Analyticity and Recursion for Multi-Loop Yang-Mills Amplitudes

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Abstract

While much progress has been made in the calculation of two-loop amplitudes in Yang-Mills theory, there remain difficulties in scaling existing methods to higher numbers of external gluons. A method of calculating two-loop all-plus Yang-Mills amplitudes using 4 dimensional unitarity and augmented recursion was previously developed that was successful in calculating amplitudes to high gluon multiplicity. This thesis presents the latest developments in extending this method to the two-loop single-minus sector, taking the previously calculated leading in color two-loop five-point single-minus amplitude as an example. A new technique for calculating the cut-constructible part of this amplitude is presented, with a focus on the 'pseudo one-loop' subsector of the cut-constructible part. We calculate this subsector using one-loop reduction methods, and present a new parameterisation that allows for the determination of the coefficients of the one- and two-mass scalar triangle integrals. The bulk of this thesis focuses on the extension of augmented recursion to the calculation of the rational part of single-minus amplitudes. The method is significantly extended to include sectors which were absent in previous calculations, and we develop novel techniques to aid in calculating Feynman integrals. Although there are still some unanswered questions, we are able to reconstruct the full rational part of the five-point amplitude using augmented recursion and universal known properties of scattering amplitudes.

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed: [redacted] Dated: 02/07/2024

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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I hereby give consent for my thesis, if accepted, to be available for electronic sharing.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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Chapter 1

Introduction

The validity of any theory in physics lives and dies on its ability to accurately predict and describe real life physical phenomena. In order to be accepted, a theory must be able to make measurable predictions which can be verified by experiment.

The area of high-energy physics - or particle physics - is concerned with the discovery of the underlying structure of the universe. This is to say, the uncovering of the most fundamental building blocks of everything we know; the particles that make up matter, and the forces through which matter particles interact with one another to build protons, atoms, molecules and everything we can see and feel.

The modern understanding of particle physics today is encapsulated in the Standard Model: a set of 12 fermions that make up the matter content along with 3 of the 4 fundamental forces through which particles interact with one another, and a Lagrangian that describes exactly how these particles interact.

While the Standard Model has been enormously successful - arguably one of the most accurate and successful theories in all of physics - it still leaves many important questions unanswered: how do we describe gravity at a quantum level? Why is there seemingly more matter than antimatter in the universe? What is the nature of dark matter and dark energy?

In order to try and reconcile these gaps in current understanding, theorists have proposed many different extensions to the Standard Model: supersymmetry, String Theory, WIMPs, SIMPs, to name just a few. All of these theories claim to solve one or another of these problems, but the question remains: which - if any - are correct?

In order to check their validity, we need to extract measurable predictions from these theories which can be verified experimentally. In the field of high - energy physics one of the most popular methods of experiment is particle scattering. In a scattering experiment such as the Large Hadron Collider (LHC), hadrons are accelerated to close to the speed of light and collide, producing swarms of particles which are detected by the numerous detectors of the collider. Many of the particles that are created in the collisions are not detected directly as they will have decayed before this is possible, and these particles must be inferred through their characteristic decay signatures. We can check the validity of a given theory by asking what it would predict the outcome of this collision to be and comparing with the experimental data. We can express the total number of detected events, N_{events}, as

$$N_{\text{events}} = \int dt \frac{dN_{\text{events}}}{dt} = \sigma \int dt \; L = \sigma L_{\text{int}} \tag{1.0.1}$$

where L_{int} is the integrated luminosity which quantifies the intensity of particle collisions within the accelerator, and σ is the cross section which may be broadly thought of as the probability that any given process will occur. Theoretically we can express this cross section as

$$\sigma = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \frac{d\sigma}{d\Omega} \sin\theta = \int d\theta \int d\phi |\mathcal{A}|^2 \sin\theta, \tag{1.0.2}$$

where $\mathcal{A}$ is the scattering amplitude. Thus we can understand the scattering amplitude as an essential bridge between theory and experiment and we can appreciate its importance in particle theory. This thesis will focus on the study of these scattering amplitudes.

Of course, things are rarely so simple. Take for example a very simple process, the scattering of two up quarks, $A(u, u \to u, u)$. There are multiple contributions to this process as the quarks can interact via photons, gluons and the Z boson, as well as loop induced processes like fermion loops. If one wanted to use this process to test for physics beyond the standard model, one would first have to subtract the contributions to the cross section due to known processes before it would be possible to know if there were discrepancies between theory and experiment, and in order to subtract these known processes their amplitudes must be calculated.

In more phenomenologically relevant processes, if one is searching for evidence of new physics, there are instances when it will be necessary to subtract the effects of pure gluon processes, and thus it necessary to calculate the amplitudes of these gluon processes.

In addition, scattering amplitudes are a worthwhile area of research in their own right from a purely theoretical perspective. Scattering amplitudes can reveal hidden structure and relationships that are not at all apparent at the level of the Lagrangian. One striking example of this is the *Double Copy* whereby one can obtain amplitudes in perturbative gravity by "squaring" the corresponding Yang-Mills amplitude [3]. Since gravity amplitudes are in general more complicated to calculate, this correspondence provides a key tool in simplifying such calculations.

Clearly there are many reasons to be interested in the calculation of gluon amplitudes and thus this will be the focus of this thesis. We will be considering in particular pure SU(N) gauge theory, that is to say an SU(N) gauge theory with no matter content. A crucial aspect of such a theory is *asymptotic freedom* [4, 5], which tells us that at high energies the coupling of such a theory becomes weak and thus in the high energy limit particles may scatter via this force and we can use perturbation theory in our calculations.

The scattering amplitude of an n-gluon interaction in SU(N) gauge theory can express as an expansion in the coupling constant, g, as

$$\mathcal{A}_n = ig^{n-2} \sum_{l \geq 0} a \mathcal{A}_n^{(l)} \tag{1.0.3}$$

where $a = \frac{g^2 e^{-\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}$, γ_E is the Euler-Mascheroni constant, ϵ is the dimensional regulator [6], and $\mathcal{A}_n^{(l)}$ is the contribution to the amplitude $\mathcal{A}_n$ and loop order l. This expansion is only possible at high energies where the coupling constant is small, $g << 1$, so that the expansion makes sense, hence the importance of asymptotic freedom. This also means that higher loop calculations provide progressively smaller corrections to the full amplitude so that we can truncate the series with a meaningful understanding of the error this truncation causes.

The traditional method of calculating amplitudes uses so-called Feynman diagrams [7] which are a diagrammatic way of representing the numerous mathematical terms that arise in an amplitude. Starting from the Lagrangian of the theory in question, one may derive "Feynman rules" which dictate the composition of diagrams in a given theory, encoding the various conservation laws of charge, momentum etc and the various particle interactions that are permitted by the theory. To calculate the amplitude of a given process one writes down all possible Feynman diagrams according to the Feynman rules, and adds up the corresponding mathematical expressions.

This method was hugely successful upon its introduction as it provided an easier mnemonic to aid in the often long and cumbersome calculations of scattering amplitudes. It also provided a physical interpretation of these amplitudes: particles will interact in every allowed way and all these possibilities are summed up (with different weightings that roughly correspond to how likely a given interaction is) with internal lines corresponding to virtual "off-shell" particles which do not obey all the rules that measurable particles do, and external physical particles which can be directly measured and are thus constrained by all the physical conservation laws and properties of fields. Feynman diagrams are an excellent pedagogical tool for students of quantum field theory as they provide a more physical interpretation of a generally rather abstract topic.

Despite these successes the method has a number of drawbacks. As one increases the multiplicity (number of external particles in a process) the number of diagrams that need to be summed increases exponentially. For a tree-level $n-$gluon amplitude, the number of diagrams at $n = 4$ is 3, at $n = 7$ is 2485, and at $n = 10$ is 10,525,900. If we could write down one Feynman diagram a minute, we would need 3 minutes for $n = 4$, 41.4 hours for $n = 7$, and over 20 years for $n = 10$. We also note an increase in the number of diagrams as we increase loop-level: as we go from tree to one-loop level, the number of diagrams for the $n = 7$ amplitude increases from 2485 to 227,585. Compounding on this problem is the fact that, as we will discuss later, at loop level Feynman diagrams actually give integrals rather than simple mathematical expressions which must be integrated, and in many cases these integrals actually diverge and so we have the added complication of regulating these divergences.

One might expect that due to the number of diagrams increasing so rapidly, that the final resultant amplitudes must also similarly increase in complexity with increased multiplicity, however this is not the case. When performing these calculations using Feynman diagrams it was noted that while intermediate expressions may be enormous, that the individual diagrams seemingly conspire so that massive cancellations occur and the final expression is far simpler than it has any right to be.

We can understand this cancellation to be a result of gauge dependence. As we touch upon later, we must choose a specific gauge condition to define our Feynman rules, and different choices of gauge change the value of an individual diagram. Since anything measurable must be independent of the choice of gauge, we know that the gauge dependence of individual diagrams must finally cancel in the sum, leading to huge cancellations and simplification.

Clearly then, there is scope for improving on the traditional methods of Feynman diagrams. One possibility is to work as much as possible with gauge invariant substructures of the full amplitude. While the full amplitude is necessarily gauge independent, it may be the case that one could factorise the amplitude into smaller pieces which are individually gauge invariant. This line of thinking was first realised via string theory with the emergence of Chan-Paton factors [8], after which an analogous decomposition was found for tree-level gluon amplitudes [9, 10]. In this decomposition, one could write the full amplitude as a sum of color-ordered amplitudes (also known as partial amplitudes and color-stripped amplitudes) multiplied by a basis of color structures,

$$\mathcal{A}_n^{(0)}(1^{\lambda_1}, 2^{\lambda_2}, ..., n^{\lambda_n}) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma_1}} T^{a_{\sigma_2}}, ..., T^{a_{\sigma_n}}) A_n^{(0)}(\sigma(1)^{\lambda_{\sigma_1}}, \sigma(2)^{\lambda_{\sigma_2}}, ..., \sigma(n)^{\lambda_{\sigma_n}}) \tag{1.0.4}$$

where k^{λ_k} denotes gluon k with momentum p_k and helicity λ_k. The sum is over the non-cyclic permutations of S_n, and T^a are the generators of SU(N) in the adjoint representation. This decomposition of the amplitude splits the color-dependent and kinematic-dependent parts and allows us to consider them separately. As we will see later on similar decompositions exist at loop level. The partial amplitudes are individually gauge invariant, making them easier to work with, and we will later see that they enjoy properties that further aid in the calculation of amplitudes. Due to their gauge invariance, we also do not need to worry about gauge artefacts such as Fadeev-Popov ghosts. For these reasons, this thesis will focus on the calculation of partial amplitudes, and unless stated otherwise we will refer to partial amplitudes simply as amplitudes.

Since these partial amplitudes are now only dependent on momentum and helicity, it is natural to try and express them in the spinor-helicity formalism, introduced in [11–16] for QED calculations and extended to non-Abelian theories in [17], where we will see the amplitudes taken on a strikingly simple form.

1.1 Spinor Helicity Formalism

As we are working with gluons which are spin 1 particles, traditionally we would work in the vector representation . The momentum p of Particle 1 can be written in this representation as a 4-vector

$$p_1^\mu = (p^0, p^1, p^2, p^3)^\mu.$$

The virtues of the 4-vector formalism are too numerous to be stated here, however when we work with massless particles, we find that expressions are often much more compact if we consider a different representation of the momentum.

For a massless particle, we can always boost to a frame

$$p^\mu = (k, 0, 0, k)^\mu$$

so that $p^2 = 0$, and from this frame it is clear that is a mismatch in the number of degrees of freedom in a 4-vector (4), and of a massless particle (2). From this representation of the momentum of a massless particle we see that there is an $SO(2) \cong U(1)$ subgroup of the Lorentz group that leaves the momentum invariant. We refer to this subgroup as the *little group* of the particle. In general, this little group invariance is obscured in the 4-vector formalism.

With this motivation, we introduce a different representation, the *spinor helicity formalism*. In this subsection we will closely follow [18], and all stated results may be found there. We define $p_{\alpha\dot{\alpha}}$ as

$$p_{\alpha\dot{\alpha}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}, \tag{1.1.1}$$

where $\sigma^\mu = (1, \sigma^i)$ and σ^i are the Pauli Matrices. We note that the determinant of the matrix $p_{\alpha\dot{\alpha}}$ is

$$\det(p_{\alpha\dot{\alpha}}) = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m^2, \tag{1.1.2}$$

which vanishes in the case of a massless momentum. A 2x2 matrix can have at most rank two, so in the case of a 2x2 matrix with determinant zero, we know the rank must be one. This means that we can write the matrix p as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{1.1.3}$$

The objects $\lambda, \tilde{\lambda}$ are the *helicity spinors*, and are the objects we will use to represent massless momenta. There are generalisations of the spinor helicity formalism to massive particles [19], but they will not be the focus of this section or thesis.

For real momenta, we require the condition $(\lambda_\alpha)^* = \tilde{\lambda}_{\dot{\alpha}}$ (up to a sign depending on the sign of the energy), however we will see later on that there are times when it is useful to relax the reality condition and consider complex momenta which allows the spinors to be independent. An explicit representation of the spinors is as follows

$$\lambda^\alpha = \frac{1}{\sqrt{p^0+p^3}} \begin{pmatrix} p^0+p^3 \\ p^1+ip^2 \end{pmatrix}, \; \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0+p^3}} \begin{pmatrix} p^0+p^3 \\ p^1-ip^2 \end{pmatrix}, \tag{1.1.4}$$

although one almost never needs to result to using an explicit representation of the spinors and we shall not in this thesis.

The spinors themselves transform in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group, which is locally isomorphic to SL(2,**C**)xSL(2,**C**) as shown in (1.1.1) , so each spinor transforms in SL(2,**C**). In the case of complexified momenta, these are independent, but when restricted to real momenta, the two copies must be related by complex conjugation.

The only invariant tensor of SL(2,**C**) is the totally antisymmetric Levi-Civita symbol, so this is used to raise and lower the spinor indices

$$\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta, \; \tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}, \tag{1.1.5}$$

and also to form Lorentz invariant quantities

$$\begin{aligned} \langle ij\rangle &= \lambda_i^\alpha \lambda_{j\alpha} = \lambda_i^\alpha \lambda_j^\beta \epsilon_{\alpha\beta} \\ [ij] &= \tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}, \end{aligned} \tag{1.1.6}$$

These two objects form the basic building blocks of our amplitudes. Since we are almost always working with these contracted spinors, we drop the λ and just label the particles by i, j etc. Note that due to the antisymmetry of the Levi-Civita symbol, that $[ij] = -[ji]$ and $\langle ij\rangle = -\langle ji\rangle$. Consequently $\langle ii\rangle = [ii] = 0$.

We can use the trace properties of the Pauli matrices to allow us to convert between spinor helicity and 4-vector formalisms, in particular

$$\mathrm{Tr}(\bar{\sigma}_\mu \sigma_\nu) = \epsilon_{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\sigma}_\mu^{\alpha\dot{\alpha}}\sigma_\nu^{\beta\dot{\beta}} = 2\eta_{\mu\nu} \tag{1.1.7}$$

lets us write the Mandelstam variables in terms of spinors as

$$s_{ij} = (p_i + p_j)^2 = p_i^2 + p_j^2 + 2p_i \cdot p_j = \langle ij \rangle [ji] \tag{1.1.8}$$

for massless momenta p_i, p_j. We can also see that

$$2p_i^2 = \langle ii \rangle [ii] = 0$$

so the massless condition of the momenta is encoded into the spinors at a fundamental level.

Immediately, we can see that the spinor representations are not unique, and that $p_{\alpha\dot{\alpha}}$ is invariant under the transformation

$$\lambda \to e^{i\phi}\lambda, \ \tilde{\lambda} \to e^{-i\phi}\tilde{\lambda}. \tag{1.1.9}$$

This symmetry is generated by the U(1) helicity generator

$$h = \frac{1}{2}\sum_{i=0}^{n}\left(-\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}}\right) \tag{1.1.10}$$

where we have assigned the helicity -$\frac{1}{2}$ to λ and +$\frac{1}{2}$ to $\tilde{\lambda}$. In this formalism, the U(1) little group symmetry which was often obscured in the 4-vector formalism is made manifest. We will see later that this is useful as we can construct 3-point amplitudes based solely on little group covariance, which is to say without reference to the Lagrangian of the underlying theory.

We can interpret these spinors as solutions of the massless Dirac equation in momentum space, $\not{p}\psi = 0$. We will work in the chiral representation of the Gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{1.1.11}$$

so

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_{\alpha\dot{\alpha}} \\ p^{\alpha\dot{\alpha}} & 0 \end{pmatrix}, \tag{1.1.12}$$

then

$$|p\rangle = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \text{ and } |p] = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{1.1.13}$$

are solutions which we can identity with the traditional Dirac spinor solutions

$$u_+(p) = v_-(p) = |p\rangle, \; u_-(p) = v_+(p) = |p],$$

where the u and v solutions are identified due to the massless limit.

Due to the two-dimensional space in which the spinors live, any three spinors must be linearly dependent. This means that one can write, say spinor 1, as linear combination of spinors 2 and 3,

$$\lambda_1 = \frac{\langle 13\rangle}{\langle 23\rangle}\lambda_2 - \frac{\langle 12\rangle}{\langle 23\rangle}\lambda_3. \tag{1.1.14}$$

Rearranging this gives rise to the Schouten identity

$$\langle 12\rangle\lambda_3 + \langle 23\rangle\lambda_1 + \langle 31\rangle\lambda_2 = 0, \tag{1.1.15}$$

with an equivalent identity for the conjugate spinors. Finally, using the identity $\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$, we have the Fierz identity,

$$[i|\sigma^\mu|j\rangle\langle k|\sigma_\mu|l] = 2[il]\langle jk\rangle. \tag{1.1.16}$$

It is worth noting that the derivation and discussion above is very much particular to four dimensions. While the spinor helicity formalism has been extended to three, five, six, and ten dimensions [20–22], there is no a priori method to extend the formalism to arbitrary - in particular non-integer - dimensions. While this will pose no issues for tree-level amplitudes, we will have to be careful at loop-level, when we carry out dimensional regularisation in $D = 4 - 2\epsilon$ dimensions.

1.2 Polarization Vectors

Writing down a gluon amplitude requires polarization vectors, so our first step will be to express polarization vectors in the spinor helicity formalism.

Polarization vectors have the following properties

$$\begin{aligned} p\cdot\epsilon_\pm(p) = 0,\ \epsilon_+(p)\cdot\epsilon_-(p) = -1, \\ \epsilon_-(p)\cdot\epsilon_-(p) = \epsilon_+(p)\cdot\epsilon_+(p) = 0,\ (\epsilon_\pm(p))^* = \epsilon_\mp(p), \end{aligned} \tag{1.2.1}$$

which are satisfied by the following vectors in the spinor helicity representation [18]

$$\epsilon_+^{\alpha\dot\alpha}(p) = -\sqrt{2}\frac{\tilde\lambda^{\dot\alpha}\mu^\alpha}{\langle\lambda\mu\rangle},\ \epsilon_-^{\alpha\dot\alpha}(p) = \sqrt{2}\frac{\lambda^\alpha\tilde\mu^{\dot\alpha}}{[\lambda\mu]}, \tag{1.2.2}$$

where $p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$ and $\mu, \tilde\mu$ are arbitrary spinors. By acting on the polarization vectors with the helicity operator (1.1.10) we can also confirm the helicity of each.

The spinors $\mu, \tilde\mu$ we have introduced are totally arbitrary and need not be related to each other, the only restriction being that the denominator of the polarization vector does not vanish. As we are totally free to choose any spinor for these references, the final value of the amplitude cannot depend on the choice of reference spinor.

We can see this in action if we take, say, the negative helicity polarization vector and vary the $\mu \to \mu + \delta\mu$, then

$$\begin{aligned} \epsilon_- &\to \epsilon_- + \delta\epsilon_-, \\ &= \epsilon_- + \sqrt{2}\frac{\lambda^\alpha}{[\lambda\mu]^2}([\lambda\mu]\delta\mu^{\dot\alpha} - [\lambda\delta\mu]\mu^{\dot\alpha}), \\ &= \epsilon_- - p^{\alpha\dot\alpha}\sqrt{2}\frac{[\mu\delta\mu]}{[\lambda\mu]^2}, \end{aligned} \tag{1.2.3}$$

where we use Schouten to get the third line. Considering the effect of this change upon the amplitude, $\mathcal{A} \to \delta\mathcal{A}$, we see that since $\mathcal{A} = \epsilon_\mu\mathcal{A}^\mu$, that under this transformation

$$\delta\mathcal{A} = \delta\epsilon_\mu\mathcal{A}^\mu \propto p_\mu\mathcal{A}^\mu = 0$$

by Ward Identity. Thus this spinor choice corresponds to the freedom of gauge transformations.

1.3 Yang-Mills Theory

Let us now bring our discussion to the specific case of SU(N) Yang-Mills theory, which shall be the focus of this thesis. The Langrangian of pure Yang-Mills theory [23] is

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}], \tag{1.3.1}$$

where F is the field strength tensor due to the gluon field A_μ,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{1.3.2}$$

Both F and A have a color component which we can show explicitly as

$$A_\mu = T^a A^a_\mu, \tag{1.3.3}$$

and thus

$$F^a_{\mu\nu} = \partial\mu A^a_\nu - \partial_\mu A^a_\nu + if^{abc}A^b_\mu A^c_\nu, \tag{1.3.4}$$

where T^a are the generators of SU(N) in the adjoint representation and and f^{abc} are the structure constants of the Lie Algebra

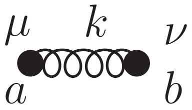
$$[T^a, T^b] = if^{abc}\, T^c, \tag{1.3.5}$$

and by definition, the adjoint generators are $(T^a)^{bc} = f^{abc}$. These generators are a set of N^2-1 matrices of dimension $N \times N$ matrices which are traceless and hermitian. There are times when it may be useful to promote the gauge group from SU(N) to U(N). As we will discuss later, this will give us “decoupling identities” which provide a linear relation between various SU(N) amplitudes.

To proceed from the Lagrangian towards the calculation scattering amplitudes, traditionally we would now derive Feynman rules. To do so we would add a gauge-fixing term to the Lagrangian, for example Lorenz gauge,

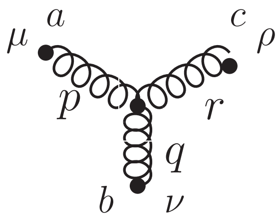
$$\mathcal{L}_{gf} = -\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2$$

as well as Fadeev-Popov ghosts. The resultant Feynman rules in Lorenz gauge and in momentum space are



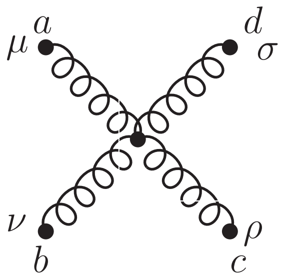
$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 + i\delta}\left(\eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2 + i\varepsilon}\right), \tag{1.3.6}$$

for the gluon propagator, along with



$$iV^{abc}_{\mu\nu\rho}(p, q, r) = gf^{abc}\left[(q - r)_\mu \eta_{\nu\rho} + (r - p)_\nu \eta_{\rho\mu} + (p - q)_\rho \eta_{\mu\nu}\right] \tag{1.3.7}$$

and



$$\begin{aligned} iV^{abcd}_{\mu\nu\rho\sigma} = &- ig^2\Big[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ &+ f^{ace}f^{dbe}(\eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\ &+ f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})\Big]. \end{aligned} \tag{1.3.8}$$

for the three and four-point gluon vertices respectively.

In principle there are also Feynman rules for the Fadeev-Popov ghost propagator and ghost-gluon interaction vertices, however we shall omit them here as they are not required for the purposes of this discussion. Indeed the ghosts and gauge fixing terms are necessary only because individual Feynman diagrams break gauge symmetry and the final gauge invariant amplitude is independent of these terms. In this thesis we shall be working with the spinor helicity formalism throughout. This formalism removes the extra degrees of freedom that ghosts usually account for, and so there is no need to discuss ghosts any further.

To separate the color and kinematic parts of the above Feynman rules we can rewrite the structure constants as a trace over generators,

$$f^{abc} = \mathrm{Tr}\left(T^a\left[T^b, T^c\right]\right) \tag{1.3.9}$$

with normalisation,

$$\mathrm{Tr}[T^a, T^b] = \delta^{ab}. \tag{1.3.10}$$

Then by summing over all diagrams the amplitude decomposes into the form (1.0.4) written above.

1.4 Properties of Partial Amplitudes

As stated previously, we can rewrite the amplitude in the following form at tree level,

$$\mathcal{A}_n^{(0)}(1^{\lambda_1}, 2^{\lambda_2}, ..., n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma_1}}T^{a_{\sigma_2}}, ..., T^{a_{\sigma_n}})A_n^{(0)}(\sigma(1)^{\lambda_{\sigma_1}}, \sigma(2)^{\lambda_{\sigma_2}}, ..., \sigma(n)^{\lambda_{\sigma_n}}) \tag{1.4.1}$$

and in similar forms at loop level, where we have written the expression as a product of a color part, and a 'partial amplitude' which only depends on the momentum and helicity of the particles. A cursory glance at the definition of partial amplitudes might lead us to believe that we have only taken one large problem, calculating the amplitude, and reduced it to a sum of smaller problems, calculating the partial amplitudes, however these partial amplitudes enjoy certain relations among themselves that greatly reduce the number of independent partial amplitudes that must be calculated in a given amplitude. Once again in this section we will follow the discussion in [18].

1. Cyclicity

$$A(1, 2, ..., n) = A(2, ..., n, 1)$$

2. Parity

$$A(\bar{1},\bar{2},...,\bar{n}) = A(1,2,...,n)$$

3. Reflection

$$A^{\text{tree}}(1,2,...,n-1,n) = (-1)^n A^{\text{tree}}(n,n-1,...,2,1)$$

4. Photon Decoupling

$$A^{\text{tree}}(1,2,3,...,n) + A^{\text{tree}}(2,1,3,...,n) + ... + A^{\text{tree}}(2,3,...,n-1,1,n) = 0$$

Let us comment on the above properties. Cyclicity is evident from the definition of partial amplitudes due to the cyclic nature of the color trace. In the parity property, the overline means that the helicity of the particle is flipped. Flipping all of them amounts to a complex conjugation and the amplitude is unchanged. Reflection - or flip symmetry - follows from the anti-symmetry of the Feynman rules. Photon decoupling follows from considering U(N) gauge theory amplitudes. In pure U(N) gauge theory the interactions stem from the commutator in the Lagrangian (1.3.1), however the U(1) generator is proportional to the identity matrix and thus its commutator always vanishes. This means that the total amplitude must vanish if we include a photon and thus we have a sum of partial amplitudes that must vanish. Finally there are the Kleiss-Kuijf relations [24], and BCJ relations [25], which further relate partial amplitudes and hence reduce the numbers of independent partial amplitudes.

There are also certain classes of amplitudes that always vanish. At tree-level all all-plus and single-minus amplitudes (likewise their parity conjugates) must vanish. This can be thought of as a consequence of supersymmetry. Supersymmetric Yang-Mills theories have a gluon, plus extra matter fields defined in such a way that the theories have symmetries that are not present in regular SU(N) theory. Supersymmmetric Ward identities require that at all-loop orders, all-plus and single-minus pure gluon amplitudes must vanish [26]. Since at tree level there are - by definition - no loops, all particles involved in an amplitude must at some point be external, that is to say if there is an electron propagator somewhere in a Feynnman diagram of the amplitude then that electron must exit the diagram as an external particle as the only other option is for it to close back on itself as a loop. The upshot of this that tree level pure gluon amplitudes in supersymmetric and non-supersymmetric theories are indistinguishable and thus must have identical properties, thus our gluon amplitudes at tree level must satisfy supersymmetric Ward identities and thus at tree-level all-plus and single-minus amplitudes must vanish.

For these reasons, the first non-zero tree level pure gluon amplitude as 2 particles of one helicity and $n-2$ particles of the other helicity. We refer to such amplitudes as Maximally Helicity Violating (MHV) amplitudes. We can understand this name as if we consider incoming particles to have incoming momenta then their helicities flip relative to our convention and so we could have, say, all positive helicity particles incoming and all but two outgoing particles having negative helicity. This is the "most" helicity violation we can have without the amplitude vanishing, hence the name. The naming convention is that amplitudes with two negative helicities (-,+,...,+,-,+,...+) are called MHV, while amplitudes with two positive helicities (+,-,...,-,+,-...,-) are called $\overline{\text{MHV}}$ or googly amplitudes.

The n-gluon MHV and googly tree amplitudes can be written as

$$A_n(1^+,2^+,...,i^-,...,j^-,...,n^+) = i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle n-1n\rangle\langle n1\rangle}, \tag{1.4.2}$$

$$A_n(1^-,2^-,...,i^+,...,j^+,...,n^-) = i\frac{[ij]^4}{[12][23]...[n-1,n][n,1]} \tag{1.4.3}$$

and these are the famous Parke-Taylor amplitudes first presented in [27] and proven in [28]. While they were not originally written in the form above using spinor helicity, they serve as a clear example of how the formalism reveals the underlying simplicity of gluon amplitudes. While these could be derived from the Feynman rules, there is in fact another way to do this. Later in this chapter we will construct MHV three-point amplitudes using general properties of amplitudes, and then we will see how these can be used to recursively generate all other tree-level gluon amplitudes.

1.5 Factorization Properties of Scattering Amplitudes

Despite the complexity of amplitudes, there are in fact certain universal properties that all amplitudes satisfy, namely collinear and soft limits. In these limits, the amplitudes will reduce to a lower point amplitude multiplied a term which is universal and dependent only on the helicities of the particles involved and the loop-level.

One such limit is the *collinear limit*. This happens when we consider what happens when we take two adjacent momenta in an amplitude, say p_i and p_j in an Amplitude $A(1,2,...,p-$

$j, p_j, ..., n)$ to be collinear, that is $p_i \to kp_j$ where k is some constant of proportionality. Generally we parameterise this limit by defining $P = p_i + p_j$ and setting $p_i = zP,\ p_j = (1-z)P$, and we write the spinors as $\lambda_i = \sqrt{z}\lambda_P,\ \lambda_j = \sqrt{1-z}\lambda_P$, likewise for the conjugate spinors. In this limit at tree level we have

$$A_n^{(0)}(1, 2, ..., p_i^{h_i}, p_j^{h_j}, ..., n) \to \sum_{\lambda=\pm} A_{n-1}^{(0)}(1, 2, ..., P^{\lambda}, ..., n)\text{Split}_{-\lambda}^{(0)}(z, i^{h_i}, j^{h_j}). \tag{1.5.1}$$

The function "Split" is what is known as a splitting function, and is only dependent on the helicities of the particles in the collinear limit and the helicity of P which of course we must sum over. As it is totally agnostic to the number and helicities of the other particles in the amplitude, these splitting functions are universal.

The tree level splitting functions are [27, 29]

$$\begin{aligned}
\text{Split}_{-}^{(0)}(z, i^-, j^-) &= 0, \\
\text{Split}_{-}^{(0)}(z, i^+, j^+) &= \frac{1}{\sqrt{z(1-z)}\langle ij\rangle}, \\
\text{Split}_{-}^{(0)}(z, i^+, j^-) &= -\frac{z^2}{\sqrt{z(1-z)}[ij]}, \\
\text{Split}_{-}^{(0)}(z, i^-, j^+) &= -\frac{(1-z)^2}{\sqrt{z(1-z)}[ij]}.
\end{aligned} \tag{1.5.2}$$

The P^+ splitting functions can be found from the above through parity

$$\text{Split}_{\lambda}^{(0)}(z, i^{-h_i}, j^{-h_j}) = -\text{Split}_{-\lambda}^{(0)}(z, i^{h_i}, j^{h_j})|_{\langle ij\rangle \leftrightarrow [ij]}. \tag{1.5.3}$$

At loop level the situation is expectedly more complex but still we have a sum of lower-point amplitudes multiplied by universal functions, at one loop we have

$$\begin{aligned}
A_n^{(1)}(1, 2, ..., p_i^{h_i}, p_j^{h_j}, ..., n) \to \sum_{\lambda=\pm} \Big(& A_{n-1}^{(1)}(1, 2, ..., P^{\lambda}, ..., n)\text{Split}_{-\lambda}^{(0)}(z, i^{h_i}, j^{h_j}) \\
& + A_{n-1}^{(0)}(1, 2, ..., P^{\lambda}, ..., n)\text{Split}_{-\lambda}^{(1)}(z, i^{h_i}, j^{h_j}) \Big),
\end{aligned} \tag{1.5.4}$$

and unsurprisingly at two-loops we have

$$\begin{aligned}A_n^{(2)}(1,2,...,p_i^{h_i},p_j^{h_j},...,n) \to \sum_{\lambda=\pm}\Big(&A_{n-1}^{(2)}(1,2,...,P^\lambda,...,n)\text{Split}_{-\lambda}^{(0)}(z,i^{h_i},j^{h_j})\\ &+A_{n-1}^{(1)}(1,2,...,P^\lambda,...,n)\text{Split}_{-\lambda}^{(1)}(z,i^{h_i},j^{h_j})\\ &+A_{n-1}^{(0)}(1,2,...,P^\lambda,...,n)\text{Split}_{-\lambda}^{(2)}(z,i^{h_i},j^{h_j})\Big).\end{aligned} \tag{1.5.5}$$

The one and two-loop splitting functions are too complicated to quote, but can be found in [30].

Another example is the *soft limit*, wherein we send the momentum of an external momentum to zero. In this limit we also see some universal behaviour. In the soft limit

$$A_n^{(0)}(1,2,...,a,s^h,b,...,n) \to A_{n-1}^{(0)}(1,2,...,a,b,...,n)\text{Soft}^{(0)}(a,s^h,b), \tag{1.5.6}$$

where the soft factors are [29, 31, 32]

$$\begin{aligned}\text{Soft}^{(0)}(a,s^+,b) &= \frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle},\\ \text{Soft}^{(0)}(a,s^-,b) &= -\frac{[ab]}{[as][sb]}.\end{aligned} \tag{1.5.7}$$

These properties can be used as powerful consistency checks as in general the universal functions and lower-point amplitudes have already been calculated.

1.6 Three-Point Amplitudes

With the knowledge of how to represent momenta and polarization vectors in spinor-helicity formalism, we are ready to construct amplitudes. One method would be to proceed from the color-stripped Feynman rules as shown above, however in the case of three-point amplitudes we can actually recover the amplitude purely from knowledge of little group scaling. In this section we follow the argument that we can be found in [33], and previously discussed in [34].

First let us recall that by our convention of all momenta outgoing, conservation of momentum for a three particle amplitude tells us that $p_1+p_2+p_3=0$, and since we are working with massless momenta, this means that $s_{12}=s_{23}=s_{31}=0$. Since $s_{ij}=\langle ij\rangle[ji]=0$, and for real momenta $\langle ij\rangle=[ji]^*$, all of the Lorentz invariant quantities vanish and thus the three-point amplitude vanishes.

How then do we proceed? If we *complexify* the momenta, then the two spinors will no longer be connected by complex conjugation and are totally independent. Then by having either $\langle ij\rangle = 0\ \forall i, j$ *or* by having $[ij] = 0\ \forall i, j$, we can satisfy $s_{ij} = [ij]\langle ji\rangle = 0$, while still having non-zero Lorentz invariants to play with.

Now we have two ansatze for our three point amplitudes,

$$\mathcal{M} = c\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2},\ \mathcal{M} = c[12]^{m_3}[23]^{m_1}[31]^{m_2},$$

where the c is a constant made up of coupling constants, normalization factors etc. In the case of complex momentum $p = \lambda\tilde{\lambda}$, p is invariant under

$$\lambda \to z\lambda,\ \tilde{\lambda} \to z^{-1}\tilde{\lambda},$$

which is a generalisation of the the case for real momenta when z was simply a phase. The polarization vectors are momentum dependent and so will also transform as

$$\epsilon_-(p) \to z^2\epsilon_-(p),\ \epsilon_+(p) \to z^{-2}\epsilon_+(p), \tag{1.6.1}$$

so that for a particle i with helicity h_i, there is a rescaling of z^{-2h_i}. Considering this rescaling for all particle, a full amplitude has little group scaling

$$\mathcal{M}(1^{h_1}, \cdots, n^{h_n}) \to \prod_{i=1}^{n} z_i^{-2h_i}\mathcal{M}(1^{h_1}, \cdots, n^{h_n}). \tag{1.6.2}$$

For the case of three point amplitudes, we can perform this rescaling on the above ansatze and the amplitude $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3})$ to obtain the following simultaneous equations

$$\begin{aligned} n_1 &= -m_1 = h_1 - h_2 - h_3, \\ n_2 &= -m_2 = h_2 - h_3 - h_1, \\ n_3 &= -m_3 = h_3 - h_1 - h_2. \end{aligned} \tag{1.6.3}$$

From, say, the formula for the differential cross section we know that the mass dimension of an n-point amplitude is $4-n$, so with these two facts in mind we can show that $A(1^+, 2^+, 3^+)$

and $A(1^-,2^-,3^-)$ have no solutions with the correct mass dimension, and that

$$
\begin{aligned}
A(1^-,2^-,3^-) &\propto \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle},\\
A(1^+,2^+,3^-) &\propto \frac{[12]^3}{[23][31]}.
\end{aligned}
\tag{1.6.4}
$$

Then since s_{ij} has mass dimension two the spinor pairs each have mass dimension 1 so the constants of proportionality are dimensionless. So finally,

$$
\begin{aligned}
A(1^-,2^-,3^-) &= i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle},\\
A(1^+,2^+,3^-) &= i\frac{[12]^3}{[23][31]}.
\end{aligned}
\tag{1.6.5}
$$

where the factor of i is a normalization convention. Immediately we see that these are in total agreement with the Parke-Taylor formula for MHV amplitudes (1.4.2).

Of course these particular forms of the amplitude are far from unique as one can always multiply top and bottom by another spinor pair and use conservation of momentum, Schouten's identity etc to rewrite the expression in any number of ways. We are free to confirm this using the Feynman rules and see that they are in total agreement.

With these tools in hand we are now able to construct all the partial amplitudes we like.

1.7 BCFW Recursion

We can see from Feynman diagrams that tree-level amplitudes are purely rational functions with only simple poles which occur when an internal propagator on-shell. The knowledge of this analytical structure allows us to use recursive methods in order to generate larger-point amplitudes from sums of products of smaller-point amplitudes. Such a process was first introduced by Berends and Giele [28] for the derivation of gluon amplitudes, however this was later improved upon with the introduction of the BCFW (Britto-Cachazo-Feng-Witten) recursion relations which we shall now discuss. This recursion relation was first introduced in [35], and proved in [36] using complex analysis and the properties of tree amplitudes only. The remainder of this section will outline this proof. It is useful as later we will discuss a method of generating the rational parts of loop amplitudes (to be defined later) which is a variation on the theme of BCFW recursion that we will call *augmented recursion*.

Let us begin this discussion with an n-point tree-level gluon amplitude, $A(1,2,...,n)$ where we will leave the helicities undefined for now. Now let us deform the momenta of two of the particles. We choose the legs 1 and n,

$$\begin{aligned}\lambda_1 &\to \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n \\ \tilde{\lambda}_n &\to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1,\end{aligned} \tag{1.7.1}$$

where $z \in \mathbb{C}$. Importantly, we still have momentum conservation as

$$\hat{p}_1^2 = \hat{p}_n^2 = 0, \text{ and } \hat{p}_1 + \hat{p}_n = p_1 + p_n.$$

Now we have defined a complex function $A(\hat{1},2,...,n-1,\hat{n})(z)$ which still satisfies momentum conservation and with massless momenta that admit the spinor helicity formalism. Having defined a complex function the natural next step is to explore its analytic structure: where are the poles of this function?

By considering the amplitude as a sum of Feynman diagrams, we can see that firstly there are no logs and square roots in tree level diagrams and so no branch cuts; secondly that we will only have simple poles in z that arise from propagators that have either p_1 or p_n in them, propagators like

$$\frac{1}{(\hat{p}_1(z) + p_2 + ...)^2}, \tag{1.7.2}$$

but importantly, propagators that have both p_1 and p_n do not have poles as the z dependence cancels. If we define

$$\hat{P}_i(z) = \hat{p}_1(z) + p_2 + ... + p_{i-1}, \tag{1.7.3}$$

then our propagators with poles in z will be of the form

$$\frac{1}{\hat{P}_i(z)^2} = \frac{1}{(\hat{p}_1(z) + p_2 + ... + p_{i-1})^2} = \frac{1}{P_i^2 - z[1|P_i|n\rangle}, \tag{1.7.4}$$

and the corresponding poles in z will be at

$$z_{P_i} = \frac{P_i^2}{[1|P_i|n\rangle}, \tag{1.7.5}$$

keeping in mind that due to momentum conservation we can also write poles involving p_n in the above form.

Now that we have classified the analytic structure of the function and we know the location of the poles, the next natural question to ask is what the residues are at each pole. As we move toward the pole, we are setting $\hat{P}_i(z)^2$ to zero, ie we are putting this propagator on-shell. This causes the amplitude to factorise into two lower point subamplitudes where the propagator now becomes an external state and we must sum over the helicities of the particles being propagated, in this case we sum over the positive and negative helicities of the gluon. One can think of this as inserting a complete set of states in place of the on-shell propagator. So we have

$$\lim_{z \to z_{P_i}} A(z) = \frac{1}{z - z_{P_i}} \frac{-1}{[1|P_i|n\rangle} \sum_s A^L(\hat{1}(z_{P_i}), 2, ..., i-1, -\hat{P}^s(z_{P_i})) A^R(\hat{P}^{-s}(z_{P_i}), i, ..., \hat{n}(z_{P_i})), \tag{1.7.6}$$

where all the momenta with hats are explicitly z dependent and are evaluated at $z = z_{P_i}$. As mentioned above we are summing over the helicity states $s = \pm 1$, and since our convention is to always have outgoing momentum, the helicities - a projection of spin onto momenta - will have opposite signs in each subamplitude.

Our end goal is of course to recover the original amplitude free of deformation, $A(z = 0)$, which we can write as

$$A(1, 2, ..., n) = A(z = 0) = \oint_C \frac{dz}{2\pi i} \frac{A(z)}{z} \tag{1.7.7}$$

using the residue theorem, where our contour encircles the pole at the origin $z = 0$ but does not encircle any of the poles $z = z_{P_i}$ of $A(z)$. We can now expand this circle off to infinity, and since we will encircle the poles of $A(z)$, we can subtract off the residues of these poles. In other words

$$A(z = 0) = \text{Res}\left(\frac{A(z)}{z}, z = \infty\right) - \sum_{i=2}^{n-1} \text{Res}\left(\frac{A(z)}{z}, z = z_{P_i}\right) \tag{1.7.8}$$

We know that at the poles $z = z_{P_i}$, we have

$$\lim_{z = z_{P_i}} \frac{A(z)}{z} = \frac{1}{z - z_{P_i}} \frac{-1}{P_i^2} \sum_s A^L(\hat{1}(z_{P_i}), 2, ..., i-1, -\hat{P}^s(z_{P_i})) A^R(\hat{P}^{-s}(z_{P_i}), i, ..., \hat{n}(z_{P_i})), \tag{1.7.9}$$

so if we assume that $A(z) \to 0$ as $z \to \infty$ (we will come back to this later), then we have the BCFW recursion relation [36]

$$A(1,2,...,n) = \sum_{i=2}^{n-1} \sum_s A^L(\hat{1}(z_{P_i}),2,...,i-1,-\hat{P}^s(z_{P_i}))\frac{1}{P_i^2}A^R(\hat{P}^{-s}(z_{P_i}),i,...,\hat{n}(z_{P_i})). \tag{1.7.10}$$

Note that while in principle there will be $n-2$ poles which each contribute to the sum and two states to consider in each resultant factorisation, in many cases the actual number of calculations to be performed will be much lower as many of the subamplitudes will vanish, eg since tree amplitudes need at least two legs of a given helicity, we need not even consider factorisations that would give us an all-plus or single-minus subamplitude as this would automatically vanish.

Let us now return the residue at infinity. In order for the recursion relation to work we need this term to be zero. If we deform momenta $(\lambda_i, \tilde{\lambda}_j)$ as in Eq 1.7.1, where we denote the helicities of these momenta as (h_i, h_j), then in the same paper BCFW showed that in Yang-Mills Theory $A(z)$ vanishes at infinity for $(h_i, h_j) = (-,+),(+,+),(-,-)$ and so the recursion relations hold for these choices of deformations. While not directly relevant to this thesis, it has been shown in [37] that gauge and gravity tree amplitudes can be calculated by BCFW recursion in any number of dimensions as a helicity choice can always be found in which $A(z)$ vanishes at infinity.

The successes of this method cannot be overstated. For example one can prove the Parke-Taylor formula for n-point MHV tree amplitudes (1.4.2) as a simple exercise in induction using BCFW recursion. The method has also been extended to recursively calculate the integrands of loop amplitudes [38], but more relevant to this thesis is that this method can also be extended to calculating the finite rational parts of loop amplitudes (parts with no branch cuts that are finite in the dimensional regulator). When we apply this method to two-loop Yang-Mills amplitudes we will come across a stumbling block in the form of double poles in $A(z)$. We will later discuss why this is an issue, and the bulk of thesis will discuss how we can augment the technique to get around this problem.

1.8 Loop-Level Amplitudes

So far our discussion has been restricted to tree level amplitudes. Such amplitudes have a particularly simple analytic structure, with only simple poles that arise when propagators go on-shell. This situation is more complicated at the loop level, where in addition to poles we now also have branch cuts. Additionally the presence of a closed loop means that we have an internal momentum which we must integrate over. This often leads to loop integrals diverging due to particular parts of the integration region: ultraviolet (UV) divergences in the high momentum regions, and infrared (IR) divergences that occur when the loop momentum vanishes and in regions where the loop momentum goes collinear to external momenta. We therefore need to understand how to regulate these divergences to obtain sensible physical results which we will do using Dimensional Regularization (DimReg) [6]. This is of course all in addition to the original problem of the sheer number of diagrams, each which would need to be calculated but now with the added complexity of having to regulate each diagram in a consistent way. This section will deal with the above issues, beginning with a discussion on divergences and how to deal with them, before moving on to various ways of simplifying the evaluation of loop amplitudes. In this thesis we will be working in the mostly-minus signature of the Minkowski metric, $\eta = (+,-,-,-)$

1.9 Loop-Level Color Structures

In the introduction we showed that at tree level we can factorise the amplitude into a color piece and a kinematic piece (1.0.4). Such a factorisation also exists at loop level, albeit in a more complex form. At one-loop level we have [39],

$$\mathcal{A}_n^{(1)}(1^{\lambda_1},2^{\lambda_2},...,n^{\lambda_n}) = \sum_{\sigma\in S_5/Z_5} N_c \mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_n})A_{n:1}^{(1)}(a_1^{\lambda_1},a_2^{\lambda_2},...,a_n^{\lambda_n})$$
$$+\sum_{r=2}^{\lfloor n/2+1\rfloor}\sum_{\sigma\in S_5/(Z_{r-1}\times Z_{n-r+1})}\mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_{r-1}})\mathrm{Tr}(T^{b_r}T^{b_{r+1}}...T^{b_n})A_{n:r}^{(1)}(a_1^{\lambda_1},...,a_{r-1}^{\lambda_{r-1}};b_r^{\lambda_r},...,b_n^{\lambda_n}) \tag{1.9.1}$$

If n is even, then there is an additional Z_2 symmetry to be factored out to prevent double counting of this term.

The term $A_{n:1}^{(1)}$ is often referred to as the "leading in color" term, as in the Large N_c

limit this will be the dominant term, and it is generally the simplest of the subamplitudes to calculate.

The amplitudes again have a cyclic symmetry due to the cyclicity of the traces, in the leading term the amplitude is cyclic across all the momenta, whereas in the subleading terms the cyclicity is among the a momenta and b momenta separately.

Of particular importance are the decoupling identities [39]. If we once more consider a U(N) gauge theory and set leg 1 to be a photon, then the vanishing of the coefficient of $\mathrm{Tr}[T^2T^3...T^n]$ tells us

$$A^{(1)}_{n:2}(1,2,3,...,n)+A^{(1)}_{n:1}(1,2,3,...,n)+A^{(1)}_{n:1}(2,1,3,...,n)+...+A^{(1)}_{n:1}(2,3,...,1,n)=0 \quad (1.9.2)$$

and thus we can express $A^{(1)}_{n:2}$ in terms of $A^{(1)}_{n:1}$.

By taking more legs to be U(1) we can express each of the $A^{(1)}_{n:r}$ in terms of linear combinations of $A^{(1)}_{n:1}$, so we can take legs $\{1...r-1\}$ to be U(1) then extract the coefficient of $\mathrm{Tr}[T^r...T^n]$ to get

$$A^{(1)}_{n:r}(1,...,r-1;r,...,n)=(-1)^{r-1}\sum_{\sigma\in COP\{\alpha\}\{\beta\}} A^{(1)}_{n:1}(\sigma), \quad (1.9.3)$$

where $\alpha_i \in \{\alpha\} = \{r-1, r-2, ..., 2, 1\}$ (note the reverse order), and $\beta_i \in \{\beta\} = \{r, r+1, ..., n\}$. The argument $COP\{\alpha\}\{\beta\}$ is defined to be the set of all permutations of $(1,...,n)$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and likewise with $\{\beta\}$, while allowing for all possible relative orderings between $\{\alpha\}$ and $\{\beta\}$.

The upshot of this is that despite looking more complicated than the tree level factorisation, we are in fact in the same position where we only need to calculate a single subamplitude, $A_{n:1}(1,2,...,n)$ and we have all the information we need to calculate the full color SU(N) amplitude. Even more convenient is the fact that, as mentioned earlier, this is often the simplest of the subamplitudes to calculate anyway.

At two loops the color trace basis is

$$
\begin{aligned}
&\mathcal{A}_n^{(2)}(1,2,...,n) = N_c^2 \sum_{S_n/Z_n} \mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_n})A_{n:1}^{(2)}(a_1,a_2,...,a_n)\\
&+ N_c \sum_{r=2}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_5/(Z_{r-1}\times Z_{n-r+1})} \mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_{r-1}})\mathrm{Tr}(T^{b_r}T^{b_{r+1}}...T^{b_n})A_{n:r}^{(2)}(a_1,...,a_{r-1};b_r,...,b_n)\\
&+ \sum_{s=1}^{\lfloor n/3\rfloor} \sum_{t=s}^{\lfloor (n-s)/2\rfloor} \sum_{\sigma\in S_5/(Z_s\times Z_t\times Z_{n-s-t})} \mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_s})\mathrm{Tr}(T^{b_{s+1}}T^{b_{s+2}}...T^{b_{s+t}})\mathrm{Tr}(T^{c_{s+t+1}}T^{c_{s+t+2}}...T^{c_n})\\
&\times A_{n:r}^{(1)}(a_1,...,a_s;b_{s+1},...,b_{s+t};c_{s+t+1}...c_n4)\\
&+ \sum_{S_n/Z_n} \mathrm{Tr}(T^{a_1}T^{a_2}...T^{a_n})A_{n:1B}^{(2)}(a_1,a_2,...,a_n),
\end{aligned}
\tag{1.9.4}
$$

where again whenever there are two traces of the same length, we factor out an extra Z_2 symmetry to prevent double counting, and an S_3 symmetry in the case that the triple trace term has all three trace terms of the same size.

Once again the the subamplitudes inherit all the flip symmetry and cyclicity due to their color trace structures, and while there are U(1) decoupling identities it is no longer the case that all subamplitudes are linearly dependent, and thus for the first time we have no choice but to calculate terms which are subleading in color if we want to calculate the full color amplitude. In this thesis we will only focus on the leading in color term, however there have been two-loop amplitudes calculated to all orders in color [40–43].

1.10 Ultraviolet and Infrared Divergences

Divergences occur in loop amplitudes such as the scalar bubble,

$$
\int \frac{d^4 l}{(2\pi)^4}\frac{1}{l^2(l-p_1)^2}, \tag{1.10.1}
$$

which is divergent at large l but finite at small l due to its $dl/l \sim \log(l)$ scaling, and the scalar box,

$$
\int \frac{d^4 l}{(2\pi)^4}\frac{1}{l^2(l-p_1)^2(l-p_{12})^2(l-p_{123})^2}, \tag{1.10.2}
$$

which is finite for large l but diverges for small l due to its $dl/l^5 \sim 1/l^4$ scaling. Additionally there are collinear divergences in the regions where l is collinear to a massless external momentum scale, for example in the above integral if p_1 is massless then when l is collinear with p_1, so $l = kp_1$, then $(l-p_1)^2 = (k-1)^2 p_1^2 = 0$. We are using the Feynman prescription to take the poles off the real axis, $\frac{1}{p^2} \to \frac{1}{p^2+i\varepsilon}$, however we will not explicitly write this term unless relevant.

In order to make sense of these integrals and extract physically relevant information, we need to regularise them. The procedure we will use is Dimensional regularisation [6], where we analytically continue the integrals to be evaluated not in 4 dimensions, but in $D = 4-2\epsilon$ dimensions. This allows us to represent the divergences in the integrals as poles in the regulator ϵ. This will however mix the UV and IR divergences as both will appear as poles in ϵ. There are some subtleties as to exactly which momenta to continue to D dimensions but for now we will gloss over this and assume that all vectors are now in D dimensions.

We now add counterterms to the Lagrangian to cancel the poles from the amplitude. There are a number of ways to define such counterterms, so we use the $\overline{\text{MS}}$ scheme. The counterterm for the coupling constant g can be thought of as an expansion of g in ϵ,

$$g \to g + \frac{\delta g}{\epsilon} + ...,$$

and expanding an amplitude with this coupling constant to two loops,

$$\begin{aligned}\mathcal{A}_n = &\left(\frac{g}{4\pi}\right)^{n-2} \mathcal{A}_n^{(0)} + \left(\frac{g}{4\pi}\right)^{n} \mathcal{A}_n^{(1)} + \left(\frac{g}{4\pi}\right)^{n+2} \mathcal{A}_n^{(2)} \\ &+ (n-2)\left(\frac{g}{4\pi}\right)^{n-3} \frac{\delta g}{\epsilon} \mathcal{A}_n^{(0)} + n\left(\frac{g}{4\pi}\right)^{n-1} \frac{\delta g}{\epsilon} \mathcal{A}_n^{(1)} + \cdots ,\end{aligned} \tag{1.10.3}$$

then at one-loop the UV divergent part of the amplitude is

$$\mathcal{A}^{(1)}|_{\text{UV}} = -\frac{(n-2)(4\pi)^{\epsilon}}{2\epsilon\Gamma(1-\epsilon)} \left(\frac{g}{4\pi}\right)^2 \beta_0 \mathcal{A}^{(0)}, \tag{1.10.4}$$

where $\beta_0 = \frac{11N_c}{3}$ [44]. From the perturbative expansion we see that in order for this divergence to cancel we require

$$\mathcal{A}^{(1)}|_{\text{UV}} + (n-2)\left(\frac{g}{4\pi}\right)^{-3} \frac{\delta g}{\epsilon} \mathcal{A}_n^{(0)} = 0, \tag{1.10.5}$$

which fixes

$$\delta g = \frac{(4\pi)^{\epsilon}}{2\Gamma(1-\epsilon)} \left(\frac{g}{4\pi}\right)^5 \beta_0 \tag{1.10.6}$$

We should note that divergences will appear for the first time at Next-To-Leading Order

(NLO), and that this need not be at the one-loop level. In fact in the case of the single-minus pure gluon amplitude - the focus of this thesis - the tree-level amplitude vanishes, so that the leading order term is the one-loop amplitude which is rational and the two-loop correction is the first term to have poles which must be removed with counterterms proportional to the one-loop amplitude. This means

$$\mathcal{A}^{(2)}|_{\mathrm{UV}} + n\left(\frac{g}{4\pi}\right)^{-3}\frac{\delta g}{\epsilon}\mathcal{A}_n^{(1)} = 0, \tag{1.10.7}$$

and we have already calculated δg so we know that

$$\mathcal{A}^{(2)}|_{\mathrm{UV}} = -n\frac{(4\pi)^{\epsilon}}{2\epsilon\Gamma(1-\epsilon)}\left(\frac{g}{4\pi}\right)^{2}\beta_0\,\mathcal{A}^{(1)}. \tag{1.10.8}$$

The IR singularities come in two forms: soft and collinear. The leading collinear singularities for the all-plus and single-minus amplitudes take the form [44]

$$\mathcal{A}^{(2)}|_{\mathrm{collinear}} = -n\left(\frac{g}{4\pi}\right)^{2}\frac{\beta_0}{2\epsilon}\mathcal{A}^{(1)}. \tag{1.10.9}$$

Notably, upon expanding in ϵ, we see that the UV and IR collinear singularities cancel each other out and so all-plus and single -minus amplitudes contain only soft IR divergences. An analysis of the soft IR divergences of two-loop QCD amplitudes was carried out in [45] which shows that the leading IR divergence of an amplitude is

$$\mathcal{A}^{L} \sim \left(\frac{1}{\epsilon}\right)^{2L} \tag{1.10.10}$$

where L signifies the L^{th} order correction rather than the number of loops. In our case, the single-minus amplitude, the leading order amplitude is one-loop so $L = 0$ denotes the one-loop amplitude and so at two loops we have at most ϵ^{-2} poles.

Explicitly, the IR divergent piece at first order correction can be written as

$$\mathcal{A}_n^{1}|_{\mathrm{IR}} = \mathcal{A}_n^{0} \times I_n^{1}, \tag{1.10.11}$$

where

$$I_n^{1} = \frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{s_{i,i+1}}\right)^{\epsilon}, \tag{1.10.12}$$

and $s_{n,n+1} = s_{n,1}$. Since we know a priori the divergent structure of the amplitude, it remains to calculate the finite part. Later we will use the fact that we know the divergent structure as a check on the cut-constructible part of the amplitude.

1.11 Integral Reduction

From Passarino-Veltman reduction [46], we know that any one-loop tensor integral in $D = 4-2\epsilon$ dimensions can be expressed up to order $\mathcal{O}(\epsilon)$ as linear combination of scalar box, triangle, bubble, and tadpole integrals, plus a rational piece. That is, for any one loop integral I, we can say

$$I = \sum_{j_4} c_{4(j_4)} I_4^{(j_4)} + \sum_{j_3} c_{3(j_3)} I_3^{(j_3)} + \sum_{j_2} c_{2(j_2)} I_2^{(j_2)} + \sum_{j_1} c_{1(j_1)} I_1^{(j_1)} + \mathcal{R} + \mathcal{O}(\epsilon) \qquad (1.11.1)$$

where I_n denotes a scalar n-point integral. These are

$$\begin{aligned} I_1 &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1}, \\ I_2 &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2}, \\ I_3 &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2 d_3}, \\ I_4 &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2 d_3 d_4}, \end{aligned} \qquad (1.11.2)$$

where $d_i = (l+q_i) - m^2 + i\varepsilon$ and $q_i = \sum_{i=0}^{N} k_i$, k_i being the outgoing momentum at each vertex.

Note that the sum is over all possible boxes, triangles etc. For example, If I is a five-point integral there are four ways to sort five momenta into four sets, and so there are four scalar boxes to be summed over, each with a different coefficient.

While the tadpole integral I_1 is included for full generality, it vanishes in theories where all the particles are massless as there are no mass scales. Since we are working in pure gauge theory with a massless gluon we will not encounter the tadpole and thus from here on we shall not mention it.

Thus one loop amplitudes in their capacity as one loop integrals may be expressed as a linear combination of scalar box, triangle and bubble integrals, plus a rational piece,

$$\mathcal{A} = \sum_{j_4} c_{4(j_4)} I_4^{(j_4)} + \sum_{j_3} c_{3(j_3)} I_3^{(j_3)} + \sum_{j_2} c_{2(j_2)} I_2^{(j_2)} + \sum_{j_1} c_{1(j_1)} I_1^{(j_1)} + \mathcal{R} + \mathcal{O}(\epsilon) \tag{1.11.3}$$

Since these standard integrals have already been computed, we have reduced the problem of calculating an amplitude to finding the values of the rational coefficients. The next section discusses how this is done. It is worth remarking that the above set of integrals is valid up to $\mathcal{O}(\epsilon)$. If one is interested in retaining a finite ϵ - ie in the case of D dimensional unitarity - then one must also include the scalar pentagon integral,

$$I_5 = \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2 d_3 d_4 d_5}. \tag{1.11.4}$$

The above integrals have been solved for various cases of massive and massless external momenta. We shall state the results that will be relevant in this thesis below, namely the massive bubble integral (the massless bubble vanishes in dimensional regularisation), the one- and two-mass triangles, and the one-mass box integral [47]. In this context by 'massive' we mean that a single vertex will have more than one outgoing gluon.

Beginning with the bubble, we have

$$I_2(k_1^2) = \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2 (l-k_1)^2} = i \frac{c_\Gamma}{\epsilon(1-2\epsilon)} (-k_1^2)^{-\epsilon} = i c_\Gamma \left(\frac{1}{\epsilon} - \ln(-k_1)^2 + 2 \right). \tag{1.11.5}$$

For the triangles, the integral

$$I_3(k_1^2, k_2^2) = \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2 (l-k_1)^2 (l-k_2)^2}, \tag{1.11.6}$$

where we assume that $k_1^2 \neq 0$, can be solved in three separate cases: the one-mass case in which $k_2^2 = (k_1+k_2)^2 = 0$ and we have

$$I_3(k_1^2, k_2^2 = 0) = -i \frac{c_\Gamma}{\epsilon^2} (-k_1^2)^{-1-\epsilon} = -i \frac{c_\Gamma}{(-k_1^2)} \left(\frac{1}{\epsilon^2} - \frac{\ln(-k_1^2)}{\epsilon} + \frac{\ln^2(-k_1^2)}{2} \right), \tag{1.11.7}$$

the two-mass case where $k_2^2 \neq 0$ but $(k_1+k_2)^2 = 0$ and we have

$$\begin{aligned} I_3(k_1^2, k_2^2) =& - i\frac{c_\Gamma}{\epsilon^2}\frac{(-k_1^2)^{-\epsilon} - (-k_2^2)^{-\epsilon}}{(-k_1^2) - (-k_2^2)} \\ =& - i\frac{c_\Gamma}{(-k_1^2) - (-k_2^2)}\left(-\frac{\ln(-k_1^2) - \ln(-k_2^2)}{\epsilon} + \frac{\ln^2(-k_1^2) - \ln^2(-k_2^2)}{2}\right), \end{aligned} \tag{1.11.8}$$

and the three-mass case in which k_1, k_2 and (k_1+k_2) are all masssive, which will not appear at five points.

Finally we have the one-mass box where we shall assume k_4 to be massive [48],

$$\begin{aligned} I_4(s_{12}, s_{23}, k_4^2) =& \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{1}{l^2(l-k_1)^2(l-k_1-k_2)^2(l+k_4)^2} \\ =& i\frac{c_\Gamma}{s_{12}s_{23}}\Bigg[\frac{2}{\epsilon^2}\left((s_{12})^{-\epsilon} + (s_{23})^{-\epsilon} + (k_4^2)^{-\epsilon}\right) \\ & - 2\mathrm{Li}_2\left(1 - \frac{k_4^2}{s_{12}}\right) - 2\mathrm{Li}_2\left(1 - \frac{k_4^2}{s_{23}}\right) - \ln^2\left(\frac{-s_{12}}{s_{23}}\right) - \frac{\pi^2}{3}\Bigg]. \end{aligned} \tag{1.11.9}$$

In the above expression, Li_2 denotes the dilogarithm function, defined as

$$\mathrm{Li}_2(z) = -\int_0^z dt\, \frac{\ln(1-t)}{t}. \tag{1.11.10}$$

1.12 Unitarity

Since we have a known basis of integrals to describe a one-loop amplitude, we can ask if it is possible to leverage the analytical properties of these integrals to isolate individual integrals and thus evaluate their coefficients. The unitarity method [48, 49], does exactly this.

First we must introduce the concept of the S-Matrix, S [50], which we can think of as a matrix whose elements give the scattering amplitudes in a theory. If we have a state of incoming particles, $|\mathrm{in}\rangle$, and a state of outgoing particles, $|\mathrm{out}\rangle$, then the scattering amplitude associated to this scattering event is

$$\mathcal{A}(\mathrm{in} \to \mathrm{out}) = \langle \mathrm{out}|S|\mathrm{in}\rangle. \tag{1.12.1}$$

The conservation of probability implies that the S-Matrix is unitary, or

$$S^\dagger S = \mathbb{I}, \tag{1.12.2}$$

where $\mathbb{I}$ is the identity matrix. We can write $S = 1 + iT$, where T is the non-trivial interacting part of the S matrix, and expand T perturbatively in the coupling constant to get

$$T = g^2 T^{(0)} + g^4 T^{(1)} + g^6 T^{(2)}, \tag{1.12.3}$$

the superscript denoting the loop order. We can now insert this expansion of T into the unitarity condition for S. Extracting coefficients of the coupling gives us

$$T^{(0)\dagger} = T^{(0)}, \qquad -i(T^{(1)} - T^{(1)\dagger}) = T^{(0)} T^{(0)}. \tag{1.12.4}$$

We can introduce states to describe matrix elements $\langle \text{out}|T|\text{in}\rangle = T_{oi}$, then

$$\langle \text{out}|T^\dagger|\text{in}\rangle = \langle \text{in}|T^\dagger|\text{out}\rangle^* = T^*_{io} \tag{1.12.5}$$

and thus we see that the second relation above gives us Cutkosky's rule [51],

$$-i(T^{(1)}_{oi} - T^{(1)*}_{io}) = \int d\mu \; T^{(0)}_{o\mu} T^{(0)}_{\mu i} \tag{1.12.6}$$

where the integral $d\mu$ stands for the integration over phase space, as well as the sum over all possible helicities and particle species of on-shell states in the given theory. From time translation invariance, we can say $T^*_{io} = T^*_{oi}$, and thus the left hand side of the above equation is $2\text{Im}T_{oi}$, the imaginary part of the matrix element.

We know that loop level amplitudes are more complicated than those at tree level, instead of being a set of rational functions, loop level amplitudes are generally expressed in terms of logarithms, polylogarithms and other special functions of the kinematical invariants of the theory. These functions are characterised by the presence of branch cut discontinuities, the most elementary example of this being $\text{Log}(x)$ which has a discontinuity of $2\pi i$ when crossing the branch cut. We can understand this discontinuity as arising from regions of the loop-momentum integration space where a virtual gluon goes on-shell and thus its propagator becomes fully imaginary. It is from these discontinuities that the imaginary part of the amplitude arises.

We can better understand this statement with the distributional relation

$$\frac{1}{p^2 \pm i\varepsilon} = P\left(\frac{1}{P^2}\right) \mp i\pi\delta(p^2), \tag{1.12.7}$$

where P denotes the principle part. Thus by replacing the two propagators by delta functions we can see the discontinuity for the loop amplitude. This is expressed in terms of products of tree amplitudes, and thus we are able to leverage our knowledge of tree amplitudes to calculate loop amplitudes. It is in this sense, that we talk about 'cutting' propagators.

In general a single one-loop amplitude will have branch cuts in all the Mandelstam invariants, for example the one loop-four point amplitude $A_4^{(1)}(1,2,3,4)$ has both an $s = (p_1 + p_2)^2$ and a $t = (p_2 + p_3)^2$ channel and thus we would expect discontinuities in both channels. Thus the unitarity program works by considering cuts in all possible momentum channels, and using that information to reconstruct the loop amplitude. In [48, 49], the authors describe this method and use it to calculate a number of one-loop amplitudes in supersymmetric Yang-Mills.

While this has undoubtedly been successful, it does have some limitations. Let us recall from earlier that all one-loop amplitudes can be written as a sum of box, triangle, and bubble integrals plus a rational piece

$$\mathcal{A} = \sum_{j_4} c_{4(j_4)} I_4^{(j_4)} + \sum_{j_3} c_{3(j_3)} I_3^{(j_3)} + \sum_{j_2} c_{2(j_2)} I_2^{(j_2)} + \sum_{j_1} c_{1(j_1)} I_1^{(j_1)} + \mathcal{R} + \mathcal{O}(\epsilon) \tag{1.12.8}$$

and with the integrals having already been calculated, evaluating a given integral is simply a matter of evaluating the rational coefficients of each integral. Naturally, one method of doing this might be to compare discontinuities in the amplitude via the unitarity method, with discontinuities in the integral functions due to branch cuts. By taking various double cuts one could *in principle* determine the coefficients and thus evaluate the amplitude. There is, however, a catch. In general each integral will be a function of multiple Mandelstam invariants and thus a single integral may appear in multiple cuts, conversely a single cut will generally receive contributions from multiple integrals. So having performed these cuts one would therefore have to try and disentangle these various contributions which can quickly become every complicated. This issue can be resolved by the introduction of generalized unitarity.

1.13 Generalized Unitarity

The method of unitarity involved the cutting of propagators across a single kinematic channel, and in [52], the authors developed the method of *generalized unitarity*, which takes the method a step further by considering the cutting of multiple propagators simultaneously across different kinematic channels. The authors used this method to calculate one-loop amplitudes in $\mathcal{N} = 4SYM$ which are fully described by scalar box functions. Of course the method can also be used to calculate amplitudes in non-supersymmetric Yang-Mills as we will show. The idea is as follows: we begin with a *maximal cut*, cutting as many propagators simultaneously as we can - in the case of one-loop amplitudes a quadruple cut - then we do triple and double cuts subtracting residual contributions from higher order cuts at each stage. We will simply sketch the outline of the idea here and will return to specifics when it comes to performing a concrete calculation.

If we think of the cut as replacing a propagator with a delta function in the integrand then performing four cuts will isolate the specific box integral that contains the four cut propagators. If an integral does not have all four of the cut propagators it will vanish, for example if we have a scalar box with propagators (D_1, D_2, D_3, D_4) and we cut (D_1, D_2, D_3, D_5), then

$$\left.\frac{1}{D_1 D_2 D_3 D_4}\right|_{cut} = \left.\frac{D_5}{D_1 D_2 D_3 D_4 D_5}\right|_{cut} \to \frac{\delta(D_1)\delta(D_2)\delta(D_3)D_5\delta(D_5)}{D_4} = 0, \tag{1.13.1}$$

since $x\delta(x) = 0$. The same is of course true for triangle and bubble diagrams which necessarily will not have all the propagators of a maximal cut.

For the box integral that is isolated we have

$$\begin{aligned} A^{(1)}(1,2,...,n)|_{cut} = &\sum_{\lambda} A^{(0)}(-l_4^{-\lambda_4}, 1, ..., i-1, l_1^{\lambda_1}) A^{(0)}(-l_1^{-\lambda_1}, i, ..., j-1, l_2^{\lambda_2}) \\ &\times A^{(0)}(-l_2^{-\lambda_2}, j, ..., k-1, l_3^{\lambda_3}) A^{(0)}(-l_3^{-\lambda_3}, k, ..., n, l_4^{\lambda_4}) \\ &= c_{4(i)} I_4^{(i)}[P^2_{i...i-1}, P^2_{i,...,j-1}, P_{j,...,k-1}, P^2_{k,...,n}]\big|_{cut} \end{aligned} \tag{1.13.2}$$

where l_i are the on-shell momenta of the cut propagators which can be evaluated, and the sum is over the possible helicities of each particle. The particular choice of cuts partitions the n external momenta in four sets, and for each set $\{a, ..., b\}$, we have $P^2_{a,...,b} = (p_a + p_{a+1} + ..., +p_b^2)$. Thus we have evaluated the coefficient $c_{4(i)}$. By considering all the possible partitions of the

n momenta we can perform all the possible cuts and evaluate the box coefficients. At first glance it may sound like the number of cuts will rise quickly with increasing n it is worth remembering that in many cases at least one of the tree amplitudes will vanish and thus the coefficient will be zero. This means that we only need to worry about the subset of partitions where the tree amplitudes are non vanishing, ie all are at least MHV.

