Large deviation for slow-fast McKean-Vlasov stochastic differential equations driven by fractional Brownian motions and Brownian motions

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Abstract

In this article, we consider slow-fast McKean-Vlasov stochastic differential equations driven by fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$ and Brownian motions. We give a definition of the large deviation principle (LDP) on the product space related to fractional Brownian motion and Brownian motion, which is different from the traditional definition for LDP. Under some proper assumptions on coefficients, LDP is investigated for this type of equations by using the weak convergence method.

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1 Introduction

The LDP theory is used to solve the asymptotic behaviour of rare events and has a wide range of applications such as in finance, statistic mechanics, biology, etc., see, e.g., [9, 37, 30, 31] and references therein. From the literature, we can see that there exist two main methods to study the LDP, one is based on contraction principle, that is, it relies on approximation arguments and exponential-type probability estimates, see, e.g., [12, 22, 32] and references therein. The other one is the weak convergence method due to [10]. This method has been proved to be very effective for slow-fast stochastic models, etc., see, e.g., [38, 13, 14, 29, 41, 16].

On the other hand, many researchers are interested in Mckean-Vlasov stochastic differential equations (SDEs) driven by Brownian motion , in which the coefficients depend not only on the state process but also on its distribution. McKean-Vlasov SDEs, being clearly more involved than Itô's SDEs, arise in McKean [25], who was inspired by Kac's Programme in Kinetic Theory [19], as well as in some other areas of high interest such as propagation of chaos phenomenon, PDEs, stability, invariant probability measures, social science, economics, engineering, etc. (see e.g. [5, 8, 6, 24, 26, 43, 15, 36]). Later, the averaging principle and LDP for slow-fast McKean-Vlasov SDEs driven by Brownian motions have been studied in many papers. We refer the reader to [17, 18, 21, 39].

Recently, the averaging principle and LDP for McKean-Vlasov SDEs driven by fractional Brownian motions have been investigated. Fan et al. [11] studied the asymptotic behaviors for distribution dependent SDEs driven by fractional Brownian motions. Shen et al. [35] and Zhang et al. [42] built the the averaging principle for distribution dependent SDEs driven by fractional Brownian motions. Aidara et al. [1] analyzed the averaging principle for BSDEs driven by two mutually independent fractional Brownian motions. As far as we are concerned, there is no literature to discuss the LDP for slow-fast McKean-Vlasov SDEs driven by fractional Brownian motions and Brownian motions.

In the paper, we shall consider the following slow-fast McKean-Vlasov SDEs driven by fractional Brownian motions and Brownian motions:

$$\left\{\begin{array}{l} \mathrm{d}X_t^\delta = f_1(X_t^\delta, \mathscr{L}_{X_t^\delta}, Y_t^\delta)\mathrm{d}t + g_1(X_t^\delta, \mathscr{L}_{X_t^\delta})\mathrm{d}W_t^1 + l(\mathscr{L}_{X_t^\delta})\mathrm{d}B_t^H, X_0^\delta = x, \\ \mathrm{d}Y_t^\delta = \frac{1}{\delta}b(X_t^\delta, \mathscr{L}_{X_t^\delta}, Y_t^\delta)\mathrm{d}t + \frac{1}{\sqrt{\delta}}\sigma_1(X_t^\epsilon, \mathscr{L}_{X_t^\delta}, Y_t^\delta)\mathrm{d}W_t^1 \\ \qquad\quad +\frac{1}{\sqrt{\delta}}\sigma_2(X_t^\delta, \mathscr{L}_{X_t^\delta}, Y_t^\delta)\mathrm{d}W_t^2, Y_0^\delta = y, \end{array}\right. \tag{1.1}$$

where $b, f_1 : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$, $g_1 : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d_1}$, $l : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d_1}$, $\sigma_1 : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$, $\sigma_2 : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d\times d_2}$, W^1, W^2 are d_1-dimensional Brownian motion and d_2-Brownian motion, respectively, B^H is a d_1-dimensional fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. W^1, W^2, B^H are mutually independent. Moreover, by similar reasons in [11], the integral with respect to B^H is interpreted in the Wiener sense due to the fact that $l(\mathscr{L}_{X_t^\delta})$ is deterministic. In slow-fast system, the small parameter $0 < \delta \ll 1$ represents the separation of time scales between the slow component X_t^δ (which can be thought of as the mathematical model for a phenomenon appearing at the natural time scale) and the fast motion Y_t^δ (which can be interpreted as the fast varying environment). Thus, it characterizes the ratio of timescales between processes X_t^δ and Y_t^δ. Obviously, it contains some essential differences compared with conventional SDEs. Since the extensive separation of time scales and the cross interaction between slow and fast motion, the slow-fast MVSDEs has been proven that it is more difficult to handle. Hence, a simplified equation which controls the evolution of the system over the long time scale is highly desirable. In this direction, the theory of averaging principle provides a good tool in research on convergence rate of rare events for the slow component.

The main contributions are as follows:

- We give the definition of LDP on the product space related to fractional Brownian motion and Brownian motion, which is different from the traditional definition for LDP.
- Two LDP criteria are given on the product space related to fractional Brownian motion and Brownian motion.
- The LDP is derived for slow-fast McKean-Vlasov SDEs (1.1) by using weak convergence method.

We close this part by giving our organization for this article. In Section 2, we introduce some necessary notations and assumptions. In Section 3, We give our main results. Throughout this paper, we make the following convention: the letter $C(\eta)$ with or without indices will denote different positive constants which depends on η, whose value may vary from one place to another. The letter C with or without subscripts denotes an unimportant positive constant and its value may vary in different cases.

2 Preliminaries

2.1 Notations

Throughout this paper, denote $C_0([0,T],\mathbb{R}^d)$ by the continuous functions vanishing at 0 equipped with the supremum norm. Let $\Omega_1 = C_0(0,T;\mathbb{R}^{d_1}), \Omega_2 = C_0(0,T;\mathbb{R}^{d_2}), \Omega_3 = C_0(0,T;\mathbb{R}^{d_1})$. $\mathscr{F}_i, i = 1,2,3$ are the Borel $\sigma-$algrebra, $\mathbb{F}_i := \{\mathscr{F}_t^i, t \in [0,T]\}$ are the $\sigma-$algebra filtration and $P^i, i = 1,2,3$ are the probability on $\Omega_i,, i = 1,2,3$ such that $W_\cdot^1$ is a d_1-Brownian motion on $(\Omega_1, \mathscr{F}_1, \mathbb{F}_1, P^1)$, $W_\cdot^2$ is a d_2-Brownian motion on $(\Omega_2, \mathscr{F}_2, \mathbb{F}_2, P^2)$ and $B_\cdot^H$ is a d_1-fractional Brownian motion $(\Omega_3, \mathscr{F}_3, \mathbb{F}_3, P^3)$. Let $(\Omega := \Omega_1 \times \Omega_2 \times \Omega_3, \mathscr{F} := \mathscr{F}_1 \times \mathscr{F}_2 \times \mathscr{F}_3, \mathbb{F} := \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3, P := P^1 \times P^2 \times P^3)$, where $\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 := \{\mathscr{F}_t^1 \times \mathscr{F}_t^2 \times \mathscr{F}_t^3, t \in [0,T]\}$, be the product space. Then, W^1, W^2, B^H are mutually independent. If $x, y \in \mathbb{R}^d$, we use $|x|$ to denote the Euclidean norm of x, and use $\langle x, y\rangle$ or xy to denote the Euclidean inner product. If A is a matrix, A^T is the transpose of A, and $|A|$ represents $\sqrt{\mathrm{Tr}(AA^T)}$. Moreover, let $\lfloor a \rfloor$ be the integer parts of a. Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel $\sigma-$algebra on $\mathbb{R}^d$, $C(\mathbb{R}^d)$ denotes all continuous functions on $\mathbb{R}^d$ and $C^k(\mathbb{R}^d)$ denotes all continuous functions on $\mathbb{R}^d$ with continuous partial derivations of order up to k. Let $\mathscr{P}(\mathbb{R}^d)$ be the space of all probability measures on $\mathscr{B}(\mathbb{R}^d)$, and $\mathscr{P}_p(\mathbb{R}^d)$ denotes the space of all probability measures defined on $\mathscr{B}(\mathbb{R}^d)$ with finite pth moment:

$$[\mu(|\cdot|^p)]^{\frac{1}{p}} := \left(\int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x)\right)^{\frac{1}{p}} < \infty.$$

For $\mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$, we define the Wasserstein distance for $p \geq 1$ as follows:

$$W_p(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left\{\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \pi(\mathrm{d}x, \mathrm{d}y)\right\}^{\frac{1}{p}},$$

where $\Pi(\mu,\nu)$ is the family of all coupling for μ, ν.

2.2 Fractional integral and derivative

This aim of this section is to introduce some notion and notation of fractional calculus involving fractional integral and derivative, Wiener space associated to fractional Brownian motion.

We mention that $B^H = (B^{H,1}, \cdots, B^{H,d})$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with the covariance function $\mathbb{E}(B_t^{H,i} B_s^{H,j}) = R_H(t,s)\delta_{i,j}$, where

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,T].$$

Then, the following results hold:

1° $\mathbb{E}(|B_t^{H,i} - B_s^{H,i}|^q) = C_q|t-s|^{qH}$ for every $q \geq 1$ and $i = 1, \cdots, d$.

2° B^H is $(H-\varepsilon)$-order Hölder continuous a.s. for any $\varepsilon \in (0, H)$ and is an H-self similar process.

For any $a, b \in \mathbb{R}$ with $a < b$, $f \in L^1([a,b], \mathbb{R})$ and $\alpha > 0$, the right-sided (respectively left-sided) fractional Riemann-Liouville integral of f of order α on $[a,b]$ is defined as

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} \mathrm{d}y \tag{2.1}$$

$$\left(\text{respectively } \; I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y \right),$$

where $x \in (a,b)$ a.e., $(-1)^{-\alpha} = e^{-i\alpha\pi}$ and Γ denotes the Gamma function.

Fractional differentiation may be given as an inverse operation. Let $\alpha \in (0,1)$ and $p \geq 1$. If $f \in I_{a+}^{\alpha}(L^p([a,b], \mathbb{R}))$ (respectively $I_{b-}^{\alpha}(L^p([a,b], \mathbb{R}))$), there exists a unique function g in $L^p([a,b], \mathbb{R})$ satisfying $f = I_{a+}^{\alpha} g$ (respectively $f = I_{b-}^{\alpha} g$) and it coincides with the left-sided (respectively right-sided) Riemann-Liouville derivative of f of order α shown by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} \mathrm{d}y$$

$$\left(\text{respectively } D_{b-}^{\alpha} f(x) = \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} \mathrm{d}y \right).$$

The corresponding Weyl representation is as follows:

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} \mathrm{d}y \right) \tag{2.2}$$

$$\left(\text{respectively } \; D_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} \mathrm{d}y \right) \right).$$

Obviously, from [34], the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all x if $p = 1$ and in the L^p sense if $p > 1$.

2.3 Wiener space associated to fractional Brownian motion

Denote by $\mathscr{E}$ the set of step functions on $[0,T]$ and the Hilbert space $\mathcal{H}$ is the closure of $\mathscr{E}$ with respect to the scalar product

$$\left\langle (\mathbb{I}_{[0,t_1]}, \cdots, \mathbb{I}_{[0,t_{d_1}]}), (\mathbb{I}_{[0,s_1]}, \cdots, \mathbb{I}_{[0,s_{d_1}]}) \right\rangle_{\mathcal{H}} = \sum_{i=1}^{d_1} R_H(t_i, s_i).$$

In order to give the integral representation of $R_H(t,s)$, we set

$$K_H(t,s) := \Gamma\left(H + \frac{1}{2}\right)^{-1} (t-s)^{H-\frac{1}{2}} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right),$$

in which $F(\cdot,\cdot,\cdot,\cdot)$ is the Gauss hypergeometric function (see [7] or [27]). K_H is a square integrable kernel and $R_H(t,s)$ has the following integral representation from [7]:

$$R_H(t,s)=\int_0^{t\wedge s}K_H(t,r)K_H(s,r)\mathrm{d}r.$$

Furthermore, the mapping $(\mathbb{I}_{[0,t_1]},\cdots,\mathbb{I}_{[0,t_{d_1}]})\mapsto\sum_{i=1}^{d_1}B_{t_i}^{H,i}$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B^H. Denote this isometry by $\varphi\mapsto B^H(\varphi)$. Consequently, due to [2] again, B^H has the following Volterra-type representation

$$B_t^H=\int_0^t K_H(t,s)\mathrm{d}W_s,\quad t\in[0,T], \tag{2.3}$$

where W is some d_1-dimensional Wiener process defined on $(\Omega_3,\mathscr{F}_3,P^3)$.

