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DOCTORAL THESIS

Holography and Dualities in Quantum Field Theories

by:

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- This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended. In particular, this thesis is based on the following publications:
 1. M. Barbosa, H. Nastase, C. Nunez, and R. Stuardo, *Penrose limits of I-branes, twist-compactified D5-branes, and spin chains*, *Phys. Rev. D* **110** (2024), no. 4 046015, [arXiv:2405.08767](#) [hep-th].
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*"I copied - don't be afraid to copy - and eventually something that I
suppose that I would call my own appeared."*

David Gilmour, 2019

Abstract

The strong coupling regime of Quantum Field theories is studied by using two different frameworks in String Theory. First, by using brane configurations and studying its dynamics, we propose an electric-magnetic duality for three dimensional Yang-Mills Chern-Simons with and without SUSY. By analysing the infrared structure we check the validity of this duality. Then, using the Gauge/Gravity correspondance, we study the strong coupling regime of SQFT in different dimensions, which may or not be conformal. We compute some observables and propose that the theories confine.

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Chapter 1

Introduction

Non-abelian Yang-Mills (YM) theories coupled to matter are at the cornerstone of fundamental particle physics. For example, $SU(3)$ Yang-Mills theory with fundamental fermions, also known as Quantum Chromodynamics (QCD), provides an adequate framework to describe the interactions between gluons and quarks. In the high energy regime (UV), in which the coupling constant is small, the theory can be described by the fundamental fields in the Lagrangian. At low energies, the theory becomes strongly coupled and there is no continuous Wilsonian method to obtain an effective Lagrangian that describes the light infrared (IR) degrees of freedom - mesons and baryons. This behaviour of the coupling constant with respect of the energy scale is characteristic of an **asymptotically free** theory.

As in the previous example, YM and QCD-like theories are well understood in some energy regime in which there is a small parameter (typically the coupling constant) which allows for a perturbative expansion and the use of Feynman diagrams. This energy scale receives the name of **weak coupling regime**, and it depends on the gauge group, number of flavours and the dimension of the spacetime, among others. For example, while pure $SU(N_c)$ YM in $D \leq 3$ is asymptotically free, in 5D the coupling constant is small at lower energies. On the other hand, in the **strong coupling regime** (typically the opposite energy regime) the coupling constant becomes very large and it is not possible to use perturbation theory.

There is also a special class of theories in which the coupling constants do not run with the energy scale, which receive the name Conformal Field Theories (CFTs). Even if the couplings do not depend on the energy scale, still they can be small, allowing the use of perturbation theory as explained before, or large, where the theory is strongly coupled. Although the enhancement of the Poincare symmetry to the conformal one allows to constraint some correlation functions of the theory, it is not enough to have a complete

control over the dynamics of the strongly coupled regime. In fact, there is no standard procedure for the study of strong coupling regime of any Quantum Field Theory (QFT).

However, for the supersymmetric (SUSY) extensions of YM and QCD-like theories, the strongly coupled dynamics becomes tractable. Due to the extra restrictions imposed by SUSY, it is possible to obtain some information about the strong coupling regime of the theory analytically. For example, in 4D $\mathcal{N} = 1$ $SU(N_c)$ Super Yang-Mills (SYM) theories it is possible to understand R-symmetry breaking by studying the gaugino condensate, and how coupling the theory to N_f (with $N_f < N_c$) flavours generates a non-perturbative superpotential leading to dynamical SUSY breaking [1]. In the case of $\mathcal{N} = 2$ $SU(N_c)$ theories it is possible to completely characterise the IR strongly coupled theory entirely in terms of genus $N_c - 1$ Riemann surfaces [2–4]. But even with these non-perturbative results, the computation of others observables, e.g. Wilson, Polyakov and 't Hooft loops, meson and glueball spectra and aspects of confinement or the presence of a mass-gap remains a difficult task. Here after we refer as SQCD to the class of SUSY extensions of QCD-like theories. See [5, 6] and references therein for more details.

Since the discovery of Dp -branes as dynamical objects in Type II String Theory [7], and from the fact SYM or SQCD can be embedded as the effective theory on the worldvolume of a Dp -brane or an intersection of them, there has been a lot of progress in the study of the strong coupling regime of SYM and SQCD through two different frameworks within String Theory: Brane Dynamics and the Gauge/Gravity Correspondence

At the core of both of the frameworks mentioned above lies the idea of a **duality**: two seemingly unrelated theories are good descriptions of the same dynamics. Moreover, the dualities studied in this thesis generically exchange the strong coupling regime of one of the theories with the weak coupling regime of the other, therefore allowing to obtain, via perturbation theory, information of the theory of interest using its dual description. These type dualities receive the name **strong-weak dualities**.

Brane Dynamics relies on the construction of brane configurations whose low energy worldvolume dynamics corresponds to SYM or SQCD with different amounts of supersymmetry. These were first introduced in [8], where a configuration of D3, D5 and NS5-branes was used to study the Coulomb branch of 3D $\mathcal{N} = 4$ three-dimensional SQCD. Then, similar configurations with D4 and NS5-branes were used to study the Coulomb branch of 4D $\mathcal{N} = 2$ SYM theories [9], where it was shown that, by lifting the configuration to M-theory, one can obtain the same Riemann surface that characterises the IR of the QFT. This is one of the ways in which String Theory provides a geometric description of field theory phenomena.

Moreover, by modifying these configurations to realise theories preserving four Poincare supercharges, it is possible to derive a **strong-weak duality** between two QCD-like the-

ories in the same dimension, see [10] for a review. Examples of these dualities are [11] for the String Theory realisation of 4D $\mathcal{N} = 1$ Seiberg duality [12], and [13] for a 3D version of it including a Chern-Simons term. In both cases, it can be shown geometrically that the theories have the same moduli space of vacua.

In 3D, these configurations can also be modified to study theories with two supercharges and non-SUSY theories [14]. This amount of SUSY is not enough to constraint the dynamics of the QFT, but it is still possible to study the phase structure of the deep IR of the theory, since in that regime, the Chern-Simons term dominates over the Yang-Mills interactions. Interestingly, the duality correctly captures the three phases of adjoint 3D QCD [15] and its $\mathcal{N} = 1$ version [16] via level-rank duality [17]. These results can be extended to the presence of flavours [18], which is the main purpose of Chapter 2.

It is important to note that these dualities can in principle be obtained only through QFT techniques, as was the case for 4D Seiberg duality [12], or the 3D QCD phases [15]. What String Theory does is to provide a formal setting to derive a duality between two QFTs, but the duality does not involve String Theory itself.

This is not the case for the Gauge/Gravity Correspondence. First proposed for Superconformal Field theories (SCFT) in [19] and then refined in [20, 21], the AdS/CFT conjecture proposes a strong-weak duality between a SCFT in dimension d and String Theory in 10D on a background with an AdS_{d+1} factor (it is also possible to consider M-theory on such backgrounds). The two theories that are part of the duality, the d -dimensional SCFT and String Theory/M-theory, are formulated in **different dimensions**. Such dualities receive the name **holographic dualities**.

These holographic dualities are also strong-weak. In its more usual form, one uses the classical and weakly coupled limit of String Theory, i.e. 10D Supergravity, to study the strong coupling limit of the dual SCFT. This allows to obtain information about non-perturbative aspects of the QFT.

The complete Type II Supergravity background is of the form $\text{AdS}_{d+1} \times \mathcal{M}_{9-d}$, which is supported by a Dilaton and Ramond-Ramond (R-R) and Neveu-Schwarz-Neveu-Schwarz (NS-NS) fluxes. The configuration geometrically encodes information about global symmetries, gauge group ranks, anomalies and non-perturbative phenomena. For the canonical example of 4D $\mathcal{N} = 4$ $SU(N_c)$ SYM, the internal space $\mathcal{M}_5 = S^5$ realises the $SO(6)$ R-symmetry of the field theory. There is also a R-R 5-form which flux on the S^5 gives precisely the rank of the gauge group N_c , and fluctuations of the Type IIB background fields encode SYM correlators. Moreover, string probes on the geometry can be used to compute the expectation value of the Wilson Loop in the strongly coupled SCFT [22]. For a more elaborate discussion, see [23, 24].

