbb{E}$ *(which it* *isn't obvious we do), then we could construct the diagonal map as $h_t = \alpha_t \circ \mathbb{E}$. This is a different method to our more general connection-based one, but it relies on a non-trivial assumption so it isn't clear how general it is.*

42.2 Example: Heisenberg Group and Quantum Circle

[Algebras: See Example 3.13 for $\mathbb{C}Hg$ and its calculus, and Example 3.2 for $\mathbb{C}_q[S^1]$ and its calculus]

The following is another example of group algebras, except here the fibration isn't just the inclusion map.

Take $A = \mathbb{C}Hg$ and $B = \mathbb{C}_1[S^1] = \mathbb{C}[t, t^{-1}]$, i.e. we equip the quantum circle with its classical calculus so 1-forms commute with algebra elements, then there is a differential graded algebra map

$$\iota : \mathbb{C}[t, t^{-1}] \to \mathbb{C}Hg, \qquad \iota(t) = w, \qquad \iota^*(\mathrm{d}t) = we^w$$

which is a differential fibre bundle. (See [10] Example 4.67)

Proposition 42.3. *There is a left bimodule connection on ${}_\iota\Omega^1(\mathbb{C}Hg)$ specified by the $\mathbb{C}[t, t^{-1}]$-$\mathbb{C}Hg$ bimodule map $\sigma_A(1 \otimes -) : {}_\iota\Omega^1(\mathbb{C}Hg) \to \Omega^1(\mathbb{C}[t, t^{-1}]) \otimes_{\mathbb{C}[t,t^{-1}]} {}_\iota\mathbb{C}Hg$ given by*

$$\sigma_A(1 \otimes e^x) = \begin{cases} t^{-1}\mathrm{d}t \otimes 1, & \text{if } x = w \\ 0, & \text{if } x \in \{u, v\} \end{cases}$$

which satisfies $\sigma_A(1 \otimes \iota^*(\xi).a) = \xi \otimes a$ for all $\xi \in \Omega^1(\mathbb{C}[t, t^{-1}])$, $a \in \mathbb{C}Hg$.

Proof. **(1)** Firstly we show that σ_A is well-defined. The first thing to check is:

$$\sigma_A(1 \otimes e^u).v = \sigma_A(1 \otimes v(e^u + \frac{1}{2}e^w))$$

But since v is not in the image of ι so it cannot move across the tensor product, so it doesn't introduce any problems. The second thing to check is:

$$\sigma_A(1 \otimes e^v).u = \sigma_A(1 \otimes u(e^v - \frac{1}{2}e^w))$$

But again, u is not in the image of ι so it cannot move across the tensor product.

(2) Secondly, we show the connection σ_A satisfies the required property. A general element of the calculus $\Omega^1(\mathbb{C}[t,t^{-1}])$ takes the form $\mathrm{d}t.f(t)$ for f some polynomial in t. Then using that ι is a differential graded algebra map and σ_A a bimodule map, we calculate:

$$\sigma_A(1\otimes\iota^*(\mathrm{d}t.f(t)))=\sigma_A(1\otimes we^w)f(w)=\mathrm{d}t\otimes f(w)=\mathrm{d}t.f(t)\otimes 1$$

as required. □

Proposition 42.4. *A general right vector field X on $\mathbb{C}[t,t^{-1}]$ takes the form $X = f(t)(it\frac{\partial}{\partial t})$ and is real if f is real. It lifts to a right vector field Y on $\mathbb{C}Hg$ given by*