After the box coefficients, we now need to evaluate the triangle and bubble coefficients. We can isolate a single triangle integral from the rest by the specific choice of triple cut, much in the same way as with boxes, and again for the same reasons as before all bubbles will vanish. The catch is however that there will be (in general multiple) boxes which have the three propagators we choose to cut and thus will survive the cutting process. This means that performing a triple cut will give

$$\begin{aligned} A^{(1)}(1,2,...,n)|_{triplecut} = & \sum_{\lambda} A^{(0)}(-l_3^{-\lambda_3}, 1, ..., i-1, l_1^{\lambda_1}) A^{(0)}(-l_1^{-\lambda_1}, i, ..., j-1, l_2^{\lambda_2}) \\ & \times A^{(0)}(-l_2^{-\lambda_2}, j, ..., n, l_3^{\lambda_3}) \\ & = \sum_i c_{4(i)} I_4^{(i)}\big|_{triplecut} + c_{3(j)} I_3[P^2_{i...i-1}, P^2_{i,...,j-1}, P^2_{j,...,n}] \end{aligned} \tag{1.13.3}$$

and we will have to subtract the contributions from various boxes to reach the desired triangle coefficient. Similarly double cuts will isolate a single bubble integral but will also have contributions from triangles and boxes. Various methods have been proposed and indeed successfully used to extract the bubble and triangle coefficients directly and we shall consider some of these later.

1.14 Moving to Higher Loops

The generalized unitarity method works particularly well at one-loop due to the existence of a known basis for one-loop integrals, however no such general basis exists for higher loop orders. How then does one proceed? There are various approaches to this, the most popular being the *master integral* approach. This approach can be summarised as follows

1. Find an integral basis for the given amplitude, the *master integrals*

2. Evaluate the integrals

3. Evaluate the rational coefficients of these integrals using, for example, unitarity cuts

In general it is highly non-trivial to find such a basis that is amenable to cuts and evaluation. While much progress has been made to develop techniques that streamline this process such as integral reduction by Integration by Parts (IBP) relations, among others, [53–64], this process is far from straightforward. Another disadvantage is that the process is specific to the single amplitude and so in general to move to a higher multiplicity effectively means that one has to start the whole process again, albeit perhaps with some gained insight. Note that while the leading-in-color single-minus amplitude at five points was published in 2018 [1], and in full color in 2023 [42, 43], there is yet to be published a full result for the six-point case, despite progress [65].

To that end, there is clearly strong motivation to develop a different method, that is more systematic and generalises much better to higher multiplicity. Such a method was developed in the case of all-plus amplitudes which allowed the calculation of five-, six-, and seven-point amplitudes in full color, [40, 41, 66–68] and the purpose of this thesis is to make progress towards generalising this method to single-minus amplitudes.

1.15 Rational Pieces

Until now we have not discussed the rational part of one-loop integrals. By definition these parts have no branch cuts and thus they cannot be seen by unitarity as discussed until now, so how can we evaluate this piece? One possibility is to use *D-dimensional unitarity* [69]. Until now we have performed cuts in $D = 4$ dimensions, that is to say when putting a propagator on-shell we kept the loop momentum l to be in four dimensions, known as $4D$ unitarity. In D-dimensional unitarity, instead ones writes the loop momentum as a sum of a 4 dimensional component and a -2ϵ dimensional component and sets this $D = 4 - 2\epsilon$ dimensional momentum on-shell

$$l = l^{[4]} + l^{[-2\epsilon]} = \tilde{l} + \mu, \qquad l^2 = \tilde{l}^2 - \mu^2. \tag{1.15.1}$$

The benefit of D-dimensional unitarity is that it also captures the rational piece but at the expense of complicated algebra and larger integral basis. We will use 4-dimensional unitarity in this thesis and thus we will have to find another way to calculate the rational piece. Using

4D unitarity means that we are able to use the spinor helicity formalism even at loop level, but at the cost of $\mathcal{O}(\epsilon)$ errors.

1.16 Augmented Recursion

Previously, we introduced BCFW recursion as a method for building tree amplitudes in terms of products of lower-point tree amplitudes. In fact this process also extends to the rational parts of loop amplitudes, but for one caveat: rational parts of two loop Yang-Mills amplitudes have double poles.

To make this statement explicit let us consider the rational part of the two-loop four-point all-plus amplitude [70]

$$R_4^{(2)}(a^+, b^+, c^+, d^+) = \frac{i}{9}\frac{[ab][cd]}{\langle ab\rangle\langle cd\rangle}\left(\frac{s_{bd}^2}{s_{bc}s_{cd}} + 8\right). \tag{1.16.1}$$

If we rewrite this amplitude using four-point kinematics then we would have $\langle ab\rangle^2$ in the denominator, and if we were to shift say the a momentum $\lambda_a \to \lambda_a - z\lambda_e$, then the total shifted rational piece $R(z)$ would have a resultant double pole in z from $(\langle ab\rangle - z\langle eb\rangle)^2$ in the denominator.

So why is this a problem? In principle it is not. The BCFW recursion method is a simple application of Cauchy's theorem where we apply a shift to two momenta to turn our rational piece into a complex function of a variable z then evaluate $R(z)$ using

$$\frac{1}{2\pi i}\oint dz \frac{R(z)}{z} = R(0) + \sum_{z_j \neq 0} \text{Res}\left[\frac{R(z)}{z}\right]\Big|_{z_j} \tag{1.16.2}$$

where z_j are the poles of $R(z)$. By taking the contour on the left to infinity this integral will vanish (assuming the appropriate shift) and so our rational piece R can be expresssed as

$$R = R(0) = -\sum_{z_j \neq 0} \text{Res}\left[\frac{R(z)}{z}\right]\Big|_{z_j} \tag{1.16.3}$$

Now we can expand a function $f(z)$ with double poles around around the pole z_i

$$f(z) = \frac{a_{-2}}{(z - z_i)^2} + \frac{a_{-1}}{z - z_i} + \cdots . \tag{1.16.4}$$

Setting $\delta = z - z_i$ and Taylor expanding,

$$\frac{f(z)}{z} = \frac{a_2}{(\delta + z_i)\delta^2} + \frac{a_1}{(\delta + z_i)\delta} + \mathcal{O}(\delta^0) = \frac{a_2}{\delta^2} + \frac{1}{\delta}\left(-\frac{a_2}{z_i^2} + \frac{a_1}{z_i}\right)\mathcal{O}(\delta^0), \tag{1.16.5}$$

we see that

$$\text{Res}\left[\frac{f(z)}{z}\right]\bigg|_{z_i} = -\frac{a_{-2}}{z_i^2} + \frac{a_{-1}}{z_i}. \tag{1.16.6}$$

Now in our previous discussion of BCFW recursion we used factorisation theorems of amplitudes to evaluate the residues of our function, however these theorems can only tell us the *leading singularities* and in general there are no theorems that can tell us about sub-leading poles. We therefore have no way to write down a_{-1}, which we refer to as the *pole under the pole* and it is precisely for this reason that we employ our method of *augmented recursion*.

This method uses an axial gauge formalism [71, 72], which allows us to generalise amplitudes to vertices with off-shell external momenta and assigned helicities. Thus we can express loop integrals as products of such vertices with internal helicity labels and scalar propagators. We can express an off-shell momentum K as a sum of two null momenta

$$K = K^\flat + K^\# = K^\flat + \frac{K^2}{2K\cdot q}q, \tag{1.16.7}$$

where q is an arbitrary null reference momentum restricted only in that $2K \cdot q \neq 0$.

In this formalism there are three-point vertices,

$$\begin{aligned} A_3(1^+, 2^+, 3^-) =& i\frac{[12]\langle 3q\rangle^2}{\langle 1q\rangle\langle 2q\rangle}, \\ A_3(1^-, 2^-, 3^+) =& i\frac{\langle 12\rangle[3q]^2}{[1q][2q]}, \end{aligned} \tag{1.16.8}$$

and four-point vertices

$$\begin{aligned} A_4(1^+, 2^+, 3^-, 4^-) =& i\frac{[1q][2q]\langle 3q\rangle\langle 4q\rangle}{\langle 1q\rangle\langle 2q\rangle[3q][4q]}\left(1 + \frac{[q|2-3|q\rangle[q|4-1|q\rangle}{[q|2+3|q\rangle[q|4+1|q\rangle}\right), \\ A_4(1^+, 2^-, 3^+, 4^-) =& i\frac{[1q]\langle 2q\rangle[3q]\langle 4q\rangle}{\langle 1q\rangle[2q]\langle 3q\rangle[4q]}\left(\frac{[q|1-2|q\rangle[q|3-4|q\rangle}{[q|1+2|q\rangle[q|3+4|q\rangle} + \frac{[q|2-3|q\rangle[q|4-1|q\rangle}{[q|2+3|q\rangle[q|4+1|q\rangle} - 2\right). \end{aligned} \tag{1.16.9}$$

Again, q is the reference momentum which is arbitrary, with the caveat that the above amplitudes cannot have a zero in their denominators.

As motivation, we should ask whence double poles arise? One place they can arise is in

BCFW channels wherein a two-loop amplitude factorises into a one-loop three-point all-plus vertex, and a one-loop amplitude (in the single-minus case, this amplitude will be an MHV). We can express this one-loop all-plus vertex in the axial gauge formalism as

$$A_3^{(1)}(a^+, b^+, k^+) = \frac{i}{3} \frac{[ab][bk^\flat][k^\flat a]}{k^2} \tag{1.16.10}$$

so double poles appear in factorisations of the type

$$A_3^{(1)}(a^+, b^+, k^+) \frac{1}{s_{ab}} A^{(1)}(-k^-, c^+, ..., n^-) \tag{1.16.11}$$

where we get one pole from the vertex on the left and one from the propagator as shown in figure 1.1.

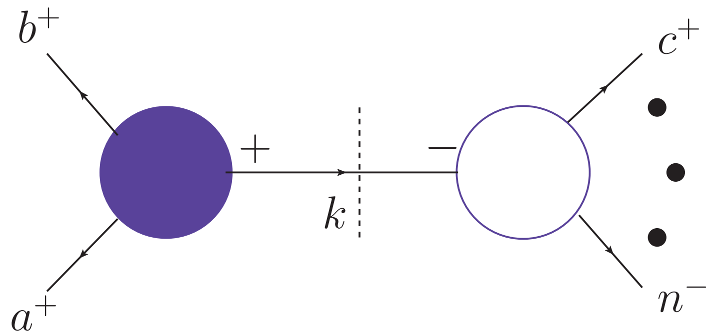


Figure 1.1: The one-loop to one-loop factorisation of a two-loop amplitude that results in a double pole

The idea of augmented recursion is as follows: we 'open' the propagator in the factorisation so our diagram becomes a tree structure connected to a one-loop current with two off-shell legs as shown in figure 1.2. We then construct a 'good enough' current $\tau^{(1)}(c^+, ..., n^-, \alpha, \beta)$ that is able to capture the pole structure of our final rational piece, and integrate this diagram before performing the BCFW shift on this integrated structure which returns the leading double-pole and also the the subleading 'pole under the pole'.

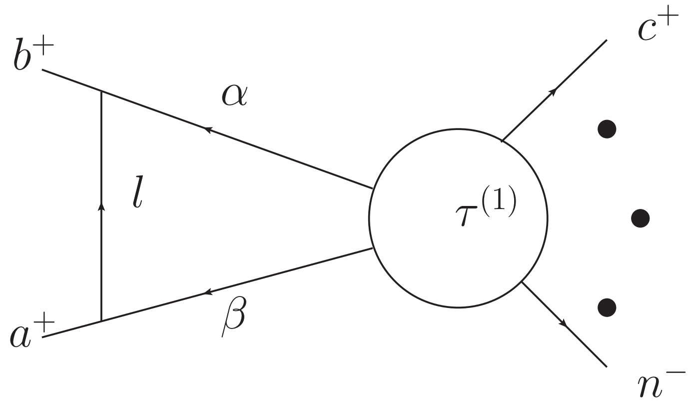


Figure 1.2: This depicts the generic structure that arises from 'opening' the propagator from the double pole factorisation on which we perform augmented recursion

To define a 'good enough' current, remember that we are only interested in this current insofar as it allows us to uncover the leading and sub-leading pole structure of the original tree to two-loop factorisation, and so we do not need the full expression for an amplitude with two off-shell legs. We require that that the current satisfies two conditions:

C1: As $s_{\alpha\beta} \to$ with $\alpha^2, \beta^2 \neq 0$, the current must reproduce the leading singularity,

C2: As $\alpha^2, \beta^2 \to 0$, with $s_{\alpha\beta} \neq 0$, we recover the on-shell amplitude $A^{(1)}(c^+, ..., n^-, \alpha, \beta)$.

These conditions ensure that we derive our current in a consistent manner that reproduces the leading and sub-leading singularities that we need.

This method has been used successfully in the past in the computation of one-loop gravity amplitudes [73], as well as in two-loop Yang-Mills amplitudes [40, 41, 66–68]. The bulk of this thesis will focus on extending this method to the calculation of two-loop single-minus amplitudes, where as we will see later, the presence of a single extra negative external helicity means that in fact there are many more structures that need to be considered in the single-minus case than in the all-plus case, and there is a whole new double pole factorisation channel to be considered due to a two-loop three-point vertex.

1.17 Thesis Outline

The remainder of this thesis is organised as follows. In Chapter 2 we introduce the two-loop five-point single-minus Yang-Mills amplitude and employ methods of $4D$ unitarity to calculate the cut-constructible part of the amplitude, focusing in particular on the psuedo one-loop subsector of the cut-constructible terms. In Chapter 3 we apply the previously developed method of augmented recursion to calculate part of the remaining rational piece of the two-loop five-point single-minus amplitude, which we name the 'tree on the left' part. We see that in fact this is not the full rational part so in Chapter 4 we extend the method of augmented recursion to include the new 'loop on the left' pieces, as well as rational contributions from the cut-constructible part of the amplitude. These are new to this particular amplitude and have not appeared in previous augmented recursion calculations. At the end of this chapter we fully reconstruct the rational part of the two-loop five-point single-minus Yang-Mills amplitude. Finally in Chapter 5 we summarise the work done in this thesis and discuss further avenues to extend the research.

Chapter 2

Two Loop Five-Point Single-Minus Amplitude: Unitarity

In [66], the two-loop five-point all-plus amplitude was calculated. While this calculation had already been completed in [74] to leading order in colour, and in [75] to all orders in colour, the cited paper was novel in its techniques. Rather than use the master integral approach and D-dimensional unitarity, the paper used used $4D$ unitarity and recursion. The use of $4D$ unitarity rather than D-dimensional unitarity leads to drastic simplifications: namely, any cuts that split the amplitude into tree $\times$ tree vanish.

The upshot of all this was that one could consider one of the loops to be an insertion into a vertex of the second loop. This is to say that this loop will act as an effective vertex. Thus the problem was reduced to effectively a one-loop problem where incidentally one of the vertices is a one-loop amplitude rather than a tree. This simplification was useful for a number of reasons

1. It cuts down on the number of contributing Feynman diagrams

2. The integral basis for one-loop integrals is well known

3. There are well developed methods for calculating one-loop amplitudes which are at our disposal.

This simplification sadly fails in the single-minus case. In the presence of an additional particle of negative helicity, there now exist non-vanishing cuts from two-loop to tree$\times$ tree.

The use of $4D$ unitarity still simplifies the number of possible cuts and thus the complexity of the calculation, however this comes at the cost losing rational pieces. To remedy this, we will need to use augmented recursion to recover the rational part of the amplitude in the following chapters.

With all this in mind, we can begin with the main focus of this thesis: to develop a general method of calculating two-loop single-minus Yang-Mills amplitudes.

We will take as an example the leading-in-color two-loop five-point single-minus Yang-Mills amplitude, which has been previously calculated in [1]. Our goal as stated previously is to develop a method that is more amenable to generalisation to higher multiplicity. To start, we split the amplitude in two

$$A_5^{(2)}(a^-, b^+, c^+, d^+, e^+) = P_5^{(2)}(a^-, b^+, c^+, d^+, e^+) + R_5^{(2)}(a^-, b^+, c^+, d^+, e^+), \tag{2.0.1}$$

where P refers to the cut-constructible piece, and R to the rational part.

The cut-constructible part of the amplitude, P can be further divided into two parts,

$$P_5^{(2)}(a^-, b^+, c^+, d^+, e^+) = U_5^{(2)}(a^-, b^+, c^+, d^+, e^+) + F_5^{(2)}(a^-, b^+, c^+, d^+, e^+), \tag{2.0.2}$$

where F is finite and U contains all of the divergences due to ϵ. As discussed earlier, we know that the divergent piece takes the form [45]

$$U_5^{(2)}(a^-, b^+, c^+, d^+, e^+) = A_5^{(1)}(a^-, b^+, c^+, d^+, e^+) \times \frac{1}{\epsilon^2} \sum_{i=1}^{5} \left(\frac{\mu^2}{s_{i,i+1}} \right)^{\epsilon}, \tag{2.0.3}$$

so in principle it remains only to calculate the finite part, F, however the methods we employ will reconstruct the full P term including divergent piece which we can use as a check for our calculations.

To calculate the cut-constructible part of the amplitude, we divide this into two parts which we will refer to as "genuine" two-loop parts, a and one-loop subsector. The genuine two-loop pieces will be outlined briefly for completion - and the reader can refer to [2] for details on calculation. To understand the division between the two-loop pieces and the one-loop subsector, we first consider all the possible configurations of propagators in the two-loop five-point case and split these into two categories: '(one-loop)^2' diagrams which can be split into two distinct diagrams by cutting (in a topological sense) a single propagator, and 'genuine two-loop' diagrams which require more than one topological cut as shown in figure 2.1.

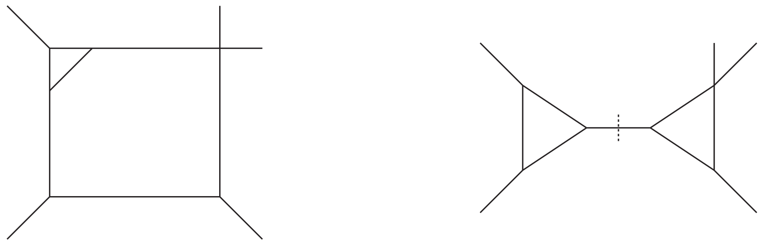


Figure 2.1: On the left is an example of a propagator configuration which we refer to as 'genuine two-loop', and on the right is an example of a (one-loop)^2 configuration known as the 'bow tie'.

To each of the possible propagator configurations we assign a numerator $\mathcal{N}$, so that for a given diagram we have the integrand

$$I^{(2)} = \frac{\mathcal{N}}{\mathcal{P}},$$

where $\mathcal{P}$ are the propagators for the diagram in question. We then compare cuts of $A_5^{(2)}(a^-, b^+, c^+, d^+, e^+)$ with cuts on the diagrams to determine the numerators such that the cuts match up. In other words, we determine numerators by imposing that

$$A_5^{(2)}(a^-, b^+, c^+, d^+, e^+)|_{\text{cuts}} = \sum_i \frac{\mathcal{N}_i}{\mathcal{P}_i}|_{\text{cuts}} \tag{2.0.4}$$

where the sum is over all diagrams. The work to calculate the numerators was completed prior to the beginning of this thesis [76], and thus we shall outline the method and present the relevant results for completion.

First we can consider triple cuts of the amplitude into two tree diagrams, for example

$$\sum_{\lambda_1,\lambda_2,\lambda_3} A^{(0)}(a^-, l_1^{\lambda_1}, l_2^{\lambda_2}, l_3^{\lambda_3}) \times A_3^{(0)}(-l_1^{-\lambda_1}, -l_2^{-\lambda_2}, -l_3^{-\lambda_3}, b^+, c^+, d^+, e^+). \tag{2.0.5}$$

The (one-loop)^2 diagrams will all vanish on such cuts and so one can isolate first the two-loop diagrams, and by applying various triple cuts to tree $\times$ tree, determine the numerators corresponding to such diagrams.

Next, we consider double cuts on the amplitude into a one-loop amplitude and a tree, for example

$$\sum_{\lambda_1,\lambda_2} A^{(1)}(a^-, b^+, l_1^{\lambda_1}, l_2^{\lambda_2}) \times A_3^{(0)}(-l_1^{-\lambda_1}, -l_2^{-\lambda_2}, c^+, d^+, e^+). \quad (2.0.6)$$

For each cut, we will have contributions from the two-loop diagrams which we know, and from the (one-loop)^2 pieces whose numerators are to be determined, and by considering various such double cuts we can determine the remaining numerators.

The (one-loop)^2 structures are written in terms of a single-integral over a numerator and its corresponding propagators. In this sense they can be regarded as pseudo one-loop integrals with an insertion at one of the vertices which corresponds to the second loop which has already been integrated. Let us take the 'bow-tie' diagram from the right of figure 2.1 for example. The contribution of such a diagram will be included in the one-loop structures, and so we consider it as the loop which is 'uncut' in the double cut fit to have been integrated out and inserted into the remaining integral as a vertex. Since there is more than one way to do this, the one-loop^2 diagrams will be split across different one-loop structures in general.

It is for this reason we refer to these terms as the one-loop subsector, and in this chapter will detail how we evaluate such integrals. The full list of these one-loop structures can be found in the appendix A.

To summarise, triple cuts on the amplitude to tree $\times$ tree determine the numerators on the two-loop structures, while double cuts to one-loop $\times$ tree determine the numerators of the one-loop structures. Together these structures satisfy all cuts of the full amplitude.

2.1 Two-Loop Structures

For completion, let us briefly discuss the two-loop pieces. As stated above, the method is to write down all possible propagator configurations, $\mathcal{P}_\eta$, and then to assign numerators, $\mathcal{N}_\eta$, to these configurations such that the sum of all such expressions can recreate the triple cuts on the full amplitude. The numerators are written in terms of external momenta $\mathcal{N}_\eta(a, k, \overline{a}, i, j, \omega)$, where η is a label that matches propagators to their numerator. The first a indicates that a is the sole external negative helicity momentum, and the remaining labels are kept generic such that one can simultaneously calculate a diagram and its flip with the caveat that the label $\overline{a}$ is special in that whenever $a = \overline{a}$ the numerator vanishes. Now we shall list the propagator configurations with their associated numerators [76]. In all the below diagrams, the arrows on propagators denote the flow of momentum. Momentum is conserved at each vertex.

$$\mathcal{N}_{tbx} = \langle a\overline{a}\rangle[\overline{a}|(-L_3)|a\rangle\langle\omega|QL_2|a\rangle[\omega|L_1PL_2|a\rangle, \tag{2.1.1}$$

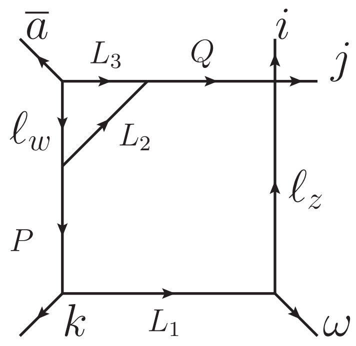


Figure 2.2: Tricorner box. $\eta = tbx$

$$\mathcal{N}_{fbbx} = \langle a\overline{a}\rangle[\overline{a}|\ell_\omega|\omega\rangle\langle a|L_AL_B|a\rangle[\omega|\ell_z\ell_\omega L_A|a\rangle. \tag{2.1.2}$$

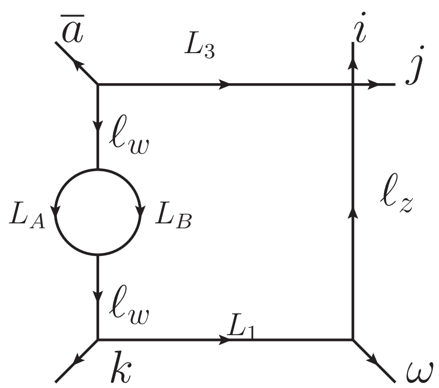


Figure 2.3: Bubble in box between two null corners. $\eta = nfbbx$.

There are in fact further diagrams that contain bubbles which are required to satisfy the cut conditions, but that vanish upon integration. These have been omitted but further details may be found in [2]. The remaining all-plus triangle diagrams are as follows:

$$\begin{aligned}\mathcal{N}_{stt} =& - \langle a\overline{a}\rangle[\overline{a}|(-L_3)|a\rangle\langle a|L_2Q|k\rangle[k|L_2|a\rangle \\ &+ \delta_a^j\langle a\overline{a}\rangle[\overline{a}|(-L_3)|a\rangle\langle a.\rangle k[k|L_1Q(-L_3)|a\rangle,\end{aligned} \tag{2.1.3}$$

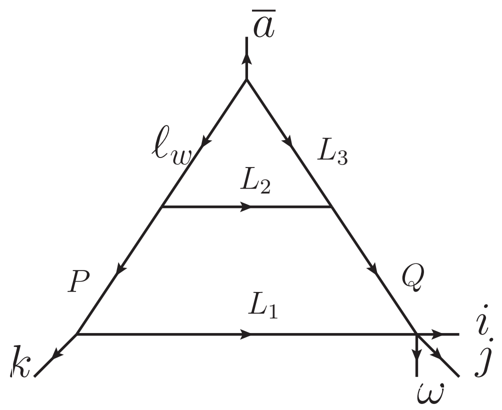


Figure 2.4: Triangle in triangle, $\eta = stt$. There is an extra term when the negative helicity leg is in the middle of the massive corner $a = j$.

where here δ_a^j indicates an additional term only present when $a = j$ is in the middle of massive corner. Pressing on we have

$$\begin{aligned}\mathcal{N}_{K21} =& \langle a\overline{a}\rangle[\overline{a}|\ell_w|a\rangle\langle ak\rangle[k|P|a\rangle \\ &+\delta_a^j\langle a\overline{a}\rangle[\overline{a}|P|a\rangle\langle ak\rangle[k|P|a\rangle \\ &+\delta_a^j\langle a\overline{a}\rangle[\overline{a}|\ell_w|a\rangle\langle ak\rangle[k|\ell_w|a\rangle,\end{aligned} \tag{2.1.4}$$

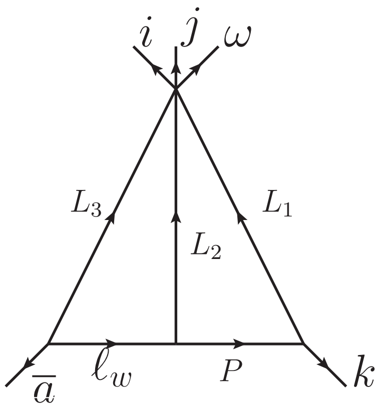


Figure 2.5: Kite diagram with two null corners and one massive corner, with $\eta = K21$. There are extra terms when the negative helicity leg is in the middle of the massive corner $a = j$.

$$\mathcal{N}_{K31} = \langle a\overline{a}\rangle[\overline{a}|\ell_w|a\rangle\langle a\omega\rangle[\omega|L_1|a\rangle, \tag{2.1.5}$$

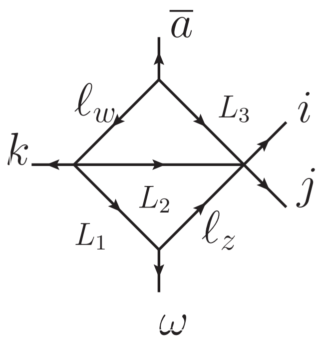


Figure 2.6: Kite diagram with three null corners and one massive corner, with $\eta = K31$.

$$\mathcal{N}_{K12} = \langle a\overline{a}\rangle[\overline{a}|(-L_3)|a\rangle\langle a|L_2(-L_3)|a\rangle, \tag{2.1.6}$$

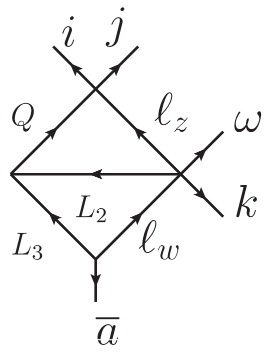


Figure 2.7: Kite diagram with one null corner and two massive corners, with $\eta = K12$.

and finally,

$$\mathcal{N}_{K1Q} = \frac{1}{2}\langle a|\overline{a}L_3|a\rangle\langle a|\overline{a}Q|a\rangle. \tag{2.1.7}$$

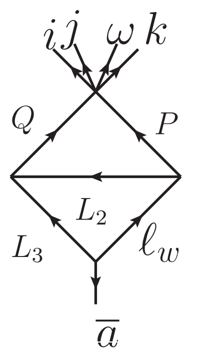


Figure 2.8: Kite diagram with one null corner and one massive corner, with $\eta = K1Q$.

In total the integrand $\mathcal{I}$ is

$$\mathcal{I}^{(2)}(a^-, b^+, c^+, d^+, e^+) =$$
$$= \frac{2}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle de\rangle\langle ea\rangle} \sum_{\{k,\overline{a},i,j,w\}} \sum_{\eta} \left(\frac{\mathcal{N}_\eta(a,k,\overline{a},i,j,\omega)}{\mathcal{P}_\eta(a,k,\overline{a},i,j,\omega)} + \frac{\mathcal{N}_\eta(a,k,\omega,j,i,\overline{a})}{\mathcal{P}_\eta(a,k,\omega,j,i,\overline{a})} \right) \tag{2.1.8}$$

where the sum over $\{k, \overline{a}, i, j, w\}$ is over $Z_5(a, b, c, d, e)$. One must also be careful to add in symmetry factors as there are cases in which the flip of a diagram is not distinct and thus we must prevent double counting. Finally then the task is to evaluate each of the two-loop integrals above, which is beyond the scope of this thesis and details can again be found in [2].

2.2 One-Loop Structures

Moving onto the one-loop subsector, since these are one-loop integrals we can exploit the very well-established one-loop integral methods in their evaluation. We know from Passarino-Veltman reduction that one-loop integrals can be expressed in terms of a basis of scalar box, triangle, and bubble integrals as discussed in Section 1.11 on integral reduction.

We then perform various cuts in accordance with the methods of generalised unitarity to calculate each term as a linear combination the one-loop scalar integrals with rational coefficients.

It is important to note however that we are not treating the diagrams as collections of Feynman diagrams on which we will cut. In fact, if we did so we would find that the cuts all vanish as it to be expected for a one-loop single-minus amplitude which we know to be fully rational. Instead, we are considering pre-determined numerators with propagators in the denominator, and the diagrams only serve to remind us of the relationship between the various propagators in terms of the external momenta. With this in mind then, we shall define a cut of the propagator D to be the substitution

$$\frac{1}{D} \to (2\pi)\delta(D). \tag{2.2.1}$$

on the integrand.

There are box and two-mass triangle structures which contribute to multiple different cuts of the two-loop diagrams, thus one must be careful not to double count. The remaining structures are one-mass triangles or bubble which only contribute to a single cut. There are also flip symmetry relations between pairs of these which we can use as a check on our results after calculating. In the end this set of 'genuine' two loop integrals and pseudo-one loop integrals fully recreate the cuts of the amplitude. A full list of these one-loop structures can be found in the appendix A.

As detailed in Section 1.13, the strategy for one-loop calculations is to begin with the maximal cut - in our case a quadruple cut - to isolate the coefficients of the scalar box integrals, then consider triple cuts to determine the coefficients of the triangle integrals and finally double cuts for the bubble integrals. Naturally, in the case of the triangle and bubble structures, we do not need to consider quadruple cuts as these will vanish. In the case of bubbles we can also skip the triple cuts. The cuts on each of these is depicted in figure 2.9.

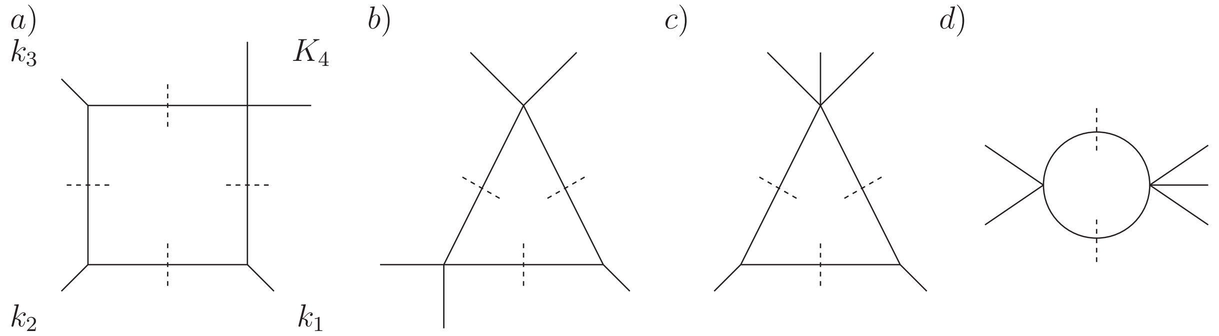


Figure 2.9: Four dimensional cuts of the one-loop structures.

2.2.1 Boxes

In 4D unitarity, we treat our loop momentum as 4 dimensional, and 4 cuts imposes 4 constraints and thus completely determines the loop momentum up to complex conjugation as first demonstrated by Britto et al in [52].

Let us take the example of a numerator with propagator configuration as depicted in figure 2.10. As there are 4 propagators we know there is exactly one choice of quadruple cut and thus there is only one scalar box integral, $I_4(s_{ab}, s_{cd}, s_{de})$ whose coefficient is to be determined.

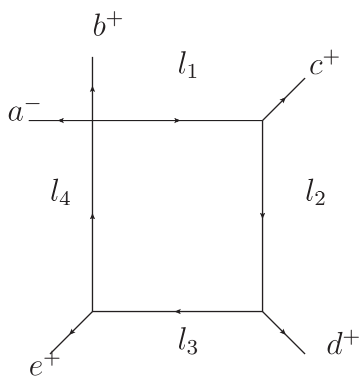


Figure 2.10: A diagram describing an example propagator configuration for a one-loop box structure.

We can choose to solve for any one of the four propagators; from there it is trivial to solve for the rest. Let us solve for l_2, with 4 cut constraints,

$$\delta(l_1)^2,\ \delta(l_2)^2,\ \delta(l_3)^2,\ \delta(l_4)^2.$$

By cutting l_2 we have the constraint $l_2^2 = 0$ from the delta function. We know that any massless momentum may be expressed via spinor helicity, so we can write

$$l_2 = X\lambda_2\tilde{\lambda}_2, \tag{2.2.2}$$

where X is a rational factor to be determined. Next we can cut l_1 which gives the constraint $l_1^2 = 0 = (l_2 + c)^2 = 2l_2 \cdot c$. We now have two options, each of which will give us a solution of l_2. We have

$$\begin{aligned} l_2^{(1)} &= X_1\lambda_c\tilde{\lambda}_2, \\ l_2^{(2)} &= X_2\lambda_2\tilde{\lambda}_c. \end{aligned} \tag{2.2.3}$$

We can cut l_3 to determine the remaining spinor as $l_3^2 = (l_2 - d)^2 = 2l_2 \cdot d = 0$, so

$$\begin{aligned} l_2^{(1)} &= X_1\lambda_c\tilde{\lambda}_d, \\ l_2^{(2)} &= X_2\lambda_d\tilde{\lambda}_c. \end{aligned} \tag{2.2.4}$$

Finally, by cutting l_4 we can determine X for each solution. We have $l_4^2 = 0 = (l_2 - P_{de})^2 = s_{de} - [l_2|d + e|l_2\rangle = s_{de} - [l_2|e|l_2\rangle$as both solutions are proportional to d.

$$\begin{aligned} X_1 &= \frac{\langle de\rangle}{\langle ce\rangle}, \\ X_2 &= \frac{[de]}{[ce]}, \end{aligned} \tag{2.2.5}$$

and so finally we have

$$\begin{aligned} l_2^{(1)} &= \frac{\langle de\rangle}{\langle ce\rangle}\lambda_c\tilde{\lambda}_d, \\ l_2^{(2)} &= \frac{[de]}{[ce]}\lambda_d\tilde{\lambda}_c. \end{aligned} \tag{2.2.6}$$

Then it remains to substitute the two values above into the numerator of the integral and average to obtain the coefficient of the scalar function $I_4(s_{ab})$ for this structure. The procedure is identical for all boxes.

2.2.2 Triangles

As the number of cuts decreases the number of choices increases. For each box there are four propagators and there are 4 ways to choose three of them to cut to find the coefficient of the corresponding triangle integral.

It is clear when distributing five external momenta across the three vertices of a triangle, that we will be dealing with one- and two-mass triangles. At higher multiplicity we will also have three-mass triangles. Initially, we had intended to use the method laid out by Forde in [77], however it became apparent that the method is a three-mass triangle method and there is some subtlety in taking the one-mass or two-mass limits. In these limits we found that the calculated coefficient differed depending on which which of the three cut propagators we parameterised the integral in terms of. The reasons for this are not entirely clear, it may be that since Forde's method was designed for use on one-loop amplitudes rather than more general one-loop integrals, that there is something in the factorisation theorems that keep the Forde method stable in the massless limit of vertices, which does not track to more general one-loop integrals.

Due to this obstacle, we present a new parameterisation that allows us to calculate the scalar triangle coefficient in the case of one- or two-mass triangle structures. At more than five external momenta, one would encounter three-mass triangles for which the Forde method would suffice.

We begin with a triangle with one massive and one massless vertex, this process is blind to the mass of the third corner and so this method works for one- and two-mass triangles. We label the loop momentum l_0 flowing from a massless vertex labelled e to a massive vertex P as shown in figure 2.11. As per Eq 1.16.7 we can write P as a sum of two null momenta,

$$P = P^\flat + P^\# = P^\flat + \frac{P^2}{2P \cdot e} e, \tag{2.2.7}$$

which holds regardless of whether P is sum of two, three, or indeed any number of massless momenta.

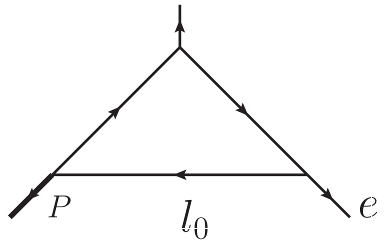


Figure 2.11: Triple cut configuration, with l_0 flowing from null leg $e^2 = 0$ and towards $P^2 \neq 0$.

We can use this to parameterise the loop momentum l_0

$$\begin{aligned} l_0 = A(P^\flat + P^\sharp) + B(P^\flat - P^\sharp) + C\left(f\frac{\langle qP^\flat\rangle}{\langle qP^\sharp\rangle}\tilde{\lambda}_{P^\flat}\lambda_{P^\sharp} + f\frac{[qP^\flat]}{[qP^\sharp]}\tilde{\lambda}_{P^\sharp}\lambda_{P^\flat}\right) \\ + iD\left(f\frac{\langle qP^\flat\rangle}{\langle qP^\sharp\rangle}\tilde{\lambda}_{P^\flat}\lambda_{P^\sharp} - f\frac{[qP^\flat}{[qP^\sharp}\tilde{\lambda}_{P^\sharp}\lambda_{P^\flat}\right). \end{aligned} \tag{2.2.8}$$

where q is some arbitrary null vector. In order that l_0 is real, we restrict all of the quantities in the above definition to be real. Then

$$\begin{aligned} l_0^2 &= P^2\big(A^2 - B^2 + f^2\frac{[q|P^\flat|q\rangle}{[q|P^\sharp|q\rangle}(C^2 + D^2)\big) \\ &= P^2\big(A^2 - B^2 + C^2 + D^2\big) \end{aligned} \tag{2.2.9}$$

where we judiciously defined f to simplify the above, and

$$(l_0 - P)^2 = P^2\big((A-1)^2 - B^2 + C^2 + D^2\big) \tag{2.2.10}$$

$$(l_0 + e)^2 = P^2\big(A^2 - B^2 + C^2 + D^2\big) + 2(A+B)e\cdot P^\flat, \tag{2.2.11}$$

the last line being true due to our choice of e as the reference spinor when defining $P^\sharp$.

Now we change the integration variables,

$$\int d^D l_0 \to (P^2)^2 \int dA dB dC dD. \tag{2.2.12}$$

We still have not imposed the cut constraints. We map the cut constraints onto our new integration variables using

$$\delta|g(x)| = \sum_i \frac{\delta(x-x_i)}{|\frac{\mathrm{d}}{\mathrm{d}x}g(x)|_{x=x_i}} = \sum_i \frac{\delta(x-x_i)}{|\delta_i'|}, \tag{2.2.13}$$

where x_i are the roots of $g(x)$. From this identity, we can use the cut constraint $\delta(l_0)^2$ to perform the B integration by imposing $B^2 = A^2 + C^2 + D^2$, with the factor

$$\frac{1}{|\delta'|} = \frac{1}{|2\sqrt{A^2+C^2+D^2}P^2|}. \tag{2.2.14}$$

Next we impose the cut constraint $(l_0 - P)^2 = 0 = P^2\big((A-1)^2 - B^2 + C^2 + D^2\big) = P^2((A-1) - A^2)$. Performing the A integration sets the value of $A = \frac{1}{2}$ with the additional factor

$$\frac{1}{|\delta'|} = \frac{1}{|2P^2|}. \tag{2.2.15}$$

Proceeding to the final cut, we run into an issue, imposing $\delta((l_0+e)^2)$ we have

$$\delta((l_0+e)^2) = \frac{\sqrt{1+4C^2+4D^2}}{|4Ce\cdot P^\flat|}\delta\left(e\cdot P^\flat\left(1-\sqrt{1+4C^2+4D^2}\right)\right) \tag{2.2.16}$$

which sets $C = D = 0$, causing a divergence when performing the integral. In fact switching to polar coordinates for the final two integrals,

$$C = \rho\cos\theta,\ D = \rho\sin\theta, \int dCdD \to \int \rho d\rho d\theta$$

we see that in fact this is a co-ordinate singularity as

$$\int dC \frac{\sqrt{1+4C^2+4D^2}}{|4Ce\cdot P^\flat|}\delta\left(e\cdot P^\flat\left(1-\sqrt{1+4C^2+4D^2}\right)\right) = \int \rho\ d\rho \frac{\sqrt{1+4\rho^2}\delta(\rho)}{4\rho|e\cdot P^\flat|} \tag{2.2.17}$$

and the divergence cancels with the Jacobian, thus we can perform the final integral which sets $\rho = 0$. What remains then is a single angular integral with an integrand that has no angular dependence.

Schematically, we have

$$\int d^4l\delta_1\delta_2\delta_3\frac{\mathcal{N}(l)}{P_4} = \int d^4l\delta_1\delta_2\delta_3\left(\frac{c_{\text{box}}}{P_4} + c_{\text{tri}}\right), \tag{2.2.18}$$

where in our labelling, 1,2 and 3 are the cut propagators and 4 is the remaining uncut propagator, so using our parameterisation, we finally have

$$c_{\text{tri}} = \left.\frac{\mathcal{N}(l) - c_{\text{box}}}{P_4}\right|_{l=P^\sharp} \tag{2.2.19}$$

where c_{box} is already known.

In the case of the box structures and two-mass triangle structures, there was an unexpected result in that in each case the numerator evaluated at $l = P^\sharp$ vanished. The upshot of this that for these terms the scalar triangle coefficient is equal to

$$c_{\text{tri}} = -\left.\frac{c_{\text{box}}}{\mathcal{N}(l)}\right|_{l=P^\sharp}$$

and in fact the ϵ^{-2} for the box and two-mass triangle structures, the IR contributions from the box cuts cancelled with the triple cuts. Perhaps even more surprising is the fact that the one-mass triangle structures fully recreate the IR piece, in other words

$$c_{\text{tri}}^{i,i+1} I_{3;1m}^{(1)}(s_{i,i+1}) = A_5(a^-, b^+, c^+, d^+, e^+) \times -\frac{1}{\epsilon^2}(-s_{i,i+1})^{-\epsilon}. \tag{2.2.20}$$

Summing over the one-mass triangle inserts, there is one of the above for each consecutive pair of momenta, and we recover the Catani IR factor (1.10.11).

2.2.3 Bubbles

Finally, we must determine the bubble contributions. For this part of the calculation, we used the canonical basis approach introduced in [78]. The idea is as follows: we take our one-loop integral and perform a double cut on it so the cut integral looks like

$$\int d^d l \delta(l^2)\delta((l-P)^2)\frac{\mathcal{N}^{R+N}(l)}{\prod_{i=1}^{R}(l+Q_i)^2}, \tag{2.2.21}$$

where the numerator $\mathcal{N}$ is a polynomial of order $N + R$. In general the Q_i may be massive or massless, however P will necessarily be massive otherwise the cut would be finding the coefficient of a massless scalar bubble which vanishes anyway. By Passarino-Veltman reduction we know that for individual terms in $\mathcal{N}$ where the order of l has $N < 0$, these terms correspond to scalar triangle and box integrals which we are not interested in, consequently

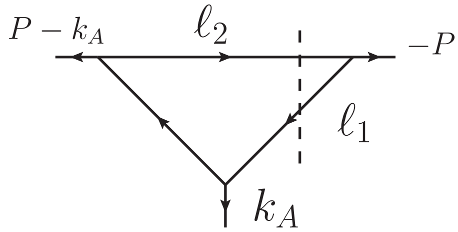


Figure 2.12: A double cut on the linear triangle used to calculate H_1.

we are only interested in terms such that $N \geq 0$.

Schematically we can write the cut integral as

$$\int d^d l \delta(l_1^2)\delta(l_2^2) \frac{\mathcal{N}^{R+N}(l)}{\prod_{i=1}^{R}(l+Q_i)^2} = \sum c_i \mathcal{F}_i(l_j), \tag{2.2.22}$$

where c_i are coefficients independent of l_j, and the canonical forms $\mathcal{F}_i$ must have zero spinor weight in l_j, ie invariant under $|l_j\rangle \to e^{i\phi_j}|l_j\rangle, |l_j] \to e^{-i\phi_j}|l_j]$. The aim of this method is to determine the possible canonical forms which appear in cut bubble integrals and determine how each contributes to the scalar bubble coefficient. The original paper was concerned with one-loop QCD amplitudes and evaluated forms up to order l^2, however in applying this method to this one-loop subsector of a larger two-loop calculation, we encountered terms of order l^3 and thus we extended the method by calculating the contributions that such terms would have towards the scalar bubble coefficient.

Let us begin with the simplest example,

$$\mathcal{H}_1(A; B; l_1) = \frac{\langle l_1 B\rangle}{\langle l_1 A\rangle} = -\frac{[A|l_1|B\rangle}{(l_1 - k_A)^2} \tag{2.2.23}$$

where k_A is taken to be real and massless. Looking at the above as a double cut integral we can see that it would arise from a linear triangle shown in figure 2.12. We can then promote this to a full covariant integral, and integrate the linear triangle, extracting the coefficient of $\log(-P^2)$ as the bubble coefficient which gives

$$H_1(A; B; P) - \frac{[A|P|B\rangle}{[A|P|A\rangle} \tag{2.2.24}$$

This can be generalised to

$$\mathcal{H}_n(A_i;B_i;l) = \sum_i c_i \frac{\langle B_1 l\rangle}{\langle A_i l\rangle} = \sum_i c_i \mathcal{H}_1(A_i;B_1;l) \tag{2.2.25}$$

by partial fractioning, where coefficients c_i are

$$c_i = \frac{\prod_{j=2}^{n}\langle B_j A_i\rangle}{\prod_{j\neq i}^{n}\langle A_j A_i\rangle}. \tag{2.2.26}$$

There are also cases of forms with massive propagators such as

$$\mathcal{G}_0(B;D;Q;l_1) = \frac{[D|l_1|B\rangle}{(l_1+Q)^2} \tag{2.2.27}$$

where $Q^2 \neq 0$. To express the above in terms of $\mathcal{H}_1$ we first introduce the following null momenta

$$\hat{P}^\mu = \frac{1}{2\sqrt{\Delta_3}}\left(P^2 Q^\mu - \left(P\cdot Q - \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right), \quad \hat{Q}^\mu = \frac{1}{2\sqrt{\Delta_3}}\left(-P^2 Q^\mu + \left(P\cdot Q + \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right), \tag{2.2.28}$$

where $\Delta_3 = 4(P\cdot Q)^2 - 4P^2Q^2$ is the Gram determinant of a three-mass triangle with external momenta P, Q, and $-P-Q$.By substituting in these null definitions into the canonical form we can write it in terms of $\mathcal{H}_1$ so that this form returns the bubble contribution

$$G_0[B;D;Q;P] = \frac{[D|P\,[Q,P]\,|B\rangle}{\Delta_3}, \tag{2.2.29}$$

where $[Q,P] = QP - PQ$ is the commutator. We also have

$$\mathcal{G}_1[A;B_0,B_1;D;Q;l_1] = \frac{[D|l_1|B_0\rangle}{(l_1+Q)^2}\frac{\langle l_1 B_1\rangle}{\langle l_1 A\rangle} \tag{2.2.30}$$

which gives

$$\begin{aligned} G_1[A;B_0,B_1;D;Q;P] = &-\frac{[D|P\,[P,Q]\,|A\rangle\langle B_1|\,[P,Q]\,|B_0\rangle}{2\Delta_3\langle A|PQ|A\rangle} \\ &+\frac{[D|P|A\rangle\left(\langle B_0 A\rangle[A|P|B_1\rangle + \langle B_1 A\rangle[A|P|B_0\rangle\right)}{2\langle A|PQ|A\rangle[A|P|A\rangle} \end{aligned} \tag{2.2.31}$$

and this can be extended to

$$\mathcal{G}_n[A_i; B_0, B_i; D; Q; l_1] = \frac{[D|l_1|B_0\rangle \prod i = 1^n \langle l_1 B_i\rangle}{(l_1+Q)^2 \prod i = 1^n \langle l_1 A_i\rangle} = \sum_i c_i \mathcal{G}_n[A_i; B_0, B_n; D; Q; l_1], \quad (2.2.32)$$

where

$$c_i = \frac{\prod_{j<n}\langle A_i B_j\rangle}{\prod_{j\neq i}\langle A_i A_j\rangle}. \quad (2.2.33)$$

The above forms are of order l^0, but there are also forms of order l^1, l^2 which can be seen in the paper.

Until now, all of the above can be found in the cited paper [78], and is presented here for completeness, however as mentioned above it was necessary to extend the method to terms of order l^3 and in fact we can generalise the massless propagator form, $\mathcal{H}_1$ to arbitrary powers of l.

We begin by defining the following form of order l^n,

$$\mathcal{H}_1^n = \frac{\prod_{i=1}^{n+1}\langle a_i l\rangle \prod_{j=1}^{n}[b_j l]}{\langle A l\rangle}, \quad (2.2.34)$$

on a double cut $\delta(l^2) = \delta((l-P)^2) = 0$. Then we define

$$P = P^\flat + P^\sharp = P^\flat + \frac{P^2}{[A|P|A\rangle} A, \quad (2.2.35)$$

and we use this to parameterise the cut momentum l in polar co-ordinates as

$$l = \frac{P^\flat + P^\sharp}{2} + \cos\theta \frac{P^\flat - P^\sharp}{2} + \sin\theta e^{i\phi}\frac{\lambda_{P^\flat}\tilde{\lambda}_{P^\sharp}}{2} + \sin\theta e^{-i\phi}\frac{\lambda_{P^\sharp}\tilde{\lambda}_{P^\flat}}{2} \quad (2.2.36)$$

which can be written in terms of spinors due to being on-shell

$$\lambda_l = \cos\frac{\theta}{2}\lambda_{P^\flat} + \sin\frac{\theta}{2}e^{-i\phi}\lambda_{P^\sharp}, \quad \tilde{\lambda}_l = \cos\frac{\theta}{2}\tilde{\lambda}_{P^\flat} + \sin\frac{\theta}{2}e^{i\phi}\tilde{\lambda}_{P^\sharp}. \quad (2.2.37)$$

Rewriting the original form, we get

$$\begin{aligned}\mathcal{H}_1^n =& \frac{P^2}{[A|P|A\rangle\langle P^\flat P^\sharp\rangle^{n+1}} \frac{\prod_{i=1}^{n+1}\left(\langle a_i P^\sharp\rangle\langle P^\flat l\rangle + \langle P^\flat a_i\rangle\langle P^\sharp l\rangle\right)\prod_{j=1}^{n}[b_j l]}{\langle P^\sharp l\rangle} \\ =& \frac{P^2}{[A|P|A\rangle\langle P^\flat P^\sharp\rangle^{n+1}} \frac{\prod_{i=0}^{n+1}\mathcal{A}_i\left(\langle P^\flat l\rangle^i + \langle P^\sharp l\rangle^{n+1-i}\right)\prod_{j=1}^{n}[b_j l]}{\langle P^\sharp l\rangle}\end{aligned} \quad (2.2.38)$$

where we define the combinatoric coefficients $\mathcal{A}_i$ according to the above expansion. Next we set $z = e^{-i\phi}$ and subsitute in our parameterisation of l to get

$$\begin{aligned}\mathcal{H}_1^n =& \frac{P^2}{[A|P|A\rangle\langle P^\flat P^\sharp\rangle} \sum_{i=0}^{n+1} \mathcal{A}_i z^i \sin^i\frac{\theta}{2}\left(-\cos\frac{\theta}{2}\right)^{n-i} \prod_{j=1}^{n}\left(\cos\frac{\theta}{2}[b_j P^\flat] + \sin\frac{\theta}{2} z^{-1}[b_j P^\flat]\right)\\ =& \frac{P^2}{[A|P|A\rangle\langle P^\flat P^\sharp\rangle} \sum_{i=0}^{n+1}\sum_{j=0}^{n} \mathcal{A}_i\mathcal{B}_j z^{i+j-n}(-1)^{n-i}\sin^{n+i-j}\frac{\theta}{2}\cos^{n+i-j}\frac{\theta}{2},\end{aligned} \tag{2.2.39}$$

where again we define $\mathcal{B}_j$ by expanding the product in the first line. To extract the bubble coefficient, we integrate z and θ for which we have the integration measure

$$-\frac{\sin\theta}{2(2\pi i)z}$$

which also sets the normalisation so that 1 integrates to 1, and finally the we have the bubble contribution

$$H_1^n = \frac{1}{\langle P^\flat A\rangle} \sum_{i=0}^{n} \mathcal{A}_i \mathcal{B}_{n-i} (-1)^{n-i} \sum_{p=0}^{n-i} \frac{(n-i)!}{(n-i-p)!} \frac{(-1)^p}{(1+p+i)}. \tag{2.2.40}$$

We do not present a closed form expression for the coefficients $\mathcal{A}_i\mathcal{B}_j$, however for any given n are easily generated on Mathematica.

The above formula successfully recreates the results of the known H forms of orders l^0, l^1 and l^2, and gives us the necessary H_0^3 form at l^3 by setting one of the $\{a_i\}$ to be equal to A. We have

$$\begin{aligned}&H_0^3[A_1, A_2, A_3; B_1, B_2, B_3; P]\\ &= \frac{1}{4}[A_1|P|B_1\rangle[A_2|P|B_2\rangle[A_3|P|B_3\rangle - \frac{P^2}{12}[A_2A_3]\langle B_2B_3\rangle[A_1|P|B_1\rangle\\ &- \frac{P^2}{12}[A_3A_1]\langle B_3B_1\rangle[A_2|P|B_2\rangle - \frac{P^2}{12}[A_1A_2]\langle B_1B_2\rangle[A_3|P|B_3\rangle.\end{aligned} \tag{2.2.41}$$

Finally we must also consider conjugate forms such as

$$\overline{\mathcal{H}}_1[A; B; l_1] = \frac{[l_1 B]}{[l_1 A]} \to \overline{H}_1[A; B; P] = \frac{[B|P|A\rangle}{[A|P|A\rangle}, \tag{2.2.42}$$

which are obtained by complex conjugation, and forms expressed in terms of l_2 which can be

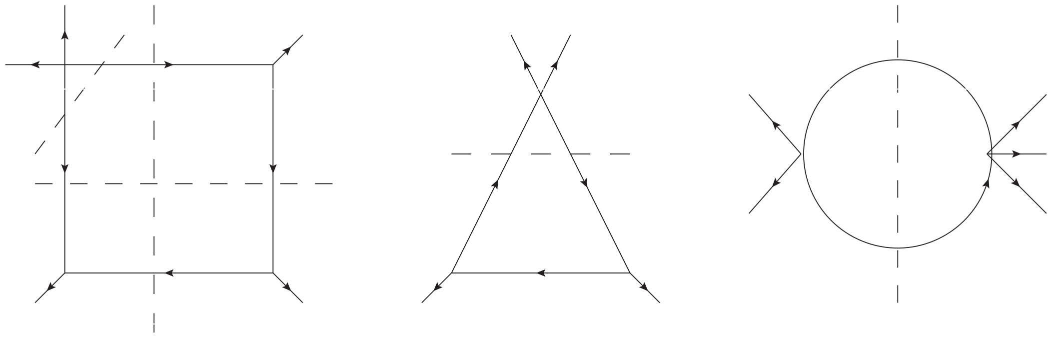


Figure 2.13: The possible double cuts on box, one-mass triangle, and bubble integrals

evaluated by using the identity

$$\frac{\langle al_2\rangle}{\langle bl_2\rangle} = \frac{\langle al_1\rangle}{\langle bl_1\rangle} - P^2 \frac{\langle ab\rangle}{\langle bl_1\rangle[l_1|P|b\rangle}. \tag{2.2.43}$$

Since the second term is of order l^{-1}, one can repeatedly use this to write all expressions of l_2 in terms of l_1 and terms of order l^N where $N < 0$ and thus can be discarded.

With all the pieces in place, it is simply a matter of writing each cut integrand in terms of the canonical forms via algebraic manipulation and then replacing the form with its bubble contribution. The algebraic manipulations are simply involve repeated use of 'partial fractioning' by writing for example

$$\frac{\langle l_1c\rangle}{\langle l_1a\rangle\langle l_1b\rangle} = \frac{\langle l_1c\rangle}{\langle l_1a\rangle\langle l_1b\rangle}\frac{\langle ab\rangle}{\langle ab\rangle} = \frac{\langle ac\rangle}{\langle l_1a\rangle} - \frac{\langle bc\rangle}{\langle l_1b\rangle}.$$

With the box integrals there 3 non-vanishing bubble cuts, and with the one-mass triangles and bubbles there is a single non-vanishing cut as depicted in figure 2.13.

2.3 Summarising the Results

With the scalar integral coefficients now fully determined we can assemble the contributions from all one-loop structures. The list of coefficients is too long to be listed explicitly in this thesis, however we can discuss the general properties of the results. As previously stated, certain pairs of structures are related by a flip symmetry by sending $\{a, b, c, d, e\}$ to $\{a, e, d, c, b\}$ and this was satisfied by the final result when checked. This is a non trivial check as it brings together various different parts of the calculation.

Let us consider the singularity structure of the full one-loop structures along with the genuine two-loop double cut pieces. We shall focus on the one-loop structures and state results of the two-loop parts when necessary.

As stated in the section on triangles, the use of our new parameterisation massively simplified computation of the triangle coefficients. We found that the leading IR poles from the box and two-mass structures cancel, and that the one-mass triangle structures gave us the full leading IR pole of the amplitude. Expanding the Catani factor to order ϵ, we see that the ϵ^{-1} singularities should have logs. The scalar bubble has ϵ^{-1} singularities with rational coefficients and so we need these to cancel with terms from the two-loop pieces for consistency. In fact this is exactly what happens [2] and so the subleading singularities are also correct.

The following table 2.1 summarises the results of the one-loop and genuine two-loop pieces, categorising terms by their order in epsilon, as well as transcendental weight. The results are compared with the literature results in [1], where a tick indicates agreement and a cross disagreement [2].

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
τ^3	✓	✓	✓	✓	✓
τ^2	✓	✓	✓	✓	✓
τ^1	✓	✓	✓	✓	X
τ^0	✓	✓	✓	✓	

Table 2.1: This table compares the current results of our method to compared to those presented in [1]. ϵ refers to the dimensional regulator and τ^i indicates transcendental weight i.

As we can see the results are almost exactly in agreement. The sole discrepancy is with are logarithms which are finite in epsilon and are functions of Mandelstam variables of positive helicity legs as shown in the table 2.2 below.

π	Log[s_{ab}]	Log[s_{ea}]	Log[s_{bc}]	Log[s_{cd}]	Log[s_{de}]
✓	✓	✓	X	X	X

Table 2.2: A table showing the present agreement with the results given in [1] for ϵ^0 and transcendental weight 1 [2].

If the source of this disagreement came from the one-loop subsector, it could only be from the bubble cut terms. We would need contributions of the form,

$$C(p_a, p_b, p_c, p_d, p_e) \times \big(I_2(s_{bc}) + I_2(s_{de}) - 2I_2(s_{cd})\big), \tag{2.3.1}$$

or indeed any combination of bubble integrals that have cancelling $\frac{1}{\epsilon}$ terms but non-cancelling logarithmic terms, where C is a common kinematic term which can be factored out. This would provide transcendental one terms at finite order without effecting the already correct IR singular terms. When calculating the bubble coefficients, there are two propagators being cut and thus two choices of loop parameterisations. We performed the calculations using both choices which uses very different algebraic manipulations, involves different canonical forms, and in both cases the results agreed. There are also flip symmetries between bubbles which are all satisfied.

For these reasons we are confident that the scalar bubble coefficients are correct. The boxes and triangles are all intricately involved in reproducing the correct singularities and so we can be confident that - despite the above describe discrepancies - the one-loop terms have all been calculated correctly.

We are confident that given the complexity of the calculations involved in the genuine two loop terms and the extent to which our result matches the literature result, we expect that the problem lies in a bug in the code. For further discussion of the various tests and checks carried out on these results, we refer the reader to [2].

Finally there is the blank box, the terms of transcendental weight 0 which are finite in epsilon. We refer to such terms as finite and rational. Due to our use of 4 dimensional unitarity methods, we are unable to reproduce these terms using unitarity methods, and the remainder of this thesis will be dedicated to the reconstruction of these rational pieces of the amplitude. It was hoped that we would be able to go back and fix the errors in the genuine two-loop part of the calculation, however as the calculation of the rational part grew in scope, we were not able to do this.

2.4 Conclusion

In this chapter we introduced the general structure and outline of our method to calculate the two-loop five-point single-minus Yang-Mills amplitude. In particular we used four dimensional unitarity to define a set of one and two loop tensor integrals that match the cuts of the full amplitude, and used one-loop integral methods to write the one loop tensor integrals as linear combinations of scalar box, triangle and bubble integrals.

We ran into some problems when using the method outlined by Forde [77] to evaluate triangle coefficients and thus we developed a new parameterisation that allows us to determine coefficients of one- and two-mass triangles. This parameterisation revealed an unexpected link between box and triangle coefficients of some one-loop structures which suggests that there may be some redundancy in our method and thus presents an avenue of future study to explore the possibility of increasing the efficiency of the process.

We used the method of canonical forms to calculate bubble coefficients. We encountered new terms of order l^3 that had not been previously calculated and thus we extended the method to arbitrarily high order in the loop momentum.

Checks on the one-loop structures such as flip symmetries and comparing the IR singularities of such terms with those of the two-loop structures and the expected IR singularities of the amplitude all indicate that these one-loop structure have been correctly calculated.

The goal of developing this new method of calculating single-minus amplitudes is that it should be easier to generalise the results to higher multiplicity. The numerators of both the one-loop and two-loop pieces have been calculated to n-points.

Calculating the one-loop structures should present no difficulties, with the only new structures that would appear are two-, three- and four-mass boxes and three mass triangles. The quadruple cuts that determine box integral coefficients will be no more difficult with the addition of extra massive vertices, and while our new triangle parameterisation requires at least one massless vertex, the Forde method is able to calculate three mass triangles and the lack of a massless vertex means that we do not expect the same problems that arose for one and two mass triangles.

In terms of the two-loop pieces, at higher multiplicity we would require a form of the 'two-mass easy' box integral that is compact and amenable to ϵ expansion. This is the only extra term and its calculation and reduction to a compact form is a work in progress.

Chapter 3

Calculating the Rational Part: Augmented Recursion I

As discussed, the trade-off for using 4-dimensional unitarity methods rather than D-dimensional methods is that - while the cut-constructible pieces are easier to calculate - we lose rational pieces. In order to reconstruct the rational pieces we use a recursive method.

In the case of the single-minus amplitude $R_5^{(2)}(a^-, b^+, c^+, d^+, e^+)$, we will shift the first and final legs

$$\begin{aligned} \lambda_e \to \lambda_{\hat{e}} &= \lambda_e - z\lambda_a, \\ \tilde{\lambda}_a \to \tilde{\lambda}_{\hat{a}} &= \tilde{\lambda}_a + z\tilde{\lambda}_e. \end{aligned} \tag{3.0.1}$$

It is essential that the shifted amplitude vanishes at $z \to \infty$, and while it is not obvious a priori that this will happen, in this case result is already known, so we can perform the shift on the known result and see that it does indeed vanish. This shift will excite channels that we can divide into three camps: the tree to two-loop channels, the one-loop to one-loop channels, and the new two-loop to tree channel which is new to the single-minus amplitude. The tree to two -loop channels are the easier of the recursion channels to calculate as they deal only in simple poles and thus we can directly perform BCFW recursion on them. These are

$$\begin{aligned} &R_4^{(2)}(\hat{a}^-, b^+, c^+, \hat{K}^+)\frac{1}{s_{d\hat{e}}}A_3^{(0)}(-\hat{K}^-, d^+, \hat{e}^+), \\ &R_4^{(2)}(c^+, d^+, \hat{e}^+, \hat{K}^+)\frac{1}{s_{\hat{a}b}}A_3^{(0)}(-\hat{K}^-, \hat{a}^-, b^+), \\ &R_4^{(2)}(c^+, d^+, \hat{e}^+, \hat{K}^-)\frac{1}{s_{\hat{a}b}}A_3^{(0)}(-\hat{K}^+, \hat{a}^-, b^+). \end{aligned} \tag{3.0.2}$$

The one-loop to one-loop factorisation channel is

$$R_3^{(1)}(d^+, \hat{e}^+, \hat{k}^+)\frac{1}{s_{d\hat{e}}}R_4^{(1)}(\hat{a}^-, b^+, c^+, \hat{k}^+) \tag{3.0.3}$$

which has a double pole and will thus require augmented recursion.

It was later realised that in fact there is now a third factorisation that must be considered in the single-minus case. This is

$$R_3^{(2)}(d^+, \hat{e}^+, \hat{k}^+)\frac{1}{s_{d\hat{e}}}A_4^{(0)}(\hat{a}^-, b^+, c^+, -\hat{k}^-) \tag{3.0.4}$$

where we interpret $R_3^{(2)}$ and the propagator as being the rational part of the two-loop all-plus splitting function [30],

$$R_3^{(2)}(d^+, \hat{e}^+, \hat{k}^+) = \frac{235}{108}\frac{[de][ek][ke]}{s_{de}}. \tag{3.0.5}$$

This channel did not contribute in the all-plus case due to the vanishing of the single-minus tree and thus it a completely novel generalisation of the augmented recursion technique which we will discuss in the next chapter.

In addition to these factorisation channels, there will also be contributions from rational terms that emerge from the cut-constructible part of the amplitude. We dub such terms *rational descendants* of the cut-constructible terms and we shall analyse this at the end of the following chapter. Once again, such terms have not yet featured in an augmented recursion calculation and hence this is another extension of the method.

To summarise, the rational piece is divided into

rational = tree to two-loop easy channels + double-pole channels + rational descendants.

There are two double pole channels as shown above, which we will later subdivide into two categories.

3.1 Tree to Two-Loop Factorisation Channels

We first need to know the rational parts of the all-plus and single-minus two-loop four point amplitudes. These can be found in [70], to be

$$R_2^{(2)}(a^+,b^+,c^+,d^+) = -\frac{i}{9}\frac{[ab][cd]}{\langle ab\rangle\langle cd\rangle}\left(\frac{s_{ac}^2}{s_{ab}s_{bc}}+8\right)$$
$$R_2^{(2)}(a^-,b^+,c^+,d^+) = -5i\frac{\langle ab\rangle\langle ad\rangle[bd]}{\langle cd\rangle\langle bc\rangle\langle bd\rangle}\left(\frac{s_{ab}^2 - s_{ac}s_{bc}}{s_{ab}s_{bc}}\right). \tag{3.1.1}$$

The first channel has a s_{de} pole, so we have

$$\langle d\hat{e}\rangle = 0 = \langle de\rangle - z_{de}\langle da\rangle$$
$$z_{de} = \frac{\langle de\rangle}{\langle da\rangle}. \tag{3.1.2}$$

Evaluating the first channel with this, we have

$$-5\frac{\langle ac\rangle[bc]}{\langle ea\rangle\langle bc\rangle^2\langle bd\rangle\langle cd\rangle^2\langle de\rangle}(\langle ac\rangle\langle ad\rangle\langle bc\rangle\langle bd\rangle + \langle ab\rangle^2\langle cd\rangle^2). \tag{3.1.3}$$

The second and third channels have an s_{ab} pole, and the residue on this pole z_{ab} is

$$[\hat{a}b] = 0 = [ab] + z_{ab}[eb]$$
$$z_{ab} = \frac{[ab]}{[be]}. \tag{3.1.4}$$

The second channel gives

$$-\frac{1}{9}\frac{[be]^2([bd]^2[ce]^2 + 8[bc][cd][de][be])}{[ab][bc][cd][de][ea]\langle cd\rangle^2}, \tag{3.1.5}$$

while the third vanishes.

3.2 One-Loop to One-Loop Factorisation Channel

We move on to the one loop - one loop channel,

$$R_3^{(1)}(d^+,\hat{e}^+,\hat{k}^+)\frac{1}{s_{d\hat{e}}}R_4^{(1)}(\hat{a}^-,b^+,c^+,\hat{k}^+) \tag{3.2.1}$$

which has a double pole. For reasons stated earlier, we will have to perform augmented recursion. In this case, this means we must evaluate the following integral represented in figure 3.1:

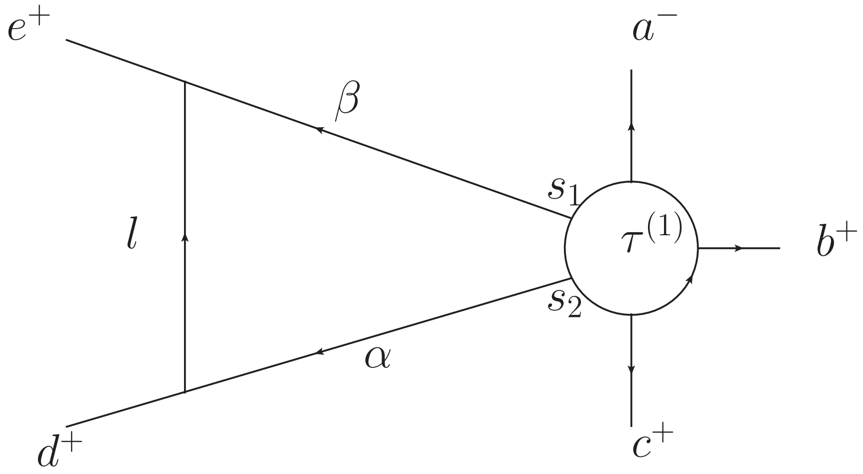


Figure 3.1: This image depicts the overall structure of the integrals that we will solve in this section. There are three integrals, depending on s_1 and s_2.

where there are three helicity choices for (s_1, s_2) which will contribute. The first two are $(s_1, s_2) = (-,+)$ and $(+,-)$ which will give adjacent and non-adjacent MHV currents respectively. The third is $(s_1, s_2) = (+,+)$ which will give a single-minus current, but does not arise from the one-loop to one-loop channel and will be dealt with later in this chapter. This third current is the first additional structure that appears in the augmented recursion of single-minus amplitudes that did not appear in the all-plus case. Finally the $(s_1, s_2) = (-,-)$ cannot contribute as it would force at least one of the tree vertices on the left to be all-plus and thus vanish. Our chosen convention is to draw diagrams such as above (3.1) with the current on the right. The left hand side corresponds to two tree vertices and so we will collectively refer to such structures as 'tree on the left' contributions to the rational part of the amplitude. This is in contrast to the 'loop on the left' contributions which we shall discuss in the next chapter.