In the case of SCFTs with eight Poincaré Supercharges, in the recent years there has been a systematic classification of the holographic duals of these theories in $d = 1$ [25–28], $d = 2$ [29–37], $d = 3$ [38–46], $d = 4$ [47–52], $d = 5$ [53–61] and $d = 6$ [62–66] dimensions, among others. These Supergravity backgrounds are dual to the strong coupling regime of configurations of Dp , $D(p + 2)$ and NS5-branes studied in [8, 9] and their generalisations. These geometries are completely characterised by a potential function. This function satisfies the Laplace equation sourced by a distribution usually called Rank function, which contains the information about the brane configuration on the geometry. We review the $d = 3$ and $d = 6$ backgrounds in chapters 3 and 4 respectively, where the former will be used to study the non-Abelian T-duals of the reduction to Type IIA of $\text{AdS}_4 \times S^7$ [44], and the later for the study of compactifications of a family of 6D SCFT down to 4D [67].

Soon after the proposal of the AdS/CFT conjecture, the holographic duality was extended to non-conformal maximally supersymmetric theories in [68, 69]. This led to the study of different set-ups, exploring the duality in string backgrounds dual to minimally SUSY or SUSY-breaking QFTs, which are more phenomenologically appealing scenarios. In the literature there are three different approaches for this matter (see [70] for a review)

- Consider an AdS Supergravity background dual to a known conformal theory. We can consider deformations of this background such that the geometry is asymptotically AdS in the UV but such that the conformal symmetry is broken in the IR (only preserving the Poincare symmetry). In the dual field theory, this corresponds to deforming the theory with a relevant operator which breaks conformal invariance. An example of this approach is the Polchinski-Strassler configuration [71], which consists of a deformation of $\mathcal{N} = 4$ $SU(N)$ SYM that flows to $\mathcal{N} = 1$ SYM in the IR.
- Starting from the solution of D3-branes on the tip of conifold [72], which is dual to a two-node $\mathcal{N} = 1$ conformal field theory, we can introduce fractional branes and other R-R fluxes to deform the theory in such a way that in the IR it flows (after the "duality-cascade") to $\mathcal{N} = 1$ SYM. This corresponds to the Klebanov-Strassler solution [73].
- We can start from a solution for Dp -branes with $p \geq 4$ and wrap some directions of the brane on a compact manifold, such that the compactified lower dimensional QFT is either non-SUSY or with small amount of SUSY preserved. Let us discuss this point separately.

There are two canonical examples of theories that arise from wrapped branes. First, continuing with holographic duals to $\mathcal{N} = 1$ SYM, the Maldacena-Nunez solution [74] describes 6D SYM on $\mathbb{R}^{3,1} \times S^2$, such that in the IR flow to 4D $\mathcal{N} = 1$ $SU(N_c)$ SYM. The second example are SUSY field theories with a circle direction with SUSY-breaking spin

structure. The compactification of the theory on the S^1 is realised by a cigar geometry: shrinking of the circle coordinate along which the boundary field theory is compactified. For example, the dual of maximally SUSY Yang-Mills in $(p+1)$ dimensions compactified on a circle with anti-periodic (thermal) boundary conditions for the fermions are double Wick rotations of non-extremal Dp -brane geometries [75]. In Chapter 5, we study modifications of these backgrounds with cigar geometries that allow to preserve some amount of SUSY, which were analysed in [76]. Then, in Chapters 6 and 7 we use this method to resolve IR singularities of two different Type IIB backgrounds [77, 78], to then compute some observables on these geometries.

This thesis is organised as follows: In Chapter 2 we focus on 3D $\mathcal{N} = 1$ YM-CS theories with matter. We start by reviewing how to embed SUSY field theories on different brane constructions as in [10]. Then, we use Giveon-Kutasov duality to obtain a magnetic dual of these family of theories, which allows us to study the IR dynamics of the electric theory, which are the results of [18]. In Chapter 3, we start by briefly discussing aspects of Abelian and non-Abelian T-duality. Afterwards we cover the results of [44], that is, we study a family of 3D $\mathcal{N} = 4$ SCFTs using the electrostatic formulation of their Supergravity duals. In particular we focus on the theory dual to the NATD of reduction to 10D of $\text{AdS}_4 \times S^7$, to study the effects of the duality transformation on its dual field theory. Chapter 4 is dedicated to the study the RG-flow across dimensions starting in the UV from an infinite family of 6D $\mathcal{N} = (1, 0)$ SCFTs on a Riemann surface to 4D $\mathcal{N} = 1$ SCFTs in the IR. We then propose various observables that count the number of degrees of freedom across the flow. This flow and observables were studied in [67]. The last three chapters of this thesis are dedicated to the study of different aspects of the twisted circle compactification of different SUSY field theories. Chapter 5 starts with a review of SUSY breaking boundary conditions when the spacetime has a circle direction. The rest of the chapter covers the results in [76], giving a detailed derivation of the twisted circle compactification of 4D $\mathcal{N} = 4$ SYM, and how it allows to recover $\mathcal{N} = 1$ worth of SUSY due to the shift in the spectrum induced by a background gauge field. Different vacua of the theory are discussed, and the holographic dual to them is also presented. Chapter 6 and Chapter 7 deal with the twisted circle compactification of two different theories constructed using D5-branes or an intersection of them (in the later one, the first section reviews some results of [79]), which are related to [77] and [78] respectively. Different observables are computed, suggesting that both theories are confining.

Chapter 2

Dualities of 3D $\mathcal{N} = 1$ SQCD from Branes and non-SUSY deformations

The aim of this chapter is to study the dynamics of 3D gauge theories with Chern-Simons (CS) terms. These theories have attracted a lot of attention, following [80, 81], due to the fact that, thanks to the CS term, this class of theories admit a rich structure where the IR dynamics exhibits phases with either a CFT, a gapped phase, or a broken flavour symmetry phase, etc. Often the IR theory contains a TQFT. The IR dynamics is constrained by matching global anomalies.

We study on gauge theories that contain one adjoint fermion and N_f flavours of quarks in the fundamental representation of the gauge group. In particular, we focus on an 'electric'¹ $\mathcal{N} = 1$ 3D $U(N_c)_{k, k + \frac{N_c}{2}}$ SQCD theory. In Section 2.1, we give a briefly discuss how to embed these theories in String Theory as the effective theory on the worldvolume of a modification of the Hanany-Witten brane configurations [8]. In Section 2.2, we propose a 'magnetic' dual, as in [11, 13], and analyse low energy dynamics of the 'electric' theory using its dual. The proposed duality for the $\mathcal{N} = 1$ theories is checked by the computation of the Witten index [82]. In Section 2.3 we show that both theories flow to the same IR fixed point when considering large quark mass. In Section 2.4 we extend this duality to a non-supersymmetric case, generalising the result of [15, 83] to the presence of fundamental matter. Section 2.5 is dedicated to the generalisation to SO and USp groups.

¹Recall that for the $U(N_c) = SU(N_c) \times U(1)$ theory, the CS level of the $SU(N_c)$ part is independent of the $U(1)$ level. Here, the two subscripts in $U(N_c)_{k, k'}$ denote the level of the $SU(N_c)$ and $U(1)$ part of the gauge group respectively, so that the Lagrangian takes the form $\mathcal{L} = \frac{k}{4\pi} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) + \frac{k'}{4\pi} \text{Tr}(A) \wedge \text{Tr}(dA)$.

2.1 SYM Embeddings in Dp-Brane Worldvolumes

Let us start by reviewing Dp-branes, see [10] for a complete discussion of the material covered in this section. These are hypersurfaces in 10D flat space defined in String Theory by the boundary conditions for open strings. The directions in which the string endpoint satisfies Neumann boundary conditions corresponds the worldvolume of the Dp-brane, which we take to be extended in the $x^0, x^1, \dots, x^p$. On the other hand, the directions in which the string satisfies Dirichlet boundary conditions define the transverse space to the Dp-brane, $x^{p+1}, \dots, x^9$, where the Dp-brane is seen as a pointlike object. Also, the presence of a Dp-brane breaks the 10D Lorentz symmetry $SO(9, 1) \rightarrow SO(p, 1) \times SO(9 - p)$.

These objects are charged under the A_{p+1} R-R potential of Type II String Theory. The charge of these objects is equal to its tension, so that Dp-branes are BPS objects preserving 16 out of the 32 supercharges of the theory.

The massless bosonic open string spectrum in the presence of a Dp-brane consists of a $(p + 1)$ -dimensional $U(1)$ gauge field A_a and $(9 - p)$ scalars X^I . Here $a = 0, 1, \dots, p$ and $I = p+1, \dots, 9$. These scalars parametrise the fluctuations of the Dp-brane in the transverse space and transform as a vector under the $SO(9 - p)$ symmetry. The massless fermionic part of the massless spectrum can be obtained by imposing SUSY.