Besides, define the operator $K_H: L^2([0,T],\mathbb{R}^{d_1})\to I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^{d_1}))$ by

$$(K_Hf)(t)=\int_0^t K_H(t,s)f(s)\mathrm{d}s.$$

According to [7], we obtain that it is an isomorphism and for each $f\in L^2([0,T],\mathbb{R}^{d_1})$,

$$(K_Hf)(s)=\begin{cases} I_{0+}^{2H}s^{1/2-H}I_{0+}^{1/2-H}s^{H-1/2}f, & H\in(0,1/2),\\ I_{0+}^{1}s^{H-1/2}I_{0+}^{H-1/2}s^{1/2-H}f, & H\in(1/2,1).\end{cases}$$

Thus, for any $g\in I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^{d_1}))$, the inverse operator of K_H is given by

$$(K_H^{-1}g)(s)=\begin{cases} s^{1/2-H}D_{0+}^{1/2-H}s^{H-1/2}D_{0+}^{2H}g, & H\in(0,1/2),\\ s^{H-1/2}D_{0+}^{H-1/2}s^{1/2-H}g', & H\in(1/2,1).\end{cases} \tag{2.4}$$

In particular, if g is absolutely continuous, we have

$$(K_H^{-1}g)(s)=s^{H-\frac{1}{2}}I_{0+}^{\frac{1}{2}-H}s^{\frac{1}{2}-H}g',\quad H\in(0,1/2). \tag{2.5}$$

Now, define the linear operator $K_H^*:\mathscr{E}\to L^2([0,T],\mathbb{R}^{d_1})$ as follows

$$(K_H^*\varphi)(s)=K_H(T,s)\varphi(s)+\int_s^T(\varphi(r)-\varphi(s))\frac{\partial K_H}{\partial r}(r,s)\mathrm{d}r.$$

According to [2], we have the following the relation for any $\varphi,\phi\in\mathscr{E}$,

$$\langle K_H^*\varphi,K_H^*\phi\rangle_{L^2([0,T],\mathbb{R}^{d_1})}=\langle\varphi,\phi\rangle_{\mathcal{H}}=H(2H-1)\int_0^T\int_0^T|t-s|^{2H-1}\langle\varphi(s),\phi(t)\rangle\mathrm{d}s\mathrm{d}t. \tag{2.6}$$

and then the bounded linear transform theorem implies that K_H^* can be extended to an isometry between $\mathcal{H}$ and $L^2([0,T],\mathbb{R}^{d_1})$.

Note that the injection $R_H = K_H \circ K_H^* : \mathcal{H} \to \Omega_3$ embeds $\mathcal{H}$ densely into Ω_3 and for every $\psi \in \Omega_3^* \subset \mathcal{H}$ it holds that $\mathbb{E}e^{i\langle B^H,\psi\rangle} = \exp(-12\|\psi\|_{\mathcal{H}}^2)$. By (2.6) in [11], for any $\varphi \in \mathcal{H}$, we derive that

$$(R_H\varphi)(t) = \int_0^t \int_0^s \frac{\partial K_H}{\partial s}(s,r)(K_H^*\varphi)(r)\mathrm{d}r\mathrm{d}s. \tag{2.7}$$

Next, we introduce some results about the Malliavin calculus for fractional Brownian motion. Let $\mathscr{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \cdots, B^H(\varphi_n)),$$

where $n \geq 1, f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}, 1 \leq i \leq n$. The Malliavin derivative of F is given as follows:

$$\mathbb{D}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \cdots, B^H(\varphi_n))\varphi_i,$$

and denoted by $\mathbb{D}F$.

For any $p \geq 1$, we denote by $\mathbb{D}^{1,p}$ the Sobolev space which is the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|\mathbb{D}F\|_{\mathcal{H}}^p.$$

Denote by δ and Domδ the dual operator of $\mathbb{D}$ and its domain, respectively. The following results are needed later. By [28, Proposition 5.2.1] and [28, Proposition 5.2.2], one has

Proposition 2.1. *Denote by $\mathbb{D}^W$ the derivative operator with respect to the underlying Wiener process W in* (2.3)*, and $\mathbb{D}_W^{1,2}$ the corresponding Sobolev space. For every $F \in \mathbb{D}_W^{1,2} = \mathbb{D}^{1,2}$, we obtain that*

$$K_H^*\mathbb{D}F = \mathbb{D}^W F,$$

Proposition 2.2. Dom$\delta = (K_H^*)^{-1}(\text{Dom}\delta_W)$*, and it holds that $\delta(u) = \delta_W(K_H^* u)$ for any $\mathcal{H}$-valued random variable u in* Domδ*, where δ_W represents the divergence operator corresponding to the underlying Wiener process W in* (2.3).

Remark 2.1. *The above proposition and [28, Proposition 1.3.11] implies that $u \in$* Domδ *if $K_H^* u \in L_a^2([0,T] \times \Omega, \mathbb{R}^{d_1})$ (the closed subspace of $L^2([0,T] \times \Omega, \mathbb{R}^{d_1})$ formed by the adapted processes).*

Finally, we complete this section by giving the following notation for future use.

$$\mathcal{A} := \left\{ \bar{h} \text{ is } \mathbb{R}^{d_1} \text{ valued } \mathscr{F}_t - \text{predictable process such that } \|\bar{h}\|_{\mathcal{H}} < \infty \ P-a.e. \right\}.$$
$$\mathcal{S}_N := \{\bar{h} \in \mathcal{H} : \|\bar{h}\|_{\mathcal{H}}^2 \leq N\}.$$
$$\mathcal{A}_N := \left\{ \bar{h} \in \mathcal{A} : \bar{h}(\omega) \in \mathcal{S}_N \ P-a.e. \right\}.$$

$\mathbb{H} := \Big\{ h = \int_0^\cdot \dot{h}(s)\mathrm{d}s : \dot{h} \in L^2(0,T;\mathbb{R}^{d_1+d_2})$ with the norm

$$\|h\|_{\mathbb{H}} := \Big(\int_0^T |\dot{h}(s)|^2 \mathrm{d}s \Big)^{\frac{1}{2}} < \infty \Big\}.$$

$\mathbb{A} := \Big\{ h$ is $\mathbb{R}^{d_1+d_2}$ valued $\mathscr{F}_t -$ predictable process such that $\int_0^T |\dot{h}(s)|^2 \mathrm{d}s < \infty \ P - a.e. \Big\}$.

$\mathbb{S}_N := \{ h \in \mathbb{H} : \int_0^T |\dot{h}(s)|^2 \mathrm{d}s \leq N \}$.

$\mathbb{A}_N := \Big\{ h \in \mathbb{A} : h(\omega) \in \mathbb{S}_N \ P - a.e. \Big\}$.

For $h_\epsilon \in \mathbb{A}_N, h \in \mathbb{A}_N$, $h_\epsilon \Rightarrow h$ means h_ϵ converges in distribution to h as $\epsilon \to 0$.
For $\bar{h}_\epsilon \in \mathcal{A}_N, \bar{h} \in \mathcal{A}_N$, $\bar{h}_\epsilon \Rightarrow \bar{h}$ means $\bar{h}_\epsilon$ converges in distribution to $\bar{h}$ as $\epsilon \to 0$.

2.4 Assumptions

We now impose the following condition on coefficients:

(H1) Let Λ and λ be two positive constants such that the following conditions hold for any $x, x_1, x_2 \in \mathbb{R}^d, \mu, \mu_1, \mu_2 \in \mathscr{P}_2(\mathbb{R}^d), y, y_1, y_2 \in \mathbb{R}^d$:

$$\begin{aligned} & |f_1(x_1, \mu_1, y_1) - f_1(x_2, \mu_2, y_2)| + |b(x_1, \mu_1, y_1) - b(x_2, \mu_2, y_2)| \\ & \ + |\sigma_1(x_1, \mu_1, y_1) - \sigma_1(x_2, \mu_2, y_2)| + |\sigma_2(x_1, \mu_1, y_1) - \sigma_2(x_2, \mu_2, y_2)| \\ & \ \leq \Lambda(|x_1 - x_2| + |y_1 - y_2| + W_2(\mu_1, \mu_2)), \qquad (2.8) \\ & |g_1(x_1, \mu_1) - g_1(x_2, \mu_2)| \leq \Lambda(|x_1 - x_2| + W_2(\mu_1, \mu_2)), \qquad (2.9) \\ & |l(\mu_1) - l(\mu_2)| \leq \Lambda W_2(\mu_1, \mu_2), \qquad (2.10) \end{aligned}$$

and

$$\begin{aligned} & 4\langle b(x, \mu, y_1) - b(x, \mu, y_2), y_1 - y_2 \rangle + 6|\sigma_1(x, \mu, y_1) - \sigma_1(x, \mu, y_2)|^2 \\ & \quad + 6|\sigma_2(x, \mu, y_1) - \sigma_2(x, \mu, y_2)|^2 \leq -\lambda |y_1 - y_2|^2. \qquad (2.11) \end{aligned}$$

Firstly, we give several uniform estimations $w.r.t.$ $\delta \in (0,1)$ for the 4th moment of solution (X^δ, Y^δ) of Eq. (1.1).

Lemma 2.3. *Assume* (H1). *Then the following inequalities hold:*

$$\sup_{\delta \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}[|X_t^\delta|^4] \leq C(T)(1 + |x|^4 + |y|^4),$$

$$\sup_{\delta \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}[|Y_t^\delta|^4] \leq C(T)(1 + |x|^4 + |y|^4).$$

Proof. By Eq. (1.1), we have

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}|X_s^\delta|^4] &\le C|x|^4 + C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s f_1(X_r^\delta,\mathscr{L}_{X_r^\delta},Y_r^\delta)\mathrm{d}r\bigg|^4\bigg]\\ &\quad + C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s g_1(X_r^\delta,\mathscr{L}_{X_r^\delta},Y_r^\delta)\mathrm{d}W_r\bigg|^4\bigg]\\ &\quad + C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s l(\mathscr{L}_{X_r^\delta})\mathrm{d}B_r^H\bigg|^4\bigg]\\ &=: C|x|^4 + I_1 + I_2 + I_3. \end{aligned}\tag{2.12}$$

where C is a constant.

For the term I_1, by the fact $W_2^4(\mathscr{L}_{X_t},\delta_0)\le \mathbb{E}[|X_t|^4]$ and (H1), we derive

$$\begin{aligned} I_1 &= C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s f_1(X_r^\delta,\mathscr{L}_{X_r^\delta},Y_r^\delta)]\mathrm{d}r\bigg|^4\bigg]\\ &\le C(T)\mathbb{E}\int_0^t(|X_s^\delta|^4+|Y_s^\delta|^4)\mathrm{d}s + C(T). \end{aligned}$$

Next, we look at I_2. The Burkholder-Davis-Gundy(BDG) inequality and (H1) yield that

$$\begin{aligned} I_2 &= C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s g_1(X_r^\delta,\mathscr{L}_{X_r^\delta})\mathrm{d}W_r\bigg|^4\bigg]\\ &\le C(T)\mathbb{E}\bigg[\bigg|\int_0^t |g_1(X_t^\delta,\mathscr{L}_{X_r^\delta})|^2\mathrm{d}s\bigg|^2\bigg]\\ &\le C(T)\mathbb{E}\int_0^t |X_s^\delta|^4\mathrm{d}s + C(T). \end{aligned}$$

Finally, for I_3, it follows from (4.1) in [11] and (H1), we obtain

$$\begin{aligned} I_3 &= C\mathbb{E}\bigg[\sup_{0\le s\le t}\bigg|\int_0^s l(\mathscr{L}_{X_r^\delta})\mathrm{d}B_r^H\bigg|^4\bigg]\\ &\le C(T,H)\mathbb{E}\bigg[\bigg|\int_0^t |l(\mathscr{L}_{X_s^\delta})|^4\mathrm{d}s\bigg|\bigg]\\ &\le C(T,H)\mathbb{E}\int_0^t |X_s^\delta|^4\mathrm{d}s + C(T). \end{aligned}$$

These imply

$$\mathbb{E}[\sup_{0\le s\le t}|X_s^\delta|^4] \le C_1|x|^4 + C(T) + C(T,H)\int_0^t (\mathbb{E}[\sup_{0\le r\le s}|X_r^\delta|^4] + \mathbb{E}[\sup_{0\le r\le s}|Y_r^\delta|^4])\mathrm{d}s.$$

An application of Itô's formula yields that

$$\mathbb{E}[|Y_t^\delta|^4] = |y|^4 + \frac{4}{\delta}\mathbb{E}\int_0^t |Y_s^\delta|^2\langle Y_s^\delta, b(s,X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)\rangle\mathrm{d}s$$

$$
\begin{aligned}
&+\frac{2}{\delta}\mathbb{E}\int_0^t |Y_s^\delta|^2|\sigma_1(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)|^2\mathrm{d}s\\
&+\frac{4}{\delta}\mathbb{E}\int_0^t |\langle Y_s^\delta,\sigma_1(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)\rangle|^2\mathrm{d}s\\
&+\frac{2}{\delta}\mathbb{E}\int_0^t |Y_s^\delta|^2|\sigma_2(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)|^2\mathrm{d}s\\
&+\frac{4}{\delta}\mathbb{E}\int_0^t |\langle Y_s^\delta,\sigma_2(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)\rangle|^2\mathrm{d}s. \qquad (2.13)
\end{aligned}
$$

By (2.11), there exists $\alpha' > 0$ such that for any $t \in [0,T]$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|Y_t^\delta|^4] &\le \frac{1}{\delta}\mathbb{E}[4|Y_t^\delta|^2\langle Y_t^\delta, b(X_t^\delta,\mathscr{L}_{X_t^\delta},Y_t^\delta)\rangle + 6|Y_s^\delta|^2|\sigma_1(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)|^2]\\
&\quad + 6|Y_s^\delta|^2|\sigma_2(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)|^2]\\
&\le -\frac{\alpha'}{\delta}\mathbb{E}[|Y_t^\delta|^4] + \frac{C(T)}{\delta}(\mathbb{E}[|X_t^\delta|^4]+1).
\end{aligned}
$$

