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September 8, 2023

Abstract

In this paper, we consider the optimal dividend policy for an insurance company under a contagious insurance market, where the occurrence of a claim can trigger sequent claims. This clustering effect is modelled by a self-exciting Hawkes process where the intensity of claims depends on its historical path. In addition, we include the concept of bankruptcy to allow the insurance company to operate with a temporary negative surplus. The objective of the management is to obtain the optimal dividend strategy that maximises the expected discounted dividend payments until bankruptcy. The Hamilton–Jacobi–Bellman variational inequalities (HJBVIs) are derived rigorously. When claim sizes follow exponential distributions and the bankruptcy rate is a positive constant, the value function can be obtained based on the Gerber-Shiu penalty function and the optimal dividend barrier can be solved numerically. Finally, numerical examples are demonstrated to show the impact of key parameters on the optimal dividend strategy.

Key Words: Dynamic programming, self-exciting Hawkes process, Gamma-Omega model, optimal dividend strategy

JEL classification: C61, G22, G35

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June 14, 2024

Abstract

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1 Introduction

Determining the optimal dividend policy has been a critical challenge in insurance and corporate finance. One approach is to assume that the cash surplus follows a classical Cramér-Lundberg model, where insurance claims satisfy a compound Poisson process. Early research by Gerber (1969) explored optimal dividend strategy, particularly the band strategy under the classical Cramér-Lundberg model. Moreover, the optimal strategy can be reduced to a barrier strategy, when claims sizes are exponential. Other early works include Højgaard (2002), Zhou (2005), Lin and Pavlova (2006), and Lin and Sendova (2008). More recently, Gerber and Shiu (2006) illustrate the explicit optimal dividend policy in a problem with the Brownian motion generalised to a compound Poisson process. Jin et al. (2022) and Miao et al. (2022), delve into maximising expected utility for insurers through investment, liability ratios, and dividend strategies. Liu et al. (2023) study the optimal band dividend strategy in a dual model with a piecewise-deterministic compound Poisson process.

In addition to compound Poisson processes, spectrally negative Lévy processes are used to model surplus in various contexts. Researchers have explored threshold dividend strategies, optimal periodic dividend strategies with fixed transaction costs, and optimal dividend problems for an insurance company under an exponentially Lévy processes (Yang et al. (2020), Avanzi et al. (2021), and Eisenberg and Palmowski (2021)). Under general Lévy processes, increments of claim arrivals are assumed to be independent and stationary, which is challenged by empirical studies in multiple insurance contexts and cannot allow any serial dependence or clustering features.

Empirical studies have shown that non-life insurance claims can be correlated to external extreme events such as natural disasters and economic recessions (Khan (2015) and Babuna et al. (2020)). To address this correlation and contribute to the literature, we apply the Hawkes process to model claim processes. The Hawkes process is a path-dependent point process, where jump intensities depend on historical paths. It has been applied in various fields. For instance, Aït-Sahalia et al. (2015) and Aït-Sahalia and Hurd (2015) introduce a model for asset return dynamics using Hawkes processes, considering contagion effects in portfolio management. Cao et al. (2020) study the optimal reinsurance-investment problem for compound dynamic contagion processes under the mean-variance criterion, allowing for self-exciting and externally-exciting clustering effects. Liu et al. (2021) explore optimal control problems in households, modelling financial markets with mutual-exciting Hawkes processes. However, the literature on optimal dividend policies under singular control frameworks remains limited. Chen and Bian (2021) prove the existence and uniqueness of the viscosity solution for the optimal dividend problem under self-exciting Hawkes processes, while Qiu et al. (2023) study dividend optimisation of insurance groups in insurance groups subject to exogenous default risk modelled by a Markov chain.

Chen and Bian (2021) simplify the free boundary problem by incorporating the classical ruin model and using a finite difference numerical scheme to solve it. However, the classical ruin model is impractical in real markets and the finite difference numerical scheme lacks robustness without a boundary condition. Qiu et al. (2023) apply the diffusion approximation to the Cramér-Lundberg model and only consider a Markov chain with a constant intensity. To enhance their work, we apply the Gamma-Omega model to the Hawkes process, extending the concept from ruin to bankruptcy. In this extended framework, insurance companies can continue to operate with a temporary negative surplus. For further applications of this model, refer to Albrecher et al. (2011), Gerber et al. (2012), and Cui and Nguyen (2016).

In this paper, we explore the optimal dividend strategy for an insurance company in a contagious insurance market. The surplus follows a self-exciting Hawkes process, where the occurrence of one claim can increase the likelihood of future claims. Additionally, the Gamma-Omega model is incorporated such that the insurance company is allowed to continue operating despite a temporary negative surplus. We rigorously derive the corresponding Hamilton-Jacobi-Bellman variational inequalities (HJBVIs) and provide a verification theorem. When claims are exponentially distributed and the bankruptcy rate remains a positive constant, the value function can be obtained using the Gerber-Shiu penalty function. Numerically, we solve the optimal dividend barrier using Monte Carlo simulations. Finally, numerical examples are presented to show the effect of different parameters on the optimal dividend strategy.

The main contributions of this paper are as follows. First, instead of considering the classical Cramér-Lundberg model, we incorporate the self-exciting Hawkes process to represent the insurance company's claim process, which can capture the contagious effect of insurance claims during pandemics. Including the Hawkes process increases the dimension of the value function and introduces challenges in solving the HJBVIs. When claims are exponentially distributed and the bankruptcy rate remains a positive constant, the semi-closed solution of the value function is derived. Additionally, we express the value function as a Gerber-Shiu penalty function, which can be efficiently computed by simulations after changing measures.

Second, including the Gamma-Omega model and the Hawkes process makes the optimal dividend strategy significantly different from those in the current literature. The optimal dividend strategy follows a non-constant barrier form, depending on the claim intensity. Hence, introducing a proper buffer for the negative surplus of the insurer helps the regulator deal with negative impacts during pandemics. Ignoring the self-exciting effect of insurance claims could significantly impact the dividend strategy and the investment choice of different stakeholders.

The rest of this paper is organised as follows. In Section 2, we formally introduce the problem, including a brief introduction of the self-exciting Hawkes process, the Gamma-Omega model, and the objective function. A comprehensive formulation of the HJBVIs and the verification theorem are derived. Section 3 presents the main result, where the derivation of solutions is included. In Section 4, numerical examples are presented to illustrate the impact of different parameters on the optimal dividend strategy. Section 5 concludes this paper. Appendices include all the technical proofs of this paper.

2 Problem formulation

In this section, we first construct the main problem and then derive the corresponding HJBVIs. Finally, a verification theorem is presented.