The previously mentioned excitations couple to massive open string excitations and bulk closed strings. In order to obtain a low energy effective theory on the Dp-brane worldvolume, we need to decouple the massless open string excitations from the rest. This is achieved by taking the string length to zero $l_s^2 = \alpha' \rightarrow 0$. Then, the effective action is $(p + 1)$ -dimensional maximally supersymmetric $U(1)$ Yang-Mills theory. This is to be expected since: the Dp-brane preserves 16 supercharges, which is the maximum amount for a SUSY QFT without gravity. Also, the $(9 - p)$ scalars transform as a vector under $SO(9 - p)$ which is precisely the number of scalars and the R-symmetry of a maximally SUSY QFT. Note that the global symmetry of the transverse space to the Dp-brane corresponds to the R-symmetry of the effective theory on its worldvolume. This will be a recurring feature when we discuss brane constructions for theories with less SUSY.

Stacking N_c Dp-branes on top of each other leads to $SU(N_c)$ maximally supersymmetric Yang-Mills theory

$$S_{SYM} = \int d^{p+1}x \operatorname{Tr} \left(-\frac{1}{4g_{YM}^2} F_{ab} F^{ab} - \frac{1}{2} \delta_{IJ} D_a X^I D^a X^J + \frac{g_{YM}^2}{4} [X, X]^2 + \text{fermions} \right) \quad (2.1.1)$$

Before moving to theories with less supersymmetries, it is important to mention the relation between the effective $(p + 1)$ -dimensional Yang-Mills coupling g_{YM} and the String

Theory couplings α' and g_s

$$g_{YM}^2 = (2\pi)^{p-2} g_s \alpha'^{\frac{p-3}{2}}. \quad (2.1.2)$$

In order to obtain as the low energy theory on a Dp -brane worldvolume a theory with less SUSY, we need to consider branes suspended between branes. For this we need to introduce a different type of brane: the NS5-brane, which is a $(5 + 1)$ -dimensional hypersurface charged *magnetically* under the NS-NS 2-form B_2 . Note that, while strings can not end on a NS5-brane, Dp -branes can. Also we can have Dp -branes ending on other Dp' -branes.

We are interested in a brane configuration of a N_c D3-brane suspended, in the x^6 direction, between two NS5-branes, which are separated by a distance L . From now on we refer to these stack of D3-branes as **colour branes**. These configurations were studied in [8]. The boundary conditions for the fields of the effective D3-brane theory on an interval are discussed in [10]. Here we present the massless spectrum arising from those boundary conditions. The directions in which the branes are extended or pointlike are displayed in Table 2.1.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	-	-	-	·	·	·	-	·	·	·
NS5	-	-	-	-	-	-	·	·	·	·

Table 2.1: Brane configuration for N_c D3-branes suspended between two NS5-branes. Here we use $-$ to show that the brane is extended in that direction, while $\cdot$ means that the brane is pointlike.

Here the 10D Lorentz symmetry is broken to $SO(2, 1) \times SO(3) \times SO(3)$, where one $SO(3)$ factor corresponds to rotations in the x^3, x^4, x^5 space, while the other one are rotations in the x^7, x^8, x^9 space. These $SO(3) \sim SU(2)$ symmetries correspond to the R-symmetry of the effective theory on the worldvolume of the D3-branes. Also the amount of SUSY preserved by this brane configuration is reduced to eight supercharges. This is due to the fact that each different type of brane imposes different projections on the SUSY spinor, depending on the directions in which the brane is extended. Compatibility of the two conditions imposed by the D3-branes and the NS5-branes only preserves eight supercharges. See [10] for a more detailed discussion, including explicit expressions for the projections.

Both types of branes share the x^0, x^1, x^2 , which in this context we will refer as the field theory directions. The directions x^7, x^8 and x^9 are transverse to both branes and they do not play a role in the analysis below. Then, all the information is encoded in the x^3, x^4, x^5, x^6 space. In that space, we can pictorially represent the brane configuration as in Figure 2.1.

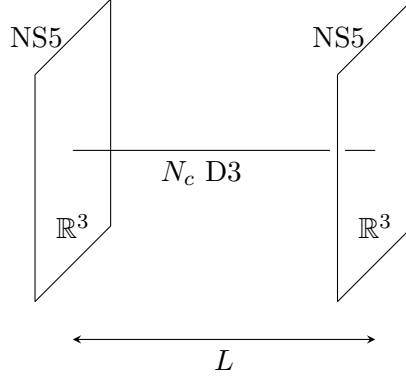


Figure 2.1: Brane configuration in the x^3, x^4, x^5, x^6 space. Here $\mathbb{R}^3$ corresponds to the x^3, x^4, x^5 space.

The massless open string spectrum on this brane configuration consists of: a 3D $U(N_c)$ gauge field, corresponding to open string excitations on the x^0, x^1, x^2 directions, and three real adjoint scalars, corresponding to transverse D3-brane fluctuations that are parallel to the NS5-branes. This is completed by fermions that make the QFT SUSY. The gauge field component along x^6 (recall that the D3-brane is extended in x^6) and the transverse excitations to both branes are massive, with mass proportional to $1/L^2$, so that in the limit $L \rightarrow 0$, the effective theory low energy theory is 3D $\mathcal{N} = 4$ $U(N_c)$ SYM.

We can include matter in the fundamental representation of $U(N_c)$ by including N_f D3-branes at the left of the left-most NS5-brane, which end at the right of N_f D3-branes. We refer to these branes as **flavours branes**. Through out this thesis, we take the flavour branes to end on the left of the left most NS5-brane in Figure 2.1. We take the flavour D5-branes to be extended in the field theory directions, and the (x^7, x^8, x^9) space. Note that the inclusion of these flavours branes does not break the $SO(2, 1) \times SU(2) \times SU(2)$ symmetry. The brane configuration is described in Table 2.2

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	-	-	-	.	.	.	-	.	.	.
NS5	-	-	-	-	-	-	.	.	.	.
D5	-	-	-	.	.	.	.	-	-	-

Table 2.2: Brane configuration for 3D $\mathcal{N} = 4$ $U(N_c)$ SYM with fundamental matter.

Pictorially, we now suppress the $\mathbb{R}^3$ plane to a line. Then, the brane configuration including flavour branes, together with the possible string excitations is shown in Figure 2.2

The massless open string spectrum in this configuration consists of: open strings stretching between colour branes, which realise the $\mathcal{N} = 4$ $U(N_c)$ vector multiplet. We also have

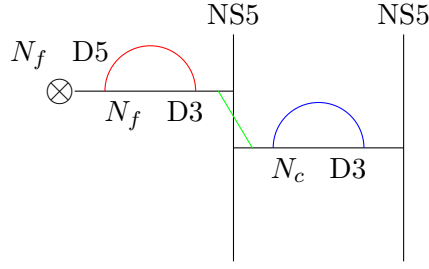


Figure 2.2: Brane configuration for 3D $\mathcal{N} = 4$ $U(N_c)$ SYM with fundamental matter. Here the blue line represents an open string that starts and ends in the colour branes. The green line represents a string connecting the colour brane with the flavour ones, while the red one are strings that start and end on the flavour branes. These are all the possible open string excitations.

open strings connecting colour branes and flavour branes, which realise N_f hypermultiplets. The reader might wonder if there is an extra gauge multiplet coming from string with both ends on flavour branes. Due to the boundary conditions imposed by the D5-brane on the left and the NS5-brane on the right, there are no massless degrees of freedom coming from those strings. In total, the effective theory on the brane configuration is 3D $\mathcal{N} = 4$ $U(N_c)$ SYM with N_f hypermultiplets.

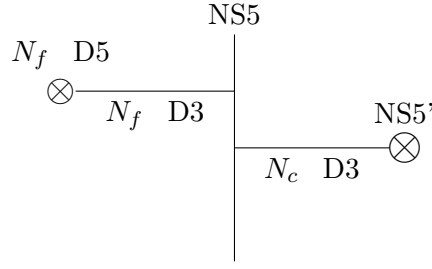
In order to obtain theories preserving four supercharges, we need to rotate one of the NS5-branes, so that now it extends in the (x^8, x^9) plane, we refer to this brane as NS5'-brane. We now focus on the theory arising from rotating the right most NS5-brane. We discuss the other case later. The brane configuration is depicted in the table below

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	-	-	-	.	.	.	-	.	.	.
NS5	-	-	-	-	-	-	.	.	.	.
NS5'	-	-	-	-	.	.	.	.	-	-
D5	-	-	-	.	.	.	.	-	-	-

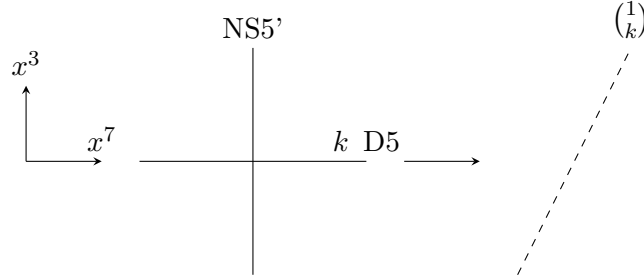
Table 2.3: Brane Configuration for the 3D $\mathcal{N} = 2$ SYM theory.

