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Stability Analysis of Degenerate Einstein Model of Brownian Motion

Isanka Garli Hevage[1, *], Akif Ibraguimov[2, 3], Zeev Sobol[4]

[1]Department of Mathematics and Statistics, Sam Houston State University, Houston, USA
[2]Department of Mathematics and Statistics, Texas Tech University, Lubbock, USA
[3]Oil and Gas Research Institute, Moscow, Russia
[4]Department of Mathematics, University of Swansea, Swansea, UK

Email address:
iug002@shsu.edu (Isanka Garli Hevage), ilya1sergey@gmail.com (Akif Ibraguimov), z.sobol@swansea.ac.uk (Zeev Sobol)
*Corresponding author

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Abstract: Recent advancements in stochastic processes have uncovered a paradox associated with the Einstein model of Brownian motion of random particles, which diffuse in the media with no boundary . The classical model developed by Einstein provide diffusion coefficient which does not depend on numbers of particles (concentration) and does not degenerate. Based on this model one can predict the propagation speed of particles movement, conflicting with the second law of thermodynamics. We justify that within Einstein paradigm this issue can be resolved. For that we revisited approach proposed by Einstein, and significantly modified his ideas by introducing inverse Kolmogorov equation, with coefficient degenerating as concentration of the particle of interest vanishes. The modified model successfully resolves paradox affiliated to classical Brownian motion model by introducing a concentration-dependent diffusion matrix, establishing a finite propagation speed. Proposed model utilize but of inverse Kolmogorov stochastic parabolic equation and propose sufficient condition (Hypotheses 1.1) for degeneracy of diffusion coefficient, which guarantee finite speed of propagation inside domain of diffusion. This paper outlines the necessary conditions for this property through a counterexample, which provide infinite speed of propagation for the solution of the equation, with diffusion coefficient, which degenerate as concentration vanishes but with lower speed than in (Hypotheses 1.1). The second part focuses on the stability analysis of the solution of the degenerate Einstein model in case when boundary condition are crucial. We considered degenerate Einstein model in the boundary domain with Dirichlet boundary conditions. Our model bridge degenerate Brownian equation in the bulk of media with boundary of the domain. We with detail investigate stability of the problem with perturbed boundary Data, which vanishes with time. A functional dependence is introduced on the solution that satisfies a specific ordinary differential inequality. The investigation explores the solution's dependence on the boundary and initial data of the original problem, demonstrating asymptotic stability under various conditions. These results have practical applications in understanding stochastic processes and its dependence on the boundary Data within bounded domains.

Keywords: Stability Analysis, Degenerate PDEs, Particle Localization, Finite Speed of Propagation

1. Introduction

In his seminal work, Einstein models the movement of a particle as a random walk, where the step time τ and the (random) displacement or *free jump* Δ are both symmetrically distributed, independent of the point and time of observation [1, 2]. In this paper, the term *free jump* is defined as the movement of particles without undergoing any collisions. It refers to the scenario where particles can move freely and independently, without encountering obstacles or interacting with other particles. The definition of *free jump* in this context is based on the classical Einstein paradigm, as presented in his famous dissertation [1]. It is worth noting that in the literature, the term *free pass* is sometimes used interchangeably with *free jump* to describe the same phenomenon [3]. Therefore, in this context, *free jump* and *free pass* are synonymous terms.

The mass conservation law constrains the random walk, expressed through the concentration function. Using Taylor's expansion, Einstein demonstrates that the concentration function u, representing the particle distribution's density, satisfies the classical heat equation. Although Einstein's paper was groundbreaking in stochastic processes and an important step toward the construction of Brownian motion, his proposed model led to a physical paradox. While u is a solution to the heat equation, it allows a void volume to reach a positive concentration of particles instantly. Moreover, since the *free jump*s process is reversible, the model permits all particles, with remarkable coherence, to concentrate instantly in a small volume. This contradicts the second law of thermodynamics and demonstrates an infinite propagation speed.

The De Giorgi–Ladyzhenskaya iteration procedure [4, 5], successfully utilized in [10], was employed to address the paradox in the classical Einstein model as stated in [11–13]. The resolution involved replacing the random walk with a diffusion process and allowing the diffusion coefficient to depend on the concentration function u, contrary to the constant diffusion coefficient in Einstein's model (see [11]). The approach to modeling various dynamic processes based on an experiment using the Einstein paradigm was very effective. The authors used this method to model different processes using a system of degenerate parabolic equations with a drift depending on the concentration of the substance and its gradient (see, for example, [12, 13, 19]). This modification aims to preserve the isotropic nature of the stationary media while removing the paradox by ensuring that the concentration function u exhibits *finite propagation speed* property, namely, if a neighborhood of a point is void of particles at time t^*, then a smaller neighborhood of the same point has been void of particles for some time before t^*. A formal definition of the finite speed of propagation from recent work in [11] is provided below.

Definition 1.1 (Finite propagation speed). A function $u \geq 0$ on $\Omega \times [0, T)$ is said to exhibit a finite propagation speed if, for any open ball of the radius R_0 $B \subset \Omega$ and any $\epsilon \in (0, 1)$, there exists $T' \in (0, T]$ which might depend on B, ϵ and u, such that, given $u(x, 0) = 0$ for all $x \in B$, one has $u(x, t) = 0$ for all $(x, t) \in \epsilon B \times [0, T')$.(Here ϵB is ball of the radius ϵR_0)

The concept of finite propagation speed was first demonstrated by G.I. Barenblatt for a degenerate porous media equation, which is different from the Einstein model, as it is based on the traditional divergent equation for fluid density which is vanishing with pressure [6]. This equation suggests that finite propagation speed occurs when a small concentration leads to a small diffusion. To reflect the idea of higher medium resistance for small numbers of particles, it is assumed in this work that the diffusion coefficient is a positive continuous function of concentration u and that it degenerates as the concentration vanishes. The concept of concentration functions finds frequent application in the analysis of stochastic processes and their associated partial differential equations (PDEs).

As mentioned in the prompt, concentration functions can be understood in two distinct manners: As a function or as a density of the distribution of particles. The first approach, where concentration functions are considered as a function, leads to the backward Kolmogorov equation, which describes the evolution of the probability distribution of a stochastic process from a given time to an earlier time. Specifically, the equation describes the evolution of the concentration function $u(x, t)$ at time t, given its value at a later time T and position x. This equation is useful in analyzing the behavior of stochastic processes over time. The second approach leads to the forward Kolmogorov equation. This equation describes the evolution of the probability density function of a stochastic process from an initial time to a later time. Specifically, the equation describes the evolution of the concentration function $u(x, t)$ at time t, given its value at an earlier time s and position x. This equation is useful in predicting the future behavior of stochastic processes.

The generalized Einstein model is derived by treating the concentration function u as a function of both time and space [7]. For a space-time point of observation $Z = (x, s) \in \mathbb{R}^N \times [0, \infty)$, an $\mathbb{R}^N$-valued random *free-jumps process* is considered, describing an interaction-free displacement of a particle from Z. Then, by assuming the extended axioms as in [11, Hypothesis 1] and employing Taylor expansion as in [8] for the generalized mass conservation law, the governing partial differential inequality (3) is derived within a fixed domain Ω and over a time horizon $T > 0$ for the scenario.