It follows from the comparison theorem that

$$
\begin{aligned}
\mathbb{E}[|Y_t^\delta|^4] &\le |y|^4 e^{-\frac{\alpha' t}{\delta}} + \frac{C(T)}{\delta}\int_0^t e^{-\frac{\alpha'(t-s)}{\delta}}(\mathbb{E}[|X_s^\delta|^4]+1)\mathrm{d}s\\
&\le |y|^4 + C(T)(\sup_{0\le s\le t}\mathbb{E}[|X_t^\delta|^4]+1).
\end{aligned}
$$

This together with (2.13) implies

$$
\sup_{0\le s\le t}\mathbb{E}[|X_t^\delta|^4] \le C(T)|y|^4 + C(T)\int_0^t \sup_{0\le r\le s}\mathbb{E}[|X_r^\delta|^4]\mathrm{d}s.
$$

Gronwall's inequality gives

$$
\sup_{0<\delta<1}\sup_{0\le s\le t}\mathbb{E}[|X_t^\delta|^4] \le C(T)(1+|x|^4+|y|^4).
$$

which also implies

$$
\sup_{0<\delta<1}\sup_{0\le s\le t}\mathbb{E}[|Y_t^\delta|^4] \le C(T)(1+|x|^4+|y|^4).
$$

□

Lemma 2.4. *Assume* (H1). *For any* $0 \le t \le t+u \le T$, *we have the following inequality.*

$$
\mathbb{E}[|X_{t+u}^\delta - X_t^\delta|^2] \le C(T)(1+|x|^2+|y|^2)u.
$$

Proof. By (1.1), we have

$$
X_{t+u}^\delta - X_t^\delta = \int_t^{t+u} f_1(X_s^\delta,\mathscr{L}_{X_s^\delta},Y_s^\delta)\mathrm{d}s
$$

$$+\int_t^{t+u} g_1(X_s^\delta, \mathscr{L}_{X_s^\delta})\mathrm{d}W_s^1 + \int_t^{t+u} l(\mathscr{L}_{X_s^\delta})\mathrm{d}B_s^H.$$

Set $\hat{X}_t := X_{t+u}^\delta - X_t^\delta$. Analogous to the calculation of (2.12), it holds that

$$\begin{aligned}
\mathbb{E}[|\hat{X}(t)|^2] &\leq C\mathbb{E}\left[\left|\int_t^{t+u} f_1(X_r^\delta, \mathscr{L}_{X_r^\delta}, Y_r^\delta)\mathrm{d}r\right|^2\right] \\
&+ C\mathbb{E}\left[\left|\int_t^{t+u} g_1(X_r^\delta, \mathscr{L}_{X_r^\delta}, Y_r^\delta)\mathrm{d}W_r\right|^2\right] + C\mathbb{E}\left[\left|\int_t^{t+u} l(\mathscr{L}_{X_r^\delta})\mathrm{d}B_r^H\right|^2\right] \\
&\leq C(T)u\int_t^{t+u} (\mathbb{E}[1+|X_s^\delta|^2] + \mathbb{E}[|Y_s^\delta|^2])\mathrm{d}s + C(T)\int_t^{t+u} (\mathbb{E}[1+|X_s^\delta|^2] + \mathbb{E}[|Y_s^\delta|^2])\mathrm{d}s.
\end{aligned}$$

Then, Lemma 2.3 leads to the required assertion. □

2.5 The averaged equation.

We now introduce the following parameterized McKean-Vlasov equation: for fixed $t \geq 0, x \in \mathbb{R}^d, \mu \in \mathscr{P}_2(\mathbb{R}^d)$, let

$$\begin{aligned}
\mathrm{d}Y_s^{x,\mu,y} &= b(x,\mu,Y_s^{x,\mu,y})\mathrm{d}s + \sigma_1(x,\mu,Y_s^{x,\mu,y})\mathrm{d}\tilde{W}_s^1 \\
&\quad + \sigma_2(x,\mu,Y_s^{x,\mu,y})\mathrm{d}\tilde{W}_s^2, \quad Y_0^{x,\mu,y} = y,
\end{aligned} \tag{2.14}$$

where $\widetilde{W}_\cdot^1, \widetilde{W}_\cdot^2$ are d_1-dimensional and d_2-dimensional Brownian motions respectively and mutually independent, on another complete probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$ and $\{\widetilde{\mathscr{F}}_t\}_{t\geq 0}$ is the natural filtration generated by $\widetilde{W}_t^1, \widetilde{W}_t^2, B_t^H$. Since the coefficients of parameterized McKean-Vlasov equation satisfy the Lipschitz conditions, under (H1), similar to the Eq.(3.3) in [33], Eq.(2.14) has a unique strong solution $\{Y_s^{t,x,\mu,y}\}_{s\geq 0}$ and it is a homogeneous Markov process with the following estimate

$$\sup_{0\leq s\leq T} \widetilde{\mathbb{E}}[|Y_s^{x,\mu,y}|^2] \leq C(T)(1+|x|^2+|y|^2+\mu(|\cdot|^2)).$$

Let $\{P_s^{x,\mu}\}_{s\geq 0}$ be the transition semigroup of $Y_s^{x,\mu,y}$. By [20, Theorem 4.3.9], under (H1), $\{P_s^{x,\mu}\}_{s\geq 0}$ has a unique invariant measure $\nu^{x,\mu}$ satisfying

$$\int_{\mathbb{R}^m} |y|\nu^{x,\mu}(\mathrm{d}y) \leq C(T)(1+|x|+[\mu(|\cdot|^2)]^{\frac{1}{2}}).$$

Set

$$\bar{f}(x,\mu) := \int_{\mathbb{R}^d} f_1(x,\mu,z)\nu^{x,\mu}(\mathrm{d}z).$$

We have the following lemma.

Lemma 2.5. *$\bar{f}$ satisfies the following Lipschitz conditions, i.e.*

$$|\bar{f}(x_1,\mu_1) - \bar{f}(x_2,\mu_2)| \leq C(|x_1-x_2| + W_2(\mu_1,\mu_2)).$$

Proof. Based on the definition of $\bar{f}$, the Lipschitz continuity of f_1 and the definition of Wasserstein metric, we derive

$$
\begin{aligned}
&|\bar{f}(x_1,\mu_1)-\bar{f}(x_2,\mu_2)| \\
&= \left|\int_{\mathbb{R}^{d_1}} f_1(x_1,\mu_1,z)\nu^{x_1,\mu_1}(\mathrm{d}z) - \int_{\mathbb{R}^{d_1}} f_1(x_2,\mu_2,z)\nu^{x_2,\mu_2}(\mathrm{d}z)\right| \\
&\leq \left|\int_{\mathbb{R}^{d_1}} [f_1(x_1,\mu_1,z)-f_1(x_2,\mu_2,z)]\nu^{x_1,\mu_1}(\mathrm{d}z)\right| \\
&\quad + \left|\int_{\mathbb{R}^{d_1}} f_1(x_2,\mu_2,z)\nu^{x_1,\mu_1}(\mathrm{d}z) - \int_{\mathbb{R}^{d_1}} f_1(x_2,\mu_2,z)\nu^{x_2,\mu_2}(\mathrm{d}z)\right| \\
&\leq C(|x_1-x_2| + W_2(\mu_1,\mu_2) + W_2(\nu^{x_1,\mu_1},\nu^{x_2,\mu_2})).
\end{aligned}
$$

By [40, Theorem 3.1], we obtain

$$
\begin{aligned}
&W_2^2(\nu^{x_1,\mu_1},\nu^{x_2,\mu_2}) \\
&\leq 3W_2^2(\nu^{x_1,\mu_1},\mathscr{L}_{Y_s^{x,\mu_1,0}}) + 3W_2^2(\nu^{x_2,\mu_2},\mathscr{L}_{Y_s^{x,\mu_2,0}}) + 3W_2^2(\mathscr{L}_{Y_s^{x,\mu_1,0}},\mathscr{L}_{Y_s^{x,\mu_2,0}}) \\
&\leq Ce^{-2\lambda_0 s}(W_2^2(\nu^{x_1,\mu_1},\delta_0) + W_2^2(\nu^{x_1,\mu_1},\delta_0)) + C(|x_1-x_2| + W_2^2(\mu_1,\mu_2)).
\end{aligned}
$$

From the above calculation, we obtain

$$
|\bar{f}(x_1,\mu_1)-\bar{f}(x_2,\mu_2)| \leq C(|x_1-x_2| + W_2(\mu_1,\mu_2)).
$$

□

We now consider the following averaged equation:

$$
\mathrm{d}\bar{X}_t = \bar{f}(\bar{X}_t,\mathscr{L}_{\bar{X}_t})\mathrm{d}t + g_1(\bar{X}_t,\mathscr{L}_{\bar{X}_t})\mathrm{d}W_t^1 + l(\mathscr{L}_{\bar{X}_t})\mathrm{d}B_t^H, \bar{X}_0 = x. \tag{2.15}
$$

Since $\bar{f}_1, \bar{g}_1, l$ satisfy the Lipschitz condition, we have the following existence and uniqueness result for Eq. (2.15).

Theorem 2.6. *Assume* (H1). *Then Eq.*(2.15) *has a unique solution.*

2.6 Large deviation principle

Consider the following multi-scale McKean-Vlasov system with small perturbation.

$$
\begin{cases}
\mathrm{d}X_t^{\epsilon,\delta} = f_1(X_t^{\epsilon,\delta},\mathscr{L}_{X_t^{\epsilon,\delta}},Y_t^{\epsilon,\delta})\mathrm{d}t + \epsilon^H l(\mathscr{L}_{X_t^{\epsilon,\delta}})\mathrm{d}B_t^H \\
\qquad\qquad + \sqrt{\epsilon}g_1(X_t^{\epsilon,\delta},\mathscr{L}_{X_t^{\epsilon,\delta}})\mathrm{d}W_t^1, X_0^{\epsilon,\delta} = x, \\
\mathrm{d}Y_t^{\epsilon,\delta} = \frac{1}{\delta}b(X_t^{\epsilon,\delta},\mathscr{L}_{X_t^{\epsilon,\delta}},Y_t^{\epsilon,\delta})\mathrm{d}t + \frac{1}{\sqrt{\delta}}\sigma_1(X_t^{\epsilon,\delta},\mathscr{L}_{X_t^{\epsilon,\delta}},Y_t^{\epsilon,\delta})\mathrm{d}W_t^1 \\
\qquad\qquad + \frac{1}{\sqrt{\delta}}\sigma_2(X_t^{\epsilon,\delta},\mathscr{L}_{X_t^{\epsilon,\delta}},Y_t^{\epsilon,\delta})\mathrm{d}W_t^2, Y_0^{\epsilon,\delta} = y.
\end{cases} \tag{2.16}
$$

ϵ describes the intensity of the noise and δ describes the ratio of the time scale between the slow component $X_t^{\epsilon,\delta}$ and fast component $Y_t^{\epsilon,\delta}$. In the following, as $\epsilon \to 0$, we need $\delta/\epsilon \to 0$, thus the coefficient of the multi-scale stochastic system is averaged first, and then for the averaged equation with small noise, the large deviation principle is obtained by using the weak convergence method. Thus, due to the classical Yamada-Watanabe theorem, there exists a measurable map $\Gamma^{\epsilon}_{\mathscr{L}_{X_t^{\epsilon,\delta}}} : C([0,T];\mathbb{R}^{d_1+d_2}) \times C(0,T;\mathbb{R}^{d_1}) \to C(0,T;\mathbb{R}^d)$ such that we have the following representation

$$X_t^{\epsilon,\delta} = \Gamma^{\epsilon}_{\mathscr{L}_{X_t^{\epsilon,\delta}}}(\sqrt{\epsilon}W, \epsilon^H B^H),$$

where $W := (W^1, W^2)$. For simplicity of notation, we denote $\Gamma^\epsilon = \Gamma^{\epsilon}_{\mathscr{L}_{X_t^{\epsilon,\delta}}}$. For $\tilde{h}^\epsilon = (h^\epsilon, \bar{h}^\epsilon), h^\epsilon \in \mathbb{A}_N, \bar{h}^\epsilon \in \mathcal{A}_N$, $X^{\epsilon,\delta,\tilde{h}^\epsilon} := \Gamma^\epsilon\Big(\sqrt{\epsilon}W_\cdot + \int_0^\cdot \dot{h}^\epsilon(\cdot)\mathrm{d}s, \epsilon^H B^H_\cdot + R_H\bar{h}^\epsilon(\cdot)\Big)$ is the first part of solution of the following equation.

$$\begin{cases} \mathrm{d}X_t^{\epsilon,\delta,\tilde{h}^\epsilon} = f_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}t + g_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}})\mathcal{P}_1\dot{h}^\epsilon(t)\mathrm{d}t + l(\mathscr{L}_{X_t^{\epsilon,\delta}})\mathrm{d}(R_H\bar{h}^\epsilon)(t) \\ \qquad +\sqrt{\epsilon}g_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}})\mathrm{d}W_t^1 + \epsilon^H l(\mathscr{L}_{X_t^{\epsilon,\delta}})\mathrm{d}B_t^H, X_0^\epsilon = x, \\ \mathrm{d}Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} = \frac{1}{\delta}b(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}t + \frac{1}{\sqrt{\epsilon\delta}}\sigma_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_1\dot{h}^\epsilon(t)\mathrm{d}t \\ \qquad\qquad + \frac{1}{\sqrt{\epsilon\delta}}\sigma_2(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_2\dot{h}^\epsilon(t)\mathrm{d}t \\ \qquad + \frac{1}{\sqrt{\delta}}\sigma_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}W_t^1 + \frac{1}{\sqrt{\delta}}\sigma_2(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}W_t^2, \\ \qquad\qquad\qquad Y_0^{\epsilon,\delta,\tilde{h}^\epsilon} = y, \end{cases} \tag{2.17}$$

where $\mathcal{P}_1 : \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1}, \mathcal{P}_2 : \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2}$ are two projection operators. In this part, we will investigate the LDP for Eq. (2.16). We need some definitions of the theory of LDP.