$$Y = f(w)\big(ie_w\big) : \Omega^1(\mathbb{C}Hg)\to\mathbb{C}Hg. \tag{78}$$

If X is real then Y is real, with respect to the Haar measure

$$\int=\phi:\mathbb{C}Hg\to\mathbb{C},\qquad \phi(g)=\delta_{g,e}.$$

Proof. (1) Taking $t=e^{i\theta}$, we have $\frac{\partial}{\partial\theta}=\frac{\partial t}{\partial\theta}\frac{\partial}{\partial t}=it\frac{\partial}{\partial t}$. A general vector field is of the form $f(t)\frac{\partial}{\partial\theta}$, and hence of the form $f(t)(it\frac{\partial}{\partial t})$ and is real if f is real.
(2) We show the formula for the lifted vector field. The general formula is $ev(Y\otimes\xi)=(ev\otimes\mathrm{id})(X\otimes\sigma_A(1\otimes\xi))$. Since a general vector field on $\mathbb{C}[t,t^{-1}]$ is of the form $f(t)(i\frac{\partial}{\partial t})$, we have

$$ev(X\otimes\mathrm{d}t)=itf(t),$$

and so, looking at the one invariant 1-form e^w that σ_A does not send to zero, we calculate:

$$\begin{aligned}ev(Y\otimes e^x)&=(ev\otimes\mathrm{id})(X\otimes\sigma_A(1\otimes e^x))=\delta_{x,w}(ev\otimes\mathrm{id})(t^{-1}\mathrm{d}t)=\delta_{x,w}\iota ev(X\otimes t^{-1}\mathrm{d}t)\\ &=if(w)\delta_{x,w}=\begin{cases} if(w), & \text{if } x=w\\ 0, & \text{if } x\in\{u,v\}\end{cases}.\end{aligned}$$

This is equivalent to $Y=f(w)\big(ie_w\big)$.
(3) Next we show that if the vector field X on $\mathbb{C}[t,t^{-1}]$ is real then its lift Y to $\mathbb{C}Hg$ is real. A general X is of the form $f(t)\big(-it\frac{\partial}{\partial t}\big)$, and is real if and only if f is a real function on the circle, i.e. if $f(t)=f(t)^*$. Expanding $f(t)=\sum_k f_k t^k$, we calculate

$$f(t)^*=\sum_k f_k^*(t^k)^*=\sum_k f_k^* t^{-k}.$$

Hence $f(t) = f(t)^*$ if and only if $f_k = f^*_{-k}$.

We calculate the differential on an element of Hg as:

$$\mathrm{d}(w^n u^m v^p) = (ne^w + me^u + pe^v \triangleleft u^{-m}) w^n u^m v^p = (ne^w + me^u + p(e^v + \frac{m}{2} e^w)) w^n u^m v^p.$$

The evaluation of the lifted vector field Y on this is then:

$$Y(\mathrm{d}(w^n u^m v^p)) = if(w)(n + p\frac{m}{2}) w^n u^m v^p = if(w)(n + \frac{1}{2}mp) w^n u^m v^p$$

The Haar measure $\phi : \mathbb{C}Hg \to \mathbb{C}$ is given by $\phi(g) = \delta_{g,e}$.

Using the expansion $f(w) = \sum\limits_k f_k w^k$ for some $f_k \in \mathbb{C}$, we calculate:

$$\phi Y(\mathrm{d}(w^n u^m v^p)) = inf_{-n}\delta_{m,0}\delta_{p,0}$$

Since $(w^n u^m v^p)^* = v^{-p} u^{-m} w^{-n}$, we get

$$\int Y(\xi^*) = \phi Y(\mathrm{d}(v^{-p} u^{-m} w^{-n})) = -inf_{-n}\delta_{-m,0}\delta_{-p,0},$$

while on the other hand we have

$$\Big(\int Y(\xi)\Big)^* = \Big(inf_{-n}\delta_{m,0}\delta_{p,0}\Big)^* = -inf^*_{-n}\delta_{m,0}\delta_{p,0}.$$

For reality of Y we need these to coincide, i.e. that $f_n = f^*_{-n}$, which is precisely the condition for X to be real. □