The upcoming hypothesis is introduced to delve into the core of the phenomenon, assuming neither drift nor consumption. The notion of a lower diffusion speed corresponding to a decreased concentration of particles is pivotal. This concept, crucial for achieving a finite propagation speed, suggests heightened medium resistance in scenarios with fewer particles.

Hypotheses 1.1. The diffusion matrix

$$a_{ij}(Z) = 2a(u(Z))\delta_{ij},$$

for $i, j = 1, 2, \ldots, N$, and for some scalar function $a \in C([0, \infty))$ with $a(0) = 0$ and $a(s) > 0$ for $s > 0$. Define

$$I(s) \triangleq \int_s^\infty \frac{d\tau}{\tau\, a(\tau)}. \tag{1}$$

Then $I(s)$ is finite for all $s > 0$ and

$$\limsup_{s\to\infty} a(s)I(s) < \infty. \tag{2}$$

Note that $s \to a(s)$ is not necessarily differentiable.

Remark 1.1. $I(s) \to \infty$ as $s \to 0$. Moreover, from equations (2) and (7), the function $s \to a(s)I(s)$ remains bounded and continuous.

Under Hypothesis 1.1, the novel governing model is reformulated as the following inequality

$$u_t \leq a(u)\Delta u, \text{in } \Omega_T. \tag{3}$$

To further analyze the above equation, (3) is multiplied by a weight function, $h(v) > 0$, where $h \in C((0,\infty)) \cap \mathrm{L}^1_{\mathrm{loc}}([0,\infty))$. Let

$$H(v) \triangleq \int_0^v h(r)dr, \text{ for } v \geq 0, \tag{4}$$

and introduce a locally Lipschitz continuous function

$$F(v) \triangleq h(v)a(v), \text{ for } v \geq 0, \tag{5}$$

which is nondecreasing. In addition require $H(0) = 0$ and $F(0) = 0$. Consequently, the following partial differential inequality is derived.

$$[H(u)]_t - F(u)\Delta u \leq 0 \quad \text{in } \Omega_T. \tag{6}$$

It is also important to assume that $F(v) \to 0$ when $v \to 0^+$. This regularization step allows us to analyze (6) and utilize a rich set of techniques for its weak solutions.

In this context, a solution to the partial differential inequality (3) is sought, as a weak positive bounded solution to (6) satisfies the finite speed of propagation property. This property ensures that the solution does not propagate information faster than a certain speed, resolving the paradox associated with the violation of the second law of thermodynamics. The main result in [11, Theorem 5.1] establishes the existence of an *a priori* finite propagation speed for a weak non-negative the bounded solution to the concentration equation (9). It states that the concentration u demonstrates a finite propagation speed if the diffusion coefficient a is defined in Hypothesis 1.1 satisfies the following two constraints

$$\limsup_{s\to 0} a(s)I(s) < \infty, \tag{7}$$

and there exists $c, \mu > 0$ such that

$$a(s)I^\mu(s) \geq c\, a(v)I^\mu(v), \tag{8}$$

for $0 < s < v < 1$. These assumptions hold for $a(s) = ks^\rho$, with k and ρ being positive constants, as proven in [12, 13]. They also hold for more general cases, such as regularly varying functions with a strictly positive lower index.

It is worth noting that an example of a degenerate function $a(u)$ is constructed in such a way that a corresponding self-similar solution to the inequality

$$\iint_{\Omega_T} \nabla u \cdot \nabla (F(u)\phi)\, dxdt \leq \iint_{\Omega_T} H(u)\phi_t\, dxdt, \tag{9}$$

for $\phi \in \mathrm{Lip_c}(\Omega_\mathrm{T})$, exhibits an infinite propagation speed, while the constraint (7) is not satisfied. Detailed examples offering further insight into the latter phenomena can be found in Section 2.

In Section 3, the stability analysis of the degenerate Einstein model is explored by studying the initial boundary value problem (IBVP) outlined in (38)-(40) with homogeneous boundary conditions. This investigation spans both bounded and unbounded strong solutions, focusing on the central quantity of the analysis, $Y(t)$ in the equation 19. Several estimates for this crucial quantity are derived in various situations. The exploration begins with the fundamental degeneracy case of $u^{-\gamma}$, which provides insights and establishes a foundation for the subsequent analysis. To ensure the validity of the analysis, essential assumptions are imposed on the functions H and F and demonstrate that the estimate $Y(t)$ remains bounded concerning the initial data. This serves as a fundamental basis for establishing asymptotic stability in forthcoming results.

Furthermore, a thorough analysis of the parameters is conducted within the framework of extended Assumption 3.6. This analysis enables us to gain deeper insights into the stability conditions under different values of β. The detailed findings and conclusions of this extension are presented in Theorem 3.6. Through these rigorous analyses and derived results, a comprehensive understanding of the stability aspects of the degenerate Einstein model is provided.

A stochastic approach was employed to transition from a model in terms of partial differential equations to a system of equations and ultimately to an integro-differential stochastic PDE with a noise source (see [21–23]). This study adopts the opposite direction, using Einstein's random walk framework to derive a PDE model of the process. Consequently, the PDE formulation of Brownian motion is utilized for the density function. This approach leverages the machinery of PDEs to provide a counterexample to the condition in Hypothesis 1.1 regarding the speed of degeneration and to investigate the stability of the system when subject to perturbations at the boundary.

2. Necessary Condition for Finite Speed of Propagation

In this section, it is shown that the constraint 7 is essential for the property of finite speed of propagation in the sense of Definition 1.1. Specifically, a diffusion coefficient function a that degenerates at $u = 0$ but violates the condition (7) is presented, demonstrating that the corresponding inequality's solution exhibits the property of infinite propagation speed. Define

$$Lu \triangleq u_t - a(u)\Delta u. \tag{10}$$

The objective is to identify a non-negative solution u of the inequality $Lu \leq 0$. This sup-solution possesses the following properties: It is positive for each moment of time $t > 0$ and each x, while being equal to zero for $t = 0$ outside of a compact set K. Assume u has the form

$$u(x,t) = \phi(t) f\left(\frac{|x|}{\theta(t)}\right), \text{ where } \theta(t) \neq 0. \tag{11}$$

Then

$$u_t = \phi'(t)f(s) - \frac{\phi(t)\theta'(t)}{\theta(t)} sf'(s), \Delta u = \frac{\phi(t)}{\theta^2(t)}\left[f''(s) + \frac{(N-1)}{s}f'(s)\right], \text{ where } s = |x|/\theta(t).$$