2.1 The objective function

In this subsection, we consider an insurance company with an unrestricted dividend policy (singular control framework). Furthermore, the insurance market is assumed to have a self-exciting feature of whether the arrival of a claim can trigger future claims. Additionally, following the work of Albrecher et al. (2011), we assume that the insurance company refrains from paying dividends when the surplus is non-positive. Formally, we work within a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying usual conditions and a filtration $\{\mathcal{F}_t\}$ representing information up to time t.

Definition 2.1. A nonnegative $\mathcal{F}_t$-adapted process $L(t) : [0, +\infty) \to [0, +\infty)$ is said to be an admissible stochastic control if $L(t)$ is càdlàg and non-decreasing. Furthermore, $dL(t) = 0$ when the corresponding surplus is non-positive. Therefore, $L(t)$ measures the cumulative dividend payment the company has paid until time t. Let $\mathcal{A}$ denote all the admissible controls.

Given an admissible control $L(t)$, we define $\Lambda = \{t \geq 0 : L(t) \neq L(t^-)\}$, as the set of times when $L(t)$ has jumps. The jump size at time t is defined by $\Delta L(t) = L(t) - L(t^-)$, and the continuous part of $L(t)$ is defined by $L^c(t) = L(t) - \sum_{0\leq s\leq t} \Delta L(s)$. Let the adapted process $X(t)$ represent the surplus process of the insurance company. Given any admissible stochastic control $L(t) \in \mathcal{A}$, the surplus process satisfies

$$X(t) = x + pt - \sum_{j=1}^{N(t)} Y_j - L(t), \tag{2.1}$$

where $p > 0$ denotes the insurance premium rate and Y_j denotes claim sizes which are *i.i.d.* random variables with distribution function $F_Y(\cdot)$. $F_Y(\cdot)$ is assumed to be continuous and concentrated on $(0, \infty)$. The problem will be constructed for general $F_Y(\cdot)$ and semi-explicit solutions of the problem will be derived for exponential claims. To capture the contagious insurance market, we assume that $N(t)$ follows a self-exciting Hawkes process, where the claim intensity is no longer a constant but satisfies the following stochastic point process.

$$d\lambda(t) =\alpha(\bar{\lambda} - \lambda(t))dt + \eta dN(t), \tag{2.2}$$

where $\alpha > 0$ denotes the mean-reverting rate of the claim intensity, $\bar{\lambda} > 0$ denotes the baseline of the claim intensity, and $\eta > 0$ denotes the instantaneous increment when a claim arrives. To have a stationary process, we need to further assume that $\eta < \alpha$. For the net profit condition, we need

$$\lim_{t\to\infty} \frac{pt - \sum_{j=1}^{N(t)} Y_j}{t} > 0. \tag{2.3}$$

According to the work of Dassios and Zhao (2012), which shows that $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{\alpha\bar{\lambda}}{\alpha-\eta}$, we can easily prove that (2.3) implies

$$p = \frac{(1+\theta)\alpha\bar{\lambda}}{\alpha - \eta}\mathbb{E}[Y_j], \tag{2.4}$$

where $\theta > 0$ denotes some non-negative risk-loading factor.

Let τ denote the stopping time. We follow the work of Albrecher and Lautscham (2013) to assume that the insurer may be allowed to continue the business even with a temporary negative surplus. Given a continuously differentiable function $\omega(X) \geq 0$ where X denotes the surplus level on $(-\infty, 0)$, the probability of bankruptcy on the time interval $[s, s + dt]$ with a negative surplus $X(s) < 0$ and no prior bankruptcy event is $\omega(X(s))dt$. The problem will be constructed for a general $\omega(X)$ and semi-explicit solutions of the problem will be derived for a positive constant bankruptcy rate.

Given any admissible control $L(t)$, we define the cost function as the expected discounted payoff until bankruptcy as follows

$$J(x, \lambda; L(t)) = \mathbb{E}\left[\int_0^\tau e^{-rt} dL(t) \Big| X(0) = x, \lambda(0) = \lambda\right], \tag{2.5}$$

where $r > 0$ is the discount factor, x is the initial surplus level, and λ is the initial claim intensity. Furthermore, motivated by Dassios and Zhao (2012), we need to assume that

$$\ln\left(\frac{r + \bar{\lambda}}{\bar{\lambda}}\right) > \frac{r}{\bar{\lambda}}\frac{\eta}{\alpha}, \tag{2.6}$$

which can ensure that the integral in (2.5) is finite. See also for the Remark 2.4 in Azcue and Muler (2010).

Problem 2.2. The management wants to obtain the optimal dividend policy $L^*(t)$ and the value function $V(x, \lambda)$ defined by

$$V(x,\lambda) = J(x,\lambda; L^*(t)) = \sup_{L(t)\in\mathcal{A}} J(x,\lambda; L(t)). \tag{2.7}$$

Note that $V(-\infty, \cdot) = 0$ and $V(\cdot, +\infty) = 0$, which implies that an insurance company with a surplus $x = -\infty$ or a claim intensity $\lambda = +\infty$ is treated as certain bankruptcy. Then, we state two basic properties for value function $V(x, \lambda)$ and the proofs are presented in the Appendix.

Lemma 2.3. *Value function $V(x, \lambda)$ is increasing in x, suggesting that a larger initial surplus implies a larger expected discounted payoff until bankruptcy.*

Lemma 2.4. *Value function $V(x, \lambda)$ is linearly bounded when $x \geq 0$, $x \leq V(x, \lambda) \leq x + \frac{p}{r}$ and $\lim_{x\to+\infty} V(x, \lambda) = x + \frac{p}{r}$ for any finite λ, suggesting that $V(x, \lambda)$ is bounded blow by distributing the initial surplus and bounded above by distributing the initial surplus and all premium incomes.*

Now, the main theorem of this section is as follows.

Theorem 2.5. *Assuming the optimal control strategies exist, the value function $V(x, \lambda)$ is a solution of the following HJB variational inequalities (HJBVIs).*

$$\begin{aligned} 0 = \sup\Big\{&p\partial_x V + \alpha(\bar{\lambda} - \lambda)\partial_\lambda V - (\lambda + r)V \\ &+\lambda \int_0^\infty V(x - y, \lambda + \eta)dF_Y(y), 1 - \partial_x V\Big\}, \quad \text{for } x \geq 0; \\ 0 = p\partial_x V &+ \alpha(\bar{\lambda} - \lambda)\partial_\lambda V - (\lambda + \omega(x) + r)V \\ &+ \lambda \int_0^\infty V(x - y, \lambda + \eta)dF_Y(y), \quad \text{for } x < 0; \end{aligned} \tag{2.8}$$

where $\partial_x V = \frac{\partial V(x,\lambda)}{\partial x}$ and $\partial_\lambda V = \frac{\partial V(x,\lambda)}{\partial \lambda}$. Furthermore, $V(x, \lambda)$ is continuous for any point (x, λ). Especially, when $x = 0$, we have

$$V(0^-, \lambda) = V(0^+, \lambda) = V(0, \lambda). \tag{2.9}$$

□

The proof of Theorem 2.5 is provided in Appendix C.[1]

We can divide the set of the surplus and the claim intensity into three regions according to HJBVIs (2.8).