Proposition 42.5. *The zeroes of the connection $\nabla(e) = \dot{e} + K(\mathrm{d}e) + be$ on the bimodule corresponding to the diagonal map satisfy*

$$\dot{e} = -\frac{1}{2}\mathrm{div}Y.e - ev(Y \otimes \mathrm{d}e).$$

where

$$\mathrm{div}Y = iw\frac{\partial f}{\partial w}. \tag{79}$$

By the KSGNS construction this gives the following positive map across the diagonal:

$$h : A \to \mathbb{C} \otimes C^\infty(\mathbb{R}), \qquad h(a)(t) = \langle \overline{e(t)}, ae(t) \rangle. \tag{80}$$

We can further describe this as $e = \sum\limits_{n,m,p} s(w,t) u^m v^p$, where

$$\dot{s} = -\frac{1}{2} iw\frac{\partial f}{\partial w} s - if(w)(\frac{\partial s}{\partial w} + \frac{1}{2}mps).$$

Proof. **(1)** First we calculate the divergence of Y. This is defined by the equation

$$\int (\mathrm{div}Y.a) + \int Y(\mathrm{d}a) = 0$$

which is equivalent to the following equation holding for all n, m, p:

$$\phi\big(\mathrm{div}Y.w^n u^m v^p\big) = i\phi\big(f(w)(n + \frac{1}{2}mp)w^n u^m v^p)\big)$$

If m and p are not both equal to zero then the right hand side vanishes, in which case for all p, q, the divergence $\mathrm{div}Y$ must be the product of a function of w with $u^{-m}v^{-p}$, which is a contradiction. Hence the divergence must be purely a function of w, and so $\mathrm{div}Y = \sum g_j w^j$ for coefficients $g_j \in \mathbb{C}$. Next, we set $m = p = 0$ to get the equation

$$\int \big(\mathrm{div}Y.w^n\big) = -in \int (f(w)w^n)$$

which implies $g_{-n} = -inf_{-n}$, or equivalently $g_n = inf_n$. But this is equivalent to

$$\mathrm{div}Y = iw\frac{\partial f}{\partial w}.$$

(2) Next we look at the differential equation. First we expand:

$$e = \sum_{n,m,p} e_{nmp}(t) w^n u^m v^p$$

If we fix m and p, we get $e = s(w,t)u^m v^p$. We can solve the equation for each of these and then add them together to get the general solution. Hence we get a differential equation for s:

$$\dot{s}u^m v^p = -\frac{1}{2}\mathrm{div}Y s u^m v^p - Y(\mathrm{d}(su^m v^p)).$$

Using the Leibniz rule and the fact that $ue^u = e^u.u$ and $ve^v = e^v.v$, and also that $e^v \triangleleft u^{-m} = e^v + \frac{m}{2}e^w$, we calculate

$$\begin{aligned}\mathrm{d}(su^m v^p) &= \mathrm{d}s.u^m v^p + mse^u u^m v^p + psu^m e^v v^p \\ &= \mathrm{d}s.u^m v^p + me^u su^m v^p + p(e^v \triangleleft u^{-m})su^m v^p \\ &= \big(\mathrm{d}s.u^m v^p + me^u su^m v^p + p(e^v + \frac{m}{2}e^w)s\big)u^m v^p\end{aligned}$$

Also, expanding $s = \sum_j s_j(t)w^j$, we have $\mathrm{d}s = \sum_j je^w s_j(t)w^j$, so $ev(\frac{\partial}{\partial w} \otimes \mathrm{d}s) = w\frac{\partial s}{\partial w}$.

Hence the differential equation becomes

$$\dot{s}u^m v^p = -\frac{1}{2}(-iw\frac{\partial f}{\partial w})su^m v^p - iw^{-1}f(w).ev\big(\frac{\partial}{\partial w}\otimes(\mathrm{d}s+\frac{1}{2}mpe^w s)\big)u^m v^p$$

Simplifying and applying the calculations above, we get:

$$\dot{s} = -\frac{1}{2}iw\frac{\partial f}{\partial w}s - if(w)(\frac{\partial s}{\partial w}+\frac{1}{2}mps)$$

If we expand this into powers of w, we can compare the coefficients to get a first order linear differential equation involving $s_j(t)$ and $s_{j+1}(t)$. The solutions of the general case don't look to be nice, but perhaps there might be an existence result. □