Set $\phi(t) = 1$ and $\theta(t) = \sqrt{2t}$. Note that

$$\theta'(t)/\theta(t) = 1/\theta^2(t),$$

and using (10) one can write

$$-sf'(s) = a(f(s))\left[f''(s) + \frac{(N-1)}{s}f'(s)\right] = a(f(s))\frac{1}{s^{N-1}}\left[s^{N-1}f'(s)\right]',$$

which yields

$$-\frac{s}{a(f(s))} = \frac{d}{ds}\Big(\ln|s^{N-1}f'(s)|\Big).$$

Let f be the form

$$f(s) = e^{-s^\lambda},$$

for $s > 0$, and some $\lambda > 0$. Then $f'(s) < 0$. Consequently,

$$a(e^{-s^\lambda}) = \frac{s^2}{\lambda s^\lambda - (\lambda + N - 2)}, \tag{12}$$

and the above is positive for

$$s \geq s_0 \triangleq 2\left(\frac{\lambda + N - 2}{\lambda}\right)^{\frac{1}{\lambda}}.$$

Note that $u = e^{-s^\lambda}$. Let

$$u_* \triangleq e^{-s_0{}^\lambda} \text{ and } a_* \triangleq \frac{s_0{}^2}{\lambda s_0{}^\lambda - (\lambda + N - 2)}, \tag{13}$$

for fixed $\lambda > 2$. Then, one can define the function $a : [0, \infty) \to [0, \infty)$ by

$$a(u) = 0\mathbb{1}_0 + \frac{|\ln u|^{\frac{2}{\lambda}}}{\lambda|\ln u| - (\lambda + N - 2)}\mathbb{1}_{(0,u_*)} + \frac{a_*}{u_* + 1}(u + 1)\mathbb{1}_{[u_*,\infty)}, \tag{14}$$

where $u_* \in (0, 1)$ and $a_* > 0$. Then the function $a(\cdot)$ is continuous on $[0, \infty))$ and

$$\int_{u_*}^{\infty} \frac{d\tau}{\tau a(\tau)} \triangleq c_* < \infty. \tag{15}$$

Also, there is $C_1 > 0$ such that

$$a(u) \geq C_1|\ln u|^{\frac{2}{\lambda}-1} \text{ for } u \in (0, u_*]. \tag{16}$$

By (14) and using (15), the following integral is computed.

$$I(u) = \frac{\lambda}{2 - \frac{2}{\lambda}}\left(|\ln(u)|^{2-\frac{2}{\lambda}} - |\ln(u_*)|^{2-\frac{2}{\lambda}}\right) - \frac{\lambda + N - 2}{1 - \frac{2}{\lambda}}\left(|\ln(u)|^{1-\frac{2}{\lambda}} - |\ln(u_*)|^{1-\frac{2}{\lambda}}\right) + c_*, \tag{17}$$

for $u \in (0, u_*]$. Combining (17) and (16), implies

$$a(u)I(u) \geq C_2|\ln(u)|^{\frac{2}{\lambda}-1+2-\frac{2}{\lambda}} = C_2|\ln(u)|, \tag{18}$$

for all $u \in (0, u_*]$. Therefore,

$$a(u)I(u) \to \infty \text{ when } u \to 0.$$

Hence, $a(u)$ does not satisfy (7).

For the condition $s = \dfrac{|x|}{\sqrt{2t}} > s_0$, the choices $|x| > \sqrt{2}s_0$ and $t < 1$ are made.

With such (x, t), it follows that $u(x,t) = e^{-\left(\frac{|x|}{\sqrt{2t}}\right)^\lambda} < u_*$.

In summary, $u(x,t) = e^{-\left(\frac{|x|}{\sqrt{2t}}\right)^\lambda}$, with $\lambda > 2$, satisfies equation

$$u_t = a(u)\Delta u,$$

for $0 < t < 1$ and $|x| > \sqrt{2}s_0$, and the function $a(u)$ given by (14) does not satisfy (7). Thus, for all $x \neq 0$,

$$\lim_{t \to 0^+} u(x,t) = 0, \; u(x,t) > 0 \text{ for any } t > 0,$$

i.e., $u(x,t)$ has infinite speed of propagation.

3. Stability Analysis of Degenerate Einstein Model

A simple observation is first made for a basic degenerate model with degeneration of the power degree, and the solution of the problem is defined. This observation is then generalized for more complex non-linearity. In the stability analysis of the degenerate Einstein model under homogeneous boundary conditions, the emphasis is on the central quantity.

$$Y(t) = \int_\Omega (H(u(x,t)))^\nu dx, \text{ with } \nu \geq 1, \tag{19}$$

for $t \geq t_0$. This function was utilized to study the stability of the solution of p-Laplacian type equations, which occur in the modeling of filtration in porous media (see [14–18]). To ensure coherence in this work, the following fundamental estimate is consistently applied throughout this section.

Lemma 3.1. Let $k > 0$ and $\beta > 1$ be constants. Let $Y(t) \in AC([t_0, \infty))$ be a nonnegative solution of

$$\frac{d}{dt}Y(t) + k\,(Y(t))^\beta \leq 0, \text{ for } t > t_0, \tag{20}$$

with $Y(t_0) \triangleq Y_0$.

(i) If $Y_0 = 0$ then $Y(t) = 0$ for all $t \geq t_0$.

(ii) If $Y_0 > 0$ then

$$Y(t) \leq \left[k(\beta - 1)(t - t_0) + Y_0^{-(\beta-1)}\right]^{-\frac{1}{\beta-1}}, \tag{21}$$

for $t \geq t_0$. Consequently,

$$\lim_{t \to \infty} Y(t) = 0. \tag{22}$$

Proof Part (i). Using the inequality (20), it follows that

$$\frac{d}{dt}(Y(t)) \leq -k(Y(t))^\beta. \tag{23}$$

Then $\dfrac{d}{dt}(Y(t)) \leq 0$.

Consequently, if $Y_0 = 0$ then $Y(t) = 0$ for all $t \geq t_0$.

Part (ii). Let $Y_0 > 0$.

1. Case 1: $Y(t) > 0$ for all $t \geq t_0$. By the inequality (20),

$$\frac{d}{dt}(Y(t))^{-\beta+1} \leq -k(1-\beta), \tag{24}$$

Integrating both sides over (t_0, t), yields,

$$(Y(t))^{-\beta+1} - Y_0^{-\beta+1} \geq -k(1-\beta)(t-t_0), (Y(t))^{-\beta+1} \geq k(\beta-1)(t-t_0) + Y_0^{-\beta+1}. \tag{25}$$

From above (21) follows.

2. Case 2: There is $t^* > t_0$ such that $Y(t^*) = 0$. Then there exists $t' \in (t_0, t^*]$ so that $Y(t') = 0$ and $Y(t) > 0$ for all $t \in [t_0, t')$ and $Y(t) = 0$ for all $t \geq t'$. Hence, the inequality (21) holds for all for all $t \geq t_0$.

Consequently, in both cases, the inequality (21) holds for all $t \geq t_0$.

Remark 3.1. The result of the above Lemma is under some assumption on the speed of convergence of function $\mathcal{Y}(s) \to \infty$, as $s \to \infty$. One can consider more general ordinary differential inequality

$$\frac{d}{dt}(Y(t)) + \mathcal{Y}(Y(t)) \leq 0, \tag{26}$$

for $t > t_0$. This will make it possible to consider the stability of a more sophisticated non-linear Einstein model.