(i) Continuation regions

$$\begin{aligned} \mathcal{C}_1 :=\{&(x,\lambda) \in [0, +\infty) \times [0, +\infty) : p\partial_x V + \alpha(\bar{\lambda} - \lambda)\partial_\lambda V - (\lambda + r)V \\ &+ \lambda \int_0^\infty V(x - y, \lambda + \eta)dF_Y(y) = 0 \text{ and } 1 - \partial_x V < 0\}. \\ \mathcal{C}_2 :=\{&(x,\lambda) \in (-\infty, 0) \times [0, +\infty) : p\partial_x V + \alpha(\bar{\lambda} - \lambda)\partial_\lambda V - (\lambda + \omega(x) + r)V \\ &+ \lambda \int_0^\infty V(x - y, \lambda + \eta)dF_Y(y) = 0\}. \end{aligned} \tag{2.10}$$

[1]Note that according to the HJBVIs (2.8), the value function is not required to be a C^2 function.

(ii) Intervention region

$$\begin{aligned}\mathcal{D}_1 :=\{(x,\lambda)\in[0,+\infty)\times[0,+\infty): &\, p\partial_x V+\alpha(\bar{\lambda}-\lambda)\partial_\lambda V-(\lambda+r)V \\ &+\lambda\int_0^\infty V(x-y,\lambda+\eta)dF_Y(y)<0 \text{ and } 1-\partial_x V=0\}.\end{aligned} \tag{2.11}$$

Now, we conjecture that value function $V(x,\lambda)$ is concave for $x\geq 0$. Let $b^*(\lambda)=\inf\{x\geq 0 : \partial_x V(x,\lambda)=1\}$, then we know that $\mathcal{C}_1=[0,b^*(\lambda)]\times[0,+\infty)$, $\mathcal{C}_2=(-\infty,0)\times[0,+\infty)$, and $\mathcal{D}_1=(b^*(\lambda),+\infty)\times[0,+\infty)$. Given the three regions, we can construct an admissible stochastic control $L^V(t)$ associated with the function $V(x,\lambda)$ in HJBVIs (2.8) as follows.

Definition 2.6. A nonnegative, $\{\mathcal{F}_t\}$-adapted, and nondecreasing control process $L^V(t)$ is associated with function $V(x,\lambda)$ if the following conditions hold.

(i) the surplus process $X^{L^V}(t)$ associated with the control $L^V(t)$ is defined as

$$X^{L^V}(t)=x+pt-\sum_{j=1}^{N(t)}Y_j-L^V(t),\quad \text{for all } t\in[0,\tau^{L^V}), \tag{2.12}$$

(ii) For a given $V(x,\lambda)$, we can determine Continuation regions and Intervention region according to (2.10) and (2.11). Then, the surplus process defined in (2.12) and the claim intensity should satisfy

$$(X^{L^V}(t),\lambda(t))\in\overline{\mathcal{C}_1} \text{ or } (X^{L^V}(t),\lambda(t))\in\overline{\mathcal{C}_2} \text{ for all } t\in[0,\tau^{L^V}),\mathbb{P}\text{-}a.s., \tag{2.13}$$

$$\int_0^{\tau^{L^V}}\mathbb{I}_{\{(X^{L^V}(t),\lambda(t))\in\mathcal{C}_1 \text{ or } (X^{L^V}(t),\lambda(t))\in\mathcal{C}_2\}}dL^V(t)=0,\mathbb{P}\text{-}a.s., \tag{2.14}$$

where τ^{L^V} denotes the bankruptcy time associated with the surplus process $X^{L^V}(t)$ generated by $L^V(t)$. □

2.2 The verification theorem

Theorem 2.7. *Let $v_1(x,\lambda)\in C^2((-\infty,0)\times[0,+\infty))$ be an increasing function on $x\in(-\infty,0)$, and $v_2(x,\lambda)\in C^2([0,+\infty)\times[0,+\infty))$ be an increasing and concave function on $x\in[0,+\infty)$. Let $\partial_x v_1(x,\lambda)$ be bounded on $x\in(-\infty,0)$ for any λ. Let $\partial_x v_2(x,\lambda)$ be bounded on $x\in[0,+\infty)$ for any λ. Define function $v(x,\lambda)$ by*

$$v(x,\lambda)=\begin{cases} v_1(x,\lambda), & x<0,\\ v_2(x,\lambda), & x\geq 0.\end{cases} \tag{2.15}$$

Assume that function $v(x,\lambda)$ satisfies the HJBVIs (2.8) and the continuity condition (2.9), we can define an admissible stochastic control $L^v(t)$ associated with the function $v(x,\lambda)$ according to Definition 2.6. Then, the new process $L^*(t)$ defined by $L^*(t)=L^v(t)$ for $t\in[0,\tau^{L^v})$ and $dL^*(t)=0$ for $t\in[\tau^{L^v},+\infty)$ is the optimal dividend policy, and function $v(x,\lambda)$ is the value function $V(x,\lambda)$ for Problem 2.2. □

The proof of Theorem 2.7 is provided in Appendix D.

3 Analytic solutions

Our objective in this section is to construct the solution for the value function and the optimal dividend policy. Therefore, we want to find two functions $v_1(x,\lambda)$ and $v_2(x,\lambda)$ satisfying conditions in Theorem 2.2. When the claim size follows exponential distributions and the bankruptcy rate is a positive constant, the semi-closed solution of the value function can be derived. To construct the solution, we assume that $v_2(x,\lambda)$ is concave for $x \geq 0$ to have $\partial_x v_2(x,\lambda)$ non-increasing for $x \geq 0$. Then, we need to consider the following two cases.