42.3 Example: Functions on S_3

[Algebras: See Example 3.10 for $C(G)$ and its calculi]

Let G be a normal subgroup of a finite group X, so the quotient X/G is a group. The algebra $\mathbb{C}X$ has left covariant calculus $\Omega^1(X) = C(X).\Lambda^1_X$, where the vector space Λ^1_X of left-invariant 1-forms has basis elements e_a for $a \in \mathcal{C} \subset X\backslash\{e\}$ satisfying

$$e_a.f = R_a(f)e_a, \qquad \mathrm{d}f = \sum_{a\in\mathcal{C}}(R_a(f)-f)e_a,$$

where $R_a(f)(x) = f(xa)$. This implies $e_a.\delta_x = \delta_{xa^{-1}}.e_a$. The calculus is inner by $\theta = \sum_a e_a$, and is a star-calculus by $e_a^* = -e_{a^{-1}}$ if $\mathcal{C}$ has inverses.

The projection $p : X \to X/G$ is a homomorphism and induces an algebra map

$$i : C(X/G) \to C(X), \qquad \delta_y \mapsto \sum_{x\in X:p(x)=y}\delta_x.$$

This is our candidate for a fibration. Now we look at a specific example. The only non-trivial normal subgroup of $X = S_3$ is the alternating group $G = A_3 = \{e,(123),(132)\}$. The quotient $\frac{S_3}{A_3}$ is isomorphic to C_2, the algebra with a single generator x.

Equip $A = C(S_3)$ with calculus given by $\mathcal{C} = \{(12),(123),(132)\}$. The set $\mathcal{C}$ is closed under inverses, so $\Omega^1(S_3)$ is a *-calculus with:

$$e^*_{(12)} = -e_{(12)}, \qquad e^*_{(123)} = -e_{(132)}, \qquad e^*_{(132)} = -e_{(123)}.$$

For the higher calculi on $C(S_3)$, we quotient by the relations $e_{(12)} \wedge e_{(12)} = 0$ and $\theta \wedge \theta = 0$. The latter of these relations is equivalent to having for all $y \in S_3$,

$$\sum_{a,b \in \mathcal{C}: ab=y} e_a \wedge e_b = 0.$$

Written out in full, the six relations are:

$$0 = e_{(12)} \wedge e_{(12)}, \qquad 0 = e_{(132)} \wedge e_{(132)}, \qquad 0 = e_{(123)} \wedge e_{(123)}$$
$$0 = e_{(12)} \wedge e_{(123)} + e_{(132)} \wedge e_{(12)}, \qquad 0 = e_{(123)} \wedge e_{(12)} + e_{(12)} \wedge e_{(132)}$$
$$0 = e_{(123)} \wedge e_{(132)} + e_{(132)} \wedge e_{(123)}$$

In a general monomial element, without loss of generality we can slide all $e_{(12)}$ terms to the right of all other terms. Since $e_{(123)}$ and $e_{(132)}$ anticommute, we can put all $e_{(132)}$ terms to the right of all $e_{(123)}$ terms. But since every type of term squares to zero, we are left with the conclusion that every nonzero element of $\Omega^3(S_3)$ is equal to a scalar multiple of $e_{(123)} \wedge e_{(132)} \wedge e_{(12)}$, and that all higher calculi are zero. The calculus $\Omega^2(S_3)$ is then spanned by the three elements $e_{(123)} \wedge e_{(132)}$ and $e_{(123)} \wedge e_{(12)}$ and $e_{(132)} \wedge e_{(12)}$.
For the algebra $B = C(C_2)$, we equip it with calculus determined by the set $\mathcal{D} = \{x\}$, which is closed under inverses, making $\Omega(C_2)$ a *-calculus with $e_x^* = -e_x$. For the higher calculi, we impose the relation $\theta \wedge \theta = 0$, which means $e_x \wedge e_x = 0$, and so for all $n \geq 2$ we have $\Omega^n(C_2) = 0$.