3.1. Basic Degenerate Model

As a model let us start with the basic non-linear function $a(u) = Ku^\gamma$, for constants $K > 0$ and $\gamma \in (0,1)$. Suppose $u \geq 0$ is a solution of the following IBVP

$$(u^{1-\gamma})_t - K\Delta u \leq 0 \text{ in } \Omega \times (0, \infty), \tag{27}$$

$$u(x, 0) = u_0(x) \text{ on } \Omega, \tag{28}$$

$$u(x,t) = 0 \text{ on } \partial\Omega \times (0,\infty), \tag{29}$$

where $K > 0$ is a constant. The following asymptotic stability result is obtained for the solution of the above IBVP in the following sense.

Definition 3.1. $u(x,t)$ is called a solution of the IBVP (27)-

(29) if $(u^{1-\gamma})_t \in \mathrm{L}^2(\Omega \times (0,\infty))$, $\Delta u \in \mathrm{L}^2(\Omega)$, and the divergent formula is applicable for the vector field ∇u, such that inequality (27) holds alomat everywhere.

Theorem 3.1. Assume $u \geq 0$ and be as in Definition 3.1. Let $m \geq 2-\gamma$. Define

$$Y(t) \triangleq \int_\Omega (u(x,t))^m dx, \tag{30}$$

for $t \geq t_0$, and $Y(t_0) \triangleq Y_0$. Then there exists a constant $k_1 > 0$ independent of the solution $u(x,t)$ such that

$$(Y(t))' + k_1 (Y(t))^{\frac{m+\gamma}{m}} \leq 0, \text{ for all } t > t_0. \tag{31}$$

If $Y_0 = 0$ then $Y(t) = 0$ for all $t \geq t_0$. If $Y_0 > 0$ then

$$Y(t) \leq \left[\frac{k_1\gamma}{m}(t-t_0) + Y_0^{-\frac{\gamma}{m}}\right]^{-\frac{m}{\gamma}} \text{ for all } t \geq t_0, \tag{32}$$

and consequently,

$$\lim_{t\to\infty} Y(t) = 0.$$

Proof First of all, note that $m+\gamma-1 \geq 1$ and

$$(u^{1-\gamma})_t \cdot u^{m+\gamma-1} = \left(u^{1-\gamma}\right)_t \cdot \left(u^{1-\gamma}\right)^{\frac{m}{1-\gamma}-1} = \frac{1-\gamma}{m}\left(u^m\right)_t.$$

Multiplying (27) by $u^{m+\gamma-1}$, and using integration by parts over Ω, yields that

$$\frac{1-\gamma}{m} \cdot \frac{d}{dt}\int_\Omega u^m \, dx + K(m+\gamma-1)\int_\Omega u^{m+\gamma-2}|\nabla u|^2 \, dx \leq 0. \tag{33}$$

Computing

$$\int_\Omega u^{m+\gamma-2}|\nabla u|^2 dx = \frac{4}{(m+\gamma)^2}\int_\Omega \left|\nabla u^{\frac{m+\gamma}{2}}\right|^2 dx \, .$$

Then (33) becomes

$$\frac{d}{dt}\int_\Omega u^m \, dx + \frac{4Km(m+\gamma-1)}{(1-\gamma)(m+\gamma)^2}\int_\Omega \left|\nabla u^{\frac{m+\gamma}{2}}\right|^2 dx \leq 0 \, . \tag{34}$$

Let

$$q = 2m/(m+\gamma). \tag{35}$$

Note that $q \in (1,2)$. By Poincaré-Sobolev inequality [9] with power q in (35), it is obtained that

$$\left[\int_\Omega \left(u^{\frac{m+\gamma}{2}}\right)^q dx\right]^{\frac{2}{q}} \leq c_0^2 \int_\Omega \left|\nabla u^{\frac{m+\gamma}{2}}\right|^2 dx,$$

where c_0 is a positive constant that depends on N. Hence,

$$c_0^{-2}(Y(t))^{\frac{m+\gamma}{m}} \leq \int_\Omega \left|\nabla u^{\frac{m+\gamma}{2}}\right|^2 dx, \tag{36}$$

Using (36) in (34), it follows that

$$\frac{d}{dt}Y(t) \; + \; \frac{4Km(m+\gamma-1)}{c_0^2(1-\gamma)(m+\gamma)^2}\,(Y(t))^{\frac{m+\gamma}{m}} \; \leq 0 \, . \tag{37}$$

Note that $\frac{m+\gamma}{m} > 1$. By the virtue of Lemma 3.1, Theorem 3.1 holds.

Remark 3.2. One can easily consider the case when $\gamma = 0$ in Theorem 3.1 and obtain the asymptotic exponential stability of the classical heat equation.

Next, the stability of the generalized Einstein model with the inequality (6) will be investigated.

3.2. Generalized Degenerate Model

$$(H(u))_t - F(u)\Delta u \leq 0 \text{ in } \Omega \times (t_0, \infty), \tag{38}$$

$$u(x, t_0) = u_0(x) \text{ on } \Omega, \tag{39}$$

$$u(x, t) = 0 \text{ on } \partial\Omega \times (t_0, \infty). \tag{40}$$

Recall that H and F are defined by equation (4) and equation (5), respectively, with $H(0) = 0$ and $F(0) = 0$.

3.2.1. Stability Analysis of Bounded Solutions

The stability of the bounded solution of the above IBVP is studied in the following sense.

Definition 3.2. $u(x, t)$ is called a bounded solution of the IBVP (38)-(40) if

$$\sup_{\Omega\times(0,\infty)} u(x, t) \triangleq M < \infty, \tag{41}$$

$(H(u))_t \in L^2(\Omega \times (0, \infty))$ and $F(u)\Delta u \in L^2(\Omega)$, such that (38) holds almost everywhere.

Then, the following assumptions are made for the functions H and F.

Assumption 3.1. There exists $\gamma_1 > 0$ such that $H^{\gamma_1+1} \in C^1([0, \infty))$, and for all $M > 0$, there is $c_1 = c_1(M) > 0$ such that

$$(H(s)^{\gamma_1+1})' \leq c_1, \text{ for all } s \in [0, M]. \tag{42}$$

It follows from (42) that

$$H(s) \leq c_1^{\frac{1}{\gamma_1+1}} s^{\frac{1}{\gamma_1+1}}.$$

Evidently by Definition 4, it is obtained that

$$(H(s)^{\gamma_1+1})' \geq 0. \tag{43}$$

Assumption 3.2. For any $M > 0$, there exists

$$c_2 = c_2(M) > 0$$

such that

$$(F(s)H^{\gamma_1+1}(s))' \geq c_2, \text{ for all } s \in [0, M].$$

Based on the above, the theorem on asymptotic stability is now stated.