We will begin by deriving and integrating the two MHV currents as these are most easily compared to the prior uses of augmented recursion in the all-plus cases. We will then use this knowledge to tackle the single-minus current and begin the extension of the method to tackle the new structures that appear in the single minus case.

First, the current with adjacent momenta of negative helicity, $\tau^{(1)}(\alpha^-, a^-, b^+, c^+, \beta^-)$.The rational part of the base amplitude is found in [79]:

$$R_5^{(1)}(\alpha^-,a^-,b^+,c^+,\beta^+) = -\frac{64i}{9}\frac{\langle\alpha a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle\langle\beta\alpha\rangle} + \frac{i}{3}\frac{[ac][b\beta]^3}{\langle bc\rangle[ab]^2[\alpha a][\beta\alpha]} + \frac{2i}{3}\frac{\langle\alpha a\rangle\langle c\alpha\rangle[b\beta][bc]}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle[ab]^2}$$
$$\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle c\alpha\rangle[bc][\beta\alpha]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2} + \frac{i}{6}\frac{\langle ac\rangle\langle\alpha a\rangle\langle c\alpha\rangle[bc][c\beta]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2}$$
$$-\frac{i}{2}\frac{\langle ab\rangle\langle ac\rangle\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2} + \frac{i}{6}\frac{\langle ab\rangle\langle ac\rangle s_{\beta\alpha}\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}$$
$$-\frac{i}{2}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2} + \frac{i}{6}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle s_{\beta\alpha}\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}, \tag{3.2.2}$$

To derive the current, we first need to know the poles of the current. The current has two poles in the $s_{\alpha\beta}\to 0$ limit as shown in diagram 3.2:

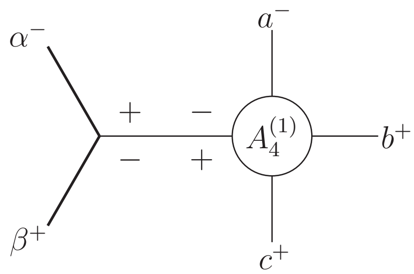


Figure 3.2: Factorisations of the adjacent MHV current on the $s_{\alpha\beta}\to 0$ pole.

In the axial gauge formalism, and taking the rational parts of the 4-point amplitudes below from [70], we have the leading pole

$$A_4^{(1)}(k^-,a^-,b^+,c^+)\frac{i}{s_{\alpha\beta}}A_3^{(0)}(-k^+,\beta^+,\alpha^-) = -\frac{64i}{9}\frac{\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}, \tag{3.2.3}$$

and subleading

$$A_4^{(1)}(a^-,b^+,c^+,k^+)\frac{i}{s_{\alpha\beta}}A_3^{(0)}(\alpha^-,-k^-,\beta^+) = \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{[ka][kq]\langle ck\rangle}\frac{[\beta q]^2}{[\alpha q]}\frac{\langle\alpha k\rangle}{s_{\alpha\beta}}. \tag{3.2.4}$$

We can rewrite the subleading pole as

$$\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[b\beta]^2[\beta q]}{[\alpha q]\langle ck\rangle[ka]}\frac{\langle\alpha\beta\rangle}{s_{\alpha\beta}}+\frac{\delta}{s_{\alpha\beta}}\left(-\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[b\beta][\beta q][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka]}-\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[\beta q]^2[bk][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka][kq]}\right) \tag{3.2.5}$$

where

$$\delta=\frac{\alpha^2}{2q\cdot\alpha}+\frac{\beta^2}{2q\cdot\beta}-\frac{k^2}{2k\cdot q}, \tag{3.2.6}$$

so this pole term can be rewritten as

$$\begin{aligned}\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{[ka][kq]\langle ck\rangle}\frac{[\beta q]^2}{[\alpha q]}\frac{\langle\alpha k\rangle}{s_{\alpha\beta}}&=\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[\beta q]}{[\alpha q]}\frac{[b\beta]^2}{\langle ck\rangle[ka]}\frac{\langle\beta\alpha\rangle}{s_{\alpha\beta}}\\&+\frac{1}{2k\cdot q}\left(\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[b\beta][\beta q][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka]}+\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[\beta q]^2[bk][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka][kq]}\right)+\mathcal{O}(\alpha^2,\beta^2).\end{aligned} \tag{3.2.7}$$

As the base amplitude only has one term with $\langle\alpha\beta\rangle$ in the denominator, the leading pole will be simple to incorporate. The same is not true for the subleading square pole term. There are multiple terms with $[\alpha\beta]$ in the denominator and only one pole, so we will need to combine these terms into a single term with this in the denominator plus other terms which lack it. Repeated use of conservation of momentum and Schouten's identity allows us to rewrite the amplitude as

$$\begin{aligned}R_5^{(1)}(\alpha^-,a^-,b^+,c^+,\beta^+)=&-\frac{64i}{9}\frac{\langle\alpha a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle\langle\beta\alpha\rangle}+\frac{i}{3}\frac{[ac][b\beta]^3}{\langle bc\rangle[ab]^2[\alpha a][\beta\alpha]}+\frac{2i}{3}\frac{\langle\alpha a\rangle\langle c\alpha\rangle[b\beta][bc]}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle[ab]^2}\\&\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle c\alpha\rangle[bc][\beta\alpha]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2}+\frac{i}{6}\frac{\langle ac\rangle\langle\alpha a\rangle\langle c\alpha\rangle[bc][c\beta]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2}\\&-\frac{i}{2}\frac{\langle ab\rangle\langle ac\rangle\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}+\frac{i}{6}\frac{\langle ab\rangle\langle ac\rangle s_{\beta\alpha}\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}\\&-\frac{i}{2}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}+\frac{i}{6}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle s_{\beta\alpha}\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}.\end{aligned} \tag{3.2.8}$$

To derive the current we need to take the amplitude sufficiently off-shell so that we can incorporate the above derived poles. To do this we use two very important identities. First we can take the $\langle\alpha\beta\rangle$ pole off-shell using

$$\frac{1}{\langle\alpha\beta\rangle\langle\beta a\rangle}=\frac{1}{\langle\alpha q\rangle\langle\beta q\rangle^2}\frac{1}{[q|P_{\alpha\beta}|a\rangle}\left(\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}[q|P_{\alpha\beta}|q\rangle+\frac{\langle q\beta\rangle\langle qa\rangle[q|\alpha|q\rangle}{\langle\beta a\rangle}\right), \tag{3.2.9}$$

and use the following identities to massage various terms into a favourable form, either to make the pole explicit, or later on to ease in integration

$$\begin{aligned}\frac{[\beta|P^\flat_{\alpha\beta}|b\rangle}{[\beta|P_{\alpha\beta}|q\rangle} &= \frac{[q|P_{\alpha\beta}|b\rangle}{[q|P_{\alpha\beta}|q\rangle} + s_{\alpha\beta}\frac{\langle qb\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2),\\ \frac{[\alpha|P^\flat_{\alpha\beta}|b\rangle}{[\alpha|P_{\alpha\beta}|q\rangle} &= \frac{[q|P_{\alpha\beta}|b\rangle}{[q|P_{\alpha\beta}|q\rangle} + s_{\alpha\beta}\frac{\langle qb\rangle[\alpha q]}{[\alpha|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2),\end{aligned} \tag{3.2.10}$$

where in this case $P^\flat_{\alpha\beta} = \alpha^\flat + \beta^\flat$. Using both of these we can extract the first pole term from the first term in the amplitude

$$\begin{aligned}-\frac{64i}{9}\frac{\langle\alpha a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle\langle\beta\alpha\rangle} \to &-\frac{64i}{9}\frac{\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\left(1+s_{\alpha\beta}\frac{\langle qa\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^3\\ &\times\left(\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}} + \frac{\langle q\beta\rangle\langle qc\rangle}{\langle\beta c\rangle}\frac{[q|\alpha|q\rangle}{[q|P_{\alpha\beta}|q\rangle}\right).\end{aligned} \tag{3.2.11}$$

and we rewrite the second term as

$$\begin{aligned}\frac{i}{3}\frac{[ac][b\beta]^3}{\langle bc\rangle[ab]^2[\alpha a][\beta\alpha]} \to &-\frac{i}{3}\frac{[ac][b\beta]^2[bq]}{\langle bc\rangle[ab]^2[\alpha a][\alpha q]} - \frac{i}{3}\frac{\langle\beta q\rangle[ac][b\beta]^2[\beta q]}{\langle bc\rangle\langle kq\rangle[ab][\alpha a][\alpha q][ka]}\\ &+\frac{i}{3}\frac{s_{ac}[b\beta]^2[\beta q]}{\langle bc\rangle\langle ck\rangle[ab][\alpha q][\beta\alpha][ka]} - \frac{i}{3}\frac{\langle\alpha\beta\rangle\langle cq\rangle[ac][b\beta]^2[bq][\beta q]}{\langle bc\rangle\langle ck\rangle[ab]^2[\alpha q][ka][q|P_{\alpha\beta}|q\rangle}.\end{aligned} \tag{3.2.12}$$

By replacing the third term above with the pole term, we have our current,

$$\begin{aligned}\tau_5^{(1)}(\alpha^-,a^-,b^+,c^+,\beta^+) = &-\frac{64i}{9}\frac{\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\left(1+s_{\alpha\beta}\frac{\langle qa\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^3\\ &\times\left(\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}} + \frac{\langle q\beta\rangle\langle qc\rangle}{\langle\beta c\rangle}\frac{[q|\alpha|q\rangle}{[q|P_{\alpha\beta}|q\rangle}\right) + \frac{2i}{3}\frac{\langle\alpha a\rangle\langle c\alpha\rangle[b\beta][bc]}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle[ab]^2} + \frac{i}{3}\frac{\langle\alpha a\rangle^2\langle c\alpha\rangle[bc][\beta\alpha]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2}\\ &+\frac{i}{6}\frac{\langle ac\rangle\langle\alpha a\rangle\langle c\alpha\rangle[bc][c\beta]}{\langle ab\rangle^2\langle bc\rangle\langle c\beta\rangle[ab]^2} - \frac{i}{2}\frac{\langle ab\rangle\langle ac\rangle\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2} + \frac{i}{6}\frac{\langle ab\rangle\langle ac\rangle s_{\beta\alpha}\langle c\alpha\rangle^2[bc]^2[c\beta]}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2}\\ &-\frac{i}{2}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2} + \frac{i}{6}\frac{\langle ac\rangle^2\langle\beta\alpha\rangle s_{\beta\alpha}\langle c\alpha\rangle[bc][c\beta]^2}{s_{ab}^2\langle bc\rangle\langle c\beta\rangle(s_{ab}-s_{\alpha\beta})^2} - \frac{i}{3}\frac{[ac][b\beta]^2[bq]}{\langle bc\rangle[ab]^2[\alpha a][\alpha q]}\\ &-\frac{i}{3}\frac{\langle\beta q\rangle[ac][b\beta]^2[\beta q]}{\langle bc\rangle[ab][\alpha a][\alpha q][a|P_{de}|q\rangle} + \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{[ka][kq]\langle ck\rangle}\frac{[\beta q]^2}{[\alpha q]}\frac{\langle\alpha k\rangle}{s_{\alpha\beta}}\\ &-\frac{1}{[q|P_{de}|q\rangle}\left(\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[b\beta][\beta q][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka]} + \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[\beta q]^2[bk][b|q|\alpha\rangle}{[\alpha q]\langle ck\rangle[ka][kq]}\right)\\ &-\frac{i}{3}\frac{\langle\alpha\beta\rangle\langle cq\rangle[ac][b\beta]^2[bq][\beta q]}{\langle bc\rangle\langle ck\rangle[ab]^2[\alpha q][ka][q|P_{\alpha\beta}|q\rangle},\\ &= \tau^{dp}_{adj} + \tau^{sq}_{adj} + \tau^{r}_{adj},\end{aligned} \tag{3.2.13}$$

where we define

$$\tau_{adj}^{dp} = -\frac{64i}{9}\frac{\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}, \tag{3.2.14}$$

and

$$\tau_{adj}^{sq} = \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{[ka][kq]\langle ck\rangle}\frac{[\beta q]^2}{[\alpha q]}\frac{\langle\alpha k\rangle}{s_{\alpha\beta}} \tag{3.2.15}$$

since the first term gives the double pole, while the second will be integrated separate to the rest of the current.

The second current is longer and the separation of the pole terms are more subtle. We begin with the rational piece of the one-loop five point non-adjacent MHV amplitude also found in [79],

$$\begin{aligned}
R_5^{(1)}(\beta^-,\alpha^+,a^-,b^+,c^+) = &-\frac{64}{9}\frac{\langle\beta a\rangle^4}{\langle\beta\alpha\rangle\langle\alpha a\rangle\langle ab\rangle\langle bc\rangle\langle c\beta\rangle} + \frac{1}{3}\frac{[\alpha b]^2[\alpha c]^2}{[\beta\alpha][\alpha a][ab]\langle bc\rangle[c\beta]} \\
&- \frac{1}{3}\frac{\langle\beta\alpha\rangle\langle b\beta\rangle^2[\alpha b]^3}{\langle bc\rangle\langle c\beta\rangle\langle\alpha b\rangle[\alpha a][ab]s_{c\beta}} + \frac{1}{3}\frac{\langle a\alpha\rangle\langle ca\rangle^2[\alpha c]^3}{\langle cb\rangle\langle ba\rangle\langle\alpha c\rangle[\alpha\beta][\beta c]s_{ab}} \\
&+ \frac{1}{6}\frac{\langle\beta a\rangle^2[\alpha b][\alpha c]}{s_{ab}\langle bc\rangle s_{c\beta}} - \frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{ab})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle} \\
&- \frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{a\alpha})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle} - \frac{1}{3}\frac{\langle\alpha a\rangle^2 s_{a\alpha}\langle b\beta\rangle^3[\alpha b]^3}{s_{c\beta}(s_{c\beta}-s_{a\alpha})^3\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} \\
&+ \frac{1}{3}\frac{\langle\alpha a\rangle^2 s_{c\beta}\langle b\beta\rangle^3[\alpha b]^3}{s_{a\alpha}(s_{c\beta}-s_{a\alpha})^3\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} + \frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{ab}-s_{\alpha\beta})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle} \\
&+ \frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{ab}-s_{c\beta})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle} - \frac{1}{3}\frac{s_{ab}+s_{\alpha\beta}}{(s_{ab}-s_{\alpha\beta})^2 s_{ab}s_{\alpha\beta}}\frac{\langle\alpha\beta\rangle^2\langle ca\rangle^3[\alpha c]^3}{\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle} \\
&+ \left(\frac{s_{c\beta}}{s_{ab}} - \frac{s_{ab}}{s_{c\beta}}\right)\frac{1}{6(s_{c\beta}-s_{ab})^3}\left(\frac{2\langle ab\rangle^2\langle b\beta\rangle\langle\beta\alpha\rangle^2[\alpha b]^3}{\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} - \right. \\
&\left.\frac{2\langle a\alpha\rangle^2\langle\beta c\rangle^2\langle ca\rangle[\alpha c]^3}{\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle} + \frac{\langle\beta a\rangle[\alpha b][\alpha c](-\langle ab\rangle\langle\alpha\beta\rangle[b\alpha] + \langle\alpha a\rangle\langle\beta c\rangle[c\alpha])}{\langle bc\rangle}\right).
\end{aligned} \tag{3.2.16}$$

The poles of the current are taken from the following diagram 3.3:

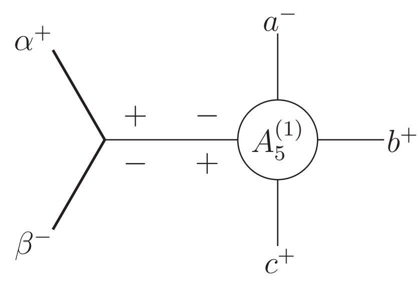


Figure 3.3: Factorisations of the non-adjacent MHV current on the $s_{\alpha\beta} \to 0$ pole.

The leading pole will take the form

$$A_4^{(1)}(k^-, a^-, b^+, c^+)\frac{i}{s_{\alpha\beta}}A_3^{(0)}(\alpha^+, -k^+, \beta^-) = -\frac{64i}{9}\frac{\langle\beta q\rangle^2}{\langle\alpha q\rangle^2}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}\frac{1}{\langle ab\rangle\langle bc\rangle}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}, \tag{3.2.17}$$

and the subleading pole looks like

$$A_4^{(1)}(a^-, b^+, c^+, k^+)\frac{i}{s_{\alpha\beta}}A_3^{(0)}(-k^-, \beta^-, \alpha^+) = \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{\langle ck\rangle[ka]}\frac{[\alpha q]^2}{[\beta q][kq]}\frac{\langle k\beta\rangle}{s_{\alpha\beta}}, \tag{3.2.18}$$

which we can expand as

$$\begin{aligned}
&\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{1}{\langle ck\rangle[ka]}\frac{[\alpha q]}{[\beta q]}\frac{1}{s_{\alpha\beta}}\left([b\alpha]^2\langle\alpha\beta\rangle - \delta\langle\beta q\rangle[bq]\big([b\alpha] + \frac{[\alpha q][bk]}{[kq]}\big)\right)\\
&= \frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{1}{\langle ck\rangle[ka]}\frac{[\alpha q]}{[\beta q]}\left(\frac{[b\alpha]^2\langle\alpha\beta\rangle}{s_{\alpha\beta}} + \frac{1}{2k\cdot q}\langle\beta q\rangle[bq]\big([b\alpha] + \frac{[\alpha q][bk]}{[kq]}\big)\right) + \mathcal{O}(\alpha^2, \beta^2)
\end{aligned} \tag{3.2.19}$$

Again we see that there is already only a single term with the angle pole in the denominator, however there is work to be done in order to isolate and consolidate the square pole term. We first rewrite the amplitude so that it has a single square pole term using the same tricks of Schouten's identity and conservation of momentum,

$$
\begin{aligned}
R_5^{(1)}(\beta^-,\alpha^+,a^-,b^+,c^+) &= -\frac{64}{9}\frac{\langle\beta a\rangle^4}{\langle\beta\alpha\rangle\langle\alpha a\rangle\langle ab\rangle\langle bc\rangle\langle c\beta\rangle} - \frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{ab})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle} \\
&- \frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{a\alpha})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle} - \frac{1}{3}\frac{\langle\alpha a\rangle^2\langle b\beta\rangle^3[\alpha b]^3}{s_{\alpha a}s_{c\beta}(s_{c\beta}-s_{a\alpha})\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} + \frac{1}{3}\frac{\langle ab\rangle^2\langle b\beta\rangle\langle\beta\alpha\rangle^2[\alpha b]^3}{s_{ab}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} \\
&+ \frac{1}{3}\frac{\langle ab\rangle^2\langle b\beta\rangle\langle\beta\alpha\rangle^2[\alpha b]^3}{s_{c\beta}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle} - \frac{1}{3}\frac{\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^3}{s_{c\beta}\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle[ab][\alpha a]} + \frac{1}{6}\frac{\langle\beta a\rangle^2[\alpha b][\alpha c]}{s_{ab}\langle bc\rangle s_{c\beta}} \\
&+ \frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{ab}-s_{\alpha\beta})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle} + \frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{c\beta}-s_{ab})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle} - \frac{1}{3}\frac{\langle\alpha a\rangle^2\langle\beta c\rangle^2\langle ca\rangle[\alpha c]^3}{s_{ab}(s_{c\beta}-s_{ab})^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle} \\
&- \frac{1}{3}\frac{\langle\alpha a\rangle^2\langle\beta c\rangle^2\langle ca\rangle[\alpha c]^3}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle} - \frac{\langle\alpha\beta\rangle^2\langle ca\rangle^3[\alpha c]^3}{s_{ab}(s_{ab}-s_{\alpha\beta})^2\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle} + \frac{1}{3}\frac{s_{\alpha\beta}\langle\alpha\beta\rangle^2\langle ca\rangle^3[\alpha c]^3}{s_{ab}^2(s_{ab}-s_{\alpha\beta})^2\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle} \\
&- \frac{1}{6}\frac{\langle ab\rangle\langle\alpha\beta\rangle\langle\beta a\rangle[\alpha b][\alpha c][b\alpha]}{s_{ab}(s_{c\beta}-s_{ab})^2\langle bc\rangle} - \frac{1}{6}\frac{\langle ab\rangle\langle\alpha\beta\rangle\langle\beta a\rangle[\alpha b][\alpha c][b\alpha]}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle bc\rangle} - \frac{1}{3}\frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle ca\rangle^2[\alpha c]^3}{s_{ab}^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle[\beta c]} \\
&+ \frac{1}{6}\frac{\langle\alpha a\rangle\langle\beta a\rangle\langle\beta c\rangle[\alpha b][\alpha c][c\alpha]}{s_{ab}(s_{c\beta}-s_{ab})^2\langle bc\rangle} + \frac{1}{6}\frac{\langle\alpha a\rangle\langle\beta a\rangle\langle\beta c\rangle[\alpha b][\alpha c][c\alpha]}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle bc\rangle} + \frac{1}{3}\frac{\langle\alpha\beta\rangle[\alpha b]^2[\alpha c]^2}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][c\beta]} \\
&- \frac{1}{3}\frac{\langle ac\rangle\langle\beta a\rangle[\alpha c]^2[bc]}{s_{ab}\langle bc\rangle\langle ab\rangle[ab][c\beta]} - \frac{1}{3}\frac{\langle c\beta\rangle[\alpha b][\alpha c]^2[bc]}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][c\beta]} - \frac{1}{3}\frac{s_{ac}[\alpha b]^2[\alpha c]^2}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][\beta\alpha][c\beta]}.
\end{aligned}
\tag{3.2.20}
$$

We can then rewrite the square term as

$$
\begin{aligned}
-\frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha c]^2}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][\beta\alpha][c\beta]} \to &-\frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha q]\langle\alpha\beta\rangle}{\langle bc\rangle\langle ab\rangle[ab]^2[\beta q]\langle ck\rangle[ka]} + \frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha q]\langle\beta q\rangle[ac]}{\langle bc\rangle\langle ab\rangle[ab]^2[\beta q]\langle ck\rangle[ka][\alpha a]} \\
&+ \frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha q]}{\langle bc\rangle[ab][\beta\alpha][\beta q]\langle ck\rangle[ka]} - \frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha c][qc]}{\langle ab\rangle\langle bc\rangle[ab]^2[\alpha a][\beta q][c\beta]} \\
&- \frac{i}{3}\frac{s_{ac}[\alpha b]^2[\alpha q]\langle\alpha\beta\rangle[q|c|q\rangle}{\langle bc\rangle\langle ab\rangle[ab]^2[\beta q]\langle ck\rangle[ka][q|P_{\alpha\beta}|q\rangle},
\end{aligned}
\tag{3.2.21}
$$

so we can continue the amplitude to the current in the standard manner. We have our second current

$$
\begin{aligned}
\tau_5^{(1)}(\beta^-,\alpha^+,a^-,b^+,c^+) &= -\frac{64i}{9}\frac{\langle\beta q\rangle^2}{\langle\alpha q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\left(\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}+\frac{\langle q\beta\rangle\langle qc\rangle}{\langle\beta c\rangle}\frac{[q|\alpha|q\rangle}{[q|P_{\alpha\beta}|q\rangle}\right)\\
&\times\left(1+s_{\alpha\beta}\frac{\langle qa\rangle[\alpha q]}{[\alpha|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^4\left(1+s_{\alpha\beta}\frac{\langle qa\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^{-1}-i\frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{ab})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle}\\
&-\frac{i}{3}\frac{\langle c\beta\rangle[\alpha b][\alpha c]^2[bc]}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][c\beta]}-i\frac{\langle ab\rangle\langle\alpha a\rangle\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^2}{s_{c\beta}(s_{c\beta}-s_{a\alpha})\langle\alpha b\rangle^2\langle bc\rangle\langle c\beta\rangle}+\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle b\beta\rangle^3[\alpha b]^3}{s_{\alpha a}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle}\\
&+\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle b\beta\rangle^3[\alpha b]^3}{s_{c\beta}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle}+\frac{i}{3}\frac{\langle ab\rangle^2\langle b\beta\rangle\langle\beta\alpha\rangle^2[\alpha b]^3}{s_{ab}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle}+\frac{i}{3}\frac{\langle ab\rangle^2\langle b\beta\rangle\langle\beta\alpha\rangle^2[\alpha b]^3}{s_{c\beta}(s_{c\beta}-s_{a\alpha})^2\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle}\\
&-\frac{i}{3}\frac{\langle b\beta\rangle^2\langle\beta\alpha\rangle[\alpha b]^3}{s_{c\beta}\langle\alpha b\rangle\langle bc\rangle\langle c\beta\rangle[ab][\alpha a]}+\frac{i}{6}\frac{\langle\beta a\rangle^2[\alpha b][\alpha c]}{s_{ab}\langle bc\rangle s_{c\beta}}+i\frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{ab}-s_{\alpha\beta})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle}\\
&+i\frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle\beta c\rangle\langle ac\rangle^2[\alpha c]^2}{s_{ab}(s_{c\beta}-s_{ab})\langle\alpha c\rangle^2\langle ab\rangle\langle bc\rangle}-\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle\beta c\rangle^2\langle ca\rangle[\alpha c]^3}{s_{ab}(s_{c\beta}-s_{ab})^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle}-\frac{i}{3}\frac{\langle\alpha a\rangle^2\langle\beta c\rangle^2\langle ca\rangle[\alpha c]^3}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle}\\
&-i\frac{\langle\alpha\beta\rangle^2\langle ca\rangle^3[\alpha c]^3}{s_{ab}(s_{ab}-s_{\alpha\beta})^2\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle}+\frac{i}{3}\frac{s_{\alpha\beta}\langle\alpha\beta\rangle^2\langle ca\rangle^3[\alpha c]^3}{s_{ab}^2(s_{ab}-s_{\alpha\beta})^2\langle\alpha c\rangle\langle ab\rangle\langle bc\rangle}-\frac{i}{6}\frac{\langle ab\rangle\langle\alpha\beta\rangle\langle\beta a\rangle[\alpha b][\alpha c][b\alpha]}{s_{ab}(s_{c\beta}-s_{ab})^2\langle bc\rangle}\\
&-\frac{i}{6}\frac{\langle ab\rangle\langle\alpha\beta\rangle\langle\beta a\rangle[\alpha b][\alpha c][b\alpha]}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle bc\rangle}-\frac{i}{3}\frac{\langle a\alpha\rangle\langle\alpha\beta\rangle\langle ca\rangle^2[\alpha c]^3}{s_{ab}^2\langle\alpha c\rangle\langle bc\rangle\langle ab\rangle[\beta c]}+\frac{i}{6}\frac{\langle\alpha a\rangle\langle\beta a\rangle\langle\beta c\rangle[\alpha b][\alpha c][c\alpha]}{s_{ab}(s_{c\beta}-s_{ab})^2\langle bc\rangle}\\
&+\frac{i}{6}\frac{\langle\alpha a\rangle\langle\beta a\rangle\langle\beta c\rangle[\alpha b][\alpha c][c\alpha]}{s_{c\beta}(s_{c\beta}-s_{ab})^2\langle bc\rangle}+\frac{i}{3}\frac{\langle\alpha\beta\rangle[\alpha b]^2[\alpha c]^2}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][c\beta]}-\frac{i}{3}\frac{\langle ac\rangle\langle\beta a\rangle[\alpha c]^2[bc]}{s_{ab}\langle bc\rangle\langle ab\rangle[ab][c\beta]}\\
&+\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{\langle ck\rangle[ka]}\frac{[\alpha q]^2}{[\beta q][kq]}\frac{\langle k\beta\rangle}{s_{\alpha\beta}}-\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{1}{\langle ck\rangle[ka]}\frac{[\alpha q]}{[\beta q]}\frac{1}{2k\cdot q}\langle\beta q\rangle[bq]\big([b\alpha]+\frac{[\alpha q][bk]}{[kq]}\big).\\
&=\tau_{na}^{dp}+\tau_{na}^{sq}+\tau_{na}^{r},
\end{aligned}
\tag{3.2.22}
$$

where

$$
\tau_{na}^{dp}=-\frac{64i}{9}\frac{\langle\beta q\rangle^2}{\langle\alpha q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\frac{[q|P_{\alpha\beta}|a\rangle^3}{[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}},\tag{3.2.23}
$$

and

$$
\tau_{na}^{sq}=\frac{i}{3}\frac{s_{ac}}{[ab]\langle bc\rangle}\frac{[bk]^2}{\langle ck\rangle[ka]}\frac{[\alpha q]^2}{[\beta q][kq]}\frac{\langle k\beta\rangle}{s_{\alpha\beta}},\tag{3.2.24}
$$

for the same reasons as in the adjacent case.

3.3 Integrating the Currents

In order to integrate the currents, we would first like to ‘massage’ the terms into forms most amenable to integration. Since α and β are the integration variables, anything we can do to reduce the number of times these occur will simplify the integral.

In accordance with our diagrams the integrals are

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{\langle dq\rangle\langle eq\rangle}\frac{\langle \beta q\rangle^2}{\langle \alpha q\rangle^2}\tau^{(1)}(\beta^+,\alpha^-,a^-,b^+,c^+), \tag{3.3.1}$$

and

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{\langle dq\rangle\langle eq\rangle}\frac{\langle \alpha q\rangle^2}{\langle \beta q\rangle^2}\tau^{(1)}(\beta^-,\alpha^+,a^-,b^+,c^+). \tag{3.3.2}$$

In both cases the integration measure is

$$\frac{d^D l}{l^2\alpha^2\beta^2}$$

and the loop momentum can be parameterised as

$$l = A\lambda_d\tilde{\lambda}_d + B\lambda_e\tilde{\lambda}_e + C\lambda_d\tilde{\lambda}_e + D\lambda_e\tilde{\lambda}_d$$

so that, in terms of scales,

$$\frac{d^D l}{l^2\alpha^2\beta^2} \sim \frac{1}{s_{de}}.$$

Since the integration measure will give one power of s_{de} in the denominator, we are free to expand the currents to order s_{de}^0 as any higher powers of s_{de} will not have a pole and thus will not take part in recursion and will not contribute to the final piece. This fact helps us both reduce the number of integrals, but also helps us simplify terms as we will see later. This also means that any terms in the current with $\langle\alpha\beta\rangle$ in the numerator can be immediately discarded as they will have no pole after integration and thus cannot contribute to the final answer.

Integration of the τ^{dp} pieces pose no particular difficulty. Taking the dp piece of the adjacent current for example and remembering that $\alpha+\beta = d+e$, we have the integral

$$-\frac{64i}{9}\frac{[q|P_{de}|a\rangle^3}{[q|P_{de}|c\rangle[q|P_{de}|q\rangle^2}\frac{1}{\langle ab\rangle\langle bc\rangle}\frac{1}{s_{de}}\frac{1}{\langle dq\rangle\langle eq\rangle}\int\frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\langle q|\alpha\beta|q\rangle, \tag{3.3.3}$$

and the equivalent piece in the non-adjacent current will give an identical integral. The dp pieces in the currents are the terms which give us the double pole terms. Extracting the integral part of this, we use the identity

$$\frac{1}{D_1^{\nu_1}D_2^{\nu_2}D_3^{\nu_3}\cdots D_N^{\nu_N}} = \frac{\Gamma\left(\sum_{i=1}^N \nu_i\right)}{\prod_{i=1}^N \Gamma\left(\nu_i\right)} \int_0^\infty \prod_{i=1}^N dx_i \, x_i^{\nu_i-1} \frac{\delta\left(1-\sum_{j=1}^N x_j\right)}{\left[x_1D_1+x_2D_2+\cdots+x_ND_N\right]^{\sum_{i=1}^N \nu_i}} \tag{3.3.4}$$

to perform the Feynman parameterisation

$$\frac{1}{l^2\alpha^2\beta^2} = \frac{1}{l^2(l-e)^2(l+d)^2} = \Gamma(3)\int \frac{[du]}{(p^2-\Delta)^3}, \tag{3.3.5}$$

where $p = l-(u_2e-u_3d)$, and $\Delta = -u_2u_3s_{de}$ and we define the shorthand $[du] = du_1...du_n\delta(1-u_1-...-u_n)$, We then use

$$\begin{aligned}\int \frac{d^D\ell}{\pi^{\frac{D}{2}}} \frac{\ell^{\mu_1}\cdots\ell^{\mu_{2m}}}{\left[\ell^2-R^2+i\delta\right]^\sigma} &= i(-1)^\sigma\left[\left(g^{\cdot\cdot}\right)^{\otimes m}\right]^{\{\mu_1\cdots\mu_{2m}\}}\left(-\frac{1}{2}\right)^m \frac{\Gamma\left(\sigma-m-\frac{D}{2}\right)}{\Gamma\left(\sigma\right)}\left(R^2-i\delta\right)^{-\sigma+m+D/2} \\ &= i(-1)^{D/2}\left[\left(g^{\cdot\cdot}\right)^{\otimes m}\right]^{\{\mu_1\cdots\mu_{2m}\}}\left(\frac{1}{2}\right)^m \frac{\Gamma\left(\sigma-m-\frac{D}{2}\right)}{\Gamma\left(\sigma\right)}\left(-R^2+i\delta\right)^{-\sigma+m+D/2}\end{aligned}$$

where the μ_i indices are distributed over the m copies of $g^{\mu\nu}$ in all possible ways. Both of the above stated identities can be found in [80].

Using this parameterisation, the numerator of the integral is

$$[d|p+u_2e|q\rangle[e|p-u_3d|q\rangle\langle q|P_{de}(p+u_2e+(1-u_3)d)|q\rangle. \tag{3.3.6}$$

In each case the p comes in the form of a term like $[x|p|q\rangle$ and since in Feynman integration $p^\mu p^\nu \sim p^2g^{\mu\nu}$ we can use the Fierz identity (1.1.16) to see $[x|p|q\rangle[y|p|q\rangle \sim p^2[xy]\langle qq\rangle = 0$. Thus only the 'scalar' part of the integral survives, and the integral is

$$\begin{aligned}\int \frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\langle q|\alpha\beta|q\rangle =& i\Gamma(1+\epsilon)\frac{[d|e|q\rangle[e|d|q\rangle\langle q|de|q\rangle}{s_{de}^{1+2\epsilon}}\int [du]u_1u_2^{-\epsilon}u_3^{-\epsilon} \\ =& \frac{ic_\Gamma}{6}\frac{[d|e|q\rangle[e|d|q\rangle\langle q|de|q\rangle}{s_{de}}\end{aligned} \tag{3.3.7}$$

We will come back to the sq terms at the end, and now we turn to the remainder of the terms in the current. We can use the results,

$$\begin{aligned}\frac{\langle \alpha x\rangle}{\langle \alpha q\rangle} &= \frac{[q|P_{de}|x\rangle}{[q|P_{de}|q\rangle} + s_{\alpha\beta}\frac{[\beta q]\langle qx\rangle}{[q|P_{de}|q\rangle[\beta^\flat|P_{de}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2)\\ \frac{\langle \beta x\rangle}{\langle \beta q\rangle} &= \frac{[q|P_{de}|x\rangle}{[q|P_{de}|q\rangle} + s_{\alpha\beta}\frac{[\alpha q]\langle qx\rangle}{[q|P_{de}|q\rangle[\alpha^\flat|P_{de}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2).\end{aligned} \tag{3.3.8}$$

to get rid of the angle bracket terms that have alpha and beta in them since the second term will vanish in recursion. Unfortunately the same cannot be done for the square bracket terms like $[x\alpha]$ etc and so we have no choice but to multiply them by $\langle \alpha q\rangle$ and directly integrate them.

We can use algebraic manipulation to rewrite each integral into as simple a form as possible for integration. Let us first examine the non-adjacent current which has terms with $(s_{c\beta} - s_{ab})$ and $(s_{c\beta} - s_{a\alpha})$ in their denominators. How do we deal with these and rewrite them so as to be amenable to integration? The first can be rewritten as

$$\frac{1}{s_{c\beta} - s_{ab}} = -\frac{1}{s_{\alpha\beta} + s_{\alpha c}}$$

and this comes from the denominator of a subleading term we can expand this in powers of $s_{\alpha\beta}$ leaving only $s_{\alpha c} = [\alpha c]\langle c\alpha\rangle$ in the denominator which we can work with.

The second is more complex, there are two terms with $(s_{c\beta} - s_{a\alpha})^2$ in their denominator. Since we have both alpha and beta in the bracket but on different terms in the current, there is no obvious way to write this as $(s_{xy}+s_{de})$. Instead we write this as $(s_{c\beta}-s_{a\alpha})^2 = [b|\alpha+a|b\rangle^2$, and the two resultant integrals are of the form

$$\int \frac{d^Dl}{l^2\alpha^2\beta^2}\frac{[b|\alpha|q\rangle^3[d|l|q\rangle[e|l|q\rangle}{[a|\alpha|q\rangle[b|\alpha+a|b\rangle^2} \quad \text{and} \quad \int \frac{d^Dl}{l^2\alpha^2\beta^2}\frac{[b|\alpha|q\rangle^3[d|l|q\rangle[e|l|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle[b|\alpha+a|b\rangle^2} \tag{3.3.9}$$

In both cases we have $[b|\alpha|q\rangle^3$ which we can rewrite and expand as

$$[b|\alpha|q\rangle = [b|\alpha|b\rangle\frac{\langle \alpha q\rangle}{\langle \alpha b\rangle} = [b|\alpha|b\rangle\frac{[q|P_{\alpha\beta}|q\rangle}{[q|P_{\alpha\beta}|b\rangle} + \mathcal{O}(s_{\alpha\beta}) \tag{3.3.10}$$

which is acceptable as these two integrals have a single pole. Then we can rewrite $[b|\alpha|b\rangle = [b|\alpha+a|b\rangle - s_{ab}$ and substitute this. This gives a series of integrals which are free of this awkward $[b|\alpha+a|b\rangle$ term, and terms which - for reasons that we will discuss below - do not give rational terms. We now have all current terms in a form that is amenable to integration.

Now let us speak of the general method by which we manipulate our integrand with goal of having the simplest possible set of integrals to solve. We shall illustrate this with an example: let us work with the following term from the non-adjacent current

$$-\frac{1}{3}\frac{\langle c\beta\rangle[\alpha b][\alpha c]^2[bc]}{\langle bc\rangle\langle ab\rangle[ab]^2[\alpha a][c\beta]}$$

which will allow us to demonstrate all the methods at our disposal. The full integral associated with this term is

$$\frac{i}{3}\frac{[bc]}{\langle dq\rangle\langle eq\rangle\langle bc\rangle\langle ab\rangle[ab]^2}\int\frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}\frac{\langle\beta c\rangle[\alpha b][\alpha c]^2}{[\alpha a][c\beta]}. \tag{3.3.11}$$

We shall declare the integrable part of the above expression as I_1 and rewrite it as

$$I_1=\int\frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[c|\alpha|q\rangle^2[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle}\frac{\langle\beta c\rangle}{\langle\beta q\rangle}. \tag{3.3.12}$$

We can use the results of Eq 3.2.10 to write

$$\frac{\langle\beta c\rangle}{\langle\beta q\rangle}=\frac{[q|P_{\alpha\beta}|c\rangle}{[q|P_{\alpha\beta}|q\rangle}+\mathcal{O}(s_{\alpha\beta})$$

so now, remembering that $\alpha+\beta=d+e$,

$$I_1=\frac{[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle}\int\frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[c|\alpha|q\rangle^2[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle}. \tag{3.3.13}$$

Next we use the identity

$$\frac{[c|\alpha|q\rangle}{[c|\beta|q\rangle}=\frac{[c|P_{\alpha\beta}-\beta|q\rangle}{[c|\beta|q\rangle}=\frac{[c|P_{de}|q\rangle}{[c|\beta|q\rangle}-1, \tag{3.3.14}$$

and so

$$\begin{aligned}I_1=&\frac{[c|P_{de}|q\rangle[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle}\int\frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[c|\alpha|q\rangle[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle}\\&-\frac{[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle}\int\frac{d^Dl}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[c|\alpha|q\rangle[b|\alpha|q\rangle}{[a|\alpha|q\rangle}.\end{aligned} \tag{3.3.15}$$

Let us now recall from the Section 1.11 on integral reduction, we know that we can write one-loop integrals as a linear combination of scalar box, triangle and bubble integrals. We are only interested in the finite rational parts of these integrals in this instance and we know that these only arise in the case of scalar bubbles. Use of the identity 3.3.14 took I_1, a quintic pentagon integral (5 powers of the loop momentum in the numerator and 5 propagators) to the sum of a quartic pentagon integral, and a quartic box integral. Thus by repeated use of this identity we can eventually reduce each integral to a sum of integrals with only a single $[x|\alpha|q\rangle$-like term in the denominator, and terms that will not contribute a rational part.

We can repeat this process with the the first of the two integrals above, so that

$$\begin{aligned} I_1 =& \frac{[c|P_{de}|q\rangle^2[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle \frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle} \\ &- \frac{[c|P_{de}|q\rangle[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle \frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle} \\ &- \frac{[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle \frac{[c|\alpha|q\rangle[b|\alpha|q\rangle}{[a|\alpha|q\rangle}. \end{aligned} \tag{3.3.16}$$

Focusing on the first of the above integrals, the integrand has the following

$$\frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle}.$$

To proceed further we can use Schouten's identity to rewrite this as

$$\begin{aligned} \frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle} =& \frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle}\frac{[ac]}{[ac]} = \frac{[ab][c|\alpha|q\rangle}{[ac][a|\alpha|q\rangle[c|\beta|q\rangle} + \frac{[bc]}{[ac][c|\beta|q\rangle} \\ =& \frac{[ab][c|P_{de}|q\rangle}{[ac][a|\alpha|q\rangle[c|\beta|q\rangle} - \frac{[ab]}{[ac][a|\alpha|q\rangle} + \frac{[bc]}{[ac][c|\beta|q\rangle} \end{aligned} \tag{3.3.17}$$

so that finally

$$\begin{aligned} I_1 =& \frac{[c|P_{de}|q\rangle^3[ab][q|P_{de}|c\rangle}{[q|P_{de}|q\rangle[ac]} \int \frac{d^D l}{l^2\alpha^2\beta^2} \frac{[d|l|q\rangle[e|l|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle} \\ &- \frac{[c|P_{de}|q\rangle^2[q|P_{de}|c\rangle[ab]}{[ac][q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2} \frac{[d|l|q\rangle[e|l|q\rangle}{[a|\alpha|q\rangle} \\ &+ \frac{[c|P_{de}|q\rangle^2[q|P_{de}|c\rangle[bc]}{[ac][q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2} \frac{[d|l|q\rangle[e|l|q\rangle}{[c|\beta|q\rangle} \\ &- \frac{[c|P_{de}|q\rangle[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle \frac{[b|\alpha|q\rangle}{[a|\alpha|q\rangle} \\ &- \frac{[q|P_{de}|c\rangle}{[q|P_{de}|q\rangle} \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle \frac{[c|\alpha|q\rangle[b|\alpha|q\rangle}{[a|\alpha|q\rangle}. \end{aligned} \tag{3.3.18}$$

Of the above integrals, the first is now a quadratic pentagon integral. By simple power counting, we have two powers of the loop momentum in the numerator and five propagators so after integral reduction there can be no scalar bubble contribution and this integral be discarded. The remaining integrals are now in their simplest forms and will in general give rational contributions.

By performing the steps which have been outlined above, we can reduce all integrals to having at most one $[x|\alpha|q\rangle$-like term in the denominator as promised, and so the following are the integrals that we must evaluate:

$$\begin{aligned}
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{\langle q|\alpha\beta|q\rangle}{[y|\alpha|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle^2}{[y|\alpha|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle[y|\alpha|q\rangle}{[z|\beta|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle}{[y|\alpha|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle^2}{[y|\beta|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\beta|q\rangle^2}{[y|\alpha|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\beta|q\rangle^2}{[y|\beta|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{[y|\alpha|q\rangle},\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{[y|\beta|q\rangle},
\end{aligned} \tag{3.3.19}$$

as well as some integrals that have no such term in the denominator

$$\begin{aligned}
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle[x|\alpha|q\rangle,\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle[x|\beta|q\rangle,\\
&\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle.
\end{aligned} \tag{3.3.20}$$

The next question to ask is how to integrate something with a $[x|\beta|q\rangle$-like term in the denominator. Previously, in the all-plus case such as [66], it was noted that if there is a term like $[x|\beta|q\rangle$ in the denominator, it can be written as $2\beta\cdot X$ where X is a complex momentum, $X = \lambda_q\tilde{\lambda}_x$, then use

$$\frac{1}{2\beta\cdot X} = \frac{1}{(\beta+X)^2} + \frac{\beta^2}{2\beta\cdot X(\beta+X)^2}. \tag{3.3.21}$$

These integrals are all subleading, meaning they have no s_{de} pole of their own and thus they require a pole to come from the integration measure. This can only happen when all three propagators l^2, α^2, β^2 all go on-shell and thus the second term is small and can dropped. Thus we can integrate $[x|\beta|q\rangle$ as if it were a regular propagator. With this we would then have integrals of the form

$$\int d^D l \frac{[d|l|q\rangle[e|l|q\rangle}{l^2\alpha^2\beta^2(\beta+X)^2} \times (\text{other numerator terms}).$$

To further ease computation we can use the identity

$$[d|l|q\rangle[e|l|q\rangle = \frac{\beta^2\langle q|le|q\rangle + \alpha^2\langle q|ld|q\rangle + l^2\langle q|\alpha\beta|q\rangle}{\langle ed\rangle} \tag{3.3.22}$$

to write a quadratic box as a sum of three linear triangles.

While this has been successfully implemented in the past, there are some subtleties associated with the dimensions of the momenta. The $l^2\alpha^2\beta^2$ propagators are D dimensional momenta, whereas the $l^2\alpha^2\beta^2$ in the numerator of the identity 3.3.22 above arise from intermediate terms like $\langle x|ll|y\rangle = l^2\langle xy\rangle$, so this l^2 is 4 dimensional. Cancelling the two of these introduces a potential $\mathcal{O}(\epsilon)$ ambiguity. Additionally this method relies on the use of the identity 3.3.21 and that one is able to drop the second term. In the next section when we deal with the single-minus current we will come across $[x|\beta|q\rangle$-like terms in the denominator of leading order integrals as well as integrals that have poles in ϵ, meaning that the second term above could potentially survive as a subleading pole and the errors in the regulator may have finite effects, so this approximation is invalid.

In any case it is clear that it would be desirable to find a method of integration that does not rely on such methods. We note that for a term like $[x|\beta|q\rangle$ in the denominator, it could be written as $[x|\beta|q\rangle = (\beta+X)^2 - \beta^2)$ where X is the same complex momentum as before, hence this term is actually the difference of two propagators. With this observation the most natural next step would be to consider the Mellin-Barnes representation [81, 82], writing $[x|\beta|q\rangle$ as

$$\frac{1}{[x|\beta|q\rangle^N} = \frac{1}{((\beta+X)^2-\beta^2)^N} = \frac{1}{\Gamma(N)}\int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\Gamma(-z)\Gamma(z+N)(\beta+X)^{2(z)}\beta^{2(-N-z)}, \tag{3.3.23}$$

where $-a < c < 0$.

This representation effectively allows us to binomially expand this denominator while delaying the problem of convergence - which of α^2 of $(\alpha + q)^2$ is larger - until after the loop integral is completed. It is however a problem that one must eventually deal with when choosing a contour to evaluate the Mellin-Barnes integral. Additionally, while a term like $[x|\beta|q\rangle$ can be interpreted as a difference of two propagators, it is more natural to see such a term as part of the numerator of a Feynman integral - as a tensor part of the integral - and thus it makes sense to think of a method that allows us to exploit this more natural interpretation.

3.3.1 The A Trick

We can write

$$\frac{1}{x^N} = \frac{1}{(A - A + x)^N} \tag{3.3.24}$$

where A is arbitrary, and then write this as a geometric sum. Because A is arbitrary we can let it be as large as we like, hence we do not need to use the Mellin-Barnes representation explicitly, as there is not issue of convergence. Instead we write

$$\frac{1}{x^N} = \frac{1}{A^N} \sum_{n=0}^{\infty} \frac{(A - x)^n}{A^n} \frac{(N, n)}{n!}, \tag{3.3.25}$$

and binomially expand $(A - x)$,

$$\frac{1}{x^N} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(N, n)}{m!(n - m)!} \frac{(-1)^m}{A^{N+m}} x^m. \tag{3.3.26}$$

In this representation, the denominator term this looks like

$$\frac{1}{[x|\beta|q\rangle^N} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(N, n)}{m!(n - m)!} \frac{(-1)^m}{A^{N+m}} [x|\beta|q\rangle^m. \tag{3.3.27}$$

We have thus managed to raise the term from the denominator to the numerator where we know how to work with it. We can now perform our integral and resum at the end.

While it is not explored in this thesis, it is worth noting that there may be some connection between this method and the *expansion by regions* method of Smirnov [83]. In the A trick we argue that convergence is not an issue as we can take A to be arbitrarily large, and one could potentially interpret this in the sense of taking the limit

$$t = \frac{[x|\beta|q\rangle}{A}, \quad t \to 0 \tag{3.3.28}$$

as in the expansion by region method. Exploring this connection further could be an avenue for future study.

3.3.2 Integration and Resummation

To demonstrate how to use the A trick we will evaluate the following integral

$$I = \int \frac{d^D l}{l^2 \alpha^2 \beta^2} \frac{\langle q|\alpha\beta|q\rangle [d|l|q\rangle [e|l|q\rangle}{[y|\alpha|q\rangle} \tag{3.3.29}$$

which can be checked using the previous method outlined above. We begin by applying the A trick to raise $[y|\alpha|q\rangle$ to the numerator,

$$I = \int \frac{d^D l}{l^2 \alpha^2 \beta^2} \frac{\langle q|\alpha\beta|q\rangle [d|l|q\rangle [e|l|q\rangle}{[y|\alpha|q\rangle} = \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \frac{(-1)^a}{A^{1+a}} \int \frac{d^D l}{l^2 \alpha^2 \beta^2} \langle q|\alpha\beta|q\rangle [d|l|q\rangle [e|l|q\rangle [y|\alpha|q\rangle^a. \tag{3.3.30}$$

Feynman parameterising, we retain only the scalar part due to all factors of the loop momentum being contracted by λ_q,

$$\begin{aligned} I =& i\Gamma(1+\epsilon) \frac{\langle q|de|q\rangle [d|e|q\rangle [e|d|q\rangle}{s_{de}^{1+\epsilon} A} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[y|e|q\rangle}{A}\right)^a \\ & \int [du](1-u_2-u_3) u_2^{-\epsilon} u_3^{-\epsilon} \left((1-u_2) + u_3 \frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^a . \end{aligned} \tag{3.3.31}$$

We then use the substitution $u_3 = v(1-u_2)$,

$$\begin{aligned} I =& i\Gamma(1+\epsilon) \frac{\langle q|de|q\rangle [d|e|q\rangle [e|d|q\rangle}{s_{de}^{1+\epsilon} A} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[y|e|q\rangle}{A}\right)^a \\ & \int du_2 \, (1-u_2)^{2-\epsilon+a} u_2^{-\epsilon} \int dv \, v^{-\epsilon}(1-v) \left(1 + v\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^a . \end{aligned} \tag{3.3.32}$$

The u_2 integral is the integral representation of the beta function, $\beta(3-\epsilon+a, 1-\epsilon)$, while the v integral can be evaluated as a Hypergeometric function using the integral representation

$$ {}_2F_1[\alpha,\beta,\gamma,x] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dv\ v^{\beta-1}(1-v)^{\gamma-\beta-1}(1-vx)^{-\alpha}, \tag{3.3.33} $$

where $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma-\beta) > 0$. Thus the full expression is

$$ \begin{aligned} I =& i\Gamma(1+\epsilon)\frac{\Gamma^2(1-\epsilon)}{\Gamma(4-2\epsilon)}\frac{\langle q|de|q\rangle[d|e|q)[e|d|q\rangle}{s_{de}^{1+\epsilon}A} \\ & \sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[y|e|q\rangle}{A}\right)^a \frac{(3-\epsilon,a)}{(4-2\epsilon,a)} {}_2F_1[-a,1-\epsilon,3-\epsilon,-\frac{[y|d|q\rangle}{[y|e|q\rangle}], \\ =& i\Gamma(1+\epsilon)\frac{\Gamma^2(1-\epsilon)}{\Gamma(4-2\epsilon)}\frac{\langle q|de|q\rangle[d|e|q\rangle[e|d|q\rangle}{s_{de}^{1+\epsilon}A} I_a. \end{aligned} \tag{3.3.34} $$

Now we isolate and deal with the sum I_a. First we can use the 'half-Gauss' identity for hypergeometrics

$$ {}_2F_1[\alpha,\beta,\gamma,z] = (1-z)^{-\beta}{}_2F_1[\gamma-\alpha,\beta,\gamma,\frac{z}{z-1}], \tag{3.3.35} $$

to rewrite the sum as

$$ I_a = \left(1+\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^{-(1-\epsilon)} \sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[y|e|q\rangle}{A}\right)^a \frac{(3-\epsilon,a)}{(4-2\epsilon,a)} {}_2F_1[3-\epsilon+a,1-\epsilon,3-\epsilon,\frac{[y|d|q\rangle}{[y|P_{de}|q\rangle}]. \tag{3.3.36} $$

Now if we write the hypergeometric function in its series representation,

$$ {}_2F_1[\alpha,\beta,\gamma,x] = \sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(\gamma,m)}\frac{x^m}{m!}, \tag{3.3.37} $$

where (a,n) is the Pochhammer symbol defined as

$$ (a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{3.3.38} $$

then I_a can be written as

$$ \begin{aligned} I_a =& \left(1+\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^{-(1-\epsilon)} \sum_w\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[y|e|q\rangle}{A}\right)^a \\ & \frac{(3-\epsilon,a+w)}{(4-2\epsilon,a)}\frac{(1-\epsilon,w)}{(3-\epsilon,w)w!}\left(\frac{[y|d|q\rangle}{[y|P_{de}|q\rangle}\right)^w. \end{aligned} \tag{3.3.39} $$

Now let us consider the sums over a and a_1. These are

$$\sum_{a_1=0}^{\infty}\sum_{a=0}^{a_1}.$$

If we sketch this on a graph

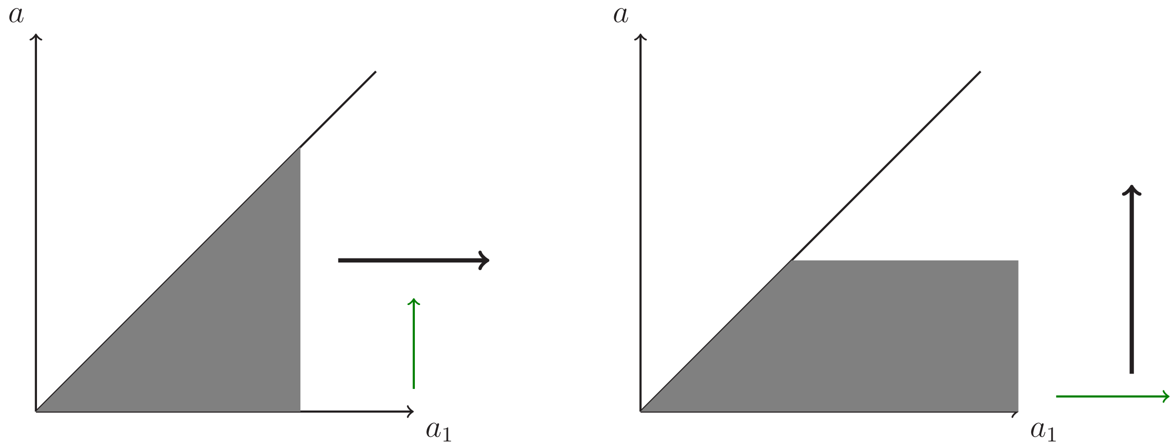


Figure 3.4: Visualising the change of sum variables. In each figure we use the thinner green arrow to demonstrate the first inner sum, and the thicker black arrow to demonstrate the second outer sum.

We can think of filling this shaded area by going from the a_1 axis ($a = 0$) vertically to the drawn line $a = a_1$ in the direction of the green arrow in figure 3.4. Then extending this line into an area by going from the y axis ($a = 0$) to infinity signified by the thicker black line. Alternatively one can think of this same image differently. We can instead think of it filling this shaded area by starting at the line $a_1 = a$ and then extending horizontally to $a_1 = \infty$ then extending from the x-axis $a = 0$ vertically to $a = \infty$. Thus we have the equivalence

$$\sum_{a_1=0}^{\infty}\sum_{a=0}^{a_1} = \sum_{a=0}^{\infty}\sum_{a_1=a}^{\infty} = \sum_{a=0}^{\infty}\sum_{r=a_1-a=0}^{\infty}. \tag{3.3.40}$$

Applying this change of variables to I_a gives

$$
\begin{aligned}
I_a &= \left(1+\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^{-(1-\epsilon)} \sum_{arw} \frac{(1,a+r)}{a!r!} \left(-\frac{[y|e|q\rangle}{A}\right)^a \\
&\quad \frac{(3-\epsilon,a+w)}{(4-2\epsilon,a)} \frac{(1-\epsilon,w)}{(3-\epsilon,w)w!} \left(\frac{[y|d|q\rangle}{[y|P_{de}|q\rangle}\right)^w, \\
&= \left(1+\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^{-(1-\epsilon)} \sum_r \frac{\Gamma(1+r)}{r!} \\
&\quad \sum_{aw} \frac{(3-\epsilon,a+w)}{a!w!} \frac{(1+r,a)}{(4-2\epsilon,a)} \frac{(1-\epsilon,w)}{(3-\epsilon,w)} \left(-\frac{[y|e|q\rangle}{A}\right)^a \left(\frac{[y|d|q\rangle}{[y|P_{de}|q\rangle}\right)^w .
\end{aligned}
\tag{3.3.41}
$$

We identity the double sum in the second line as the Appell function of the second kind, F_2, defined as

$$
F_2[\alpha;\beta,\beta';\gamma,\gamma';x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{3.3.42}
$$

so that now

$$
\begin{aligned}
I_a &= \left(1+\frac{[y|d|q\rangle}{[y|e|q\rangle}\right)^{-(1-\epsilon)} \sum_r \frac{\Gamma(1+r)}{r!} \\
&\quad F_2[3-\epsilon;1+r,1-\epsilon;4-2\epsilon,3-\epsilon;-\frac{[y|e|q\rangle}{A},\frac{[y|d|q\rangle}{[y|P_{de}|q\rangle}].
\end{aligned}
\tag{3.3.43}
$$

Immediately this can be simplified using the reduction formula

$$
F_2[\alpha;\beta,\beta';\gamma,\alpha;x,y] = (1-x)^{-\beta'} F_1[\beta,\alpha-\beta',\beta',\gamma,x,\frac{x}{1-y}] \tag{3.3.44}
$$

where F_1 is the Appell function of the first kind, defined as

$$
F_1[\alpha,\beta,\beta',\gamma,x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{3.3.45}
$$

This reduces I_a to

$$
I_a = \sum_r \frac{\Gamma(1+r)}{r!} F_1[1+r,2,1-\epsilon,4-2\epsilon,-\frac{[y|e|q\rangle}{A},-\frac{[y|P_{de}|q\rangle}{A}]. \tag{3.3.46}
$$

Now we must resolve the sum in r. Writing the F_1 function above as a double sum in m,n

as in the series representation, then the full r sum in I_a is

$$\begin{aligned}\sum_r \frac{\Gamma(1+r)}{r!}(1+r,m+n)(1-\delta)^r =&(1,m+n)\sum_r \frac{(1+m+n,r)}{r!}(1-\delta)^r \\ =&(1,m+n)\delta^{-1-m-n},\end{aligned} \tag{3.3.47}$$

where we added in the $(1-\delta)$ to regulate the sum. Putting this back into I_a,

$$I_a = \frac{1}{\delta}F_1[1,2,1-\epsilon,4-2\epsilon,-\frac{[y|e|q\rangle}{A\delta},-\frac{[y|P_{de}|q\rangle}{A\delta}]. \tag{3.3.48}$$

We now use the identity

$$F_1[\alpha;\beta,\beta';\gamma;x,y] = (1-x)^{-\alpha}F_1[\alpha;\gamma-\beta-\beta',\beta';\gamma';\frac{x}{x-1},\frac{x-y}{x-1}] \tag{3.3.49}$$

to rewrite

$$I_a = \frac{1}{\delta+\frac{[y|e|q\rangle}{A}}F_1[1,1-\epsilon,1-\epsilon,4-2\epsilon,\frac{[y|e|q\rangle}{[y|e|q\rangle-A\delta},-\frac{[y|d|q\rangle}{[y|e|q\rangle+A\delta}] \tag{3.3.50}$$

which allows us to safely take $\delta \to 0$, removing the regulator, leaving

$$I_a = \frac{A}{[y|e|q\rangle}F_1[1,1-\epsilon,1-\epsilon,4-2\epsilon,1,-\frac{[y|d|q\rangle}{[y|e|q\rangle}]. \tag{3.3.51}$$

This can still be further simplified using the identity

$$F_1[\alpha;\beta,\beta';\gamma,\alpha;1,y] = {}_2F_1[\alpha,\beta,\gamma,1]{}_2F_1[\alpha,\beta',\gamma-\beta,y], \tag{3.3.52}$$

to write

$$I_a = \frac{A}{[y|e|q\rangle}2F_1[1,1-\epsilon,4-2\epsilon,1]{}_2F_1[1,1-\epsilon,3-\epsilon,-\frac{[y|d|q\rangle}{[y|e|q\rangle}] \tag{3.3.53}$$

and the identity

$${}_2F_1[\alpha,\beta,\gamma,1] = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{3.3.54}$$

to finally arrive at

$$I_a = \frac{A}{[y|e|q\rangle}\frac{\Gamma(4-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(3-2\epsilon)}{}_2F_1[1,1-\epsilon,3-\epsilon,-\frac{[y|d|q\rangle}{[y|e|q\rangle}] \tag{3.3.55}$$

so that the full integral is

$$I = i\Gamma(1+\epsilon)\frac{\Gamma^3(1-\epsilon)}{\Gamma(3-2\epsilon)\Gamma(3-\epsilon)}\frac{\langle q|de|q\rangle[d|e|q\rangle[e|d|q\rangle}{s_{de}^{1+\epsilon}[y|e|q\rangle}\,{}_2F_1[1,1-\epsilon,3-\epsilon,-\frac{[y|d|q\rangle}{[y|e|q\rangle}]. \quad (3.3.56)$$

Immediately we note that the final result is independent of A as it should be. Taking $\epsilon \to 0$ and throwing away non-rational pieces, we are left with

$$I = -\frac{ic_\Gamma}{2}\frac{\langle q|de|q\rangle[d|e|q\rangle[e|d|q\rangle}{s_{de}[y|d|q\rangle}. \quad (3.3.57)$$

Let us now repeat this calculation using the previous method. We first use identities 3.3.21 and 3.3.22 to rewrite the integral I as

$$\begin{aligned} I =& \int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle[d|l|q\rangle[e|l|q\rangle}{[y|\alpha|q\rangle} \\ =& \frac{[de]}{s_{de}}\int \frac{d^D l}{\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^2}{(\alpha+Y)^2} \\ +& \frac{[de]}{s_{de}}\int \frac{d^D l}{l^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle\langle q|ld|q\rangle}{(\alpha+Y)^2} \\ +& \frac{[de]}{s_{de}}\int \frac{d^D l}{l^2\alpha^2}\frac{\langle q|\alpha\beta|q\rangle\langle q|le|q\rangle}{(\alpha+Y)^2}, \end{aligned} \quad (3.3.58)$$

where $Y = \lambda_q\tilde{\lambda}_y$. Now let us look at the third of these integrals. If we Feynman parameterise, we get $p = \alpha - (u_2 e - u_3 Y)$. Substituting this into the numerator term $q\langle q|le|q\rangle$ we see that both e and Y vanish, and since both loop momenta in the numerator are contracted by λ_q so too does p. Thus this integral evaluates to zero. The same argument applies to the first integral so only one remains, that is

$$I = \frac{[de]}{s_{de}}\int \frac{d^D l}{l^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle\langle q|ld|q\rangle}{(\alpha+Y)^2}. \quad (3.3.59)$$

Feynman parameterising the above propagators, we get $p = l - (u_2 e - u_2 Y - u_3 d)$, and $\Delta = -u_2\left((1-u_2)[y|e|q\rangle + u_3[y|d|q\rangle + u_3 s_{de}\right)$. From here the integral proceeds as usual

$$I = i\Gamma(1+\epsilon)\frac{[de]\langle q|de|q\rangle^2}{s_{de}^{1+\epsilon}}\int [du]\ (1-u_2-u_3)u_2^{-\epsilon}\left((1-u_2)[y|e|q\rangle + u_3\left(s_{de}+[y|d|q\rangle\right)\right)^{-1-\epsilon} \quad (3.3.60)$$

and using the substitution $u_3 = v(1-u_2)$

$$\begin{aligned}
&i\Gamma(1+\epsilon)\frac{[de]\langle q|de|q\rangle^2}{s_{de}^{1+\epsilon}[y|e|q\rangle^{1+\epsilon}}\int du_2\ u_2^{-\epsilon}(1-u_2)^{1-\epsilon}\int dv\ (1-v)\left(1+v\frac{s_{de}+[y|d|q\rangle}{[y|e|q\rangle}\right)^{-1-\epsilon}\\
=&\frac{i}{2}\Gamma(1+\epsilon)\frac{[de]\langle q|de|q\rangle^2}{s_{de}^{1+\epsilon}[y|e|q\rangle^{1+\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}{}_2F_1[1+\epsilon,1,3,-\frac{s_{de}+[y|d|q\rangle}{[y|e|q\rangle}]\\
=&-\frac{ic_\Gamma}{2}\frac{[de]\langle q|de|q\rangle^2}{s_{de}\left(s_{de}+[y|d|q\rangle\right)}+\mathcal{O}(\epsilon)+\text{Logs}\\
=&-\frac{ic_\Gamma}{2}\frac{[de]\langle q|de|q\rangle^2}{s_{de}[y|d|q\rangle}+\mathcal{O}(s_{de})
\end{aligned} \tag{3.3.61}$$

which agrees with the evaluation of I using the A trick.

While the A trick is very effective in allowing one to evaluate integrals with terms like $[y|\alpha|q\rangle$ in the denominator without resorting to the Mellin-Barnes representation, it may appear at first glance that the above resummation method relies on the third argument in the Hypergeometric function being equal to the Pochhammer in a in Eq 3.3.34. In fact, this is not the case. As we shall later see there will be times when these two terms differ and we shall introduce a derivative operator which circumvents this issue.

Throughout this thesis we will use this method to aid in integration. We will generalise the techniques used above to cases where the power of the $[y|\alpha|q\rangle$ is greater then 1, and to cases where there is more than one such term in the denominator.

Finally let us state the rational parts of the list of integrals to be evaluated

$$\begin{aligned}
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{\langle q|\alpha\beta|q\rangle}{[y|\alpha|q\rangle} &\to -\frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle\langle q|de|q\rangle}{s_{de}[y|d|q\rangle},\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle^2}{[y|\alpha|q\rangle} &\to -\frac{ic_\Gamma}{6}\frac{[d|e|q\rangle[e|d|q\rangle[x|d|q\rangle}{s_{de}[y|d|q\rangle^2}\\
&\qquad (4[x|e|q\rangle[y|d|q\rangle + [x|d|q\rangle([y|d|q\rangle - 2[y|e|q\rangle))\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle[y|\alpha|q\rangle}{[z|\beta|q\rangle} &\to \frac{ic_\Gamma}{6}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}[z|e|q\rangle}\\
&([x|d|q\rangle[y|e|q\rangle + [x|e|q\rangle[y|d|q\rangle + 5[x|e|q\rangle[y|e|q\rangle + 2[x|e|q\rangle[y|e|q\rangle\frac{[z|d|q\rangle}{[z|e|q\rangle}),\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle}{[y|\alpha|q\rangle} &\to -\frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}\frac{[xd]}{[yd]},\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle}{[y|\beta|q\rangle} &\to \frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}\frac{[xe]}{[ye]},\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\beta|q\rangle}{[y|\alpha|q\rangle} &\to \frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}\frac{[xd]}{[yd]},\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\beta|q\rangle}{[y|\beta|q\rangle} &\to -\frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}\frac{[xe]}{[ye]},\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{[y|\alpha|q\rangle} &\to 0,\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{[y|\beta|q\rangle} &\to 0,
\end{aligned} \tag{3.3.62}$$

and of course the remaining integrals that do not require the A trick,

$$\begin{aligned}
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle[x|\alpha|q\rangle &\to -\frac{ic_\Gamma}{6}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}(2[x|e|q\rangle + [x|d|q\rangle),\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle[x|\beta|q\rangle &\to -\frac{ic_\Gamma}{6}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}([x|e|q\rangle + 2[x|d|q\rangle),\\
\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle &\to -\frac{ic_\Gamma}{2}\frac{[d|e|q\rangle[e|d|q\rangle}{s_{de}}.
\end{aligned} \tag{3.3.63}$$

We note that this set of integrals in in fact overcomplete since for example,

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\beta|q\rangle}{[y|\alpha|q\rangle} = [x|P_{de}|q\rangle\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{[d|l|q\rangle[e|l|q\rangle}{[y|\alpha|q\rangle} - \int \frac{d^D l}{l^2\alpha^2\beta^2}[d|l|q\rangle[e|l|q\rangle\frac{[x|\alpha|q\rangle}{[y|\alpha|q\rangle}$$

among other similar relations. These are useful however in allowing us to confirm the accuracy of the results via internal consistency. The above integrals were evaluated using both the A trick and triangle decomposition methods and are in agreement.