Definition 2.1. *A nonnegative function I is called a rate function on $C(0,T;\mathbb{R}^d)$ if it is lower semicontinuous. Moreover, I is a good rate function if for each constant $M < \infty$, the level set $\{x \in C(0,T;\mathbb{R}^d) : I(x) \le M\}$ is a compact subset of $C(0,T;\mathbb{R}^d)$.*

Definition 2.2. *Let I be a rate function on $C(0,T;\mathbb{R}^d)$. The family $\{X^{\epsilon,\delta} := \Gamma^\epsilon(\sqrt{\epsilon}W_\cdot, \epsilon^H B^H_\cdot)\}_{\epsilon>0}$ of $C(0,T;\mathbb{R}^d)-$valued random variables is said to be satisfied a LDP on $C(0,T;\mathbb{R}^d)$ with speed $l(\epsilon) := (\epsilon, \epsilon^{2H})$ and rate function I if the following two conditions hold:*

1° *(Upper bound) For each closed subset $F \in C(0,T;\mathbb{R}^d)$,*

$$\limsup_{\epsilon\to 0} \epsilon^{2H}\log(P(X^{\epsilon,\delta} \in F)) \le -\inf_{x\in F} I(x). \tag{2.18}$$

2° *(Lower bound) For each open subset $G \in C(0,T;\mathbb{R}^d)$,*

$$\liminf_{\epsilon\to 0} \epsilon\log(P(X^{\epsilon,\delta} \in G)) \ge -\inf_{x\in G} I(x). \tag{2.19}$$

Definition 2.3. *Let I be a rate function on $C(0,T;\mathbb{R}^d)$. $\{X^{\epsilon,\delta} := \Gamma^\epsilon(\sqrt{\epsilon}W_\cdot, \epsilon^H B^H_\cdot)\}_{\epsilon>0}$ is said to be satisfied the Laplace principle upper bound (respectively, lower bound) on $C(0,T;\mathbb{R}^d)$ with speed $l(\epsilon) := (\epsilon, \epsilon^{2H})$ and rate function I if for any bounded continuous function ρ on $C(0,T;\mathbb{R}^d)$,* 1°

$$\limsup_{\epsilon\to 0} -\epsilon^{2H}\log\mathbb{E}\Big[\exp\Big(-\frac{\rho(X^{\epsilon,\delta})}{\epsilon^{2H}}\Big)\Big] \le \inf_{x\in C(0,T;\mathbb{R}^d)}\{\rho(x)+I(x)\}. \tag{2.20}$$

2° *(Lower bound) For any open subset* $G_i \in C(0,T;\mathbb{R}^d), i=1,2,$

$$\liminf_{\epsilon\to 0} -\epsilon\log\mathbb{E}\Big[\exp\Big(-\frac{\rho(X^{\epsilon,\delta})}{\epsilon}\Big)\Big] \ge \inf_{x\in C(0,T;\mathbb{R}^d)}\{\rho(x)+I(x)\}. \tag{2.21}$$

Remark 2.2. *In these definitions, we use different speeds for the upper bound and lower bound, i.e. the speed in upper bound* (2.18) *is* ϵ^{2H}*, while the speed in lower bound* (2.19) *is* ϵ*. In fact, this is reasonable. Noting that* $H \ge \frac{1}{2}$ *and* $\epsilon\in(0,1)$*, we therefore have*

$$\begin{aligned}\limsup_{\epsilon\to 0}\epsilon\log(P(X^{\epsilon,\delta}\in F)) &\le \limsup_{\epsilon\to 0}\epsilon^{2H}\log(P(X^{\epsilon,\delta}\in F)),\\ \liminf_{\epsilon\to 0}\epsilon\log(P(X^{\epsilon,\delta}\in G)) &\le \liminf_{\epsilon\to 0}\epsilon^{2H}\log(P(X^{\epsilon,\delta}\in G)).\end{aligned}$$

This implies the consistency the definition of the LDP of SDEs driven by Brownian motion or fraction Brownian motion, respectively.

We give the following sufficient condition for LDP criteria, which is a version of [4, Theorem 4.2] on the product space.

Lemma 2.7. *Assume the following conditions hold:*

1° *For any* $\epsilon>0$, $\Gamma^\epsilon : C([0,T];\mathbb{R}^{d_1+d_2})\times C(0,T;\mathbb{R}^{d_1}) \to C(0,T;\mathbb{R}^d)$ *is a measurable mapping.*

2° *Let* $\Gamma^0 : C(0,T;\mathbb{R}^{d_1+d_2})\times I_{0+}^{H+\frac{1}{2}}(L^2(0,T;\mathbb{R}^{d_1}))\to C(0,T;\mathbb{R}^d)$ *be a measurable mapping.*

3° *For every* $N>0$*, and any family* $\{h_\epsilon;\epsilon>0\}\subset\mathbb{A}_N$, $\{\bar h_\epsilon;\epsilon>0\}\subset\mathcal{A}_N$ *satisfying that* $h_\epsilon\Rightarrow h, \bar h_\epsilon\Rightarrow\bar h, \epsilon\to 0$*, then*

$$\Gamma^\epsilon\Big(\sqrt{\epsilon}W_\cdot+\int_0^\cdot \dot h_\epsilon(\cdot)\mathrm{d}s, \epsilon^H B^H_\cdot + R_H\bar h_\epsilon(\cdot)\Big)\Rightarrow\Gamma^\circ\Big(\int_0^\cdot \dot h(s)\mathrm{d}s, R_H\bar h(\cdot)\Big), \epsilon\to 0.$$

4° *For every* $N>0$*, the set* $\{\Gamma^\circ\Big(\int_0^\cdot \dot h(s)\mathrm{d}s, R_H\bar h(\cdot)\Big); h\in\mathbb{S}_N, \bar h\in\mathcal{S}_N\}$ *is a compact subset of* $C(0,T;\mathbb{R}^d)$.

Then the family $\{\Gamma^\epsilon(\sqrt{\epsilon}W_\cdot, \epsilon^H B^H_\cdot)\}$ *satisfies a large deviation principle in* $C(0,T;\mathbb{R}^d)$ *with the rate function* I *given by*

$$I(g) := \inf_{\{(h,\bar h)\in(\mathbb{H},\mathcal{H}); g=\Gamma^0(\int_0^\cdot \dot h(s)\mathrm{d}s, R_H\bar h)\}}\Big\{\frac{1}{2}\int_0^T|\dot h(s)|^2\mathrm{d}s+\frac{1}{2}\|\bar h\|^2_{\mathcal{H}}\Big\}, g\in C(0,T;\mathbb{R}^d), \tag{2.22}$$

with $\inf\emptyset=\infty$ *by convention.*

Proof. The proof is placed in the appendix. □

The following lemma is equivalent to the above one.

Lemma 2.8. *Assume the following conditions hold:*

1° *Let* $\{h^\epsilon : \epsilon > 0\} \subset \mathbb{A}_N, \{\bar{h}^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$. *For any* $\varepsilon_0 > 0$,

$$\lim_{\epsilon \to 0} \Bigg[d\Bigg(\Gamma^\epsilon\Bigg(\sqrt{\epsilon} W_\cdot + \int_0^\cdot \dot{h}_\epsilon(\cdot)\mathrm{d}s, \epsilon^H B_\cdot^H + R_H \bar{h}_\epsilon(\cdot) \Bigg), \\ \Gamma^\circ\Bigg(\int_0^\cdot \dot{h}^\epsilon(s)\mathrm{d}s, R_H \bar{h}^\epsilon(\cdot) \Bigg) \Bigg) > \varepsilon_0 \Bigg] \\ = 0,$$

where $d(\cdot, \cdot)$ *stands for the metric on* $C(0, T; \mathbb{R}^d)$.

2° *Let* $\{h^\epsilon\} \subset \mathbb{S}_N, \{\bar{h}^\epsilon\} \subset \mathcal{S}_N$. *If* h^ϵ *converges to some element* h *in* $\mathbb{S}_N$ *and* $\bar{h}^\epsilon$ *converges to some element* $\bar{h}$ *in* $\mathcal{S}_N, n \to \infty$, *then*

$$\Gamma^\circ\Bigg(\int_0^\cdot \dot{h}^\epsilon(s)\mathrm{d}s, R_H \bar{h}^\epsilon(\cdot) \Bigg) \to \Gamma^\circ\Bigg(\int_0^\cdot \dot{h}(s)\mathrm{d}s, R_H \bar{h}(\cdot) \Bigg) \, in \, C(0, T; \mathbb{R}^d).$$

Set $X_\cdot^\epsilon = \Gamma^\epsilon(\sqrt{\epsilon} W_\cdot, \epsilon^H B_\cdot^H)$. *Then* $\{X^\epsilon, \epsilon > 0\}$ *satisfies the Laplace principle (hence the LDP) on* $C(0, T; \mathbb{R}^d)$ *with the rate function* I *given by* (2.22).

Proof. Since the proof is similar to that in [23, Theorem 3.2], we omit it here. □

Assume $\bar{X}^0$ satisfies the following equation.

$$\mathrm{d}\bar{X}_t^0 = \bar{f}(\bar{X}_t^0, \mathscr{L}_{\bar{X}_t^0})\mathrm{d}t. \tag{2.23}$$

The following skeleton equation w.r.t. the slow component of stochastic system (2.16) will be used later.

$$\mathrm{d}\bar{X}_t^{\tilde{h}} = \bar{f}(\bar{X}_t^{\tilde{h}}, \mathscr{L}_{\bar{X}_t^0})\mathrm{d}t + g_1(\bar{X}_t^{\tilde{h}}, \mathscr{L}_{\bar{X}_t^0})\mathcal{P}_1 \dot{h}\mathrm{d}t + l(\mathscr{L}_{\bar{X}_t^0})\mathrm{d}(R_H \bar{h})(t), \tag{2.24}$$

where $\tilde{h} = (h, \bar{h}), h \in \mathbb{A}_N, \bar{h} \in \mathcal{A}_N$. Since $\bar{f}, g_1, l$ satisfy Lipschitz condition, Eq. (2.24) has a unique solution $\bar{X}^{\tilde{h}}$. Define a mapping $\bar{X}_\cdot^{\tilde{h}} = \Gamma^\circ\Big(\int_0^\cdot \dot{h}(s)\mathrm{d}s, R_H \bar{h}(\cdot) \Big)$. Now, we are in the position to state our main result.

Theorem 2.9. *Assume* (H1) *and* $\lim_{\epsilon \to 0} \frac{\delta}{\epsilon} = 0$. *If*

$$\sup_{y \in \mathbb{R}^d} |\sigma_1(x, \mu, y)| \vee \sup_{y \in \mathbb{R}^d} |\sigma_2(x, \mu, y)| \leq C(1 + |x| + W_2(\mu, \delta_0)),$$

then $\{X^{\epsilon,\delta}, \epsilon > 0\}$ *satisfies the LDP on* $C(0, T; \mathbb{R}^d)$ *with the rate function* I *given by* (2.22).

2.7 Several priori estimates

Choose a step size $\Delta \in (0,1)$, and set $\bar{t} := \lfloor \frac{t}{\Delta} \rfloor \Delta$. Before giving the proof of the main results, we intend to show several useful estimates by using similar method as used in Lemma 2.3 and Lemma 2.4.

Lemma 2.10. *Assume* (H1). *Then it holds that*

(i) $\sup_{\epsilon,\delta\in(0,1)} \sup_{t\in[0,T]} \mathbb{E}[|X_t^{\epsilon,\delta}|^4] \leq C(T)(1+|x|^2+|y|^2)$,

(ii) $\sup_{\epsilon,\delta\in(0,1)} \sup_{t\in[0,T]} \mathbb{E}[|Y_t^{\epsilon,\delta}|^4] \leq C(T)(1+|x|^2+|y|^2)$,

(iii) $\sup_{\epsilon,\delta\in(0,1)} \mathbb{E}[|X_t^{\epsilon,\delta} - X_{\bar{t}}^{\epsilon,\delta}|^2] \leq C(T)\Delta(1+|x|^2+|y|^2)$,

(iv) $\sup_{h\in\mathbb{S}_N} \sup_{t\in[0,T]} |\bar{X}_t^{\tilde{h}}|^2 \leq C(N,T)(1+|x|^2+\sup_{t\in[0,T]} |\bar{X}_t^0|^2)$,

(v) $\mathbb{E}\int_0^T |\bar{X}_t^{\tilde{h}} - \bar{X}_{\bar{t}}^{\tilde{h}}|^2] \leq C(N,T)\Delta(1+|x|^2+|y|^2)$,
Furthermore, if $\lim_{\epsilon\to 0} \frac{\delta}{\epsilon} = 0$, *we have*

(vi) $\mathbb{E}[\sup_{t\in[0,T]} |Y_t^{\epsilon,\delta,\tilde{h}^\epsilon}|^2] \leq C(T,N)(1+|x|^2+|y|^2)$,

(vii) $\mathbb{E}\int_0^T |X_t^{\epsilon,\delta,\tilde{h}^\epsilon}|^2 \mathrm{d}t \leq C(T,N)(1+|x|^2+|y|^2)$,

(viii) $\mathbb{E}\int_0^T |X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}|^2] \leq C(N,T)\Delta(1+|x|^2+|y|^2)$.