Theorem 3.2. Let Assumption 3.1, and Assumption 3.2 hold. Assume $u \geq 0$ is a bounded solution for IBVP (38)-(40). Let $p \geq \gamma_1 + 1$. Define

$$Y(t) \triangleq \int_\Omega (H(u(x, t)))^{p+1}\, dx, \tag{44}$$

for $t \geq t_0$, and $Y_0 \triangleq Y(t_0)$. Then $Y(t)$ satisfies the differential inequality

$$\frac{d}{dt}Y(t) + c(Y(t))^{\frac{p+1+\gamma_1}{p+1}} \leq 0, \text{ for all } t > t_0, \tag{45}$$

where $c > 0$ is a constant depending on M. If $Y_0 = 0$ then $Y(t) = 0$ for all $t \geq t_0$. If $Y_0 > 0$ then

$$Y(t) \leq \left[c\left(\frac{\gamma_1}{p+1}\right)(t - t_0) + Y_0^{-(\frac{\gamma_1}{p+1})}\right]^{-\frac{p+1}{\gamma_1}} \text{ for } t \geq t_0, \tag{46}$$

and consequently,

$$\lim_{t\to\infty} Y(t) = 0. \tag{47}$$

Proof With M defined by (41), let $c_1 = c_1(M)$ and $c_2 = c_2(M)$ be positive constants in Assumption 3.1 and Assumption 3.2 respectively.

Note that $(H(u))^p F(u)\big|_{\partial\Omega} = 0$. Multiplying the first inequality in (38) by $(H(u))^p$, and integrating over Ω, to obtain

$$\int_\Omega (H(u))^p (H(u))_t\, dx - \int_\Omega (H(u))^p F(u)\Delta u\, dx \leq 0. \tag{48}$$

Using integration by parts for the second integral in (48), yields

$$\frac{d}{dt}\int_\Omega (H(u))^{p+1}\, dx + (p + 1)\int_\Omega \nabla u \cdot \nabla((H(u))^p F(u))\, dx \leq 0. \tag{49}$$

Note that

$$[H^p]' = \left([H^{\gamma_1+1}]^{\frac{p}{\gamma_1+1}}\right)' = \frac{p}{\gamma_1+1}(H^{\gamma_1+1})^{\frac{p}{\gamma_1+1}-1}(H^{\gamma_1+1})' = \frac{p}{\gamma_1+1}H^{p-(\gamma_1+1)}(H^{\gamma_1+1})'.$$

Recall that $p \geq \gamma_1 + 1$, and H subject to (43). Using the above, it is found that

$$\begin{aligned}
&\nabla u \cdot \nabla((H(u))^p F(u)) \\
&= |\nabla u|^2 F(u) \frac{p}{\gamma_1+1} H^{p-(\gamma_1+1)}(H^{\gamma_1+1})' + \ |\nabla u|^2 (H(u))^p F'(u) \\
&= \left[\frac{\gamma_1+1}{p} F'(u) H^{\gamma_1+1}(u) + F(u)[H^{\gamma_1+1}]'(u)\right] \cdot \frac{p}{\gamma_1+1} |\nabla u|^2 (H(u))^{p-(\gamma_1+1)} \\
&\geq (F H^{\gamma_1+1})'(u)\ |\nabla u|^2 (H(u))^{p-(\gamma_1+1)}.
\end{aligned} \tag{50}$$

By Assumption 3.2, it is shown that

$$\nabla u \cdot \nabla((H(u))^p F(u)) \geq \ c_2 \cdot |\nabla u|^2 (H(u))^{p-(\gamma_1+1)}. \tag{51}$$

Using (51) in the inequality (49) becomes

$$\frac{d}{dt}\int_\Omega (H(u))^{p+1}\, dx + c_2(p+1)\int_\Omega |\nabla u|^2 (H(u))^{p-(\gamma_1+1)}\, dx \leq 0\ . \tag{52}$$

Let

$$q = (2p+2)\big/(p+\gamma_1+1). \tag{53}$$

Note that $q \in (1,2)$. Applying Poincaré-Sobolev inequality with power q in (53), it follows that

$$\left[\int_\Omega \left(\left(H(u)^{\gamma_1+1}\right)^{\frac{\gamma_1}{(\gamma_1+1)(2-q)}}\right)^q dx\right]^{\frac{2}{q}} \leq c_0^2 \int_\Omega \left|\nabla \left(H(u)^{\gamma_1+1}\right)^{\frac{\gamma_1}{(\gamma_1+1)(2-q)}}\ \right|^2 dx, \tag{54}$$

where $c_0 > 0$ represents a constant that depends on N. Observe that

$$\left(\left(H^{\gamma_1+1}\right)^{\frac{\gamma_1}{(\gamma_1+1)(2-q)}}\right)^q = H^{p+1}.$$

Therefore, (54) becomes

$$\left[\int_\Omega (H(u))^{p+1}\, dx\right]^{\frac{2}{q}} \leq \tag{55}$$

$$K_1 \int_\Omega \left|\left(H(u)^{\gamma_1+1}\right)^{\frac{\gamma_1}{(\gamma_1+1)(2-q)}-1} (H(s)^{\gamma_1+1})'|_{s=u(x,t)} \nabla u\right|^2 dx, \tag{56}$$

for

$$K_1 = c_0^2 \gamma_1^2 / (\gamma_1+1)^2 (2-q)^2.$$

Using Assumption 3.1 in (56), gives

$$(Y(t))^{\frac{2}{q}} \leq c_1 K_1 \int_\Omega \left|\ \left(H(u)^{\gamma_1+1}\right)^{\frac{\gamma_1}{(\gamma_1+1)(2-q)}-1} \nabla u\right|^2 dx. \tag{57}$$

One can obtain that

$$\frac{\gamma_1}{(\gamma_1+1)(2-q)} - 1 = \frac{p-(\gamma_1+1)}{2(\gamma_1+1)}. \tag{58}$$

Hence, the right-hand side of (57), becomes

$$(Y(t))^{\frac{2}{q}} \leq c_1 K_1 \int_\Omega [H(u)]^{p-(\gamma_1+1)} |\nabla u|^2 \, dx. \tag{59}$$

Using (59) in (52), it is obtained that

$$(Y(t))' + \frac{c_2(p+1)}{c_1 K_1}(Y(t))^{\frac{2}{q}} \; \leq \; 0,$$

which proves the inequality (68). Consequently, the limit (47) is obtained.

3.2.2. Stability Analysis of Unbounded Solutions

Definition 3.3. Let u be a non-negative solution of the IBVP (38)-(40) provided that $(H(u))_t \in L^2(\Omega \times (0,\infty))$ and $F(u)\Delta u \in L^2(\Omega \times (0,\infty))$, such that (38) holds almost everywhere.

In this subsection, the unbounded solution in the above Definition is studied by introducing the following relaxed version of Assumption 3.2.

Assumption 3.3. Let $p \geq 0$ such that $[F(s)H^p(s)]' \geq 0$ for all $s \in [0,\infty)$.

Assumption 3.4. Assume u is such that $(H(u))^p F(u) \in W_0^{1,2}(\Omega)$, for p in Assumption (3.3).

Next, a primary property of the solution u is established.

Theorem 3.3. Let $p \geq 0$ and Assumptions 3.3 and 3.4 hold. Assume $u \geq 0$ is a solution of IBVP (38)-(40). Suppose $H(u_0(x)) \in \mathrm{L}^{\mathrm{p}+1}(\Omega)$. Define

$$Y(t) \triangleq \int_\Omega (H(u(x,t)))^{p+1} \, dx, \tag{60}$$

for $t \geq t_0$, and $Y_0 \triangleq Y(t_0)$. Then $Y(t)$ is nonincreasing (monotone), and

$$\int_\Omega (H(u(x,t)))^{p+1} \, dx \leq \int_\Omega [H(u_0(x))]^{p+1} \, dx, \tag{61}$$

for all $t \geq t_0$.