3.1 Case one: $\partial_x v_2(0^+,\lambda) > 1$

We consider $\partial_x v_2(0^+,\lambda) > 1$, where the claim intensity λ is large such that claim risks are not negligible with zero surplus. According to our conjecture that $\partial_x v_2(x,\lambda)$ is non-increasing for $x > 0$, we know that the unknown function $b^*(\lambda) > 0$ satisfies

$$\begin{cases} \partial_x v_2(x,\lambda) = 1, & \text{if } x > b^*(\lambda), \\ \partial_x v_2(x,\lambda) \geq 1, & \text{if } 0 \leq x \leq b^*(\lambda). \end{cases} \tag{3.1}$$

Then, we can further rewrite the candidate function $v(x,\lambda)$ as follows

$$v(x,\lambda) = \begin{cases} v_1(x,\lambda), \ \text{for } x < 0; \\ v_2(x,\lambda) = \begin{cases} v_{21}(x,\lambda), \ \text{for } 0 \leq x \leq b^*(\lambda); \\ v_{22}(x,\lambda), \ \text{for } x > b^*(\lambda). \end{cases} \end{cases} \tag{3.2}$$

Therefore, HJBVIs in (2.8) can be rewritten as

$$\begin{aligned} 0 &= p\partial_x v_1 + \alpha(\bar{\lambda} - \lambda)\partial_\lambda v_1 - (\lambda + \omega(x) + r)v_1 + \lambda \int_0^\infty v_1(x-y, \lambda+\eta)dF_Y(y), \quad \text{for } x < 0; \\ 0 &= p\partial_x v_{21} + \alpha(\bar{\lambda} - \lambda)\partial_\lambda v_{21} - (\lambda + r)v_{21} \\ &\quad + \lambda \left(\int_0^x v_{21}(x-y, \lambda+\eta)dF_Y(y) + \int_x^\infty v_1(x-y, \lambda+\eta)dF_Y(y) \right), \quad \text{for } 0 \leq x \leq b^*(\lambda); \\ 1 &= \partial_x v_{22}, \quad \text{for } x > b^*(\lambda). \end{aligned} \tag{3.3}$$

For the remainder of the paper, we assume that claim sizes are *i.i.d.* exponential random variables with

$$Y_j \sim Exp(\pi), \quad \pi > 0, \forall j. \tag{3.4}$$

Lemma 3.1. *Consider an operator $(d/dx+\pi)$ on an arbitrary function $\Phi(x,\lambda)$ where $(d/dx+\pi)\Phi = \frac{d\Phi(x,\lambda)}{dx} + \pi\Phi(x,\lambda)$, then we can apply the operator $(d/dx+\pi)$ to the differential equations (3.3) to eliminate integrations to get*

$$\begin{aligned} 0 = p\partial_{xx}v_1 &+ \alpha(\bar{\lambda} - \lambda)\partial_{\lambda x}v_1 + (p\pi - \lambda - \omega(x) - r)\partial_x v_1 + \alpha(\bar{\lambda} - \lambda)\pi\partial_\lambda v_1 \\ &- [\omega'(x) + (\omega(x) + r)\pi]v_1 + \lambda\pi\left(v_1(x, \lambda+\eta) - v_1(x,\lambda)\right), \ \textit{for } x < 0, \end{aligned} \tag{3.5}$$

where $\partial_{xx}v_i = \frac{\partial^2 v_i(x,\lambda)}{\partial x}$ and $\partial_{\lambda x}v_i = \frac{\partial^2 v_i(x,\lambda)}{\partial x \partial \lambda}$, $\forall i = 1, 21$. and $\omega'(x) = \frac{d\omega(x)}{dx}$. □

The proof of Lemma 3.1 is provided in Appendix E.

Note that, the delayed differential equations (3.5) can be solved explicitly by the following lemma when the bankruptcy rate is a constant. For the remainder of the paper, we further assume that $\omega(x) = \omega > 0$ for any $x < 0$.

Lemma 3.2. *Given the net profit condition (2.4) and assume that a function $h(x,\lambda)$ satisfies the following differential equation*

$$\begin{aligned}0 = p\partial_{xx}h + \alpha(\bar{\lambda}-\lambda)\partial_{\lambda x}h + (p\pi - \lambda - \omega - r)\partial_x h + \alpha(\bar{\lambda}-\lambda)\pi\partial_\lambda h \\ - (\omega + r)\pi h + \lambda\pi\left(h(x,\lambda+\eta) - h(x,\lambda)\right),\end{aligned} \tag{3.6}$$

where $\partial_{xx}h = \frac{\partial^2 h(x,\lambda)}{\partial x}$, $\partial_{\lambda x}h = \frac{\partial^2 h(x,\lambda)}{\partial x \partial\lambda}$, $\partial_x h = \frac{\partial h(x,\lambda)}{\partial x}$, and $\partial_\lambda h = \frac{\partial h(x,\lambda)}{\partial \lambda}$.

Then, it follows that

$$h(x,\lambda) = Ae^{a_1 x + c_1\lambda} + Be^{a_2 x + c_2 \lambda}, \tag{3.7}$$

where A and B are constants to be determined, $c_1 > 0$ and $c_2 < 0$ are the two unique roots of the following equation

$$(\alpha\bar{\lambda}c_i - \omega - r - p\pi)(\alpha c_i + 1) + p\pi e^{c_i\eta} = 0, \qquad i = 1, 2, \tag{3.8}$$

and

$$a_i = \frac{\omega + r - \alpha\bar{\lambda}c_i}{p}, \qquad i = 1, 2. \tag{3.9}$$

The proof of Lemma 3.2 is provided in Appendix F.

Therefore, according to Lemma 3.1 and Lemma 3.2, we know that $v_1(x,\lambda)$ follows

$$v_1(x,\lambda) = Ae^{a_1 x + c_1\lambda} + Be^{a_2 x + c_2\lambda}. \tag{3.10}$$

Furthermore, the fact that $V(\cdot,+\infty) = 0$ and $V(x,\lambda) > 0$ imply that $A = 0$ and $B > 0$. Now, we consider the differential equation (3.3) for $0 \le x \le b^*(\lambda)$. Motivated by Lemma 3.2, we consider the following ansatzes

$$v_{21}(x,\lambda) = e^{ax + c\lambda}g(x,\lambda), \tag{3.11}$$

where a and b are the roots of the following system of equation and $g(x,\lambda)$ is a function to be determined.

$$\begin{aligned}&(\alpha\bar{\lambda}c - r - p\pi)(\alpha c + 1) + p\pi e^{c\eta} = 0, \\ &a = \frac{r - \alpha\bar{\lambda}c}{p}.\end{aligned} \tag{3.12}$$

We can also easily prove that there is a unique pair of $a \in (-\pi, 0)$ and $c > 0$ for (3.12). Substituting (3.10) and (3.11) into (3.3) for $0 \le x \le b^*(\lambda)$ and dividing $e^{ax+c\lambda}$ on both sides, we have

$$\begin{aligned}0 =& p(ag(x,\lambda) + \partial_x g(x,\lambda)) + \alpha(\bar{\lambda}-\lambda)(cg(x,\lambda) + \partial_\lambda g(x,\lambda)) - rg(x,\lambda) \\ &+ \lambda\left[\int_0^x e^{-ay+c\eta}g(x-y,\lambda+\eta)dF_Y(y) + \int_x^{+\infty} Be^{(a_2-a)x - ay + (c_2-c)\lambda + c_2\eta}dF_Y(y) - g(x,\lambda)\right].\end{aligned} \tag{3.13}$$