The final terms to consider are the 'square' τ^{sq} terms in both the adjacent and non-adjacent currents. The integral for the non-adjacent square pole is

$$C^{+-:tri} = \int \frac{d^D l}{l^2\alpha^2\beta^2} \frac{[d|l|q\rangle[e|l|q\rangle}{\langle dq\rangle\langle eq\rangle} \frac{[q|\alpha\beta|q]}{[kq]^2} \frac{[q|\alpha|q\rangle^2}{[q|\beta|q\rangle^2}, \tag{3.3.64}$$

while the adjacent current gives

$$C^{-+:tri} = \int \frac{d^D l}{l^2\alpha^2\beta^2} \frac{[d|l|q\rangle[e|l|q\rangle}{\langle dq\rangle\langle eq\rangle} \frac{[q|\alpha\beta|q]}{[kq]^2} \frac{[q|\beta|q\rangle^2}{[q|\alpha|q\rangle^2}. \tag{3.3.65}$$

In this particular case, we can sidestep a direct evaluation by comparison to the one-loop $(+,+,-)$ splitting function [84], and write down the result

$$C^{+-:tri} + C^{-+:tri} = \frac{i}{3} \frac{[qd][qe][ed]}{[kq]^2} A_4^{(1)}(a^-, b^+, c^+, k^+). \tag{3.3.66}$$

The final step is that of recursion, however before we go into detail we must discuss the existence of spurious poles. In general the term spurious pole simply means that the final expression for an amplitude has terms with a pole which should not exist in the full amplitude. This may be simply be due to the method of calculating the amplitude resulting in an expression which is not in its simplest form. In our case, however we are referring to something more specific. Due to our use of the axial gauge formalism, we have to introduce an arbitrary reference vector q. When we initially assembled our result using techniques above we had terms with $[aq]$ in the denominator. This meant that when we shifted $\tilde{\lambda}_a$ we inadvertently created a pole that is q dependent and thus spurious. To remedy this we were thus forced to set $\tilde{\lambda}_q = \tilde{\lambda}_e$ such that this term was no longer excited by the BCFW shift. In fact we later managed to rewrite the integrated current such that these spurious poles cancelled, however in performing integrals in the next chapter it became necessary to retain this partial fixing of q in order to make the calculations tractable.

Setting $\tilde{\lambda}_q = \tilde{\lambda}_e$ actually makes the square pole term vanish as it is proportional to $[eq]$, hence the final result for the sum of the contributions of the adjacent and non-adjacent currents is

$$
\begin{aligned}
&-\frac{1}{18}\frac{\langle ac\rangle\langle bd\rangle\langle da\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]}+\frac{2}{9}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]}-\frac{64}{27}\frac{\langle ad\rangle^3\langle cq\rangle[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle}\\
&-\frac{1}{3}\frac{\langle ad\rangle^2[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle\langle cd\rangle}-\frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle}+\frac{1}{6}\frac{\langle ad\rangle^3[bd][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{1}{9}\frac{\langle ad\rangle^2\langle bd\rangle[bd]^2[de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]}+\frac{32}{27}\frac{\langle ad\rangle^2\langle ae\rangle[be][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}+\frac{64}{27}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}\\
&+\frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle^2\langle ae\rangle[ce][de]^2}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}-\frac{4}{9}\frac{\langle ad\rangle\langle ae\rangle\langle bd\rangle[be][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]}+\frac{7}{6}\frac{\langle ad\rangle^3\langle ae\rangle[de]^3}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{2}{9}\frac{\langle ad\rangle^2\langle bd\rangle\langle ae\rangle[bd]^2[de]^2[ea]}{s_{de}(s_{bc}-s_{ea})^2\langle bc\rangle^2\langle cd\rangle[bc]}.
\end{aligned}
\tag{3.3.67}
$$

As previously stated, the square pole vanishes due to our partial fixing of q, however we still have the double pole,

$$\frac{64}{27}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}.$$

We can compare this to the rational part of expected leading pole that comes from

$$
\begin{aligned}
A_5^{(2)}(a^-,b^+,c^+,d^+,e^+) &\to A_5^{(1)}(a^-,b^+,c^+,-k^-)\frac{i}{s_{de}}A^{(1)}(k^+,d^+,e^+)\\
&=-\frac{64i}{9}\frac{\langle ka\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle}\times\frac{i}{3}\frac{[de][ek][kd]}{s_{de}^2}\\
&=\frac{64}{27}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}
\end{aligned}
\tag{3.3.68}
$$

as desired. This is as far as augmented recursion has gone so far in implementation, and initially it was expected that this would be the end of the calculation. The initial expression that one obtains is more complicated than that presented above, and numerical testing indicated that the expression was not q independent as would be expected. This lead us to the realisation of the new pole structures that appear in the single-minus case that were absent from the all-plus case. The final section of this chapter will discuss the extra contribution from the one-loop currents with a ‘tree on the left’.

3.4 Single-Minus Current Contribution

Now let us consider the new contribution, the single-minus current contribution that arises from the helicity configuration $(s_1, s_2) = (+,+)$. There are in fact two contributions to the 'left hand side' in this case as we can see in figure 3.5.

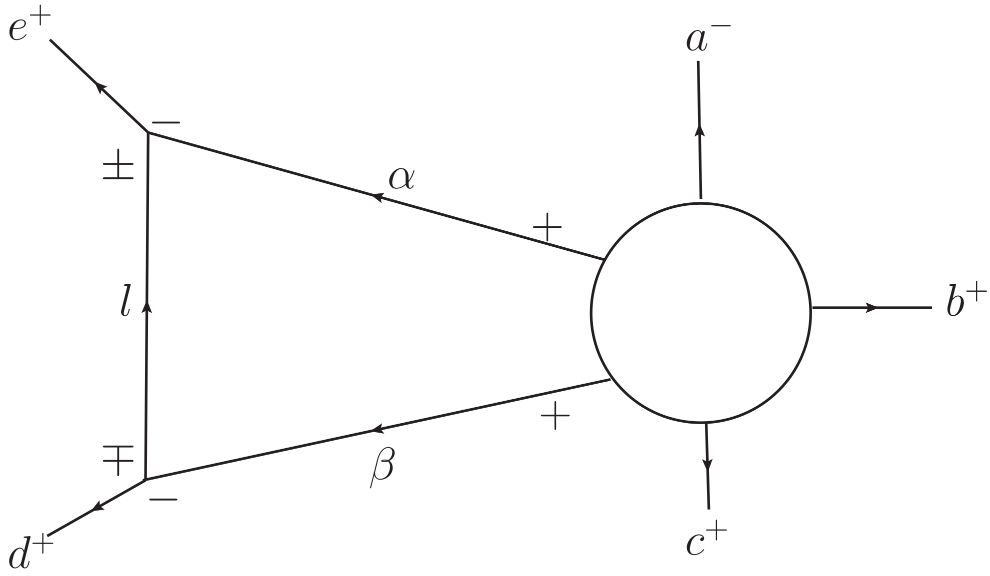


Figure 3.5: The single-minus current on the right contributes to two different helicity structures on the left which must be added together.

Beginning with this left hand side, the two configurations are

$$A_3^{(0)}(-l^+, e^+, -\alpha^-)A_3^{(0)}(l^-, -\beta^-, d^+) \tag{3.4.1}$$

$$+A_3^{(0)}(-\alpha^-, -l^-, e^+)A_3^{(0)}(d^+, l^+, -\beta^-) \tag{3.4.2}$$

which are

$$\frac{\langle\alpha q\rangle^2\langle\beta q\rangle^2[eq]^2[d|l|q\rangle[q|el|q]}{\langle dq\rangle[q|\alpha|q\rangle^2[q|l|q\rangle^2} - \frac{\langle\alpha q\rangle^2\langle\beta q\rangle^2[dq]^2[e|l|q\rangle[q|ld|q]}{\langle eq\rangle[q|\beta|q\rangle^2[q|l|q\rangle^2}. \tag{3.4.3}$$

The above terms have a $\langle el\rangle$ and $\langle ld\rangle$, respectively, in the numerator and are thus of order $s_{\alpha\beta}$. In order for this whole structure to contribute to the rational part, we require that after we integrate the current we have a s_{de} pole which means that the current must have a double pole in $s_{\alpha\beta}$. We know that the one-loop all-plus amplitude [85],

$$A_n^{(1)}(1^+, 2^+, ..., n^+) = -\frac{i}{3}\sum_{1\leq k_1<k_2<k_3<k_4\leq n}\frac{\langle k_1k_2\rangle[k_2k_3]\langle k_3k_4\rangle[k_4k_1]}{\langle 12\rangle\langle 23\rangle....\langle n1\rangle}$$

has no double poles which is why this term did not feature in the all-plus case.

The one-loop five-point single minus amplitude [79],

$$A^{(1)}(a^-,b^+,c^+,\beta^+,\alpha^+) = \frac{i}{3}\frac{1}{\langle c\beta\rangle^2}\left[-\frac{[b\alpha]^3}{[ab][\alpha a]} + \frac{\langle a\beta\rangle^3[\beta\alpha]\langle c\alpha\rangle}{\langle ab\rangle\langle bc\rangle\langle\beta\alpha\rangle^2} - \frac{\langle ac\rangle^3[cb]\langle\beta b\rangle}{\langle\alpha a\rangle\langle\alpha\beta\rangle\langle bc\rangle^2}\right], \quad (3.4.4)$$

does have such a double pole, and so we must now derive the corresponding current. Looking at the above amplitude we see that the second term gives the leading double pole, the third term gives the subleading single pole, while the first has no pole. In fact since the first term has no pole we know a priori that this term will not contribute to the final rational part.

To get our current we need to insert the poles depicted in figure 3.4 into the single-minus amplitude,

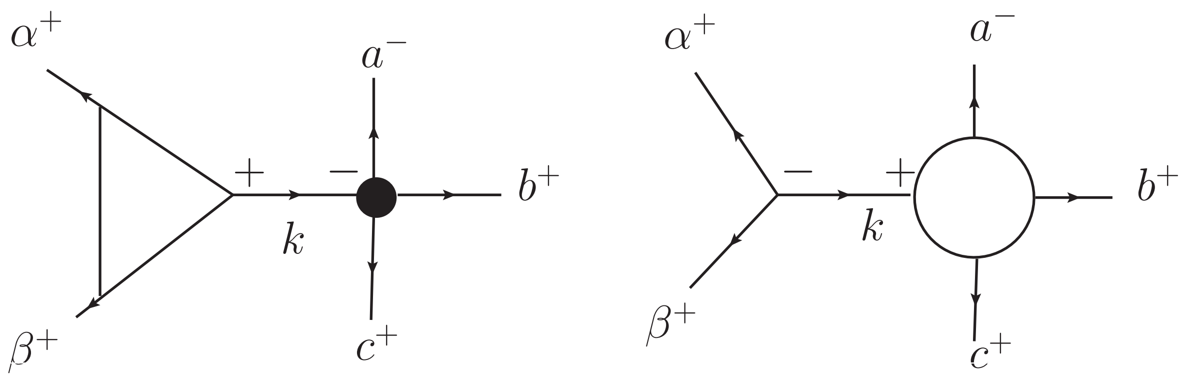


Figure 3.6: The figure on the left give the double pole of the single-minus current where the black dot signifies a tree vertex, while the figure on the right gives the right gives the subleading pole where the circle signifies a loop vertex.

where on the left we have

$$A_3^{(1)}(\alpha^+\beta^+,k^+)\frac{i}{s_{\alpha\beta}}A_4^{(0)}(-k^-,a^-,b^+,c^+) = \frac{i}{3}\frac{\langle q|\alpha\beta|q\rangle^3}{s_{\alpha\beta}^2}\frac{1}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{[q|P_{de}|a\rangle^3}{[q|P_{de}|q\rangle[q|P_{de}|c\rangle\langle ab\rangle\langle bc\rangle} \quad (3.4.5)$$

and on the right [70],

$$A_3^{(0)}(\beta^+,\alpha^+,k^-)\frac{i}{s_{\alpha\beta}}A_4^{(1)}(a^-,b^+,c^+,-k^+) = \frac{i}{3}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}\frac{1}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{[b|P_{de}|q\rangle^2[q|P_{de}|q\rangle s_{ac}}{[ab]\langle bc\rangle[a|P_{de}|q\rangle[q|P_{de}|c\rangle}. \quad (3.4.6)$$

Using the same method as before we arrive at the final current,

$$\begin{aligned}
&\tau^{(1)}(a^-,b^+,c^+,\beta^+,\alpha^+,q)\\
&=\frac{i}{3}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}\frac{1}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{[b|P_{de}|q\rangle^2[q|P_{de}|q\rangle s_{ac}}{[ab]\langle bc\rangle[a|P_{de}|q\rangle[q|P_{de}|c\rangle}+\frac{i}{3}\frac{\langle q|\alpha\beta|q\rangle^3}{s_{\alpha\beta}^2}\frac{1}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{[q|P_{de}|a\rangle^3}{[q|P_{de}|q\rangle[q|P_{de}|c\rangle\langle ab\rangle\langle bc\rangle}\\
&+\frac{i}{3}\frac{\langle q|\alpha\beta|q\rangle^2}{s_{\alpha\beta}}\frac{(2[q|\alpha|q\rangle+[q|\beta|q\rangle)}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{\langle cq\rangle[q|P_{de}|a\rangle^3}{\langle ab\rangle\langle bc\rangle[q|P_{de}|c\rangle^2[q|P_{de}|q\rangle^2}\\
&-i\frac{\langle q|\alpha\beta|q\rangle^2}{s_{\alpha\beta}}\frac{[q|\alpha|q\rangle}{\langle\alpha q\rangle^2\langle\beta q\rangle^2}\frac{\langle aq\rangle[q|P_{de}|a\rangle^2}{\langle ab\rangle\langle bc\rangle[q|P_{de}|c\rangle[q|P_{de}|q\rangle^2}+\mathcal{O}(s_{\alpha\beta}^0).
\end{aligned} \tag{3.4.7}$$

If we multiply this current by the 'left hand side' (3.4.3), we can use

$$\frac{1}{[q|\beta|q\rangle[q|l|q\rangle}=\frac{1}{[q|d|q\rangle}\left(\frac{1}{[q|l|q\rangle}-\frac{1}{[q|\beta|q\rangle}\right),$$

and

$$\frac{1}{[q|\alpha|q\rangle[q|l|q\rangle}=\frac{1}{[q|e|q\rangle}\left(\frac{1}{[q|l|q\rangle}+\frac{1}{[q|\alpha|q\rangle}\right),$$

as well as Eq 3.3.14 to partial fraction out the integrand until we have four integrals which we need to evaluate:

$$\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[d|l|q\rangle[q|el|q]}{[q|\alpha|q\rangle^n} \tag{3.4.8}$$

$$\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[d|l|q\rangle[q|el|q]}{[q|l|q\rangle^n} \tag{3.4.9}$$

$$\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[e|l|q\rangle[q|ld|q]}{[q|\beta|q\rangle^n} \tag{3.4.10}$$

$$\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[e|l|q\rangle[q|ld|q]}{[q|l|q\rangle^n}, \tag{3.4.11}$$

where $m=1,2,3$ and $n=1,2$. Let us go into detail with the evaluation of the first integral as an example of an integral that uses the A trick and has tensor contributions and with generic powers of the 'propagator' $[q|\alpha|q\rangle$.

First we use the A trick

$$\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\frac{(-1)^a}{A^{a+n}}\int\frac{d^Dl}{l^2\alpha^2\beta^2}\langle q|\alpha\beta|q\rangle^m[d|l|q\rangle[q|el|q][q|\alpha|q\rangle^a. \tag{3.4.12}$$

Using the Feynman parameterisation $p=l-(u_2e-u_3d)$ and $\Delta=-u_2u_3s_{de}$, and the numerator is

$$\langle q|P_{de}(p+d(1-u_3)+u_2e)|q\rangle^m[d|p+u_2e|q\rangle[q|e(p-u_3d)|q][q|p+e(1-u_2)+u_3d|q\rangle^a \quad (3.4.13)$$

which gives one tensor and one scalar term

1. $\langle q|de|q\rangle^m(-u_1)^m u_2[d|e|q\rangle u_3[q|de|q][q|e(1-u_2)+u_3d|q\rangle^a$

2. $m\langle q|de|q\rangle^{m-1}(-u_1)^{m-1}u_2[d|e|q\rangle[q|e(1-u_2)+u_3d|q\rangle^a[q|e|q\rangle[q|P_{de}|q\rangle p^2/D$

Let us deal first with the scalar term

$$\begin{aligned}
&i\Gamma(1+\epsilon)\frac{(-1)^m\langle q|de|q\rangle^m[d|e|q\rangle[q|de|q]}{A^n s_{de}^{1+\epsilon}}\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\frac{(-1)^a}{A^a}\\
&\int[du]u_1^m u_2^{-\epsilon}u_3^{-\epsilon}[q|e(1-u_2)+u_3d|q\rangle^a\\
=&i\Gamma(1+\epsilon)\frac{(-1)^m\langle q|de|q\rangle^m[d|e|q\rangle[q|de|q]}{A^n s_{de}^{1+\epsilon}}\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\frac{(-1)^a}{A^a}\\
&\int[du](1-u_2-u_3)^m u_2^{-\epsilon}u_3^{-\epsilon}[q|e(1-u_2)+u_3d|q\rangle^a\\
=&i\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon+m)}\Gamma(1+m)\frac{(-1)^m\langle q|de|q\rangle^m[d|e|q\rangle[q|de|q]}{A^n s_{de}^{1+\epsilon}}\\
&\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\left(-\frac{[q|e|q\rangle}{A}\right)^a\frac{(2-\epsilon+m,a)}{(3-2\epsilon+m,a)}{}_2F_1[-a,1-\epsilon,2-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].
\end{aligned} \tag{3.4.14}$$

We can pull out the last line as I_a, using the Half-Gauss transformation C.5.2 on the hypergeometric,

$$\begin{aligned}
I_a=&\left(1+\frac{[q|d|q\rangle}{[q|e|q\rangle}\right)^{-(1-\epsilon)}\sum_r\frac{(n,r)}{r!}\\
&F_2[2-\epsilon+m;n+r,1-\epsilon;3-2\epsilon+m,2-\epsilon+m,-\frac{[q|e|q\rangle}{A},\frac{[q|d|q\rangle}{[q|P_{de}|q\rangle}]\\
=&\sum_r\frac{(n,r)}{r!}F_1[n+r,1+m,1-\epsilon,3-2\epsilon+m,-\frac{[q|e|q\rangle}{A},-\frac{[q|P_{de}|q\rangle}{A}].
\end{aligned} \tag{3.4.15}$$

Now it would be prudent to take the opportunity to generalise one of the derivations we used when introducing the A trick. We will make use of the following result many times throughout this thesis,

$$\sum_r \frac{(a,r)}{r!} F_1[a+r,b,c,d,x,y] = (-x)^{-a} \frac{\Gamma(d)\Gamma(b+c-a)}{\Gamma(b+c)\Gamma(d-a)} {}_2F_1[a,c,b+c,1-\frac{y}{x}]. \tag{3.4.16}$$

To prove this, let us first resolve the sum in r. Writing the F_1 function as a double sum in m,n the r sum is

$$\sum_r \frac{(a,r)}{r!}(a+r,m+n)(1-\delta)^r = (a,m+n)\sum_r \frac{(a+m+n,r)}{r!}(1-\delta)^r = (a,m+n)\delta^{-a-m-n},$$

where again we temporarily add in a regulator δ. Putting this back into the F_1 we have

$$\begin{aligned}\sum_r \frac{(a,r)}{r!} F_1[a+r,b,c,d,x,y] =& \frac{1}{\delta^a} F_1[a,b,c,d,\frac{x}{\delta},\frac{y}{\delta}] \\ =& \frac{1}{(\delta-x)^a} F_1[a,d,c,b,c,d,\frac{x}{x-\delta},\frac{x-y}{x-\delta}].\end{aligned} \tag{3.4.17}$$

where we used the identity C.5.4. Now we can remove the regulator,

$$\begin{aligned}\sum_r \frac{(a,r)}{r!} F_1[a+r,b,c,d,x,y] =& (-x)^{-a} F_1[a,d-c-b,c,d,1,1-\frac{x}{y}] \\ =& (-x)^{-a} {}_2F_1[a,d-c-b,d,1] {}_2F_1[a,c,b+c,1-\frac{y}{x}] \\ =& (-x)^{-a} \frac{\Gamma(d)\Gamma(b+c-a)}{\Gamma(b+c)\Gamma(d-a)} {}_2F_1[a,c,b+c,1-\frac{y}{x}].\end{aligned} \tag{3.4.18}$$

where we used the reduction formula C.6.3 and the identity C.3.1.

Thus we can evaluate I_a

$$I_a = \frac{A^n}{[q|e|q\rangle^n} \frac{\Gamma(3-2\epsilon+m)}{\Gamma(3-2\epsilon-n+m)} \frac{\Gamma(2-\epsilon-n+m)}{\Gamma(2-\epsilon+m)} {}_2F_1[n,1-\epsilon,2-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}], \tag{3.4.19}$$

and substitute this back into the rest of the expression to get the final result for the scalar term

$$\begin{aligned}& i\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\Gamma(1+m)\frac{(-1)^m\langle q|de|q\rangle^m[d|e|q\rangle[q|de|q]}{[q|e|q\rangle^n s_{de}^{1+\epsilon}} \\ & \frac{\Gamma(2-\epsilon-n+m)}{\Gamma(2-\epsilon+m)} {}_2F_1[n,1-\epsilon,2-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].\end{aligned} \tag{3.4.20}$$

Next, the tensor contribution

$$
\begin{aligned}
&m(-1)^{m-1}i\Gamma(\epsilon)\frac{\langle q|de|q\rangle^{m-1}[d|e|q\rangle[q|e|q\rangle[q|P_{de}|q\rangle}{A^n s_{de}^{\epsilon}}\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\frac{(-1)^a}{A^a}\\
&\int[du]u_1^{m-1}u_2^{1-\epsilon}u_3^{-\epsilon}[q|e(1-u_2)+u_3 d|q\rangle^a\\
=&\Gamma(1+m)(-1)^{m-1}i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon+m)}\frac{\langle q|de|q\rangle^{m-1}[d|e|q\rangle[q|e|q\rangle[q|P_{de}|q\rangle}{A^n s_{de}^{\epsilon}}\\
&\sum_{a,a_1}\frac{(n,a_1)}{a!(a_1-a)!}\left(-\frac{[q|e|q\rangle}{A}\right)^a\frac{(1-\epsilon+m,a)}{(3-2\epsilon+m,a)}{}_2F_1[-a,1-\epsilon,1-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].
\end{aligned}
\tag{3.4.21}
$$

Again we pull out the last line as I_a

$$
\begin{aligned}
I_a=&\left(1+\frac{[q|d|q\rangle}{[q|e|q\rangle}\right)^{-(1-\epsilon)}\sum_r\frac{(n,r)}{r!}\\
&F_2[1-\epsilon+m;n+r,1-\epsilon;3-2\epsilon+m,1-\epsilon+m;-\frac{[q|e|q\rangle}{A},\frac{[q|d|q\rangle}{[q|P_{de}|q\rangle}]\\
=&\sum_r\frac{(n,r)}{r!}F_1[n+r,m,1-\epsilon,3-2\epsilon+m,-\frac{[q|e|q\rangle}{A},-\frac{[q|P_{de}|q\rangle}{A}]\\
=&\frac{A^n}{[q|e|q\rangle^n}\frac{\Gamma(3-2\epsilon+m)}{\Gamma(3-2\epsilon-n+m)}\frac{\Gamma(1-\epsilon+m-n)}{\Gamma(1-\epsilon+m)}{}_2F_1[n,1-\epsilon,1-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].
\end{aligned}
\tag{3.4.22}
$$

and sub this back in

$$
\begin{aligned}
&\Gamma(1+m)(-1)^{m-1}i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\langle q|de|q\rangle^{m-1}[d|e|q\rangle[q|P_{de}|q\rangle}{[q|e|q\rangle^{n-1}s_{de}^{\epsilon}}\\
&\frac{\Gamma(1-\epsilon+m-n)}{\Gamma(1-\epsilon+m)}{}_2F_1[n,1-\epsilon,1-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].
\end{aligned}
\tag{3.4.23}
$$

In total then,

$$
\begin{aligned}
&\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[d|l|q\rangle[q|el|q]}{[q|\alpha|q\rangle^n}\\
&\quad=i\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\Gamma(1+m)\frac{(-1)^m\langle q|de|q\rangle^m[d|e|q\rangle[q|de|q]}{[q|e|q\rangle^n s_{de}^{1+\epsilon}}\\
&\quad\frac{\Gamma(2-\epsilon-n+m)}{\Gamma(2-\epsilon+m)}{}_2F_1[n,1-\epsilon,2-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}]\\
&\quad+\Gamma(1+m)(-1)^{m-1}i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\langle q|de|q\rangle^{m-1}[d|e|q\rangle[q|P_{de}|q\rangle}{[q|e|q\rangle^{n-1}s_{de}^{\epsilon}}\\
&\quad\frac{\Gamma(1-\epsilon+m-n)}{\Gamma(1-\epsilon+m)}{}_2F_1[n,1-\epsilon,1-\epsilon+m,-\frac{[q|d|q\rangle}{[q|e|q\rangle}].
\end{aligned}
\tag{3.4.24}
$$

Evaluating the other three integrals employs the same methods and so we can simply skip to the result for these

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m [d|l|q\rangle[q|el|q]}{[q|l|q\rangle^n}$$
$$=i\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\Gamma(1+m)(-1)^m\langle q|de|q\rangle^m [d|e|q\rangle[q|de|q]}{[q|e|q\rangle^n s_{de}^{1+\epsilon}}$$
$$\frac{\Gamma(2-2\epsilon-n)}{\Gamma(2-2\epsilon)}{}_2F_1[n,1-\epsilon,2-2\epsilon,\frac{[q|P_{de}|q\rangle}{[q|e|q\rangle}] \tag{3.4.25}$$
$$+\Gamma(1+m)(-1)^{m-1}i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\langle q|de|q\rangle^{m-1}[d|e|q\rangle[q|P_{de}|q\rangle}{[q|e|q\rangle^{n-1}s_{de}^{\epsilon}}$$
$$\frac{\Gamma(3-2\epsilon-n)}{\Gamma(3-2\epsilon)}{}_2F_1[n,1-\epsilon,3-2\epsilon,\frac{[q|P_{de}|q\rangle}{[q|e|q\rangle}],$$

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m [e|l|q\rangle[q|ld|q]}{[q|\beta|q\rangle^n}$$
$$=i\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\langle q|de|q\rangle^m [e|d|q\rangle[q|de|q]}{[q|d|q\rangle^n s_{de}}(-1)^m\Gamma(1+m)$$
$$\frac{\Gamma(2-\epsilon-n+m)}{\Gamma(2-\epsilon+m)}{}_2F_1[n,1-\epsilon,2-\epsilon+m,-\frac{[q|e|q\rangle}{[q|d|q\rangle}] \tag{3.4.26}$$
$$+i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\Gamma(1+m)(-1)^{m-1}\frac{\langle q|de|q\rangle^{m-1}[e|d|q\rangle[q|P_{de}|q\rangle}{s_{de}[q|d|q\rangle^{n-1}}$$
$$\frac{\Gamma(1-\epsilon+m-n)}{\Gamma(1-\epsilon+m)}{}_2F_1[n,1-\epsilon,1-\epsilon+m,-\frac{[q|e|q\rangle}{[q|d|q\rangle}],$$

and finally

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m [e|l|q\rangle[q|ld|q]}{[q|l|q\rangle^n}$$
$$=i\Gamma(1+m)\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\frac{\langle q|de|q\rangle^m [e|d|q\rangle[q|de|q]}{[q|e|q\rangle^n s_{de}}(-1)^m$$
$$\frac{\Gamma(2-2\epsilon-n)}{\Gamma(2-2\epsilon)}{}_2F_1[n,1-\epsilon,2-2\epsilon,\frac{[q|P_{de}|q\rangle}{[q|e|q\rangle}]$$
$$+i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon-n+m)}\Gamma(1+m)(-1)^{m-1}\frac{[q|de|q\rangle^{m-1}[e|d|q\rangle[q|P_{de}|q\rangle[q|d|q\rangle}{s_{de}^{\epsilon}[q|e|q\rangle^n}$$
$$\frac{\Gamma(3-2\epsilon-n)}{\Gamma(3-2\epsilon)}{}_2F_1[n,2-\epsilon,3-2\epsilon,\frac{[q|P_{de}|q\rangle}{[q|e|q\rangle}]. \tag{3.4.27}$$

We can then assemble the final result and perform recursion, however this process is greatly simplified when we choose to set $\tilde{\lambda}_q = \tilde{\lambda}_e$. We include both derivations to emphasise this point, and to demonstrate the power of the A trick.

For a start, on the left hand side one of the two terms vanishes, and thus we have

$$-\frac{\langle \alpha q\rangle^2\langle \beta q\rangle^2[de]^2[e|ld|e]}{\langle eq\rangle[e|\beta|q\rangle^2[e|l|q\rangle}, \tag{3.4.28}$$

while the current becomes

$$\begin{aligned}
&\tau_e^{(1)}(a^-,b^+,c^+,\beta^+,\alpha^+)\\
&=-\frac{i}{3}\frac{\langle ac\rangle^3\langle db\rangle\langle dq\rangle^2[cb]\langle q|\alpha\beta|q\rangle}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle da\rangle\langle \alpha q\rangle^2\langle \beta q\rangle^2}-\frac{i}{3}\frac{\langle da\rangle^3\langle q|\alpha\beta|q\rangle^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle\langle \alpha q\rangle^2\langle \beta q\rangle^2}\\
&\quad-\frac{2i}{3}\frac{\langle cq\rangle\langle da\rangle^3\langle q|\alpha\beta|q\rangle^2[e|\alpha|q\rangle}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle^2[de]\langle \alpha q\rangle^2\langle \beta q\rangle^2}+i\frac{\langle da\rangle^2\langle qa\rangle\langle q|\alpha\beta|q\rangle^2[e|\alpha|q\rangle}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2[de]\langle \alpha q\rangle^2\langle \beta q\rangle^2}\\
&\quad-\frac{i}{3}\frac{\langle cq\rangle\langle da\rangle^3\langle q|\alpha\beta|q\rangle^2[e|\beta|q\rangle}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle^2[de]\langle \alpha q\rangle^2\langle \beta q\rangle^2}.
\end{aligned} \tag{3.4.29}$$

Combining the two gives a set of terms with the following ‘typical integral’

$$I_s=\int\frac{d^Dl}{l^2\alpha^2\beta^2}\frac{\langle q|\alpha\beta|q\rangle^m[e|\alpha\beta|e]}{[e|\alpha|q\rangle^t[e|\beta|q\rangle^n} \tag{3.4.30}$$

where $m = 1, 2, 3$ and $n = 1, 2$. Using the same methods as before, we evaluate the integral to be

$$I_s=\Gamma(1+m)(-1)^{m-1}i\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(1-\epsilon-t)}{\Gamma(2-2\epsilon-n-t+m)}\frac{\Gamma(1-\epsilon+m-n)}{\Gamma(1-\epsilon+m)}\frac{\langle q|de|q\rangle^{m-1}}{[e|d|q\rangle^{n+t-2}s_{de}^{\epsilon}}. \tag{3.4.31}$$

Putting everything back together, the finite part of the result is

$$\begin{aligned}
&-\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle\langle dq\rangle[cb][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle da\rangle\langle eq\rangle}-\frac{7}{12}\frac{\langle cq\rangle\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle}\\
&+\frac{7}{2}\frac{\langle da\rangle^2\langle qa\rangle[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle}-\frac{223}{108}\frac{\langle da\rangle^3\langle eq\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle}.
\end{aligned} \tag{3.4.32}$$

Once again we run into the issue of spurious poles, as in the first term we have $\langle eq\rangle$ in the denominator. When we perform recursion we shift λ_e which excites a pole in $\langle eq\rangle$ despite q being an artefact of our calculation method. We therefore have a non-physical pole and thus we cannot determine its residue. To remedy this we have two options. If we simply fix $\lambda_q = \lambda_a$ then this pole will vanish. The downside of this is that since we have now fully fixed

q we lose the ability to use q independence of our final result as a check.

Another possibility may be to use Schouten's identity

$$\frac{\langle dq\rangle}{\langle eq\rangle}\frac{\langle eX\rangle}{\langle eX\rangle} = s_{de}\frac{\langle Xq\rangle}{[de]} + \frac{\langle dX\rangle}{\langle eX\rangle} \tag{3.4.33}$$

to rewrite this term as two terms, one with a spurious pole but no physical pole and one with a physical pole but no spurious pole. If we drop the term with no physical pole, then we can compare three terms. We can recurse the term without worrying out the spurious pole, and we can use the above prescription to remove it then recurse and compare. In both cases we get the same result and in fact the λ_q dependence drops out of its own accord. Thus we can also set $\lambda_q = \lambda_a$ then perform the recursion and compare with the first two results. In all three cases the result is the same.

The final result after recursion is

$$\begin{aligned}&\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle[bc][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle ea\rangle} - \frac{7}{2}\frac{\langle aq\rangle\langle da\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle}\\ &- \frac{7}{12}\frac{\langle cq\rangle\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle} - \frac{223}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle},\end{aligned} \tag{3.4.34}$$

however given that we have not fully explored the consequences of this prescription and we will not be able to check q independence for the full result in any case as we will see in the next chapter, we will instead choose to fix q such that the result simplifies to

$$\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle[bc][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle ea\rangle} - \frac{7}{12}\frac{\langle ca\rangle\langle da\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2} - \frac{223}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}. \tag{3.4.35}$$

Again, we have a leading double pole which we can compare to the expected double poles from the factorisations of the full amplitude. Indeed, looking at the two loop splitting function and tree MHV limit, the rational part of this expression [30] is

$$\frac{235}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}$$

which is a strong indicator of the calculation being correct, while also indicating that there are further contributions to these terms to be expected.

3.5 Conclusion

In the preamble to this chapter we divided the rational part of the amplitude into

rational = tree to two-loop easy channels + double-pole channels + rational descendants.

In this chapter we calculated the tree to two-loop easy channels, and three contributions from the double-pole channels which we dub the 'tree on the left' contributions. We saw that the two one-loop MHV currents came from the the one-loop to one-loop channel, and the new single-minus current came from the two-loop to tree channel. From a calculational perspective, it is more natural to divide the double-pole channel into "tree on the left + loop on the left" rather than by the two factorisation channels. Thus the rational part is now written as

rational = tree to two-loop easy channels +double-pole channels + rational descendants,

where

double-pole channels = tree on the left + loop on the left.

The rational contribution from the tree to two-loop easy channels are

$$-5\frac{\langle ac\rangle[bc]}{\langle ea\rangle\langle bc\rangle^2\langle bd\rangle\langle cd\rangle^2\langle de\rangle}(\langle ac\rangle\langle ad\rangle\langle bc\rangle\langle bd\rangle + \langle ab\rangle^2\langle cd\rangle^2), \tag{3.5.1}$$

and

$$-\frac{1}{9}\frac{[be]^2([bd]^2[ce]^2 + 8[bc][cd][de][be])}{[ab][bc][cd][de][ea]\langle cd\rangle^2}. \tag{3.5.2}$$

The rational part due to the two loop MHV currents is

$$\begin{aligned}
&-\frac{1}{18}\frac{\langle ac\rangle\langle bd\rangle\langle da\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]} + \frac{2}{9}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]} - \frac{64}{27}\frac{\langle ad\rangle^3\langle cq\rangle[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle}\\
&-\frac{1}{3}\frac{\langle ad\rangle^2[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle\langle cd\rangle} - \frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle} + \frac{1}{6}\frac{\langle ad\rangle^3[bd][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{1}{9}\frac{\langle ad\rangle^2\langle bd\rangle[bd]^2[de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]} + \frac{32}{27}\frac{\langle ad\rangle^2\langle ae\rangle[be][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]} + \frac{64}{27}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}\\
&+\frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle^2\langle ae\rangle[ce][de]^2}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]} - \frac{4}{9}\frac{\langle ad\rangle\langle ae\rangle\langle bd\rangle[be][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]} + \frac{7}{6}\frac{\langle ad\rangle^3\langle ae\rangle[de]^3}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{2}{9}\frac{\langle ad\rangle^2\langle bd\rangle\langle ae\rangle[bd]^2[de]^2[ea]}{s_{de}(s_{bc}-s_{ea})^2\langle bc\rangle^2\langle cd\rangle[bc]},
\end{aligned} \tag{3.5.3}$$

and the single-minus current contributes

$$\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle[bc][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle ea\rangle} - \frac{7}{12}\frac{\langle ca\rangle\langle da\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2} - \frac{223}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}. \tag{3.5.4}$$

As stated earlier, we were unaware at the beginning of the calculation that the method would require a significant extension, however there were signs such as the failure of the final result to be independent of the reference vector q as is expected. Nonetheless we were successful in carrying out this part of the calculation and in the case of the MHV currents, the double pole matches exactly what we had expected, while in the single-minus case the double pole has the correct spinor content that one would expect. We introduced a new method of integration that we dubbed the 'A trick' and used the previous integration method to compare and verify the accuracy of its results. We then moved on to the single-minus current and the A trick allowed us to calculate much more complicated integrals that we would likely not have been able to calculate solely with the old method of promoting denominator terms to propagators. This demonstrates the power and utility of the technique.

The aim with our entire method is that it should be easily generalised to higher multiplicity. In the calculations we carried out in this chapter, higher multiplicity will mean the addition of particles with positive helicity, which would take us from five-point MHV currents, with adjacent negative helicities and with a positive helicity between them, to the same at six points [86, 87], and from a five-point single-minus current to a six-point single-minus current [88], and so on for higher multiplicity. In all cases, we expect these calculations to be much the same as the five-point case, albeit longer and with more algebra.

Chapter 4

Calculating the Rational Part: Augmented Recursion II

Before we begin our calculations, we should briefly clarify the relationship between the one-loop (3.0.3) and two-loop (3.0.4) factorisation channels that require augmented recursion, and how they relate to the various diagrams that will be tackled in the previous chapter and this one. Naively one might assume that the one-loop to one-loop factorisation channel will be the source of the 'tree on the left' diagrams that we dealt with in the last chapter, while the two-loop splitting function to tree amplitude channel will give all the 'loop on the left' diagrams that we will see now. This does not quite add up, however, as we saw in the last chapter that the two MHV currents together gave the full leading double pole that was predicted by the one-loop splitting function, and yet there was an additional third current, a single-minus current, which contributed a double pole.

We can better understand this with an example. From the derivation of the two-loop splitting function in [30] we can see that there are many contributions to this function, one of which is

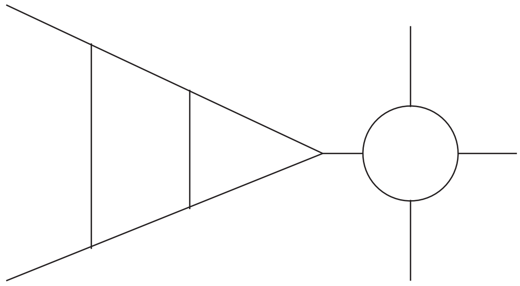


Figure 4.1: A contribution to the two-loop gluon splitting function

and while at first glance one might think that this is an extra 'loop on the left' term that must be calculated, we can see that actually this is already included in the 'tree on the left' part as shown below.

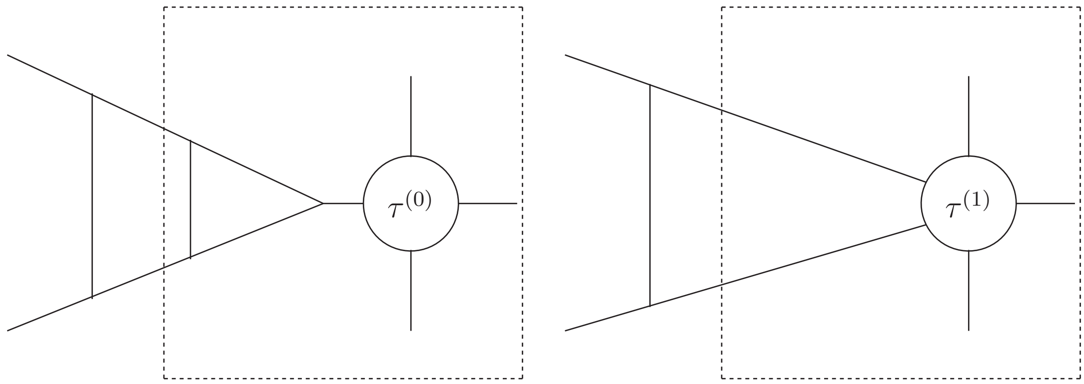


Figure 4.2: This figure shows how elements of the two-loop splitting function are incorporated into the 'tree on the left' parts of recursion.

The box on the left of figure 4.2 contains what is in fact a particular pole of the one-loop current on the right. This is exactly the type of pole we see in the single-minus current as shown in figure 3.6.

This observation means that the only 'new' terms we have to consider in the 'loop on the left' case are those that affect the diagrams such that a box cannot be drawn in the manner above to split the diagram into a 'tree on the left' and a loop diagram within. These are diagrams that alter the d or e corners, or the propagator that connects them.

With these conditions in mind, there are three structures to consider: one which is a bubble insertion on the connecting propagator, and two structures which have currents on the right that have three off-shell legs.

We will name this diagram the 'l-bubble' diagram,

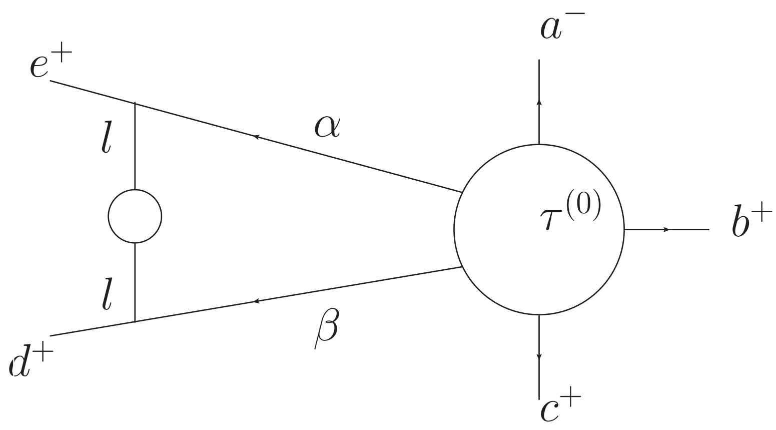


Figure 4.3: This diagram shows a bubble insertion on the l propagator. We name this the 'l-bubble' diagram.

Bubbles on either of the α or β legs would be contained within the one-loop currents of the previous chapter.

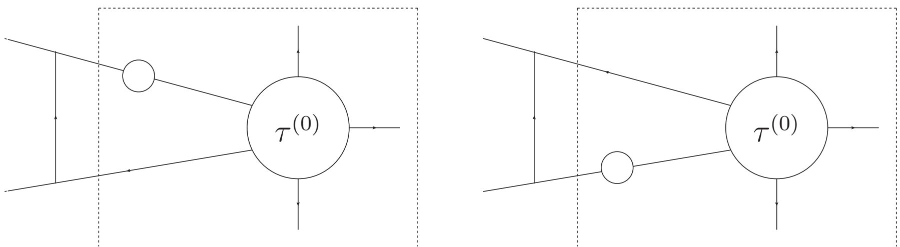


Figure 4.4: Diagrams with bubbles inserted on the α or β propagators are subsumed within one-loop currents

Then there is the 'double-box' structure,

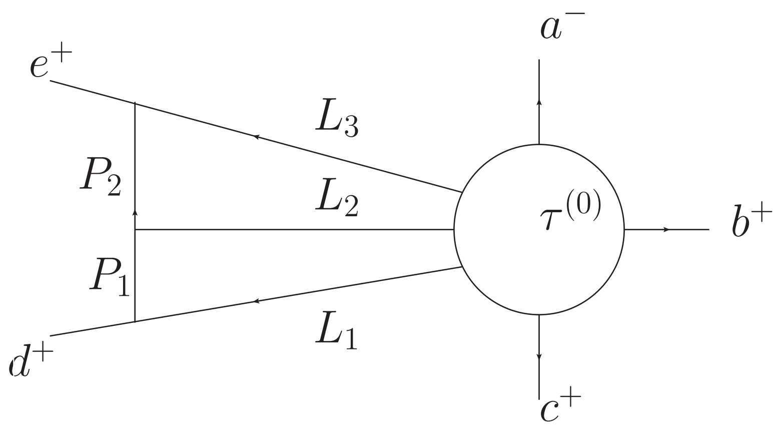


Figure 4.5: The two-loop ‘double-box’ structure with all three-point vertices and three off-shell momenta coming from the current

and the ‘box-triangle’ structure,

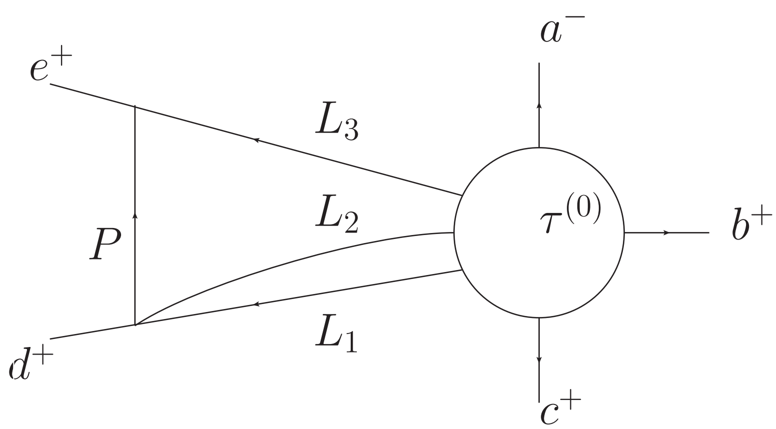


Figure 4.6: The two-loop ‘box-triangle’ structure with a four-point vertex and three off-shell momenta coming from the current.

There are structures such as the ‘embedded triangles’ which ostensibly need to be included, however as shown in figure 4.7 with the use of the dashed box, we see that these are a collinear limit of the current in the ‘double-box’ diagram, the first when L_2 goes collinear with L_3, and the second when L_2 goes collinear with L_1.

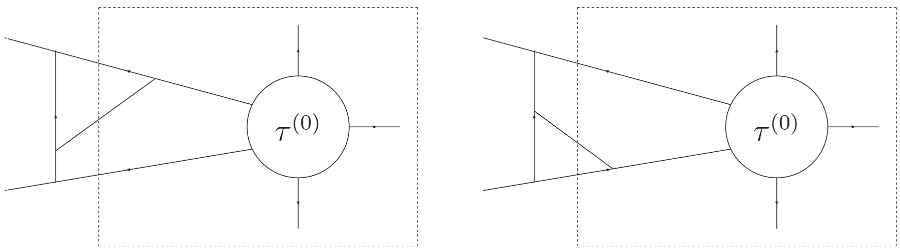


Figure 4.7: These are the 'embedded triangle' limits obtained from figure 4.5 when two of the inner propagators are collinear. The dashed box shows how this term is contained within the current of the 'double box' structure.

The same is true for the 'box-triangle' structures (4.6) where we obtain the below structures, the first when L_2 goes collinear with L_3, and the second when L_2 goes collinear with L_1.

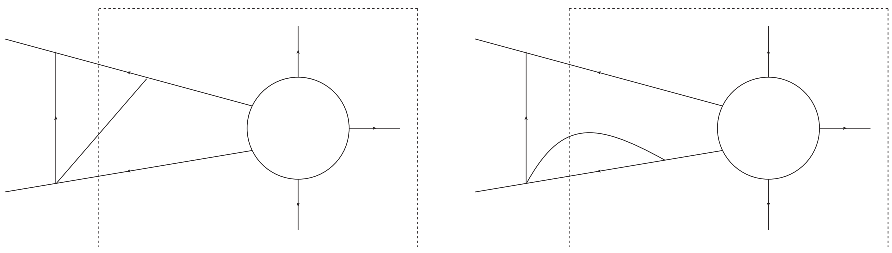


Figure 4.8: These are the two limits we obtain from figure 4.6 when two of the inner propagators are collinear. The dashed box shows how this term is contained within the current of the 'double box' structure.

In principle there are two further contributions that involve a four-point vertex in the e corner, however as we will later see this vertex is proportional to $[eq]$ and thus vanishes due to our choice of q.

Finally then, the full decomposition of the rational part is

$$\text{rational} = \text{tree to two-loop easy channels} + \text{double-pole channels} + \text{rational descendants},$$

where

$$\text{double-pole channels} = \text{tree on the left} + \text{loop on the left},$$

and

$$\text{loop on the left} = \text{l-bubble} + \text{double-box} + \text{box-triangle}.$$

The MHV currents in the tree on the left sector arise from the one-loop to one-loop channel, while the single-minus current in the tree on the left sector and the full loop on the left sector arise from the two-loop to tree channel. This two-loop to tree channel did not exist in the all-plus case which is why such terms have not been seen until now.

The easy channels and tree on the left parts were calculated in the previous chapter, and the rational descendants will be calculated at the end of this chapter, so now we focus on the loop on the left parts.

4.1 The Double-Box and Box-Triangle Diagrams

Let us consider the double-box (4.5) and the box-triangle diagrams (4.6). Beginning with the box-triangle diagram, this contains a four-point vertex which in the axial gauge formalism is

$$A_4(1^+,2^+,3^-,4^-) = i\frac{[1q][2q]\langle 3q\rangle\langle 4q\rangle}{\langle 1q\rangle\langle 2q\rangle[3q][4q]}\left(1+\frac{[q|2-3|q\rangle[q|4-1|q\rangle}{[q|2+3|q\rangle[q|4+1|q\rangle}\right), \tag{4.1.1}$$

or

$$A_4(1^+,2^-,3^+,4^-) = i\frac{[1q]\langle 2q\rangle[3q]\langle 4q\rangle}{\langle 1q\rangle[2q]\langle 3q\rangle[4q]}\left(\frac{[q|1-2|q\rangle[q|3-4|q\rangle}{[q|1+2|q\rangle[q|3+4|q\rangle}+\frac{[q|2-3|q\rangle[q|4-1|q\rangle}{[q|2+3|q\rangle[q|4+1|q\rangle}-2\right). \tag{4.1.2}$$

As an example, let us consider a single contribution to this diagram,

$$\int\frac{d^DL_1d^DL_2}{P^2L_1^2L_2^2L_3^2}\frac{[q|P|q\rangle[e|L_3|q\rangle}{[q|L_1|q\rangle[q|L_2|q\rangle}\frac{\langle L_1q\rangle^2\langle L_2q\rangle^2}{\langle L_3q\rangle^2\langle dq\rangle\langle eq\rangle}\left(1-\frac{[q|P+L_2|q\rangle[q|L_1+d|q\rangle}{[q|P-L_2|q\rangle[q|d-L_1|q\rangle}\right) \\ \tau^{(0)}(L_3^-,a^-,b^+,c^+,L_1^+,L_2^+). \tag{4.1.3}$$

The complexity of the 4-point vertex in the context of a double Feynman integral means that we are unfortunately unable to compute this integral as of now.

Even with the lack of a 4-point vertex, unfortunately the same is true of the double box diagram. In this case however there is some progress we can make in parts of this diagram. As stated earlier, in the limit where the L_2 propagator in figure 4.5 goes collinear with either L_1 or L_3, then we get the 'embedded triangle' diagrams as shown in figure 4.7. These embedded triangles will contain all the leading order contributions from the double-box, and some subleading terms. There will necessarily be some further subleading terms that cannot be reached as they would come from the finite parts of the 6pt currents. In this chapter we will analyse the embedded triangle terms and obtain as much information as we can.

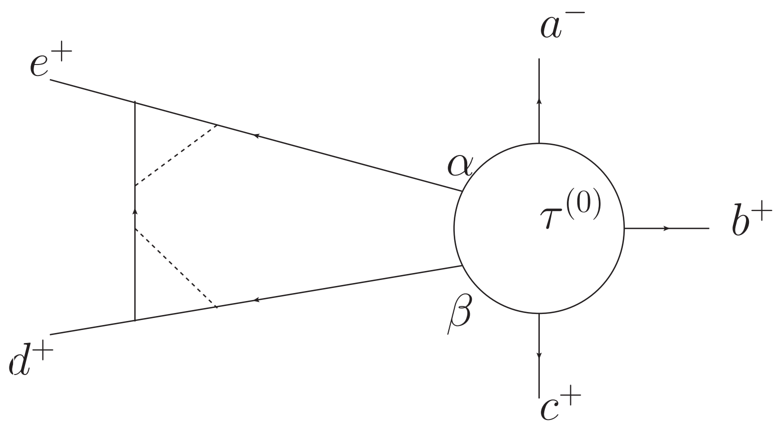


Figure 4.9: The embedded triangles diagram. The dashed lines indicate there are two cases, one where each of the vertices has an embedded loop triangle

4.1.1 Sources of Embedded triangles

Let us first look at these embedded triangles from the perspective of the double-box structure. First, we focus on the left hand side. Since there are five propagators and two loop integrals on the left of the double-box, overall we expect a factor of s_{de}^{-1} from these. In order for there to be an overall pole, all the three-point vertices on the left must be googly. The options for this are shown in figure 4.10:

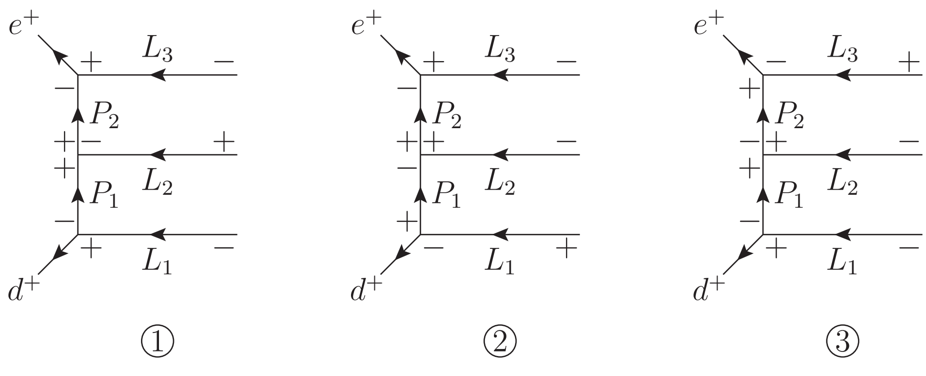


Figure 4.10: The possible helicity configurations on the left hand side of the 'double-box' diagram 4.5.

We first note from the above diagrams that when L_1 and L_2 we get a triangle in the $K = d$ corner, and when L_2 and L_3 go collinear we get a triangle in the $K = e$ corner.

As we can see from the helicities on the right of each diagram in figure 4.10, the currents we require will be 6-point tree NMHV. As stated above, the triangles emerge from the collinear limits of the propagators and so we shall list below the various collinear poles of the 6pt NMHV currents. All of the below 6pt amplitudes can be found in [89].

Let us start with the 6pt all-adjacent NMHV amplitude

$$
\begin{aligned}
A_6(b^+, c^+, L_1^+, L_2^-, L_3^-, a^-) =& i\frac{[L_1|L_2+L_3|a\rangle^3}{\langle ab\rangle\langle bc\rangle[L_1L_2][L_2L_3]s_{L_1L_2L_3}[L_3|L_2+L_1|c\rangle} \\
&+ i\frac{[b|L_3+a|L_2\rangle^3}{\langle cL_1\rangle\langle L_1L_2\rangle[L_3a][ab]s_{L_3ab}[L_3|a+b|c\rangle},
\end{aligned}
\tag{4.1.4}
$$

which has poles as shown below in figure 4.11.

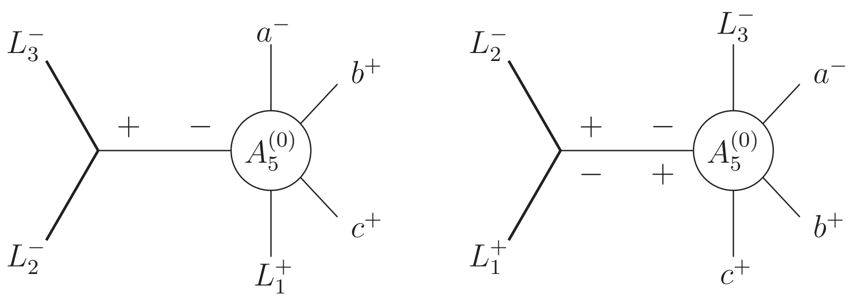


Figure 4.11: Factorisations of the all-adjacent NMHV current on the L_1L_2 and L_2L_3 collinear pole.

In the limit where L_2 and L_3 are collinear these terms can be identified with embedded triangles in the $K = e$ corner. This pole on the left of figure 4.11 is

$$
A_3(L_2^-, L_3^-, k^+)\frac{i}{s_{L_2L_3}}A_5(-k^-, a^-, b^+, c^+, L_1^+) = i\frac{\langle L_2L_3\rangle[kq]^2}{[L_2q][L_3q]}\frac{i}{s_{L_2L_3}}i\frac{\langle ka\rangle^3}{\langle ab\rangle\langle bc\rangle\langle cL_1\rangle\langle L_1k\rangle}.
\tag{4.1.5}
$$

This pole actually splits across two contributions as shown in figure 4.12,

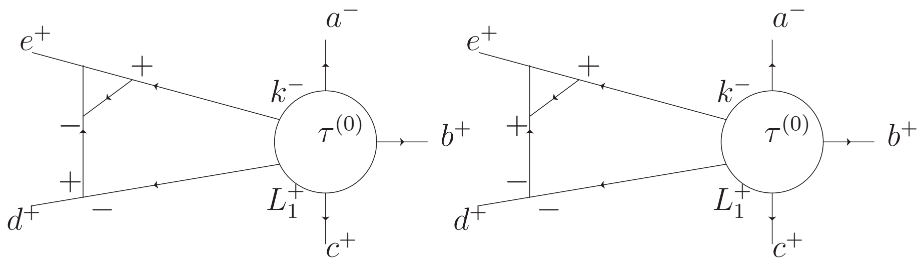


Figure 4.12: The pole 4.1.5 contributes to two different embedded triangle structures shown above.

where the first contributes to the single-minus embedded triangle with an adjacent MHV current, while the second is the sole contribution that has the all-plus triangle on the left and the adjacent MHV current on the right.

When L_2 goes collinear with L_1 instead we recover embedded triangles in the $K = d$ corner, and we get the following two poles as shown on the right of figure 4.11.

$$A_3(L_2^-, k^-, L_1^+)\frac{i}{s_{L_1L_2}}A_5(L_3^-, a^-, b^+, c^+, -k^+) = i\frac{\langle L_2k\rangle[L_1q]^2}{[L_2q][kq]}\frac{i}{s_{L_1L_2}}i\frac{\langle L_3a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle\langle kL_3\rangle} \quad (4.1.6)$$

$$A_3(k^+, L_1^+, L_2^-)\frac{i}{s_{L_1L_2}}A_5(b^+, c^+, -k^-, L_3^-, a^-) = i\frac{[kL_1]\langle L_2q\rangle^2}{\langle kq\rangle\langle L_1q\rangle}\frac{i}{s_{L_1L_2}}i\frac{[bc]^3}{[ck][kL_3][L_3a][ab]}. \quad (4.1.7)$$

The first can be linked to the single-minus triangle with the adjacent MHV current, while the second can be linked to the all-plus triangle with the googly current.

The next amplitude is

$$\begin{aligned} A_6(b^+, c^+, L_1^-, L_2^+, L_3^-, a^-) = {} & i\frac{\langle L_3a\rangle^2[bc]^2[L_2|b+c|L_1\rangle^2}{s_{bc}s_{cL_1}s_{L_2L_3}s_{L_3a}t_{bcL_1}} + i\frac{\langle L_3a\rangle^2[cL_2]^2[b|c+L_2|L_1\rangle^2}{s_{cL_1}s_{L_1L_2}s_{L_3a}s_{ab}t_{cL_1L_2}} \\ & + i\frac{\langle L_1L_3\rangle\langle L_3a\rangle[bc][cL_2][b|c+L_2|L_1\rangle[L_2|b+c|a\rangle t_{bcL_1}}{s_{bc}s_{cL_1}s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}} \\ & - i\frac{\langle L_1L_3\rangle\langle L_3a\rangle[bc]^2[L_2|b+c|L_1\rangle[L_2|b+c|a\rangle t_{cL_1L_2}}{s_{bc}s_{cL_1}s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}} + i\frac{\langle L_1L_3\rangle^2[bc]^2[L_2|b+c|a\rangle^2}{s_{bc}s_{L_1L_2}s_{L_2L_3}s_{ab}t_{L_1L_2L_3}} \\ & - i\frac{\langle L_3a\rangle^2[bc][cL_2][b|c+L_2|L_1\rangle[L_2|b+c|L_1\rangle t_{L_1L_2L_3}}{s_{bc}s_{cL_1}s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}}, \end{aligned} \quad (4.1.8)$$

There are four poles for this term, shown in figure 4.13.

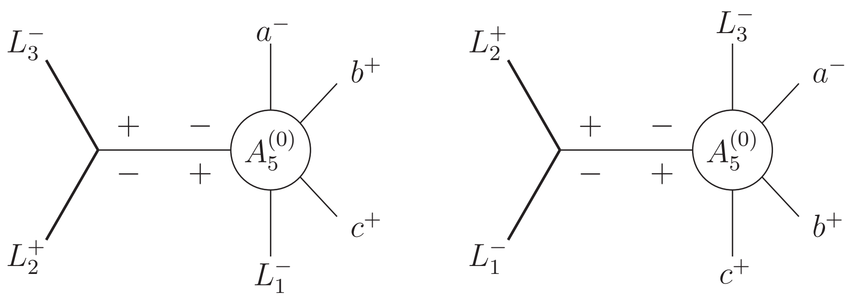


Figure 4.13: Factorisations of the second NMHV current on the L_1L_2 and L_2L_3 collinear pole.

First on the L_1L_2 limit on the left of figure 4.13 which gives the $K = d$ embedded triangle,

$$A_3(k^-, L_1^-, L_2^+)\frac{i}{s_{L_1L_2}}A_5(L_3^-, a^-, b^+, c^+, -k^+) = i\frac{\langle kL_1\rangle[L_2q]^2}{[kq][L_1q]}\frac{i}{s_{L_1L_2}}i\frac{\langle L_3a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle\langle kL_3\rangle}, \tag{4.1.9}$$

$$A_3(L_2^+, k^+, L_1^-)\frac{i}{s_{L_1L_2}}A_5(b^+, c^+, -k^-, L_3^-, a^-) = i\frac{[L_2k]\langle L_1q\rangle^2}{\langle kq\rangle\langle L_2q\rangle}\frac{i}{s_{L_1L_2}}i\frac{[bc]^3}{[ck][kL_3][L_3a][ab]}, \tag{4.1.10}$$

where the first contributes to the single-minus triangle and adjacent MHV current, the second contributes to the all-plus triangle and googly current.

Then in the L_2L_3 limit shown on the right of figure 4.13 which gives the $K = e$ embedded triangle,

$$A_3(k^+, L_2^+, L_3^-)\frac{i}{s_{L_2L_3}}A_5(b^+, c^+, L_1^-, -k^-, a^-) = i\frac{[KL_2]\langle L_3q\rangle^2}{\langle kq\rangle\langle L_2q\rangle}\frac{i}{s_{L_2L_3}}i\frac{[bc]^3}{[cL_1][L_1k][ka][ab]} \tag{4.1.11}$$

$$A_3(L_3^-, k^-, L_2^+)\frac{i}{s_{L_2L_3}}A_5(L_1^-, -k^+, a^-, b^+, c^+) = i\frac{\langle L_3k\rangle[L_2q]^2}{[L_3q][kq]}\frac{i}{s_{L_2L_3}}i\frac{\langle L_1a\rangle^4}{\langle L_1k\rangle\langle ka\rangle\langle ab\rangle\langle bc\rangle\langle cL_1\rangle}. \tag{4.1.12}$$

The first contributes to the all-plus triangle with googly current, and the second contributes the single-minus triangle and non-adjacent MHV current.

The final amplitude is

$$A_6(L_1^-, L_2^-, L_3^+, a^-, b^+, c^+) = i\frac{\langle L_1L_2\rangle^2[bc]^2[L_3|L_1+L_2|a\rangle^2}{s_{L_1L_2}s_{L_2L_3}s_{ab}s_{bc}t_{L_1L_2L_3}} + i\frac{\langle L_2a\rangle^2[bc]^2[L_|L_2+a|L_1\rangle^2}{s_{L_2L_3}s_{L_3a}s_{bc}s_{cL_1}t_{L_2L_3a}}$$
$$+ i\frac{\langle L_1L_2\rangle\langle L_2a\rangle[L_3b][bc][L_3|L_2+a|L_1\rangle[c|L_1+L_2|a\rangle t_{L_1L_2L_3}}{s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}s_{bc}s_{cL_1}}$$
$$- i\frac{\langle L_1L_2\rangle^2[L_3b][bc][L_3|L_1+L_2|a\rangle[c|L_1+L_2|a\rangle t_{L_2L_3a}}{s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}s_{bc}s_{cL_1}} + i\frac{\langle L_1L_2\rangle^2[L_3b]^2[c|L_1+L_2|a\rangle^2}{s_{L_1L_2}s_{L_3a}s_{ab}s_{cL_1}t_{L_3ab}}$$
$$- i\frac{\langle L_1L_2\rangle\langle L_2a\rangle[bc]^2[L_3|L_1+L_2|a\rangle[L_3|L_2+a|L_1\rangle t_{abc}}{s_{L_1L_2}s_{L_2L_3}s_{L_3a}s_{ab}s_{bc}s_{cL_1}}, \tag{4.1.13}$$

which has poles shown in figure 4.14.

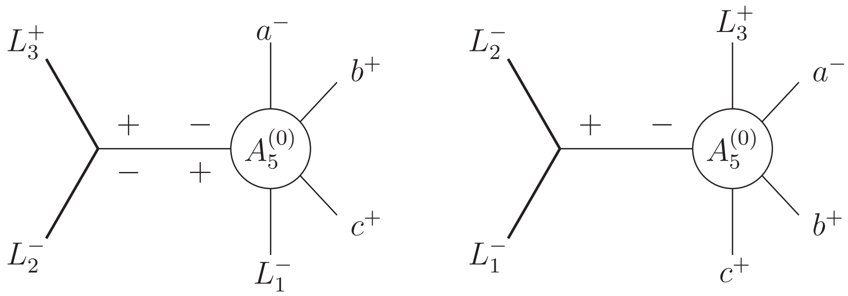


Figure 4.14: Factorisations of the second NMHV current on the L_1L_2 and L_2L_3 collinear pole.

There is one pole in the L_1L_2 limit on the right of figure 4.14

$$A_3(L_1^-, L_2^-, k^+)\frac{i}{s_{L_1L_2}}A_5(k^-, L_3^+, a^-, b^+, c^+) = i\frac{\langle L_1L_2\rangle[kq]^2}{[L_1q][L_2q]}\frac{i}{s_{L_1L_2}}i\frac{\langle ka\rangle^4}{\langle kL_3\rangle\langle L_3a\rangle\langle ab\rangle\langle bc\rangle\langle ck\rangle}, \tag{4.1.14}$$

which contributes to the single-minus triangle on the $K = d$ corner and the non-adjacent MHV current on the right.

There are two in the L_2L_3 limit on the left of figure 4.14 where the triangle would be in the $K = e$ corner

$$A_3(L_3^+,k^+,L_2^-)\frac{i}{s_{L_2L_3}}A_5(b^+,c^+,L_1^-,-k^-,a^-) = i\frac{[L_3k]\langle L_2q\rangle^2}{\langle L_3q\rangle\langle kq\rangle}\frac{i}{s_{L_2L_3}}i\frac{[bc]^3}{[cL_1][L_1k][ka][ab]}, \tag{4.1.15}$$

$$A_3(k^-,L_2^-,L_3^+)\frac{i}{s_{L_2L_3}}A_5(L_1^-,-k^+,a^-,b^+,c^+) = i\frac{\langle kL_2\rangle[L_3q]}{[kq][L_2q]}\frac{i}{s_{L_2L_3}}i\frac{\langle L_1a\rangle^4}{\langle L_1k\rangle\langle ka\rangle\langle ab\rangle\langle bc\rangle\langle cL_1\rangle} \tag{4.1.16}$$

where the first contributes to the all-plus triangle and googly current, and finally the second contributes to the single-minus triangle and non-adjacent MHV current.

To derive the full currents, we would have to incorporate these poles into the amplitude, however in this case as we are only considering the pole terms, this step is unnecessary.

Sorting the above poles from the three NMHV currents according to how they contribute to the embedded triangle calculations, we collate these in the following table 4.1 .

Embedded Triangle	5 pt current	Pole Contributions	6pt current
$K = e$ AP	Adj MHV	4.1.5	1
$K = e$ AP	googly	4.1.11, 4.1.15	2, 3
$K = d$ AP	googly	4.1.7, 4.1.10	2, 3
$K = e$ SM	adj MHV	4.1.5	1
$K = e$ SM	non-adj MHV	4.1.12, 4.1.16	2, 3
$K = d$ SM	adj MHV	4.1.6, 4.1.9	1, 2
$K = d$ SM	non-adj MHV	4.1.14	3

Table 4.1: In this table we summarise how the different collinear poles of the 6pt NMHV currents of the double-box structure correspond to the embedded triangle structures.

We label embedded triangles by which corner they are in, and AP for all-plus or SM for single-minus. The 5pt current denotes the current associated from the embedded triangles calculation. We show which poles contribute to which triangle in the third diagram. In the final column we label which of the 6pt NMHV amplitudes each pole comes from. To aid in readability, we label these amplitudes by which of the three off-shell legs has positive helicity, so $A_6(b^+,c^+,L_1^+,L_2^-,L_3^-,a^-)$ is labelled 1 above, $A_6(b^+,c^+,L_1^-,L_2^+,L_3^-,a^-)$ is 2, and $A_6(L_1^-,L_2^-,L_3^+,a^-,b^+,c^+)$ is 3.

4.2 Tree-level Currents

There are three possible non-zero currents that will be needed in the calculation of the embedded triangles and the bubble insertion: two MHV currents, $\tau^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+)$ and $\tau^{(0)}(\beta^-, \alpha^+, a^-, b^+, c^+)$, and one googly current, $\tau^{(0)}(b^+, c^+, \beta^-, \alpha^-, a^-)$. For higher multiplicity the MHV currents will generalise quite naturally, but with the googly current, the situation is not quite so clear: at higher multiplicity this will become an NMHV current. While there exists an n-point formula for tree-level NMHV amplitudes [90], it is not in a form conducive to direct manipulation for our purposes. With that being said, a look at six-point tree level NMHV amplitudes [89] does not appear to present any new kinematic structures that are not present in the 5-point googly, thus the integration methods in both cases should be the same.

We have the following currents, with full derivations in Appendix B,

$$\begin{aligned}\tau^{(0)}(\beta^-, \alpha^+, a^-, b^+, c^+) = {} & i\frac{\langle\beta q\rangle^2\langle cq\rangle[q|\alpha|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{\langle ab\rangle\langle\alpha q\rangle^2\langle bc\rangle[q|P_{\alpha\beta}|c\rangle^2[q|P_{\alpha\beta}|q\rangle^2} - i\frac{\langle\beta q\rangle^2\langle aq\rangle[q|\beta|q\rangle[q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle\langle\alpha q\rangle^2\langle bc\rangle[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} \\ & + i\frac{\langle\beta q\rangle^2[q|P_{\alpha\beta}|a\rangle^3\langle q|\alpha\beta|q\rangle}{s_{de}\langle ab\rangle\langle\alpha q\rangle^2\langle bc\rangle[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} - 4i\frac{\langle\beta q\rangle^2\langle aq\rangle[q|\alpha|q\rangle[q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle\langle\alpha q\rangle^2\langle bc\rangle[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} \\ & + O(s_{\alpha\beta}),\end{aligned} \tag{4.2.1}$$

$$\begin{aligned}\tau^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+) = {} & i\frac{\langle\alpha q\rangle^2\langle cq\rangle[q|\alpha|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle^2[q|P_{\alpha\beta}|q\rangle^2} + 3i\frac{\langle\alpha q\rangle^2\langle aq\rangle[q|\beta|q\rangle[q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} \\ & + i\frac{\langle\alpha q\rangle^2\langle q|\alpha\beta|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{s_{de}\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} + O(s_{\alpha\beta}),\end{aligned} \tag{4.2.2}$$

and

$$\begin{aligned}\tau_5^{(0)}(b^+, c^+, \beta^-, \alpha^-, a^-) = {} & - i\frac{\langle\alpha\beta\rangle[e|P_{\alpha\beta}|a\rangle^3}{s_{de}\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[\alpha e][\beta e]} - i\frac{\langle q\alpha\rangle[e|P_{\alpha\beta}|a\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|q\rangle^2[e|P_{\alpha\beta}|c\rangle[\alpha a][c\beta][\beta e]} \\ & + i\frac{\langle\beta q\rangle[e|P_{\alpha\beta}|a\rangle^3[ea]}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[e|P_{\alpha\beta}|q\rangle[\alpha a][\alpha e][\beta e]} \\ & - i\frac{\langle\alpha\beta\rangle[bc]^2}{\langle bc\rangle[ab][c\beta][\alpha a]} - i\frac{\langle\alpha\beta\rangle[bc]^2[be]}{[ab][c\beta][\alpha a][e|P_{\alpha\beta}|c\rangle} \\ & + i\frac{\langle aq\rangle\langle\alpha\beta\rangle[bc]^2[ce]}{\langle ab\rangle[ab][\alpha a][c\beta][e|P_{\alpha\beta}|q\rangle} - i\frac{\langle\alpha\beta\rangle\langle aq\rangle[bc]^2[ea]}{\langle bc\rangle[e|P_{\alpha\beta}|q\rangle[ab][\alpha a][c\beta]} + O(s_{\alpha\beta}^2).\end{aligned} \tag{4.2.3}$$

Due to the power counting of the triangles which we shall see later, we will need to know the googly current to order $s_{\alpha\beta}$.