Proof Multiplying the first inequality in (38) by $(H(u))^p$, and integrating over Ω, to find

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (H(u))^{p+1} \, dx + \int_\Omega \nabla u \cdot \nabla\big((H(u))^p F(u)\big) \, dx \; \leq 0. \tag{62}$$

By Assumption 3.3, it is computed that

$$\nabla u \cdot \nabla\big((H(u))^p F(u)\big) = |\nabla u|^2 [H^p(s)F(s)]'\Big|_{s=u} \geq 0. \tag{63}$$

Using (63) in (62), yields

$$\frac{d}{dt}\left[\int_\Omega (H(u))^{p+1} \, dx\right] = \frac{d}{dt}Y(t) \; \leq \; 0. \tag{64}$$

From the above, the monotonicity of $Y(t)$ follows. Consequently

$$Y(t) \leq Y(t_0) \text{ for } t \geq t_0.$$

With the above, the inequality in (61) is concluded.

Moving forward, the analysis will continue with the functions u, H and u_0 as stipulated in Theorem 3.3. The following two structural conditions on the functions H and F will be introduced before delving into the next results.

Assumption 3.5. There exist $p_1 > 0,\ q_1 > 0$ and $c_3 > 0$ such that

$$\big(H^{p_1}(s)F(s)\big)' \geq c_3 (H(s))^{q_1}, \tag{65}$$

for all $s \in [0,\infty)$.

Assumption 3.6. There exist $\gamma_1, \beta > 0$ and $c_4 > 0$ such that

$$\big(H^{\gamma_1+1}(s)\big)' \leq c_4 (H(s))^{\beta} \; \forall \, s \in [0,\infty). \tag{66}$$

Remark 3.3. Example for Assumptions 3.5 and 3.6

Given the above assumptions, the following theorem is first stated when $\beta = \frac{q_1}{2}$.

Theorem 3.4. Let Assumption 3.5 and Assumption 3.6 hold for $\beta = q_1/2$. Assume $H(u_0(x)) \in \mathrm{L}^{\mathrm{p}_1+1}(\Omega)$. Define $Y(t) \triangleq \int_\Omega (H(u(x,t)))^{p_1+1}\,dx, t \geq t_0$. Let

$$\delta_1 = (p_1+1)/(\gamma_1+1). \tag{67}$$

Assume $1 \leq \delta_1 < 2$. Then there exists $c > 0$ such that

$$\frac{d}{dt}(Y(t)) + c(Y(t))^{\frac{2}{\delta_1}} \leq 0, \text{ for all } t > t_0. \tag{68}$$

If $Y_0 > 0$, then

$$Y(t) \leq \left[c\left(\frac{2-\delta_1}{\delta_1}\right)(t-t_0) + Y_0^{-\left(\frac{2-\delta_1}{\delta_1}\right)}\right]^{-\left(\frac{\delta_1}{2-\delta_1}\right)}, \tag{69}$$

for $t \geq t_0$ and consequently,

$$\lim_{t\to\infty} Y(t) = 0. \tag{70}$$

Proof Note that $(H(u))^{p_1}F(u)\big|_{\partial\Omega} = 0$. Multiplying the first inequality in (38) by $(H(u))^{p_1}$, and using integration by parts, to obtain

$$\frac{d}{dt}\int_\Omega (H(u))^{p_1+1}\,dx + (p+1)\int_\Omega \nabla u \cdot \nabla((H(u))^{p_1}F(u))\,dx \leq 0. \tag{71}$$

By Assumption 3.5, it implies that

$$\nabla u \cdot \nabla((H(u))^{p_1}F(u)) = [H^{p_1}(u)F(u)]'|\nabla u|^2 \geq c_3 \cdot (H(u))^{q_1}|\nabla u|^2. \tag{72}$$

Using (72) in (71) becomes

$$\frac{d}{dt}\left[\int_\Omega (H(u))^{p_1+1}\,dx\right] + c_3(p+1)\int_\Omega (H(u))^{q_1}|\nabla u|^2\,dx \leq 0. \tag{73}$$

Applying Poincaré-Sobolev inequality for $1 \leq \delta_1 < 2$, it follows that

$$\left[\int_\Omega \left((H(u))^{\gamma_1+1}\right)^{\delta_1}\,dx\right]^{\frac{1}{\delta_1}} \leq c_p\left[\int_\Omega \left|\nabla\left((H(u))^{\gamma_1+1}\right)\right|^2 dx\right]^{\frac{1}{2}}, \tag{74}$$

for constant $c_p > 0$, depending on N. Hence,

$$(Y(t))^{\frac{2}{\delta_1}} \leq c_p^2 \int_\Omega \left|\frac{d}{ds}\left(H(s)^{\gamma_1+1}\right)\right|^2_{s=u(x,t)} |\nabla u|^2\,dx. \tag{75}$$

By Assumption 3.6, it follows from (75) that

$$\begin{aligned}(Y(t))^{\frac{2}{\delta_1}} &\leq c_4 \cdot c_p^2 \int_\Omega (H(s))^{2\beta}|\nabla u|^2\,dx \\ &= c_4 \cdot c_p^2 \int_\Omega (H(s))^{q_1}|\nabla u|^2\,dx.\end{aligned} \tag{76}$$

Utilizing inequality (76) in (73) provides

$$(Y(t))' + (p_1+1)\frac{c_3}{c_4 \cdot c_p^2}(Y(t))^{\frac{2}{\delta_1}} \leq 0, \text{ for all } t > 0, \tag{77}$$

which implies inequality (68). Here $\delta_1 < 2$. Thus, the estimate is obtained by Lemma 3.1, and consequently, the limit (70) is derived.

The stability result for $\beta > \frac{q}{2}$ is established, starting with the next auxiliary result.

Theorem 3.5. Let Assumptions 3.5 and 3.6 hold for $\beta > q_1/2$. Let δ_1 be defined as in (67), where $1 \leq \delta_1 < 2$. Moreover, let

$$\delta_2 = 1, \text{if } 1 \leq \delta_1 \leq \frac{N}{N-1}, \tag{78}$$

or

$$\delta_2 = \frac{\delta_1 N}{N+\delta_1}, \text{if } \frac{N}{N-1} < \delta_1 < 2. \tag{79}$$

Assume that for any $t \geq t_0$. Then there exists $k > 0$ such that

$$Y'(t) + \frac{k}{(Z(t)+1)^{\frac{2-\delta_2}{\delta_2}}}(Y(t))^{\frac{2}{\delta_1}} \leq 0, \text{ for all } t > t_0. \tag{80}$$

Here $Z(t) = \int_\Omega (H(u))^{p_0} dx$, and $p_0 = \left(\beta - \frac{q_1}{2}\right) \frac{2\delta_2}{2-\delta_2}$.