Note that a and c satisfy (3.12), (3.13) can be rewritten as

$$\begin{aligned}0 =& p\partial_x g(x,\lambda) + \alpha(\bar{\lambda}-\lambda)\partial_\lambda g(x,\lambda) \\ &+ \lambda e^{c\eta}\left[\int_0^x e^{-ay}g(x-y,\lambda+\eta)dF_Y(y) + \int_x^{+\infty} Be^{(a_2-a)x - ay + (c_2-c)(\lambda+\eta)}dF_Y(y) - \frac{\pi}{a+\pi}g(x,\lambda)\right].\end{aligned} \tag{3.14}$$

Now, consider a new measure $\widehat{\mathbb{P}}$ where

$$d\widehat{F}_Y(y) = \frac{(a+\pi)e^{-ay}dF_Y(y)}{\pi}. \tag{3.15}$$

We can rewrite (3.14) under new measure $\widehat{\mathbb{P}}$ as

$$\begin{aligned}0 =&p\partial_x g(x,\lambda) + \alpha(\bar{\lambda}-\lambda)\partial_\lambda g(x,\lambda)\\ &+ (1+c\alpha)\lambda\left[\int_0^x g(x-y,\lambda+\eta)d\widehat{F}_Y(y) + \int_x^{+\infty} Be^{(a_2-a)x+(c_2-c)(\lambda+\eta)}d\widehat{F}_Y(y) - g(x,\lambda)\right].\end{aligned} \tag{3.16}$$

Consider a new process for the claim intensity $\widehat{\lambda}_t = (1+c\alpha)\lambda_t$ and define $\tilde{g}(x,\widehat{\lambda}) = g\left(x, \frac{\widehat{\lambda}}{1+c\alpha}\right) = g(x,\lambda)$, we can further rewrite (3.16) as follows

$$\begin{aligned}0 =&p\partial_x\tilde{g}(x,\widehat{\lambda}) + \alpha(\tilde{\lambda}-\widehat{\lambda})\partial_\lambda\tilde{g}(x,\widehat{\lambda})\\ &+ \widehat{\lambda}\left[\int_0^x \tilde{g}(x-z,\widehat{\lambda}+\widehat{\eta})d\widehat{F}_Y(y) + \int_x^{+\infty} Be^{(a_2-a)x+(c_2-c)\frac{\widehat{\lambda}+\widehat{\eta}}{1+c\alpha}}d\widehat{F}_Y(y) - \tilde{g}(x,\widehat{\lambda})\right],\end{aligned} \tag{3.17}$$

where $\tilde{\lambda} = (1+c\alpha)\bar{\lambda}$ and $\widehat{\eta} = (1+c\alpha)\eta$. Further, we need to assume the stationary condition for the new process such that $\widehat{\eta} < \alpha$.

Now, if we treat $Be^{(a_2-a)x+(c_2-c)\frac{\widehat{\lambda}+\widehat{\eta}}{1+b\alpha}} > 0$ as a penalty amount and define $\widehat{\tau}(x,\widehat{\lambda}) = \inf\{t|X(t) \le 0\}$ with initial surplus $X(0) = x$ and claim intensity $\widehat{\lambda}(0) = \widehat{\lambda}$, the stopping time under the classical ruin theory, based on the work of Dassios and Zhao (2012). Then, according to the Gerber-Shiu penalty function, we know that the solution of (3.17) satisfies

$$\tilde{g}(x,\widehat{\lambda}) = \mathbb{E}\left[Be^{(a_2-a)X_{\widehat{\tau}-}+(c_2-c)\frac{\widehat{\lambda}_{\widehat{\tau}}}{1+c\alpha}}\mathbb{I}_{(\widehat{\tau}(x,\widehat{\lambda})<+\infty)}\middle| X(0)=x, \widehat{\lambda}(0)=\widehat{\lambda}\right], \tag{3.18}$$

where $\mathbb{E}[\cdot|X(0)=x,\widehat{\lambda}(0)=\widehat{\lambda}]$ denotes the conditional expectation under measure $\tilde{\mathbb{P}}$ given $(x,\widehat{\lambda})$, $X_{\widehat{\tau}-}$ denotes the surplus level right before the ruin time, and $\widehat{\lambda}_{\widehat{\tau}}$ denotes the value of the new claim intensity at ruin.

Therefore, we know that $v_{21}(x,\lambda)$ for $0 \le x \le b^*(\lambda)$ follows

$$v_{21}(x,\lambda) = e^{ax+c\lambda}\mathbb{E}\left[Be^{(a_2-a)X_{\widehat{\tau}-}+(c_2-c)\frac{\widehat{\lambda}_{\widehat{\tau}}}{1+c\alpha}}\mathbb{I}_{(\widehat{\tau}(x,\widehat{\lambda})<+\infty)}\middle| X(0)=x, \widehat{\lambda}(0)=\widehat{\lambda}\right]. \tag{3.19}$$

Remark 3.3. Given exponential claims with parameter $\pi > 0$ and the stationary assumption, the net-premium principle (2.4) holds under measure $\mathbb{P}$, which implies

$$p = \frac{(1+\theta)\alpha\bar{\lambda}}{(\alpha-\eta)\pi}. \tag{3.20}$$

Under new measure $\widehat{\mathbb{P}}$, $\widehat{F}_Y(y) \sim Exp(\pi+a)$. Since $a \in (-\pi, 0)$ and $c > 0$, the net-premium principle (2.4) under new measure $\widehat{\mathbb{P}}$ suggests that

$$\frac{(1+\theta)\alpha\tilde{\lambda}}{(\alpha-\widehat{\eta})(\pi+a)} > \frac{(1+\theta)\alpha\bar{\lambda}}{(\alpha-\eta)\pi}\frac{\pi}{a+\pi}\frac{\alpha-\eta}{\alpha-(1+c\alpha)\eta}(1+c\alpha) > p, \tag{3.21}$$

which implies that ruin becomes certain under new measure $\widehat{\mathbb{P}}$. Hence, the simulation method works well in computing (3.19). □

Finally, when $x > b^*(\lambda)$, from (3.3), we define

$$v_{22}(x,\lambda) = x + K(\lambda), \tag{3.22}$$

where $K(\lambda)$ is a function to be determined.

According to the continuity of the value function at $x = b^*(\lambda)$, we need $v_{21}(b^*(\lambda),\lambda) = v_{22}(b^*(\lambda),\lambda)$, which implies that $K(\lambda) = v_{21}(b^*(\lambda),\lambda) - b^*(\lambda)$.