Proposition 42.6. *The algebra map*

$$i : C(C_2) \to C(S_3), \qquad \begin{cases} \delta_e \mapsto \delta_e + \delta_{(123)} + \delta_{(132)} \\ \delta_x \mapsto \delta_{(12)} + \delta_{(23)} + \delta_{(31)} \end{cases} \tag{81}$$

which extends to the above calculi as a differential graded algebra map

$$i^* : \Omega^1(C_2) \to \Omega^1(S_3), \qquad e_x \mapsto e_{(12)} \tag{82}$$

is a differential fibration.

Proof. We look at the following quotient

$$N_{p,q} = \frac{i\Omega^p(C_2) \wedge \Omega^q(S_3)}{i\Omega^{p+1}(C_2) \wedge \Omega^{q-1}(S_3)}$$

In the case $p = 0$ we see that g is an isomorphism:

$$C(C_2) \otimes_{C(C_2)} \frac{\Omega^q(S_3)}{e_{(12)} \wedge \Omega^{q-1}(S_3)} \to \frac{\Omega^q(S_3)}{e_{(12)} \wedge \Omega^{q-1}(S_3)}.$$

Next, in the case $p = 1$, we can see that g is an isomorphism.

$$e_x \otimes_{C(C_2)} \frac{\Omega^q(S_3)}{e_{(12)} \wedge \Omega^{q-1}(S_3)} \to e_{(12)} \wedge \Omega^q(S_3)$$

□

The Haar measure on $C(S_3)$ is given by:

$$\phi : C(S_3) \to \mathbb{C}, \qquad \delta_g \mapsto \frac{1}{6}$$

Proposition 42.7. *With the above calculi, there is precisely one left bimodule connection on ${}_iC(S_3)$ satisfying $\nabla_A(1) = 0$ and $\sigma_A(1 \otimes i^*(\xi)) = \xi \otimes 1$, and it is given by:*

$$\sigma_A(1 \otimes e_a.f) = \begin{cases} e_x \otimes f, & \text{if } a = (12) \\ 0, & \text{if } a \in \{(123), (132)\} \end{cases} \tag{83}$$

for $f \in C(S_3)$.

Proof. We show uniqueness of the bimodule connection. The fact that $\sigma_A(1 \otimes e_{(12)}) = e_x$ is automatic from the condition, but the values of $\sigma_A(1 \otimes e_{(123)})$ and $\sigma_A(1 \otimes e_{(132)})$ are non-trivial to calculate. Denote:

$$\sigma_A(1 \otimes e_{(123)}) = e_x f_{(123)} \otimes 1$$
$$\sigma_A(1 \otimes e_{(132)}) = e_x f_{(132)} \otimes 1$$

Then the fact that σ is a bimodule map implies that for $\theta \in \{e, x\}$ and $a \in \{(12), (123), (132)\}$ that

$$\delta_\theta \sigma_A(1 \otimes e_a) = \sigma_A(1 \otimes e_a) R_{a^{-1}}(i(\delta_\theta))$$

which gives the following equations:

$$\delta_e e_x f_{(123)} \otimes 1 = e_x f_{(123)} \otimes R_{(123)^{-1}}(i(\delta_e))$$
$$\delta_x e_x f_{(123)} \otimes 1 = e_x f_{(123)} \otimes R_{(123)^{-1}}(i(\delta_x))$$

$$\delta_e e_x f_{(132)} \otimes 1 = e_x f_{(132)} \otimes R_{(132)^{-1}}(i(\delta_e))$$
$$\delta_x e_x f_{(132)} \otimes 1 = e_x f_{(132)} \otimes R_{(132)^{-1}}(i(\delta_x))$$