Proof The proof of Theorem 3.4 is followed up to (73). Note that $1 \leq \delta_2 < N$ and, for both cases of δ_1 in (78)-(79),

$$\delta_1 \leq \frac{\delta_2 N}{N-\delta_2}.$$

Applying Poincaré-Sobolev inequality, to obtain

$$\left[\int_\Omega \left((H(u))^{\gamma_1+1}\right)^{\delta_1} \, dx\right]^{\frac{1}{\delta_1}} \leq c_p \left[\int_\Omega \left|\nabla \left((H(u))^{\gamma_1+1}\right)\right|^{\delta_2} dx\right]^{\frac{1}{\delta_2}}. \tag{81}$$

Here, $c_p > 0$ represents a constant that depends on N. Using Assumption 3.6, the estimate (81) implies

$$(Y(t))^{\frac{\delta_2}{\delta_1}} \leq c_p^{\delta_2} c_4 \int_\Omega (H(u))^{\beta\delta_2} |\nabla u|^{\delta_2} \, dx. \tag{82}$$

By Hölder's inequality with powers $2/(2-\delta_2)$ and $2/\delta_2$ on the right-hand-side of (82) and from Assumption 3.5, it follows that

$$\int_\Omega (H(u))^{\beta\delta_2} |\nabla u|^{\delta_2} \, dx \leq \left[\int_\Omega (H(u))^{\left(\beta - \frac{q_1}{2}\right)\frac{2\delta_2}{2-\delta_2}} dx\right]^{\frac{2-\delta_2}{2}} \left[\int_\Omega (H(u))^{q_1} |\nabla u|^2 dx\right]^{\frac{\delta_2}{2}}.$$

Hence, (82) becomes

$$(Y(t))^{\frac{\delta_2}{\delta_1}} \leq c_p^{\delta_2} c_4 (Z(t))^{\frac{2-\delta_2}{\delta_2}} \left[\int_\Omega (H(u))^{q_1} |\nabla u|^2 dx\right]^{\frac{\delta_2}{2}}. \tag{83}$$

Using () in (82), it is obtained that

$$(Z(t)+1)^{-\frac{2-\delta_2}{\delta_2}} c_p^{-2} c_4^{-2/\delta_2} (Y(t))^{\frac{2}{\delta_1}} \leq \int_\Omega (H(u))^{q_1} |\nabla u|^2 \, dx. \tag{84}$$

Combining (73) with (84) gives

$$(Y(t))' + \frac{c_3(p_1+1)}{c_4^{2/\delta_2} \cdot c_p^2 (Z(t)+1)^{\frac{2-\delta_2}{\delta_2}}}(Y(t))^{\frac{2}{\delta_1}} \leq 0, \tag{85}$$

for all $t > t_0$. Hence, (80) is derived.

Assuming all the conditions in Theorem 3.5 hold, the following analogy is presented.

Corollary 3.1. Let $Y(t)$, δ_1 and p_0 be defined as in Theorem 3.5. Let $p_* \geq 0$ be such that $p_*+1 \geq p_0$. Assume that Assumption

3.3 is satisfied for $p = p^*$, and $[H(u_0)] \in \mathrm{L}^{\mathrm{p}_*+1}(\Omega)$. Then $Y(t)$ satisfies the differential inequality

$$(Y(t))' + C_2(Y(t))^{\frac{2}{\delta_1}} \leq 0, \tag{86}$$

for all $t > t_0$, for some constant $C_2 > 0$. If $Y_0 = 0$ then $Y(t) = 0$ for all $t \geq t_0$. If $Y_0 > 0$ then

$$Y(t) \leq \left[C_2\left(\frac{2-\delta_1}{\delta_1}\right)(t-t_0) + Y_0^{-\left(\frac{2-\delta_1}{\delta_1}\right)}\right]^{-\frac{\delta_1}{2-\delta_1}}, \tag{87}$$

for $t \geq t_0$ and consequently,

$$\lim_{t\to\infty} Y(t) = 0. \tag{88}$$

Proof Applying Hölder's inequality to $Z(t)$ in Theorem 3.5 and using Theorem 3.3 , gives

$$\begin{aligned} Z(t) &= \int_\Omega (H(u(x,t)))^{p_0} dx \qquad (89) \\ &\leq \left[\int_\Omega (H(u(x,t)))^{1+p^*} dx\right]^{\frac{p_0}{1+p^*}} |\Omega|^{\frac{1+p^*-p_0}{1+p^*}} \\ &\leq \left[\int_\Omega [H(u_0(x)]^{1+p^*} dx\right]^{\frac{p_0}{1+p^*}} |\Omega|^{\frac{1+p^*-p_0}{1+p^*}} \\ &< \infty. \qquad (90) \end{aligned}$$

The relation (90) is used in (80), leading to the inequality (86). Consequently, leveraging Lemma 3.1, the estimate (87), and the limit (88) are established.

Remark 3.4. Assume $Y(t) > 0$ and $Z(t) > 0$ for any $t \in [t_0, \infty)$. Then the estimate

$$Y'(t) + \frac{k}{(Z(t))^{\frac{2-\delta_2}{\delta_2}}}(Y(t))^{\frac{2}{\delta_1}} \leq 0, \tag{91}$$

holds for all $t > t_0$. Next, if $p_0 \geq p_1 + 1$, then

$$Y(t_0) = \int (H(u(x,t_0)))^{p_0+1} dx,$$

will be picked, and as a result

$$\lim_{t\to\infty} Y(t) = 0.$$

If $Z(t_0) = 0$ or $Y(t_0) = 0$ then $u(x,t_0) = 0$ a.e.. Moreover, since $Y(t)$ monotonically nonincreasing, $u(x,t) = 0$ for all $t \geq t_0$ a.e..

The next step involves extending Assumption 3.6 by introducing an additional term $(H(s))^{\beta_2}$ as follows.

Assumption 3.7. There exist constants $\gamma_1, \beta_1, \beta_2 > 0$ and $c_5 > 0$ such that

$$\left(H^{\gamma_1+1}(s)\right)' \leq c_5\Big((H(s))^{\beta_1} + (H(s))^{\beta_2}\Big), \tag{92}$$

for all $s \in [0, \infty)$.