Therefore, we have constructed the candidate function $v(x,\lambda)$. In summary, when $\partial_x v_2(0^+,\lambda) > 1$, we have

$$v(x,\lambda) = \begin{cases} Be^{a_2 x + c_2\lambda}, \quad x < 0, \\ e^{ax+c\lambda}\mathbb{E}\left[Be^{(a_2-a)X_{\widehat{\tau}-}+(c_2-c)\frac{\widehat{\lambda}_{\widehat{\tau}}}{1+c\alpha}}\mathbb{I}_{(\widehat{\tau}(x,\widehat{\lambda})<+\infty)}\middle| X(0)=x,\widehat{\lambda}(0)=\widehat{\lambda}\right], 0 \le x \le b^*(\lambda), \\ x + v(b^*(\lambda),\lambda) - b^*(\lambda), \quad x > b^*(\lambda), \end{cases} \tag{3.23}$$

where constant B and barrier $b^*(\lambda)$ are to be determined.

Note that when $X(0) = 0$, $\widehat{\tau}$ is defined to be zero based on Dassios and Zhao (2012). Hence, there is no claim thereafter and $\widehat{\lambda}_{\widehat{\tau}} = \widehat{\lambda} = (1+c\alpha)\lambda$. Then, it is very easy to check that $v_1(0^-,\lambda) = v_{21}(0,\lambda)$. Hence, $v(x,\lambda)$ is continuous at $x = 0$. Furthermore, for $x < 0$ and $x > b^*(\lambda)$, it is very trivial to check that $v(x,\lambda)$ is increasing in x. When $0 \le x \le b^*(\lambda)$, we need to rely on the numerical solution presented in Section 4 to observe that $v(x,\lambda)$ is increasing and concave in $x \ge 0$. Hence, conditions in Theorem 2.2 are satisfied.

3.2 Case two: $\partial_x v_2(0^+,\lambda) \le 1$

We consider $\partial_x v_2(0^+,\lambda) \le 1$, where the claim intensity λ is small such that claim risks are negligible with zero surplus. According to our conjecture that $\partial_x v_2(x,\lambda)$ is non-increasing for $x > 0$, we obtain that

$$\partial_x v_2(x,\lambda) = 1, \quad \text{for } x \ge 0, \tag{3.24}$$

which also implies that the optimal barrier is $b^*(\lambda) = 0$.

Therefore, HJBVIs in (2.8) can be rewritten as

$$\begin{aligned} &0 = p\partial_x v_1 + \alpha(\bar{\lambda}-\lambda)\partial_\lambda v_1 - (\lambda+\omega(x)+r)v_1 + \lambda\int_0^\infty v_1(x-y,\lambda+\eta)dF_Y(y), \quad \text{for } x < 0; \\ &1 = \partial_x v_{22}, \quad \text{for } x \ge 0. \end{aligned} \tag{3.25}$$

Applying the same method as in Section 3.1, we can construct the candidate function $v(x,\lambda)$, when $\partial_x v_2(0^+,\lambda) \le 1$, as follows

$$v(x,\lambda) = \begin{cases} Be^{a_2 x + c_2\lambda}, \quad x < 0, \\ x + v(0,\lambda), \quad x \ge 0, \end{cases} \tag{3.26}$$

where we can easily check that $v(x,\lambda)$ is continuous at $x = 0$, increasing in x, and concave in $x \ge 0$.

3.3 The optimal dividend barrier

In this subsection, we will consider the optimal barrier strategy in more detail. First, we consider the method to determine the constant $B > 0$. Note that we know $\lim_{x\to+\infty} V(x,\lambda) = x + \frac{p}{r}$ for any finite λ, which implies that when claim risks are negligible, the value function approaches $x + \frac{p}{r}$ as the initial surplus goes to infinity. Hence, according to (3.26), we have $\lim_{x\to+\infty} v(x,0) = x + v_1(0,0) = x + \frac{p}{r}$. Hence, $v_1(0,0) = \frac{p}{r}$, which implies $B = \frac{p}{r}$.

Second, recall that the key condition for the two cases in the above subsections is the value of $\partial_x v_2(0^+, \lambda)$. Therefore, the optimal dividend barrier can be summarised as follows

$$b^*(\lambda) = \begin{cases} 0, & \partial_x v_2(0^+, \lambda) \leq 1, \\ \arg_x \partial_x v_{21}(x, \lambda) = 1, & \partial_x v_2(0^+, \lambda) > 1, \end{cases} \tag{3.27}$$

where $v_{21}(x, \lambda)$ is given by (3.19). Based on (3.27), we can see that the insurer hesitates to distribute dividends to reduce the bankruptcy probability when claim risks dominate and distributes all surplus as dividends to increase the expected discounted dividend when claim risks are small.

Now, we summarise analytic solutions of the value function and the optimal dividend strategy for Problem 2.2 by the following Corollary.

Corollary 3.4. *Suppose that the bankruptcy rate $\omega(x)$ is a positive constant ω, claim sizes follow an exponential distribution with parameter $\pi > 0$, $b^*(\lambda)$ is given by (3.27), and a, b, a_2, and c_2 are defined by (3.8), (3.9), and (3.12). Then, function $v(x, \lambda)$ is given by:*
when $\partial_x v(0^+, \lambda) > 1$,

$$v(x, \lambda) = \begin{cases} \frac{p}{r} e^{a_2 x + c_2 \lambda}, & x < 0, \\ \frac{p}{r} e^{ax + c\lambda} \mathbb{E}\left[e^{(a_2 - a) X_{\widehat{\tau}-} + (c_2 - c)\frac{\widehat{\lambda}_{\widehat{\tau}}}{1 + c\alpha}} \mathbb{I}_{(\widehat{\tau}(x, \widehat{\lambda}) < +\infty)} \middle| X(0) = x, \widehat{\lambda}(0) = \widehat{\lambda} \right], & 0 \leq x \leq b^*(\lambda), \\ x + v_{21}(b^*(\lambda), \lambda) - b^*(\lambda), & x > b^*(\lambda), \end{cases} \tag{3.28}$$

when $\partial_x v(0^+, \lambda) \leq 1$,

$$v(x, \lambda) = \begin{cases} \frac{p}{r} e^{a_2 x + c_2 \lambda}, & x < 0, \\ x + \frac{p}{r} e^{c_2 \lambda}, & x > 0, \end{cases} \tag{3.29}$$

is the value function $V(x, \lambda)$ and $b^*(\lambda)$ is the optimal dividend barrier in Problem 2.2. □