Notice that the global symmetries of the configuration are broken to $SO(2, 1) \times U(1) \times U(1)$, where the $U(1)$ factors correspond to rotations on the (x^4, x^5) and (x^8, x^9) planes. By studying the projections imposed on the 10D SUSY spinor by the different Dp and NS5-branes, one can check that the configuration preserves four supercharges. The effective theory is then 3D $\mathcal{N} = 2$ $U(N_c)$ SYM with N_f fundamentals. The configuration is showed in Figure 2.3.

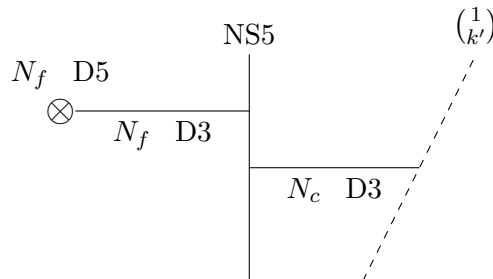
3D $\mathcal{N} = 2$ SYM admits the inclusion of Chern-Simons terms. In order to obtain


 Figure 2.3: Brane configuration for 3D $\mathcal{N} = 2$ $U(N_c)$ YM theory with flavours

a Chern-Simons term from the brane configuration, we need to introduce a tilted $(1, k)$ fivebrane. These are a recombination of a NS5-brane and k D5-branes, we explain how these branes appear now. Note that in Table 2.3 the NS5' and D5-branes share the field theory and the (x^8, x^9) directions. When a NS5'-brane intersects k D5-brane on the (x^3, x^7) plane, they can combine into a $(1, k)$ fivebrane [84]. This process is shown in Figure 2.4


 Figure 2.4: Recombination of the NS5' and k D5-branes into the tilted $(1, k)$ fivebrane

Replacing the NS5'-brane with a tilted $(1, k)$ fivebrane, leads to a Chern-Simons term on the effective theory on the worldvolume of the stretched D3-branes [85] due to the mixed boundary conditions imposed by the tilted fivebrane. Finally, the brane configuration realises the "electric" 3D $\mathcal{N} = 2$ $U(N_c)_{k,k}$ YM-CS theory with N_f fundamentals is shown in Figure 2.5


 Figure 2.5: Brane configuration for the "electric" 3D $\mathcal{N} = 2$ $U(N_c)_{k,k}$ with N_f fundamental flavours.

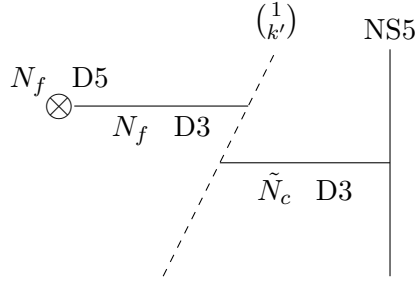


Figure 2.6: Brane configuration for the "magnetic" 3D $\mathcal{N} = 2$ $U(\tilde{N}_c)_{-k, -k}$ YM-CS theory with matter (here $\tilde{N}_c = k + N_f - N_c$).

This theory admits a "magnetic" dual [13], similar to 4D $\mathcal{N} = 1$ Seiberg duality. The dual theory is 3D $\mathcal{N} = 2$ $U(k + N_f - N_c)_{-k, -k}$ YM-CS theory coupled to N_f fundamental flavours and a $N_f \times N_f$ meson matrix which is a singlet of the gauge group. These mesons couple to the magnetic flavours via the superpotential $W = Mq\bar{q}$. The brane configuration is the same as for the electric theory, but with the NS5-brane on the right and the tilted fivebrane on the left, as in Figure 2.6

The mesons fields in the magnetic theory arise from open strings with both endpoints on the N_f D3-flavour branes. The boundary conditions imposed by the D5-branes on the left and the tilted fivebrane on the right still set the zero mode gauge field and its superpartners to zero, but now they allow to have massless excitations coming from fluctuations of the N_f D3-branes on the (x^8, x^9) directions. Note that these directions are parallel to both D5 and tilted fivebrane, as seen from Table 2.3. We will argue later that there is also a Chern-Simons term on these branes.

We now move to the main objective of this chapter, which is the study of 3D $\mathcal{N} = 1$ $U(N_c)_{k, k'}$ YM-CS theory, the proposal of its dual via Giveon-Kutasov duality and the matching of their Witten index, and the study of the IR structure.

2.2 $\mathcal{N} = 1$ Dualities from Branes

We derive the gauge theory dualities from the duality in String Theory studied in the previous section. Recall that the electric (or magnetic) configuration consist of N_c (or $\tilde{N}_c$) D3-branes suspended between an NS5-brane and a $(1, k')$ tilted fivebrane. The brane configurations for the electric and magnetic theory are represented in Figure 2.5 and Figure 2.6 respectively. There, the D3-branes span the 012 directions and an interval in the 6 direction. The NS5-brane spans the 012345 directions. The $(1, k')$ fivebrane spans the 01289 directions and it is tilted in the (37) plane. In order to obtain $U(N_f)$ flavour symmetry, we add N_f D3-branes suspended between N_f D5-branes and the NS5-brane on the electric

theory (or the $(1, k')$ fivebrane in the magnetic one). As mentioned before, this set-up is the one considered in [13] and realises 3D $\mathcal{N} = 2$ $U(N_c)_{k', k'}$ Yang-Mills-Chern-Simons (YM-CS) theory.

Both $\mathcal{N} = 2$ electric and magnetic theories can be written in terms of $\mathcal{N} = 1$ variables as follows: The $\mathcal{N} = 2$ vector multiplet splits into a $\mathcal{N} = 1$ vector multiplet (A_μ, λ) and an adjoint scalar multiplet (χ, φ) , while the $\mathcal{N} = 2$ flavours split into two $\mathcal{N} = 1$ chiral multiplets Φ and Φ' (here $\Phi = (\psi, \phi)$, similarly for Φ').

Let us begin with the electric theory. In order to obtain 3D $\mathcal{N} = 1$ $U(N_c)$ with one adjoint fermion and N_f flavours as the low energy theory, we set² $k' = k + \frac{N_c}{2} - \frac{N_f}{2}$ (with $(k' > 0)$) and then integrate out χ for large negative mass, which shifts the $SU(N_c)$ CS level by $-\frac{N_c}{2}$, and ψ' for large positive mass, shifting both CS-levels by $\frac{N_f}{2}$. This can be achieved by further rotation of the tilted fivebrane as in [14]. We then obtain 3D $\mathcal{N} = 1$ $U(N_c)_{k, k + \frac{N_c}{2}}$ with N_f flavours Φ .

To obtain the Seiberg dual, we swap the fivebranes following the prescription of [11, 13]. In order to preserve supersymmetry we consider the case $\tilde{N}_c \geq 0$. After swapping the fivebranes we obtain the $\mathcal{N} = 1$ magnetic theory $U(\tilde{N}_c)_{\tilde{k}, -k' + \frac{N_f}{2}}$, where the number of D3-branes on the magnetic side is

$$\tilde{N}_c = k' - N_c + N_f = k - \frac{N_c}{2} + \frac{N_f}{2}. \quad (2.2.1)$$

As in the electric side, the $SU(\tilde{N}_c)$ CS level is shifted by the massive adjoint fermion and the massive chiral flavour³

$$\tilde{k} = -k' + \frac{\tilde{N}_c}{2} - \frac{N_f}{2} = -\frac{k}{2} - \frac{3N_c}{4} + \frac{N_f}{4}, \quad (2.2.2)$$

while the $U(1)$ CS-level only receives a contribution from the flavours $-k' + \frac{N_f}{2} = -k - \frac{N_c}{2}$. Therefore we propose the following Seiberg duality between two $\mathcal{N} = 1$ supersymmetric YM-CS theories

$$U(N_c)_{k, k + \frac{N_c}{2}} + N_f \Phi \Leftrightarrow U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-\left(\frac{k}{2} + \frac{3N_c}{4} - \frac{N_f}{4}\right), -\left(k + \frac{N_c}{2}\right)} + N_f \tilde{\Phi}, \quad (2.2.3)$$

²We could also start with level $k'' = k - \frac{N_c}{2} + \frac{N_f}{2}$, with $k'' < 0$, and give masses to the fields as $m_\chi > 0$ and $m_{\Phi'} < 0$. This defines the electric' theory and its dual magnetic'. When the theories are SUSY, the range of k is different: $k > \frac{N_c - N_f}{2}$ for the electric-magnetic and $k < -\frac{N_c - N_f}{2}$ for the electric'-magnetic'. The pairs of theories are related by parity. In the non-SUSY case, $|k| < \frac{N_c - N_f}{2}$ in both cases. We conjecture that in the non-SUSY case, the electric and electric' theories are the same since the $U(1)$ factor decouples. Then, the magnetic and magnetic' allows us to describe three different phases of the $SU(N_c)_k$ electric theory.