4.3 The Bubble Insertion

Let us begin with the easier of the three. First we need to work out the bubble insertion. There is only one non-zero helicity configuration for the bubble insertion,

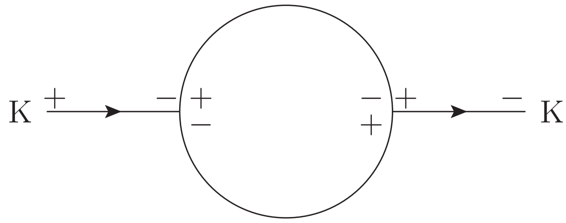


Figure 4.15: The sole non-zero contribution to the bubble insert for the l-bubble structure.

This gives the integral

$$Bub = \int \frac{d^D L_T}{L_B^2 L_T^2} \frac{[q|KL_B|q]\langle q|KL_B|q\rangle}{\langle Kq\rangle^4} = -\frac{1}{2}(K^2)^{1+\epsilon}\frac{\Gamma(-1+\epsilon)\Gamma(-\epsilon)\Gamma(4-\epsilon)}{\Gamma(4-2\epsilon)}. \tag{4.3.1}$$

The bubble can be inserted in two ways and both give the same answer.

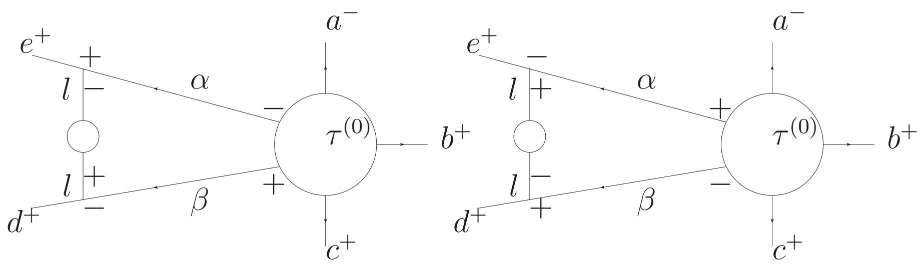


Figure 4.16: These are the two contributions from the 'l-bubble' diagram.

These are

$$\begin{aligned}&\int \frac{d^D l}{l^{2(2)}\alpha^2\beta^2} A_3(-l^+, e^+, -\alpha^-)A_3(-\beta^+, d^+, l^-)\tau^{(0)}(\beta^-, \alpha^+, a^-, b^+, c^+) \times Bub,\\ &\int \frac{d^D l}{l^{2(2)}\alpha^2\beta^2} A_3(e^+, -\alpha^+, -l^-)A_3(d^+, l^+, -\beta^-)\tau^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+) \times Bub.\end{aligned} \tag{4.3.2}$$

The details of this calculation are very similar to those from the previous chapter and so we will skip to the final result. After $\epsilon-$ expansion and recursion, we have the contributions to the final result,

$$-\frac{11}{18}\frac{\langle ac\rangle\langle ad\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2}-\frac{4}{9}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle} \tag{4.3.3}$$

which have the correct spinor content for the expected double pole.

4.4 Embedded Triangles

Ignoring particle labels for now so we can make general arguments about the triangles, we have the following structure

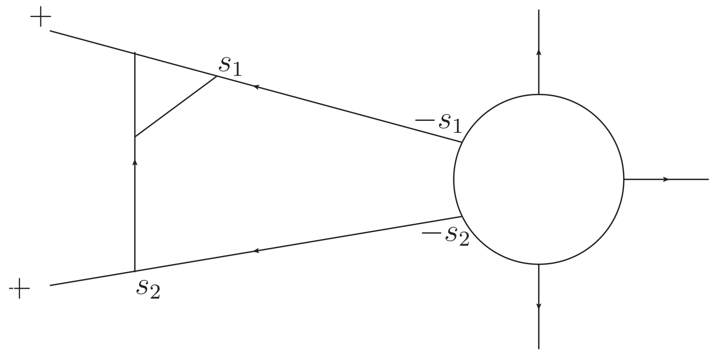


Figure 4.17: This diagram depicts the general structure of the embedded triangle diagrams that we will consider in this chapter.

where on the right we remember that the external particles have two positive helicities and one negative. From the currents, the options are that either one or both of s_1 and s_2 are positive. Below we will derive expressions for the one-loop triangle with two off-shell legs and one massless leg. To the best of our knowledge these are original results.

4.4.1 All-Plus Triangle

Zooming in on the left hand side of the above diagram we have the following options

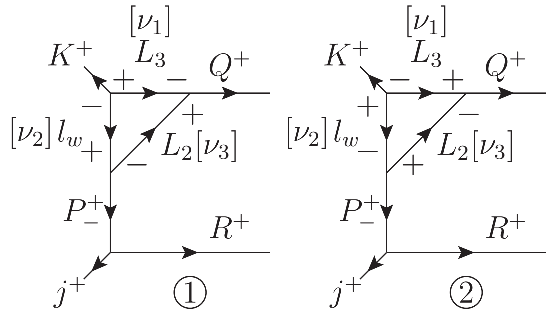


Figure 4.18: These two diagrams are all-plus embedded triangles that contribute to the diagram with the googly current on the right.

We label the external momenta as j and K so that we can obtain the triangle in both the d and the e corners by flipping. Similarly P and Q will be α and β depending on the flip in question. These are the all-plus triangles that arise from the googly current on the right, so $(s_1, s_2) = (+,+)$. These two triangles give

$$\begin{aligned}
ET_1^{++} &= + \int dL_3^d \frac{[KL_3]\langle l_w q\rangle^2}{\langle Kq\rangle\langle L_3 q\rangle} \frac{[QL_2]\langle L_3 q\rangle^2}{\langle Qq\rangle\langle L_2 q\rangle} \frac{[Pl_w]\langle L_2 q\rangle^2}{\langle Pq\rangle\langle l_w q\rangle} \frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
&= - \int dL_3^d \frac{\langle q|KL_3|q\rangle\langle q|QL_2|q\rangle\langle q|Pl_w|q\rangle}{\langle Kq\rangle^2\langle Pq\rangle^2\langle Qq\rangle^2} \frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
ET_2^{++} &= - \int dL_3^d \frac{[l_w K]\langle L_3 q\rangle^2}{\langle l_w q\rangle\langle Kq\rangle} \frac{[L_3 Q]\langle L_2 q\rangle^2}{\langle L_3 q\rangle\langle Qq\rangle} \frac{[L_2 P]\langle l_w q\rangle^2}{\langle L_2 q\rangle\langle Pq\rangle} \frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
&= + \int dL_3^d \frac{\langle q|l_w K|q\rangle\langle q|L_3 Q|q\rangle\langle q|L_2 P|q\rangle}{\langle Kq\rangle^2\langle Pq\rangle^2\langle Qq\rangle^2} \frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2}, \qquad (4.4.1)
\end{aligned}$$

where we have K is set to be null. In fact both integrals are equal and thus we can write

$$\begin{aligned}
A_3^{(1)}(K^+,P^+,Q^+) = ET_1^{++} + ET_2^{++} &= 2i\Gamma(1+\epsilon)\frac{\langle q|KQ|q\rangle^3}{\langle Kq\rangle^2\langle Pq\rangle^2\langle Qq\rangle^2} \int [du] \frac{u_2 u_3^{-\epsilon}(1-u_2-u_3)}{((1-u_3)Q^2 + u_2 2K\cdot Q)^{1+\epsilon}} \\
&= i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\langle q|KQ|q\rangle^3}{\langle Kq\rangle^2\langle Pq\rangle^2\langle Qq\rangle^2} \frac{1}{Q^{2(1+\epsilon)}} \frac{1}{3} {}_2F_1[1+\epsilon, 2, 4, -\frac{2K\cdot Q}{Q^2}] \qquad (4.4.2)
\end{aligned}$$

The all-plus triangle also features on the left with an MHV current on the right,

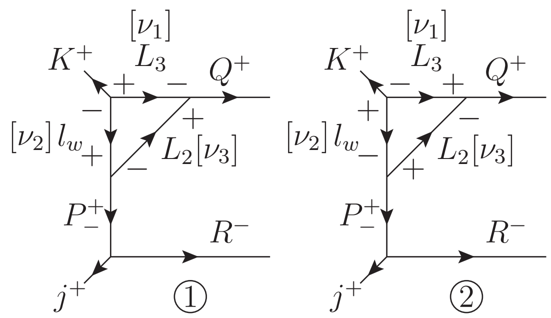


Figure 4.19: These embedded triangles are all-plus embedded triangles that contribute to the diagram with an adjacent MHV current on the right.

One check that we can perform is to take a second leg on-shell and compare it to the one-loop three point 'triangle' vertex. Rewriting $-\frac{2K\cdot Q}{Q^2} = 1 - \frac{P^2}{Q^2}$, then taking $P^2 \to 0$ the hypergeometric becomes

$$ {}_2F_1[1+\epsilon, 2, 4, 1] = \frac{\Gamma(4)\Gamma(1-\epsilon)}{\Gamma(2)\Gamma(3-\epsilon)} \tag{4.4.3} $$

So, setting $\epsilon \to 0$

$$ A_3^{(1)}(K^+, P^+, Q^+) \to -\frac{i}{3}\frac{[KP][PQ][QK]}{Q^2} \tag{4.4.4} $$

which precisely agrees with the one loop vertex result we have used thus far.

For reasons we shall discuss when it comes to integrating the current, it is useful to rewrite the hypergeometric function in a different representation.

First we use the $z \to 1-z$ continuation of the hypergeometric function (C.4.2),

$$ \begin{aligned} {}_2F_1[1+\epsilon, 2, 4, 1-\frac{P^2}{Q^2}] = &\frac{\Gamma(4)\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)\Gamma(2)} {}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{P^2}{Q^2}] \left(\frac{P^2}{Q^2}\right)^{1-\epsilon} \\ &+ \frac{\Gamma(4)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2)} {}_2F_1[1+\epsilon, 2, \epsilon, \frac{P^2}{Q^2}], \end{aligned} \tag{4.4.5} $$

and use the $z \to \frac{1}{z}$ continuation (C.4.1) to rewrite the second term above as

$$ \begin{aligned} {}_2F_1[1+\epsilon, 2, \epsilon, \frac{P^2}{Q^2}] = &\frac{\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(1+\epsilon)\Gamma(-1)} \left(-\frac{P^2}{Q^2}\right)^{-1-\epsilon} {}_2F_1[1+\epsilon, 2, \epsilon, \frac{Q^2}{P^2}] \\ &+ \frac{\Gamma(-1+\epsilon)\Gamma(\epsilon)}{\Gamma(1+\epsilon)\Gamma(\epsilon-2)} \left(-\frac{P^2}{Q^2}\right)^{-2} {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{Q^2}{P^2}]. \end{aligned} \tag{4.4.6} $$

The first term above vanishes as $\frac{1}{\Gamma(-1)} = 0$, so putting this together,

$$
\begin{aligned}
{}_2F_1[1+\epsilon, 2, 4, 1-\frac{P^2}{Q^2}] &= \frac{\Gamma(4)\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)\Gamma(2)} {}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{P^2}{Q^2}] \left(\frac{P^2}{Q^2}\right)^{1-\epsilon} \\
&+ \frac{\Gamma(4)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2)} \frac{\Gamma(-1+\epsilon)\Gamma(\epsilon)}{\Gamma(1+\epsilon)\Gamma(\epsilon-2)} \left(-\frac{P^2}{Q^2}\right)^{-2} {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{Q^2}{P^2}].
\end{aligned}
\tag{4.4.7}
$$

Using the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

we have

$$\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(\epsilon-2)} = 1,$$

so

$$
\begin{aligned}
{}_2F_1[1+\epsilon, 2, 4, 1-\frac{P^2}{Q^2}] &= 3\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} {}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{P^2}{Q^2}] \left(\frac{P^2}{Q^2}\right)^{1-\epsilon} \\
&+ 3\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{Q^2}{P^2}] \left(\frac{Q^2}{P^2}\right)^2 .
\end{aligned}
\tag{4.4.8}
$$

4.4.2 Single-Minus Triangle

In the case of the single-minus triangle, these only feature in the MHV current case.

These configurations arise when $(s_1, s_2) = (+, -)$. There are four integrals which add to give the single-minus triangle.

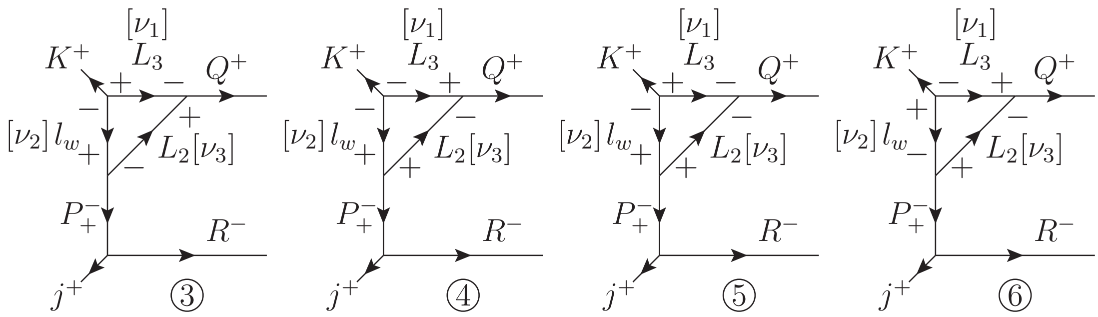


Figure 4.20: These four configurations add to the single-minus embedded triangle and contribute to the diagram with the adjacent MHV current on the right

$$
\begin{aligned}
ET_3^{+-} &= +\int dL_3^d \frac{[KL_3]\langle l_w q\rangle^2}{\langle Kq\rangle\langle L_3 q\rangle}\frac{-[QL_2]\langle L_3 q\rangle^2}{\langle Qq\rangle\langle L_2 q\rangle}\frac{\langle L_2 P\rangle[l_w q]^2}{[L_2 q][Pq]}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
&= +\int dL_3^d \frac{\langle q|KL_3|q\rangle\langle q|QL_2|q\rangle[q|L_2P|q]}{\langle Kq\rangle^2[Pq]^2\langle Qq\rangle^2}\frac{[q|l_w|q\rangle^2}{[q|L_2|q\rangle^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
&= -ET_4^{-+}\big|_{\{P\leftrightarrow Q, l_w\leftrightarrow L_3, L_2\to -L_2\}} \\
ET_4^{+-} &= -ET_2^{-+}\big|_{\{P\leftrightarrow Q, l_w\leftrightarrow L_3, L_2\to -L_2\}} \\
ET_5^{+-} &= -ET_3^{-+}\big|_{\{P\leftrightarrow Q, l_w\leftrightarrow L_3, L_2\to -L_2\}} \\
ET_6^{+-} &= -ET_1^{-+}\big|_{\{P\leftrightarrow Q, l_w\leftrightarrow L_3, L_2\to -L_2\}}
\end{aligned} \tag{4.4.9}
$$

These contributions arise when $(s_1, s_2) = (-,+)$,

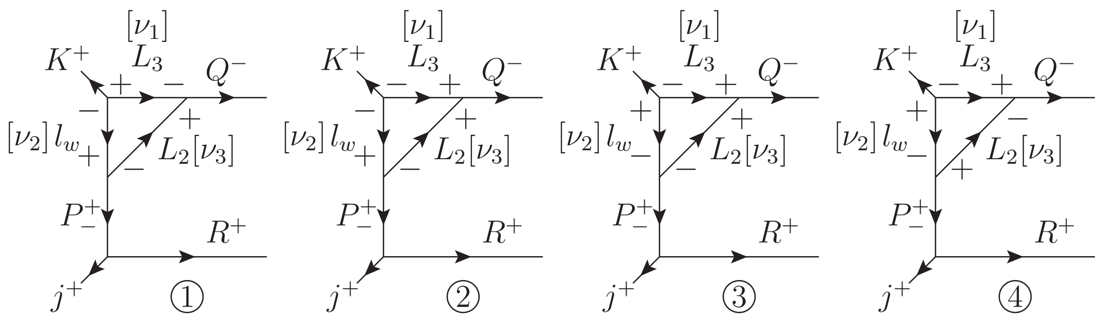


Figure 4.21: These four configurations give the single-minus embedded triangle contributions with the non-adjacent MHV current on the right.

$$
\begin{aligned}
ET_1^{-+} &= +\int dL_3^d \frac{\langle q|KL_3|q\rangle\langle q|Pl_w|q\rangle[q|QL_3|q]}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2}\frac{[q|L_2|q\rangle^2}{[q|L_3|q\rangle^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
ET_2^{-+} &= -\int dL_3^d \frac{\langle q|L_2L_3|q\rangle\langle q|Pl_w|q\rangle[q|l_wL_3|q]}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2}\frac{[q|K|q\rangle^2[q|Q|q\rangle^2}{[q|l_w|q\rangle^2[q|L_3|q\rangle^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
ET_3^{-+} &= -\int dL_3^d \frac{\langle q|l_wK|q\rangle\langle q|L_2L_3|q\rangle[q|L_2l_w|q]}{\langle kq\rangle^2\langle Pq\rangle^2[Qq]^2}\frac{[q|P|q\rangle^2[q|Q|q\rangle^2}{[q|l_w|q\rangle^2[q|L_2|q\rangle^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \\
ET_4^{-+} &= +\int dL_3^d \frac{\langle q|Kl_w|q\rangle\langle q|PL_2|q\rangle[q|QL_2|q]}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2}\frac{[q|L_3|q\rangle^2}{[q|L_2|q\rangle^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2}
\end{aligned} \tag{4.4.10}
$$

It will take more work to calculate the single-minus triangle than in the case of the all-plus triangle, however both cases involving the single-minus triangle they are the same calculation related by a flip as indicated in equation 4.4.9.

Summing the above terms we have

$$\sum_i ET_i^{-+} = \int dL_3^d \frac{C}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2}\frac{1}{L_3^2(L_3-Q)^2(L_3+K)^2} \tag{4.4.11}$$

where

$$\begin{aligned} C = &-\frac{[q|Q|q\rangle^2}{[q|L_3|q\rangle^2}\langle q|L_2P|q\rangle\Big(\langle q|KL_3|q\rangle[q|QL_3|q] + \langle q|QL_3|q\rangle[q|KL_3|q]\Big) \\ &- \frac{[q|Q|q\rangle^2}{[q|l_w|q\rangle^2}\langle q|L_2Q|q\rangle\Big(\langle q|Kl_w|q\rangle[q|Pl_w|q] + \langle q|Pl_w|q\rangle[q|Kl_w|q]\Big) \\ &- \frac{[q|Q|q\rangle^2}{[q|L_2|q\rangle^2}\langle q|l_wK|q\rangle\Big(\langle q|QL_2|q\rangle[q|PL_2|q] + \langle q|PL_2|q\rangle[q|QL_2|q]\Big) \\ &+ 2\frac{[q|Q|q\rangle}{[q|L_3|q\rangle}\Big(\langle q|KL_3|q\rangle + \langle q|QL_3|q\rangle\Big)[q|QL_3|q]\langle q|L_2P|q\rangle \\ &+ 2\frac{[q|Q|q\rangle}{[q|l_w|q\rangle}\Big(\langle q|Pl_w|q\rangle - \langle q|Kl_w|q\rangle\Big)[q|Ql_w|q]\langle q|L_2Q|q\rangle \\ &- 2\frac{[q|Q|q\rangle}{[q|L_2|q\rangle}\langle q|KL_2|q\rangle[q|QL_2|q]\langle q|l_wK|q\rangle \\ &+ 2\langle q|KL_3|q\rangle\langle q|Pl_w|q\rangle[q|QL_3|q], \end{aligned} \tag{4.4.12}$$

where we label the term in the first line C_1, the term in the second line C_2 and so on until C_7.

We can write the calculation of $\sum_i ET_i^{-+}$ in a basis of the following three integrals

$$\begin{aligned} I_g^{L_2} &= \int d^dL_3\left(\frac{[q|Q|q\rangle}{[q|L_2|q\rangle}\right)^N \frac{[\omega|L_2|q\rangle\langle q|l_wK|q\rangle\langle\zeta|L_2|q]}{L_3^2(L_3-Q)^2(L_3+K)^2} \\ I_g^{l_w} &= \int d^dL_3\left(\frac{[q|Q|q\rangle}{[q|l_w|q\rangle}\right)^N \frac{[\omega|l_w|q\rangle\langle q|L_2Q|q\rangle\langle\zeta|l_w|q]}{L_3^2(L_3-Q)^2(L_3+K)^2} \\ I_g^{L_3} &= \int d^dL_3\left(\frac{[q|Q|q\rangle}{[q|L_3|q\rangle}\right)^N \frac{[\omega|L_3|q\rangle\langle q|L_2P|q\rangle\langle\zeta|L_3|q]}{L_3^2(L_3-Q)^2(L_3+K)^2} \end{aligned} \tag{4.4.13}$$

where ω, ζ are generic spinors.

Additionally, we note that I^{L_3} and I^{l_w} are related by a flip $L_3 \leftrightarrow l_w$ and $P \leftrightarrow Q$ (apart from the $[q|Q|q\rangle^N$ which does not flip). Because we have chosen K to be null, we cannot relate L_2 by flips, so we have two distinct typical Feynman integrals,

$$\mathrm{Typ}_{L_2} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{p=0}^{m} \frac{(N)_n}{A^{N+m}(n-m)!(m-p)!p!} \int [dx] \frac{x_2^{d_2+p}(1-x_2-x_3)^{d_1+m-p}[q|Q|q\rangle^{m-p}(-[q|P|q\rangle)^p}{x_3^{1+d_d}\big([P^2-Q^2]x_2+(1-x_3)Q^2\big)^{1+d_d}}$$
$$\mathrm{Typ}_{L_3} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(N)_n}{m!(n-m)!A^{N+m}} \int [dx] \frac{x_1^{d_1}x_3^{d_3}[q|(1-x_1)K+x_3P|q\rangle^m}{\big([Q^2-P^2]x_1x_3+(1-x_3)x_3P^2\big)^{1+d_d}} \tag{4.4.14}$$

where the Feynman parameterisation is

$$p = L_3 - (x_3Q - u_2K), \qquad \Delta = (-x_3)\left((1-x_3)Q^2 + x_2(P^2-Q^2)\right). \tag{4.4.15}$$

In the above integrals, d_1, d_2 are extra factors of the Feynman parameters that may arise in the integrals, and $d_d = \epsilon$ for scalar contributions, and $d_d = \epsilon - 1$ for tensors. N is of course the power of $[q|L_2|q\rangle$ (or $[q|l_w|q\rangle$) in the denominator.

The Typ_{L_2} integral is the simpler of the two, we can write this in a symmetric manner as $\mathrm{Typ}_{L_2} = \mathrm{Typ}^A_{L_2} + \mathrm{Typ}^B_{L_2}$, where $\mathrm{Typ}^B_{L_2} = \mathrm{Typ}^A_{L_2}|_{P\leftrightarrow -Q, d_1\leftrightarrow d_2}$, where

$$\mathrm{Typ}^A_{L_2} = \frac{\Gamma(-d_d)\Gamma(d_d-d_1)\Gamma(1+d_1)\Gamma(1+d_2+d_1-d_d-N)}{\Gamma(1+d_d)\Gamma(1+d_2+d_1-2d_d-N)\big(Q^2\big)^{1+d_d}[q|P|q\rangle^N}\Big(\frac{P^2}{Q^2}\Big)^{d_1-d_d}$$
$$\times F_1\Big[1+d_1\ ;\ N\ ,\ 1+d_2+d_1-d_d-N\ ;\ 1+d_1-d_d\ ;\ -\frac{P^2}{Q^2}\frac{[q|Q|q\rangle}{[q|P|q\rangle}\ ,\ \frac{P^2}{Q^2}\Big], \tag{4.4.16}$$

and so

$$\mathrm{Typ}_{L_2}[N,d_1,d_2,d_d] = \frac{\Gamma(-d_d)\Gamma(d_d-d_1)\Gamma(1+d_1)\Gamma(1+d_2+d_1-d_d-N)}{\Gamma(1+d_d)\Gamma(1+d_2+d_1-2d_d-N)\big(Q^2\big)^{1+d_d}[q|P|q\rangle^N}\Big(\frac{P^2}{Q^2}\Big)^{d_1-d_d}$$
$$\times F_1\Big[1+d_1\ ;\ N\ ,\ 1+d_2+d_1-d_d-N\ ;\ 1+d_1-d_d\ ;\ -\frac{P^2}{Q^2}\frac{[q|Q|q\rangle}{[q|P|q\rangle}\ ,\ \frac{P^2}{Q^2}\Big]$$
$$+\frac{\Gamma(-d_d)\Gamma(d_d-d_2)\Gamma(1+d_2)\Gamma(1+d_2+d_1-d_d-N)}{\Gamma(1+d_d)\Gamma(1+d_2+d_1-2d_d-N)\big(P^2\big)^{1+d_d}(-[q|Q|q\rangle)^N}\Big(\frac{Q^2}{P^2}\Big)^{d_2-d_d}$$
$$\times F_1\Big[1+d_2\ ;\ N\ ,\ 1+d_2+d_1-d_d-N\ ;\ 1+d_2-d_d\ ;\ \frac{Q^2}{P^2}\frac{[q|P|q\rangle}{[q|Q|q\rangle}\ ,\ \frac{Q^2}{P^2}\Big]. \tag{4.4.17}$$

The expression for Typ_{L_3} is noticeably more complex

$$
\begin{aligned}
\mathrm{Typ}_{L_3}[N, d_2, d_3, d_d] = & \frac{\Gamma(d_d)\Gamma(d_3-d_d)\Gamma(-d_3+d_d+N)}{\Gamma(N)\Gamma(1+d_d)\big(Q^2\big)^{1+d_d}} \Big(-[q|Q|q\rangle\Big)^{-N} \Big(-\frac{[q|K|q\rangle}{[q|Q|q\rangle}\Big)^{d_3-d_d-N} \\
& \times (1-x)^{-[1+d_2+d_3-2d_d-N]} \Big(\frac{P^2}{Q^2}\Big)^{-d_d} \\
& F_2\Big[1+d_2+d_3-2d_d-N \ ; \ 1-N \ , \ 1 \ ; \ 1+d_3-d_d-N \ , \ 1-d_d \ ; \ \frac{x}{x-1} \ , \ \frac{y}{1-x}\Big] \\
& + \frac{\Gamma(-d_d)\Gamma(d_3-d_d)\Gamma(-d_3+d_d+N)\Gamma(1+d_2+d_3-d_d-N)}{\Gamma(N)\Gamma(1+d_2+d_3-2d_d-N)\big(Q^2\big)^{1+d_d}} \Big(-[q|Q|q\rangle\Big)^{-N} \\
\times & \Big(-\frac{[q|K|q\rangle}{[q|Q|q\rangle}\Big)^{d_3-d_d-N} (1-x-y)^{-[1+d_2+d_3-d_d-N]} \\
& {}_2F_1\Big[1+d_2+d_3-d_d-N \ ; \ 1-N \ ; \ 1+d_3-d_d-N \ ; \ \frac{x}{x+y-1}\Big] \\
& + \frac{\Gamma(d_d)\Gamma(d_3-d_d-N)\Gamma(1+d_2-d_d)}{\Gamma(1+d_d)\Gamma(1+d_2+d_3-2d_d-N)\big(Q^2\big)^{1+d_d}} (-[q|Q|q\rangle)^{-N} \Big(\frac{P^2}{Q^2}\Big)^{-d_d} \\
& \times F_2\Big[1+d_2-d_d \ ; \ N \ , \ 1 \ ; \ 1-d_3+d_d+N \ , \ 1-d_d \ ; \ -\frac{[q|K|q\rangle}{[q|Q|q\rangle}, \frac{P^2}{Q^2}\Big] \\
& + \frac{\Gamma(-d_d)\Gamma(d_3-d_d-N)\Gamma(1+d_2)}{\Gamma(1+d_2+d_3-2d_d-N)\big(Q^2\big)^{1+d_d}} (-[q|Q|q\rangle)^{-N} \\
& \qquad \times (1-y)^{-[1+d_2]} {}_2F_1\Big[1+d_2 \ ; \ N \ ; \ 1-d_3+d_d+N \ ; \ \frac{x}{1-y}\Big],
\end{aligned}
\tag{4.4.18}
$$

where $x = -\frac{[q|K|q\rangle}{[q|Q|q\rangle}$ and $y = \frac{P^2}{Q^2}$.

We will also need the special case when $[q|K|q\rangle = 0$ due to the triangle being in the $K = e$ corner. In this case the expressions are

$$
\begin{aligned}
\mathrm{Typ}_{L_2}^{A:[q|k|q\rangle=0} = & \frac{\Gamma(-d_d)\Gamma(d_d-d_1)\Gamma(1+d_1)\Gamma(1+d_2+d_1-d_d-N)}{\Gamma(1+d_d)\Gamma(1+d_2+d_1-2d_d-N)\big(Q^2\big)^{1+d_d}[q|P|q\rangle^N} \big(\frac{P^2}{Q^2}\big)^{d_1-d_d} \\
& \times \big(1-\frac{P^2}{Q^2}\big)^{-1-d_1-d_2} {}_2F_1\big[-d_d \ , \ -d_2 \ ; \ 1+d_1-d_d \ ; \ \frac{P^2}{Q^2}\big],
\end{aligned}
\tag{4.4.19}
$$

and

$$
\begin{aligned}
\mathrm{Typ}_{L_3}^{[q|k|q\rangle=0} =& \frac{1}{[q|P|q\rangle^N} \frac{\Gamma(d_3-d_d-N)}{\Gamma(1+d_1+d_3-2d_d-N)} \frac{1}{\left(P^2\right)^{1+d_d}} \\
& \times \left[\frac{\Gamma(d_d)\Gamma(1+d_1-d_d)}{\Gamma(1+d_d)} \left(\frac{Q^2}{P^2}\right)^{-d_d} {}_2F_1\big[1+d_1-d_d\ ,\ 1\ ;\ 1-d_d\ ;\ \frac{Q^2}{P^2}\big] \right. \\
& \left. \qquad + \Gamma(-d_d)\Gamma(1+d_1)\left(1-\frac{Q^2}{P^2}\right)^{-(1+d_1)} \right]. \qquad (4.4.20)
\end{aligned}
$$

With these integrals in hand, the next task is to express C in terms of the above. The first step is to write the basis integrals, (4.4.13) in terms of the typical integrals. These are

$$
\begin{aligned}
I_g^{L_3} =& i\Gamma(1+\epsilon)[q|Q|q\rangle^N \langle q|QP|q\rangle [\omega|Q|q\rangle [q|Q|\zeta\rangle \\
& \times \Big(\mathrm{Typ}_{L_3}[N,0,2,\epsilon] - \mathrm{Typ}_{L_3}[N,1,2,\epsilon] - \mathrm{Typ}_{L_3}[N,0,3,\epsilon] \Big) \\
& - i\Gamma(1+\epsilon)[q|Q|q\rangle^N \langle q|QP|q\rangle \left([\omega|Q|q\rangle [q|P|\zeta\rangle + [\omega|P|q\rangle [q|Q|\zeta\rangle \right) \\
& \times \Big(\mathrm{Typ}_{L_3}[N,1,1,\epsilon] - \mathrm{Typ}_{L_3}[N,2,1,\epsilon] - \mathrm{Typ}_{L_3}[N,1,2,\epsilon] \Big) \\
& + i\Gamma(1+\epsilon)[q|Q|q\rangle^N \langle q|QP|q\rangle [\omega|K|q\rangle [q|K|\zeta\rangle \\
& \times \Big(\mathrm{Typ}_{L_3}[N,2,0,\epsilon] - \mathrm{Typ}_{L_3}[N,0,3,\epsilon] - \mathrm{Typ}_{L_3}[N,2,1,\epsilon] \Big) \qquad (4.4.21) \\
& + i\Gamma(\epsilon)[q|Q|q\rangle^N \langle q|QP|q\rangle \langle \zeta q\rangle [\omega q] \\
& \times \big(\mathrm{Typ}_{L_3}[N,0,0,\epsilon-1] - \mathrm{Typ}_{L_3}[N,1,0,\epsilon-1] - \mathrm{Typ}_{L_3}[N,0,1,\epsilon-1] \big) \\
& + i\Gamma(\epsilon)[q|Q|q\rangle^N [q|P|q\rangle \langle \zeta q\rangle \\
& \times \big([\omega|Q|q\rangle \mathrm{Typ}_{L_3}[N,0,1,\epsilon-1] - [\omega|K|q\rangle \mathrm{Typ}_{L_3}[N,1,0,\epsilon-1] \big) ,
\end{aligned}
$$

and we get $I_g^{l_w}$ by flipping $P \leftrightarrow Q$ (not including the factor of $[q|Q|q\rangle^N$), then finally

$$\begin{aligned}
I_g^{L_2} &= [q|Q|q\rangle^N \langle q|KQ|q\rangle i\Gamma(1+\epsilon) \\
&\times \Bigg([q|Q|\zeta\rangle[\omega|Q|q\rangle \mathrm{Typ}_{L_2}[N,2,0,\epsilon] + [q|P|\zeta\rangle[\omega|P|q\rangle \mathrm{Typ}_{L_2}[N,0,2,\epsilon] \\
&- \left([q|Q|\zeta\rangle[\omega|P|q\rangle + [q|P|\zeta\rangle[\omega|Q|q\rangle\right) \mathrm{Typ}_{L_2}[N,1,1,\epsilon] \Bigg) \\
&+ i\Gamma(\epsilon)\langle q\zeta\rangle[q|K|q\rangle[q|Q|q\rangle^N \\
&\times \big([\omega|Q|q\rangle \mathrm{Typ}_{L_2}[N,1,0,\epsilon-1] - [\omega|P|q\rangle \mathrm{Typ}_{L_2}[N,0,1,\epsilon-1]\big) \\
&+ [q|Q|q\rangle^N \langle q|KQ|q\rangle i\Gamma(\epsilon)\langle \zeta q\rangle[\omega q] \\
&\big(\mathrm{Typ}_{L_2}[N,0,0,\epsilon-1] - \mathrm{Typ}_{L_2}[N,1,0,\epsilon-1] - \mathrm{Typ}_{L_2}[N,0,1,\epsilon-1]\big)\,.
\end{aligned} \tag{4.4.22}$$

Then we can write C in terms of the above integrals, beginning with

$$C_1 = -I_{N=2}^{L_3}\left(\{\omega = \langle q|K, \zeta = [q|Q\} + \{\omega = \langle q|Q, \zeta = [q|K\}\right), \tag{4.4.23}$$

and $C_2 = C_1|_{P\leftrightarrow Q}$. We can write C_3 as

$$C_3 = -I_{N=2}^{L_2}\left(\{\omega = \langle q|Q, \zeta = [q|P\} + \{\omega = \langle q|P, \zeta = [q|Q\}\right). \tag{4.4.24}$$

The next two terms are *almost* related by a flip similarly to C_1 and C_2. We define

$$C_\sigma = 2I_{N=1}^{L_3}\left(\sigma\,\{\omega = \langle q|K, \zeta = [q|Q\} + \{\langle q|Q, \zeta = [q|Q\}\right) \tag{4.4.25}$$

so that

$$C_4 = C_\sigma|_{\sigma=1}, \qquad C_5 = C_\sigma|_{\sigma=-1, P\leftrightarrow Q}, \tag{4.4.26}$$

and

$$C_6 = -2I_{N=1}^{L_2}\left(\omega = \langle q|K, \zeta = [q|Q\right). \tag{4.4.27}$$

The final term C_7 cannot be written in terms of the basis integrals but is easy to evaluate with the usual integration methods,

$$
\begin{aligned}
C_7 = &-\frac{i}{6}\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle q|KQ|q\rangle\langle q|KP|q\rangle[q|QK|q]}{Q^{2(1+\epsilon)}}{}_2F_1[1+\epsilon,2,4,\left(1-\frac{P^2}{Q^2}\right)] \\
&- i\Gamma(\epsilon)\beta(2-\epsilon,2-\epsilon)\frac{\langle q|KQ|q\rangle[q|Q|q\rangle[q|P|q\rangle}{Q^{2(\epsilon)}}{}_2F_1[\epsilon,1,2,\left(1-\frac{P^2}{Q^2}\right)] \\
&+ \frac{i}{2}\Gamma(\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle q|KP|q\rangle[q|Q|q\rangle[q|K|q\rangle}{Q^{2(\epsilon)}}{}_2F_1[\epsilon,1,3,\left(1-\frac{P^2}{Q^2}\right)].
\end{aligned}
\tag{4.4.28}
$$

Finally we note that since we are setting $\tilde{\lambda}_q = \tilde{\lambda}_e$, then in the $K = e$ corner we have $[q|K = 0$ and thus there will be simplifications. In this case, replace the typical integrals with their $[q|K|q\rangle = 0$ limits.

4.5 Integrating the Googly Current

The integral we are solving is

$$
\begin{aligned}
&\int \frac{d^D l}{l^2\alpha^2\beta^2}\ \tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-) \\
&\qquad \times \left(A_3^{(1)}(d^+,l^+,-\beta^+)A_3^{(0)}(e^+,-\alpha^+,-l^-) + A_3^{(1)}(e^+,-\alpha^+,-l^+)A_3^{(0)}(-\beta^+,d^+,l^-)\right).
\end{aligned}
\tag{4.5.1}
$$

Explicitly for each term,

$$
\begin{aligned}
&\int \frac{d^D l}{l^2\alpha^2\beta^2}\ A_3^{(1)}(d^+,l^+,-\beta^+)A_3^{(0)}(e^+,-\alpha^+,-l^-)\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-) \\
=&\frac{-1}{3}\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\int \frac{d^D l}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle q|dl|q\rangle^3}{\langle dq\rangle^2\langle\beta q\rangle^2}\frac{[e\alpha]}{\langle\alpha q\rangle\langle eq\rangle}{}_2F_1[1+\epsilon,2,4,1-\frac{l^2}{\beta^2}] \\
&\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-),
\end{aligned}
\tag{4.5.2}
$$

and

$$
\begin{aligned}
&\int \frac{d^D l}{l^2\alpha^2\beta^2}\ \tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-)A_3^{(1)}(e^+,-\alpha^+,-l^+)A_3^{(0)}(-\beta^+,d^+,l^-) \\
=&\frac{-1}{3}\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3\langle eq\rangle[d\beta]}{\langle dq\rangle\langle\alpha q\rangle^2\langle\beta q\rangle}{}_2F_1[1+\epsilon,2,4,1-\frac{l^2}{\alpha^2}] \\
&\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-).
\end{aligned}
\tag{4.5.3}
$$

Using the analytical continuation (4.4.8), we rewrite the above as

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\ A_3^{(1)}(d^+,l^+,-\beta^+)A_3^{(0)}(e^+,-\alpha^+,-l^-)\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-)$$
$$= -\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int \frac{d^D l}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle q|dl|q\rangle^3}{\langle dq\rangle^2\langle\beta q\rangle^2}\frac{[e|\alpha|q\rangle}{\langle\alpha q\rangle^2\langle eq\rangle}$$
$$\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-)\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right), \tag{4.5.4}$$

and

$$\int \frac{d^D l}{l^2\alpha^2\beta^2}\ \tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-)A_3^{(1)}(e^+,-\alpha^+,-l^+)A_3^{(0)}(-\beta^+,d^+,l^-)$$
$$= -\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3\langle eq\rangle[d|\beta|q\rangle}{\langle dq\rangle\langle\alpha q\rangle^2\langle\beta q\rangle^2}\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-)$$
$$\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right). \tag{4.5.5}$$

The googly current is

$$\begin{aligned}\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-) = &-i\frac{\langle\alpha\beta\rangle[e|P_{\alpha\beta}|a\rangle^3}{s_{de}\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[\alpha e][\beta e]} - i\frac{\langle q\alpha\rangle[e|P_{\alpha\beta}|a\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|q\rangle^2[e|P_{\alpha\beta}|c\rangle[\alpha a][c\beta][\beta e]}\\ &+i\frac{\langle\beta q\rangle[e|P_{\alpha\beta}|a\rangle^3[ea]}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[e|P_{\alpha\beta}|q\rangle[\alpha a][\alpha e][\beta e]}\\ &-i\frac{\langle\alpha\beta\rangle[bc]^2}{\langle bc\rangle[ab][c\beta][\alpha a]} - i\frac{\langle\alpha\beta\rangle[bc]^2[be]}{[ab][c\beta][\alpha a][e|P_{\alpha\beta}|c\rangle}\\ &+i\frac{\langle aq\rangle\langle\alpha\beta\rangle[bc]^2[ce]}{\langle ab\rangle[ab][\alpha a][c\beta][e|P_{\alpha\beta}|q\rangle} - i\frac{\langle\alpha\beta\rangle\langle aq\rangle[bc]^2[ea]}{\langle bc\rangle[e|P_{\alpha\beta}|q\rangle[ab][\alpha a][c\beta]} + O(s_{\alpha\beta}^2),\end{aligned} \tag{4.5.6}$$

where the first three terms are of leading order $O(s^0_{\alpha\beta})$, while the remaining four are of subleading order $O(s_{\alpha\beta})$.

4.5.1 First Leading Order Term

Beginning with the first term, we shall first consider the triangle in the $K = e$ corner. The integral is

$$\begin{aligned}
R_1^e =& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3\langle eq\rangle[d\beta]}{\langle dq\rangle\langle \alpha q\rangle^2\langle \beta q\rangle}\frac{\langle\alpha\beta\rangle\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle[\alpha e][\beta e]}\\
&\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right)\\
=& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle eq\rangle[de]^2\langle da\rangle^3}{\langle dq\rangle s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\\
&\int d^D l\frac{[e|l|q\rangle^3[d\beta]\langle\alpha\beta\rangle}{\langle\alpha q\rangle[e|\alpha|q\rangle[e|\beta|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}+\frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right)\\
=&i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle eq\rangle[de]^2\langle da\rangle^3}{\langle dq\rangle s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\\
&\int d^D l\frac{[e|l|q\rangle[d|\beta|q\rangle[e|\alpha\beta|e]}{[e|\beta|q\rangle^2}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}+\frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right).
\end{aligned} \tag{4.5.7}$$

and we label the integrals as

$$R_1^e = i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle eq\rangle[de]^2\langle da\rangle^3}{\langle dq\rangle s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle}(I_{e;1}+I_{e;2}). \tag{4.5.8}$$

Starting with $I_{e;1}$,

$$I_{e;1} =\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int d^D l\frac{[e|l|q\rangle[d|\beta|q\rangle[e|\alpha\beta|e]}{[e|\beta|q\rangle^2}\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}. \tag{4.5.9}$$

We can use the A trick to raise $[e|\beta|q\rangle$ to the numerator,

$$\begin{aligned}
I_{e;1} =&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\frac{(-1)^a}{A^{2+a}}\\
&\int\frac{d^D l}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}[e|l|q\rangle[d|\beta|q\rangle[e|\alpha\beta|e][e|\beta|q\rangle^a,
\end{aligned} \tag{4.5.10}$$

and Feynman parameterise

$$\begin{aligned}I_{e;1} =& -\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a \\ &\frac{\Gamma(4+\epsilon)}{\Gamma(\epsilon-k)\Gamma(3+k)}\int[du]u_1^{\epsilon-k-1}u_2^{2+k}u_3(1-u_3)^a\int d^Dp\frac{p^2/D}{(p^2-\Delta)^{4+\epsilon}} \\ =& -i\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a \\ &\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-k)\Gamma(3+k)}\int[du]u_1^{\epsilon-k-1}u_2^{1-2\epsilon+k}u_3^{-2\epsilon}(1-u_3)^a.\end{aligned} \tag{4.5.11}$$

The sum over k in the above expression arises from the hypergeometric function from the all-plus triangle, however to write the hypergeometric as a sum in this way is to assume that it converges in this region, in other words we are assuming that $\frac{u_2}{u_1} < 1$ which is not necessarily true for the full region of Feynman parameter integration. Thus to make our calculation more robust we shall now resum this term.

$$\begin{aligned}I_{x;1} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \\ &\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\int[du]\ u_1^{-1+\epsilon}u_2^{-2\epsilon}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ &\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\frac{\Gamma(2+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\left(\frac{u_2}{u_1}\right)^k.\end{aligned} \tag{4.5.12}$$

We shall call this sum S_k, which we can extract and write as

$$\begin{aligned}S_k\left[\frac{u_2}{u_1}\right] =& \frac{1}{\Gamma(3)\Gamma(\epsilon)}{}_3F_2[2,3-\epsilon,1-\epsilon,2-\epsilon,3,-\frac{u_2}{u_1}] \\ =& \frac{1}{\Gamma(\epsilon)(\epsilon-2)(1+\epsilon)}\left(\frac{u_1^2}{(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}}+\frac{u_1^4}{u_2^2(u_1+u_2)^2}\right. \\ &\left.-\frac{u_1^{4-\epsilon}}{u_2^2(u_1+u_2)^{2-\epsilon}}+2\frac{u_1^3}{u_2(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{3-\epsilon}}{u_2(u_1+u_2)^{2-\epsilon}}-\frac{u_1^{1-\epsilon}u_2}{(u_1+u_2)^{2-\epsilon}}\right).\end{aligned} \tag{4.5.13}$$

This expression is exact and as a check it is easy to see that it is correct both in the $u_2 \to 0$ and $\epsilon \to 0$ limits. It is therefore trivially convergent so that now our integral is

$$
\begin{aligned}
I_{e;1} = &- i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a\int[du]u_1^{\epsilon-1}u_2^{1-2\epsilon}u_3^{-2\epsilon}(1-u_3)^a \\
&\left(\frac{u_1^2}{(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}}+\frac{u_1^4}{u_2^2(u_1+u_2)^2}\right. \\
&\left.-\frac{u_1^{4-\epsilon}}{u_2^2(u_1+u_2)^{2-\epsilon}}+2\frac{u_1^3}{u_2(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{3-\epsilon}}{u_2(u_1+u_2)^{2-\epsilon}}-\frac{u_1^{1-\epsilon}u_2}{(u_1+u_2)^{2-\epsilon}}\right).
\end{aligned}
\tag{4.5.14}
$$

From the above we define the typical integral,

$$
\begin{aligned}
I_{e;1}^{\text{typ}} = &- i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a \\
&\int[du]u_1^{d_1+\epsilon-1}u_2^{1-2\epsilon+d_2}u_3^{-2\epsilon}(1-u_3)^{a+d_3},
\end{aligned}
\tag{4.5.15}
$$

where the d_1, d_2, d_3 represent the various powers of u_1, u_2 and $(1-u_3)$, respectively. We also note that in each case $d_1 + d_2 + d_3 = 0$ as expected. Moving on we use the substitution $u_2 = v(1-u_3)$,

$$
\begin{aligned}
I_{e;1}^{\text{typ}} = &- i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a \\
&\int dv\ v^{1-2\epsilon+d_2}(1-v)^{\epsilon+d_1-1}\int du_3\ u_3^{-2\epsilon}(1-u_3)^{1-\epsilon+d_{123}+a} \\
= &- i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\frac{\Gamma(2-2\epsilon+d_2)\Gamma(\epsilon+d_1)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(2-\epsilon+d_{123})}{\Gamma(3-3\epsilon+d_{123})} \\
&\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a\frac{(2-\epsilon+d_{123},a)}{(3-3\epsilon+d_{123},a)} \\
= &- i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2 s_{de}^{1+2\epsilon}}\frac{\Gamma(2-2\epsilon+d_2)\Gamma(\epsilon+d_1)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(2-\epsilon+d_{123})}{\Gamma(3-3\epsilon+d_{123})} \\
&F_2[2,U,2-\epsilon+d_{123},U,3-3\epsilon+d_{123},1-\delta,-\frac{[e|d|q\rangle}{A}]
\end{aligned}
\tag{4.5.16}
$$

where U signifies an undefined argument. This can be simplified in the same way we have done before, to reach

$$
I_{e;1}^{\text{typ}}[d_1,d_2,d_3] = - i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[de]}{s_{de}^{1+2\epsilon}}\frac{\Gamma(2-2\epsilon+d_2)\Gamma(\epsilon+d_1)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(1-3\epsilon+d_{123})}.
\tag{4.5.17}
$$

Now we can put this back together to write $I_{e;1}$ as

$$\begin{aligned}I_{e;1} =&I^{\rm typ}_{e;1}[2,0,-2]+(\epsilon-2)I^{\rm typ}_{e;1}[2-\epsilon,,0-2+\epsilon]+I^{\rm typ}_{e;1}[4,-2,-2]-I^{\rm typ}_{e;1}[4-\epsilon,-2,-2+\epsilon]\\ &+2I^{\rm typ}_{e;1}[3,-1,-2]+(\epsilon-2)I^{\rm typ}_{e;1}[3-\epsilon,-1,-2+\epsilon]-I^{\rm typ}_{e;1}[1-\epsilon,1,-2+\epsilon].\end{aligned} \tag{4.5.18}$$

Moving on to $I_{e;2}$

$$\begin{aligned}I_{e;2} =&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\frac{[e|l|q\rangle[d|\beta|q\rangle[e|\alpha\beta|e]}{[e|\beta|q\rangle^2}\\ =&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\frac{-1^a}{A^{2+a}}\\ &\int\frac{d^Dl}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}[e|l|q\rangle[d|\beta|q\rangle[e|\alpha\beta|e][e|\beta|q\rangle^a\\ =&-i\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2s_{de}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a\\ &\frac{\Gamma(1+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\int[du]u_2^{-1-2\epsilon}u_3^{-1-2\epsilon}u_1^{2+k}u_2^{\epsilon-k-1}u_3(1-u_3)^a.\end{aligned} \tag{4.5.19}$$

Now sum in k is equal to $S_k[\frac{u_1}{u_2}]$ as defined in eq 4.5.13, so the integral is

$$\begin{aligned}I_{e;2} =&-i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2s_{de}}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a\int[du]u_2^{-2-\epsilon}u_3^{-2\epsilon}u_1^2(1-u_3)^a\\ &\left(\frac{u_2^2}{(u_1+u_2)^2}+(\epsilon-2)\frac{u_2^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}}+\frac{u_2^4}{u_1^2(u_1+u_2)^2}\right.\\ &\left.-\frac{u_2^{4-\epsilon}}{u_1^2(u_1+u_2)^{2-\epsilon}}+2\frac{u_2^3}{u_1(u_1+u_2)^2}+(\epsilon-2)\frac{u_2^{3-\epsilon}}{u_1(u_1+u_2)^{2-\epsilon}}-\frac{u_2^{1-\epsilon}u_1}{(u_1+u_2)^{2-\epsilon}}\right).\end{aligned} \tag{4.5.20}$$

In the same way we define

$$\begin{aligned}I^{\rm typ}_{e;2} =&-i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[e|d|q\rangle^2[de]}{A^2s_{de}}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\left(-\frac{[e|d|q\rangle}{A}\right)^a\\ &\int[du]u_2^{-2-\epsilon+d_2}u_3^{-2\epsilon}u_1^{2+d_1}(1-u_3)^{a+d_3}\\ =&-i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{[de]}{s_{de}}\frac{\Gamma(-1-\epsilon+d_2)\Gamma(3+d_1)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(1-3\epsilon+d_{123})}\end{aligned} \tag{4.5.21}$$

and putting it back together,

$$\begin{aligned}I_{e;2} =&I^{\rm typ}_{e;2}[0,2,-2]+(\epsilon-2)I^{\rm typ}_{e;2}[0,2-\epsilon,-2+\epsilon]+I^{\rm typ}_{e;2}[-2,4,-2]-I^{\rm typ}_{e;2}[-2,4-\epsilon,-2+\epsilon]\\&+2I^{\rm typ}_{e;2}[-1,3,-2]+(\epsilon-2)I^{\rm typ}_{e;2}[-1,3-\epsilon,-2+\epsilon]-I^{\rm typ}_{e;2}[1,1-\epsilon,-2+\epsilon].\end{aligned} \tag{4.5.22}$$

Assembling the full $K=e$ contribution we get

$$R^e_1 = i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle eq\rangle[de]^2\langle da\rangle^3}{\langle dq\rangle s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle}(I_{e;1}+I_{e;2}). \tag{4.5.23}$$

We then we set $\lambda_q=\lambda_a$, expand to obtain the $\epsilon-$finite piece and perform recursion, and the final contribution is

$$R^e_1 = -\frac{85}{27}\frac{\langle ea\rangle[de]^3\langle da\rangle^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle}, \tag{4.5.24}$$

which is precisely the spinor content we expect for the leading pole.

Moving on to the $K=d$ embedded triangle, we have similarly,

$$\begin{aligned}R^d_1 =&-\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle q|dl|q\rangle^3}{\langle dq\rangle^2\langle\beta q\rangle^2}\frac{[e|\alpha|q\rangle}{\langle\alpha q\rangle^2\langle eq\rangle}\frac{\langle\alpha\beta\rangle\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle[\alpha e][\beta e]}\\&\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right),\\=&\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2\langle eq\rangle}\int d^Dl\frac{\langle q|dl|q\rangle^3[e|\alpha\beta|e]}{[e|\alpha|q\rangle[e|\beta|q\rangle^2}\\&\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}+\frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right)\\=&\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2\langle eq\rangle}(I_{e;3}+I_{e;4}).\end{aligned} \tag{4.5.25}$$

Starting with $I_{e;3}$

$$I_{e;3} =\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}\frac{\langle q|dl|q\rangle^3[e|\alpha\beta|e]}{[e|\alpha|q\rangle[e|\beta|q\rangle^2}, \tag{4.5.26}$$

again we use the A trick on both the denominator spinor terms

$$\begin{aligned}I_{e;3} =&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(2,a_1)}{a!(a_1-a)!}\sum_{b,b_1}\frac{(2,b_1)}{b!(b_1-b)!}\frac{-1^a}{A^{1+a}}\frac{-1^b}{B^{1+b}}\\&\int\frac{d^Dl}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}\langle q|dl|q\rangle^3[e|\alpha\beta|e][e|\alpha|q\rangle^a[e|\beta|q\rangle^b,\end{aligned} \tag{4.5.27}$$

and do Feynman parameterisation

$$\begin{aligned}
I_{e;3} =&3i\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle q|de|q\rangle^2[e|d|q\rangle^2}{ABs_{de}^{1+2\epsilon}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\sum_{b,b_1}\frac{(2,b_1)}{b!(b_1-b)!}\\
&\left(-\frac{[e|d|q\rangle}{A}\right)^a\left(-\frac{[e|d|q\rangle}{B}\right)^b\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-k)\Gamma(3+k)}\int[du]u_2^{1-2\epsilon}(1-u_3)^b u_1^{\epsilon-k-1}u_3^{1-2\epsilon+k+a}\\
=&3i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{\langle q|de|q\rangle^2[e|d|q\rangle^2}{ABs_{de}^{1+2\epsilon}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\sum_{b,b_1}\frac{(2,b_1)}{b!(b_1-b)!}\\
&\left(-\frac{[e|d|q\rangle}{A}\right)^a\left(-\frac{[e|d|q\rangle}{B}\right)^b\int[du]u_2^{1-2\epsilon}(1-u_3)^b u_1^{\epsilon-1}u_3^{1-2\epsilon+a}\\
&\left(\frac{u_1^2}{(u_1+u_3)^2}+(\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_3)^{2-\epsilon}}+\frac{u_1^4}{u_3^2(u_1+u_3)^2}\right.\\
&\left.-\frac{u_1^{4-\epsilon}}{u_3^2(u_1+u_3)^{2-\epsilon}}+2\frac{u_1^3}{u_3(u_1+u_3)^2}+(\epsilon-2)\frac{u_1^{3-\epsilon}}{u_3(u_1+u_3)^{2-\epsilon}}-\frac{u_1^{1-\epsilon}u_3}{(u_1+u_3)^{2-\epsilon}}\right),
\end{aligned} \tag{4.5.28}$$

where in the above expression we now get the expression $S_k[\frac{u_3}{u_1}]$. As before, this defines the typical integral, where this time we use e_1, e_2, e_3 to signify the powers of $u_1, (1-u_2)$, and u_3, respectively. Once again we note $e_1+e_2+e_3=0$.

$$\begin{aligned}
I_{e;3}^{\rm typ} =&3i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{\langle q|de|q\rangle^2[e|d|q\rangle^2}{ABs_{de}^{1+2\epsilon}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\sum_{b,b_1}\frac{(2,b_1)}{b!(b_1-b)!}\\
&\left(-\frac{[e|d|q\rangle}{A}\right)^a\left(-\frac{[e|d|q\rangle}{B}\right)^b\int[du]u_1^{\epsilon-1+e_1}u_2^{1-2\epsilon}(1-u_3)^b u_3^{1-2\epsilon+e_3+a}(1-u_2)^{e_2}\\
=&3i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{\langle q|de|q\rangle^2[e|d|q\rangle^2}{ABs_{de}^{1+2\epsilon}}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\sum_{b,b_1}\frac{(2,b_1)}{b!(b_1-b)!}\\
&\left(-\frac{[e|d|q\rangle}{A}\right)^a\left(-\frac{[e|d|q\rangle}{B}\right)^b\sum_{j=0}^{b}\binom{b}{j}(-1)^j\frac{\Gamma(2-2\epsilon)\Gamma(2-\epsilon+e_{123}+j+a)}{\Gamma(4-3\epsilon+e_{123}+j+a)}\\
&\frac{\Gamma(\epsilon+e_1)\Gamma(2-2\epsilon+e_3+j+a)}{\Gamma(2-\epsilon+e_{13}+j+a)}.
\end{aligned} \tag{4.5.29}$$

Unfortunately we are not able to complete the resummation, however we can see quite clearly from our past calculations using the A trick that the integral would evaluate to

$$\frac{\langle q|de|q\rangle^2}{[e|d|q\rangle s_{de}^{1+2\epsilon}}F(\epsilon), \tag{4.5.30}$$

where the function $F(\epsilon)$ is a function of the dimensional regulator and is free of kinematic factor.

Bringing the terms back together,

$$I_{e;3} = I^{\text{typ}}_{e;3}[2,-2,0] + (\epsilon-2)I^{\text{typ}}_{e;3}[2-\epsilon,-2+\epsilon,0] + I^{\text{typ}}_{e;3}[4,-2,-2] - I^{\text{typ}}_{e;3}[4-\epsilon,-2+\epsilon,-2]$$
$$+ 2I^{\text{typ}}_{e;3}[3,-2,-1] + (\epsilon-2)I^{\text{typ}}_{e;3}[3-\epsilon,-2+\epsilon,-1] - I^{\text{typ}}_{e;3}[1-\epsilon,-2+\epsilon,1]. \tag{4.5.31}$$

We can perform the same analysis on $I_{e;4}$ and we see the same result, that in the end the expression will evaluate to something of the form

$$R^d_1 = \frac{\langle q|de|q\rangle^2}{[e|d|q\rangle s^{1+2\epsilon}_{de}} F(\epsilon). \tag{4.5.32}$$

Recall that the full rational contribution from the $K = d$ embedded triangle is of the form

$$R^d_1 = \Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle da\rangle^3[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2\langle eq\rangle}(I_{e;3}+I_{e;4}), \tag{4.5.33}$$

so setting q as before and performing recursion we can be confident that is of the form

$$R^d_1 = \frac{\langle da\rangle^2[de]^3\langle ea\rangle}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle} F'(\epsilon), \tag{4.5.34}$$

again as expected.

While it would undoubtedly be preferable to be able to complete the computation and calculate the coefficient, we will later see that this is not necessarily an impediment.

4.5.2 Second Leading Order Term

The second term gives the following integral when we consider the triangle loop in the $K = e$ corner:

$$\begin{aligned}
R^e_2 =& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3\langle eq\rangle[d\beta]}{\langle dq\rangle\langle\alpha q\rangle^2\langle\beta q\rangle}\frac{\langle q\alpha\rangle\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2[\alpha a][c\beta][\beta e]} \\
&\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon} + {}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right) \\
=& i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle eq\rangle\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^3}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \\
&\int d^Dl\frac{[e|l|q\rangle^3[d|\beta|q\rangle}{[a|\alpha|q\rangle[e|\beta|q\rangle[c|\beta|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} + \frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right).
\end{aligned} \tag{4.5.35}$$

We can simplify this integral using

$$\frac{[d|\beta|q\rangle}{[c|\beta|q\rangle[e|\beta|q\rangle} = \frac{[cd]}{[ce][c|\beta|q\rangle} + \frac{[de]}{[ce][e|\beta|q\rangle} \tag{4.5.36}$$

so now we have

$$\begin{aligned} R_2^e =& i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle eq\rangle\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^3}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \\ & \int d^Dl\frac{[e|l|q\rangle^3}{[a|\alpha|q\rangle}\left(\frac{[cd]}{[ce][c|\beta|q\rangle}+\frac{[de]}{[ce][e|\beta|q\rangle}\right)\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}+\frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right). \end{aligned} \tag{4.5.37}$$

We shall therefore consider the following integral

$$\begin{aligned} I_x =& \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int d^Dl\frac{[e|l|q\rangle^3}{[a|\alpha|q\rangle[x|\beta|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}+\frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right), \\ =& I_{x;1}+I_{x;2}, \end{aligned} \tag{4.5.38}$$

where x is generic.

Starting with the first integral we use the A trick to raise the two $[x|y|z\rangle$ terms to the numerator and then Feynman parameterise:

$$\begin{aligned} I_{x;1} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\frac{\Gamma(2+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \\ & \sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\int[du]\ u_1^{-1+\epsilon-k}u_2^{-2\epsilon+k}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ & \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b. \end{aligned} \tag{4.5.39}$$

Resumming the k sum in the same way as Eq 4.5.13,

$$\begin{aligned}I_{x;1} =&i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\ &\int[du]\ u_1^{-1+\epsilon}u_2^{-2\epsilon}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \\ &\left(\frac{u_1^2}{(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}}+\frac{u_1^4}{u_2^2(u_1+u_2)^2}\right. \\ &\left.-\frac{u_1^{4-\epsilon}}{u_2^2(u_1+u_2)^{2-\epsilon}}+2\frac{u_1^3}{u_2(u_1+u_2)^2}+(\epsilon-2)\frac{u_1^{3-\epsilon}}{u_2(u_1+u_2)^{2-\epsilon}}-\frac{u_1^{1-\epsilon}u_2}{(u_1+u_2)^{2-\epsilon}}\right).\end{aligned} \tag{4.5.40}$$

Again this gives five integrals with varying powers of u_1, u_2 and $1-u_3$ which we will solve simultaneously by considering the "typical" integral

$$\begin{aligned}I_{x;1}^{\text{typ}} =&i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\ &\int[du]\ u_1^{-1+\epsilon+d_1}u_2^{-2\epsilon+d_2}(1-u_3)^{d_3}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b.\end{aligned} \tag{4.5.41}$$

Note that in each of the cases above the values of d_1, d_2 and d_3 sum to zero. We shall not use this fact until the end in order to keep our results as generic as possible.