The first of these equations gives, using the fact that $\delta_\theta.e_a = e_a.\delta_{\theta a}$,

$$\begin{aligned} e_x\delta_x f_{(123)} \otimes 1 &= e_x f_{(123)} \otimes R_{(123)^{-1}}(\delta_e + \delta_{(123)} + \delta_{(132)}) = e_x f_{(123)} \otimes (\delta_e + \delta_{(123)} + \delta_{(132)}) \\ &= e_x f_{(123)} \otimes i(\delta_e) = e_x f_{(123)}\delta_e \otimes 1. \end{aligned}$$

This implies

$$\delta_x f_{(123)} = f_{(123)}\delta_e$$

Since these are functions, the equality has to hold when evaluated at the points e and x. Consequently $f_{(123)}(e) = 0$ and $f_{(123)}(x) = 0$, so $f_{(123)} = 0$.
Similarly, the third equation gives:

$$e_x\delta_x f_{(132)} \otimes 1 = e_x f_{(132)}\delta_e \otimes 1.$$

This gives

$$\delta_x f_{(132)} = f_{(132)}\delta_e,$$

which for the same reasons as above implies $f_{(132)} = 0$, as required. The other two equations simply reduce to $0 = 0$. □

Proposition 42.8. *A general right vector field $X = X^i e^x = (\mu_e\delta_e + \mu_x\delta_x)e^x$ on $C(S_2)$ lifts to a right vector field*

$$Y : \Omega^1(S_3) \to C(S_3), \qquad ev(Y \otimes e_a) = \begin{cases} iX(e_x), & \text{if } a = (12) \\ 0, & \text{if } a \in \{(123), (132)\} \end{cases} \tag{84}$$

If X satisfies the reality condition

$$\int_B X(\xi^*) = \int_B X(\xi)^*.$$

where $\int_B = \int_A \circ i$, then Y satisfies the reality condition:

$$\int_A Y(\xi^*) = \int_A Y(\xi)^*.$$

If integration $\int$ *is taken to be the Haar measure*

$$\phi : C(S_3) \to \mathbb{C}, \qquad \phi(\delta_y) = \frac{1}{6},$$

the divergence divY *of the vector field* Y*, defined by*

$$\int \big(\mathrm{div}Y.a\big) + \int Y(\mathrm{d}a) = 0,$$

is given by $\mathrm{div}Y = 0$.

Proof. (1) Firstly we show the reality condition. If we take the reality condition on X, then use the definition of $\int_B = \int_A \circ i$ and the formula for Y, then the fact that i is a star-algebra map implies the reality condition for Y.
(2) Next, we calculate the divergence. For general $f \in C(S_3)$ we have

$$\mathrm{d}f = \sum_{a\in\mathcal{C}} e_a(f - R_{a^{-1}}(f)),$$

which expanding $f = \sum_{y\in S_3} \lambda_y \delta_y$ gives

$$\mathrm{d}f = \sum_{y\in S_3} \lambda_y \sum_{a\in\mathcal{C}} e_a(\delta_y - \delta_{ya})$$

A general right vector field on $C(C_2)$ is given by $X = X^x e^x$ for some $X^x \in C(C_2)$, where e^x is the dual of e_x. Using that Y is a right module map and that $ev(Y \otimes e_a) = \delta_{a,(12)} i(X^x)$ we have:

$$ev(Y \otimes \mathrm{d}f) = i(X^x).(f - R_{(12)}(f)) = i(X^x).\sum_{y\in S_3} \lambda_y(\delta_y - \delta_{y.(12)})$$

Hence for the Haar measure $\phi : C(S_3) \to \mathbb{C}$ given by $\phi(\delta_y) = \frac{1}{6}$ for all y, we have

$$\phi(Y(\mathrm{d}f)) = \sum_{y\in S_3} \lambda_y(\frac{1}{6} - \frac{1}{6}) = 0.$$