³In the magnetic theory, the masses of the adjoint fermion and the flavours have opposite signs when compared with their electric counterparts.

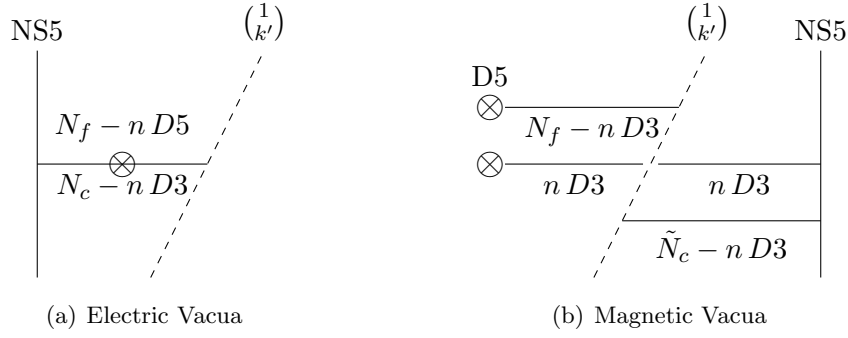


Figure 2.7: SUSY vacua on both sides of the duality

where we have denoted the flavours in the magnetic theory by $\tilde{\Phi}$. This duality was obtained from the field theory side in [86, 87]. Note that the r.h.s. of (2.2.3), namely the magnetic theory, admits also mesons and an additional CS term, arising from the intersection of the flavour branes with the tilted fivebrane which is interpreted as a WZ term in the phase where the flavour symmetry is broken.

2.2.1 Witten Index

To check the duality, we derive the Witten Index from the brane configuration. We need to count all different SUSY vacua as in [13]. On the electric side, it is convenient to move the D5-brane past the NS5-brane and set them on top of the D3-branes. On a SUSY vacuum, we allow n of the N_c D3-branes to break on the D5-brane, so that half of the n D3 stretches between the NS5 and the D5, and the other half stretches between the D5 and the $(1, k')$ fivebrane. Sending the n D5s to infinity leads to a brane configuration with $N_c \rightarrow N_c - n$ and $N_f \rightarrow N_f - n$. Then we need to count how the leftover $N_c - n$ D3-branes can end on the $(1, k')$ brane. Summing over all possible n leads to

$$I_W = \sum_{n=0}^{N_c} \binom{k'}{N_c - n} \binom{N_f}{n} = \binom{k' + N_f}{N_c} = \frac{\left(k + \frac{N_c}{2} + \frac{N_f}{2}\right)!}{N_c! \left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)!}. \quad (2.2.4)$$

On the magnetic side the argument is similar. We allow n of the $\tilde{N}_c$ D3-branes to reconnect with n of the N_f D3 flavour branes. Then we count how the leftover $\tilde{N}_c - n$ branes can end on the tilted fivebrane

$$I_W = \sum_{n=0}^{\tilde{N}_c} \binom{k'}{\tilde{N}_c - n} \binom{N_f}{n} = \binom{k' + N_f}{\tilde{N}_c} = \frac{\left(k + \frac{N_c}{2} + \frac{N_f}{2}\right)!}{N_c! \left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)!}. \quad (2.2.5)$$

Both n th SUSY vacua are shown in figure 2.7.

2.2.2 Low energy with Symmetry Breaking

When $\tilde{N}_c \equiv \frac{N_f}{2} + k - \frac{N_c}{2} \geq 0$ the dual SQCD theory admits two phases: large and small quark mass. In this section we discuss the small mass phase.

Let us consider the brane configuration of the magnetic theory, depicted in Figure 2.6. When the quark mass is small, the flavour branes are located near the colour branes along the tilted fivebrane axis. Therefore colour and flavour can be reconnected and form vacua with three kind of D3 branes: leftover colour branes, leftover flavour branes and reconnected branes. In a vacuum where n branes are reconnected we have $N_f - n$ leftover flavour branes and $\tilde{N}_c - n$ leftover colour branes, see Figure 2.7(b). In field theory those vacua correspond to a dual squark condensation

$$\langle \tilde{\phi}_\alpha^i \rangle \neq 0 \quad (2.2.6)$$

where i and α are flavour and dual colour indices, respectively.

The supersymmetric field theory that lives on the brane configuration consists of the following ingredients: $U(\tilde{N}_c - n)$ SQCD theory with a broken flavour symmetry of the following pattern

$$U(N_f) \rightarrow U(N_f - n) \times U(n) \quad (2.2.7)$$

described by the Grassmannian $U(N_f)/U(N_f - n) \times U(n)$. The $2n(N_f - n)$ Nambu-Goldstone bosons correspond to the open strings stretched between the stack of n D3-branes and the stack of $N_f - n$ D3-branes. These vacua were described in [86, 87].

In addition to the Grassmannian there is a WZ term, as argued in [88]. We claim that the source of this term in the brane setup is a CS term that lives on the intersection of the leftover flavour branes and the tilted fivebrane. Indeed, as was recently argued in [83] when a D3-brane is suspended between a D5-brane and a tilted fivebrane there is a CS term living on the intersection, even when the D5-brane is taken to infinity [89].

The WZ term is written as

$$k_f \int_{\mathcal{M}} \text{tr} \frac{1}{2\pi} F \wedge F, \quad (2.2.8)$$

where F transforms in the adjoint of $U(N_f - n)$. Let us calculate the level k_f . It is the level k' shifted by the fermions on the flavours branes and colour branes

$$k_f = k' + (N_f - n) - (\tilde{N}_c - n) = k' + N_f - \tilde{N}_c = N_c \quad (2.2.9)$$

namely, in each vacuum of the magnetic theory there is a WZ term of the form

$$N_c \int_{\mathcal{M}} \text{tr} \frac{1}{2\pi} F \wedge F. \quad (2.2.10)$$

The gauge field $F = dA$ is not independent of the mesons, but as in the standard chiral Lagrangian, $A \sim M^\dagger \partial M$.

When supersymmetry is broken by giving a small mass to the adjoint fermion, the vacuum degeneracy is lifted and one vacuum is selected. It is natural to propose that the colour group disappears, namely the chosen vacuum is $n = \tilde{N}_c$. The motivation is Coleman-Witten theorem in 3D: flavour symmetry may be broken to $U(m) \times U(N_f - m)$ without a Yang-Mills gauge theory [90]. Hence, in the vacuum with $n = \tilde{N}_c$ we have symmetry breaking of the form

$$U(N_f) \rightarrow U\left(\frac{N_f}{2} - k + \frac{N_c}{2}\right) \times U\left(\frac{N_f}{2} + k - \frac{N_c}{2}\right). \quad (2.2.11)$$

The WZ term is

$$N_c \int_{\mathcal{M}} \text{tr} \frac{1}{2\pi} F \wedge F, \quad (2.2.12)$$

where $\partial\mathcal{M}$ is the 3D spacetime. F transforms in the adjoint of $U(\frac{N_f}{2} - k + \frac{N_c}{2})$.

Note that if we integrate out the adjoint fermion in the electric side $k \rightarrow k - \frac{N_c}{2}$ and our proposal for a symmetry breaking pattern and the Grassmannian coincide with the proposal of [88] for QCD.

While we consider the case $U(N_c)_{k, k + \frac{N_c}{2}}$, note that the $SU(N_c)_k$ is special when $k = 0$, since in this case the latter theory preserves parity, so that a flavour symmetry breaking of the form $U(N_f) \rightarrow U(\frac{N_f}{2} + \frac{N_c}{2}) \times U(\frac{N_f}{2} - \frac{N_c}{2})$ is forbidden, due to Vafa-Witten theorem. We propose that in the $U(N_c)$ case with $k = 0$ squark condensation does not occur and the theory is described by a TQFT, as we discuss in the next section.