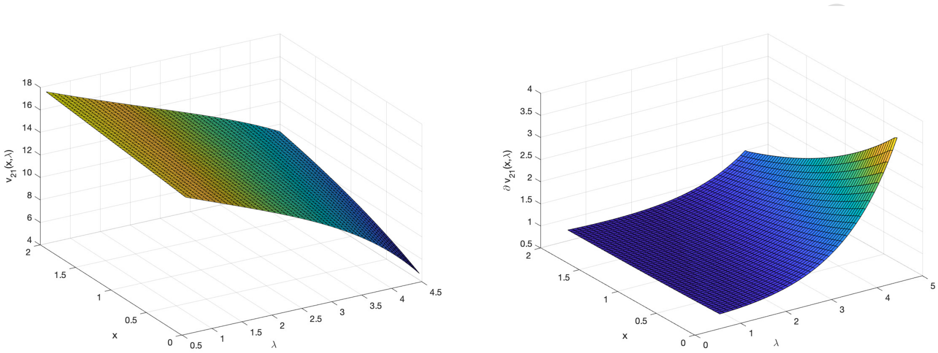
4 Numerical examples

In this section, we demonstrate numerical methods to obtain the value function and the optimal dividend barrier. We then present numerical results to gain insights into the value function and the optimal barrier curve under different parameters. Following Chen and Bian (2021), we consider an insurance company where claim occurrences follow a self-exciting Hawkes process with parameters $\alpha = 1$, $\bar{\lambda} = 2$, and $\eta = 0.5$. The risk-loading factor is $\theta = 0.1$, the discount factor is $r = 0.2$, and the bankruptcy rate is $\omega = 2$. Additionally, we follow the assumption in (3.4) and set $\pi = 0.5$.

First, based on Theorem 3.4, the key step to obtain the value function is to solve the following equation numerically.

$$v_{21}(x, \lambda) = \frac{p}{r} e^{ax + c\lambda} \mathbb{E}\left[e^{(a_2 - a) X_{\widehat{\tau}-} + (c_2 - c)\frac{\widehat{\lambda}_{\widehat{\tau}}}{1 + c\alpha}} \mathbb{I}_{(\widehat{\tau}(x, \widehat{\lambda}) < +\infty)} \middle| X(0) = x, \widehat{\lambda}(0) = \widehat{\lambda} \right], \quad 0 \leq x \leq b^*(\lambda). \tag{4.1}$$

In this section, we apply the Monte Carlo simulation to compute the conditional expectation in (4.1), where we generate 10,000 paths and exclude outliers based on the Thompson Tau method. Figure 4.1.1 demonstrates the behaviour of $v_{21}(x, \lambda)$ for suitable x and λ, from which we can verify that $v_{21}(x, \lambda)$ is increasing for $x \geq 0$. Furthermore, we can also easily obtain the partial derivative of $v_{21}(x, \lambda)$ *w.r.t.* x, which is shown by 4.1.2. Then, we can easily verify that $v_{21}(x, \lambda)$ is concave for $x \geq 0$.

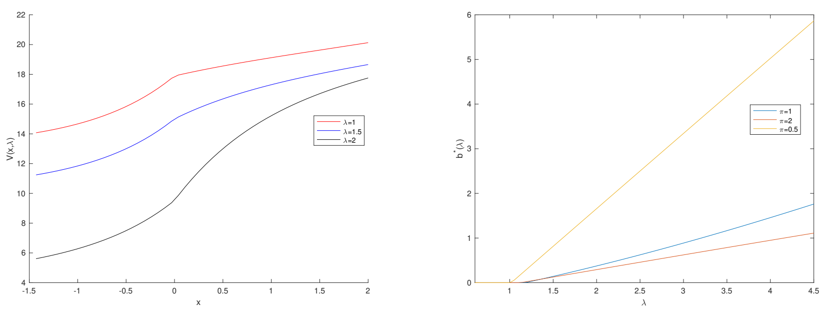


(4.1.1) $v_{21}(x, \lambda)$ (4.1.2) $\partial_x v_{21}(x, \lambda)$

Figure 4.1: $v_{21}(x, \lambda)$

Figure 4.2.1 illustrates the impact of different initial claim intensities on the value function. Notably, $V(x, \lambda)$ is decreasing in λ. This is because a higher claim intensity means a shorter expected waiting time until the next claim, leading the insurer to hesitate in distributing dividends to maintain a higher surplus for claim risks. Additionally, $V(x, \lambda)$ is not smooth at $x = 0$ due to a constant bankruptcy rate. Interestingly, there are no restrictions on the concavity of the value function for $x < 0$ and we can see that the value function is convex for $x < 0$ in Figure 4.2.1, consistent with Albrecher et al. (2011) under the Gamma-Omega model.

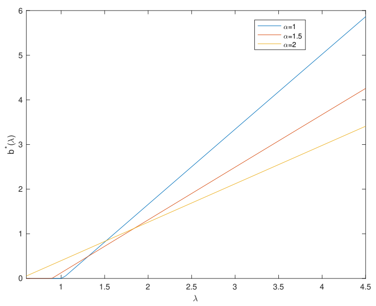
Figure 4.2.2 illustrates the behaviour of the optimal barrier curve, which aligns with our analytical results in Section 3.3. First, the three curves in Figure 4.2.2 are not linear. Second, when claim intensity λ is small, the optimal barrier is always 0. The insurer distributes everything as dividends as long as the surplus is non-negative. This is because when claim intensity λ is small, the claim risk is limited. Despite unexpected events, the insurer can operate with a temporary negative surplus. Third, as the claim intensity λ increases, the optimal barrier curve $b^*(\lambda)$ also rises. The insurer needs a larger surplus to manage higher claim risks. Finally, the optimal dividend barrier increases with larger average claim sizes.



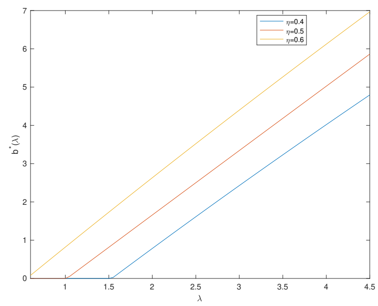
(4.2.1) The value function with different initial states (4.2.2) The optimal barrier curve with different initial states

Figure 4.2: The value function and the optimal barrier

Figure 4.3 illustrates the impact of different parameters on the optimal barrier curve. Figure 4.3.1 shows the effect of the mean-reverting rate α on the optimal dividend barrier. When λ is small, the claim intensity increases back to the long-term average more quickly as α increases. This implies higher potential claim risks, leading to an increased optimal dividend barrier. However, when λ is large, the claim intensity decreases back to the long-term average more quickly as α increases, implying smaller potential claim risks. In this case, the optimal dividend barrier decreases. As α goes to the positive infinity, the self-exciting Hawkes process theoretically converges to a standard Poisson process with a constant intensity, where the optimal dividend barrier is expected to be a positive constant. Figure 4.3.2 shows that the optimal barrier curve $b^*(\lambda)$ decreases as the instantaneous increment η decreases. This is intuitive because a higher η implies larger contagious effects and, consequently, larger claim risks. In our future works, we will relax the assumption of a constant η and consider suitable distributions of the instantaneous increment.