Consequently, for all $a \in C(S_3)$, the definition of divergence gives:

$$\phi(\mathrm{divY}.a) = 0.$$

It follows from this that $\mathrm{divY} = 0$. □

Proposition 42.9. *For* $b = \frac{1}{2}\mathrm{divY}$*, the zeroes of the connection* $\nabla(e) = \dot{e}K(\mathrm{de}) + be$ *on the bimodule corresponding to the diagonal map satisfy*

$$\dot{e} = -\frac{1}{2}\mathrm{div}Y.e - Y(\mathrm{de}), \tag{85}$$

which has solution

$$e(t) = \sum_{y \in S_3} \left(A_y + B_y e^{-2(\mu_e + \mu_x)t}\right)\delta_y \tag{86}$$

for $y \in S_3$*, and constants* $A_y, B_y, \mu_e, \mu_x \in \mathbb{C}$ *satisfying* $A_y = A_{y.(12)}$ *and* $B_y = -B_{y.(12)}$*. Via the KSGNS construction, this gives positive maps across the diagonal* $h(a) = \langle \overline{e}, ae \rangle$*.*

Proof. We solve the differential equation for zeroes of ∇.

$$-\dot{e} = Y(\mathrm{de}).$$

Expanding $-e(t) = \sum_{y \in S_3} \lambda_y(t)\delta_y$, the differential equation becomes:

$$\sum_{y \in S_3} \dot{\lambda}_y(t)\delta_y = i(X^x).\sum_{y \in S_3} \lambda_y(t)(\delta_y - \delta_{y.(12)})$$

We can expand $X^x = \mu_e\delta_e + \mu_x\delta_x$ for some constants $\mu_e, \mu_x \in \mathbb{C}$, giving

$$i(X^x) = \mu_e(\delta_e + \delta_{(123)} + \delta_{(132)}) + \mu_x(\delta_{(12)} + \delta_{(23)} + \delta_{(31)}).$$

Hence the differential the equation becomes:

$$\begin{aligned}
\sum_{y \in S_3} \dot{\lambda}_y(t)\delta_y &= \sum_{y \in \{e,(123),(132)\}} \lambda_y(t)\big(\mu_e\delta_y - \mu_x\delta_{y.(12)}\big) + \sum_{y \in \{(12),(23),(31)\}} \lambda_y(t)\big(\mu_x\delta_y - \mu_e\delta_{y.(12)}\big) \\
&= \sum_{x \in C(S_3)} \Big(\lambda_y(t)(\mu_e\delta_y - \mu_x\delta_{y.(12)}) + \lambda_{y.(12)}(t)(\mu_x\delta_{y.(12)} - \mu_e\delta_y)\Big) \\
&= \sum_{y \in C(S_3)} \big((\lambda_y(t) - \lambda_{y.(12)}(t))(\mu_e\delta_y - \mu_x\delta_{y.(12)})\big) \\
&= \sum_{y \in C(S_3)} (\mu_e + \mu_x)(\lambda_y(t) - \lambda_{y.(12)}(t))\delta_y
\end{aligned}$$

Since the functions δ_y are linearly independent, comparing their coefficients gives the following differential equation in t:

$$\dot{\lambda}_y(t) = (\mu_e + \mu_x)(\lambda_y(t) - \lambda_{y.(12)}(t)).$$

Replacing y with $y.(12)$ in this gives

$$\dot{\lambda}_{y.(12)}(t) = (\mu_e + \mu_x)(\lambda_{y.(12)}(t) - \lambda_y(t)),$$

and hence

$$\dot{\lambda}_y(t) + \dot{\lambda}_{y.(12)}(t) = 0.$$

Thus $\lambda_y(t) + \lambda_{y.(12)}(t) = C_y$, for some constant C_y and for all t. Substituting into the original equation and re-arranging, we get

$$\dot{\lambda}_y(t) - 2(\mu_e + \mu_x)\lambda_y(t) = -(\mu_e + \mu_x)C_y.$$

Multiplying both sides by $e^{-2(\mu_e+\mu_x)t}$ and using the Leibniz rule, the equation becomes

$$\frac{\partial}{\partial t}\left(e^{-2(\mu_e+\mu_x)t}.\lambda_y(t)\right) = -(\mu_e + \mu_x)e^{-2(\mu_e+\mu_x)t}C_y.$$