2.3 Low Energy Dynamics with a TQFT

Let us consider the case of large quark mass (m_ψ). In the case with $N_f = 0$, the electric theory is supersymmetric when the adjoint fermion mass is $m_\lambda = -g^2 k$. In the deep IR, this theory admits two phases described by two TQFTs. It was shown that using the magnetic dual, one is able to recover the TQFT in a neighbourhood of a SUSY point. Here we show that this match of the IR dynamics around the SUSY point also holds in the presence of flavours.

In order to obtain the IR dynamics of the electric theory, we need to integrate out the adjoint fermion λ and the flavours ψ . There are four possible TQFTs in the IR depending on the sign of the masses m_λ and m_ψ . These theories are listed in the table 2.4

Repeating the same analysis on the magnetic side leads to the TQFs presented in table 2.5

	$m_\lambda < 0$	$m_\lambda > 0$
$m_\psi > 0$	$U(N_c)_{k - \frac{N_c}{2} + \frac{N_f}{2}, k + \frac{N_c}{2} + \frac{N_f}{2}}$	$U(N_c)_{k + \frac{N_c}{2} + \frac{N_f}{2}, k + \frac{N_c}{2} + \frac{N_f}{2}}$
$m_\psi < 0$	$U(N_c)_{k - \frac{N_c}{2} - \frac{N_f}{2}, k + \frac{N_c}{2} - \frac{N_f}{2}}$	$U(N_c)_{k + \frac{N_c}{2} - \frac{N_f}{2}, k + \frac{N_c}{2} - \frac{N_f}{2}}$

Table 2.4: Possible IR theories on the electric theory

	$m_{\tilde{\lambda}} < 0$	$m_{\tilde{\lambda}} > 0$
$m_{\tilde{\psi}} > 0$	$U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-k - \frac{N_c}{2} + \frac{N_f}{2}, -k - \frac{N_c}{2} + \frac{N_f}{2}}$	$U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-N_c + N_f, -k - \frac{N_c}{2} + \frac{N_f}{2}}$
$m_{\tilde{\psi}} < 0$	$U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-k - \frac{N_c}{2} - \frac{N_f}{2}, -k - \frac{N_c}{2} - \frac{N_f}{2}}$	$U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-N_c, -k - \frac{N_c}{2} - \frac{N_f}{2}}$

Table 2.5: Possible IR theories on the magnetic theory

Since from level-rank duality⁴ we have

$$U(N_c)_{k - \frac{N_c}{2} + \frac{N_f}{2}, k + \frac{N_c}{2} + \frac{N_f}{2}} \iff U\left(k - \frac{N_c}{2} + \frac{N_f}{2}\right)_{-N_c, -k - \frac{N_c}{2} - \frac{N_f}{2}}, \quad (2.3.1)$$

then from Table 2.4 and Table 2.5 we see that we are able to recover the IR TQFT around the SUSY point in the electric theory from the magnetic dual.

Notice that the IR dualities hold when in the electric theory we have $m_\lambda < 0$ and $m_\psi > 0$, while on the magnetic theory the mass have the opposite signs, i.e. $m_{\tilde{\lambda}} > 0$ and $m_{\tilde{\psi}} < 0$ as expected. Since the SUSY point is located at $m_\lambda = -g^2 k$ and $m_\psi = g^2 k$ we claim that the IR dualities hold in a neighbourhood of the SUSY point.

In what follows we will perform a similar analysis of the IR theories for non-SUSY theories and theories with different gauge groups. There we will not present all the possible TQFTs in the IR but rather limit ourselves to show the points in which the duality holds.

2.4 Non-SUSY Dualities

Now we consider the case $\tilde{N}_c < 0$, where supersymmetry is spontaneously broken. When swapping the fivebranes we obtain $-k + \frac{N_c}{2} - \frac{N_f}{2}$ *anti*-D3-branes, therefore breaking supersymmetry⁵. Then, the magnetic theory on the branes is

$$U\left(-k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{\frac{k}{2} + \frac{3N_c}{4} - \frac{N_f}{4}, k + \frac{N_c}{2}} + N_f \tilde{\Phi}. \quad (2.4.1)$$

⁴Level-Rank Duality for $U - U$ theories is given by [17]

$$U(N)_{K, K \pm N} \iff U(K)_{-N, -N \mp K}$$

⁵The appearance of anti branes in a 3D theory with a spontaneously broken supersymmetry was first pointed out in [91].

Similarly, the magnetic' theory is $U\left(k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{\frac{k}{2} - \frac{3N_c}{4} + \frac{N_f}{4}, k - \frac{N_c}{2}}$ with N_f flavours. Note that $\tilde{\lambda}$ is the Goldstino associated with the breaking of supersymmetry in the electric side.

We proceed to study the low energy dynamics of both sides of the duality. For $N_f = 0$ it was shown in [15] that the electric theory admits three phases: in addition to the large positive mass and large negative masses, there is an intermediate quantum phase characterised by a TQFT. These phases were recovered in [14] using brane dynamics. Here we extend this dualities to the case including flavours and show that it also admits an intermediate phase.

As anticipated, in this case the electric and electric' theories share the same range for k (see footnote 1). Since the theories cover the same regime, we conjecture that the $U(1)$ factor of both theories should decouple, thus, the electric and electric' theories are the same: $SU(N_c)_k$. This theory has two magnetic duals. In order for them to correctly reproduce the IR dynamics of the electric theory, we need to ensure that their $U(1)$ levels match in the IR: we have to assign $U(1)$ level $N_c - N_f$ to the magnetic theory and $-N_c + N_f$ to the magnetic'.

The matching of the low energy dynamics goes as follows: for the magnetic theory we integrate out $\tilde{\lambda}$ and $\tilde{\psi}$ for $m_{\tilde{\lambda}} > 0$ and $m_{\tilde{\psi}} > 0$, obtaining $U\left(-k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{N_c, N_c}$, which is dual⁶ to $SU(N_c)_{k - \frac{N_c}{2} + \frac{N_f}{2}}$, while $m_{\tilde{\lambda}} < 0$ and $m_{\tilde{\psi}} < 0$ leads to $U\left(-k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{k + \frac{N_c}{2} - \frac{N_f}{2}, N_c - N_f}$. These theories cover the negative mass and the intermediate phase in the electric theory. The magnetic' covers the intermediate phase and the positive mass regime. The details are as follows: for $m_{\tilde{\lambda}} > 0$ and $m_{\tilde{\psi}} > 0$ we obtain $U\left(k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{k - \frac{N_c}{2} + \frac{N_f}{2}, -N_c + N_f}$ as TQFT in the IR, which is level-rank dual to $U\left(-k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{k + \frac{N_c}{2} - \frac{N_f}{2}, N_c - N_f}$. For $m_{\tilde{\lambda}} < 0$ and $m_{\tilde{\psi}} < 0$, we obtain $U\left(k + \frac{N_c}{2} - \frac{N_f}{2}\right)_{-N_c, -N_c}$ which is dual to $SU(N_c)_{k + \frac{N_c}{2} - \frac{N_f}{2}}$. It is important to note that both $SU(N_c)$ theories are the expected ones from the electric side when integrating out the adjoint fermion and the flavour (for opposite mass signs).

2.5 USp and SO Theories

In this section we generalise the previous analysis, to $USp(2N_c)$ or $SO(2N_c)$ (or $SO(2N_c + 1)$). We'll study both cases simultaneously. In order to realise these theories from the brane

⁶Level-Rank duality for SU - U groups is given by [17]

$$SU(N)_{\pm K} \Leftrightarrow U(K)_{\mp N, \mp N}.$$

configuration we need to add an $O3^+$ ($O3^-$) plane extending in the 0126 directions. In the presence of the $O3^+$ we need to place N_c D3-branes and their mirrors since an odd number of branes is not compatible with the orientifold. On the other hand, when we place the $O3^-$ we can have an even or odd number of branes. This leads to the theories $USp(2N_c)$ and⁷ $SO(2N_c)$. In both cases we will use a $(1, 2k')$ tilted fivebranes and $2N_f$ semi-infinite D3 flavour branes.

As in the $U(N_c)$ case, when integrating out an adjoint fermion with mass m the CS level is shifted as

$$k \rightarrow k + \frac{\text{sign}(m)}{2} h(G), \quad (2.5.1)$$

where $h(G)$ is the dual Coxeter number of the group G . In our case $G = USp(2N_c)$ or $G = SO(2N_c)$ we have $h(USp(2N_c)) = 2N_c + 2$ and $h(SO(2N_c)) = 2N_c - 2$.