We start by binomially expanding the a bracket,

$$\begin{aligned}I_{x;1}^{\text{typ}} =&i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\ &\sum_{c=0}^{a}\binom{a}{c}\left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c} \\ &\int[du]\ u_1^{-1+\epsilon+d_1}u_2^{-2\epsilon+d_2}(1-u_3)^{d_3}u_3^{1-2\epsilon+a-c}(1-u_2)^c\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b,\end{aligned} \tag{4.5.42}$$

then in turn we binomially expand $(1-u_2)^c$,

$$
\begin{aligned}
I_{x;1}^{\rm typ} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\sum_{c=0}^{a}\binom{a}{c}\left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c}\sum_{j=0}^{c}\binom{c}{j}(-1)^j \\
&\int[du]\ u_1^{-1+\epsilon+d_1}u_2^{-2\epsilon+d_2+j}(1-u_3)^{d_3}u_3^{1-2\epsilon+a-c}\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b,
\end{aligned}
\tag{4.5.43}
$$

and use the substitution $u_2=v(1-u_3)$ to rewrite the above

$$
\begin{aligned}
I_{x;1}^{\rm typ} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\sum_{c=0}^{a}\binom{a}{c}\left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c}\sum_{j=0}^{c}\binom{c}{j}(-1)^j \\
&\int du_3\ u_3^{1-2\epsilon+a-c}(1-u_3)^{-\epsilon+d_{123}+j+b}\int dv\ v^{-2\epsilon+d_2+j}(1-v)^{-1+\epsilon+d_1}\left(1+v\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \\
=& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\sum_{c=0}^{a}\binom{a}{c}\left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c}\sum_{j=0}^{c}\binom{c}{j}(-1)^j\frac{\Gamma(2-2\epsilon+a-c)\Gamma(1-\epsilon+d_{123}+j)}{\Gamma(3-3\epsilon+d_{123}+a-c+j)} \\
&\frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)} \\
&\frac{(1-\epsilon+d_{123}+j,b)}{(3-3\epsilon+d_{123}+a-c+j,b)}{}_2F_1[-b,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}], \\
=& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{c=0}^{a}\binom{a}{c}\left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c} \\
&\sum_{j=0}^{c}\binom{c}{j}(-1)^j\frac{\Gamma(2-2\epsilon+a-c)\Gamma(1-\epsilon+d_{123}+j)}{\Gamma(3-3\epsilon+d_{123}+a-c+j)}\frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)}I_b.
\end{aligned}
\tag{4.5.44}
$$

Now we take out and solve I_b,

$$
\begin{aligned}
I_b =& \sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\frac{(1-\epsilon+d_{123}+j,b)}{(3-3\epsilon+d_{123}+a-c+j,b)} \\
&{}_2F_1[-b,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}],
\end{aligned}
\tag{4.5.45}
$$

We would like to proceed as we did in the case of the single-minus one-loop current in the previous chapter, however the presence of the d_3 in the first argument of the hypergeometric presents an impediment. To remedy this we shall introduce a differential operator $\mathcal{D}_y^b$, which is defined to act on Pochhammer symbols as $\mathcal{D}_y^b\left[(a,n)y^n\right] = (a+b,n)y^n$ and $\mathcal{D}_y^b\left[\frac{y^n}{(a+b,n)}\right] = \frac{y^n}{(a,n)}$ which can then be pulled out of the sum. This operator allows us to increase the argument of a Pochhammer in the numerator, or decrease the argument of the Pochhammer in the denominator of a sum. Explicitly, if we have a function $F(y)$ with a Pochhammer in the numerator with argument a that we want to increase by b, then

$$\mathcal{D}_y^b\left[F(y)\right] = \frac{y^{1-a}}{(a,b)}\frac{d^b}{dy^b}\left(y^{a+b-1}F(y)\right). \tag{4.5.46}$$

With this, we set $X = -\frac{[x|e|q\rangle}{[x|d|q\rangle}$ and use the derivative operator to rewrite

$$\begin{aligned} I_b =& \mathcal{D}_X^{d_3}\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\frac{(1-\epsilon+d_{123}+j,b)}{(3-3\epsilon+d_{123}+a-c+j,b)} \\ & {}_2F_1[-b,1-2\epsilon+d_2+j,1-\epsilon+d_{123}+j,X], \end{aligned} \tag{4.5.47}$$

which is now in the desired form. We do a half-Gauss transformation on the hypergeometric,

$$\begin{aligned} I_b =& \mathcal{D}_X^{d_3}(1-X)^{-(1-2\epsilon+d_2+j)}\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\frac{(1-\epsilon+d_{123}+j,b)}{(3-3\epsilon+d_{123}+a-c+j,b)} \\ & {}_2F_1[1-\epsilon+d_{123}+j+b,1-2\epsilon+d_2+j,1-\epsilon+d_{123}+j,\frac{X}{X-1}], \\ =& \mathcal{D}_X^{d_3}(1-X)^{-(1-2\epsilon+d_2+j)}\sum_r \\ & F_2[1-\epsilon+d_{123}+j;1+r,1-2\epsilon+d_2+j;3-3\epsilon+d_{123}+a-c+j,1-\epsilon+d_{123}+j;-\frac{[x|d|q\rangle}{B},\frac{X}{X-1}] \\ =& \mathcal{D}_X^{d_3}\sum_r \\ & F_1[1+r,\epsilon+d_{13},1-2\epsilon+d_2+j,3-3\epsilon+d_{123}+a-c+j,-\frac{[x|d|q\rangle}{B},-\frac{[x|d|q\rangle}{B}(1-X)] \end{aligned} \tag{4.5.48}$$

where we used the identity

$$F_2[\alpha;\beta,\beta';\gamma,\alpha;x,y] = (1-y)^{-\beta'}F_1[\beta,\alpha-\beta',\beta',\gamma,x,\frac{x}{1-y}], \tag{4.5.49}$$

in the last line. From here we can use our derived result

$$\sum_r \frac{(a,r)}{r!} F_1[a+r,b,c,d,x,y] = (-x)^{-a} \frac{\Gamma(d)\Gamma(b+c-a)}{\Gamma(b+c)\Gamma(d-a)} {}_2F_1[a,c,b+c,1-\frac{y}{x}], \tag{4.5.50}$$

to write I_b as

$$\begin{aligned} I_b =& \frac{B}{[x|d|q\rangle} \frac{\Gamma(3-3\epsilon+d_{123}+a-c+j)}{\Gamma(2-3\epsilon+d_{123}+a-c+j)} \frac{\Gamma(-\epsilon+d_{123}+j)}{\Gamma(1-\epsilon+d_{123}+j)} \\ & \mathcal{D}_X^{d_3} {}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{123}+j,X] \\ =& \frac{B}{[x|d|q\rangle} \frac{\Gamma(3-3\epsilon+d_{123}+a-c+j)}{\Gamma(2-3\epsilon+d_{123}+a-c+j)} \frac{\Gamma(-\epsilon+d_{123}+j)}{\Gamma(1-\epsilon+d_{123}+j)} \\ & {}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,X], \end{aligned} \tag{4.5.51}$$

where in the last line we applied the derivative operator on the hypergeometric to get rid of it. Now we can reset $X = -\frac{[x|e|q\rangle}{[x|d|q\rangle}$ and substitute this back into the full integral,

$$\begin{aligned} I^{\text{typ}}_{x;1} =& i \frac{[e|d|q\rangle^3}{A[x|d|q\rangle s_{de}^{2+2\epsilon}} \frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[a|e|q\rangle}{A}\right)^a \\ & \sum_{c=0}^{a} \binom{a}{c} \left(\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{a-c} \sum_{j=0}^{c} \binom{c}{j} (-1)^j \frac{\Gamma(2-2\epsilon+a-c)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+a-c+j)} \\ & \frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)} {}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}]. \end{aligned} \tag{4.5.52}$$

In the previous chapter, Eq 3.3.40, for the A trick we did the change of variables

$$\sum_{a_1=0}^{\infty} \sum_{a=0}^{a_1} = \sum_{a=0}^{\infty} \sum_{r=a_1-a=0}^{\infty} .$$

Applying this same above, we would now have four sums,

$$\sum_{a=0}^{\infty} \sum_{r=0}^{\infty} \sum_{c=0}^{a} \sum_{j=0}^{c} \tag{4.5.53}$$

Doing the same change on a and c sums we get

$$\sum_{c=0}^{\infty} \sum_{r=0}^{\infty} \sum_{q=a-c=0}^{\infty} \sum_{j=0}^{c} \tag{4.5.54}$$

and finally

$$\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\sum_{t=c-j=0}^{\infty} \tag{4.5.55}$$

where

$$a = q + j + t \tag{4.5.56}$$

$$c = j + t. \tag{4.5.57}$$

Rewriting the above integral in this new form,

$$\begin{aligned} I_{x;1}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{A[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{jqrt}\frac{(1,q+r+j+t)}{q!r!t!j!}\left(-\frac{[a|e|q\rangle}{A}\right)^{j+t} \\ & \left(-\frac{[a|d|q\rangle}{A}\right)^{q}(-1)^j\frac{\Gamma(2-2\epsilon+q)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+q+j)} \\ & \frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)}{}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}] \\ =& i\frac{[e|d|q\rangle^3}{A[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{j}\frac{(1,j)}{j!}\left(\frac{[a|e|q\rangle}{A}\right)^{j} \\ & \frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+j)}\frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)} \\ & {}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}]I_q, \end{aligned} \tag{4.5.58}$$

where we pull out the q,r,t sums. These terms are

$$I_q = \sum_{qrt}\frac{(1+j,q+r+t)}{q!r!t!}\frac{(2-2\epsilon,q)}{(2-3\epsilon+d_{123}+j,q)}\left(-\frac{[a|e|q\rangle}{A}\right)^{t}\left(-\frac{[a|d|q\rangle}{A}\right)^{q}. \tag{4.5.59}$$

This sum will come up consistently throughout these calculations so let us calculate this in a more generic form. We add in a regulator for the r sum,

$$\begin{aligned} \sum_{qrt}\frac{(a,q+r+t)}{q!r!t!}\frac{(b,q)}{(c,q)}X^tY^q =& \sum_t\frac{(a,t)}{t!}X^t\sum_{qr}\frac{(a+t,q+r)}{q!r!}\frac{(b,q)}{(c,q)}Y^q(1-\delta)^r \\ =& \sum_t\frac{(a,t)}{t!}X^tF_2[a+t,U,b,U,c,1-\delta,Y] \end{aligned} \tag{4.5.60}$$

where U signifies an arbitrary argument. Then we can use the identity,

$$F_2[\alpha,\beta,\beta',\beta,\gamma',x,y] = (1-x)^{-\alpha}{}_2F_1[\alpha,\beta',\gamma',\frac{y}{1-x}], \tag{4.5.61}$$

to rewrite this as

$$\sum_{qrt}\frac{(a,q+r+t)}{q!r!t!}\frac{(b,q)}{(c,q)}X^tY^q = \frac{1}{\delta^a}\sum_t\frac{(a,t)}{t!}\frac{X}{\delta}^t{}_2F_1[a+t,b,c,\frac{Y}{\delta}] \tag{4.5.62}$$

We can write the hypergeometric in the series representation, which lets us write this as an Appell F2 function and immediately use the same identity C.6.4 as before,

$$\begin{aligned}\sum_{qrt}\frac{(a,q+r+t)}{q!r!t!}\frac{(b,q)}{(c,q)}X^tY^q =& \frac{1}{\delta^a}F_2[a,U,b,U,c,\frac{X}{\delta},\frac{Y}{\delta}]\\ =&\frac{1}{(\delta-X)^a}{}_2F_1[a,b,c,\frac{Y}{\delta-X}],\end{aligned} \tag{4.5.63}$$

Finally we remove the regulator and we have the final result

$$\begin{aligned}\sum_{rt}\frac{(a,r+t)}{r!t!}X^t{}_2F_1[a+r+t,b,c,Y] = \sum_{qrt}&\frac{(a,q+r+t)}{q!r!t!}\frac{(b,q)}{(c,q)}X^tY^q\\ =&(-X)^{-a}{}_2F_1[a,b,c,-\frac{Y}{X}].\end{aligned} \tag{4.5.64}$$

Then we can immediately evaluate

$$I_q = \left(\frac{A}{[a|e|q\rangle}\right)^{1+j}{}_2F_1[1+j,2-2\epsilon,2-3\epsilon+d_{123}+j,-\frac{[a|d|q\rangle}{[a|e|q\rangle}]. \tag{4.5.65}$$

Putting this back into the whole expression,

$$\begin{aligned}I^{\mathrm{typ}}_{x;1} =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_j\frac{(1,j)}{j!}\\ &\frac{\Gamma(1-2\epsilon+d_2+j)\Gamma(\epsilon+d_1)}{\Gamma(1-\epsilon+d_{12}+j)}{}_2F_1[1,1-2\epsilon+d_2+j,1-\epsilon+d_{12}+j,-\frac{[x|e|q\rangle}{[x|d|q\rangle}]\\ &\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+j)}{}_2F_1[1+j,2-2\epsilon,2-3\epsilon+d_{123}+j,-\frac{[a|d|q\rangle}{[a|e|q\rangle}].\end{aligned} \tag{4.5.66}$$

Now, we write the first hypergeometric function in the integral representation (C.2.6)

$$\begin{aligned}I_{x;1}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_j\frac{(1,j)}{j!}\\ &\int dt\; t^{-2\epsilon+d_2+j}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\\ &\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+j)}{}_2F_1[1+j,2-2\epsilon,2-3\epsilon+d_{123}+j,-\frac{[a|d|q\rangle}{[a|e|q\rangle}],\end{aligned}\tag{4.5.67}$$

and perform a full-Gauss transformation on the second

$$\begin{aligned}I_{x;1}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_j\left(1+\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-\epsilon+d_{123}-1}\sum_j\frac{(1,j)}{j!}\\ &\int dt\; t^{-2\epsilon+d_2+j}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123}+j)}{\Gamma(2-3\epsilon+d_{123}+j)}\\ &{}_2F_1[1-3\epsilon+d_{123},-\epsilon+d_{123}+j,2-3\epsilon+d_{123}+j,-\frac{[a|d|q\rangle}{[a|e|q\rangle}].\end{aligned}\tag{4.5.68}$$

Now we will perform the j sum,

$$\begin{aligned}I_{x;1}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\left(1+\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-\epsilon+d_{123}-1}\\ &\int dt\; t^{-2\epsilon+d_2}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})}\\ &\sum_j\frac{(1,j)}{j!}\frac{(-\epsilon+d_{123},j)}{(2-3\epsilon+d_{123},j)}{}_2F_1[1-3\epsilon+d_{123},-\epsilon+d_{123}+j,2-3\epsilon+d_{123}+j,-\frac{[a|d|q\rangle}{[a|e|q\rangle}]t^j,\\ =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\left(1+\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-\epsilon+d_{123}-1}\\ &\int dt\; t^{-2\epsilon+d_2}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})}\\ &F_1[-\epsilon+d_{123},1,1-3\epsilon+d_{123},2-3\epsilon+d_{123},t,-\frac{[a|d|q\rangle}{[a|e|q\rangle}].\end{aligned}\tag{4.5.69}$$

We can use the identity C.6.3 to reduce the Appell F1 function to a hypergeometric function,

$$\begin{aligned}I_{x;1}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{[a|P_{de}|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\\ &\int dt\; t^{-2\epsilon+d_2}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})}\\ &{}_2F_1[-\epsilon+d_{123},1,2-3\epsilon+d_{123},\frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}].\end{aligned}\tag{4.5.70}$$

Recall that $d_{123} = d_1 + d_2 + d_3 = 0$ in all cases, so

$$\begin{aligned} I^{\text{typ}}_{x;1} =& i\frac{[e|d|q\rangle^3}{[a|P_{de}|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)} \\ & \int dt\, t^{-2\epsilon+d_2}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)} \\ & {}_2F_1[-\epsilon,1,2-3\epsilon,\frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}]. \end{aligned} \tag{4.5.71}$$

Unfortunately we are unable to go further and complete this integral, however we can get to this point with the various contributions of the leading and subleading terms and we will make some comments at the end.

The second integral $I_{x;2}$

$$I_{x;2} = \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\frac{[e|l|q\rangle^3}{[a|\alpha|q\rangle[x|\beta|q\rangle}, \tag{4.5.72}$$

we start in the same way with the A trick and parameterisation

$$\begin{aligned} I_{x;2} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\frac{\Gamma(2+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \\ & \sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\int[du]u_1^{2+k}u_2^{-3-\epsilon-k}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ & \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \\ =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \\ & \sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\int[du]u_1^2u_2^{-3-\epsilon}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ & \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b\sum_{k=0}^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\frac{\Gamma(2+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\left(\frac{u_1}{u_2}\right)^k. \end{aligned} \tag{4.5.73}$$

This k sum is $S_k\left[\frac{u_1}{u_2}\right]$ as defined above (4.5.13), so the integral is

$$
\begin{aligned}
I_{x;2} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\int[du]u_1^2u_2^{-3-\epsilon}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \\
&\left(\frac{u_2^2}{(u_1+u_2)^2}+(\epsilon-2)\frac{u_2^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}}+\frac{u_2^4}{u_1^2(u_1+u_2)^2}\right. \\
&\left.-\frac{u_2^{4-\epsilon}}{u_1^2(u_1+u_2)^{2-\epsilon}}+2\frac{u_2^3}{u_1(u_1+u_2)^2}+(\epsilon-2)\frac{u_2^{3-\epsilon}}{u_1(u_1+u_2)^{2-\epsilon}}-\frac{u_2^{1-\epsilon}u_1}{(u_1+u_2)^{2-\epsilon}}\right),
\end{aligned}
\tag{4.5.74}
$$

and we define

$$
\begin{aligned}
I_{x;2}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\int[du]u_1^{2+d_1}u_2^{-3-\epsilon+d_2}u_3^{1-2\epsilon}(1-u_3)^{d_3}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b,
\end{aligned}
\tag{4.5.75}
$$

such that

$$
\begin{aligned}
I_{x;2} =& I_{x;2}^{\text{typ}}[0,2,-2]+(\epsilon-2)I_{x;2}^{\text{typ}}[0,2-\epsilon,-2+\epsilon]+I_{x;2}^{\text{typ}}[-2,4,-2]-I_{x;2}^{\text{typ}}[-2,4-\epsilon,-2+\epsilon] \\
&+2I_{x;2}^{\text{typ}}[-1,3,-2]+(\epsilon-2)I_{x;2}^{\text{typ}}[-1,3-\epsilon,-2+\epsilon]-I_{x;2}^{\text{typ}}[1,1-\epsilon,-2+\epsilon].
\end{aligned}
\tag{4.5.76}
$$

The method to solving this integral is very similar to the method used to solve $I_{x;1}^{\text{typ}}$ so we shall simply present the result:

$$
\begin{aligned}
I_{x;2}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{[a|P_{de}|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)} \\
&\int dt\, t^{-3-\epsilon+d_2}(1-t)^{2+d_1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1} \\
&{}_2F_1[1,-\epsilon,2-3\epsilon,\frac{t+\frac{[a|d|q\rangle}{[a|e|q\rangle}}{1+\frac{[a|d|q\rangle}{[a|e|q\rangle}}].
\end{aligned}
\tag{4.5.77}
$$

Again, we are similarly unable to proceed from this point.

With both integrals now defined, we bring everything back together, and the total contribution from this term would be

$$R_2^e = i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle eq\rangle\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^3}\left(\frac{[cd]}{[ce]}\left(I_{x;1}^{x=c}+I_{x;2}^{x=c}\right)+\frac{[de]}{[ce]}\left(I_{x;1}^{x=e}+I_{x;2}^{x=e}\right)\right), \tag{4.5.78}$$

where the superscript on each expression denotes the value that x takes.

Next we consider the case when the triangle is embedded in the $K=d$ corner. In this case the integral is

$$\begin{aligned}
R_2^d =& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle q|dl|q\rangle^3}{\langle dq\rangle^2\langle\beta q\rangle^2}\frac{[e|\alpha|q\rangle}{\langle\alpha q\rangle^2\langle eq\rangle}\\
&\frac{\langle q\alpha\rangle\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle^2[\alpha a][c\beta][\beta e]}\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right)\\
=& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle\langle eq\rangle}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{[d|l|q\rangle^3[e|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle[e|\beta|q\rangle}\\
&\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right)\\
=& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle da\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle\langle eq\rangle}\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!}\\
&\int d^Dl\frac{[d|l|q\rangle^3[e|\alpha|q\rangle}{[a|\alpha|q\rangle[c|\beta|q\rangle[e|\beta|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}+\frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right)
\end{aligned} \tag{4.5.79}$$

We use

$$\frac{[d|\beta|q\rangle}{[c|\beta|q\rangle[e|\beta|q\rangle}=\frac{[cd]}{[ce][c|\beta|q\rangle}+\frac{[de]}{[ce][e|\beta|q\rangle}$$

to simplify the above integral so the integral we are trying to solve is

$$\begin{aligned}
I_y =&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!}\int d^Dl\frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}+\frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right)\\
=&I_{y;1}+I_{y;2}.
\end{aligned} \tag{4.5.80}$$

The first integral is

$$I_{y;1}=\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}\frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle} \tag{4.5.81}$$

We Feynman parameterise this

$$
\begin{aligned}
I_{y;1} =& i\frac{[d|e|q\rangle^2[e|d|q\rangle}{ABs_{de}^{2+2\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!}\frac{\Gamma(2+2\epsilon)}{\Gamma(3+k)\Gamma(\epsilon-k)}\int[du]u_1^{-1+\epsilon-k}u_2^{-2\epsilon}u_3^{1-2\epsilon+k}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\
&\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b.
\end{aligned}
\tag{4.5.82}
$$

The k sum is $\mathrm{S}\left[\frac{u_3}{u_1}\right]$, and we get

$$
\begin{aligned}
I_{y;1} =& i\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{[d|e|q\rangle^2[e|d|q\rangle}{ABs_{de}^{2+2\epsilon}}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\int[du]u_1^{-1+\epsilon}u_2^{-2\epsilon}u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \\
&\left(\frac{u_1^2}{(u_1+u_3)^2}+(\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_3)^{2-\epsilon}}+\frac{u_1^4}{u_3^2(u_1+u_3)^2}\right. \\
&\left.-\frac{u_1^{4-\epsilon}}{u_3^2(u_1+u_3)^{2-\epsilon}}+2\frac{u_1^3}{u_3(u_1+u_3)^2}+(\epsilon-2)\frac{u_1^{3-\epsilon}}{u_3(u_1+u_3)^{2-\epsilon}}-\frac{u_1^{1-\epsilon}u_3}{(u_1+u_3)^{2-\epsilon}}\right).
\end{aligned}
\tag{4.5.83}
$$

This gives the typical integral

$$
\begin{aligned}
I_{y;1}^{\text{typ}} =& i\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{[d|e|q\rangle^2[e|d|q\rangle}{ABs_{de}^{2+2\epsilon}}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\int[du]u_1^{-1+\epsilon+e_1}u_2^{-2\epsilon}u_3^{1-2\epsilon+e_3}(1-u_2)^{e_2}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b,
\end{aligned}
\tag{4.5.84}
$$

where this time we label the powers of u_1, u_3 as e_1, e_3 and powers of $(1-u_2)$ as e_2 to avoid confusion.

The method to calculate this integral is very similar to the I_x cases, so let us simply state a few key differences. In the I_x^{typ} cases we had a factor of $(1-u_3)^{d_3}$ in the Feynman integrand and thus in order that the sums be amenable to the reordering (4.5.54), we must expand the $(\cdots)^b$ bracket that results from the A trick . In the case of the I_y^{typ} integrals, we instead have a factor of $(1-u_2)^{e_2}$ in the integrand and so for the same reasons, we need to instead expand the $(\cdots)^a$ bracket instead.

The remainder of the method is the same so let us proceed to the last step. Once we set

$e_{123} = 0$ we have

$$
\begin{aligned}
I_{y;1}^{\text{typ}} =& i\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)} \frac{[d|e|q\rangle^2[e|d|q\rangle}{[a|e|q\rangle[x|P_{de}|q\rangle s_{de}^{2+2\epsilon}} \frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-3\epsilon)} \\
& \int dt\ t^{1-2\epsilon+e_3}(1-t)^{-1+\epsilon+e_1} \left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} \\
& {}_2F_1[1-\epsilon, 1, 2-3\epsilon, t+\frac{[x|e|q\rangle}{[x|P_{de}|q\rangle}].
\end{aligned} \tag{4.5.85}
$$

and the second integral,

$$
I_{y;2} = \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \sum_{k=0}^{\infty} \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}} \frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}, \tag{4.5.86}
$$

can be solved with the same process, where we have

$$
\begin{aligned}
I_{y;2}^{\text{typ}} =& i\frac{[d|e|q\rangle^2[e|d|q\rangle}{[a|e|q\rangle[x|P_{de}|q\rangle s_{de}^{2+2\epsilon}} \frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)} \frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-3\epsilon+e_{123})} \\
& \int dt\ t^{-2-\epsilon+e_3}t^{2+e_1} \left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} {}_2F_1[1-\epsilon, 1, 2-3\epsilon, t+\frac{[x|e|q\rangle}{[x|P_{de}|q\rangle}].
\end{aligned} \tag{4.5.87}
$$

Note the similarities with the I_x^{typ} integrals.

4.5.3 Third Leading Order Term

The final leading order term with the triangle embedded in the $K = e$ corner is

$$
\begin{aligned}
R_3^e =& i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2} \frac{[e|l|q\rangle^3\langle eq\rangle[d|\beta|q\rangle}{\langle dq\rangle\langle\alpha q\rangle^2\langle\beta q\rangle^2} \frac{\langle\beta q\rangle\langle da\rangle^3[de][ea]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle[\alpha a][\alpha e][\beta e]} \\
& \left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right) \\
=& i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \frac{\langle eq\rangle\langle da\rangle^3[de][ea]}{\langle dq\rangle^2\langle ab\rangle\langle bc\rangle\langle cd\rangle} \sum_{k=0}^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \\
& \int d^D l \frac{[e|\alpha|q\rangle^2[d|\beta|q\rangle}{[a|\alpha|q\rangle[e|\beta|q\rangle} \left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} + \frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right).
\end{aligned} \tag{4.5.88}
$$

Thus we shall consider the following integral

$$
\begin{aligned}
I_k =& \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \sum_{k=0}^{\infty} \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} \frac{[d|\beta|q\rangle [e|\alpha|q\rangle^2}{[a|\alpha|q\rangle [e|\beta|q\rangle} \\
&+ \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \sum_{k=0}^{\infty} \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2} \frac{[d|\beta|q\rangle [e|\alpha|q\rangle^2}{[a|\alpha|q\rangle [e|\beta|q\rangle} \\
=& I_{k;1} + I_{k;2}.
\end{aligned}
\tag{4.5.89}
$$

Beginning with the first integral we see that this integral is simpler due to the presence of $[e|\beta|q\rangle$ which simplifies the Feynman parameterisation

$$
\begin{aligned}
I_{k;1} = i& \frac{[d|e|q\rangle [e|d|q\rangle^2}{s^{2+2\epsilon}AB} \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \sum_{k=0}^{\infty} \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!} \frac{\Gamma(2+2\epsilon)}{\Gamma(\epsilon-k)\Gamma(3+k)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[a|e|q\rangle}{A}\right)^a \\
&\sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!} \left(-\frac{[e|d|q\rangle}{B}\right)^b \int [du] u_1^{-1+\epsilon-k} u_2^{1-2\epsilon+k} u_3^{-2\epsilon} (1-u_3)^b \left((1-u_2) + u_3 \frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a .
\end{aligned}
\tag{4.5.90}
$$

The k sum gives $S_k[\frac{u_2}{u_1}]$ so the expression becomes

$$
\begin{aligned}
I_{k;1} = i& \frac{[d|e|q\rangle [e|d|q\rangle^2}{s^{2+2\epsilon}AB} \frac{\Gamma(-2+\epsilon)\Gamma(2+2\epsilon)}{\Gamma(2+\epsilon)\Gamma(\epsilon)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[a|e|q\rangle}{A}\right)^a \\
&\sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!} \left(-\frac{[e|d|q\rangle}{B}\right)^b \int [du] u_1^{-1+\epsilon} u_2^{1-2\epsilon} u_3^{-2\epsilon} (1-u_3)^b \left((1-u_2) + u_3 \frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\
&\left(\frac{u_1^2}{(u_1+u_2)^2} + (\epsilon-2)\frac{u_1^{2-\epsilon}}{(u_1+u_2)^{2-\epsilon}} + \frac{u_1^4}{u_2^2(u_1+u_2)^2} - \frac{u_1^{4-\epsilon}}{u_2^2(u_1+u_2)^{2-\epsilon}} \right. \\
&\left. + 2\frac{u_1^3}{u_2(u_1+u_2)^2} + (\epsilon-2)\frac{u_1^{3-\epsilon}}{u_2(u_1+u_2)^{2-\epsilon}} - \frac{u_1^{1-\epsilon}u_2}{(u_1+u_2)^{2-\epsilon}} \right).
\end{aligned}
\tag{4.5.91}
$$

and we define

$$
\begin{aligned}
I^{\text{typ}}_{k;1} = i& \frac{[d|e|q\rangle [e|d|q\rangle^2}{s^{2+2\epsilon}AB} \frac{\Gamma(-2+\epsilon)\Gamma(2+2\epsilon)}{\Gamma(2+\epsilon)\Gamma(\epsilon)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!} \left(-\frac{[a|e|q\rangle}{A}\right)^a \\
&\sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!} \left(-\frac{[e|d|q\rangle}{B}\right)^b \int [du] u_1^{-1+\epsilon+d_1} u_2^{1-2\epsilon+d_2} u_3^{-2\epsilon} (1-u_3)^{b+d_3} \left((1-u_2) + u_3 \frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a .
\end{aligned}
\tag{4.5.92}
$$

We can skip to the final result in this case as it is similar to the methods laid out above,

$$
\begin{aligned}
I_{k;1}^{\text{typ}} =& i\frac{[d|e|q\rangle[e|d|q\rangle}{s^{2+2\epsilon}[a|P_{de}|q\rangle}\frac{\Gamma(-2+\epsilon)\Gamma(2+2\epsilon)}{\Gamma(2+\epsilon)\Gamma(\epsilon)} \\
&\frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})}\int dt\ t^{1-2\epsilon+d_2}(1-t)^{-1+\epsilon+d_1} \\
&{}_2F_1[1-\epsilon,1,2-3\epsilon,\frac{t+\frac{[a|d|q\rangle}{[a|e|q\rangle}}{1+\frac{[a|d|q\rangle}{[a|e|q\rangle}}].
\end{aligned}
\tag{4.5.93}
$$

Similarly, $I_{k;2}$ is defined by the k sum $S_k[\frac{u_2}{u_1}]$ and

$$
\begin{aligned}
I_{k;2}^{\text{typ}} =& i\frac{\Gamma(1-\epsilon)\Gamma(2+2\epsilon)}{\Gamma(3-\epsilon)\Gamma(2+\epsilon)}\frac{[d|e|q\rangle[e|d|q\rangle}{s^{2+2\epsilon}[a|P_{de}|q\rangle}\frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})} \\
&\int dt\ t^{-2-\epsilon+d_2}(1-t)^{2+d_1} \\
&{}_2F_1[1-\epsilon,1,2-3\epsilon+,\frac{t+\frac{[a|d|q\rangle}{[a|e|q\rangle}}{1+\frac{[a|d|q\rangle}{[a|e|q\rangle}}],
\end{aligned}
\tag{4.5.94}
$$

In the case where the triangle is in the $K=d$ corner, the integral is

$$
\begin{aligned}
R_3^d =& i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle q|dl|q\rangle^3}{\langle dq\rangle^2\langle\beta q\rangle^2}\frac{[e|\alpha|q\rangle}{\langle\alpha q\rangle^2\langle eq\rangle} \\
&\frac{\langle\beta q\rangle\langle da\rangle^3[de][ea]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle[\alpha a][\alpha e][\beta e]}\left({}_2F_1[3-\epsilon,2,2-\epsilon,\frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon}+{}_2F_1[2,3-\epsilon,2-\epsilon,\frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right), \\
=& i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\langle da\rangle^3[de][ea]}{\langle eq\rangle\langle ab\rangle\langle bc\rangle\langle cd\rangle}\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!} \\
&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int d^Dl\frac{[d|l|q\rangle^3}{[e|\beta|q\rangle[a|\alpha|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}+\frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right)
\end{aligned}
\tag{4.5.95}
$$

where the last line defines

$$
\begin{aligned}
&\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)k!}\int d^Dl\frac{[d|l|q\rangle^3}{[e|\beta|q\rangle[a|\alpha|q\rangle}\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}}+\frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right) \\
=& I_{k;3}+I_{k;4}.
\end{aligned}
\tag{4.5.96}
$$

We can write $I_{k;3}$ as

$$\begin{aligned} I_{k;3} =& I^{\text{typ}}_{k;3}[2,-2,0] + (\epsilon-2) I^{\text{typ}}_{k;3}[2-\epsilon,-2+\epsilon,0] + I^{\text{typ}}_{k;3}[4,-2,-2] - I^{\text{typ}}_{k;3}[4-\epsilon,-2+\epsilon,-2] \\ &+ 2I^{\text{typ}}_{k;3}[3,-2,-1] + (\epsilon-2) I^{\text{typ}}_{k;3}[3-\epsilon,-2+\epsilon,-1] - I^{\text{typ}}_{k;3}[1-\epsilon,-2+\epsilon,1]. \end{aligned} \tag{4.5.97}$$

where the typical integral evaluates to

$$\begin{aligned} I^{\text{typ}}_{k;3} =& i\frac{[d|e|q\rangle^3}{[a|e|q\rangle [e|d|q\rangle s_{de}^{2+2\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)} \frac{\Gamma(2+2\epsilon)}{\Gamma(3-\epsilon)} \frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)} \\ & \int dt\ t^{-2\epsilon+e_3}(1-t)^{\epsilon+e_1-1}\left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} {}_2F_1[1,-\epsilon,2-3\epsilon,t]. \end{aligned} \tag{4.5.98}$$

and finally

$$\begin{aligned} I^{\text{typ}}_{k;4} =& i\frac{[d|e|q\rangle^3}{[a|e|q\rangle [e|d|q\rangle s_{de}^{2+2\epsilon}} \frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)} \frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)} \\ & \int dt\ t^{-3-\epsilon+e_3}(1-t)^{2+e_1}\left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} {}_2F_1[1,-\epsilon,2-3\epsilon,t] \end{aligned} \tag{4.5.99}$$

where

$$\begin{aligned} I_{k;4} =& I^{\text{typ}}_{k;4}[0,-2,2] + (\epsilon-2) I^{\text{typ}}_{k;4}[0,-2+\epsilon,2-\epsilon] + I^{\text{typ}}_{k;4}[-2,-2,4] \\ & - I^{\text{typ}}_{k;4}[-2,-2+\epsilon,4-\epsilon] + 2I^{\text{typ}}_{k;4}[-1,-2,3] + (\epsilon-2) I^{\text{typ}}_{k;4}[-1,-2+\epsilon,3-\epsilon] \\ & - I^{\text{typ}}_{k;4}[1,-2+\epsilon,1-\epsilon]. \end{aligned} \tag{4.5.100}$$

4.5.4 Subleading Terms

As stated earlier, we are aware that there will be further subleading contributions to from the double-box structure which cannot be accessed from the embedded triangle diagrams - terms that arise from the non-collinear pole terms of the 6 point current - nonetheless it is worth analysing those that we can access.

The subleading terms will return single poles in s_{de}. These terms are

$$\begin{aligned} & - i\frac{\langle\alpha\beta\rangle[bc]^2}{\langle bc\rangle[ab][c\beta][\alpha a]} - i\frac{\langle\alpha\beta\rangle[bc]^2[be]}{[ab][c\beta][\alpha a]\langle cd\rangle[de]} - i\frac{\langle aq\rangle\langle\alpha\beta\rangle[bc]^2[ce]}{\langle ab\rangle\langle dq\rangle[ab][\alpha a][c\beta][de]} + i\frac{\langle\alpha\beta\rangle\langle aq\rangle[bc]^2[ea]}{\langle bc\rangle\langle dq\rangle[ab][\alpha a][c\beta][de]} \\ =& i\frac{\langle\alpha\beta\rangle}{[c\beta][\alpha a]}\left(-\frac{[bc]^2}{\langle bc\rangle[ab]} - \frac{[bc]^2[be]}{[ab]\langle cd\rangle[de]} - \frac{\langle aq\rangle[bc]^2[ce]}{\langle ab\rangle\langle dq\rangle[ab][de]} + \frac{\langle aq\rangle[bc]^2[ea]}{\langle bc\rangle\langle dq\rangle[ab][de]}\right). \end{aligned} \tag{4.5.101}$$

First we note all terms have the same α and β dependence, so that in the end they can be integrated as one.

For the $K = e$ case the integral to be solved is

$$\begin{aligned}R_s^e = &- i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3\langle eq\rangle[d\beta]}{\langle dq\rangle\langle\alpha q\rangle^2\langle\beta q\rangle}\frac{\langle\alpha\beta\rangle}{[c\beta][\alpha a]} \\ &\left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right),\end{aligned} \tag{4.5.102}$$

where we have suppressed the terms in the bracket (4.5.101) which do not participate in the integration and can thus be added again at the end before performing BCFW recursion.

While in principle there is nothing wrong with having a tensor integral to solve, the fact that the $[x|y|q\rangle$ terms in the denominator which will be raised to the numerator via the A trick will appear in the numerator as $[x|y|q\rangle^a$. Contracting this term with $[e|\alpha\beta|e]$ as happens for tensor terms in Feynman integration will give rise to combinatorics which are best avoided if possible.

In order to rid ourselves of potential tensor contributions, we can use the following process to write the unwanted $\langle\alpha\beta\rangle$ as propagators:

$$\begin{aligned}\frac{\langle eq\rangle\langle\alpha\beta\rangle}{\langle\alpha q\rangle\langle\beta q\rangle} =&\frac{\langle\alpha e\rangle}{\langle\alpha q\rangle} - \frac{\langle\beta e\rangle}{\langle\beta q\rangle} = \frac{[e|\alpha|e\rangle}{[e|\alpha|q\rangle} - \frac{[e|\beta|e\rangle}{[e|\beta|q\rangle} \\ =&\frac{[e|\alpha|e\rangle}{[e|\alpha|q\rangle} + \frac{[e|\alpha|e\rangle}{[e|\beta|q\rangle} - \frac{s_{de}}{[e|\beta|q\rangle} \\ =&(\alpha^2 - l^2)\frac{[e|d|q\rangle}{[e|\alpha|q\rangle[e|\beta|q\rangle} - \frac{s_{de}}{[e|\beta|q\rangle}.\end{aligned} \tag{4.5.103}$$

Thus we can write the integral as

$$\begin{aligned}R_s^e =&i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int \frac{d^D l}{l^2\alpha^{2(2+\epsilon)}\beta^2}\frac{[e|l|q\rangle^3[d|\beta|q\rangle}{\langle dq\rangle[c|\beta|q\rangle[e|\beta|q\rangle[a|\alpha|q\rangle} \\ &\left((\alpha^2 - l^2)\frac{[e|d|q\rangle}{[e|\alpha|q\rangle} - s_{de}\right)\left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{\alpha^2}{l^2}]\left(\frac{\alpha^2}{l^2}\right)^2\right).\end{aligned} \tag{4.5.104}$$

One might be concerned that since the α^2 and l^2 above are 4 dimensional propagators by nature of their having been sourced from 4D spinor helicity, while those in the integral are D dimensional, that by cancelling one with the other we would be introducing an $\mathcal{O}(\epsilon)$ error. In this case however we are safe. Let us illustrate this by expressing the D dimensional momentum as a 4D piece and a non-integer piece, so

$$\alpha^{[D]} = \alpha^{[4]} + \alpha^{[-2\epsilon]}, \tag{4.5.105}$$

$$l^{[D]} = l^{[4]} + l^{[-2\epsilon]}. \tag{4.5.106}$$

Then remembering that $\alpha + l = e$ where e is an external - hence 4D - momentum, we see that

$$l^{[-2\epsilon]} = -\alpha^{[-2\epsilon]},$$

so finally

$$\begin{aligned}(\alpha^{[4]})^2 - (l^{[4]})^2 &= \left(\alpha^{[D]} - \alpha^{[-2\epsilon]}\right)^2 - \left(l^{[D]} + \alpha^{[-2\epsilon]}\right)^2 = \left(\alpha^{[D]}\right)^2 - \left(l^{[D]}\right)^2 + \left(\alpha^{[-2\epsilon]}\right)^2 - \left(\alpha^{[-2\epsilon]}\right)^2 \\ &= \left(\alpha^{[D]}\right)^2 - \left(l^{[D]}\right)^2,\end{aligned} \tag{4.5.107}$$

so no error is introduced.

We use

$$\frac{[d|\beta|q\rangle}{[c|\beta|q\rangle[e|\beta|q\rangle} = \frac{[cd]}{[ce][c|\beta|q\rangle} + \frac{[de]}{[ce][e|\beta|q\rangle}$$

to further break down the integral into more manageable pieces,

$$\begin{aligned}&i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_k^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int d^D l \frac{[e|l|q\rangle^3}{\langle dq\rangle[a|\alpha|q\rangle} \\ &\left(\frac{[cd]}{[ce][c|\beta|q\rangle} + \frac{[de]}{[ce][e|\beta|q\rangle}\right)\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} + \frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right)\left((\alpha^2 - l^2)\frac{[e|d|q\rangle}{[e|\alpha|q\rangle} - s_{de}\right).\end{aligned} \tag{4.5.108}$$

Considering first the s_{de} term in the final bracket, this integral is

$$\begin{aligned}&-\frac{s_{de}}{\langle dq\rangle}i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_k^{\infty}\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int d^D l \frac{[e|l|q\rangle^3}{[a|\alpha|q\rangle} \\ &\left(\frac{[cd]}{[ce][c|\beta|q\rangle} + \frac{[de]}{[ce][e|\beta|q\rangle}\right)\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} + \frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right).\end{aligned} \tag{4.5.109}$$

This integral is (up to prefactors) precisely I_x (4.5.38) which we have already analysed. For the remaining terms, we have integrals of the following forms to tackle:

$$\begin{aligned}
J_1 =& \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(-1+\epsilon-k)}\alpha^{2(3+k)}\beta^2} \frac{[e|l|q\rangle^2 [e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle},\\
J_2 =& \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(\epsilon-k)}\alpha^{2(2+k)}\beta^2} \frac{[e|l|q\rangle^2 [e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle},\\
J_3 =& \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(2+k)}\alpha^{2(\epsilon-k)}\beta^2} \frac{[e|l|q\rangle^2 [e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}\\
J_4 =& \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(3+k)}\alpha^{2(-1-\epsilon-k)}\beta^2} \frac{[e|l|q\rangle^2 [e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}.
\end{aligned} \tag{4.5.110}$$

If we consider the integral

$$I_{s;1} = \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(\epsilon-k-d_d)}\alpha^{2(2+k+d_d)}\beta^2} \frac{[e|l|q\rangle^2 [e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle} \tag{4.5.111}$$

we can do the first two integrals simultaneously as $d_d = 1$ gives J_1 and $d_d = 0$ gives J_2.

Feynman parameterising this gives

$$\begin{aligned}
I_{s;1} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{1+2\epsilon}} \sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d-k)\Gamma(2+d_d+k)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\\
&\sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \int [du] u_1^{-1+\epsilon-d_d-k} u_2^{k+d_d-2\epsilon} u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\\
&\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b\\
=& i\frac{[e|d|q\rangle^3}{ABs_{de}^{1+2\epsilon}} \frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b\\
&\int [du] u_1^{-1+\epsilon-d_d} u_2^{d_d-2\epsilon} u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b\\
&\sum_k^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \frac{(1-\epsilon+d_d,k)}{(2+d_d,k)}\left(-\frac{u_2}{u_1}\right)^k.
\end{aligned} \tag{4.5.112}$$

If we take $d_d = 1$, then the sum is

$$
\begin{aligned}
&{}_2F_1[3-\epsilon,2,3,-\frac{u_2}{u_1}] \\
&= \frac{1}{(\epsilon-1)(\epsilon-2)}\big(2u_1^2(1-u_3)^{-2}+2u_1^4u_2^{-2}(1-u_3)^{-2}+4u_1^3u_2^{-1}(1-u_3)^{-2} \\
&- u_1^{4-\epsilon}u_2^{-2}(1-u_3)^{-2+\epsilon}+2(\epsilon-2)u_1^{3-\epsilon}u_2^{-1}(1-u_3)^{-2+\epsilon}\big)
\end{aligned}
\tag{4.5.113}
$$

while for $d_d = 0$ we get

$$
{}_2F_1[1-\epsilon,3-\epsilon,2-\epsilon,-\frac{u_2}{u_1}] = \left(1+\frac{u_2}{u_1}\right)^{-2+\epsilon} {}_2F_1[1,-1,2-\epsilon,-\frac{u_2}{u_1}] = \left(1+\frac{u_2}{u_1}\right)^{-2+\epsilon}\left(1+\frac{u_2}{u_1(\epsilon-2)}\right)
\tag{4.5.114}
$$

In either case we can continue with the typical integral

$$
\begin{aligned}
I_{s;1}^{\mathrm{typ}} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\
&\int[du]u_1^{-1+\epsilon-d_d+d_1}u_2^{d_d-2\epsilon+d_2}u_3^{1-2\epsilon}(1-u_3)^{d_3}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a\left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b,
\end{aligned}
\tag{4.5.115}
$$

which evaluates to

$$
\begin{aligned}
I_{s;1}^{\mathrm{typ}}[d_1,d_2,d_3] =& i\frac{[e|d|q\rangle^3}{[a|e|q\rangle[x|d|q\rangle s_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)}\left(1+\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} \\
&\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)}\int dt\; t^{-2\epsilon+d_d+d_2}(1-t)^{\epsilon-d_d+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1} \\
&{}_2F_1[-\epsilon,1,2-3\epsilon,\frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}].
\end{aligned}
\tag{4.5.116}
$$

Thus the integral corresponding to $d_d = 1$ is

$$
\begin{aligned}
J_1 = I_{s;1}^{\mathrm{d_d=1}} =& \frac{1}{(1-\epsilon)(2-\epsilon)}\Big(2I_{s;1}^{\mathrm{typ,d_d=1}}[2,0,-2]+2I_{s;1}^{\mathrm{typ,d_d=1}}[4,-2,-2]+4I_{s;1}^{\mathrm{typ,d_d=1}}[3,-2,-1] \\
&- I_{s;1}^{\mathrm{typ,d_d=1}}[4-\epsilon,-2+\epsilon,-2]-2(2-\epsilon)I_{s;1}^{\mathrm{typ,d_d=1}}[3-\epsilon,-1,-2+\epsilon]\Big),
\end{aligned}
\tag{4.5.117}
$$

and for $d_d = 0$

$$
J_2 = I_{s;1}^{\mathrm{d_d=0}} = I_{s;1}^{\mathrm{typ,d_d=0}}[2-\epsilon,0,-2+\epsilon]-\frac{1}{2-\epsilon}I_{s;1}^{\mathrm{typ,d_d=0}}[1-\epsilon,1,-2+\epsilon].
\tag{4.5.118}
$$

The third and fourth integrals can be solved simultaneously with

$$I_{s;2} = \sum_{k}^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \int \frac{d^D l}{l^{2(2+k+d_d)}\alpha^{2(\epsilon-k-d_d)}\beta^2} \frac{[e|l|q\rangle^2[e|d|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}, \tag{4.5.119}$$

$d_d = 0$ giving J_3 and $d_d = 1$ giving J_4.

This time If we take $d_d = 1$, then the sum is

$$\begin{aligned} &{}_2F_1[3-\epsilon,2,3,-\frac{u_1}{u_2}] \\ &= \frac{1}{(\epsilon-1)(\epsilon-2)}\big(2u_2^2(1-u_3)^{-2} + 2u_2^4 u_1^{-2}(1-u_3)^{-2} + 4u_2^3 u_1^{-1}(1-u_3)^{-2} \\ &- u_2^{4-\epsilon}u_1^{-2}(1-u_3)^{-2+\epsilon} + 2(\epsilon-2)u_2^{3-\epsilon}u_1^{-1}(1-u_3)^{-2+\epsilon}\big) \end{aligned} \tag{4.5.120}$$

while for $d_d = 0$ we get

$${}_2F_1[1-\epsilon,3-\epsilon,2-\epsilon,-\frac{u_1}{u_2}] = \left(1+\frac{u_1}{u_2}\right)^{-2+\epsilon}\left(1+\frac{u_1}{u_2(\epsilon-2)}\right) \tag{4.5.121}$$

$$\begin{aligned} I_{s;2} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{1+2\epsilon}} \sum_{k}^{\infty} \frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!} \frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d-k)\Gamma(2+d_d+k)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \\ & \sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \int [du] u_1^{1+d_d+k} u_2^{-2-\epsilon-d_d-k} u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ & \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b \end{aligned} \tag{4.5.122}$$

In the same way as the first integral, we know that the typical integral we need to solve is of the form

$$\begin{aligned} I_{s;2}^{\text{typ}} =& i\frac{[e|d|q\rangle^3}{ABs_{de}^{1+2\epsilon}} \frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)} \sum_{a,a_1} \frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a \sum_{b,b_1} \frac{(1,b_1)}{b!(b_1-b)!}\left(-\frac{[x|d|q\rangle}{B}\right)^b \\ & \int [du] u_1^{1+d_d+d_1} u_2^{-2-\epsilon-d_d+d_2}(1-u_3)^{d_3} u_3^{1-2\epsilon}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b. \end{aligned} \tag{4.5.123}$$

Again we can get as far as

$$
\begin{aligned}
I_{s;2}^{\text{typ}}[d_1, d_2, d_3] =& i\frac{[e|d|q\rangle^3}{[a|P_{de}|q\rangle[x|d|q\rangle s_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)} \\
& \frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(2-3\epsilon+d_{123})}\int dt\ t^{-2-\epsilon-d_d+d_2}(1-t)^{1+d_d+d_1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1} \\
& {}_2F_1[-\epsilon+d_{123}, 1, 2-3\epsilon+d_{123}, \frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}]
\end{aligned}
\tag{4.5.124}
$$

Then for $d_d = 1$,

$$
\begin{aligned}
J_3 = I_{s;2}^{d_d=1} =& \frac{1}{(1-\epsilon)(2-\epsilon)}\Bigg(2I_{s;2}^{\text{typ}}[0,2,-2] + 2I_{s;2}^{\text{typ}}[-2,4,-2] + 4I_{s;2}^{\text{typ}}[-1,3,-2] \\
& - I_{s;2}^{\text{typ}}[-2, 4-\epsilon. -2+\epsilon] - 2(2-\epsilon)I_{s;2}^{\text{typ}}[-1, 3-\epsilon, -2+\epsilon]\Bigg),
\end{aligned}
\tag{4.5.125}
$$

and for $d_d = 0$,

$$
J_4 = I_{s;2}^{d_d=0} = I_{s;2}^{\text{typ}}[0, 2-\epsilon, -2+\epsilon] - \frac{1}{2-\epsilon}I_{s;2}^{\text{typ}}[1, 1-\epsilon, -2+\epsilon]. \tag{4.5.126}
$$

Now we can bring everything back together, the subleading integrals for the $K = e$ embedded triangle are

$$
\begin{aligned}
R_s^e = \frac{i}{\langle dq\rangle}\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\Bigg[& -s_{de}\left(\frac{[cd]}{[ce]}\left(I_{x;1}^{x=c} + I_{x;2}^{x=c}\right) + \frac{[de]}{[ce]}\left(I_{x;1}^{x=e} + I_{x;2}^{x=e}\right)\right) \\
& - \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}\left(\frac{[cd]}{[ce]}\left(J_4^{x=c} + J_2^{x=c}\right) + \frac{[de]}{[ce]}\left(J_4^{x=e} + J_2^{x=e}\right)\right) \\
& + \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}\left(\frac{[cd]}{[ce]}\left(J_1^{x=c} + J_3^{x=c}\right) + \frac{[de]}{[ce]}\left(J_1^{x=e} + J_3^{x=e}\right)\right)\Bigg].
\end{aligned}
\tag{4.5.127}
$$

For the $K = d$ triangle the integral is

$$
\begin{aligned}
R_s^d =& i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle dq\rangle[d|l|q\rangle^3}{\langle\beta q\rangle^2}\frac{[e\alpha]}{\langle\alpha q\rangle\langle eq\rangle}\frac{\langle\alpha\beta\rangle}{[c\beta][\alpha a]} \\
& \left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right).
\end{aligned}
\tag{4.5.128}
$$

Again, we would like to remove the tensor terms which we can do using

$$\begin{aligned}\langle\alpha\beta\rangle[\beta d] &= \frac{(\alpha^2 - l^2)}{[e\alpha]}[ed] - \delta\langle\alpha q\rangle[ed] \\ &= \langle\alpha q\rangle[de]\left(\frac{l^2}{[e|\alpha|q\rangle} + \frac{\beta^2}{[e|\beta|q\rangle}\right).\end{aligned} \tag{4.5.129}$$

Thus the integral becomes

$$\begin{aligned}R_s^d =& - i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{d^Dl}{l^2\alpha^2\beta^{2(2+\epsilon)}}\frac{\langle dq\rangle[de]}{\langle eq\rangle}\frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[c|\beta|q\rangle[a|\alpha|q\rangle} \\ &\left(\frac{l^2}{[e|\alpha|q\rangle} + \frac{\beta^2}{[e|\beta|q\rangle}\right)\left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\beta^2}]\left(\frac{l^2}{\beta^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{\beta^2}{l^2}]\left(\frac{\beta^2}{l^2}\right)^2\right) \\ =& - i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int d^Dl\frac{\langle dq\rangle[de]}{\langle eq\rangle}\frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[c|\beta|q\rangle[a|\alpha|q\rangle} \\ &\left(\frac{l^2}{[e|\alpha|q\rangle} + \frac{\beta^2}{[e|\beta|q\rangle}\right)\left(\frac{1}{l^{2(\epsilon-k)}\alpha^2\beta^{2(3+k)}} + \frac{1}{l^{2(3+k)}\alpha^2\beta^{2(\epsilon-k)}}\right).\end{aligned} \tag{4.5.130}$$

Let us expand this a little,

$$\frac{[d|l|q\rangle^2[e|\alpha|q\rangle}{[c|\beta|q\rangle[a|\alpha|q\rangle}\left(\frac{l^2}{[e|\alpha|q\rangle} + \frac{\beta^2}{[e|\beta|q\rangle}\right) = l^2\frac{[d|l|q\rangle^2}{[c|\beta|q\rangle[a|\alpha|q\rangle} + \beta^2\frac{[e|\alpha|q\rangle[d|l|q\rangle}{[a|\alpha|q\rangle}\left(\frac{[cd]}{[ce][c|\beta|q\rangle} + \frac{[de]}{[ce][e|\beta|q\rangle}\right), \tag{4.5.131}$$

thus we have four integrals:

$$\begin{aligned}J_5 =& \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(-1+\epsilon-k)}\alpha^2\beta^{2(3+k)}}\frac{[d|l|q\rangle^2}{[x|\beta|q\rangle[a|\alpha|q\rangle}, \\ J_6 =& \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(\epsilon-k)}\alpha^2\beta^{2(2+k)}}\frac{[e|\alpha|q\rangle[d|l|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}, \\ J_7 =& \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(2+k)}\alpha^2\beta^{2(\epsilon-k)}}\frac{[d|l|q\rangle^2}{[x|\beta|q\rangle[a|\alpha|q\rangle}, \\ J_8 =& \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(3+k)}\alpha^2\beta^{2(-1+\epsilon-k)}}\frac{[e|\alpha|q\rangle[d|l|q\rangle}{[a|\alpha|q\rangle[x|\beta|q\rangle}.\end{aligned} \tag{4.5.132}$$

We will once again combine the integrals as in the $K = e$ case, this time we will combine the first two integrals as

$$I_{s;3} = \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(\epsilon-k-d_d)}\alpha^2\beta^{2(2+k+d_d)}}\frac{[e|\alpha|q\rangle^{1-d_d}[d|l|q\rangle^{1+d_d}}{[a|\alpha|q\rangle[x|\beta|q\rangle}, \tag{4.5.133}$$

where $d_d = 1$ gives J_5 and $d_d = 0$ gives J_6. From this integral we get the typical integral

$$\begin{aligned} I_{s;3}^{\text{typ}} =& i\frac{[e|d|q\rangle^{1-d_d}[d|e|q\rangle^{1+d_d}}{ABs_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)}\sum_{a,a_1}\frac{(1,a_1)}{a!(a_1-a)!}\left(-\frac{[a|e|q\rangle}{A}\right)^a\sum_{b,b_1}\frac{(1,b_1)}{b!(b_1-b)!} \\ & \left(-\frac{[x|d|q\rangle}{B}\right)^b\int[du]u_1^{-1+\epsilon-d_d+e_1}u_2^{d_d-2\epsilon}u_3^{1-2\epsilon+e_3}(1-u_2)^{e_2}\left((1-u_2)+u_3\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^a \\ & \left((1-u_3)+u_2\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^b. \end{aligned} \tag{4.5.134}$$

We can skip to the last step,

$$\begin{aligned} I_{s;3}^{\text{typ}} =& i\frac{[e|d|q\rangle^{1-d_d}[d|e|q\rangle^{1+d_d}}{[a|e|q\rangle[x|P_{de}|q\rangle s_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(\epsilon-d_d)\Gamma(2+d_d)}\frac{\Gamma(1-2\epsilon+d_d)\Gamma(1-\epsilon-d_d)}{\Gamma(2-3\epsilon)} \\ & \int dt\ t^{1-2\epsilon+e_3}(1-t)^{\epsilon-d_d+e_1-1}\left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} \\ & {}_2F_1[1-\epsilon-d_d,1,2-3\epsilon,\frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}], \end{aligned} \tag{4.5.135}$$

and we see that for $d_d = 0$ we get the hypergeometric ${}_2F_1[1,1,2,x] + \mathcal{O}(\epsilon)$, while for $d_d = 1$ we get ${}_2F_1[-\epsilon,1,2-3\epsilon,x]$.

Thus

$$\begin{aligned} J_5 =& \frac{1}{(1-\epsilon)(2-\epsilon)}\bigg(2I_{s;3}^{\text{typ,d}_\text{d}=1}[2,-2,0]+2I_{s;3}^{\text{typ,d}_\text{d}=1}[4,-2,-2]+4I_{s;3}^{\text{typ,d}_\text{d}=1}[3,-1,-2] \\ & -I_{s;3}^{\text{typ,d}_\text{d}=1}[4-\epsilon,-2,-2+\epsilon]-2(2-\epsilon)I_{s;3}^{\text{typ,d}_\text{d}=1}[3-\epsilon,-2+\epsilon,-1]\bigg), \end{aligned} \tag{4.5.136}$$

and

$$J_6 = I_{s;3}^{d_d=0} = I_{s;3}^{\text{typ,d}_\text{d}=0}[0,2-\epsilon,-2+\epsilon]-\frac{1}{2-\epsilon}I_{s;3}^{\text{typ,d}_\text{d}=0}[1,1-\epsilon,-2+\epsilon]. \tag{4.5.137}$$

To get the final two integrals we can consider the integral

$$I_{s;4} = \sum_k^\infty\frac{(2,k)(3-\epsilon,k)}{(2-\epsilon,k)k!}\int\frac{d^Dl}{l^{2(2+k+d_d)}\alpha^2\beta^{2(\epsilon-k-d_d)}}\frac{[d|l|q\rangle^{2-d_d}[e|\alpha|q\rangle^{d_d}}{[x|\beta|q\rangle[a|\alpha|q\rangle}, \tag{4.5.138}$$

whence $d_d = 0$ gives J_7 and $d_d = 1$ gives J_8. This integral evaluates to

$$\begin{aligned} I^{\text{typ}}_{s;4} =& i\frac{[d|e|q\rangle^{2-d_d}[e|d|q\rangle^{d_d}}{[a|e|q\rangle[x|P_{de}|q\rangle s_{de}^{1+2\epsilon}}\frac{\Gamma(1+2\epsilon)}{\Gamma(2+d_d)\Gamma(\epsilon-d_d)}\frac{\Gamma(2-2\epsilon-d_d)\Gamma(-\epsilon+d_d)}{\Gamma(2-3\epsilon)} \\ & \int dt\, t^{-2-\epsilon+e_3}(1-t)^{1+d_d+e_1}\left(1+t\frac{[a|d|q\rangle}{[a|e|q\rangle}\right)^{-1} \\ & {}_2F_1[-\epsilon+d_d,1,2-3\epsilon,\frac{t[x|d|q\rangle+[x|e|q\rangle}{[x|P_{de}|q\rangle}]. \end{aligned} \tag{4.5.139}$$

and in this case $d_d = 0$ gives ${}_2F_1[1-\epsilon,1,2-3\epsilon],x]$ and $d_d = 1$ gives ${}_2F_1[-\epsilon,1,2-3\epsilon,x]$ for the final hypergeometric function. Then

$$J_7 = I^{\text{typ;d}_\text{d}=0}_{s;4}[0,-2+\epsilon,2-\epsilon] + \frac{1}{(\epsilon-2)}I^{\text{typ;d}_\text{d}=0}_{s;4}[1,-2+\epsilon,1-\epsilon], \tag{4.5.140}$$

and

$$\begin{aligned} J_8 =& \frac{1}{(1-\epsilon)(2-\epsilon)}\Big(2I^{\text{typ,d}_\text{d}=1}_{s;4}[2,-2,0] + 2I^{\text{typ,d}_\text{d}=1}_{s;4}[4,-2,-2] + 4I^{\text{typ,d}_\text{d}=1}_{s;4}[3,-1,-2] \\ & - I^{\text{typ,d}_\text{d}=1}_{s;4}[4-\epsilon,-2,-2+\epsilon] - 2(2-\epsilon)I^{\text{typ,d}_\text{d}=1}_{s;4}[3-\epsilon,-2+\epsilon,-1]\Big). \end{aligned} \tag{4.5.141}$$

Putting the pieces back together,

$$\begin{aligned} R^d_s =& -i\Gamma(1+\epsilon)\beta(1-\epsilon,3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle dq\rangle[de]}{\langle eq\rangle} \\ & \left(J_5^{x=c} + J_7^{x=c} + \frac{[cd]}{[ce]}(J_6^{x=c} + J_8^{x=c}) + \frac{[de]}{[ce]}(J_6^{x=e} + J_8^{x=e})\right). \end{aligned} \tag{4.5.142}$$

To get the rational part, we still need to add in the prefactor from the subleading terms (4.5.101) and perform recursion. After we set $\lambda_q = \lambda_a$, both R^d_s and R^e_s are unchanged so we need only perform recursion on the prefactor, however as we are unable to solve the integrals currently, we cannot provide a final expression with any accuracy.

4.5.5 Summary

Let us comment on the integrals we calculated above in both the leading and subleading cases. In the case of the first leading order term in the current analysed in Section 4.5.1, we were able to fully determine the rational contribution in the $K = e$ triangle case, and in the $K = d$ case we were unable to fully determine the coefficient but we were able to demonstrate that the result would have the correct kinematic structure.

In the remaining two leading order terms and in the subleading terms we notice that quite unexpectedly all of the typical integrals can be brought to a point where they can be expressed as an integral over one of two hypergeometric functions,

$$ {}_2F_1[1,-\epsilon,2-3\epsilon,x(t)], \quad \text{and} \quad {}_2F_1[1-\epsilon+e_{123},1,2-3\epsilon+e_{123},y(t)], $$

where the functions $x(t), y(t)$ are functions of the integration variable t and some spinor terms. These functions appear in two leading order terms and the subleading terms, and in both the $K = e$ and $K = d$ embedded triangles. Since all three terms used different methods to divide up the integral it seems unlikely to be an artefact of the method.

A look at table 4.1 shows us that the case of the all-plus triangles with the googly current receives contributions from poles of two different 6pt NMHV currents and so it may be that these hypergeometric functions are remnants of this fact, one from each pole.

We were unable to definitively solve or extract the spinor content of these integrals despite our best efforts. While there is no clear evidence that these terms might give unexpected kinematics such as spurious poles, there is also no definitive proof to the contrary. We can note that, taking $I^{\text{typ}}_{x;1}$ as example,

$$ \begin{aligned} I^{\text{typ}}_{x;1} =& i\frac{[e|d|q\rangle^3}{[a|P_{de}|q\rangle[x|d|q\rangle s_{de}^{2+2\epsilon}}\frac{\Gamma(2+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)} \\ & \int dt\, t^{-2\epsilon+d_2}(1-t)^{\epsilon+d_1-1}\left(1+t\frac{[x|e|q\rangle}{[x|d|q\rangle}\right)^{-1}\frac{\Gamma(2-2\epsilon)\Gamma(-\epsilon)}{\Gamma(2-3\epsilon)} \\ & {}_2F_1[-\epsilon,1,2-3\epsilon,\frac{t[a|e|q\rangle+[a|d|q\rangle}{[a|P_{de}|q\rangle}], \end{aligned} \tag{4.5.143} $$

that when we set $\lambda_q = \lambda_a$ that none of the spinors in this expression - with the exception of the pole - will feature in the recursion and so these expressions are also the final rational contribution post recursion.

4.6 Integrating the MHV current

The MHV current features with both the all-plus and single-minus embedded triangles.

4.6.1 All-Plus Triangle contribution

As shown in (4.19), contributions also arise in the case where there is an all-plus embedded triangle and a tree level MHV current. Looking at the left hand side, we have

$$A_3^{(1)}(K^+, P^+, Q^+)A_3^{(0)}(-P^-, R^-, j^+),$$

and writing out the tree amplitude

$$A_3^{(0)}(-P^-, R^-, j^+) = i\frac{\langle PR\rangle[jq]^2}{[Pq][Rq]} \tag{4.6.1}$$

we note that since we have set $\tilde{\lambda}_q = \tilde{\lambda}_e$, this amplitude vanishes when $j = e$, ie when $K = d$. Thus in this case we only need to consider the $K = e$ case. This looks like figure 4.22.

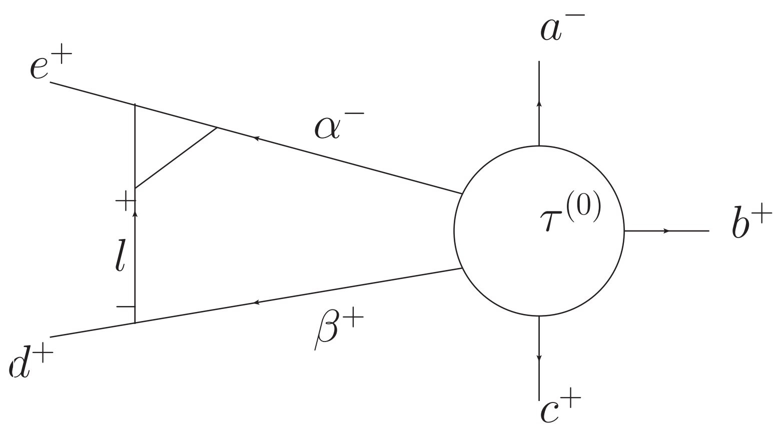


Figure 4.22: This figure depicts the contribution with the all-plus triangle embedded on the left, and a tree level adjacent MHV current on the right hand side.

which gives the integral

$$\int \frac{d^Dl}{l^2\alpha^2\beta^2}A_3^{(1)}(e^+, -\alpha^+, -l^+)A_3^{(0)}(l^-, -\beta^-, d^+)\tau^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+). \tag{4.6.2}$$

We have already stated the current in equation 4.2.2, however let us restate it in the case where $\tilde{\lambda}_q = \tilde{\lambda}_e$:

$$\tau^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+) = i\frac{\langle \alpha q\rangle^2 \langle cq\rangle [e|\alpha|q\rangle [e|d|a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle \beta q\rangle^2 [e|d|c\rangle^2 [e|d|q\rangle^2} + 3i\frac{\langle \alpha q\rangle^2 \langle aq\rangle [e|\beta|q\rangle [e|d|a\rangle^2}{\langle ab\rangle\langle bc\rangle\langle \beta q\rangle^2 [e|d|c\rangle [e|d|q\rangle^2}$$
$$+ i\frac{\langle \alpha q\rangle^2 \langle q|\alpha\beta|q\rangle [e|d|a\rangle^3}{s_{de}\langle ab\rangle\langle bc\rangle\langle \beta q\rangle^2 [e|d|c\rangle [e|d|q\rangle^2}, \tag{4.6.3}$$

where we note that the third term is the leading order term. On the left we have

$$A_3^{(1)}(e^+, -\alpha^+, -l^+)A_3^{(0)}(l^-, -\beta^-, d^+)$$
$$= -\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{\langle q|el|q\rangle^3}{\langle eq\rangle^2\langle \alpha q\rangle^2\langle lq\rangle^2}\frac{\langle l\beta\rangle[dq]^2}{[lq][\beta q]}\frac{1}{\alpha^{2(1+\epsilon)}}$$
$$\left({}_2F_1[3-\epsilon, 2, 2-\epsilon, \frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^{1-\epsilon} + {}_2F_1[2, 3-\epsilon, 2-\epsilon, \frac{l^2}{\alpha^2}]\left(\frac{l^2}{\alpha^2}\right)^2\right) \tag{4.6.4}$$

We will get a s^{-1} contribution from the loop triangle and a s^1 contribution from the tree so overall this left hand side gives no pole. Thus we need only expand the current to order s^0 as shown above.

This integral in its totality is

$$-i\Gamma(1+\epsilon)\beta(1-\epsilon, 3-\epsilon)\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\frac{[e|d|a\rangle^3[de]^2\langle eq\rangle}{[e|d|c\rangle[e|d|q\rangle^2 s_{de}\langle ab\rangle\langle bc\rangle}\sum_k \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)}$$
$$\int d^D l\left(\frac{1}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2} + \frac{1}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\right)\frac{[e|l|q\rangle\langle q|\alpha\beta|q\rangle[e|ld|e]}{[e|\beta|q\rangle^2} \tag{4.6.5}$$

We therefore define

$$I_{m;1} = \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_k \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)}\int \frac{d^D l}{l^{2(\epsilon-k)}\alpha^{2(3+k)}\beta^2}\frac{[e|l|q\rangle\langle q|\alpha\beta|q\rangle[e|ld|e]}{[e|\beta|q\rangle^2}, \tag{4.6.6}$$

and

$$I_{m;2} = \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\sum_k \frac{(3-\epsilon,k)(2,k)}{(2-\epsilon,k)}\int \frac{d^D l}{l^{2(3+k)}\alpha^{2(\epsilon-k)}\beta^2}\frac{[e|l|q\rangle\langle q|\alpha\beta|q\rangle[e|ld|e]}{[e|\beta|q\rangle^2}. \tag{4.6.7}$$

as before. Since the methods used to solve these integrals are similar to prior calculations we will simply state the results. We can write $I_{m;1}$ as

$$\begin{aligned} I_{m;1} =& I^{\text{typ}}_{m;1}[2,0,-2] + (\epsilon-2)I^{\text{typ}}_{m;1}[2-\epsilon,0,-2+\epsilon] + I^{\text{typ}}_{m;1}[4,-2,-2] - I^{\text{typ}}_{m;1}[4-\epsilon,-2,-2+\epsilon] \\ &+ 2I^{\text{typ}}_{m;1}[3,-1,-2] + (\epsilon-2)I^{\text{typ}}_{m;1}[3-\epsilon,-1,-2+\epsilon] - I^{\text{typ}}_{m;1}[1-\epsilon,1,-2+\epsilon], \end{aligned} \tag{4.6.8}$$

where

$$I^{\text{typ}}_{m;1}[d_1,d_2,d_3] = i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{\Gamma(\epsilon+d_1)\Gamma(2-2\epsilon+d_2)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(1-3\epsilon+d_{123})}\frac{[e|d|q\rangle}{s_{de}^{1+2\epsilon}}, \tag{4.6.9}$$

and

$$\begin{aligned} I_{m;2} =& I^{\text{typ}}_{m;2}[0,2,-2] + (\epsilon-2)I^{\text{typ}}_{m;2}[0,2-\epsilon,-2+\epsilon] + I^{\text{typ}}_{m;2}[-2,4,-2] - I^{\text{typ}}_{m;2}[-2,4-\epsilon,-2+\epsilon] \\ &+ 2I^{\text{typ}}_{m;2}[-1,3,-2] + (\epsilon-2)I^{\text{typ}}_{m;2}[-1,3-\epsilon,-2+\epsilon] - I^{\text{typ}}_{m;2}[1,1-\epsilon,-2+\epsilon], \end{aligned} \tag{4.6.10}$$

where

$$I^{\text{typ}}_{m;2}[d_1,d_2,d_3] = i\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+\epsilon)\Gamma(3-\epsilon)}\frac{\Gamma(3+d_1)\Gamma(-1-\epsilon+d_2)}{\Gamma(2-\epsilon+d_{12})}\frac{\Gamma(1-2\epsilon)\Gamma(-\epsilon+d_{123})}{\Gamma(1-3\epsilon+d_{123})}\frac{[e|d|q\rangle}{s_{de}^{1+2\epsilon}}. \tag{4.6.11}$$

In the end we can put this all together and the resultant double pole is

$$\frac{181}{108}\frac{\langle ad\rangle^3\langle eq\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle dq\rangle}. \tag{4.6.12}$$

Applying recursion to this, we get our final double pole contribution to the amplitude

$$-\frac{181}{108}\frac{\langle ad\rangle\langle ea\rangle[ed]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle\langle da\rangle}. \tag{4.6.13}$$

In the case of the subleading terms, the integrable spinor content from the current is

$$\frac{[e|\alpha|q\rangle\langle\alpha q\rangle^2}{\langle\beta q\rangle^2}$$

while the left hand side gives

$$\frac{\langle\beta q\rangle^2}{\langle\alpha q\rangle^2}\frac{[e|l|q\rangle[e|ld|e]}{[e|\beta|q\rangle^2}.$$

Due to the $[e|ld|e]$ term from the left hand side, when we Feynman parameterise as we have done so far with $p = l - (u_2 e - u_3 d)$, the only surviving term will be $[e|pd|e]$. We require an even number of powers of p in Feynman integrals in order to be non-zero however given that all the other terms in the integral are of the form $[e|X|q\rangle$ and so after Feynman parameterisation the p dependent parts of these terms will be of the form $[e|X|q\rangle$ however after contraction with $[e|pd|e]$ we will have $p^2[ee] = 0$. Thus we have no non-zero subleading contributions.

4.6.2 Single-Minus Triangle Contribution

Above we constructed the single-minus embedded triangle and we can see how complex these terms are. We then have to multiply this triangle by the MHV currents and perform the second 'outer' integral. As of now we are unable to fully carry out this second outer integral and fully determine the contribution from these terms. We are able to make general arguments about the spinor content that we expect such terms to have in both the leading and subleading cases which we will present below.

We begin with the MHV tree currents, (4.2.1,4.2.2), and set $\lambda_q = \lambda_a$ which leaves the currents as

$$\begin{aligned}\tau_5^{(0)}(\alpha^-, a^-, b^+, c^+, \beta^+, q) =& i\frac{\langle \alpha a\rangle^2}{\langle ab\rangle\langle bc\rangle\langle \beta a\rangle^2}\left(\frac{\langle a|P_{\alpha\beta}|a\rangle[q|P_{\alpha\beta}|a\rangle}{s_{\alpha\beta}[q|P_{\alpha\beta}|c\rangle} + i\frac{\langle ca\rangle[q|\alpha|a\rangle[q|P_{\alpha\beta}|a\rangle}{[q|P_{\alpha\beta}|c\rangle^2}\right)\\ &+ O(s_{\alpha\beta}),\end{aligned} \tag{4.6.14}$$

and

$$\begin{aligned}\tau^{(0)}(\beta^-, \alpha^+, a^-, b^+, c^+, q) = i&\frac{\langle \beta a\rangle^2}{\langle ab\rangle\langle \alpha a\rangle^2\langle bc\rangle}\left(\frac{[q|P_{\alpha\beta}|a\rangle\langle a|\alpha\beta|a\rangle}{s_{de}[q|P_{\alpha\beta}|c\rangle} + i\frac{\langle ca\rangle[q|\alpha|a\rangle[q|P_{\alpha\beta}|a\rangle}{[q|P_{\alpha\beta}|c\rangle^2}\right)\\ &+ O(s_{\alpha\beta}).\end{aligned} \tag{4.6.15}$$

Writing the currents in this way shows that the currents are the same up to the spinor weight factor. The single-minus embedded triangle on the left admits both the adjacent and non-adjacent MHV currents on the right and the triangle can be in both the $K = e$ and $K = d$ corners, so that there are four contributions to be considered. By writing the currents in a 'flip-neutral' form, labelling the off-shell legs Q and R as we did in figure 4.21, we can combine these four contributions into two contributions plus their flip.

This 'flip-neutral' form is

$$\tau_5^{(0)}(Q^-,a^-,b^+,c^+,R^+,q) = i\frac{\langle Ra\rangle^2}{\langle ab\rangle\langle bc\rangle\langle Qa\rangle^2}\left(\frac{\langle a|RQ|a\rangle[q|P_{RQ}|a\rangle}{s_{RQ}[q|P_{RQ}|c\rangle}+i\frac{\langle ca\rangle[q|R|a\rangle[q|P_{RQ}|a\rangle}{[q|P_{RQ}|c\rangle^2}\right), \tag{4.6.16}$$

and

$$\tau^{(0)}(R^-,Q^+,a^-,b^+,c^+,q) = i\frac{\langle Qa\rangle^2}{\langle ab\rangle\langle Ra\rangle^2\langle bc\rangle}\left(\frac{\langle a|RQ|a\rangle[q|P_{RQ}|a\rangle}{s_{RQ}[q|P_{RQ}|c\rangle}+i\frac{\langle ca\rangle[q|R|a\rangle[q|P_{RQ}|a\rangle}{[q|P_{RQ}|c\rangle^2}\right). \tag{4.6.17}$$

The full integral can be schematically written as

$$\int\frac{d^DP}{P^2Q^2R^2}\text{ (embedded triangle)}\times\frac{[Rj]\langle Pq\rangle^2}{\langle Rq\rangle\langle jq\rangle}\times\text{ (current)}, \tag{4.6.18}$$

and the embedded triangles are written in terms of the typical integrals Typ_{L_2} and Typ_{L_3} multiplied by the spinor prefactor

$$\begin{cases}\frac{[q|Q|q\rangle^N\langle q|KQ|q\rangle^2[q|KQ|q]}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2} & \text{scalar pieces}\\ \frac{[q|Q|q\rangle^2[q|T|q\rangle^{b_c}[q|K|q\rangle^{N-b_c}\langle q|KQ|q\rangle}{\langle Kq\rangle^2\langle Pq\rangle^2[Qq]^2} & \text{tensor pieces}\end{cases} \tag{4.6.19}$$

where T can be either P or Q depending on the specific term, b_c is also a term dependent exponent, and $N=1,2$. The tensor and scalar pieces labels above refer to those from the embedded triangle, so scalar refers to terms where the d_d argument in the Typ integral is $d_d=\epsilon$, while tensor refers to $d_d=\epsilon-1$.