2.8 The auxiliary process

Choose a step size $\Delta \in (0,1)$ and define an auxiliary process $\bar{Y}_t^{\epsilon,\delta}$ with $\bar{Y}_0^{\epsilon,\delta} = Y_0^{\epsilon,\delta,\tilde{h}^\epsilon} = y$. For $t \in [k\Delta, (k+1)\Delta], k = 1, 2, 3, \cdots$,

$$\begin{aligned}\bar{Y}_t^{\epsilon,\delta} &= \bar{Y}_{k\Delta}^{\epsilon,\delta} + \frac{1}{\delta}\int_{k\Delta}^t b(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_s^{\epsilon,\delta})\mathrm{d}s + \frac{1}{\sqrt{\delta}}\int_{k\Delta}^t \sigma_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_s^{\epsilon,\delta})\mathrm{d}W_s^1 \\ &\quad + \frac{1}{\sqrt{\delta}}\int_{k\Delta}^t \sigma_2(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_s^{\epsilon,\delta})\mathrm{d}W_s^2. \end{aligned} \tag{2.25}$$

We will show the following error estimate between the process $Y^{\epsilon,\delta,\tilde{h}^\epsilon}$ and $\bar{Y}^{\epsilon,\delta}$.

Lemma 2.11. *Assume* (H1). *For any* $N, T > 0$, *it holds that*

$$\sup_{\epsilon,\delta\in(0,1)} \sup_{t\in[0,T]} \mathbb{E}[|\bar{Y}_t^{\epsilon,\delta}|^2] \leq C(T)(1+|x|^2+|y|^2),$$

and

$$\mathbb{E}\int_0^T |Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2 \mathrm{d}t \leq C(T,N)(1+|x|^2+|y|^2)(\frac{\delta}{\epsilon}+\Delta).$$

Proof. Similar to arguments as the proof in Lemma 2.4, we can obtain the first result. Next, we prove the second statement. Note that

$$
\begin{aligned}
Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta} &= \frac{1}{\delta}\int_0^t [b(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta,\tilde{h}^\epsilon}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon}) - b(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_t^{\epsilon,\delta})]\mathrm{d}s \\
&+ \frac{1}{\sqrt{\delta}}\int_0^t [\sigma_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon}) - \sigma_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_s^{\epsilon,\delta})]\mathrm{d}W_s^1 \\
&+ \frac{1}{\sqrt{\varepsilon}}\int_0^t [\sigma_2(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon}) - \sigma_2(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_s^{\epsilon,\delta})]\mathrm{d}W_s^2 \\
&+ \frac{1}{\sqrt{\epsilon\delta}}\int_0^t \sigma_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_1\dot{h}_s^\epsilon \mathrm{d}s \\
&+ \frac{1}{\sqrt{\epsilon\delta}}\int_0^t \sigma_2(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_2\dot{h}_s^\epsilon \mathrm{d}s.
\end{aligned}
$$

By Itô's formula, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2] \\
&\leq \frac{2}{\delta}\mathbb{E}\langle Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}, b(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta,\tilde{h}^\epsilon}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon}) - b(X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_t^{\epsilon,\delta})\rangle \\
&+ \frac{2}{\sqrt{\epsilon\delta}}\mathbb{E}\langle Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}, \sigma_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_1\dot{h}^\epsilon\rangle \\
&+ \frac{2}{\sqrt{\epsilon\delta}}\mathbb{E}\langle Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}, \sigma_2(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathcal{P}_2\dot{h}^\epsilon\rangle \\
&+ \frac{1}{\varepsilon}\mathbb{E}|\sigma_1(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon}) - \sigma_1(X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_t^{\epsilon,\delta})|^2 \\
&+ \frac{1}{\delta}\mathbb{E}|\sigma_2(X_t^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_t^{\epsilon,\delta}}, Y_t^{\epsilon,\delta,\tilde{h}^\epsilon}) - \sigma_2(X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}}, \bar{Y}_t^{\epsilon,\delta})|^2 =: \sum_{i=1}^{5} M_i.
\end{aligned}
$$

For M_1, M_4, M_5, by (H1), we have

$$
\begin{aligned}
M_1 + M_4 + M_5 \leq& -\frac{\tilde{\alpha}}{\delta}\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2] + \frac{C}{\delta}\mathbb{E}[|X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}|^2] \\
&+ \frac{C}{\delta}W_2^2(\mathscr{L}_{X_t^{\epsilon,\delta}}, \mathscr{L}_{X_{\bar{t}}^{\epsilon,\delta}}),
\end{aligned} \tag{2.26}
$$

where $\tilde{\alpha} \in (0, \lambda)$. For M_2, M_3 by Young's inequality, we derive

$$
\begin{aligned}
M_2 + M_3 \leq& \frac{\tilde{\alpha}_1}{\delta}\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2] + \frac{C}{\epsilon}\mathbb{E}[(1 + |X_t^{\epsilon,\delta,\tilde{h}^\epsilon}|^2 + \mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_1\||\dot{h}_t^\epsilon|^2] \\
&+ \frac{C}{\epsilon}\mathbb{E}[(1 + |X_t^{\epsilon,\delta,\tilde{h}^\epsilon}|^2 + \mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_2\||\dot{h}_t^\epsilon|^2].
\end{aligned} \tag{2.27}
$$

where $\tilde{\alpha}_1 \in (0, \tilde{\alpha})$. By the above calculations, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2] \leq -\frac{\varsigma}{\delta}\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{Y}_t^{\epsilon,\delta}|^2] + \frac{C}{\delta}\mathbb{E}[|X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - X_{\bar{t}}^{\epsilon,\delta,\tilde{h}^\epsilon}|^2]
$$

$$
\begin{aligned}
&+\frac{C}{\delta}\mathbb{E}[|X_t^{\epsilon,\delta}-X_{\bar t}^{\epsilon,\delta}|^2]+\frac{C}{\epsilon}\mathbb{E}[(1+|X_t^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_1\||\dot h_t^\epsilon|^2]\\
&+\frac{C}{\epsilon}\mathbb{E}[(1+|X_t^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_2\||\dot h_t^\epsilon|^2],
\end{aligned}
$$

where $\varsigma=\tilde\alpha-\tilde\alpha_1$. Due to the comparison theorem, we derive

$$
\begin{aligned}
&\mathbb{E}[|Y_t^{\epsilon,\delta,\tilde h^\epsilon}-\bar Y_t^{\epsilon,\delta}|^2]\\
&\le\frac{C}{\delta}\int_0^t e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathbb{E}[|X_s^{\epsilon,\delta,\tilde h^\epsilon}-X_{\bar s}^{\epsilon,\delta,\tilde h^\epsilon}|^2]\mathrm{d}s+\frac{C}{\delta}\int_0^t e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathbb{E}[|X_s^{\epsilon,\delta}-X_{\bar s}^{\epsilon,\delta}|^2]\mathrm{d}s\\
&\quad+\frac{C}{\epsilon}\int_0^t e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathbb{E}[(1+|X_s^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_1\||\dot h_s^\epsilon|^2]\mathrm{d}s\\
&\quad+\frac{C}{\epsilon}\int_0^t e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathbb{E}[(1+|X_s^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))\|\mathcal{P}_2\||\dot h_s^\epsilon|^2]\mathrm{d}s.
\end{aligned}
$$

From Fubini's theorem, it holds that

$$
\begin{aligned}
&\mathbb{E}\int_0^T[|Y_t^{\epsilon,\delta,\tilde h^\epsilon}-\bar Y_t^{\epsilon,\delta}|^2]\\
&\le\frac{C}{\delta}\int_0^T\mathbb{E}[|X_s^{\epsilon,\delta,\tilde h^\epsilon}-X_{\bar s}^{\epsilon,\delta,\tilde h^\epsilon}|^2]\bigg(\int_t^T e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathrm{d}t\bigg)\mathrm{d}s\\
&\quad+\frac{C}{\delta}\int_0^T\mathbb{E}[|X_s^{\epsilon,\delta}-X_{\bar s}^{\epsilon,\delta}|^2]\bigg(\int_t^T e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathrm{d}t\bigg)\mathrm{d}s\\
&\quad+\frac{C}{\epsilon}\int_0^T\mathbb{E}[(1+|X_s^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))|\dot h_s^\epsilon|^2]\bigg(\int_t^T e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathrm{d}t\bigg)\mathrm{d}s\\
&\quad+\frac{C}{\epsilon}\int_0^T\mathbb{E}[(1+|X_s^{\epsilon,\delta,\tilde h^\epsilon}|^2+\mathscr{L}_{X^{\epsilon,\delta}}(|\cdot|^2))|\dot h_s^\epsilon|^2]\bigg(\int_t^T e^{-\frac{\varsigma(t-s)}{\epsilon}}\mathrm{d}t\bigg)\mathrm{d}s\\
&\le C(T,N)(1+|x|^2+|y|^2)(\Delta+\frac{\delta}{\epsilon}),
\end{aligned}
$$

as required. □

3 The proof of LDP

Lemma 3.1. *Assume* (H1) *and* $\lim_{\epsilon\to0}\frac{\delta}{\epsilon}=0$. *It holds that*

$$
\lim_{\epsilon\to0}\sup_{0\le t\le T}\mathbb{E}|X_t^{\epsilon,\delta}-\bar X_t^0|^2=0. \tag{3.1}
$$

Proof. In the same way as in [33, Theorem 2.3], we can get the desired result. □

We shall investigate the LDP by using criteria in Lemma 2.8. The criterion 2° in Lemma 2.8 will be shown in the following Theorem.

Theorem 3.2. *Assume* (H1). *Let* $\{h^n\}\subset\mathbb{S}_N,\{\bar h^n\}\subset\mathcal{S}_N$ *such that* $h^n\to h$ *in* $\mathbb{S}_N$, *and* $\bar h^n\to\bar h$ *in* $\mathcal{S}_N$, *as* $n\to\infty$, *respectively. Then* $\Gamma^\circ\Big(\int_0^\cdot\dot h^n(s)\mathrm{d}s,R_H\bar h^n(\cdot)\Big)\to\Gamma^\circ\Big(\int_0^\cdot\dot h(s)\mathrm{d}s,R_H\bar h(\cdot)\Big)$ *in* $C(0,T;\mathbb{R}^d)$.

Proof. Let $\tilde{h}^n=(h^n,\bar{h}^n)$ and $\bar{X}^{\tilde{h}^n}_\cdot=\Gamma^\circ\bigg(\int_0^\cdot \dot{h}^n(s)\mathrm{d}s, R_H\bar{h}^n(\cdot)\bigg)$. Then, $\bar{X}^{\tilde{h}^n}_t$ solves the following equation:

$$\mathrm{d}\bar{X}^{\tilde{h}^n}_t=\bar{f}(\bar{X}^{\tilde{h}^n}_t,\mathscr{L}_{\bar{X}^0_t})\mathrm{d}t+g_1(\bar{X}^{\tilde{h}^n}_t,\mathscr{L}_{\bar{X}^0_t})\mathcal{P}_1\dot{h}^n_t\mathrm{d}t+l(\mathscr{L}_{\bar{X}^0_t})\mathrm{d}(R_H\bar{h}^n)(t). \tag{3.2}$$

If $h^n\to h$ in S_N and $\bar{h}^n\to\bar{h}$ in $\mathbb{S}_N$, as $n\to\infty$, respectively, it suffices to prove that $\bar{X}^{\tilde{h}^n}$ converges strongly to $\bar{X}^{\tilde{h}}$ in $C(0,T;\mathbb{R}^d)$ with $\tilde{h}=(h,\bar{h})$, as $n\to\infty$, which solves

$$\mathrm{d}\bar{X}^{\tilde{h}}_t=\bar{f}(\bar{X}^{\tilde{h}}_t,\mathscr{L}_{\bar{X}^0_t})\mathrm{d}t+g_1(\bar{X}^{\tilde{h}}_t,\mathscr{L}_{\bar{X}^0_t})\mathcal{P}_1\dot{h}\mathrm{d}t+l(\mathscr{L}_{\bar{X}^0_t})\mathrm{d}(R_H\bar{h})(t). \tag{3.3}$$

By (3.2), we have

$$\begin{aligned}\bar{X}^{\tilde{h}^n}_t-\bar{X}^{\tilde{h}^n}_s=&\int_s^t\bar{f}(\bar{X}^{\tilde{h}^n}_r,\mathscr{L}_{\bar{X}^0_r})\mathrm{d}r\\&+\int_s^t g_1(\bar{X}^{\tilde{h}^n}_r,\mathscr{L}_{\bar{X}^0_r})\mathcal{P}_1\dot{h}^n_r\mathrm{d}r+\int_s^t l(\mathscr{L}_{\bar{X}^0_r})\mathrm{d}(R_H\bar{h}^n)(r).\end{aligned} \tag{3.4}$$