2.5.1 $USp(2N_c)$ Theory

We start by setting $k' = k + \frac{1}{2}(N_c + 1) - \frac{N_f}{2}$. On the electric side, we integrate out the adjoint fermion in the scalar multiplet for negative mass and one of the chiral flavours for positive mass. The resulting theory is $USp(2N_c)_{2k}$ with $2N_f$ flavours.

On the magnetic side, when we swap the fivebranes an additional D3 anti-brane and its mirror image are created. After integrating out the adjoint fermion and one of the chiral flavours, we obtain the $\mathcal{N} = 1$ theory $USp(2\tilde{N}_c)_{-2\tilde{k}}$, with

$$\tilde{N}_c = k' - N_c + N_f - 1 = k - \frac{1}{2}(N_c + 1) + \frac{N_f}{2}, \quad 2\tilde{k} = k + \frac{3N_c}{2} - \frac{N_f}{2} + \frac{1}{2}. \quad (2.5.2)$$

The magnetic dual is supersymmetric if $\tilde{N}_c > 0$. We propose the following $\mathcal{N} = 1$ Seiberg duality

$$USp(2N_c)_{2k} + 2N_f \Phi \Leftrightarrow USp\left(2\left(\frac{k}{2} - \frac{1}{2}(N_c + 1) + \frac{N_f}{2}\right)\right)_{-2\left(\frac{k}{2} + \frac{3N_c}{4} - \frac{N_f}{4} + \frac{1}{4}\right)} + 2N_f \tilde{\Phi}. \quad (2.5.3)$$

The dynamics of the USp theory goes as follows: for negative adjoint fermion mass and positive flavour mass the IR TQFT is $USp(2N_c)_{2k-(N_c+1)+N_f}$. In the magnetic theory, for opposite sign masses with respect to the electric theory we obtain $USp(2k - (N_c + 1) + N_f)_{-2N_c}$ TQFT. These theories agree due to level-rank duality⁸

$$USp(2N_c)_{2k-(N_c+1)+N_f} \Leftrightarrow USp(2k - (N_c + 1) + N_f)_{-2N_c}, \quad (2.5.4)$$

⁷For the SO case, N_c can be an integer or a half-integer

⁸Level-rank duality for USp CS theories is [92]

$$USp(2N)_{2K} \Leftrightarrow USp(2K)_{-2N}$$

In addition, for small quark mass the magnetic squark is expected to condense resulting in a $USp(2\tilde{N}_c - 2n)$ SQCD theory and flavour symmetry breaking

$$USp(2N_f) \rightarrow USp(2N_f - 2n) \times USp(2n), \quad (2.5.5)$$

described by the Grassmannian $USp(2N_f)/USp(2N_f - 2n) \times USp(2n)$, together with a WZ term, $2N_c \int_{\mathcal{M}} \text{tr} \frac{1}{2\pi} F \wedge F$, with F in the adjoint of $USp(2N_f - 2n)$. When we add SUSY breaking masses the selected vacuum is $n = \tilde{N}_c$.

In the non-SUSY case, $\tilde{N}_c < 0$, again we have anti-D3-branes in the magnetic configuration. In this case the theory on the branes is still $USp(2N_c)$ but the fermion transforms in the two-index antisymmetric representation, therefore when we integrate out the adjoint fermion the shift of the CS level is $\text{sign}(m_{\tilde{\lambda}})(\tilde{N}_c - 1)$ instead of $\text{sign}(m_{\tilde{\lambda}})(\tilde{N}_c + 1)$. With this consideration the magnetic dual is

$$USp(-2k + (N_c + 1) - N_f)_{2\tilde{k}} + 2N_f \tilde{\Phi}, \quad 2\tilde{k} = k + \frac{3N_c}{2} - \frac{N_f}{2} + \frac{1}{2}, \quad (2.5.6)$$

where we integrated out the fermions for $m_{\tilde{\lambda}} > 0$ and $m_{\tilde{\psi}'} > 0$. Now we study the low energy dynamics of the magnetic theory. Integrating both the adjoint fermion and the flavour for positive mass we reach the TQFT

$$USp(-2k + (N_c + 1) - N_f)_{2N_c}, \quad (2.5.7)$$

which corresponds to the large negative mass phase of the electric theory. When both fields have negative mass in the magnetic theory we obtain

$$USp(-2k + (N_c + 1) - N_f)_{2k+N_c-1+N_f} \Leftrightarrow USp(2k + N_c + 1 - N_f)_{2k-N_c+N_f-1} \quad (2.5.8)$$

which covers the intermediate phase of the electric theory. Similar to the $U(N_c)$ we can define the magnetic' theory

$$USp(2k + N_c + 1 - N_f)_{2\tilde{k}} + 2N_f \tilde{\Phi}, \quad 2\tilde{k} = k - \frac{3N_c}{2} + \frac{N_f}{2} - \frac{1}{2}, \quad (2.5.9)$$

which covers the intermediate and the large positive mass phases of the electric theory.

2.5.2 $SO(2N_c)$ Theory

The study of dualities for $SO(2N_c)$ is almost identical to the $USp(2N_c)$. We start by setting $k' = k + \frac{1}{2}(N_c - 1) - \frac{N_f}{2}$. On the electric side we have $\mathcal{N} = 1$ $SO(2N_c)_{2k}$ with $2N_f$ flavours. On the magnetic side, swapping the fivebranes a brane is created. After integrating out the adjoint fermion and one of the chiral flavours, we obtain the $\mathcal{N} = 1$ theory $SO(2\tilde{N}_c)_{-2\tilde{k}}$, with

$$\tilde{N}_c = k' - N_c + N_f + 1 = k - \frac{1}{2}(N_c - 1) + \frac{N_f}{2}, \quad 2\tilde{k} = k + \frac{3N_c}{2} - \frac{N_f}{2} - \frac{1}{2}. \quad (2.5.10)$$

As before, the theory is SUSY if $\tilde{N}_c > 0$. We propose the following $\mathcal{N} = 1$ Seiberg duality

$$SO(2N_c)_{2k} + 2N_f \Phi \Leftrightarrow SO\left(2\left(-k - \frac{3N_c}{2} + \frac{N_f}{2} + \frac{1}{2}\right)\right)_{-2\left(\frac{k}{2} - \frac{3N_c}{4} + \frac{N_f}{4} + \frac{1}{4}\right)} + 2N_f \tilde{\Phi} \quad (2.5.11)$$

In the IR, the TQFTs coincide when the adjoint fermion has negative mass and the flavour a positive one

$$SO(2N_c)_{2k-(N_c-1)+N_f} \Leftrightarrow SO(2k - (N_c - 1) + N_f)_{-2N_c} \quad (2.5.12)$$

As before, we expect squark condensation for small quark. We obtain a $SO(2N_c - 2n)$ gauge theory flavour symmetry breaking pattern

$$SO(2N_f) \rightarrow SO(2N_f - 2n) \times SO(2n), \quad (2.5.13)$$

described by the Grassmannian $SO(2N_f)/SO(2N_f - 2n) \times SO(2n)$, together with a WZ term $2N_c \int_{\mathcal{M}} \text{tr} \frac{1}{2\pi} F \wedge F$, with F in the adjoint of $SO(2N_f - 2n)$. When SUSY breaking masses are added the selected vacuum is $n = \tilde{N}_c$.

In the non-SUSY case, $\tilde{N}_c < 0$, the magnetic theory is

$$SO(-2k + (N_c + 1) - N_f)_{2\tilde{k}} + 2N_f \tilde{\Phi}, \quad 2\tilde{k} = k + \frac{3N_c}{2} - \frac{N_f}{2} - \frac{1}{2}, \quad (2.5.14)$$

where the adjoint fermion now transforms in the two-index symmetric representation because the theory lives on an anti-brane. Similarly the magnetic' theory is

$$SO(2k + N_c - 1 - N_f)_{2\tilde{k}} + 2N_f \tilde{\Phi}, \quad 2\tilde{k} = k - \frac{3N_c}{2} + \frac{N_f}{2} + \frac{1}{2}. \quad (2.5.15)$$

In the IR, these theories reproduce the three phases of the electric theory.

2.6 Summary

In this chapter, we discussed the dynamics of non-Abelian $\mathcal{N} = 1$ SQCD Chern-Simons theories. We used string theory to propose magnetic duals and using the magnetic description we were able to explore the IR dynamics of the electric theories. The dualities are valid not only at the supersymmetric point, but also in its vicinity, where supersymmetry is softly broken.