Altogether now, the integrand of the integrand of the P *outer integral* is

$$\begin{cases}\frac{[q|Q|a\rangle^{N-2}\langle q|KQ|a\rangle^2[q|KQ|q]\langle a|jR|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2P^2Q^2R^2}\frac{\text{Typ}}{\langle ab\rangle\langle bc\rangle}\left(\frac{[q|P_{RQ}|a\rangle\langle ac\rangle[q|Q|a\rangle}{[q|P_{RQ}|c\rangle^2}-\frac{\langle a|RQ|a\rangle}{s_{RQ}}\frac{[q|P_{RQ}|a\rangle}{[q|P_{RQ}|c\rangle}\right) & \text{scalar pieces }(\mathrm{d_d}=\epsilon)\\ \frac{[q|T|a\rangle^{b_c}[q|K|a\rangle^{N-b_c}\langle a|KQ|a\rangle\langle a|jR|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2P^2Q^2R^2}\frac{\text{Typ}}{\langle ab\rangle\langle bc\rangle}\left(\frac{[q|P_{RQ}|a\rangle\langle ac\rangle[q|Q|a\rangle}{[q|P_{RQ}|c\rangle^2}-\frac{\langle a|RQ|a\rangle}{s_{RQ}}\frac{[q|P_{RQ}|a\rangle}{[q|P_{RQ}|c\rangle}\right) & \text{tensor pieces }(\mathrm{d_d}=\epsilon-1),\end{cases} \tag{4.6.20}$$

where Typ signifies the typical integral. We notice that the $(d_d=\epsilon-1)$ terms from the embedded triangles will only give scalar terms in the outer integral.

Furthermore, we have a one-mass triangle integral which we can Feynman parameterise

$$x_1P^2 + x_2Q^2 + x_3R^2 = x_1P^2 + x_2(P+K)^2 + x_3(P-j)^2$$
$$\to P^2 + 2x_2P\cdot K - 2x_3P\cdot j = (P + x_2K - x_3j)^2 + x_2x_3s_{Kj} \quad (4.6.21)$$

so that

$$P \to p - x_2K + x_3j \qquad Q \to -p - (1-x_2)K - x_3j \qquad R \to p - x_2K - (1-x_3)j. \quad (4.6.22)$$

This means that when $\tilde{\lambda}_q = \tilde{\lambda}_j$ there will be no contribution from the $d_d = \epsilon$ terms due to the factor of $[q|KQ|q]$.

Finally we note that since we have taken $\tilde{\lambda}_q = \tilde{\lambda}_e$, and and e will be either j or K, that the scalar part of $[q|P|a\rangle$ and $[q|Q|a\rangle$ will be purely Feynman parameters. This will be useful when trying to separate the kinematic parts from the numerical parts.

Let us now focus on the typical integral $\mathrm{Typ}^{\mathrm{A}}_{L_2}$,

$$\begin{aligned}\mathrm{Typ}^{A}_{L_2} &= \frac{\Gamma(-d_d)\Gamma(d_d-d_1)\Gamma(1+d_1)\Gamma(1+d_2+d_1-d_d-N)}{\Gamma(1+d_d)\Gamma(1+d_2+d_1-2d_d-N)\big(Q^2\big)^{1+d_d}[q|P|q\rangle^N}\Big(\frac{P^2}{Q^2}\Big)^{d_1-d_d} \\ &\times F_1\Big[1+d_1\ ;\ N\ ,\ 1+d_2+d_1-d_d-N\ ;\ 1+d_1-d_d\ ;\ -\frac{P^2}{Q^2}\frac{[q|Q|q\rangle}{[q|P|q\rangle}\ ,\ \frac{P^2}{Q^2}\Big].\end{aligned} \quad (4.6.23)$$

where we see that the integrable parts of the expression are

$$\frac{\big(P^2\big)^{d_1-d_d+m+n}(-[q|Q|a\rangle)^m}{\big(Q^2\big)^{1+d_1+m+n}[q|P|a\rangle^{N+m}} \quad (4.6.24)$$

where m and n denote the sum variables of the Appell function in the double sum representation (C.2.4). Let us begin with the $d_d = \epsilon$ pieces of the embedded integral. Denoting the powers of $[q|Q|q\rangle$, $[q|P|q\rangle$, $[q|K|q\rangle$ coming from the tensor term in the embedded triangles by $N + b_Q$, b_P and b_k respectively, the integral over the $\{m, n\}$ term of the Appell expansion is

$$\begin{aligned}\mathrm{O}^{\mathrm{Typ}^{\mathrm{A}}_{\mathrm{L}_2}:\mathrm{ten}}_{m,n} &= \int d^dP\,\frac{[q|Q|a\rangle^{N-2+b_Q}[q|P|a\rangle^{b_P}[q|K|a\rangle^{b_k}\langle a|KQ|a\rangle\langle a|jR|a\rangle}{\langle jq\rangle^2\langle kq\rangle^2P^2Q^2R^2}\,\frac{\big(P^2\big)^{d_1-d_d+m+n}[q|Q|a\rangle^m(-1)^m}{\big(Q^2\big)^{1+d_1+m+n}[q|P|a\rangle^{N+m}} \\ &\times\frac{1}{\langle ab\rangle\langle bc\rangle}\left(\frac{[q|P_{jK}|a\rangle\langle ac\rangle[q|Q|a\rangle}{[q|P_{jK}|c\rangle^2} - \frac{\langle a|RQ|a\rangle}{s_{jK}}\frac{[q|P_{jK}|a\rangle}{[q|P_{jK}|c\rangle}\right).\end{aligned} \quad (4.6.25)$$

Note that $R+Q=j+K$. The integral will vanish if $\tilde{\lambda}_q=\tilde{\lambda}_k$ so we set $\tilde{\lambda}_q=\tilde{\lambda}_j$. First let us tackle the leading pole integral,

$$\begin{aligned}
\mathrm{O}_{m,n}^{\mathrm{Typ}^{\mathrm{A}}_{\mathrm{L}_2}:\mathrm{ten:dp}} &= \int d^dP \frac{[j|Q|a\rangle^{N-2+b_Q+m}[j|P|a\rangle^{b_P-N-m}[j|K|a\rangle^{b_k}\langle a|KQ|a\rangle\langle a|jR|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2\left(P^2\right)^{1-d_1+d_d-m-n}\left(Q^2\right)^{2+d_1+m+n}R^2} \\
&\quad \frac{\langle a|(j+K)Q|a\rangle}{\langle ab\rangle\langle bc\rangle s_{jK}}\frac{[j|K|a\rangle}{[j|K|c\rangle}(-1)^m \\
&= \frac{\Gamma(4+d_d)(-1)^m}{\Gamma(1-d_1+d_d-m-n)\Gamma(2+d_1+m+n)}\int dx_i x_1^{-d_1+d_d-m-n}x_2^{1+d_1+m+n} \\
&\quad \times \int d^dP \frac{[j|Q|a\rangle^{N-2+b_Q+m}[j|P|a\rangle^{b_P-N-m}[j|k|a\rangle^{b_k}\langle a|KQ|a\rangle\langle a|jR|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2\left(x_1P^2+x_2Q^2+x_3R^2\right)^{4+d_d}} \\
&\quad \frac{\langle a|(j+K)Q|a\rangle}{\langle ab\rangle\langle bc\rangle s_{de}}\frac{[j|K|a\rangle}{[j|K|c\rangle}.
\end{aligned} \tag{4.6.26}$$

After Feynman parameterisation only the scalar contribution survives,

$$\begin{aligned}
\mathrm{O}_{m,n}^{\mathrm{Typ}^{\mathrm{A}}_{\mathrm{L}_2}:\mathrm{ten:dp}} &= \frac{\Gamma(2+d_d+\epsilon)(-1)^m}{\Gamma(1-d_1+d_d-m-n)\Gamma(2+d_1+m+n)}\int dx_i x_1^{-d_1+d_d-m-n}x_2^{1+d_1+m+n} \\
&\times \frac{[j|K(x_2-1)|a\rangle^{N-2+b_Q+m}[j|K(-x_2)|a\rangle^{b_P-N-m}[j|K|a\rangle^{b_k}\langle a|Kj(-x_3)|a\rangle\langle a|jK(-x_2)|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2\left(x_2x_3s_{jK}\right)^{2+d_d+\epsilon}} \\
&\quad \times \frac{\langle a|jK(x_2-1)+Kj(-x_3)|a\rangle}{\langle ab\rangle\langle bc\rangle s_{jK}}\frac{[j|K|a\rangle}{[j|K|c\rangle} \\
&= \frac{[j|K|a\rangle^{b_Q+b_P+b_k-2}\langle a|jK|a\rangle^3}{\langle ja\rangle^2\langle Ka\rangle^2\langle ab\rangle\langle bc\rangle s_{jK}^{3+d_d+\epsilon}}\frac{[j|K|a\rangle}{[j|K|c\rangle}\frac{\Gamma(2+d_d+\epsilon)(-1)^m}{\Gamma(1-d_1+d_d-m-n)\Gamma(2+d_1+m+n)} \\
&\quad \times \int dx_i x_1^{-d_1+d_d-m-n}x_2^{1+d_1+m+n}\frac{(x_2-1)^{N-2+b_Q+m}(-x_2)^{b_P-N-m}x_1x_2x_3}{\left(x_2x_3\right)^{2+d_d+\epsilon}}.
\end{aligned} \tag{4.6.27}$$

While we cannot solve the Feynman integral, it is free of kinematic factors and hence a purely numeric factor.

We can simplify the spinor part of the above and thus the spinor contribution from the leading order term is

$$\frac{\langle ja\rangle\langle ka\rangle^2[jk]^{1-2\epsilon}}{\langle ab\rangle\langle bc\rangle\langle kc\rangle\langle jk\rangle^{2+2\epsilon}}. \tag{4.6.28}$$

The subleading integral is

$$\begin{aligned}
\mathrm{O}_{m,n}^{\mathrm{Typ}^{\mathrm{A}}_{\mathrm{L}_2}:\mathrm{ten}} &= \int d^dP \frac{[j|Q|a\rangle^{N-2+b_Q}[j|P|a\rangle^{b_P}[j|K|a\rangle^{b_k}\langle a|KQ|a\rangle\langle a|jR|a\rangle}{\langle ja\rangle^2\langle Ka\rangle^2P^2Q^2R^2}\frac{\left(P^2\right)^{d_1-d_d+m+n}[j|Q|a\rangle^m}{\left(Q^2\right)^{1+d_1+m+n}[j|P|a\rangle^{N+m}} \\
&\quad \times \frac{1}{\langle ab\rangle\langle bc\rangle}\frac{[j|P_{RQ}|a\rangle\langle ac\rangle[j|Q|a\rangle}{[j|P_{RQ}|c\rangle^2},
\end{aligned}$$

and given that the only difference between this integral and the last is the presence of $[j|Q|a\rangle$ (which gives $[j|K|a\rangle(1-x_2)$ after Feynman parameterisation), and the absence of $\langle a|RQ|a\rangle/s_{RQ}$(which gives $(1+x_2x_3)\langle a|jK|a\rangle/s_{de}$ after Feynman parameterisation), we can apply the same arguments as in the leading order case to see that the subleading terms will contribute the following spinors

$$\frac{\langle ka\rangle[jk]^{1-2\epsilon}\langle ac\rangle\langle ad\rangle}{\langle ab\rangle\langle bc\rangle\langle kc\rangle\langle cd\rangle\langle jk\rangle^{1+2\epsilon}}. \tag{4.6.29}$$

The above arguments rest on the ability to write the integrals as a term with no pole which can be discarded, and a term wherein the kinematics can be extracted to leave the integral as purely numeric. These arguments are general and apply to $\text{Typ}^B_{L_3}$ and indeed the more complicated Typ_{L_2} will almost no change. We are thus able to state that in the end the single-minus triangle with the MHV current will produce a contribution of the form

$$\frac{\langle ja\rangle\langle ka\rangle^2[jk]^{1-2\epsilon}}{\langle ab\rangle\langle bc\rangle\langle kc\rangle\langle jk\rangle^{2+2\epsilon}}f_1(\epsilon)+\frac{\langle ka\rangle[jk]^{1-2\epsilon}\langle ac\rangle\langle ad\rangle}{\langle ab\rangle\langle bc\rangle\langle kc\rangle\langle cd\rangle\langle jk\rangle^{1+2\epsilon}}f_2(\epsilon), \tag{4.6.30}$$

where $\{j,k\}=\{d,e\}$ and vice versa. In the leading order term above, we consider both cases for j and k and perform recursion and we find that in fact both return the same expression in terms of spinor content

$$\frac{\langle ad\rangle^2\langle ae\rangle[ed]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle} \tag{4.6.31}$$

which is precisely the spinor content of the two loop splitting function to tree MHV factorisation (3.0.4). The same thing happens in the subleading terms where we see that after recursion both terms give the same spinors

$$\frac{\langle ac\rangle\langle ad\rangle^2[ed]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2} \tag{4.6.32}$$

4.7 Rational Descendants of Cut-Constructible Pieces

Let us take a step back and consider exactly how we define 'rational pieces'. Due to our use of 4D unitarity there are some terms which we cannot reconstruct using unitarity cuts, and these remaining terms are what we refer to as rational. In other words,

$$\text{Rational} = \text{Amplitude} - \text{cut-constructible}.$$

We then apply a recursive method to reconstruct the rational part which involves complex integration. As we are only interested in the rational parts we drop any logs, dilogs etc which have branch cuts before we carry out our BCFW shift and the remainder of the outlined process. There is however the possibility of obtaining rational pieces from recursion on cut constructible functions. Recall that in the derivation of the BCFW relations, we perform a shift

$$\begin{aligned} \lambda_e \to& \lambda_e - z\lambda_a, \\ \tilde{\lambda}_a \to& \tilde{\lambda}_a + z\tilde{\lambda}_e, \end{aligned} \tag{4.7.1}$$

then any terms in the amplitude with one of these momenta in them will be affected. With a few examples we will demonstrate how these rational descendants arise, before summarising the results at the end of this section.

Let us take the example of the pole that arises when $\langle ce\rangle \to 0$.

In order to give a rational piece, a term with transcendental weight n must have a pole of at least power $n+1$. This will be justified in due course. In the $\langle ce\rangle$ case there are eight such terms: four with weight 2 and four with weight 1. The weight 2 terms are

$$\pi^2,\ \log\left(\frac{s_{cd}}{s_{ab}}\right)\log\left(\frac{s_{de}}{s_{ab}}\right),\ \mathrm{Li}_2\left(1-\frac{s_{cd}}{s_{ab}}\right),\ \mathrm{Li}_2\left(1-\frac{s_{de}}{s_{ab}}\right),$$

and the weight 1 terms are

$$\log\left(s_{ab}\right),\ \log\left(s_{cd}\right)$$

and

$$\log\left(s_{ab}\right),\ \log\left(s_{de}\right).$$

Starting with the weight 2 terms, these terms have the same coefficient, so let us consider these terms together. We perform the BCFW shift,

$$F_f(z) = -\frac{\pi^2}{6} + \log\left(\frac{s_{cd}}{s_{\hat{a}b}}\right)\log\left(\frac{s_{d\hat{e}}}{s_{\hat{a}b}}\right) + \ \mathrm{Li}_2\left(1-\frac{s_{cd}}{s_{\hat{a}b}}\right) + \ \mathrm{Li}_2\left(1-\frac{s_{d\hat{e}}}{s_{ab}}\right) \tag{4.7.2}$$

The term $\langle c\hat{e}\rangle$ has a pole at $z_0 = \frac{\langle ce\rangle}{\langle ca\rangle}$, and by the identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(z)\log(1-z) - \frac{\pi^2}{6} = 0, \tag{4.7.3}$$

we see that

$$F_f(z_0) = -\frac{\pi^2}{6} + \log\left(\frac{s_{cd}}{s_{\hat{a}b}}\right)\log\left(\frac{s_{d\hat{e}}}{s_{\hat{a}b}}\right)) + \text{ Li}_2\left(1 - \frac{s_{cd}}{s_{\hat{a}b}}\right) + \text{ Li}_2\left(1 - \frac{s_{d\hat{e}}}{s_{ab}}\right) = 0, \qquad (4.7.4)$$

Because these terms together vanish on the pole, we can expand the expression around the pole in a natural manner. We can also say that such term would not have counted in the cut-constructible part of the amplitude, so there would be no double counting.

To calculate the residue, we will shift the functions, $F_f(z)$, well as the coefficient of the functions,

$$F_c(z) = 10\frac{[cd][de]\langle ac\rangle^2\langle ae\rangle}{\langle ab\rangle[ce]\langle c\hat{e}\rangle^3\langle bc\rangle}, \qquad (4.7.5)$$

and expand the function around the pole, so we set $z = z_0 + \delta$ and Taylor expand around δ. The functions in F_f are of transcendental weight 2, they are dilogarithms and $\log^2$, and we note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{Li}_2(z) &= -\frac{\log(1-z)}{z},\\ \frac{\mathrm{d}}{\mathrm{dz}}\log(z) &= \frac{1}{z},\end{aligned} \qquad (4.7.6)$$

so rational expressions will start to appear in at $\mathcal{O}(\delta^2)$. This generalises to rational expressions appearing in an expansion weight n functions at $\mathcal{O}(\delta^n)$. From the derivation of the BCFW recursion relations in Section 1.7, we know that the rational piece from this pole will be the residue of $F_f(z)F_c(z)/z$ at $z = z_0$, hence the need for a pole of order $n+1$ from the coefficient in order to have a rational non-zero residue.

Finally to calculate the rational part, it simply remains to expand $F_f(z)F_c(z)/z$ around $z = z_0 + \delta$ and determine the coefficient of δ^{-1}.

Similarly with the weight 1 functions, the log terms $\{\log(s_{ab}),\ \log(s_{cd})\}$ and $\{\log(s_{ab}),\ \log(s_{de})\}$ can be paired up as their coefficients are equal with opposite signs, and the shifted functions

$$\log\left(\frac{s_{\hat{a}b}}{s_{cd}}\right), \quad \text{and} \quad \log\left(\frac{s_{\hat{a}b}}{s_{d\hat{e}}}\right),$$

both vanish on the pole $z_0 = \frac{\langle ce\rangle}{\langle ca\rangle}$. Again this means we can sensibly expand the functions around the pole and calculate residues. In this case the rational terms appear in the expansion at $\mathcal{O}(\delta)$ and the coefficients have double poles.

Adding all three contributions together, we can simplify the final expression and the rational contribution to the amplitude from the $\langle c\hat{e}\rangle$ pole is

$$R_{\langle ce\rangle} = -5\frac{\langle ac\rangle^3[ce]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle ce\rangle}. \tag{4.7.7}$$

Now let us consider the $\langle de\rangle$ pole. In this case, there are 6 pairs of logs with a double $\langle de\rangle$ pole that can be paired up due to their coefficients being the same with opposite sign as before. Of these pairs, one is $\log(s_{bc}/s_{ea})$ which is not shifted and thus will not give a rational part. This can be immediately discounted.

Another of the pairs is $\log(s_{ab}/s_{de})$. While this one is shifted, the presence of s_{de} means that the logarithm will diverge on the pole. Since we cannot make sense of an expansion around the pole in this case, it would be unnatural to include any resulting rational part due to this term, and so we will discard it.

The final type of pair can be exemplified by $\log(s_{ab}/s_{cd})$. This function does shift and will not diverge. The $\langle d\hat{e}\rangle$ pole is at $z_1 = \langle de\rangle/\langle da\rangle$, and on this pole the function above evaluates to

$$\log\left(\frac{s_{\hat{a}b}}{s_{cd}}\right)\bigg|_{z_1} = \log\left(-\frac{[c|b|a\rangle}{[c|d|a\rangle}\right). \tag{4.7.8}$$

Since this logarithm does not vanish on the pole, we do not include any rational descendants of this as it would have already been included in the cut-constructible part. Hence the $\langle de\rangle$ pole produces no rational descendants from the cut-constructible parts of the amplitude.

One by one we check all the various poles that could arise due, both physical and spurious. As discussed in Section 2.3, there are still some discrepancies in the two loop sector of the 4D unitarity cut constructible method presented in this thesis. For this reason we use the cut-constructible part of the published result [1] with the expectation that once resolved, the two will agree.

We find that none of the physical poles contribute anything, while there are four contributions from spurious poles, from the $\langle ce\rangle$ pole

$$R_{\langle ce\rangle} = -5\frac{\langle ac\rangle^3[ce]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle ce\rangle}, \tag{4.7.9}$$

and from the $\langle be\rangle$ pole.

$$R_{\langle be\rangle} = \frac{\langle ae\rangle\langle bc\rangle\langle bd\rangle[be]^2[cd]}{3\left(s_{ea}-s_{cd}\right)\langle be\rangle^2\langle cd\rangle^2[ae]} - 10\frac{\langle ab\rangle^2[be]}{\langle be\rangle\langle bc\rangle\langle bd\rangle\langle cd\rangle} - 5\frac{\langle ab\rangle^2\langle ae\rangle[ae][be]}{s_{ea}\langle be\rangle\langle bc\rangle\langle bd\rangle\langle cd\rangle}$$
$$- \frac{2}{9}\frac{s_{cd}\langle ea\rangle^2\langle bc\rangle\langle bd\rangle[be]^3[cd]}{\langle be\rangle s_{ea}^2\left(s_{ea}-s_{cd}\right)^2\langle cd\rangle^2} - \frac{\langle ae\rangle\langle bc\rangle\langle bd\rangle[be]^3[cd]}{3s_{ea}^2\langle be\rangle\langle cd\rangle^2[ae]}$$
$$+ \frac{[be]^2[cd]}{3\left(s_{ea}-s_{cd}\right)\langle ae\rangle\langle cd\rangle^2[ae]^3\langle be\rangle}\left(\langle ad\rangle\langle ae\rangle\langle bc\rangle[ae]^2 + \langle bd\rangle\langle cd\rangle\left(\langle bc\rangle\langle be\rangle[cd] - 2\langle ae\rangle[ae][de]\right)\right), \tag{4.7.10}$$

There are also two co planar poles,

$$R_{ab;cd} = -\frac{\langle ab\rangle^2[be]^2}{\left(s_{ab}-s_{cd}\right)s_{cd}}\left(\frac{2}{3}\frac{[be][cd]}{\langle ab\rangle\langle cd\rangle[ae]} + \frac{\langle ae\rangle\langle bc\rangle[be][cd][ce]}{9s_{ea}\langle ab\rangle^2\langle cd\rangle[ae]}\right) - \frac{2}{9}\frac{\langle ad\rangle[be]^2[cd][de]}{\left(s_{ab}-s_{cd}\right)s_{cd}\langle cd\rangle[ae]}$$
$$+ \frac{\left(s_{ab}-3s_{cd}\right)\langle ae\rangle[be]^2[cd][ce][de]}{9\left(s_{ab}-s_{cd}\right)^2 s_{cd}^2[ae]}, \tag{4.7.11}$$

and

$$R_{ab;de} = -\frac{2}{9}\frac{\langle ac\rangle^3[ce]^2(\langle ab\rangle[bc] + 2\langle ae\rangle[ce])}{\left(s_{ab}-s_{de}\right)\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}$$
$$- \frac{\langle ac\rangle^3[ce]^3}{\left(s_{ab}-s_{de}\right)\langle ab\rangle\langle cd\rangle[bc][de]}\left(-\frac{\langle ae\rangle[de]}{3\langle ab\rangle\langle bc\rangle\langle cd\rangle} + \frac{2}{3}\frac{[bc][de]}{\langle bc\rangle\langle cd\rangle[ce]}\right)$$
$$- \frac{s_{cd}\langle ac\rangle^3[ce]^2}{9\left(s_{ab}-s_{de}\right)\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^3[bc][de]}\left(\langle ab\rangle^2[ab][be] - 3\langle ab\rangle\langle cd\rangle[bc][de] + \langle ad\rangle\langle de\rangle[de]^2\right) \tag{4.7.12}$$

which arise from $s_{\hat{a}b} - s_{cd}$ and $s_{\hat{a}b} - s_{d\hat{e}}$, respectively.

4.8 Epsilon-Expansions

Through this chapter, when we were able complete the full calculations, we then had to extract the finite rational part of the expression. In all cases this involved ϵ- expansions of hypergeometric functions, ${}_2F_1$, with integer parameters. This was carried out using the HypExp package [91]. Naively inputting these expressions directly into this package, we ran into some issues of runtime and of extracting rational parts.

The main issue lies in the expansion of Gamma functions. In general the function $\Gamma(a+b\epsilon)$ expands as

$$\Gamma(a+b\epsilon) = \Gamma(a) + b\epsilon\Gamma(a)\psi^{(0)}(a) + \mathcal{O}(\epsilon^2),$$

where $\psi^{(0)}(z)$ is the Polygamma function defined as

$$\psi^{(m)}(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}} \ln \Gamma(z). \tag{4.8.1}$$

The Polygamma function is of mixed transcendentality and so it is not possible to directly extract a rational part from it.

We can understand this better by expanding $\Gamma(1 + b\epsilon)$ and $\Gamma(-1 + b\epsilon)$ as examples,

$$\begin{aligned} \Gamma(1 + b\epsilon) =& 1 - b\gamma\epsilon + \mathcal{O}(\epsilon^2), \\ \Gamma(-1 + b\epsilon) =& -\frac{1}{b\epsilon} + \gamma - 1 + b\epsilon \left(\gamma - 1 - \frac{b}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right)\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.8.2}$$

The polygamma function encapsulates both of these but it is not an ideal representation for our purposes. Additionally at two loops there were naturally a lot of gamma functions alongside hypergeometric functions which significantly increased the run time of the HypExp expansion and since we are only interested in the rational piece the extra terms are superfluous to our needs anyway.

For Gamma functions $\Gamma(a + b\epsilon)$ for negative a we used the Gamma function property $\Gamma(1 + n) = n\Gamma(n)$ to write them as a Gamma function with positive a divided by some factors, so $\Gamma(-1 - 2\epsilon) = \Gamma(1 - 2\epsilon)/(2\epsilon(1 + 2\epsilon))$, for example. This isolated the singularities and then we removed the epsilon part of the Gamma function, losing only non-rational terms. This also sped up the expansion process. At the end, we know that we will have a factor of c_Γ^2 in front which we can add in.

4.9 Results

Before we proceed to reconstruct the full rational part of the amplitude, let us first bring together the various parts of the 'loop on the left' calculations and summarise whether we were able to calculate spinor content and the full numerical coefficient in table 4.2.

Contribution	Spinors	Coefficient	Section
l-bubble	✓	✓	4.3
first LO googly $K = e$	✓	✓	4.5.1
first LO googly $K = d$	✓	X	4.5.1
Second googly LO	X	X	4.5.2
Third googly LO	X	X	4.5.3
Googly subleading	X	X	4.5.3
AP Triangle MHV	✓	✓	4.6.1
SM Triangle MHV	✓	X	4.6.2

Table 4.2: This table collates the results of our calculations of the loop on the left terms, specifically whether we could deduce the resulting spinor content, and whether we could also get the full coefficient.

While there are many individual terms that we were unable to determine, we will now demonstrate how one can still reconstruct the full rational part from the information at hand.

The full decomposition of the rational part is

rational = tree to two-loop easy channels +double-pole channels + rational descendants,

where

double-pole channels = tree on the left + loop on the left,

and

loop on the left = l-bubble + double-box + box-triangle.

From the previous chapter we were able to calculate the contributions from the tree to two-loop factorisation channels to be

$$R^1_{2l} = -5\frac{\langle ac\rangle[bc]}{\langle ea\rangle\langle bc\rangle^2\langle bd\rangle\langle cd\rangle^2\langle de\rangle}(\langle ac\rangle\langle ad\rangle\langle bc\rangle\langle bd\rangle + \langle ab\rangle^2\langle cd\rangle^2), \tag{4.9.1}$$

and

$$R^2_{2l} = -\frac{1}{9}\frac{[be]^2([bd]^2[ce]^2 + 8[bc][cd][de][be])}{[ab][bc][cd][de][ea]\langle cd\rangle^2}, \tag{4.9.2}$$

from the one-loop MHV currents to be

$$\begin{aligned}
&-\frac{1}{18}\frac{\langle ac\rangle\langle bd\rangle\langle da\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]}+\frac{2}{9}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]}-\frac{64}{27}\frac{\langle ad\rangle^3\langle cq\rangle[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle dq\rangle}\\
&-\frac{1}{3}\frac{\langle ad\rangle^2[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle\langle cd\rangle}-\frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle}+\frac{1}{6}\frac{\langle ad\rangle^3[bd][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{1}{9}\frac{\langle ad\rangle^2\langle bd\rangle[bd]^2[de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]}+\frac{32}{27}\frac{\langle ad\rangle^2\langle ae\rangle[be][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}+\frac{64}{27}\frac{\langle ad\rangle^2\langle ae\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}\\
&+\frac{1}{6}\frac{\langle ac\rangle\langle ad\rangle^2\langle ae\rangle[ce][de]^2}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}-\frac{4}{9}\frac{\langle ad\rangle\langle ae\rangle\langle bd\rangle[be][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]}+\frac{7}{6}\frac{\langle ad\rangle^3\langle ae\rangle[de]^3}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}\\
&-\frac{2}{9}\frac{\langle ad\rangle^2\langle bd\rangle\langle ae\rangle[bd]^2[de]^2[ea]}{s_{de}(s_{bc}-s_{ea})^2\langle bc\rangle^2\langle cd\rangle[bc]},
\end{aligned} \tag{4.9.3}$$

and from the single-minus current to be

$$\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle[bc][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle ea\rangle}-\frac{7}{12}\frac{\langle ca\rangle\langle da\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2}-\frac{223}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}. \tag{4.9.4}$$

If we collate the above results with the parts of the loop on the left terms that we could determine, we have a total of 11 linearly independent functions that we can use to express the terms we derived from augmented recursion, plus 4 terms due to rational descendants. This linear independence is confirmed by numerics. We note that those terms of the loop on the left- sector that we were able to calculate all had the same two spinor structures

$$\frac{\langle ad\rangle^2\langle ae\rangle[ed]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle},\quad \text{and}\quad \frac{\langle ac\rangle\langle ad\rangle^2[ed]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2} \tag{4.9.5}$$

which are already present in the tree on the left functions.

From these 15 functions we can create an ansatz for the full rational part of the amplitude. If we impose the $A(a,b,c,d,e)=-A(a,e,d,c,b)$ flip symmetry on this ansatz we fix the coefficients of all but two of the functions. These two terms are

$$\frac{[be]^3}{\langle cd\rangle^2[ab][ae]}\left(8+\frac{[bd]^2[ce]^2}{[bc][be][cd][de]}\right)\quad \text{and}\quad \frac{\langle ad\rangle^2\langle ea\rangle][ed]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}.$$

The first is the tree to two-loop channel contribution in Eq 4.9.2 whose coefficient is known. The second is the double pole term in $\langle de\rangle$ whose coefficient can be fixed by imposing consistency in the (de) collinear limit.

Putting all the coefficients in the ansatz, the result is

$$
\begin{aligned}
R_5^{(2)}(a^-,b^+,c^+,d^+,e^+) &= 5\frac{\langle ac\rangle(\langle ac\rangle\langle ad\rangle\langle bc\rangle\langle bd\rangle+\langle ab\rangle^2\langle cd\rangle^2)}{\langle ae\rangle\langle bc\rangle^2\langle bd\rangle\langle cd\rangle^2\langle de\rangle}+\frac{1}{3}\frac{\langle ae\rangle\langle bc\rangle\langle bd\rangle[be]^2[cd]}{(s_{ea}-s_{cd})\langle be\rangle^2\langle cd\rangle^2[ae]}\\
&-\frac{2}{9}\frac{[ac]^3\langle ce\rangle^2(-\langle ab\rangle[bc]+2\langle ae\rangle[ce])}{(s_{ab}-s_{de})\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}+\frac{1}{9}\frac{[be]^3}{\langle cd\rangle^2[ab][ae]}\left(8+\frac{[bd]^2[ce]^2}{[bc][be][cd][de]}\right)\\
&-\frac{\langle ab\rangle^2[be]^2}{s_{cd}(s_{ab}-s_{cd})}\left(\frac{2}{3}\frac{[be][cd]}{\langle ab\rangle\langle cd\rangle[ae]}+\frac{1}{9}\frac{\langle ae\rangle\langle bc\rangle[be][cd][ce]}{s_{ae}\langle ab\rangle^2\langle cd\rangle[ae]}\right)-5\frac{\langle ac\rangle^3[ce]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle^2\langle ce\rangle}\\
&-\frac{1}{9}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][be][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2[bc]}-\frac{2}{9}\frac{\langle ad\rangle[be]^2[cd][de]}{(s_{ab}-s_{cd})s_{cd}\langle cd\rangle[ae]}-\frac{1}{9}\frac{\langle ac\rangle\langle ad\rangle\langle bd\rangle[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle}\\
&-\frac{2}{3}\frac{\langle ac\rangle\langle ad\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2}+\frac{1}{3}\frac{\langle ad\rangle^2[bd][de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle\langle cd\rangle}+\frac{1}{9}\frac{(s_{ab}-3s_{cd})\langle ae\rangle[be]^2[cd][ce][de]}{(s_{ab}-s_{cd})^2s_{cd}^2[ae]}\\
&-\frac{1}{3}\frac{\langle ad\rangle^3[bd][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}-\frac{1}{3}\frac{\langle ad\rangle^3[bd][de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2[bc]}+\frac{1}{3}\frac{\langle ad\rangle^2\langle bd\rangle[bd]^2[de]^2}{s_{de}(s_{bc}-s_{ea})\langle bc\rangle^2\langle cd\rangle[bc]}\\
&+\frac{2}{9}\frac{\langle ad\rangle^2\langle ae\rangle\langle bd\rangle[ae][bd]^2[de]^2}{s_{de}(s_{bc}-s_{ea})^2\langle bc\rangle^2\langle cd\rangle[bc]}+\frac{2}{9}\frac{\langle ad\rangle^3\langle ae\rangle[de]^3}{s_{de}\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^2[bc]}+5\frac{\langle ad\rangle^2\langle ea\rangle][ed]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}\\
&-\frac{\langle ac\rangle^3[ce]^3}{(s_{ab}-s_{de})\langle ab\rangle[bc]\langle cd\rangle[de]}\left(-\frac{1}{3}\frac{\langle ae\rangle[de]}{\langle ab\rangle\langle bc\rangle\langle cd\rangle}+\frac{2}{3}\frac{[bc][de]}{\langle bc\rangle\langle cd\rangle[ce]}\right)-10\frac{\langle ab\rangle^2[be]}{\langle be\rangle\langle bc\rangle\langle bd\rangle\langle cd\rangle}\\
&-\frac{1}{9}\frac{s_{cd}\langle ac\rangle^3[ce]^2(\langle ab\rangle^2][ab][be]-3\langle ab\rangle\langle cd\rangle[bc][de]+\langle ad\rangle\langle de\rangle[de]^2)}{(s_{ab}-s_{de})^2\langle ab\rangle^2\langle bc\rangle\langle cd\rangle^3[bc][de]}\\
&-5\frac{\langle ab\rangle^2\langle ae\rangle[ae][be]}{\langle be\rangle s_{ea}\langle bc\rangle\langle bd\rangle\langle cd\rangle}-\frac{2}{9}\frac{s_{cd}\langle ae\rangle^2\langle bc\rangle\langle bd\rangle[be]^3[cd]}{s_{ea}^2\langle be\rangle(s_{ea}-s_{cd})^2\langle cd\rangle^2}-\frac{1}{3}\frac{\langle ae\rangle\langle bc\rangle\langle bd\rangle[be]^3[cd]}{s_{ea}^2\langle be\rangle\langle cd\rangle^2[ae]}\\
&+\frac{1}{3}\frac{[be]^2[cd](\langle ad\rangle\langle ae\rangle\langle bc\rangle[ae]^2+\langle bd\rangle\langle cd\rangle(\langle bc\rangle[be][cd]-2\langle ae\rangle[ae][de]))}{\langle be\rangle(s_{ea}-s_{cd})\langle ae\rangle\langle cd\rangle^2[ae]^3}.
\end{aligned}
\tag{4.9.6}
$$

Comparing this with the rational part calculated in [1] we find that the two are in exact agreement and so we have managed to fully reconstruct the rational part of the two-loop five-point single-minus Yang-Mills amplitude.

4.10 Discussion

Looking more closely, we see that in fact the tree to two-loop easy channels and the two one-loop MHV currents calculated in Section (3.2), along with the rational descendants found at the end of section 4.7 generated all the distinct kinematic terms necessary to reconstruct the final result.

Starting with the tree to two-loop factorisation channels, the final coefficients on these terms in the ansatz are exactly those that were calculated. At least in the *ab* channel this is to be expected as it lacks a $\langle de\rangle$ pole hence there can be no further contributions from the

(de) channels. This means that the unknown terms from the loop on the left terms do not add to these kinematic functions.

The MHV contributions are the two that arise from the one-loop to one-loop factorisation channel. Focusing only on these terms, we see that while the kinematic functions are correct, the numerical coefficients are not, which indicates that the remaining contributions that arise from the two-loop splitting function to tree factorisation are still required to reconstruct the full result, however the final contribution from these will have the same kinematic functions.

What happens then to the two-loop splitting function terms? In the case of the one-loop single-minus current contribution, we have three terms,

$$-\frac{7}{12}\frac{\langle ca\rangle\langle da\rangle^2[de]^2}{s_{de}\langle ab\rangle\langle bc\rangle\langle cd\rangle^2}-\frac{223}{108}\frac{\langle da\rangle^2\langle ea\rangle[de]^3}{s_{de}^2\langle ab\rangle\langle bc\rangle\langle cd\rangle}+\frac{2}{3}\frac{\langle ac\rangle^3\langle db\rangle[bc][de]}{s_{de}\langle bc\rangle^2\langle cd\rangle^2\langle ea\rangle}. \tag{4.10.1}$$

The first two of these are kinematic functions which also appear in the one-loop MHV case and thus contribute to the final answer, while the third does not appear in the MHV functions, nor does it contribute to the final answer.

Moving on to the 'loop on the left terms', in both the l bubble case in Section 4.3 and the single-minus embedded triangle case, we only have two functions contribute which are the same two that contribute in the single-minus current case above. The contribution with the all-plus embedded triangle on the left and MHV current on the right in Section 4.6.1 only contributes to the leading double pole term in s_{de}.

Finally the rational descendants have the same coefficients in the final result relative to one another as were calculated so we do not expect the loop on the left terms to contribute to these kinematic functions. Since the rational descendants are taken from the published result of Badger et al [1], this sets the normalization of the final result presented above to be that of Badger et al. Many of the amplitudes used in this chapter and the previous use the normalisation set by Bern et al in [70], so there is some discrepancy between conventions, however since we are fitting coefficients in this chapter, it is of little consequence to the calculation at hand.

With all this in mind, we expect that the remaining unknown 'loop on the left' contributions should add give the same kinematic functions as were found in the one-loop to one-loop channel with the requisite coefficients that add to give the final result. As stated above, we were unable to fully determine the the contributions from the double-box and box-triangle diagrams, and there are some comments that should be made that point the road towards future work.

Using spinor helicity : the augmented recursion method presented in this thesis uses an axial gauge formalism to describe off-shell currents and vertices, and thus both it and the spinor helicity formalism are inherently 4 dimensional. Thus when we cast our loop integrals into this formalism, we lose the (-2ϵ) part of the loop momentum and thereby introduce $\mathcal{O}(\epsilon)$ errors. This was not a problem when this technique was used to calculate the all-plus amplitudes as all of the integrals were finite in ϵ and thus one could take $\epsilon \to 0$ and the errors would vanish. This is also true in the case of the one-loop MHV currents in the single-minus case. In both cases, there is however one exception. When calculating the 'square' terms in equations 3.3.64 and 3.3.65, we do not directly calculate them, but rather we sidestep this by comparing to the one-loop splitting function. Initially this was a matter of convenience as it was not clear how to directly evaluate the integral, however with the development of the A trick we can now proceed, We find that

$$C^{+-tri} + C^{-+tri} = -\frac{i}{3}c_\Gamma \frac{[de][dq][eq]}{[kq]^2} + ic_\Gamma \frac{91}{9}\frac{[de][q|P_{de}|q\rangle^2}{\langle dq\rangle\langle eq\rangle[kq]^2}, \tag{4.10.2}$$

where for this evaluation we left q completely unfixed.

The first term is what one obtains from the one-loop splitting function and what we used in our result, however we also get this second term. Given that this approximation using the splitting function has been successfully used in the past to reconstruct the rational part, we are confident that this term should not be present. The essential difference between this integral and the others in the one-loop MHV current section is the presence of $[q|\alpha\beta|q]$ in the numerator. In the other integrals, all terms in the numerator were of the form $[x|l|q\rangle$ where l is the loop momentum, so the integrals were effectively 'scalar' as after Feynman parameterisation there were no terms with p^2 in the numerator since they would be proportional to $\langle qq\rangle = 0$. This is no longer true for the square terms. In a triangle integral, terms of order p^2 in the numerator will produce a factor of $\Gamma(\epsilon)$ which diverges as $\epsilon \to 0$ and thus requires expansion. The errors of order ϵ due to use of spinor helicity can now multiply by the ϵ pole of the Gamma function and become finite, contributing to the rational piece. Furthermore, when carrying out the calculation, the desired correct term come the scalar p^0 part of the integral, and the undesirable term from the tensor p^2 parts. Similarly, if we take the single-minus embedded triangle and take the 'unembedded limit' $P^2 \to 0$, then the 'scalar parts' of the integrals $\{C_1, ..., C_7\}$ reproduce the correct splitting function term.

Looking at the contributions from the single-minus current we quoted above, we see a third term that does not feature in the final result. The similarity between the square pole integral and the integrals from the single-minus calculation suggest that this extra term may also be an artefact of the spinor helicity formalism, but without access to the full completed calculation it is difficult to say for certain. Of course this also shows that the presence of poles in certain integrals is not necessarily an insurmountable obstacle if we can isolate and control them possibly evaluating them by other means such as in the square pole case above. If we can find a way to rewrite the loop on the left calculation so that the terms we require for the rational part are written in terms of integrals that are finite in epsilon (plus potentially some terms like the square pole term above that can be evaluated by appealing to other known calculations) then we may be able to see more clearly the errors that spinor helicity introduces and potentially be able to work around it.

Complexity of loop on the left calculations: the two new structures that appear in the loop on the left sector of the calculation are too complex for us to be able to solve, and yet our analysis suggests that the final result should be as simple as the one-loop MHV current calculations, which we can solve easily. This hints that there may be some underlying structure or some way of rewriting these terms to simplify the calculation. One notable aspect of the calculation is that the currents now have three off-shell legs. This means we now need to consider poles due to triple collinear limits in which all three legs go collinear as well as the sub-leading poles in which two of the legs go collinear, and incorporate these into a base tree amplitude to generate the 'good enough current'. While the technology required to incorporate poles from double collinear limits into currents is by now well understood and developed to a near algorithmic process, the same is definitely not true in the triple collinear case, and given that triple collinear limits are noticeably more complex than their double counterparts, this is far from a trivial generalisation. One avenue of future work would be to develop a process by which we incorporate these poles into the triply off-shell tree currents. A better understanding of the full current - and thus the full integral - may be helpful in finding ways to tackle the full calculation.

Integration methods: We introduced the 'A trick' in the previous chapter and put it to great use in this chapter to allow us to solve or uncover the structure of many of the loop on the left integrals. Terms such as $[x|\beta|q\rangle$ will appear in the denominator of complex integrals and while Mellin-Barnes methods have been used for years in the evaluation of Feynman integral, the method we have developed allows us to treat such a term as being in the numerator rather than by splitting it as two propagators. It is far more natural to treat such a term as a numerator in Feynman integral and so this method allows us to solve such integrals without having to get caught in the details of the Mellin-Barnes contour integrals. The method swaps the integral with infinite sums that must be resolved and in this chapter and the previous, we have shown in many cases how one can use special functions such as hypergeometric and Appell functions and their properties to simplify these expressions down to a point where we are able to extract the rational part. We derived new hypergeometric identities in Eqs 3.4.16 and 4.5.64 that allow us to resolve these sums and are of course generic results that to the best of our knowledge are new identities.

There is of course still work to be done. As of now the double-box and box-triangle integrals, as well as the integrals over the single-minus triangle are too complex to be directly tackled, and in particular the presence of the four-point axial gauge vertex in the box-triangle diagram poses a challenge. Thus a worthwhile further avenue of research would be to study such integrals and extend the methods of integration to allow the evaluation of such integrals. We are currently forced to fully fix the axial reference vector q in order to avoid spurious poles at the recursion stage, and to simplify integrals enough to be workable. A further development in integration methods, or finding a way to remove such spurious poles before recursion could allow for q to remain at least partially unfixed such that it then may be possible to fix it in order to simplify or fully eliminate the four-point vertices. Being able to leave q at least partially unfixed for the full calculation would also provide a very powerful check on the calculation as was done in the past as q independence is highly non-trivial and brings together many different contributions across the full calculation.

Scaling to higher multiplicity: In going from the five-point single-minus to six or seven points would be add additional external legs with positive helicity. This would affect the currents but leave the 'left hand side' unchanged. We would go from a googly 5 pt current to a 6 pt NMHV current, while the MHV currents would simply become 6 pt MHV currents and so on. In the NMHV cases again one would simply go from a 6 pt NMHV to a 7 pt NMHV. In moving to tree currents with higher multiplicity, we see that while the expressions become longer, no new kinematic structures emerge. When carrying out the integrals in this chapter we see that the limiting factor in terms of complexity is the presence of terms like $[x|\alpha|q\rangle$ and $[y|\beta|q\rangle$ in the denominator. While we can use the A trick to raise as many of these as we like to the denominator, each use of this method introduces a further two infinite sums to the expression which must be resolved. We have demonstrated in Eq 4.5.36 that we can use Schouten's identity to 'partial fraction' such terms in the denominator until we are left with one $[x|\alpha|q\rangle$ and one $[y|\beta|q\rangle$. In this chapter we have made in progress in tackling such an integral, and thus at higher multiplicity these methods should still hold up. We showed that one only needs the one-loop MHV channel terms plus the rational descendants to reconstruct the full rational part using universal properties of the amplitudes, and it would be interesting to see if this holds at higher multiplicity. To analyse this we would first need the cut-constructible part of the higher point amplitudes and so far none of these have been calculated.

4.11 Conclusion

In this chapter we continued our calculation of the rational part of the two-loop five-point single-minus Yang-Mills amplitude. We introduced the 'loop on the left' sector of the augmented recursion method which is new to this calculation and showed the three structures that exist in this sector to be calculated. We were able to fully determine one - the l bubble - while we were unable to calculate the final two - the double-box and the box-triangle - due to the complexity to these structures. Nonetheless we made progress by evaluating parts of a particular limit of the double-box structure, which we dubbed the 'embedded triangles'. With all these various pieces, we completed the process of BCFW recursion to get their contributions to the rational part of the amplitude. We then moved on to the last piece of the puzzle by calculating the rational descendants of the cut-constructible parts of the amplitude.

We further developed and demonstrated the power of the A trick which was introduced in the previous chapter as a method of integration. We demonstrate its usefulness and derive new identities concerning hypergeometric functions that aid in the use of this method.

While we were unable to complete the entire calculation, we showed that in fact a small subsection of the augmented recursion calculation - the loop MHV current contributions from the previous chapter - which stems from the one-loop to one-loop factorisation of amplitude, along with the rational descendants generated all the necessary kinematic functions needed to fully reconstruct the rational part of the amplitude by imposing flip symmetry and collinear limits, both of which are known properties of the amplitude. This is a surprising result as it implies an unexpected simplicity in the contributions from the terms that stem from the two-loop splitting function to tree amplitude factorisation channel, which comprises the single-minus one-loop current contribution plus the loop on the left sector.

We finish by discussing the method of augmented recursion as it applies in the context of this calculation and suggest further avenues of research which are unfortunately beyond the scope of this thesis due to time constraints. We argue that - once complete - all signs indicate that this method should scale well at higher multiplicity as is the goal in developing this method.

Chapter 5

Conclusion

In this thesis we took major steps towards the development of a method to calculate two-loop single-minus Yang-Mills amplitudes using 4 dimensional unitarity and recursion. This method, once fully completed should provide a pathway to ready calculation of two-loop single-minus amplitudes at higher multiplicities.

Taking the five-point amplitude as an example, we explored the cut-constructible part of the amplitude using 4D unitarity. We summarised the full method, but with a particular focus on the pseudo one-loop subsector of this part of the amplitude. In particular we note that this one-loop subsector alone generates the leading IR singularities of the full amplitude. In this calculation we also developed a new parameterisation that allows the determination of one- and two-mass triangle coefficients in one-loop integral reduction, and we extended the method of canonical forms that allow us to calculate the coefficients of scalar bubble integrals.

The bulk of the thesis focused on the extension of the augmented recursion method to calculate the rational part of the amplitude. We began by calculating the 'tree on the left' subsector, and introduced the A trick as a method of solving Feynman integrals. We then moved on to the 'loop on the left' subsector, which we were able to partially calculate, further developing the A trick in the process. Finally we calculated the rational descendants of the cut-constructible part of the amplitude. From the various pieces we were able to reconstruct the rational part of the five-point amplitude by imposing universal known properties of amplitudes.

The method developed in this thesis was an extension of a method previously employed to calculate two-loop all-plus amplitudes. We saw in this thesis that the necessary extensions were significant, requiring the addition of the two-loop sector in the cut-constructible part of the method, and the addition of a second double pole factorisation channel in the rational part which added to the tree on the left subsector, and generated the new loop on the left subsector. We also had to include rational descendants of the cut-constructible part for the first time. The addition of these new parts of the calculation clearly displays the gulf in complexity when moving from the all-plus to the single-minus sectors.

Given the complexity of these amplitudes, and the time it takes to calculate these using the master integral method - as stated earlier the 5 point result was first published in 2018 while the 6 point amplitude has still not been completed - it is clear that there is much appetite for a method that is amenable to generalisation to higher numbers of external gluons as we have developed in this thesis. In extending this method to six points and beyond, there would be one additional two-loop structure in the cut-constructible part of the amplitude, whose calculation is in progress. For the rational part, there are no indications that any new or more complex structures will arise at higher multiplicity and the integration methods developed in this thesis are expected to hold. It will be of particular interest to see if it is still possible to reconstruct the full rational part of the amplitude without fully calculating the loop on the left structures, however work is in progress to directly calculate these terms which would further strengthen the arguments of the method.

Appendix A

One-Loop Structures

The following structures contribute to multiple double cuts of the genuine two-loop parts of the amplitude, with the loop insert being indicated by the vertex in the corresponding figure. While the loop insertion is not directly relevant to the calculation of such terms, it is useful to keep track of when checking the cuts and flip symmetries. The propagators are left out of the equations but are indicated by the corresponding figure. There is also a Parke-Taylor denominator factorised out $1/(\langle a\,b\rangle\,\langle b\,c\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle e\,a\rangle)$.

$$\begin{aligned} Ib_1 = & \frac{(P_1+d)^2\,\langle a\,d\rangle^2\,\langle a\,c\rangle\,[a|P_1|c\rangle[c|(P_1+P_{de})|a\rangle\,\langle e\,a\rangle}{6\,\langle c\,d\rangle\,\langle d\,e\rangle} \\ & - \frac{\langle a\,b\rangle\,\langle e\,a\rangle\,s_{bc}[b|c|d\rangle[c|P_1|a\rangle[b|(P_1+P_{de})|c\rangle}{6\,[a\,b]\,\langle d\,e\rangle} \end{aligned} \tag{A.0.1}$$

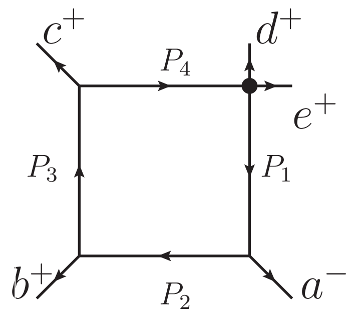


Figure A.1: Insert diagram corresponding to Ib_1. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_2 = -\frac{\langle a\,b\rangle^2 \langle a\,c\rangle^2 [a|P_1|c\rangle[c|(P_1+P_{de})|a\rangle\,[b\,c]}{6\,\langle b\,c\rangle} + \frac{\langle a\,b\rangle\,\langle a\,c\rangle^2 \langle a\,d\rangle\,[a|P_1|c\rangle[c|(P_1+P_{de})|a\rangle\,[b\,c]}{6\,\langle c\,d\rangle} \tag{A.0.2}$$

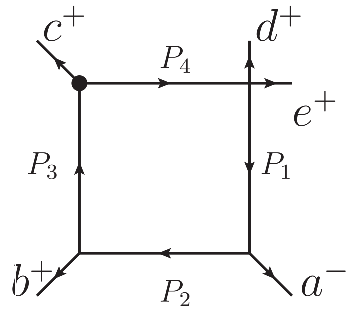


Figure A.2: Insert diagram corresponding to Ib_2. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_3 = \frac{(P_1+b)^2 \langle a\,c\rangle^2 \langle a\,d\rangle\,[a|(P_1+P_{bc})|d\rangle[d|P_1|a\rangle\,\langle b\,a\rangle}{6\,\langle b\,c\rangle\,\langle c\,d\rangle} - \frac{\langle e\,a\rangle\,\langle a\,b\rangle\,s_{de}[e|d|c\rangle[d|(P_1+P_{bc})|a\rangle[e|P_1|d\rangle}{6\,[e\,a]\,\langle b\,c\rangle} \tag{A.0.3}$$

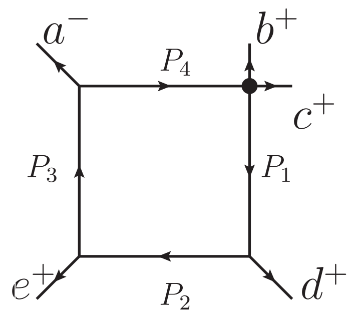


Figure A.3: Insert diagram corresponding to Ib_3. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_4 = -\frac{\langle e\,a\rangle^2 \langle a\,d\rangle^2 [d|P_1|a\rangle [a|(P_1+P_{bc})|d\rangle [d\,e]}{6\,\langle d\,e\rangle} + \frac{\langle e\,a\rangle \langle a\,d\rangle^2 \langle a\,c\rangle [a|(P_1+P_{bc})|d\rangle [d|P_1|a\rangle [d\,e]}{6\,\langle c\,d\rangle} \tag{A.0.4}$$

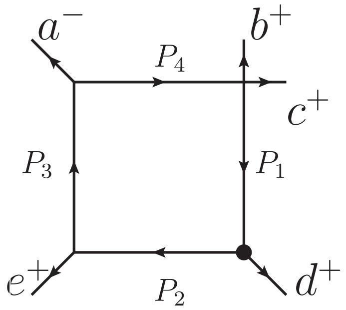


Figure A.4: Insert diagram corresponding to Ib_4. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_5 = \frac{\langle e\,a\rangle^2 \langle c\,e\rangle [e|(P_1+P_{ab}|a\rangle [e|P_1+P_{ab}|a\rangle [c|P_1|d\rangle}{6\,\langle d\,e\rangle} \tag{A.0.5}$$

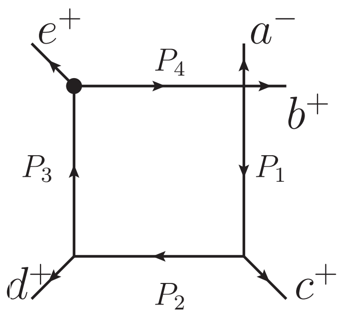


Figure A.5: Insert diagram corresponding to Ib_5. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_6 = -\frac{\langle a\, c\rangle^2 \langle d\, e\rangle\, [c|P_1|a\rangle[c|P_1|a\rangle[e|(P_1+P_{ab})|c\rangle}{6\,\langle c\, d\rangle} + \frac{\langle a\, b\rangle \langle a\, c\rangle \langle d\, e\rangle\, [d|P_1|a\rangle[c|P_1|a\rangle[e|(P_1+P_{ab})|c\rangle}{6\,\langle b\, c\rangle} \quad \text{(A.0.6)}$$

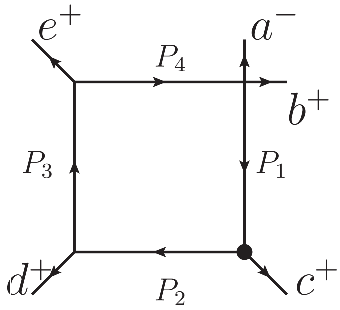


Figure A.6: Insert diagram corresponding to Ib_6. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_7 = -\frac{\langle a\, b\rangle \langle a\, c\rangle \langle e\, a\rangle\, [a|P_1|a\rangle[c|P_1|a\rangle[e|(P_1+P_{ab})|c\rangle}{6\,\langle b\, c\rangle} \quad \text{(A.0.7)}$$

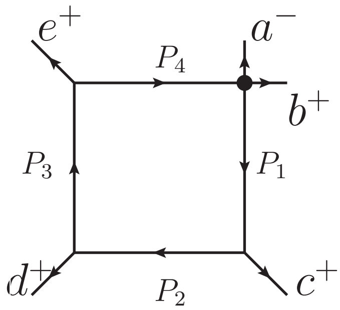


Figure A.7: Insert diagram corresponding to Ib_7. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_8 = \frac{\langle a\, b\rangle^2 \langle d\, b\rangle\, [b|P_1|a\rangle[b|P_1|a\rangle[d|(P_1+P_{ea})|c\rangle}{6\,\langle b\, c\rangle} \quad \text{(A.0.8)}$$

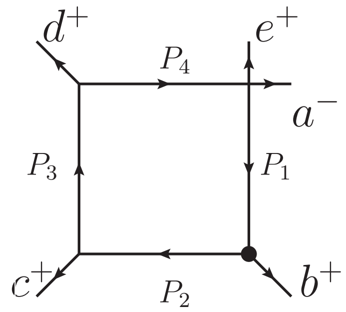


Figure A.8: Insert diagram corresponding to Ib_8. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_9 = \frac{\langle a\,d\rangle^2 \langle b\,c\rangle\,[d|(P_1 + P_{ea})|a\rangle[d|(P_1 + P_{ea})|a\rangle[b|P_1|d\rangle}{6\,\langle c\,d\rangle} + \frac{\langle e\,a\rangle\,\langle a\,d\rangle\,\langle b\,c\rangle\,[c|(P_1 + P_{ea})|a\rangle[d|(P_1 + P_{ea})|a\rangle[b|P_1|d\rangle}{6\,\langle d\,e\rangle} \tag{A.0.9}$$

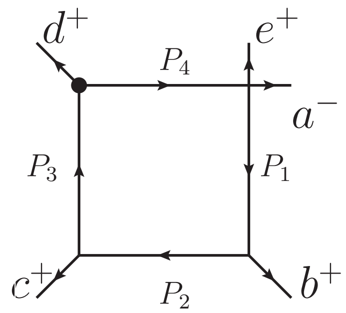


Figure A.9: Insert diagram corresponding to Ib_9. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{10} = -\frac{\langle e\,a\rangle\,\langle a\,d\rangle\,\langle a\,b\rangle\,[a|(P_1 + P_{ea})|a\rangle[d|(P_1 + P_{ea})|a\rangle[b|P_1|d\rangle}{6\,\langle d\,e\rangle} \tag{A.0.10}$$

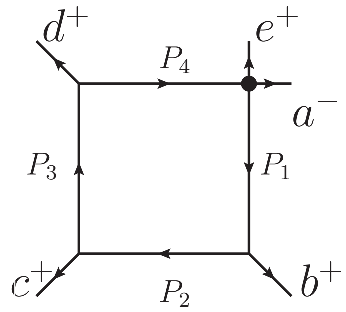


Figure A.10: Insert diagram corresponding to Ib_{10}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{11} = \frac{\langle a\, b\rangle^2}{6\,\langle b\, c\rangle}\Bigg(\langle e\, c\rangle\, [e|P_1|a\rangle[a|b|a\rangle[b|(P_1+P_{cd})|a\rangle + [e|P_1|e\rangle\, \langle e\, c\rangle\, [e|b|a\rangle[b|(P_1+P_{cd})|a\rangle$$
$$- (P1)^2\, \langle e\, c\rangle\, [e|b|a\rangle[b|P_1+P_{cd}|a\rangle + [b|(P_1+P_{cd})|b\rangle\, \langle e\, c\rangle\, [e|P_1|a\rangle[b|(P_1+P_{cd})|a\rangle$$
$$- [e|P_1|e\rangle\, \langle b\, c\rangle\, [b|(P_1+P_{cd})|a\rangle[b(P_1+P_{cd}|a\rangle\Bigg) \tag{A.0.11}$$

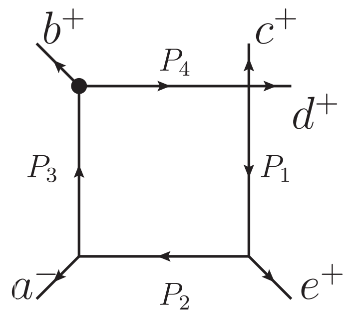


Figure A.11: Insert diagram corresponding to Ib_{11}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{12} = \frac{\langle a\,b\rangle^2 \langle e\,c\rangle\,[b|(P_1+P_{cd})|a\rangle[b|(P_1-e)|c\rangle[c|P_1|d\rangle[e|P_1|c\rangle}{6\,\langle b\,c\rangle\,\langle c\,d\rangle\,[a\,b]}$$
$$+\frac{\langle e\,a\rangle^2 \langle b\,d\rangle\,[e|P_1|a\rangle[e|(P_1-a)|d\rangle}{12\,\langle c\,d\rangle\,\langle d\,e\rangle\,[e\,a]}$$
$$\times\Bigg(-(P_4)^2[b|P_1|c\rangle + \big((P_1+c)^2-[d|P_1|d\rangle\big)[b|(P_1+P_{cd})|c\rangle + [b|(P_1+P_{cd})P_1 d|c\rangle\Bigg)$$
$$-\frac{\langle a\,b\rangle\,\langle e\,a\rangle\,\langle b\,c\rangle\,\langle d\,e\rangle\,[b\,e]^3}{6\,\langle c\,d\rangle\,[a\,b]\,[e\,a]}\Bigg(((P_1)^2-(P_2)^2)[b|P_1|b\rangle-(P_1)^2[e|b|e\rangle+[b|P_1|e\rangle[e|P_{cd}|b\rangle\Bigg) \tag{A.0.12}$$

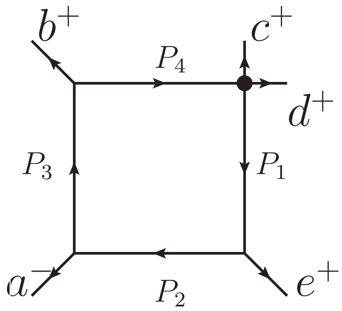


Figure A.12: Insert diagram corresponding to Ib_{12}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{13} = \frac{\langle e\,a\rangle^2\,[e|P_1|a\rangle}{6\,\langle d\,e\rangle}\times\Bigg(-(P_1)^2\,\langle a\,b\rangle\,[b|(P_1+P_{cd})|d\rangle$$
$$+(P_2)^2\big(\langle b\,d\rangle\,\langle e\,a\rangle\,[e\,b]+\langle a\,b\rangle\,[b|(P_1+P_{cd})|d\rangle\big)$$
$$-\langle e\,a\rangle\,\big(\langle b\,d\rangle\,[a\,b]\,[e|P_1|a\rangle+((P_3)^2-(P_4)^2)[e|P_1|d\rangle\Bigg) \tag{A.0.13}$$

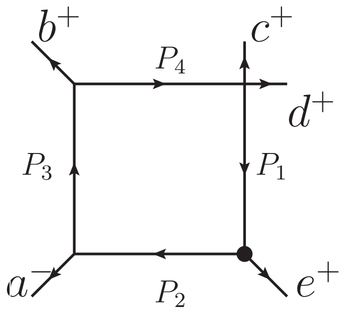


Figure A.13: Insert diagram corresponding to Ib_{13}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{14} = \frac{\langle a\,c\rangle^2\,[c|P_1|a\rangle}{6\,\langle b\,c\rangle\,\langle c\,d\rangle}\Big(\,\langle b\,c\rangle\,(-\,\langle c\,d\rangle\,[c|P_1|a\rangle + \langle d\,a\rangle\,[c|(P_1 + P_{ab}|c\rangle$$

$$+\,\langle a\,b\rangle\,\langle c\,d\rangle\,([d|(P_1 + P_{ab})|d\rangle + [e|(P_1 + P_{ab}|e\rangle\Big) \tag{A.0.14}$$

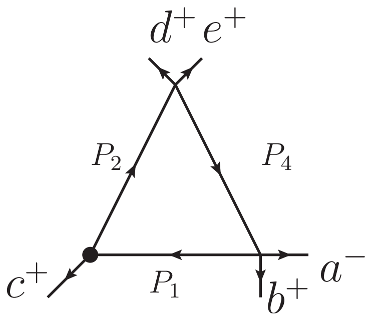


Figure A.14: Insert diagram corresponding to Ib_{14}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{15} = -\frac{\langle e\,a\rangle\,[c|P_1|a\rangle}{6s_{de}\,\langle d\,e\rangle\,[a\,b]}\Big(2\,\langle d\,e\rangle\,[b|(P_1 + P_{ab})|c\rangle[e|(P_1 + P_{ab}|a\rangle[e|(P_1 + P_{ab})|e\rangle$$

$$-\,s_{de}\,\langle a\,c\rangle\,[a\,b]\,\big(\,\langle d\,e\rangle\,[e|(P_1 + P_{ab}|a\rangle - 2\,\langle e\,a\rangle\,[e|(P_1 + P_{ab})|d\rangle$$

$$+\,\langle d\,a\rangle\,(s_{de} + [e|(P_1 + P_{ab})|e\rangle)\big)$$

$$+\,s_{de}\,\langle a\,d\rangle\,(\langle d\,a\rangle\,\langle e\,c\rangle\,[a\,b]\,[d\,e] + 2[b|(P_1 + P_{ab})|c\rangle[e|(P_1 + P_{ab})|e\rangle))\Big) \tag{A.0.15}$$

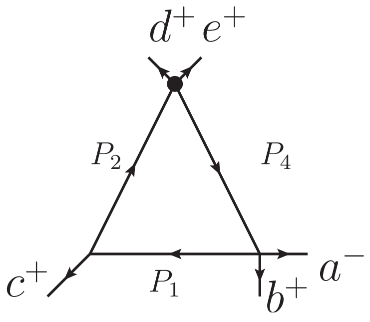


Figure A.15: Insert diagram corresponding to Ib_{15}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{16} = \frac{\langle e\,a\rangle}{6\,\langle c\,d\rangle\,\langle d\,e\rangle\,[a\,b]}\Big(\langle a\,c\rangle\,\langle a\,d\rangle^2\,\langle a\,e\rangle\,[a\,b]\,[d\,e]\,[a|P_1|d\rangle + s_{bc}\,\langle a\,b\rangle\,\langle c\,d\rangle^2\,[b\,c]\,[b|P_1|a\rangle\Big) \tag{A.0.16}$$

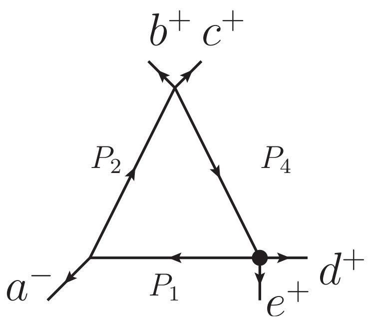


Figure A.16: Insert diagram corresponding to Ib_{16}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{17} = -\frac{\langle a\,b\rangle}{6\,\langle b\,c\rangle}\left(\frac{\langle a\,b\rangle\,\langle a\,c\rangle^2\,\langle a\,d\rangle\,[b\,c]\,[a|P_1|c\rangle}{\langle c\,d\rangle} + \frac{s_{de}\,\langle e\,a\rangle\,\langle c\,d\rangle\,[d\,e]\,[e|P_1|a\rangle}{[e\,a]}\right) \tag{A.0.17}$$

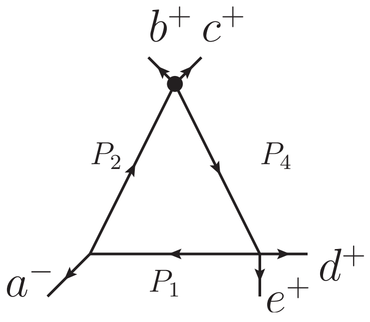


Figure A.17: Insert diagram corresponding to Ib_{17}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{18} = \frac{\langle e\,a\rangle^2\,[e|P_1|a\rangle^2}{6} \tag{A.0.18}$$

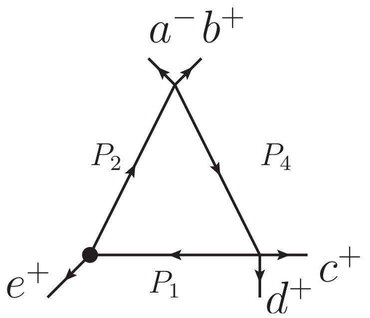


Figure A.18: Insert diagram corresponding to Ib_{18}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{19} = \frac{\langle a\,b\rangle\,\langle e\,a\rangle\,[b|(P_1 + P_{cd})|a\rangle[e|P_1|a\rangle}{6} \tag{A.0.19}$$

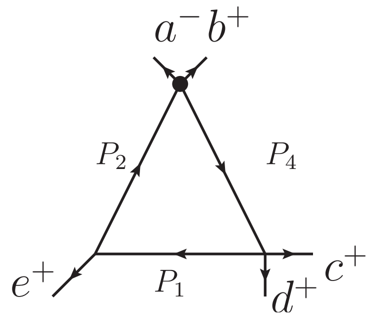


Figure A.19: Insert diagram corresponding to Ib_{19}. The vertex on the massive corner indicates the one-loop insertion.

$$
\begin{aligned}
Ib_{20} = &-\frac{1}{6s_{cd}\,\langle b\,c\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,[a\,b]\,[e\,a]} \\
&\Big(\,\langle a\,c\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,[e\,a]\,[b|(P_1+P_{cd})|e\rangle[d|(P_1+P_{cd}|a\rangle[e|P_1|d\rangle \\
&\qquad\times(\langle b\,c\rangle\,[c|(P_1+P_{cd}|a\rangle-\langle a\,b\rangle\,[c|(P_1+P_{cd}|c\rangle) \\
&+s_{cd}\,\langle b\,c\rangle\,[b|(P_1+P_{cd})|d\rangle[e|P_1|a\rangle \\
&\qquad\times\big(\,\langle d\,a\rangle\,s_{ea}\,\langle d\,e\rangle\,[d|(P_1+P_{cd}|c\rangle-\langle c\,d\rangle\,\langle e\,a\rangle\,(\langle d\,e\rangle\,[c\,e]\,[d|P_1|c\rangle+s_{ea}[d|(P_1+P_{cd})|d\rangle)\big) \\
&+s_{cd}\,\langle c\,a\rangle\,\langle d\,e\rangle\,[a\,b]\,[e\,a] \\
&\qquad\times\big(\,\langle a\,d\rangle\,\langle a\,b\rangle\,\langle c\,e\rangle\,[d|(P_1+P_{cd})|a\rangle[e|P_1|d\rangle \\
&\qquad\quad+\langle e\,a\rangle\,\langle c\,d\rangle\,(\langle b\,c\rangle\,[c|(P_1+P_{cd})|a\rangle[e|P_1|a\rangle+\langle a\,b\rangle\,[d|(P_1+P_{cd})|a\rangle[e|P_1|d\rangle)\big)\Big)
\end{aligned}
\tag{A.0.20}
$$

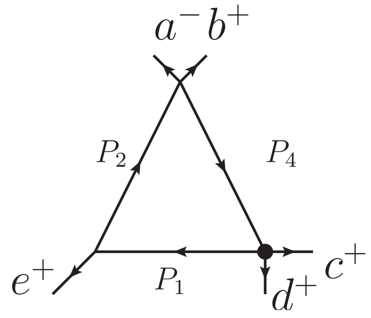


Figure A.20: Insert diagram corresponding to Ib_{20}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{21} = \frac{\langle d\,a\rangle^2 [d|P_1|a\rangle(\langle e\,a\rangle \langle c\,d\rangle ([b|(P_1 + P_{bc})|b\rangle + [c|(P_1 + P_{bc}|c\rangle)}{6\,\langle c\,d\rangle \langle d\,e\rangle} - \frac{\langle d\,a\rangle^2 [d|P_1|a\rangle(\langle c\,d\rangle [d|P_1|a\rangle + \langle c\,a\rangle [d|(P_1 + P_{bc})|d\rangle)}{6\,\langle c\,d\rangle} \tag{A.0.21}$$

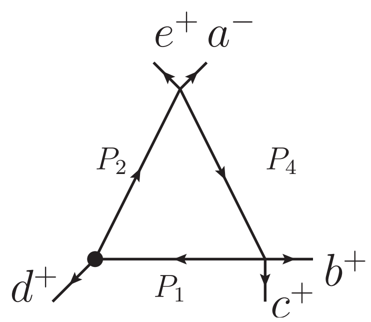


Figure A.21: Insert diagram corresponding to Ib_{21}. The vertex on the massive corner indicates the one-loop insertion.

$$\begin{aligned} Ib_{22} = \frac{\langle a\,b\rangle [d|P_1|a\rangle}{6s_{bc}\,\langle b\,c\rangle [e\,a]}\Big(& - 2\,\langle b\,c\rangle [b|(P_1 + P_{bc}|a\rangle[b|(P_1 + P_{bc}|b\rangle[e|(P_1 + P_{bc})|d\rangle \\ & - s_{bc}\,\langle a\,c\rangle^2 \langle b\,d\rangle [e\,a] [b\,c] \\ & - s_{bc}\,\langle a\,d\rangle [e\,a] (\langle b\,c\rangle [b|(P_1 + P_{bc}|a\rangle - 2\,\langle a\,b\rangle [b|(P_1 + P_{bc})|c\rangle) \\ & + s_{bc}\,\langle a\,c\rangle \left(\langle a\,d\rangle [e\,a] (s_{bc} - [b|(P_1 + P_{bc})|b\rangle) + 2[b|(P_1 + P_{bc})|b\rangle[e|(P_1 + P_{bc})|d\rangle \right)\Big) \end{aligned} \tag{A.0.22}$$

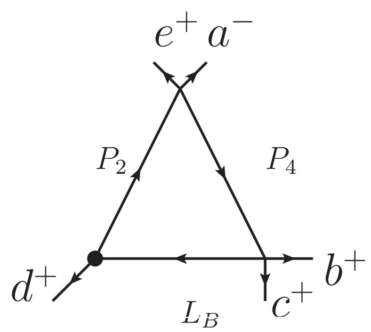


Figure A.22: Insert diagram corresponding to Ib_{22}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{23} = -\frac{\langle a\, b\rangle^2\, [b|P_1|a\rangle^2}{6} \tag{A.0.23}$$

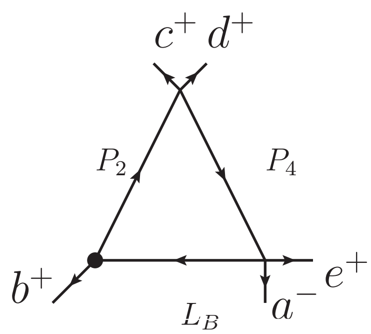


Figure A.23: Insert diagram corresponding to Ib_{23}. The vertex on the massive corner indicates the one-loop insertion.

$$Ib_{24} = \frac{\langle a\, b\rangle\, \langle e\, a\rangle\, [b|P_1|a\rangle[e|P_1|a\rangle}{6} \tag{A.0.24}$$

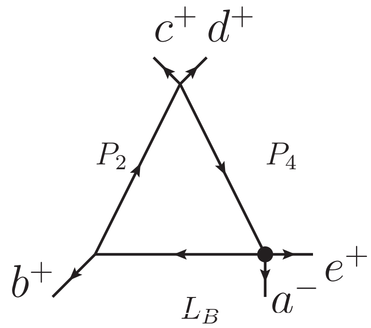


Figure A.24: Insert diagram corresponding to Ib_{24}. The vertex on the massive corner indicates the one-loop insertion.

$$
\begin{aligned}
Ib_{25} = &-\frac{1}{6s_{cd}\langle b\,c\rangle\langle c\,d\rangle\langle d\,e\rangle[a\,b][e\,a]} \\
&\Big(\langle a\,d\rangle\langle b\,c\rangle\langle c\,d\rangle[a\,b][b|P_1|e\rangle[c|(P_1+P_{ea}|a\rangle[e|(P_1+P_{ea})|b\rangle \\
&\qquad\times(\langle d\,e\rangle[d|(P_1+P_{ea}|a\rangle-\langle e\,a\rangle[d|(P_1+P_{ea}|d\rangle) \\
&+s_{cd}\langle c\,a\rangle\langle b\,c\rangle[a\,b] \\
&\qquad\times\big(\langle a\,d\rangle\langle b\,d\rangle s_{ea}[b|P_1|c\rangle[c|(P_1+P_{ea})|a\rangle \\
&\qquad+\langle a\,b\rangle\langle d\,e\rangle[b|P_1|a\rangle[c|(P_1+P_{ea})|d\rangle[e|(P_1+P_{ea})|c\rangle\big) \\
&-s_{cd}\langle c\,d\rangle\langle a\,b\rangle\langle a\,d\rangle\langle b\,c\rangle[a\,b][e\,a] \\
&\qquad\times(\langle e\,a\rangle[b|P_1|c\rangle[c|(P_1+P_{ea})|a\rangle+\langle d\,e\rangle[b|P_1|a\rangle[d|(P_1+P_{ea})|a\rangle) \\
&+s_{cd}\langle c\,d\rangle\langle a\,b\rangle\langle d\,e\rangle[b|P_1|a\rangle \\
&\qquad\times(s_{ab}[c|(P_1+P_{ea})|c\rangle[e|(P_1+P_{ea})|c\rangle+\langle b\,c\rangle[b\,d][c|(P_1+P_{ea})|d\rangle[e|(P_1+P_{ea}|c\rangle)\Big)
\end{aligned}
\tag{A.0.25}
$$

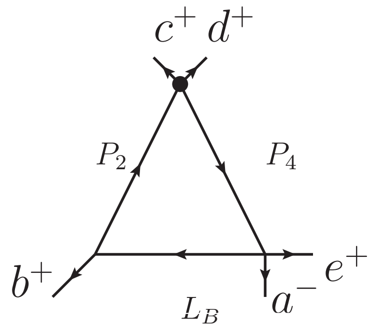


Figure A.25: Insert diagram corresponding to Ib_{25}. The vertex on the massive corner indicates the one-loop insertion.