From (H1) and Lemma 2.10, we have

$$\begin{aligned}&\bigg|\int_s^t\bar{f}(\bar{X}^{\tilde{h}^n}_r,\mathscr{L}_{\bar{X}^0_r})\mathrm{d}r\bigg|\le C(T,N)\int_s^t(|\bar{X}^{\tilde{h}^n}_r|+|\bar{X}^0_r|)\mathrm{d}r+C(T,N)(t-s)\\&\le C(T,N)(1+\sup_{t\in[0,T]}|\bar{X}^0_t|)(t-s),\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\bigg|\int_s^t g_1(\bar{X}^{\tilde{h}}_t,\mathscr{L}_{\bar{X}^0_t})\mathcal{P}_1\dot{h}^n_r\mathrm{d}r\bigg|\le\int_s^t|g_1(\bar{X}^{\tilde{h}}_t,\mathscr{L}_{\bar{X}^0_r})||\mathcal{P}_1\dot{h}^n_r|\mathrm{d}r\\&\le\bigg(\int_s^t|g_1(\bar{X}^{\tilde{h}}_t,\mathscr{L}_{\bar{X}^0_t})|^2\mathrm{d}r\bigg)^{\frac12}\bigg(\int_s^t|\mathcal{P}_1\dot{h}^n_r|^2\mathrm{d}r\bigg)^{\frac12}\\&\le C(T,N)(1+\sup_{t\in[0,T]}|\bar{X}^0_t|)(t-s)^{\frac12}.\end{aligned} \tag{3.6}$$

In view of (H1) and (2.7), one has

$$\begin{aligned}&\bigg|\int_s^t l(\mathscr{L}_{\bar{X}^0_r})\mathrm{d}(R_H\bar{h}^n)(r)\bigg|=\bigg|\int_s^t l(\mathscr{L}_{\bar{X}^0_r})\int_0^r\frac{\partial K_H}{\partial r}(r,u)(K^*_Hh^n)(u)\mathrm{d}u\mathrm{d}r\bigg|\\&\le C(T)(1+\sup_{r\in[0,T]}|\bar{X}^0_r|)\int_s^t\int_0^r(\frac{r}{u})^{H-\frac12}(r-u)^{H-\frac32}|(K^*_Hh^n)(u)|\mathrm{d}u\mathrm{d}r\\&\le C(T)(1+\sup_{r\in[0,T]}|\bar{X}^0_r|)\bigg[\int_0^s u^{\frac12-H}|(K^*_Hh^n)(u)|\bigg(\int_s^t r^{H-\frac12}(r-u)^{H-\frac32}\mathrm{d}r\bigg)\mathrm{d}u\\&\qquad+\int_s^t u^{\frac12-H}|(K^*_Hh^n)(u)|\bigg(\int_u^t r^{H-\frac12}(r-u)^{H-\frac32}\mathrm{d}r\bigg)\mathrm{d}u\bigg].\end{aligned} \tag{3.7}$$

By the relation $\|K_H^* h^n\|_{L^2} = \|h^n\|_{\mathcal{H}}$, we derive that

$$\begin{aligned}
&\int_0^s u^{\frac{1}{2}-H}|(K_H^* h^n)(u)|\bigg(\int_s^t r^{H-\frac{1}{2}}(r-u)^{H-\frac{3}{2}}\mathrm{d}r\bigg)\mathrm{d}u \\
&\leq T^{H-\frac{1}{2}}\int_s^t (r-s)^{H-\frac{3}{2}}\mathrm{d}r\int_0^s u^{\frac{1}{2}-H}|(K_H^* h^n)(u)|\mathrm{d}u \\
&\leq \frac{T^{\frac{1}{2}}}{(H-\frac{1}{2})\sqrt{2(1-H)}}\bigg(\int_0^T |(K_H^* h^n)(u)|^2\mathrm{d}u\bigg)^{\frac{1}{2}}(t-s)^{H-\frac{1}{2}} \\
&= \frac{T^{\frac{1}{2}}}{(H-\frac{1}{2})\sqrt{2(1-H)}}\|h^n\|_{\mathcal{H}}(t-s)^{H-\frac{1}{2}}, \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^t u^{\frac{1}{2}-H}|(K_H^* h^n)(u)|\bigg(\int_u^t r^{H-\frac{1}{2}}(r-u)^{H-\frac{3}{2}}\mathrm{d}r\bigg)\mathrm{d}u\bigg] \\
&\leq \frac{T^{H-\frac{1}{2}}}{H-\frac{1}{2}}\int_s^t u^{\frac{1}{2}-H}(t-u)^{H-\frac{1}{2}}|(K_H^* h^n)(u)|\mathrm{d}u \\
&\leq \frac{\sqrt{\mathcal{B}(2-2H,2H)}T^{H-\frac{1}{2}}}{H-\frac{1}{2}}\|h^n\|_{\mathcal{H}}\sqrt{t-s}. \qquad (3.9)
\end{aligned}$$

By $(3.1)-(3.6)$ and the fact that $\|h^n\|_{\mathcal{H}} \leq \sqrt{2M}$, one can see that $\{\bar{X}^{\tilde{h}^n}\}$ is equi-continuous and bounded in $C(0,T;\mathbb{R}^d)$, which implies $\{\bar{X}^{\tilde{h}^n}\}$ is relatively compact in $C(0,T;\mathbb{R}^d)$. Thus, there exists a subsequence still denoted by $\{\bar{X}^{\tilde{h}^n}\}$ such that $\{\bar{X}^{\tilde{h}^n}\}$ converges to some $\bar{X} \in C(0,T;\mathbb{R}^d)$.

Next, it suffices to prove $\bar{X} = \bar{X}^{\tilde{h}}$. By (H1), firstly, we have

$$\begin{aligned}
\bigg|\int_0^t \bar{f}(\bar{X}_s^{\tilde{h}^n},\mathscr{L}_{\bar{X}_s^0})\mathrm{d}s - \int_0^t \bar{f}(\bar{X}_s,\mathscr{L}_{\bar{X}_s^0})\mathrm{d}s\bigg| &\leq C\int_0^t |\bar{X}_s^{\tilde{h}^n} - \bar{X}_s|\mathrm{d}s \\
&\leq C(T)\sup_{t\in[0,T]}|\bar{X}_s^{\tilde{h}^n} - \bar{X}_s| \to 0, n\to\infty.
\end{aligned}$$

Thus, for any $t \in [0,T]$, we have

$$\lim_{n\to\infty}\int_0^t \bar{f}(\bar{X}_s^{\tilde{h}^n},\mathscr{L}_{\bar{X}_s^0})\mathrm{d}s = \int_0^t \bar{f}(\bar{X}_s,\mathscr{L}_{\bar{X}_s^0})\mathrm{d}s. \qquad (3.10)$$

Secondly,

$$\begin{aligned}
&\bigg|\int_0^t g_1(\bar{X}_s^{\tilde{h}^n},\mathscr{L}_{\bar{X}_s^0})\mathcal{P}_1\dot{h}_s^n\mathrm{d}s - \int_0^t g_1(\bar{X}_s^{\tilde{h}},\mathscr{L}_{\bar{X}_s^0})\mathcal{P}_1\dot{h}_s\mathrm{d}s\bigg| \\
&\leq \bigg(\int_0^t |g_1(\bar{X}_s^{\tilde{h}},\mathscr{L}_{\bar{X}_s^0}) - g_1(\bar{X}_s^{\tilde{h}},\mathscr{L}_{\bar{X}_s^0})|^2\mathrm{d}s\bigg)^{\frac{1}{2}}\bigg(\int_0^t |\dot{h}_s^n - \dot{h}_s|^2\mathrm{d}s\bigg)^{\frac{1}{2}} \\
&\leq C(T,N)\sup_{t\in[0,T]}|\bar{X}_t^{\tilde{h}^n} - \bar{X}_t| \to 0, n\to\infty. \qquad (3.11)
\end{aligned}$$

Then, for any $t \in [0,T]$, we have

$$\lim_{n\to\infty}\int_0^t g_1(\bar{X}_s^{\tilde{h}^n}, \mathscr{L}_{\bar{X}_s^0})\mathcal{P}_1\dot{h}_s^n \mathrm{d}s\mathrm{d}s = \int_0^t g_1(\bar{X}_s^{\tilde{h}}, \mathscr{L}_{\bar{X}_s^0})\mathcal{P}_1\dot{h}_s \mathrm{d}s. \tag{3.12}$$

Finally, for any $t \in [0,T]$, we intend to prove

$$\lim_{n\to\infty}\int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h}^n)(s) = \int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h})(s) \tag{3.13}$$

By Fubini's theorem, we have

$$\begin{aligned}
&\int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h}^n)(s) - \int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h})(s)\\
&= \int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}s\bigg(\int_0^s \frac{\partial K_H}{\partial s}(s,r)[(K_H^*h^n)(r) - (K_H^*h)(r)]\mathrm{d}r\bigg)\\
&= C(H)\int_0^T \bigg[1_{[0,t]}(r)r^{\frac{1}{2}-H}\bigg(\int_r^t l(\mathscr{L}_{\bar{X}_s^0})s^{H-\frac{1}{2}}(s-r)^{H-\frac{3}{2}}\mathrm{d}s\bigg)\bigg][(K_H^*h^n)(r) - (K_H^*h)(r)]\mathrm{d}r.
\end{aligned}$$

For any unit vector $e \in \mathbb{R}^d, t \in [0,T]$, let

$$\rho_t(r) := 1_{[0,t]}(r)r^{\frac{1}{2}-H}\bigg(\int_r^t l(\mathscr{L}_{\bar{X}_s^0})s^{H-\frac{1}{2}}(s-r)^{H-\frac{3}{2}}\mathrm{d}s\bigg)e, r \in [0,T].$$

From (H1), one has

$$|\rho_t(r)| \le \frac{C(T,H)}{H+\frac{1}{2}}(1+\sup_{s\in[0,T]}|X_s^0|)r^{\frac{1}{2}-H}.$$

This implies $\rho_t(\cdot) \in L^2(0,T;\mathbb{R}^d)$. Combining (3.27) and the condition that $\bar{h}^n \to \bar{h}$ in $\mathbb{S}_N$, we derive

$$\lim_{n\to\infty}\int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h}^n)(s) = \int_0^t l(\mathscr{L}_{\bar{X}_s^0})\mathrm{d}(R_H\bar{h})(s). \tag{3.14}$$

Taking $n \to \infty$ in (3.2), one has $\bar{X}$ solves (3.3). By a standard subsequential argument, we can conclude that the sequence $\{\bar{X}^{\tilde{h}^n}\}$ to $\bar{X}^{\tilde{h}}$ in $\mathcal{E}$, which implies $\bar{X} = \bar{X}^{\tilde{h}}$. The proof is therefore complete. □

We now prove the criterion 1° in Lemma 2.8.

Theorem 3.3. *Assume* (H1). *Then we have*

$$\lim_{\epsilon\to 0}P\bigg\{d\bigg(\Gamma^\epsilon\bigg(\sqrt{\epsilon}W_\cdot + \int_0^\cdot \dot{h}^\epsilon(\cdot)\mathrm{d}s, \epsilon^H B_\cdot^H + \frac{\epsilon^H}{\epsilon^{\frac{1}{2}}}R_H\bar{h}^\epsilon(\cdot)\bigg), \Gamma^\circ\bigg(\int_0^\cdot \dot{h}^\epsilon(s)\mathrm{d}s, R_H\bar{h}^\epsilon(\cdot)\bigg)\bigg) > \varepsilon_0\bigg\}$$
$$= 0.$$

Proof. Note that

$$\begin{aligned}X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{X}_t^{\tilde{h}^\epsilon} &= \int_0^t [f_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon}) - \bar{f}(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})]\mathrm{d}s \\ &+ \int_0^t [g_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}) - g_1(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})]P_1 h^\epsilon(s)\mathrm{d}s \\ &+ \sqrt{\epsilon}\int_0^t g_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}W_s^1 \\ &+ \int_0^t [l(\mathscr{L}_{X_s^{\epsilon,\delta}}) - l(\mathscr{L}_{\bar{X}_s^0})]\mathrm{d}(R_H\bar{h}^\epsilon)(s) \\ &+ \epsilon^H \int_0^t l(\mathscr{L}_{X_s^{\epsilon,\delta}})\mathrm{d}B_s^H.\end{aligned}$$

Then, we have

$$\begin{aligned}|X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{X}_t^{\tilde{h}^\epsilon}|^2 &\le \left|\int_0^t [f_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon}) - \bar{f}(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})]\mathrm{d}s\right|^2 \\ &+ \epsilon\left|\int_0^t g_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon})\mathrm{d}W_s^1\right|^2 \\ &+ \left|\int_0^t [l(\mathscr{L}_{X_s^{\epsilon,\delta}}) - l(\mathscr{L}_{\bar{X}_s^0})]\mathrm{d}(R_H\bar{h}^\epsilon)(s)\right|^2 \\ &+ \epsilon^{2H}\left|\int_0^t l(\mathscr{L}_{X_s^{\epsilon,\delta}})\mathrm{d}B_s^H\right|^2 \\ &+ \left|\int_0^t [g_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}) - g_1(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})]P_1 h^\epsilon(s)\mathrm{d}s\right|^2 \\ &=: \sum_{i=1}^5 K_i(t).\end{aligned}$$

For $K_2(t)$, applying BDG's inequality and Lemma 2.10, we have

$$\mathbb{E}\sup_{0\le s\le t} K_2(s) \le C(T)\epsilon(1 + |x|^2 + |y|^2). \tag{3.15}$$

For $K_3(t)$, using (H1), (2.7), the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2(0,T;\mathbb{R}^{d_1})$, and the same way as used in (3.8), (3.9), we arrive at

$$\mathbb{E}[\sup_{0\le s\le t} K_3(s)] \le C(T,N)\sup_{0\le s\le t}\mathbb{E}[|X_s^{\epsilon,\delta} - \bar{X}_s^0|^2]. \tag{3.16}$$