(4.3.1) The optimal barrier curve with different α-s



(4.3.2) The optimal barrier curve with different η-s

Figure 4.3: The optimal barrier curve with different parameters

5 Conclusions

This paper explores the optimal dividend strategy for an insurance company in a contagious insurance market. The insurance company's surplus follows a modified Cramér-Lundberg process where the claim intensity follows a self-exciting Hawkes process. Instead of the classical ruin concept, we consider the concept of bankruptcy, allowing the insurer to continue with a temporary negative surplus. The probability of business closure increases as the surplus becomes more negative. We rigorously derive the HJBVIs and show that the value function can be solved using the Gerber-Shiu penalty function when the bankruptcy rate is a positive constant and claim sizes follow i.i.d. exponential distributions. Notably, the optimal dividend barrier is no longer a constant but a curve depending on the claim intensity. For a low claim intensity, the optimal barrier could be zero, indicating full dividend distribution for a non-negative surplus. As the claim intensity increases, the optimal barrier rises, necessitating a larger surplus to manage growing claim risks. Additionally, the optimal dividend barrier depends on other parameters, including average claim sizes, the mean-reverting rate, and the instantaneous increment of the self-exciting Hawkes process.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendices

Appendices related to this article can be found in another document.

References

Albrecher, H., Gerber, H. U., & Shiu, E. S. (2011). The optimal dividend barrier in the Gamma–Omega model. *European Actuarial Journal*, 1(1), 43-55.

Albrecher, H., & Lautscham, V. (2013). From ruin to bankruptcy for compound Poisson surplus processes. *ASTIN Bulletin: The Journal of the IAA*, 43(2), 213-243.

Aït-Sahalia, Y., Cacho-Diaz, J., & Laeven, R. J. A. (2015). Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117(3), 585-606.

Aït-Sahalia, Y., & Hurd, T. R. (2015). Portfolio choice in markets with contagion. *Journal of Financial Econometrics*, 14(1), 1-28.

Avanzi, B., Lau, H., & Wong, B. (2021). Optimal periodic dividend strategies for spectrally negative Lévy processes with fixed transaction costs. *Scandinavian Actuarial Journal*, 2021(8), 645-670.

Azcue, P., & Muler, N. (2010). Optimal investment policy and dividend payment strategy in an insurance company. *The Annals of Applied Probability*, 20(4), 1253-1302.

Babuna. P., Yang, X., Gyilbag, A., Awudi, D. A., Ngmenbelle, D., & Bian, D. (2020). The impact of Covid-19 on the insurance industry. *International Journal of Environmental Research and Public Health*, 17(16), 5766.

Cao, J., Landriault, D., & Li, B. (2020). Optimal reinsurance-investment strategy for a dynamic contagion claim model. *Insurance: Mathematics and Economics*, 93, 206-215.

Chen, Y., & Bian, B. (2021). Optimal dividend policy in an insurance company with contagious arrivals of claims. *Mathematical Control & Related Fields*, 11(1), 1-22.

Cui, Z., & Nguyen, D. (2016). Omega diffusion risk model with surplus-dependent tax and capital injections. *Insurance: Mathematics and Economics*, 68, 150-161.

Dassios, A., & Zhao, H. (2012). Ruin by dynamic contagion claims. *Insurance: Mathematics and Economics*, 51(1), 93-106.

Eisenberg, J., & Palmowski, Z. (2021). Optimal dividends paid in a foreign currency for a Lévy insurance risk model. *North American Actuarial Journal*, 25(3), 417-437.

Gerber, H, U. (1969). Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. *Schweizerische Vereinigung der Versicherungsmathematike. Mitteilungen*, 69, 185-228.

Gerber, H, U., & Shiu, E. S. (2006). On optimal dividend strategies in the compound Poisson model. *North American Actuarial Journal*, 10(2), 76-93.

Gerber, H. U., Shiu, E. S., & Yang, H. (2012). The Omega model: from bankruptcy to occupation times in the red. *European Actuarial Journal*, 22, 259-272.

Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1), 83-90.

Højgaard, B. (2002). Optimal dynamic premium control in non-life insurance. Maximising dividend pay-outs. *Scandinavian Actuarial Journal*, 2002(4), 225-245.

Jin, Z., Quan Xu, Z., & Zou, B. (2022). A perturbation approach to optimal investment, liability ratio, and dividend strategies. *Scandinavian Actuarial Journal*, 2022(2), 165-188.

Khan, S. A. (2015). Determinants of the non life insurance performance: the Portuguese case. PhD. Dissertation. Instituto Superior de Economia e Gestão.

Lin. X. S., & Pavlova, K. P. (2006). The compound Poisson risk model with a threshold dividend strategy. *Insurance: Mathematics and Economics*, 38(1), 57-80.

Lin. X. S., & Sendova, K. P. (2008). The compound Poisson risk model with multiple thresholds. *Insurance: Mathematics and Economics*, 42(2), 617-627.

Liu, G., Jin, Z., & Li, S. (2021). Optimal investment, consumption, and life insurance strategies under a mutual-exciting contagious market. *Insurance: Mathematics and Economics*, 101, 508-524.

Liu, S., Liu, Z., & Liu, G. (2023). Optimal dividend strategy for the dual model with surplus-dependent expense. *Communications in Statistics-Theory and Methods*, 52(3), 543-566.

Miao, Y., Sendova, K. P., & Jones, B. L. (2022). On a risk model with dual seasonalities. *North American Actuarial Journal*, 1-19.

Qiu, M., Jin, Z., & Li, S. (2023). Optimal risk sharing and dividend strategies under default contagion: A semi-analytical approach. *Insurance: Mathematics and Economics*, 113, 1-23.

Yang, C., Sendova, K. P., & Li, Z. (2020). Parisian ruin with a threshold dividend strategy under the dual Lévy risk model. *Insurance: Mathematics and Economics*, 90, 135-150.

Yao, D., Yang, H., & Wang, R. (2011). Optimal dividend and capital injection problem in the dual model with proportional and fixed transaction costs. *European Journal of Operational Research*, 211(3), 568-576.

Zhou, X. (2005). On a classical risk model with a constant dividend barrier. *North American Actuarial Journal*, 9(4), 95-108.

Highlight

- The optimal dividend problem under a contagious insurance market has been considered.
- Contagious claims show a clustering effect modelled by a self-exciting Hawkes process.
- The bankruptcy concept allows the insurer to operate with a temporary negative surplus.
- A numerical method is established rigorously to solve the problem.
- The optimal dividend barrier is a curve depending on the claim intensity.

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Guo Liu: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data Curation, Writing – Original Draft, Writing -Review & Editing, Visualization, Project administration.

Zhuo Jin: Supervision

Shuanming Li: Supervision