Integrating and re-arranging gives the solution

$$\lambda_y(t) = \frac{1}{2}C_y + B_y e^{-2(\mu_e+\mu_x)t},$$

so

$$-e(t) = \sum_{y\in S_3}\left(A_y + B_y e^{-2(\mu_e+\mu_x)t}\right)\delta_y$$

for constants $A_y, B_y \in \mathbb{C}$ satisfying $A_y = A_{y.(12)}$ and $B_y = -B_{y.(12)}$. We note that due to these relations, although it appears there are 12 constants, there are actually only 6, which is the same number as the number of elements of S_3. Lastly we absorb the minus sign into the constants to get the result in the proposition. □

42.4 Example: $\mathbb{C}_q[SU_2]$ and $\mathbb{C}_q[S^2]$ (Using Hopf Fibration)

[Algebras: See Example 3.5 for $\mathbb{C}_q[SU_2]$ and its 3D calculus, and Example 3.6 for $\mathbb{C}_q[S^2]$ and its calculus]

There is a Haar measure on $\int : \mathbb{C}_q[SU_2] \to \mathbb{C}$ given on elements of the form $(bc)^n$ as

$$\int (bc)^n = \frac{(-1)^n q^n}{[n+1]_{q^2}},$$

and zero on all basis elements not of this form. Here the square brackets denote q-integers, defined as

$$[n]_q = \frac{1-q^n}{1-q}$$

The Haar measure on $\mathbb{C}_q[SU_2]$ restricts to $\mathbb{C}_q[S^2]$ as

$$\int x^n = \frac{1}{[n+1]_{q^2}}.$$

The inclusion map $i : \mathbb{C}_q[S^2] \to \mathbb{C}_q[SU_2]$ is a differential fibration called the Hopf fibration, and it extends to the calculi. (See [10] Example 4.68)

Proposition 42.10. *The* $\mathbb{C}_q[S^2]$-$\mathbb{C}_q[SU_2]$ *bimodule map* $\sigma_A(1\otimes -) : {}_i\Omega^1_{\mathbb{C}_q[SU_2]} \to \Omega^1_{\mathbb{C}_q[S^2]} \otimes_{\mathbb{C}_q[S^2]} {}_i\mathbb{C}_q[SU_2]$ *for all* $\eta \in \Lambda^1_{\mathbb{C}_q[S^2]}$ *defined by*

$$\sigma_A(1\otimes\eta) = \begin{cases} \eta\otimes 1, & \textit{if } |\eta| = 0 \\ 0, & \textit{else} \end{cases}.$$

satisfies $\sigma_A(1\otimes i^*(\xi).a) = \xi\otimes a$ *for all* $\xi \in \Omega^1_{\mathbb{C}_q[S^2]}$ *and* $a \in \mathbb{C}_q[SU_2]$.

Proof. We see that the bimodule map σ_A satisfies the condition, because $|i^*(\xi)| = 0$ and $\sigma_A(1\otimes -)$ is a bimodule map. □

43 Future Ideas and Discussion

These are just the first few ideas in the direction of differential cofibrations. Our focus here has been primarily on the vector fields specifying the α_t rather than the maps themselves. In order to replace $\mathbb{C}$ in the diagram, we would need to generalise the geodesics methods that describe connections in terms of vector fields.

Moreover, it is not even clear at this stage what the right category in which to look at the problem is. We might also look at how much can still be done if ι is no longer an injective algebra map but an injective completely positive map, and whether this relates to our notion of fibre bundle in Part IV. We could even consider a category of algebras whose morphisms are Hilbert C* bimodules with connections.

Also, the state path lifting problem we solved may have applications aside from the study of cofibrations. For example, we found a formula for lifting paths, but we might investigate the question of under what conditions noncommutative geodesics lift to noncommutative geodesics, i.e. whether if the initial connection satisfying $\mathbb{T}(\sigma_E) = 0$ implies that the lifted connection also satisfies this.

Part VII

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