For $K_4(t)$, by (H1), it yields

$$\mathbb{E}[\sup_{0\le s\le t} K_4(s)] \le C(T)\epsilon^{2H}(1 + |x|^2 + |y|^2). \tag{3.17}$$

For $K_5(t)$, (H1) implies

$$\mathbb{E}\left|\int_0^t [g_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X^{\epsilon,\delta}}) - g_1(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})]\mathcal{P}_1 h^\epsilon(s)\mathrm{d}s\right|^2$$

$$\leq C(T,N)\mathbb{E}\int_0^t |X_s^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{X}_s^{\tilde{h}^\epsilon}|^2 \mathrm{d}s + C(N,T)\mathbb{E}\int_0^t |X_s^{\epsilon,\delta} - \bar{X}_s^0|^2 \mathrm{d}s. \tag{3.18}$$

Next, we intend to estimate $K_1(t)$. By Lemma 2.11, one can derive that

$$\begin{aligned}
\mathbb{E}[\sup_{0\leq s\leq t} K_1(s)] &\leq C(T)\mathbb{E}\int_0^t |f_1(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}, Y_s^{\epsilon,\delta,\tilde{h}^\epsilon}) - f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta})|^2 \mathrm{d}s \\
&\quad + C(T)\left|\int_0^t [f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta}) - \bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})]\mathrm{d}s\right|^2 \\
&\quad + C(T)\mathbb{E}\int_0^t |\bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}) - \bar{f}(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}})|^2 \mathrm{d}s \\
&\quad + C(T)\mathbb{E}\int_0^t |\bar{f}(X_s^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_s^{\epsilon,\delta}}) - \bar{f}(\bar{X}_s^{\tilde{h}^\epsilon}, \mathscr{L}_{\bar{X}_s^0})|^2 \mathrm{d}s \\
&\leq C(T,N)(1+|x|^2+|y|^2)(\delta + \frac{\delta}{\epsilon}) + C(T,N)\mathbb{E}\int_0^t |X_s^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{X}_s^{\tilde{h}^\epsilon}|^2 \mathrm{d}s \\
&\quad + C(T)\mathbb{E}\left|\int_0^t [f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_{\bar{s}}^{\epsilon,\delta}) - \bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})]\mathrm{d}s\right|^2 \\
&\quad + C(N,T)\mathbb{E}\int_0^t |X_s^{\epsilon,\delta} - \bar{X}_s^0|^2 \mathrm{d}s.
\end{aligned} \tag{3.19}$$

Now, we need to estimate $C(T)\mathbb{E}\int_0^t |f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta}) - \bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})|^2 \mathrm{d}s$.

$$\begin{aligned}
&C(T)\mathbb{E}\left|\int_0^t [f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta}) - \bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})]\mathrm{d}s\right|^2 \\
&\quad \leq \frac{C(T)}{\Delta}\mathbb{E}\sum_{k=0}^{k=\lfloor t/\Delta\rfloor -1}\left|\int_{k\Delta}^{(k+1)\Delta} [f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta}) - \bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})]\mathrm{d}s\right|^2 \\
&\qquad + C(T)\mathbb{E}\left|\int_{\bar{t}}^t [f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}}, \bar{Y}_s^{\epsilon,\delta}) - \bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})]\mathrm{d}s\right|^2 \\
&\quad \leq \frac{C(T)\delta^2}{\Delta^2}\max_{0\leq k\leq \lfloor T/\Delta\rfloor -1}\left\{\int_0^{\frac{\Delta}{\delta}}\int_r^{\frac{\Delta}{\delta}} \Upsilon_k(s,r)\mathrm{d}s\mathrm{d}r\right\} + C(T,N)\Delta(1+|x|^2+|y|^2),
\end{aligned} \tag{3.20}$$

where for any $0 \leq r \leq s \leq \frac{\Delta}{\delta}$,

$$\begin{aligned}
\Upsilon_k(s,r) := \mathbb{E}[\langle & f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}, \bar{Y}_{\delta s+k\Delta}^{\epsilon,\delta}) - \bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}), \\
& f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}, \bar{Y}_{\delta r+k\Delta}^{\epsilon,\delta}) - \bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}, \mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle].
\end{aligned} \tag{3.21}$$

For any $s > 0, \mu \in \mathscr{P}_2(\mathbb{R}^d), x, y \in \mathbb{R}^d$, consider the following equation:

$$\begin{aligned}
\tilde{Y}_t^{\delta,s,x,\mu,y} &= y + \frac{1}{\delta}\int_s^t b(x,\mu,\tilde{Y}_r^{\delta,s,x,\mu,y})\mathrm{d}r + \frac{1}{\sqrt{\delta}}\int_s^t \sigma_1(x,\mu,\tilde{Y}_r^{\delta,s,x,\mu,y})\mathrm{d}W_r^1 \\
&\quad + \frac{1}{\sqrt{\delta}}\int_s^t \sigma_2(x,\mu,\tilde{Y}_r^{\delta,s,x,\mu,y})\mathrm{d}W_r^2, \quad t \geq s.
\end{aligned} \tag{3.22}$$

Then, we have

$$\bar{Y}_t^{\epsilon,\delta}=\tilde{Y}_t^{\delta,k\Delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}},\quad t\in[k\Delta,(k+1)\Delta].$$

It follows from the definition of $\Upsilon_k(s,r)$ that

$$\begin{aligned}\Upsilon_k(s,r)=\mathbb{E}[\langle f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},\tilde{Y}_{\delta s+k\Delta}^{\delta,k\delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta},\tilde{h}^\epsilon},\bar{Y}_{k\Delta}^{\epsilon,\delta}})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}),\\ f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},\tilde{Y}_{\delta r+k\Delta}^{\epsilon,k\Delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle].\end{aligned}$$

Since $\tilde{Y}^{\epsilon,k\Delta,x,\mu,y}$ is independent of $\mathscr{F}_{k\Delta}$, and $X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\bar{Y}_{k\Delta}^{\epsilon,\delta}$ are $\mathscr{F}_{k\Delta}-$measurable, we have

$$\begin{aligned}\Upsilon_k(s,r)&=\mathbb{E}\Bigg\{\mathbb{E}\Bigg[\langle f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},\tilde{Y}_{\delta s+k\delta}^{\delta,k\Delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}),\\ &\quad f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},\tilde{Y}_{\delta r+k\Delta}^{\varepsilon,k\Delta,X_{k\Delta}^{\epsilon,\varepsilon,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\varepsilon}})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle\Bigg|\mathscr{F}_{k\Delta}\Bigg](\omega)\Bigg\}\\ &=\mathbb{E}\Bigg\{\mathbb{E}\Bigg[\langle f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\delta}^{\epsilon,\delta}},\tilde{Y}_{\delta s+k\Delta}^{\delta,k\Delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}),\\ &\quad f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},\tilde{Y}_{\delta r+k\Delta}^{\delta,k\Delta,X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle\Bigg]\Bigg\}.\end{aligned}$$

From the definition of $\tilde{Y}^{\epsilon,s,x,\mu,y}$, we know that

$$\begin{aligned}\tilde{Y}_{\delta s+k\Delta}^{\epsilon,k\Delta,x,\mu,y}&=y+\int_0^s b(x,\mu,\tilde{Y}_{\delta r+k\Delta}^{\epsilon,k\Delta,x,\mu,y})\mathrm{d}r+\int_0^s\sigma_1(x,\mu,\tilde{Y}_{\delta r+k\Delta}^{\epsilon,k\Delta,x,\mu,y})\mathrm{d}W_r^{1,k\Delta}\\ &\quad+\int_0^s\sigma_2(x,\mu,\tilde{Y}_{\delta r+k\Delta}^{\epsilon,k\Delta,x,\mu,y})\mathrm{d}W_r^{2,k\Delta},\end{aligned}\tag{3.23}$$

where $W_r^{1,k\Delta}=\frac{1}{\sqrt{\delta}}(W_{\delta r+k\Delta}^1-W_{k\Delta}^1),W_r^{2,k\Delta}=\frac{1}{\sqrt{\delta}}(W_{\delta r+k\Delta}^2-W_{k\Delta}^2)$. Note that

$$\begin{aligned}Y_s^{x,\mu,y}&=y+\int_0^s b(x,\mu,Y_r^{x,\mu,y})\mathrm{d}r+\int_0^s\sigma_1(x,\mu,Y_r^{x,\mu,y})\mathrm{d}\tilde{W}_r^1\\ &\quad+\int_0^s\sigma_2(x,\mu,Y_r^{x,\mu,y})\mathrm{d}\tilde{W}_r^2,\end{aligned}\tag{3.24}$$

By the uniqueness of the solutions of (3.23) and (3.24), we have that $\{\tilde{Y}_{\delta s+k\Delta}^{\epsilon,k\Delta,x,\mu,y})\}_{0\le s\le\frac{\Delta}{\delta}}$ and $\{Y_s^{x,\mu,y}\}_{0\le s\le\frac{\Delta}{\delta}}$ have the same distribution. By [33, Proposition 3.7], it holds that

$$\begin{aligned}&\Upsilon_k(s,r)\\ &=\mathbb{E}\Bigg\{\tilde{\mathbb{E}}\Bigg[\langle f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},Y_s^{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}),\end{aligned}$$

$$f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},Y_r^{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle\Big]\Big\}$$

$$=\mathbb{E}\Big\{\tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}[\langle f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},Y_s^{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})|\tilde{\mathscr{F}}_r](\tilde{\omega})$$

$$-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}),$$

$$f_1(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}},Y_r^{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)})-\bar{f}(X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}})\rangle\Big]\Big\}$$

$$\leq C(T)\mathbb{E}\Big\{\tilde{\mathbb{E}}\Big[1+|X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega)|^2+|Y_r^{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}(\omega),\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}},\bar{Y}_{k\Delta}^{\epsilon,\delta}(\omega)}(\tilde{\omega})|^2+\mathscr{L}_{X_{k\Delta}^{\epsilon,\delta}}(|\cdot|^2)\Big]e^{-\frac{(s-r)\alpha}{2}}\Big\}$$

$$\leq C(T)e^{-\frac{(s-r)\alpha}{2}}\mathbb{E}[1+|X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}|^2+|\hat{Y}_{k\Delta}^{\delta,\delta}|^2+\mathbb{E}(|X_{k\Delta}^{\epsilon,\delta}|^2)+\mathbb{E}(|X_{k\Delta}^{\epsilon,\delta,\tilde{h}^\epsilon}|^2)]$$

$$\leq C(T,N)(1+|x|^2+|y|^2)e^{-\frac{(s-r)\alpha}{2}}. \tag{3.25}$$

Applying (3.20) and (3.11), we conclude

$$C(T)\Big|\int_0^t[f_1(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}},\bar{Y}_s^{\epsilon,\delta})-\bar{f}(X_{\bar{s}}^{\epsilon,\delta,\tilde{h}^\epsilon},\mathscr{L}_{X_{\bar{s}}^{\epsilon,\delta}})]\mathrm{d}s\Big|^2$$

$$\leq C(T,N)(1+|x|^2+|y|^2)(\frac{\delta^2}{\Delta^2}+\frac{\delta}{\Delta}+\Delta). \tag{3.26}$$

Thus,

$$\mathbb{E}[\sup_{0\leq s\leq t}|X_s^{\epsilon,\delta,\tilde{h}^\epsilon}-\bar{X}_s^{\tilde{h}^\epsilon}|^2]$$

$$\leq C(T,N)(1+|x|^2+|y|^2)(\frac{\delta^2}{\Delta^2}+\frac{\delta}{\Delta}+\frac{\delta}{\epsilon}+\Delta+\epsilon)+C(T,N)\sup_{0\leq t\leq T}\mathbb{E}[|X_s^{\epsilon,\delta}-\bar{X}_s^0|^2]$$

$$+\int_0^t\mathbb{E}[\sup_{0\leq r\leq s}|X_r^{\epsilon,\delta,\tilde{h}^\epsilon}-\bar{X}_r^{\tilde{h}^\epsilon}|^2]\mathrm{d}s.$$

Taking $\Delta=\epsilon^{\frac{2}{3}}$, it yields

$$\mathbb{E}[\sup_{0\leq s\leq t}|X_s^{\epsilon,\delta,\tilde{h}^\epsilon}-\bar{X}_s^{\tilde{h}^\epsilon}|^2]$$

$$\leq C(T,N)(1+|x|^2+|y|^2)(\frac{\delta}{\epsilon}+\epsilon^{\frac{2}{3}}+\epsilon)+C(T,N)\sup_{0\leq t\leq T}\mathbb{E}[|X_s^{\epsilon,\delta}-\bar{X}_s^0|^2]$$

$$+\int_0^t\mathbb{E}[\sup_{0\leq r\leq s}|X_r^{\epsilon,\delta,\tilde{h}^\epsilon}-\bar{X}_r^{\tilde{h}^\epsilon}|^2]\mathrm{d}s. \tag{3.27}$$

By (3.27), Gronwall's inequality and Lemma 3.1, we have for any $\varepsilon_0>0$

$$P\Big\{d\Big(\Gamma^\epsilon\Big(\sqrt{\epsilon}W_\cdot+\int_0^\cdot\dot{h}_\epsilon(\cdot)\mathrm{d}s,\epsilon^H B_\cdot^H+\frac{\epsilon^H}{\epsilon^{\frac{1}{2}}}R_H h_\epsilon(\cdot)\Big),\Gamma^\circ\Big(\int_0^\cdot\dot{h}^\epsilon(s)\mathrm{d}s,R_H\bar{h}^\epsilon(\cdot)\Big)\Big)>\varepsilon_0\Big\}$$

$$=P\{|X_t^{\epsilon,\delta,\tilde{h}^\epsilon}-\bar{X}_t^{\tilde{h}^\epsilon}|>\varepsilon_0\}$$

$$\leq \frac{\mathbb{E}[\sup_{0\leq t\leq T}|X_t^{\epsilon,\delta,\tilde{h}^\epsilon} - \bar{X}_t^{\tilde{h}^\epsilon}|^2]}{\varepsilon_0} \to 0, \text{ as } \epsilon \to 0. \tag{3.28}$$

□

4 Appendix

4.1 Proof of Lemma 2.7

We now borrow the method in [3, Theorem 4.4] to prove Lemma 2.7. The key is to use the following a variational representation for random functional by making a slight change to that of [44, Theorem 3.2].