The remaining inserts are either one-mass triangles or one-mass bubbles, both of which only contribute to one channel on a four-dimensional double cut. As an example, for $A_4^{(0)}(a^-, b^+, \ell_1, \ell_2) \times \ldots$ we denote the insert It_{ab} whereas the $A_4^{(1)}(a^-, b^+, \ell_1, \ell_2) \times \ldots$ is denoted $I\ell_{ab}$. We define ℓ_1 and ℓ_2 to be going away from the massless corners. We present the insert as they appear on the cut, with cut propagators removed and, in the case of the one-mass triangle, the uncut propagator remaining. We still factorise out the Parke-Taylor denominator.

$$I\ell_{ab} = -\frac{\langle a\,b\rangle^2 \langle a\,\ell_1\rangle \langle \ell_2\,c\rangle [\ell_2|\ell_1|a\rangle}{6\,\langle b\,c\rangle \langle b\,\ell_1\rangle} \tag{A.0.26}$$

$$
\begin{aligned}
&It_{ab}\\
&= \frac{1}{6}\Bigg(\frac{\langle b\,a\rangle\,\langle c\,a\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_2\,d\rangle\,[b\,\ell_1]}{\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle} + \frac{\langle a\,c\rangle\,\langle a\,e\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,[c\,e]}{\langle d\,e\rangle}\\
&- \frac{\langle a\,c\rangle\,\langle b\,a\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_2\,c\rangle\,[c\,\ell_1]}{\langle b\,c\rangle\,\langle \ell_1\,\ell_2\rangle} - \frac{\langle a\,c\rangle\,\langle b\,a\rangle\,\langle d\,b\rangle\,\langle \ell_1\,a\rangle^3\,\langle \ell_2\,c\rangle^2\,[c\,\ell_1]}{\langle b\,\ell_1\rangle\,\langle c\,b\rangle\,\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,a\rangle}\\
&+ \frac{\langle b\,c\rangle\,\langle c\,d\rangle\,\langle e\,a\rangle\,\langle \ell_2\,a\rangle\,[b\,\ell_2]\,[c\,b]\,[c\,\ell_2]}{\langle d\,e\rangle\,[a\,b]\,[a\,\ell_2]} + \frac{\langle a\,c\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_2\,d\rangle^2\,[d\,c]}{\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle}\\
&- \frac{\langle a\,c\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,d\rangle\,[d\,e]}{\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle} - \frac{\langle a\,c\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_2\,d\rangle^2\,[d\,e]}{\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle}\\
&+ \frac{\langle b\,a\rangle\,\langle c\,e\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle^3\,\langle \ell_2\,d\rangle^3\,[d\,e]}{\langle b\,\ell_1\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,a\rangle\,\langle \ell_2\,\ell_1\rangle} + \frac{\langle a\,d\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,[d\,\ell_1]}{\langle d\,e\rangle}\\
&- \frac{\langle a\,c\rangle\,\langle d\,a\rangle^2\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,d\rangle\,[d\,\ell_1]}{\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_2\,a\rangle} + \frac{\langle b\,d\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,\langle \ell_1\,c\rangle\,\langle \ell_1\,d\rangle\,\langle \ell_2\,a\rangle\,[b\,e]\,[d\,\ell_1]}{\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle\,[a\,e]}\\
&- \frac{\langle a\,c\rangle\,\langle b\,a\rangle\,\langle c\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[d\,\ell_1]}{\langle b\,c\rangle\,\langle c\,d\rangle\,[c\,d]} - \frac{\langle a\,b\rangle\,\langle a\,\ell_1\rangle\,\langle e\,a\rangle\,\langle \ell_2\,a\rangle\,[e\,\ell_2]}{\langle b\,\ell_1\rangle}\\
&+ \frac{\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle^2\,[e\,\ell_2]}{\langle \ell_1\,\ell_2\rangle} + \frac{\langle b\,e\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,d\rangle\,\langle \ell_2\,a\rangle^2\,[e\,\ell_2]}{\langle b\,\ell_1\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle}\\
&+ \frac{\langle a\,d\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,\langle \ell_2\,e\rangle\,[e\,\ell_2]}{\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle} - \frac{\langle c\,a\rangle^2\,\langle \ell_1\,a\rangle^2\,\langle \ell_2\,d\rangle\,[\ell_1\,c]}{\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle}\\
&- \frac{\langle b\,c\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,d\rangle^3\,\langle \ell_2\,a\rangle^2\,[\ell_1\,d]}{\langle b\,\ell_1\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle^2} + \frac{\langle a\,d\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_1\,\ell_2]}{\langle d\,e\rangle}\\
&+ \frac{\langle a\,b\rangle^2\,\langle c\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,c\rangle\,\langle \ell_2\,d\rangle\,[\ell_2\,b]}{\langle b\,c\rangle\,\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle} + \frac{\langle a\,\ell_2\rangle\,\langle c\,a\rangle\,\langle e\,a\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[\ell_2\,b]}{\langle d\,e\rangle\,[a\,b]}\\
&- \frac{\langle c\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[d\,\ell_1]\,[\ell_2\,b]}{\langle c\,d\rangle\,[a\,b]\,[c\,d]} - \frac{\langle b\,a\rangle\,\langle c\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,c\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[d\,\ell_1]\,[\ell_2\,b]}{\langle b\,c\rangle\,\langle c\,d\rangle\,[a\,b]\,[c\,d]}\\
&- \frac{\langle b\,c\rangle\,\langle e\,a\rangle\,\langle \ell_1\,d\rangle^3\,\langle \ell_2\,a\rangle^3\,[\ell_2\,d]}{\langle b\,\ell_1\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle^2} - \frac{\langle b\,c\rangle\,\langle c\,d\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle^2\,[c\,\ell_2]^2}{\langle b\,\ell_1\rangle\,\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle\,[\ell_2\,\ell_1]}\\
&+ \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_2\,\ell_1]}{\langle b\,c\rangle} + \frac{\langle a\,b\rangle\,\langle b\,c\rangle\,\langle d\,e\rangle\,\langle e\,a\rangle\,[e\,b]^3\,[b|\ell_2|b\rangle}{[b|\ell_1|b\rangle\,\langle c\,d\rangle\,[a\,b]\,[e\,a]}\\
&- \frac{\langle b\,c\rangle\,\langle d\,e\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,(3[e|b|a\rangle^2 + 3[e|b|a\rangle[e|\ell_2|a\rangle + [e|\ell_2|a\rangle^2}{\langle b\,\ell_1\rangle\,\langle c\,d\rangle\,\langle \ell_1\,\ell_2\rangle\,[\ell_2\,\ell_1]}\\
&+ \frac{2\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_1\,e]\,[\ell_2\,b]\,[e|\ell_2|e\rangle}{\langle d\,e\rangle\,[a\,b]\,[d\,e]} + \frac{\langle a\,c\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,\langle \ell_1\,e\rangle\,\langle \ell_2\,a\rangle\,[c\,e]}{\langle d\,e\rangle\,\langle \ell_1\,\ell_2\rangle}\Bigg) \qquad \text{(A.0.27)}
\end{aligned}
$$

$$
\begin{aligned}
I\ell_{bc} = \frac{1}{6}\Bigg(& \frac{\langle a\,b\rangle\,\langle c\,a\rangle\,\langle a|\ell_2\ell_1|a\rangle}{\langle b\,c\rangle} + \frac{\langle a\,b\rangle^2\,\langle a\,c\rangle^2\,[b\,c]}{\langle b\,c\rangle} \\
& - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,d\rangle\,\langle \ell_2\,c\rangle\,[c\,\ell_2]\,[b|c|a\rangle}{s_{bc}\,\langle c\,d\rangle} - \frac{\langle a\,b\rangle\,\langle a\,c\rangle^2\,\langle d\,a\rangle\,[b|\ell_1|b\rangle}{\langle b\,c\rangle\,\langle c\,d\rangle} \\
& + \frac{\langle a\,b\rangle\,\langle c\,a\rangle\,\langle \ell_1\,b\rangle\,[e\,\ell_1]\,[b|\ell_2|a\rangle}{\langle c\,b\rangle\,[e\,a]} + \frac{2\,\langle a\,b\rangle^2\,\langle \ell_1\,c\rangle\,[e\,\ell_1]\,[b|\ell_2|a\rangle}{\langle c\,b\rangle\,[e\,a]} \\
& + \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,d\rangle\,[b|\ell_1|b\rangle[b|\ell_2|a\rangle}{s_{bc}\,\langle c\,d\rangle} + \frac{\langle a\,b\rangle\,\langle \ell_1\,a\rangle\,[e\,\ell_1]\,[b|\ell_1|b\rangle[b|\ell_2|a\rangle}{s_{bc}\,[e\,a]} \\
& - \frac{\langle a\,b\rangle\,\langle b\,a\rangle\,[b|\ell_2|a\rangle^2}{s_{bc}} - \frac{\langle a\,b\rangle\,\langle \ell_1\,b\rangle\,[e\,\ell_1]\,[b|\ell_2|a\rangle^2}{s_{bc}\,[e\,a]} \\
& + \frac{2\,\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,\ell_1\rangle\,[e\,\ell_1]\,[b|\ell_2|b\rangle}{\langle b\,c\rangle\,[e\,a]} - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle c\,a\rangle\,[c|\ell_1|a\rangle}{\langle b\,c\rangle} \\
& + \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,d\rangle\,\langle \ell_2\,c\rangle\,[b\,\ell_2]\,[c|\ell_1|a\rangle}{s_{bc}\,\langle c\,d\rangle} - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle c\,b\rangle\,[b|c|a\rangle[c|\ell_1|a\rangle}{[c|\ell_1|c\rangle\,\langle b\,c\rangle} \\
& - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,[b|\ell_1|a\rangle[c|\ell_1|a\rangle}{s_{bc}} + \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,[b|\ell_2|a\rangle[c|\ell_1|a\rangle}{[c|\ell_1|c\rangle} \\
& + \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,d\rangle\,[b|\ell_2|c\rangle[c|\ell_2|a\rangle}{[c|\ell_1|c\rangle\,\langle c\,d\rangle} + \frac{\langle a\,b\rangle\,\langle c\,b\rangle\,[b|\ell_2|a\rangle^2[c|\ell_2|a\rangle}{s_{bc}[b|\ell_2|b\rangle}\Bigg)
\end{aligned}
\tag{A.0.28}
$$

$$
\begin{aligned}
It_{bc} = \frac{1}{6}\Bigg(&\\
&- \frac{\langle a\,b\rangle\,\langle a\,d\rangle\,\langle a\,e\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,[d\,e]}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle} - \frac{\langle a\,d\rangle^2\,\langle a\,e\rangle\,\langle c\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,[d\,e]}{\langle c\,d\rangle\,\langle d\,e\rangle\,\langle \ell_2\,b\rangle}\\
&- \frac{\langle a\,d\rangle^3\,\langle c\,b\rangle\,\langle c\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,e\rangle\,[d\,e]}{\langle c\,d\rangle\,\langle c\,\ell_1\rangle\,\langle d\,e\rangle\,\langle \ell_2\,b\rangle} - \frac{s_{bc}\,\langle a\,b\rangle\,\langle c\,d\rangle\,\langle e\,a\rangle\,[b\,c]\,[e\,b]}{\langle d\,e\rangle\,[a\,b]\,[e\,a]}\\
&+ \frac{\langle a\,b\rangle\,\langle d\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_1\,d]}{\langle \ell_2\,b\rangle} + \frac{2\,\langle a\,b\rangle\,\langle a\,\ell_1\rangle\,\langle e\,a\rangle\,\langle \ell_2\,d\rangle\,[e\,d]\,[\ell_1\,e]}{\langle \ell_2\,b\rangle\,[e\,a]}\\
&+ \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle^2\,[\ell_1\,a]\,[\ell_1\,e]}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle\,[a\,e]} + \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,\ell_1\rangle^2\,\langle c\,\ell_2\rangle\,\langle d\,e\rangle\,\langle d\,\ell_2\rangle\,[\ell_1\,\ell_2]}{\langle c\,d\rangle\,\langle c\,\ell_1\rangle\,\langle e\,d\rangle\,\langle \ell_2\,b\rangle}\\
&- \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_2\,d\rangle\,[e\,d]^2\,[\ell_2\,e]}{\langle \ell_2\,b\rangle\,[e\,a]^2} - \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,d\rangle\,[d\,e]\,[\ell_1\,a]\,[\ell_2\,e]}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^2}\\
&- \frac{\langle a\,b\rangle\,\langle c\,\ell_2\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,[\ell_1\,e]\,[\ell_2\,e]}{\langle c\,d\rangle\,\langle \ell_2\,b\rangle\,[e\,a]} - \frac{\langle a\,b\rangle\,\langle c\,\ell_2\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_2\,d\rangle\,[d\,e]\,[\ell_2\,e]^2}{\langle c\,d\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^2}\\
&+ \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,d\rangle\,[d\,e]\,[\ell_1\,a]\,[\ell_2\,e]^2}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^3} - \frac{\langle a\,b\rangle\,\langle c\,\ell_2\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,\ell_2\rangle\,[\ell_1\,e]\,[\ell_2\,e]^2}{\langle c\,d\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^2}\\
&+ \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,\ell_2\rangle^2\,[\ell_1\,a]\,[\ell_1\,e]\,[\ell_2\,e]^2}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^3} + \frac{\langle a\,d\rangle\,\langle a\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,[\ell_2\,\ell_1]}{\langle d\,e\rangle}\\
&+ \frac{\langle a\,b\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,\langle \ell_1\,\ell_2\rangle^2\,[\ell_1\,e]\,[\ell_2\,e]\,[\ell_2\,\ell_1]}{\langle d\,e\rangle\,\langle \ell_2\,b\rangle\,[e\,a]^2} - \frac{s_{bc}\,\langle a\,b\rangle\,\langle a\,d\rangle\,\langle e\,a\rangle\,[e\,b]\,[c|\ell_2|c\rangle}{[c|\ell_1|c\rangle\,\langle d\,e\rangle\,[e\,a]}\\
&- \frac{\langle a\,b\rangle\,\langle b\,c\rangle\,\langle e\,a\rangle\,[b\,e]\,[c\,b]\,[e\,b]\,[c|\ell_2|c\rangle}{[c|\ell_1|c\rangle\,[a\,b]\,[e\,a]} - \frac{s_{bc}\,\langle a\,b\rangle\,\langle a\,d\rangle\,\langle e\,a\rangle\,[b\,c]\,[e|\ell_2|c\rangle}{[c|\ell_1|c\rangle\,\langle d\,e\rangle\,[e\,a]}\\
&- \frac{\langle a\,c\rangle\,\langle a\,d\rangle^2\,\langle e\,a\rangle\,[c|\ell_1|e\rangle[e|\ell_2|c\rangle}{[c|\ell_1|c\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle} - \frac{\langle a\,b\rangle\,\langle d\,a\rangle\,\langle e\,a\rangle\,[b\,c]\,[e|\ell_2|c\rangle^2}{[c|\ell_1|c\rangle\,\langle c\,d\rangle\,[e\,a]}\Bigg)
\end{aligned}
\tag{A.0.29}
$$

$$
\begin{aligned}
I\ell_{cd} = \frac{1}{6}\Bigg(& \\
&- \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,\langle a|\ell_2\ell_1|a\rangle}{\langle c\,d\rangle} + \frac{\langle a\,d\rangle^2\,\langle a\,e\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,c\rangle\,[c\,\ell_1]\,[d\,\ell_1]}{s_{cd}\,\langle d\,e\rangle} \\
&- \frac{\langle a\,b\rangle\,\langle b\,c\rangle\,\langle d\,e\rangle\,\langle e\,a\rangle\,[b\,e]^3}{\langle c\,d\rangle\,[a\,b]\,[e\,a]} + \frac{\langle a\,b\rangle\,\langle e\,c\rangle\,\langle \ell_1\,c\rangle\,\langle \ell_1\,d\rangle\,\langle \ell_2\,a\rangle\,[c\,\ell_1]\,[d\,\ell_2]\,[e\,b]\,[e\,\ell_1]}{s_{cd}\,\langle c\,b\rangle\,[a\,b]\,[e\,a]} \\
&+ \frac{\langle a\,b\rangle\,\langle e\,c\rangle\,\langle \ell_1\,c\rangle\,\langle \ell_2\,a\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[d\,\ell_1]\,[e\,b]\,[e\,\ell_2]}{s_{cd}\,\langle c\,b\rangle\,[a\,b]\,[e\,a]} + \frac{\langle a\,d\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_1\,c\rangle\,[c\,\ell_1]\,[d\,\ell_1]\,[\ell_1\,b]}{s_{cd}\,[a\,b]} \\
&- \frac{\langle a\,e\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,c\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_2]\,[d\,\ell_2]\,[e\,c]\,[\ell_1\,b]}{\langle d\,e\rangle\,[a\,b]\,[c\,d]\,[e\,a]} - \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_1\,\ell_2]}{\langle c\,d\rangle} \\
&+ \frac{\langle a\,d\rangle\,\langle c\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,c\rangle\,[c\,\ell_1]\,[\ell_2\,b]}{\langle c\,d\rangle\,[a\,b]} + \frac{\langle a\,d\rangle\,\langle \ell_1\,a\rangle^2\,\langle \ell_1\,c\rangle\,[c\,\ell_1]\,[d\,\ell_1]\,[\ell_2\,b]}{s_{cd}\,[a\,b]} \\
&+ \frac{\langle a\,e\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_1\,c\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_1]\,[d\,\ell_2]\,[e\,c]\,[\ell_2\,b]}{\langle d\,e\rangle\,[a\,b]\,[c\,d]\,[e\,a]} - \frac{\langle a\,e\rangle^2\,\langle \ell_1\,a\rangle\,\langle \ell_2\,d\rangle\,[c\,\ell_1]\,[e\,b]\,[\ell_2\,d]}{\langle d\,e\rangle\,[a\,b]\,[c\,d]} \\
&+ \frac{\langle a\,d\rangle^2\,\langle a\,e\rangle\,\langle \ell_1\,c\rangle\,[d\,\ell_2]\,[c|\ell_1|a\rangle}{s_{cd}\,\langle d\,e\rangle} - \frac{\langle a\,b\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[d\,\ell_2]\,[e\,\ell_1]\,[c|\ell_1|c\rangle}{\langle c\,b\rangle\,[d\,c]\,[e\,a]} \\
&- \frac{3\,\langle c\,a\rangle^2\,\langle d\,a\rangle\,[d\,c]\,[c|\ell_2|a\rangle}{[c|\ell_2|c\rangle} - \frac{3\,\langle c\,a\rangle\,\langle d\,a\rangle^2\,[c\,d]\,[d|\ell_1|a\rangle}{[c|\ell_1|c\rangle} - \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,[c|\ell_2|a\rangle[d|\ell_2|a\rangle}{s_{cd}} \\
&- \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,[c|\ell_1|a\rangle[d|\ell_1|a\rangle}{s_{cd}} - \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,\langle d\,c\rangle\,[c|\ell_1|a\rangle[d|\ell_1|a\rangle}{s_{cd}\,\langle d\,e\rangle} \\
&+ \frac{\langle a\,d\rangle\,\langle \ell_1\,c\rangle\,[e\,\ell_2]\,[c|\ell_1|a\rangle[d|\ell_1|a\rangle}{s_{cd}\,[e\,a]} + \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,\langle \ell_1\,c\rangle\,[e\,\ell_2]\,[c|\ell_1|a\rangle[d|\ell_1|d\rangle}{s_{cd}\,\langle d\,e\rangle\,[e\,a]} \\
&+ \frac{\langle a\,b\rangle\,\langle a\,c\rangle^2\,\langle \ell_1\,d\rangle\,[c\,\ell_1]\,[d|\ell_2|a\rangle}{s_{cd}\,\langle c\,b\rangle} - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle \ell_1\,d\rangle\,[\ell_1\,b]\,s_{cd}[d|\ell_2|a\rangle}{s_{cd}\,\langle c\,b\rangle\,[a\,b]} \\
&- \frac{\langle a\,b\rangle\,\langle \ell_2\,a\rangle\,[e\,\ell_2]\,[c|\ell_1|c\rangle[d|\ell_2|a\rangle)}{\langle c\,b\rangle\,[d\,c]\,[e\,a]} - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle \ell_1\,d\rangle\,[\ell_1\,b]\,[c|\ell_1|c\rangle[d|\ell_2|a\rangle}{s_{cd}\,\langle c\,b\rangle\,[a\,b]} \\
&- \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle c\,d\rangle\,[c|\ell_2|a\rangle[d|\ell_2|a\rangle}{s_{cd}\,\langle c\,b\rangle} + \frac{\langle a\,c\rangle\,\langle \ell_1\,d\rangle\,[\ell_1\,b]\,[c|\ell_2|a\rangle[d|\ell_2|a\rangle}{s_{cd}\,[a\,b]}\Bigg)
\end{aligned}
\tag{A.0.30}
$$

$$
\begin{aligned}
It_{cd} = \frac{1}{6}\Bigg(& \\
& - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,\ell_2\rangle^2\,\langle d\,b\rangle\,\langle e\,a\rangle\,\langle \ell_1\,c\rangle\,[b\,\ell_2]}{\langle a\,e\rangle\,\langle c\,b\rangle\,\langle d\,\ell_1\rangle\,\langle \ell_1\,c\rangle} - \frac{\langle a\,b\rangle\,\langle b\,c\rangle\,\langle d\,e\rangle\,\langle e\,a\rangle\,\langle \ell_1\,\ell_2\rangle\,[b\,e]^3}{\langle d\,\ell_1\rangle\,\langle \ell_1\,c\rangle\,[a\,b]\,[e\,a]} \\
& + \frac{\langle a\,d\rangle\,\langle a\,\ell_1\rangle^2\,\langle c\,e\rangle\,\langle d\,\ell_2\rangle\,\langle e\,a\rangle\,[e\,\ell_1]}{\langle d\,e\rangle\,\langle d\,\ell_1\rangle\,\langle \ell_1\,c\rangle} + \frac{\langle a\,c\rangle\,\langle e\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_1\,e]}{\langle \ell_1\,c\rangle} \\
& - \frac{\langle a\,b\rangle\,\langle a\,c\rangle\,\langle a\,\ell_2\rangle^2\,\langle \ell_1\,c\rangle\,[\ell_1\,\ell_2]}{\langle c\,b\rangle\,\langle \ell_1\,c\rangle} + \frac{\langle a\,d\rangle\,\langle b\,a\rangle\,\langle \ell_1\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_2\,b]}{\langle \ell_1\,d\rangle} \\
& - \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,\langle a\,\ell_1\rangle^2\,\langle \ell_2\,d\rangle\,[\ell_2\,\ell_1]}{\langle d\,e\rangle\,\langle \ell_1\,d\rangle}\Bigg)
\end{aligned} \tag{A.0.31}
$$

$$
\begin{aligned}
I\ell_{de} = \frac{1}{6}\Bigg(& \\
& - \frac{\langle a\,c\rangle\,\langle a\,d\rangle^2\,\langle a\,e\rangle\,\langle \ell_1\,d\rangle\,[d\,\ell_1]}{\langle d\,c\rangle\,\langle e\,d\rangle} + \frac{\langle a\,d\rangle\,\langle a\,e\rangle^2\,\langle d\,a\rangle\,[e\,d]}{\langle d\,e\rangle} \\
& + \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,\langle a\,e\rangle\,\langle \ell_1\,d\rangle\,[e\,\ell_1]\,[d|\ell_2|a\rangle)}{s_{de}\,\langle d\,c\rangle} + \frac{2\,\langle a\,e\rangle\,\langle \ell_2\,a\rangle\,[\ell_2\,b]\,[e|d|a\rangle}{[a\,b]} \\
& + \frac{2\,\langle a\,e\rangle\,\langle \ell_2\,a\rangle\,[\ell_2\,b]\,[e|\ell_1|a\rangle}{[a\,b]} + \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,[d|\ell_2|a\rangle[e|\ell_1|a\rangle}{[d|l2|d\rangle} \\
& - \frac{\langle a\,e\rangle\,\langle e\,a\rangle\,[e|\ell_1|a\rangle^2}{s_{de}} + \frac{\langle a\,e\rangle\,\langle d\,e\rangle\,[d|\ell_1|a\rangle[e|\ell_1|a\rangle^2}{s_{de}[e|\ell_1|e\rangle} \\
& + \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,\langle a\,e\rangle\,[d|\ell_2|a\rangle[e|\ell_1|d\rangle}{[d|l2|d\rangle\,\langle d\,c\rangle} - \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,[d|e|a\rangle[d|\ell_2|a\rangle[e|\ell_1|d\rangle}{s_{de}[d|l2|d\rangle} \\
& - \frac{\langle a\,c\rangle\,\langle a\,d\rangle\,\langle a\,e\rangle\,\langle \ell_2\,e\rangle\,[e\,\ell_2]\,[e|\ell_2|a\rangle}{s_{de}\,\langle d\,c\rangle} - \frac{2\,\langle a\,e\rangle\,\langle \ell_2\,a\rangle\,\langle \ell_2\,e\rangle\,[e\,\ell_2]\,[\ell_2\,b]\,[e|\ell_2|a\rangle}{s_{de}\,[a\,b]} \\
& + \frac{\langle a\,d\rangle\,\langle a\,e\rangle\,[d|\ell_1|d\rangle[d|\ell_2|a\rangle[e|\ell_2|a\rangle}{s_{de}[d|l2|d\rangle} - \frac{\langle a\,e\rangle\,\langle d\,a\rangle\,\langle \ell_2\,a\rangle\,[\ell_2|\ell_1|a\rangle}{\langle d\,e\rangle}\Bigg)
\end{aligned} \tag{A.0.32}
$$

$$
\begin{aligned}
It_{de} = \frac{1}{6}\Bigg(&\\
&- \frac{\langle a\,b\rangle \langle a\,c\rangle \langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle [c\,b]}{\langle c\,b\rangle \langle \ell_1\,e\rangle} - \frac{\langle a\,b\rangle \langle a\,c\rangle^2 \langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,e\rangle [c\,b]}{\langle c\,b\rangle \langle d\,c\rangle \langle \ell_1\,e\rangle}\\
&- \frac{\langle a\,c\rangle^3 \langle b\,a\rangle \langle d\,e\rangle \langle d\,\ell_1\rangle \langle \ell_2\,b\rangle [c\,b]}{\langle c\,b\rangle \langle d\,c\rangle \langle d\,\ell_2\rangle \langle \ell_1\,e\rangle} - \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,a\rangle \langle d\,\ell_2\rangle [e\,d]\,[e\,\ell_2]}{\langle c\,b\rangle [e\,a]}\\
&+ \frac{\langle a\,c\rangle^2 \langle a\,d\rangle \langle b\,a\rangle \langle d\,\ell_1\rangle \langle \ell_2\,b\rangle [\ell_1\,b]}{\langle c\,b\rangle \langle d\,c\rangle \langle d\,\ell_2\rangle} - \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_1\,c\rangle [b\,c]^2 [\ell_1\,b]}{\langle \ell_1\,e\rangle [b\,a]^2}\\
&- \frac{s_{de} \langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,\ell_1\rangle [e\,b]\,[\ell_1\,b]}{\langle d\,c\rangle \langle \ell_1\,e\rangle [b\,a]\,[e\,a]} - \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,\ell_1\rangle \langle \ell_1\,c\rangle [c\,b]\,[\ell_1\,b]^2}{\langle d\,c\rangle \langle \ell_1\,e\rangle [b\,a]^2}\\
&+ \frac{s_{de} \langle a\,e\rangle \langle b\,a\rangle \langle c\,b\rangle \langle d\,\ell_1\rangle^2 [e\,b]\,[\ell_1\,b]^2}{[e|\ell_1|e\rangle \langle d\,c\rangle \langle d\,e\rangle [b\,a]\,[e\,a]} + \frac{\langle a\,c\rangle^2 \langle a\,d\rangle \langle a\,\ell_1\rangle [\ell_1\,c]}{\langle d\,c\rangle}\\
&- \frac{\langle a\,d\rangle \langle a\,\ell_1\rangle \langle c\,a\rangle^2 [\ell_1\,c]}{\langle d\,c\rangle} + \frac{\langle a\,c\rangle \langle a\,\ell_1\rangle \langle b\,a\rangle \langle \ell_2\,a\rangle [\ell_1\,\ell_2]}{\langle c\,b\rangle}\\
&+ \frac{s_{de} \langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_2\,\ell_1\rangle [e\,b]\,[\ell_1\,\ell_2]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [b\,a]\,[e\,a]} - \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle [c\,b]\,[\ell_1\,b]\,[\ell_2\,a]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [a\,b]^2}\\
&+ \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_1\,c\rangle \langle \ell_2\,\ell_1\rangle [c\,b]\,[\ell_1\,b]^2 [\ell_2\,a]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [b\,a]^3} + \frac{2 \langle a\,e\rangle \langle a\,\ell_2\rangle \langle b\,a\rangle \langle \ell_1\,c\rangle [b\,c]\,[\ell_2\,b]}{\langle \ell_1\,e\rangle [b\,a]}\\
&+ \frac{s_{de} \langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_2\,\ell_1\rangle [\ell_1\,b]\,[\ell_2\,b]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [b\,a]^2} - \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,\ell_1\rangle \langle \ell_2\,a\rangle [\ell_1\,b]\,[\ell_2\,b]}{\langle d\,c\rangle \langle \ell_1\,e\rangle [b\,a]}\\
&- \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,\ell_1\rangle \langle \ell_2\,\ell_1\rangle [\ell_1\,b]^2 [\ell_2\,b]}{\langle d\,c\rangle \langle \ell_1\,e\rangle [b\,a]^2} + \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_2\,a\rangle^2 [\ell_2\,a]\,[\ell_2\,b]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [a\,b]}\\
&+ \frac{\langle a\,e\rangle \langle b\,a\rangle \langle c\,\ell_1\rangle \langle \ell_2\,\ell_1\rangle^2 [\ell_1\,b]^2 [\ell_2\,a]\,[\ell_2\,b]}{\langle c\,b\rangle \langle \ell_1\,e\rangle [b\,a]^3} + \frac{\langle a\,e\rangle \langle c\,a\rangle \langle \ell_1\,a\rangle \langle \ell_2\,a\rangle [\ell_1\,c]}{\langle \ell_1\,e\rangle}\\
&+ \frac{\langle a\,d\rangle \langle a\,e\rangle \langle a\,\ell_2\rangle^2 \langle c\,b\rangle \langle c\,\ell_1\rangle \langle d\,\ell_1\rangle [\ell_2\,\ell_1]}{\langle b\,c\rangle \langle d\,c\rangle \langle d\,\ell_2\rangle \langle \ell_1\,e\rangle} - \frac{s_{de} \langle a\,c\rangle \langle a\,e\rangle \langle b\,a\rangle \langle d\,\ell_1\rangle [\ell_1\,b]\,[e|\ell_2|d\rangle}{[e|\ell_1|e\rangle \langle d\,c\rangle \langle d\,e\rangle [e\,a]}\\
&+ \frac{s_{de} \langle a\,e\rangle \langle b\,a\rangle \langle c\,a\rangle \langle d\,\ell_1\rangle [\ell_1\,d]\,[e|\ell_2|d\rangle}{[e|\ell_1|e\rangle \langle c\,b\rangle \langle d\,e\rangle [e\,a]}\Bigg) \qquad \text{(A.0.33)}
\end{aligned}
$$

$$
I\ell_{ea} = -\frac{\langle a\,e\rangle^2 \langle e\,a\rangle \langle \ell_1\,d\rangle [e\,\ell_1]\,[e|\ell_2|a\rangle}{6 \langle e\,d\rangle \langle e\,\ell_2\rangle [\ell_2\,e]} \qquad \text{(A.0.34)}
$$

$$
\begin{aligned}
It_{ea} = \frac{1}{6}\Bigg(& \\
&\frac{\langle a\,b\rangle^2 \langle c\,\ell_2\rangle \langle e\,d\rangle \langle \ell_1\,a\rangle^2 \langle \ell_1\,c\rangle^2 [b\,c]}{\langle c\,b\rangle \langle d\,c\rangle \langle e\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle^2} - \frac{\langle a\,e\rangle \langle a\,\ell_2\rangle \langle b\,a\rangle \langle \ell_1\,a\rangle [b\,\ell_1]}{\langle e\,\ell_2\rangle} \\
&+ \frac{\langle b\,a\rangle \langle e\,c\rangle \langle \ell_1\,a\rangle^2 \langle \ell_2\,a\rangle \langle \ell_2\,b\rangle [b\,\ell_1]}{\langle c\,b\rangle \langle e\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle} + \frac{\langle a\,b\rangle \langle a\,c\rangle \langle b\,a\rangle \langle \ell_2\,a\rangle [b\,\ell_2]}{\langle c\,b\rangle} \\
&+ \frac{\langle a\,c\rangle \langle b\,a\rangle \langle d\,a\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle [c\,b]}{\langle d\,c\rangle \langle \ell_1\,\ell_2\rangle} + \frac{\langle b\,a\rangle \langle d\,a\rangle \langle \ell_1\,c\rangle^2 \langle \ell_2\,a\rangle^2 [c\,b]}{\langle d\,c\rangle \langle \ell_1\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle} \\
&+ \frac{\langle b\,a\rangle \langle d\,b\rangle \langle e\,a\rangle \langle \ell_1\,c\rangle^3 \langle \ell_2\,a\rangle^3 [c\,b]}{\langle c\,b\rangle \langle d\,c\rangle \langle e\,\ell_2\rangle \langle \ell_1\,a\rangle \langle \ell_1\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle} - \frac{\langle a\,c\rangle \langle b\,a\rangle \langle d\,a\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle [c\,d]}{\langle c\,b\rangle \langle \ell_1\,\ell_2\rangle} \\
&+ \frac{\langle a\,c\rangle^2 \langle b\,a\rangle \langle d\,a\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle [c\,\ell_2]}{\langle c\,b\rangle \langle d\,c\rangle \langle \ell_1\,a\rangle} - \frac{\langle b\,a\rangle \langle d\,a\rangle \langle e\,d\rangle \langle \ell_1\,a\rangle^2 \langle \ell_1\,c\rangle^3 [d\,c]}{\langle c\,b\rangle \langle d\,c\rangle \langle e\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle^2} \\
&- \frac{2\,\langle a\,c\rangle \langle b\,a\rangle \langle d\,a\rangle \langle \ell_2\,a\rangle [d\,\ell_2]}{\langle c\,b\rangle} - \frac{\langle a\,d\rangle \langle e\,a\rangle \langle \ell_1\,d\rangle \langle \ell_2\,a\rangle^2 [d\,\ell_2]}{\langle e\,d\rangle \langle \ell_2\,\ell_1\rangle} \\
&- \frac{\langle a\,d\rangle \langle c\,e\rangle \langle e\,a\rangle \langle \ell_1\,d\rangle^2 \langle \ell_2\,a\rangle^3 [d\,\ell_2]}{\langle d\,c\rangle \langle d\,e\rangle \langle e\,\ell_2\rangle \langle \ell_1\,a\rangle \langle \ell_2\,\ell_1\rangle} + \frac{\langle b\,a\rangle \langle d\,a\rangle \langle \ell_1\,c\rangle^2 \langle \ell_2\,a\rangle [c\,d] [e\,\ell_1]}{\langle c\,b\rangle \langle \ell_1\,\ell_2\rangle [e\,a]} \\
&+ \frac{\langle b\,a\rangle \langle d\,c\rangle \langle e\,d\rangle \langle \ell_1\,a\rangle^2 \langle \ell_2\,a\rangle [d\,\ell_1]^2}{\langle c\,b\rangle \langle e\,a\rangle \langle \ell_2\,\ell_1\rangle [\ell_1\,a]} - \frac{2\,\langle d\,a\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle^2 [d\,\ell_2] [e\,\ell_2] [\ell_1\,c]}{\langle d\,c\rangle [d\,c] [e\,a]} \\
&- \frac{\langle d\,a\rangle^2 \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle \langle \ell_2\,e\rangle [d\,\ell_2] [e\,\ell_2] [\ell_1\,c]}{\langle d\,c\rangle \langle e\,d\rangle [d\,c] [e\,a]} + \frac{\langle a\,\ell_1\rangle \langle b\,a\rangle \langle \ell_1\,b\rangle \langle \ell_2\,a\rangle [b\,\ell_2] [\ell_1\,b] [\ell_1\,e]}{\langle c\,b\rangle [c\,b] [e\,a]} \\
&+ \frac{\langle a\,d\rangle \langle a\,e\rangle \langle \ell_1\,a\rangle \langle \ell_2\,a\rangle [\ell_1\,\ell_2]}{\langle e\,d\rangle} + \frac{\langle d\,a\rangle^2 \langle e\,a\rangle \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle [\ell_1\,\ell_2]}{\langle d\,c\rangle \langle e\,d\rangle} \\
&- \frac{s_{de} \langle b\,a\rangle \langle d\,c\rangle \langle \ell_1\,a\rangle \langle \ell_2\,a\rangle [d\,\ell_1] [\ell_1\,\ell_2]}{\langle c\,b\rangle \langle e\,a\rangle [a\,e] [\ell_1\,a]} + \frac{\langle b\,a\rangle \langle c\,a\rangle \langle \ell_1\,b\rangle \langle \ell_2\,a\rangle [\ell_1\,e] [\ell_2\,b]}{\langle c\,b\rangle [e\,a]} \\
&+ \frac{\langle a\,d\rangle \langle b\,a\rangle \langle \ell_1\,b\rangle \langle \ell_2\,a\rangle [b\,d] [\ell_1\,e] [\ell_2\,b]}{\langle c\,b\rangle [c\,b] [e\,a]} - \frac{\langle a\,\ell_2\rangle \langle b\,a\rangle \langle \ell_1\,b\rangle \langle \ell_2\,a\rangle [b\,\ell_2] [\ell_1\,e] [\ell_2\,b]}{\langle c\,b\rangle [c\,b] [e\,a]} \\
&- \frac{\langle d\,a\rangle^2 \langle \ell_1\,c\rangle \langle \ell_2\,a\rangle^2 [\ell_2\,d]}{\langle d\,c\rangle \langle \ell_2\,\ell_1\rangle} + \frac{\langle b\,a\rangle \langle c\,a\rangle \langle \ell_1\,b\rangle \langle \ell_2\,a\rangle [b\,\ell_1] [\ell_2\,e]}{\langle c\,b\rangle [e\,a]} \\
&- \frac{\langle b\,a\rangle \langle c\,\ell_1\rangle \langle d\,a\rangle \langle \ell_2\,a\rangle [\ell_1\,d] [\ell_2\,e]}{\langle c\,b\rangle [e\,a]} \\
&- \frac{\langle b\,a\rangle \langle c\,b\rangle \langle e\,d\rangle \langle \ell_2\,a\rangle \big(3[b|e|a\rangle^2 + 3[b|e|a\rangle[b|\ell_1|a\rangle + [b|\ell_1|a\rangle^2\big)}{\langle d\,c\rangle \langle e\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle [\ell_1\,\ell_2]} \\
&+ \frac{\langle a\,b\rangle \langle a\,c\rangle \langle e\,c\rangle \langle \ell_2\,d\rangle [b\,e] [e\,\ell_2] [c|\ell_2|c\rangle}{\langle c\,b\rangle \langle d\,c\rangle [a\,b] [a\,e]} - \frac{\langle b\,a\rangle \langle c\,b\rangle \langle e\,a\rangle \langle e\,d\rangle [b\,e]^3 [e|\ell_1|e\rangle}{[e|\ell_2|e\rangle \langle d\,c\rangle [a\,b] [e\,a]}\Bigg)
\end{aligned}
\tag{A.0.35}
$$

Appendix B

Deriving the Tree-Level Currents

B.1 Deriving the googly current

We need to derive the googly current up and including terms of order $\langle\alpha\beta\rangle$ since the left hand side with the all -plus integral has a double pole.

Starting with the tree level googly amplitude,

$$A(b^+, c^+, \beta^-, \alpha^-, a^-) = i\frac{[bc]^3}{[c\beta][\beta\alpha][\alpha a][ab]} \tag{B.1.1}$$

there is a square pole that needs to be incorporated into the amplitude to get the current.

The pole structure is

$$\begin{aligned} &A_3(\beta^-, \alpha^-, k^+)\frac{i}{s_{\alpha\beta}}A_4(k^-, a^-, b^+, c^+) \\ &= i\frac{\langle\alpha\beta\rangle[kq]^2}{[\alpha q][\beta q]}\frac{i}{s_{\alpha\beta}}i\frac{\langle ka\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle}. \end{aligned} \tag{B.1.2}$$

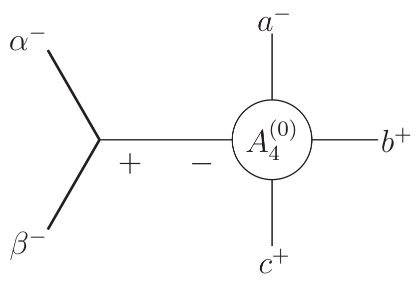


Figure B.1: Factorisations of the googly current on the $s_{\alpha\beta} \to 0$ pole.

Now we begin to rewrite this term. We can write

$$\langle\alpha\beta\rangle = \frac{\langle\beta k\rangle[ka]}{[\alpha a]} - \delta\frac{\langle\beta q\rangle[qa]}{[\alpha a]}$$

where

$$\delta = \frac{\beta^2}{2\beta\cdot q} + \frac{\alpha^2}{2\alpha\cdot q} - \frac{k^2}{2k\cdot q}.$$

Subsituting this back into the pole expression, we then use

$$\langle\beta k\rangle[kq] = -\langle\alpha\beta\rangle[\alpha q]$$

to now write the pole term as

$$-i\frac{\langle\alpha\beta\rangle\langle ka\rangle^3[ka][kq]}{s_{\alpha\beta}\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\beta q]} + i\frac{\delta}{s_{\alpha\beta}}\frac{\langle\beta q\rangle\langle ka\rangle^3[kq]^2[qa]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\beta q][\beta q]}. \qquad \text{(B.1.3)}$$

Next we do something similar, writing

$$\langle\alpha\beta\rangle = \frac{[ck]\langle k\alpha\rangle}{[c\beta]} - \delta\frac{[cq]\langle q\alpha\rangle}{[c\beta]}$$

followed by

$$\langle k\alpha\rangle[kq] = -\langle\alpha\beta\rangle[\beta q].$$

Now we have

$$i\frac{\langle\alpha\beta\rangle\langle ka\rangle^3[ck][ka]}{s_{\alpha\beta}\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][c\beta]}+i\frac{\delta}{s_{\alpha\beta}}\left(\frac{\langle ka\rangle^3\langle q\alpha\rangle[cq][ka][kq]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][c\beta][\beta q]}+\frac{\langle\beta q\rangle\langle ka\rangle^3[kq]^2[qa]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\alpha q][\beta q]}\right). \tag{B.1.4}$$

Now we modify the first term above using

$$[ck][ka]=[c|k^\flat|a\rangle=-\langle ab\rangle[bc]-\frac{s_{\alpha\beta}}{[q|P_{\alpha\beta}|q\rangle}[cq]\langle qa\rangle$$

and

$$\langle ka\rangle=\frac{[k^\flat|a|b\rangle}{[ab]}=-\frac{[b|c|k^\flat\rangle}{[ab]}-s_{\alpha\beta}\frac{[bq]}{[kq][ab]}.$$

Now we have

$$\begin{aligned}&i\frac{\langle\alpha\beta\rangle\langle ka\rangle[bc]^2[ka]}{s_{\alpha\beta}[ab]\langle bc\rangle[\alpha a][c\beta]}-i\frac{\langle\alpha\beta\rangle\langle ka\rangle[bc][bq][ka]}{\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq]}-i\frac{\langle\beta\alpha\rangle\langle ka\rangle\langle qa\rangle[bc][cq][ka]}{\langle ab\rangle\langle bc\rangle[ab][\alpha a][c\beta][q|P_{\alpha\beta}|q\rangle}\\&-i\frac{s_{\alpha\beta}\langle\alpha\beta\rangle\langle ka\rangle\langle qa\rangle[bq][cq][ka]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq][q|P_{\alpha\beta}|q\rangle}+i\frac{\delta}{s_{\alpha\beta}}\left(\frac{\langle ka\rangle^3\langle q\alpha\rangle[cq][ka][kq]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\beta q][c\beta]}+\frac{\langle\beta q\rangle\langle ka\rangle^3[kq]^2[qa]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\alpha q][\beta q]}\right).\end{aligned} \tag{B.1.5}$$

Finally, on the first term we substitute

$$\langle ka\rangle[ka]=[bc]\langle bc\rangle+s_{\alpha\beta}\left(1+\frac{[q|a|q\rangle}{[q|P_{\alpha\beta}|q\rangle}\right)$$

to get

$$\begin{aligned}&+i\frac{\langle\alpha\beta\rangle[bc]^3}{s_{\alpha\beta}[ab][\alpha a][c\beta]}+i\frac{\langle\alpha\beta\rangle[bc]^2[q|\alpha|q\rangle}{[ab]\langle bc\rangle[\alpha a][c\beta][q|P_{\alpha\beta}|q\rangle}+i\frac{\langle\alpha\beta\rangle[bc]^2}{[ab]\langle bc\rangle[\alpha a][c\beta]}\\&-i\frac{\langle\alpha\beta\rangle\langle ka\rangle[bc][bq][ka]}{\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq]}-i\frac{\langle\beta\alpha\rangle\langle ka\rangle\langle qa\rangle[bc][cq][ka]}{\langle ab\rangle\langle bc\rangle[ab][\alpha a][c\beta][q|P_{\alpha\beta}|q\rangle}\\&-i\frac{s_{\alpha\beta}\langle\alpha\beta\rangle\langle ka\rangle\langle qa\rangle[bq][cq][ka]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq][q|P_{\alpha\beta}|q\rangle}+i\frac{\delta}{s_{\alpha\beta}}\left(\frac{\langle ka\rangle^3\langle q\alpha\rangle[cq][ka][kq]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\beta q][c\beta]}+\frac{\langle\beta q\rangle\langle ka\rangle^3[kq]^2[qa]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\alpha q][\beta q]}\right).\end{aligned} \tag{B.1.6}$$

We can identify the first term as the googly amplitude up to corrections of $\mathcal{O}(\alpha^2,\beta^2)$, so that we can write the current as

$$\begin{aligned}\tau^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-) = & i\frac{\langle\alpha\beta\rangle[kq]^2}{[\alpha q][\beta q]}\frac{i}{s_{\alpha\beta}}i\frac{\langle ka\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle} - i\frac{\langle\alpha\beta\rangle[bc]^2}{[ab]\langle bc\rangle[\alpha a][c\beta]} \\ & + i\frac{\langle\alpha\beta\rangle\langle ka\rangle[bc][bq][ka]}{\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq]} + i\frac{\langle\beta\alpha\rangle\langle ka\rangle\langle qa\rangle[bc][cq][ka]}{\langle ab\rangle\langle bc\rangle[ab][\alpha a][c\beta][q|P_{\alpha\beta}|q\rangle} + i\frac{s_{\alpha\beta}\langle\alpha\beta\rangle\langle ka\rangle\langle qa\rangle[bq][cq][ka]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[ab][\alpha a][c\beta][kq][q|P_{\alpha\beta}|q\rangle} \\ & - i\frac{\delta}{s_{\alpha\beta}}\left(\frac{\langle ka\rangle^3\langle q\alpha\rangle[cq][ka][kq]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\beta q][c\beta]} + \frac{\langle\beta q\rangle\langle ka\rangle^3[kq]^2[qa]}{\langle ab\rangle\langle bc\rangle\langle ck\rangle[\alpha a][\alpha q][\beta q]}\right).\end{aligned} \tag{B.1.7}$$

Recall that in the case of the all-plus triangle that we only require terms up to $\mathcal{O}(s_{\alpha\beta})$ and up to $\mathcal{O}(\alpha^2,\beta^2)$, so we can slim the current down to

$$\begin{aligned}\tau_5^{(0)}(b^+,c^+,\beta^-,\alpha^-,a^-,q) = & -i\frac{\langle\alpha\beta\rangle[e|P_{\alpha\beta}|a\rangle^3}{s_{de}\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[\alpha e][\beta e]} - i\frac{\langle q\alpha\rangle[e|P_{\alpha\beta}|a\rangle^3[ce][a|P_{de}|q\rangle}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|q\rangle^2[e|P_{\alpha\beta}|c\rangle[\alpha a][c\beta][\beta e]} \\ & + i\frac{\langle\beta q\rangle[e|P_{\alpha\beta}|a\rangle^3[ea]}{\langle ab\rangle\langle bc\rangle[e|P_{\alpha\beta}|c\rangle[e|P_{\alpha\beta}|q\rangle[\alpha a][\alpha e][\beta e]} \\ & - i\frac{\langle\alpha\beta\rangle[bc]^2}{\langle bc\rangle[ab][c\beta][\alpha a]} - i\frac{\langle\alpha\beta\rangle[bc]^2[be]}{[ab][c\beta][\alpha a][e|P_{\alpha\beta}|c\rangle} \\ & + i\frac{\langle aq\rangle\langle\alpha\beta\rangle[bc]^2[ce]}{\langle ab\rangle[ab][\alpha a][c\beta][e|P_{\alpha\beta}|q\rangle} - i\frac{\langle\alpha\beta\rangle\langle aq\rangle[bc]^2[ea]}{\langle bc\rangle[e|P_{\alpha\beta}|q\rangle[ab][\alpha a][c\beta]} + O(s_{\alpha\beta}^2).\end{aligned} \tag{B.1.8}$$

B.2 Deriving the MHV currents

Deriving the MHV currents is more straightforward. Let us use the adjacent MHV current as an example as the procedure is identical for the non-adjacent equivalent.

Beginning with the amplitude

$$A_5(\alpha^-,a^-,b^+,c^+,\beta^+) = i\frac{\langle\alpha a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle c\beta\rangle\langle\beta\alpha\rangle}, \tag{B.2.1}$$

which has the pole

$$\begin{aligned}&A_3(\beta^+,\alpha^-,k^+)\frac{i}{s_{\alpha\beta}}A_4(-k^-,a^-,b^+,c^+) \\ &= i\frac{[k\beta]\langle\alpha q\rangle^2}{\langle kq\rangle\langle\beta q\rangle}\frac{i}{s_{\alpha\beta}}i\frac{\langle ka\rangle^3}{\langle ab\rangle\langle bc\rangle\langle ck\rangle}.\end{aligned} \tag{B.2.2}$$

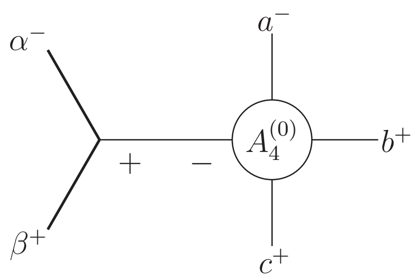


Figure B.2: Factorisations of the adjacent MHV current on the $s_{\alpha\beta} \to 0$ pole.

We can uncover the leading pole using

$$\frac{1}{\langle\alpha\beta\rangle\langle\beta a\rangle} = \frac{1}{\langle\alpha q\rangle\langle\beta q\rangle^2}\frac{1}{[q|P_{\alpha\beta}|a\rangle}\left(\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}[q|P_{\alpha\beta}|q\rangle + \frac{\langle q\beta\rangle\langle qa\rangle[q|\alpha|q\rangle}{\langle\beta a\rangle}\right), \tag{B.2.3}$$

and expand as before using

$$\begin{aligned}\frac{[\beta|P^\flat_{\alpha\beta}|b\rangle}{[\beta|P_{\alpha\beta}|q\rangle} &= \frac{[q|P_{\alpha\beta}|b\rangle}{[q|P_{\alpha\beta}|q\rangle} + s_{\alpha\beta}\frac{\langle qb\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2),\\ \frac{[\alpha|P^\flat_{\alpha\beta}|b\rangle}{[\alpha|P_{\alpha\beta}|q\rangle} &= \frac{[q|P_{\alpha\beta}|b\rangle}{[q|P_{\alpha\beta}|q\rangle} + s_{\alpha\beta}\frac{\langle qb\rangle[\alpha q]}{[\alpha|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|q\rangle} + \mathcal{O}(\alpha^2,\beta^2).\end{aligned} \tag{B.2.4}$$

The current is now

$$\begin{aligned}\tau_5^{(0)}(\alpha^-,a^-,b^+,c^+,\beta^+) =& -i\frac{\langle\alpha q\rangle^2\langle qc\rangle[q|\alpha|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle^2[q|P_{\alpha\beta}|q\rangle^2}\left(1 + s_{\alpha\beta}\frac{\langle qa\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^3\\ &\times\left(1 + s_{\alpha\beta}\frac{\langle qc\rangle[\alpha q]}{[\alpha|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|c\rangle}\right)^{-1}\\ &+ i\frac{\langle\alpha q\rangle^2\langle q|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{s_{\alpha\beta}\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2}\left(1 + s_{\alpha\beta}\frac{\langle qa\rangle[\beta q]}{[\beta|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle}\right)^3.\end{aligned} \tag{B.2.5}$$

At this point we recall that we require the current up to order $\mathcal{O}(s_{\alpha\beta})$, so we expand the bracket in the denominator using $(1-x)^{-1} = 1 + x + \mathcal{O}(x^2)$ to get

$$\begin{aligned}\tau_5^{(0)}(\alpha^-,a^-,b^+,c^+,\beta^+,q) =& i\frac{\langle\alpha q\rangle^2\langle q|P_{\alpha\beta}|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{s_{\alpha\beta}\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} - i\frac{\langle\alpha q\rangle^2\langle qc\rangle[q|\alpha|q\rangle[q|P_{\alpha\beta}|a\rangle^3}{\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle^2[q|P_{\alpha\beta}|q\rangle^2}\\ &+ 3i\frac{\langle\alpha q\rangle^2\langle qa\rangle[q|\beta|q\rangle[q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle\langle bc\rangle\langle\beta q\rangle^2[q|P_{\alpha\beta}|c\rangle[q|P_{\alpha\beta}|q\rangle^2} + O(s_{\alpha\beta}).\end{aligned} \tag{B.2.6}$$

The non-adjacent MHV current derivation is identical so we shall simply state the final result,

$$\begin{aligned}\tau^{(0)}(\beta^-, \alpha^+, a^-, b^+, c^+, q) = {} & i\frac{\langle \beta q\rangle^2 \langle cq\rangle [q|\alpha|q\rangle [q|P_{\alpha\beta}|a\rangle^3}{\langle ab\rangle \langle \alpha q\rangle^2 \langle bc\rangle [q|P_{\alpha\beta}|c\rangle^2 [q|P_{\alpha\beta}|q\rangle^2} - i\frac{\langle \beta q\rangle^2 \langle aq\rangle [q|\beta|q\rangle [q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle \langle \alpha q\rangle^2 \langle bc\rangle [q|P_{\alpha\beta}|c\rangle [q|P_{\alpha\beta}|q\rangle^2} \\ & + i\frac{\langle \beta q\rangle^2 [q|P_{\alpha\beta}|a\rangle^3 \langle q|\alpha\beta|q\rangle}{s_{de}\langle ab\rangle \langle \alpha q\rangle^2 \langle bc\rangle [q|P_{\alpha\beta}|c\rangle [q|P_{\alpha\beta}|q\rangle^2} - 4i\frac{\langle \beta q\rangle^2 \langle aq\rangle [q|\alpha|q\rangle [q|P_{\alpha\beta}|a\rangle^2}{\langle ab\rangle \langle \alpha q\rangle^2 \langle bc\rangle [q|P_{\alpha\beta}|c\rangle [q|P_{\alpha\beta}|q\rangle^2} \\ & + O(s_{\alpha\beta}).\end{aligned} \tag{B.2.7}$$

Appendix C

Hypergeometric functions and Identities

Given the ubiquity of hypergeometric functions and their various identities, we present a summary of these in this appendix. The results can be found in many places, such as [92], online at [93], and the appendix of [94] was particularly useful.

C.1 Pochhammer Symbols

The basic structure in the series representations of hypergeometric functions is the Pochhammer symbol, defined as

$$(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{C.1.1}$$

The most useful identity for Pochhammer symbols is

$$(a,n) = \frac{(-1)^n}{(1-a,-n)} \tag{C.1.2}$$

C.2 Series and Integral Representations

The hypergeometric function of a single variable can be written as a sum of Pochhammer symbols,

$$ {}_2F_1[\alpha,\beta,\gamma,x] = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)} \frac{x^m}{m!}, \tag{C.2.1}$$

$$ {}_3F_2[\alpha,\beta,\beta';\gamma,\gamma',x] = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)(\beta',m)}{(\gamma,m)(\gamma',m)} \frac{x^m}{m!}, \tag{C.2.2} $$

and this can naturally be extended to arbitrary numbers of arguments,

$$ {}_{p+1}F_p[\alpha,\beta_1,...,\beta_p;\gamma_1,...,\gamma_p,x] = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta_1,m)...(\beta_p,m)}{(\gamma_1,m)...(\gamma_p,m)} \frac{x^m}{m!}. \tag{C.2.3} $$

All of these series converge when $|x| < 1$.

There are two hypergeometric functions of two variables which we shall use extensively in this thesis: these are the first two of the four Appell functions:

$$ F_1[\alpha,\beta,\beta',\gamma,x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{C.2.4} $$

and

$$ F_2[\alpha;\beta,\beta';\gamma,\gamma';x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{C.2.5} $$

The first function converges when $|x| < 1, |y| < 1$ and the second when $|x| + |y| < 1$.

All of the above functions also admit an integral representation, however we shall only state the representation of ${}_2F_1$ as it is the only one we will use.

$$ {}_2F_1[\alpha,\beta,\gamma,x] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dv\ v^{\beta-1}(1-v)^{\gamma-\beta-1}(1-vx)^{-\alpha}, \tag{C.2.6} $$

where $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma-\beta) > 0$.

C.3 Hypergeometric Functions at specific values

At certain specific values - usually 1 - the hypergeometric will simplify. These are

$$ {}_2F_1[\alpha,\beta,\gamma,1] = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{C.3.1} $$

where $\gamma - \alpha - \beta > 0$, and

$$ F_1[\alpha;\beta,\beta';\gamma;1,y] = {}_2F_1[\alpha,\beta,\gamma,1]{}_2F_1[\alpha,\beta',\gamma-\beta,y]. \tag{C.3.2} $$

In any case where one of the arguments is 0, the corresponding series evaluates to 1, so

$$ {}_2F_1[\alpha, \beta, \gamma, 0] = 1, \tag{C.3.3} $$

$$ F_1[\alpha; \beta, \beta'; \gamma, \alpha; 0, y] = {}_2F_1[\alpha, \beta', \gamma, y] \tag{C.3.4} $$

and

$$ F_2[\alpha; \beta, \beta'; \gamma, \gamma'; x, 0] = {}_2F_1[\alpha, \beta, \gamma, x]. \tag{C.3.5} $$

If one of the Pochammers in a hypergeometric function is 0, say $(0, n)$ then the n series terminates, so

$$ F_2[\alpha; \beta, 0; \gamma, \gamma'; x, y] = {}_2F_1[\alpha, \beta, \gamma, x], \tag{C.3.6} $$

and if the Pochhammer with both series variables is 0, so $(0, m+n)$ then both series terminates,

$$ F_2[0; \beta, \beta'; \gamma, \gamma'; x, y] = 1. \tag{C.3.7} $$

A Pochhammer with a negative argument can be written as

$$ (-a, n) = \frac{(-1)^n \Gamma(1+a)}{\Gamma(1+a-n)!}, \tag{C.3.8} $$

so if a is an integer then for $n \geq a$, the Gamma function in the denominator will cause the term to vanish, thus negative integer arguments of Pochhammer symbols in the numerator of a series will return a polynomial.

Should a Pochhammer in the denominator of a series be non-positive, then that function will diverge, and the series is undefined.

C.4 Analytical Continuations

While the functions as defined above have a limited region of convergence, there are many formulae that extend the region of convergence for each. While there are truly numerous such functions, we shall restrict ourselves to list the expressions that have been used in this thesis.

For the hypergeometric function,

$$ {}_2F_1[\alpha,\beta,\gamma,z] = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\ {}_2F_1[\alpha,\alpha-\gamma+1,\alpha-\beta+1,\frac{1}{z}] + \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}\ {}_2F_1[\beta,\beta-\gamma+1,\beta-\alpha+1,\frac{1}{z}], \tag{C.4.1} $$

where $\beta-\alpha\neq\mathbb{Z}$ so that neither of $\Gamma(\beta-\alpha)$ and $\Gamma(\alpha-\beta)$ diverge.

$$ {}_2F_1[\alpha,\beta,\gamma,z] = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}{}_2F_1[\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z](1-z)^{\gamma-\alpha-\beta} + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{}_2F_1[\alpha,\beta,\alpha+\beta-\gamma+1,1-z] \tag{C.4.2} $$

where $\gamma-\alpha-\beta\neq\mathbb{Z}$ for the same reasons as above.

C.5 Identities

For the hypergeometric functions we will extensively use the two “half-Gauss” transformations, and the combined “full-Gauss” transformation:

$$ {}_2F_1[\alpha,\beta,\gamma,z] = (1-z)^{-\alpha}{}_2F_1[\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}], \tag{C.5.1} $$

$$ {}_2F_1[\alpha,\beta,\gamma,z] = (1-z)^{-\beta}{}_2F_1[\gamma-\alpha,\beta,\gamma,\frac{z}{z-1}], \tag{C.5.2} $$

and

$$ {}_2F_1[\alpha,\beta,\gamma,z] = (1-z)^{\gamma-\alpha-\beta}{}_2F_1[\gamma-\alpha,\gamma-\beta,\gamma,z] \tag{C.5.3} $$

For the Appell functions,

$$ F_1[\alpha;\beta,\beta';\gamma;x,y] = (1-x)^{-\alpha}F_1[\alpha;\gamma-\beta-\beta',\beta';\gamma';\frac{x}{x-1},\frac{x-y}{x-1}] \tag{C.5.4} $$

C.6 Reduction Formulae

In certain special cases, more complicated hypergeometric functions can reduce to simpler functions. Again while there are many such examples we will state those that are explicitly used in this thesis

$$F_2[\alpha;\beta,\beta';\gamma,\alpha;x,y] =(1-x)^{-\beta'}F_1[\beta,\alpha-\beta',\beta',\gamma,x,\frac{x}{1-y}], \tag{C.6.1}$$

$$F_2[\alpha;\beta,\beta';\alpha,\gamma';x,y] =(1-x)^{-\beta}F_1[\beta',\beta,\alpha-\beta,\gamma',\frac{y}{1-x},y], \tag{C.6.2}$$

$$F_1[\alpha,\beta,\beta',\beta+\beta',x,y] =(1-y)^{-\alpha}{}_2F_1[\alpha,\beta,\beta+\beta',\frac{x-y}{1-y}] \tag{C.6.3}$$

$$F_2[\alpha,\beta,\beta',\beta,\gamma',x,y] =(1-x)^{-\alpha}{}_2F_1[\alpha,\beta',\gamma',\frac{y}{1-x}] \tag{C.6.4}$$

C.7 Derived Identities

The following identities will appear many times over the course of our calculations in the context of resummation when using the A trick in integration.

$$\sum_r \frac{(a,r)}{r!}F_1[a+r,b,c,d,x,y] = (-x)^{-a}\frac{\Gamma(d)\Gamma(b+c-a)}{\Gamma(b+c)\Gamma(d-a)}{}_2F_1[a,c,b+c,1-\frac{y}{x}], \tag{C.7.1}$$

and

$$\begin{aligned}\sum_{rt} \frac{(a,r+t)}{r!t!}X^t{}_2F_1[a+r+t,b,c,Y] =& \sum_{qrt}\frac{(a,q+r+t)}{q!r!t!}\frac{(b,q)}{(c,q)}X^tY^q \\ =&(-X)^{-a}{}_2F_1[a,b,c,-\frac{Y}{X}].\end{aligned} \tag{C.7.2}$$

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