Theorem 3.6. Let Assumptions 3.5 and 3.7 hold with $\beta_1 > \beta_2 > q_1/2$. Let δ_1 and δ_2 be defined by Theorem 3.1. Define

$$Y(t) \triangleq \int_\Omega (H(u(x,t)))^{p_1+1} dx, t \geq t_0.$$

Then there exists $K > 0$ such that

$$(Y(t))' + \frac{K}{[Z_1(t) + Z_2(t) + 1]^{\frac{2-\delta_2}{\delta_2}}}(Y(t))^{\frac{2}{\delta_1}} \leq 0, \tag{93}$$

for all $t > t_0$. Here,

$$Z_1(t) \triangleq \int_\Omega (H(u(x,t)))^{\left(\beta_1 - \frac{q_1}{2}\right)\frac{2\delta_2}{2-\delta_2}} dx, \tag{94}$$

and

$$Z_2(t) \triangleq \int_\Omega (H(u(x,t)))^{\left(\beta_2 - \frac{q_1}{2}\right)\frac{2\delta_2}{2-\delta_2}} dx. \tag{95}$$

Proof The proof of Theorem 3.1 is followed up to (81). Using Assumptions 3.6 in (81), provides

$$\begin{aligned}
&(Y(t))^{\frac{\delta_2}{\delta_1}} \\
&\le c_p^{\delta_2} c_5 \int_\Omega \left[(H(u))^{\beta_1} + (H(u))^{\beta_2}\right]^{\delta_2} |\nabla u|^{\delta_2}\, dx \\
&\le c_p^{\delta_2} c_5 2^{\delta_2} \int_\Omega (H(u))^{\beta_1\delta_2}|\nabla u|^{\delta_2} + (H(u))^{\beta_2\delta_2}|\nabla u|^{\delta_2}\, dx.
\end{aligned} \tag{96}$$

Applying () for $\beta = \beta_1$ and $\beta = \beta_2$, to obtain

$$\int_\Omega (H(u))^{\beta_1\delta_2}|\nabla u|^{\delta_2}\, dx \le\ (Z_1(t))^{\frac{2-\delta_2}{2}} \left[\int_\Omega (H(u))^{q_1}|\nabla u|^2 dx\right]^{\frac{\delta_2}{2}}, \tag{97}$$

and

$$\int_\Omega (H(u))^{\beta_2\delta_2}|\nabla u|^{\delta_2}\, dx \le\ (Z_2(t))^{\frac{2-\delta_2}{2}} \left[\int_\Omega (H(u))^{q_1}|\nabla u|^2 dx\right]^{\frac{\delta_2}{2}}. \tag{98}$$

Combining (96) with (97) and (98) yields

$$(Y(t))^{\frac{\delta_2}{\delta_1}} \le c_p^{\delta_2} c_5 2^{\delta_2+1} \left[Z_1(t) + Z_2(t) + 1\right]^{\frac{2-\delta_2}{2}} \cdot \left[\int_\Omega (H(u))^{q_1}|\nabla u|^2 dx\right]^{\frac{\delta_2}{2}}. \tag{99}$$

Raising both sides of (99) to the power $\frac{2}{\delta_2}$, gives

$$(Y(t))^{\frac{2}{\delta_1}} \le c_6 \left[Z_1(t) + Z_2(t) + 1\right]^{\frac{2-\delta_2}{\delta_2}} \int_\Omega (H(u))^{q_1}|\nabla u|^2 dx, \tag{100}$$

where $c_6 = 2^{2+2/\delta_2} c_p^2 c_5^{2/\delta_2}$. Then, combining (73) with (100) to obtain

$$(Y(t))' + \frac{c_3(p+1)}{c_6\left[Z_1(t) + Z_2(t) + 1\right]^{\frac{2-\delta_2}{\delta_2}}} (Y(t))^{\frac{2}{\delta_1}}\ \le\ 0,$$

for all $t > t_0$.

Assuming all the conditions in Theorem 3.6 hold, the following result is obtained.

Corollary 3.2. Let $Y(t), \delta_1$, and p_2 be defined as in Theorem 3.6. Let $p' \ge 0$ be such that $p' + 1 \ge p_2$. Assume that the Assumption 3.3 is satisfied for $p = p'$, and $[H(u_0)] \in \mathrm{L}^{\mathrm{p}'+1}(\Omega)$. Then $Y(t)$ satisfies the differential inequality

$$(Y(t))' + C_1 (Y(t))^{\frac{2}{\delta_1}} \le 0, \text{ for all } t > t_0,$$

for some constant $C_1 > 0$. If $Y_0 = 0$ then $Y(t) = 0$ for all $t \ge t_0$. If $Y_0 > 0$ then

$$Y(t) \le \left[C_1\left(\frac{2-\delta_1}{\delta_1}\right)(t - t_0) + Y_0^{-\left(\frac{2-\delta_1}{\delta_1}\right)}\right]^{-\frac{\delta_1}{2-\delta_1}}, \tag{101}$$

for $t \geq t_0$ and consequently,

$$\lim_{t\to\infty} Y(t) = 0. \tag{102}$$

Proof Recalling (94) and (95), note that $p_2 > p_3$ due to $\beta_1 > \beta_2$. Similarly, by applying Hölder's inequality and utilizing Theorem 3.3, one can show that

$$Z_1(t),\ Z_2(t) < \infty,\ \text{for all } t > t_0. \tag{103}$$

Hence, for some constant $C_1 > 0$, the differential inequality (93) yields the form

$$(Y(t))' + C_0(Y(t))^{\frac{2}{\delta_1}} \leq 0,\ \text{for all } t > t_0.$$

Leveraging the insights from Lemma 3.1, the estimate (101) is derived, and the limit (102) is established.

Remark 3.5. One can consider the generalization of Assumption 3.7, by assuming

$$(H(s)^{\gamma_1+1})' \leq \sum c_i (H(s))^{\beta_i}, \tag{104}$$

for $\beta_i > 0$, and obtain the ODI of the form similar to (93), and corresponding corollary. Moreover, in analogous to Remark (3.4), one can obtain conclusions for $u(x,t)$ for $t \geq t_0$ under Assumption 3.7.

4. Conclusions

This paper presents significant advancements in the understanding of the degenerate Einstein-Brownian model with a diffusion matrix depending on the concentration of particles u, addressing the paradox in the classical Einstein framework of Brownian motion. By introducing Hypothesis 1.1, the paradox was resolved, and the necessity of two critical conditions for the model's validity was demonstrated. Additionally, a counterexample was provided that illustrates the infinite propagation speed of the solution when these conditions are violated. The analogs obtained in this study hold substantial potential for applications across various fields of mathematics and science, particularly within physics and mathematical sciences. These findings offer new insights that could refine and enhance the behavior of related models, paving the way for more accurate physical solutions to numerous natural phenomena.

Furthermore, this paper delves into the stability of the degenerate model via the central quantity

$$Y(t) = \int_\Omega (H(u(x,t)))^\lambda dx,\ \text{for } \lambda \geq 1.$$

It was established that the above quantity remains bounded relative to the initial data under different scenarios. This boundedness forms the foundation for introducing the concept of asymptotic stability for the solutions to the initial-boundary value problem (38)-(40), applicable to both bounded and unbounded cases of u, with more generic H and F functions.

In addition to establishing fundamental and generic stability theorems, Assumption 3.6 was expanded by incorporating additional parameters β_1 and β_2, thus enhancing the flexibility and comprehensiveness of the results on asymptotic stability. The stability analysis is anticipated to be extended to the model incorporating nonlinear force terms and in homogeneous boundary conditions for both bounded and unbounded domains. These developments advance the theoretical framework of the degenerate Einstein-Brownian model and open new avenues for practical applications in various scientific and mathematical contexts.

ORCID

0000-0002-5520-4030 (Isanka Garli Hevage)
0000-0001-6827-8007 (Akif Ibraguimov)
0000-0003-4862-427X (Zeev Sobol)

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Conflicts of Interest

The authors declare no conflicts of interest.

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