Lemma 4.1. *Let f be a bounded Borel measurable function on Ω. Then it holds that*

$$-\log \mathbb{E}(e^{-f}) = \inf_{\tilde{h}=(h,\bar{h})\in\mathcal{A}\times\mathbb{A}} \mathbb{E}(f(\cdot + h(\cdot), \cdot + R_H\bar{h}(\cdot)) + \frac{1}{2}\|\bar{h}\|_{\mathcal{H}}^2 + \frac{1}{2}\|h\|_{\mathbb{H}}^2).$$

Proof of Lemma 2.7. Replacing $f(\cdot)$ by $\frac{\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}\cdot,\epsilon^H\cdot)}{l(\epsilon)}$ in Lemma 4.1, where ρ is real-valued, bounded and continuous function on $\mathcal{E} := C(0,T;\mathbb{R}^d)$, $l(\epsilon) := \epsilon \text{ or } \epsilon^{2H}$, we have

$$\begin{aligned}
&-l(\epsilon)\log\mathbb{E}\left[e^{-\frac{\rho(X^{\epsilon,\varepsilon})}{l(\epsilon)}}\right] = -l(\epsilon)\log\mathbb{E}\left[e^{-\frac{\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}W_\cdot,\epsilon^H B_\cdot^H)}{l(\epsilon)}}\right] \\
&= \inf_{\tilde{h}=(h,\bar{h})\in\mathcal{A}\times\mathbb{A}} \mathbb{E}[\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}(W_\cdot + h(\cdot)), \epsilon^H(B_\cdot^H + R_H\bar{h}(\cdot))) + \frac{1}{2}l(\epsilon)\|\bar{h}\|_{\mathcal{H}}^2 + \frac{1}{2}l(\epsilon)\|h\|_{\mathbb{H}}^2].
\end{aligned}\tag{4.1}$$

The rest of the proof will be divided into two steps.

Step1 : The upper bound. Without lost of generality, we assume that $\inf_{x\in\mathcal{E}}\{\rho(x) + I(x)\} < \infty$, where I is a rate function given in (2.22). Taking $\gamma > 0$, then there exists $x_0 \in C(0,T;\mathbb{R}^d)$ satisfying

$$\rho(x_0) + I(x_0) \leq \inf_{x\in\mathcal{E}}\{\kappa(x) + I(x)\} + \frac{\gamma}{2} \tag{4.2}$$

From (2.22), there exists $(h_1, \bar{h}_1) \in \mathcal{H}\times\mathbb{H}$ such that $\Gamma^0(\int_0^\cdot \dot{h}_1(s)\mathrm{d}s, R_H\bar{h}_1) = x_0$ and

$$\frac{1}{2}\|h_1\|_{\mathbb{H}} + \frac{1}{2}\|\bar{h}_1\|_{\mathcal{H}} \leq I(x_0) + \frac{\gamma}{2}.$$

This together with (4.1) implies

$$\begin{aligned}
&-\epsilon^{2H}\log\mathbb{E}\left[e^{-\frac{\rho(X^{\epsilon,\delta})}{\epsilon^{2H}}}\right] \\
&\leq \inf_{\tilde{h}=(h,\bar{h})\in\mathcal{A}\times\mathbb{A}} \mathbb{E}[\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}(W_\cdot + h(\cdot)), \epsilon^H(B_\cdot^H + R_H\bar{h}(\cdot))) + \frac{1}{2}\epsilon^{2H}\|\bar{h}\|_{\mathcal{H}}^2 + \frac{1}{2}\epsilon\|h\|_{\mathbb{H}}^2]
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{\tilde{h}=(h,\bar{h})\in\mathcal{A}\times\mathbb{A}} \mathbb{E}[\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}(W_\cdot + h(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}(\cdot)/\sqrt{\epsilon^{2H}})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}\|\bar{h}\|_{\mathcal{H}}^2 + \frac{1}{2}\|h\|_{\mathbb{H}}^2] \\
&\le \mathbb{E}[\rho\circ\Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h_1(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}_1(\cdot)/\sqrt{\epsilon^{2H}}))] + I(x_0) + \frac{\gamma}{2}.
\end{aligned}
$$

By the fact that ρ is bounded and continuous and taking $\epsilon\to 0$, we derive

$$
\begin{aligned}
\limsup_{\epsilon\to 0} -\epsilon^{2H}\log\mathbb{E}\left[e^{-\frac{\rho(X^{\epsilon,\delta})}{\epsilon^{2H}}}\right] &\le \rho\circ\Gamma^0(\int_0^\cdot \dot{h}_1(s)\mathrm{d}s, R_H\bar{h}_1)) + I(x_0) + \frac{\gamma}{2} \\
&= \rho(x_0) + I(x_0) + \frac{\gamma}{2} \\
&\le \inf_{x\in\mathcal{E}}\{\rho(x) + I(x)\} + \frac{\gamma}{2}.
\end{aligned}
$$

Combining this and the fact that γ being arbitrary, we finish the proof of the upper bound.

Step2 : The lower bound. Taking $\gamma > 0$, by (4.1), for every $\epsilon > 0$, there exist $(h^\epsilon, \bar{h}^\epsilon) \in A\times\mathbb{A}$ such that

$$
\begin{aligned}
&-\epsilon\log\mathbb{E}\left[e^{-\frac{\rho(X^{\epsilon,\delta})}{\epsilon}}\right] \\
&\ge \inf_{\tilde{h}=(h,\bar{h})\in\mathcal{A}\times\mathbb{A}} \mathbb{E}[\rho\circ\Gamma^\epsilon(\sqrt{\epsilon}(W_\cdot + h(\cdot)), \epsilon^H(B_\cdot^H + R_H\bar{h}(\cdot))) + \frac{1}{2}\epsilon^{2H}\|\bar{h}\|_{\mathcal{H}}^2 + \frac{1}{2}\epsilon\|h\|_{\mathbb{H}}^2] \\
&\ge \mathbb{E}[\rho\circ\Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) + \frac{1}{2}\|\bar{h}^\epsilon\|_{\mathcal{H}}^2 + \frac{1}{2}\|h^\epsilon\|_{\mathbb{H}}^2] - \gamma, \quad (4.3)
\end{aligned}
$$

which also implies

$$
\sup_{\epsilon>0}\mathbb{E}\left[\frac{1}{2}\|\bar{h}^\epsilon\|_{\mathcal{H}}^2 + \frac{1}{2}\|h^\epsilon\|_{\mathbb{H}}^2\right] \le 2\|\rho\|_\infty + \gamma. \tag{4.4}
$$

For a given constant $M > 0$, define the following stopping times.

$$
\sigma_M^\epsilon := \inf_{t\in[0,T]}\left\{\frac{1}{2}\|\bar{h}^\epsilon 1_{[0,t]}\|_{\mathcal{H}}^2 + \frac{1}{2}\|1_{[0,t]}h^\epsilon\|_{\mathbb{H}}^2 \ge M\right\}\wedge T.
$$

Let $h^{\epsilon,M}(t) := h^\epsilon(t)1_{[0,\sigma_M^\epsilon]}(t), \bar{h}^{\epsilon,M}(t) := \bar{h}^\epsilon(t)1_{[0,\sigma_M^\epsilon]}(t)$. It holds that $h^{\epsilon,M}\in\mathbb{A}, \bar{h}^{\epsilon,M}\in\mathcal{A}$. From the Markov inequality and (4.4), one has

$$
P(h^\epsilon \ne h^{\epsilon,M}, h^\epsilon \ne h^{\epsilon,M}) \le P\left(\frac{1}{2}\|\bar{h}^\epsilon\|_{\mathcal{H}}^2 + \frac{1}{2}\|h^\epsilon\|_{\mathbb{H}}^2 \ge M\right) \le \frac{2(\|\kappa\|_\infty + \gamma)}{M}. \tag{4.5}
$$

Moreover, we derive that

$$
\begin{aligned}
&\rho\circ\Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) \\
&= \rho\circ\Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^{\epsilon,M}(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}^{\epsilon,M}(\cdot)/\sqrt{\epsilon^{2H}})) \\
&+ \Bigg[\rho\circ\Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}), \epsilon^H(B_\cdot^H + R_H\bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}}))
\end{aligned}
$$

$$- \rho \circ \Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^{\epsilon,M}(\cdot)/\sqrt{\epsilon}), \epsilon^H(B^H_\cdot + R_H \bar{h}^{\epsilon,M}(\cdot)/\sqrt{\epsilon^{2H}}))\Big] 1_{\{\bar{h}^\epsilon \neq \bar{h}^{\epsilon,M}, h^\epsilon \neq h^{\epsilon,M}\}}$$

$$\geq \rho \circ \Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^{\epsilon,M}(\cdot)/\sqrt{\epsilon}), \epsilon^H(B^H_\cdot + R_H \bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) - 2\|\rho\|_\infty 1_{\{\bar{h}^\epsilon \neq \bar{h}^{\epsilon,M}, h^\epsilon \neq h^{\epsilon,M}\}}, \quad (4.6)$$

By (2.6) , one can see that

$$\begin{aligned} &\|\bar{h}^\epsilon\|^2_{\mathcal{H}} = \|K^*_H \bar{h}^\epsilon\|^2_{L^2} \geq \|K^*_H \bar{h}^{\epsilon,M}\|^2_{L^2} = \|\bar{h}^{\epsilon,M}\|^2_{\mathcal{H}}, \\ &\|h^\epsilon\|^2_{\mathcal{H}} = \|\dot{h}^\epsilon\|^2_{L^2} \geq \|\dot{h}^{\epsilon,M}\|^2_{L^2} = \|h^{\epsilon,M}\|^2_{\mathcal{H}}. \end{aligned} \quad (4.7)$$

From (4.3) (4.5), (4.6) and (4.7), we therefore have

$$\begin{aligned} &- \epsilon \log \mathbb{E}\left[e^{-\frac{\kappa(X^{\epsilon,\delta})}{\epsilon}}\right] \\ &\geq \mathbb{E}[\rho \circ \Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}), \epsilon^H(B^H_\cdot + R_H \bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) \\ &\quad + \frac{1}{2}\|\bar{h}\|^2_{\mathcal{H}} + \frac{1}{2}\|h\|^2_{\mathbb{H}}] - \frac{2\|\rho\|_\infty(2\|\kappa\|_\infty + \gamma)}{M} - \gamma. \end{aligned} \quad (4.8)$$

Since M, γ are arbitrary, in order to prove the lower bound, it suffices to show

$$\begin{aligned} &\liminf_{\epsilon \to 0} \mathbb{E}[\kappa \circ \Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}), \epsilon^H(B^H_\cdot + R_H \bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) + \frac{1}{2}\|\bar{h}\|^2_{\mathcal{H}} + \frac{1}{2}\|h\|^2_{\mathbb{H}}] \\ &\geq \inf_{x \in \mathcal{E}} \{\kappa(x) + I(x)\}. \end{aligned} \quad (4.9)$$

Since

$$\frac{1}{2}\|\bar{h}^{\epsilon,M}\|^2_{\mathcal{H}} \leq M, \frac{1}{2}\|h^{\epsilon,M}\|^2_{\mathbb{H}} \leq M,$$

we can extract a (not relabelled) subsequence such that $\bar{h}^{\epsilon,M}$ converges to $\bar{h}$ in distribution and $h^{\epsilon,M}$ converges to h in distribution. Then, we obtain

$$\begin{aligned} &\liminf_{\epsilon \to 0} \mathbb{E}[\kappa \circ \Gamma^\epsilon(\epsilon^{\frac{1}{2}}(W_\cdot + h^\epsilon(\cdot)/\sqrt{\epsilon}, \epsilon^H(B^H_\cdot + R_H \bar{h}^\epsilon(\cdot)/\sqrt{\epsilon^{2H}})) + \frac{1}{2}\|\bar{h}\|^2_{\mathcal{H}} + \frac{1}{2}\|h\|^2_{\mathbb{H}}] \\ &\geq \mathbb{E}[\kappa \circ \Gamma^0(\dot{h}(\cdot), R_H \bar{h}(\cdot)) + \frac{1}{2}\|\bar{h}\|^2_{\mathcal{H}} + \frac{1}{2}\|h\|^2_{\mathbb{H}}] \\ &\geq \inf_{(x,h,\bar{h}) \in \mathcal{E} \times \mathbb{H} \times \mathcal{H}} \mathbb{E}[\kappa(x) + \frac{1}{2}\|\bar{h}\|^2_{\mathcal{H}} + \frac{1}{2}\|h\|^2_{\mathbb{H}}] \\ &\geq \inf_{x \in \mathcal{E}} \{\kappa(x) + I(x)\}. \end{aligned}$$

The proof is complete.

□

Competing interests

The author declares they have no competing interests.

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