The analysis revealed known as well as new phenomena, using string dynamics. In particular for $\tilde{N}_c \geq 0$ we found, apart from the semi-classical large quark mass phase, a quantum phase with symmetry breaking.

An interesting novelty is the identification of a WZ term in the magnetic description of the gauge theory. We argued that when a D3-brane is suspended between a D5-brane and a tilted fivebrane there is a Chern-Simons term localised on the fivebrane. While the presence of this term was argued in [88], it was overlooked in previous analysis in both field theory and string theory [86, 87, 93].

Our dualities include the regime $\tilde{N}_c < 0$, where Seiberg duality was assumed to be invalid. Nevertheless, we showed that the IR theory of the electric and the magnetic theories is the same.

Chapter 3

Comments on non-Abelian T-duals and Their Holographic Description

The main objective in this chapter is to provide a geometric and field theoretical interpretation of the field theory dual to a Type IIB background obtained after performing a non-Abelian T-duality (NATD) transformation to a background with a well defined holographic dual. We focus our attention in the NATD of the reduction of $\text{AdS}_4 \times S^7$ to Type IIA Supergravity, which was obtained in [42], where it was shown to have a brane-configuration given by an unbounded Hanany-Witten set-up, which suggests that the dual field theory is not well defined.

We start this Chapter by reviewing aspects of Abelian T-Duality (ATD) and NATD and how they affect or not the dynamics of the dual theory, which is done in Section 3.1. Then, in Section 3.2, we show how to obtain the ATD and NATD of the background discussed above within the "electrostatic description" of the infinite family of AdS_4 backgrounds, which are dual to 3D $\mathcal{N} = 4$ linear quiver SCFTs [46]. In Section 3.3, using the formalism of [41, 50], we argue that there is a consistent way of completing the field theory into a linear quiver, such that the non-Abelian T-dual can be understood as a zoom-in of a more general background with a well defined holographic dual. The ATD interpretation is also discussed.

3.1 Brief Review of ATD and NATD

Let us start by briefly reviewing T-Duality at the level of bosonic String Theory on $\mathbb{R}^{8,1} \times S^1_R$, with R the radius of the S^1 direction. For a more detailed discussion see for example [94]. T-duality is a symmetry of the String Theory spectrum that, for closed strings, exchanges momentum and winding along the S^1 , while for open strings, it ex-

changes Dirichlet with Neumann boundary conditions, while also changing the radius of the S^1 as $R \rightarrow R' = \alpha'/R$. Therefore, T-duality is a duality between a String Theory formulated on $\mathbb{R}^{8,1} \times S_R^1$, with a different one formulated on $\mathbb{R}^{8,1} \times S_{R'}^1$. Note that the $U(1)$ isometry along the circle S_R^1 is "exchanged" by another $U(1)$ isometry along the $S_{R'}^1$ direction on the dual theory. Since T-duality related two different $U(1)$ isometries, from now on we refer to this duality as Abelian T-duality. When considering Super String Theory, ATD also acts on the R-R-spectrum, and the duality relates Type IIA with Type IIB Super String Theory.

This String Theory duality is realised at the level of the effective 10D Type II Supergravity via the Buscher rules [95, 96]. Consider a Type II Supergravity background that has a $U(1)$ isometry, parametrised by y . Here ∂_y is not only an isometry of the metric, but also a symmetry of the whole solution, that is, the Lie derivative of all the fields along ∂_φ vanishes. Such background configurations can be written as (here we split the 10D spacetime coordinates as $x^\mu = (x^m, y)$, $m = 0, \dots, 8$)

$$ds_{10D}^2 = ds_{9D}^2 + e^{2C}(dy + A_1)^2, \quad B = B_2 + B_1 \wedge dy, \quad F = F_\perp + F_\parallel \wedge E_y, \quad (3.1.1)$$

where $E_y = e^C(dy + A_1)$, F is a polyform comprising all the possible RR-forms that the background could contain, $F_\perp$ are R-R-forms that do not have components along E_y , while $F_\parallel \wedge E_y$ is an explicit way of writing the forms that do have components along E_y . Also, the forms A_1 , B_1 and B_2 and the function C are valued in ds_{9D}^2 . In order to obtain the T-dual solution, we follow the Buscher rules, which we present in the form presented in [97]

$$\begin{aligned} d\tilde{s}_{9D}^2 &= ds_{9D}^2, \quad \tilde{\Phi} = \Phi - C, \quad \tilde{C} = -C \\ \tilde{B}_2 &= B_2 + A_1 \wedge B_1, \quad \tilde{A}_1 = -B_1, \quad \tilde{B}_1 = -A_1 \\ \tilde{F}_\perp &= e^C F_\perp, \quad \tilde{F}_\parallel = e^C F_\parallel, \end{aligned} \quad (3.1.2)$$

here, the tilded quantities correspond to the new background fields. Note that after ATD, the resulting R-R-forms are one degree higher or lower than the ones present in the seed configuration. Thus, had we started in Type IIA (B), the resulting configuration is a solution of Type IIB (A). Also, as was in the case of ATD in String Theory, the $U(1)$ isometry is preserved after dualising.

On the other hand, non-Abelian T-duality transformation is also a duality between Type II Supergravities, where instead of a $U(1)$ isometry, the transformation is performed along a non-abelian isometry of the background. These transformations, firstly developed for pure NS-NS fields in [98] and then extended for backgrounds containing R-R fields that have a $SU(2)$ isometry subgroup in [99] (the extension beyond $SU(2)$ was done in [100] and an explicit flux transformation was given in [97]), do not preserve the symmetries of the compact manifold along which one performs the transformations. Take for example 10D flat spacetime. Focusing on a $\mathbb{R}^3$ section of it, which we write in spherical coordinates

$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$. Performing a NATD transformation along the $SO(3)$ isometry of the S^2 , we obtain [98]

$$d\tilde{s}^2 = dr^2 + \frac{1}{r^2} \frac{1}{x^2 - y^2} (x^2 dx^2 + r^4 dy^2), \quad \tilde{\Phi} = -\log(r^2(x^2 - y^2)). \quad (3.1.3)$$

When considering the Abelian T-duality transformation of a Supergravity solution with a certain holographically dual field theory, since it is a symmetry of Supergravity, both geometries describe the same dynamics in different ways. Therefore both geometries are holographically dual to the same field theory.

The previous statement is no longer valid if one considers non-Abelian T-duality (NATD) transformations. Since the isometries of the compact manifold along which the transformation is performed are broken the NATD geometry describes different dynamics with respect to its seed. In this manner NATD, at the level of Supergravity¹, is not a symmetry, but rather a solution generating method, see e.g. [102, 103].

The effect that NATD has on the dual field theory was first studied in [104, 105], where it was shown that it is indeed possible to describe properties of the NATD geometry in terms of those of its predecessor, e.g. observables charged under the dualised isometries changed after the transformation while non-charged (neutral) observables and the holographic central charge remain the same. However, no precise proposal for a dual field theory to NATD geometries was given until the work of [50] in $\text{AdS}_5 \times S^5$, where the dual theory came in the form of a ‘long quiver’ CFT preserving $\mathcal{N} = 2$ supersymmetry.

Now we proceed with the study of the corresponding dual field theory of a background obtained after NATD transformations. In particular, we focus on the NATD of the geometry studied in [42]. This background was obtained by starting from $\text{AdS}_4 \times S^7$ in 11D Supergravity

$$ds_{11D}^2 = ds^2(\text{AdS}_4) + 4\ell^2 \left(\frac{1}{4} d\mu^2 + \sin^2\left(\frac{\mu}{2}\right) ds^2(S_1^3) + \cos^2\left(\frac{\mu}{2}\right) ds^2(S_2^3) \right), \quad (3.1.4)$$

$$G_4 = \frac{3}{L} \text{Vol}(\text{AdS}_4)$$

then taking the $\mathbb{Z}_k$ orbifold of S_2^3 , that is replacing $S_2^3 \rightarrow S_2^3/\mathbb{Z}_k$. The metric on the orbifold reads

$$ds^2(S_2^3/\mathbb{Z}_k) = \frac{1}{4} (d\theta_2^2 + \sin^2(\theta_2)d\phi_2) + \frac{1}{k^2} \left(d\psi_2 + \frac{k}{2} \cos(\theta_2)d\phi_2 \right). \quad (3.1.5)$$

Choosing $k = 1$ recovers the Hopf fibration parametrisation of the 3-sphere. Then reducing

¹It is an open question whether NATD is a symmetry at the level of String Theory, however there is evidence that the broken symmetries of the background can be realised in a nonlocal way, see e.g. [101].

from 11D to Type IIA in 10D leads to the following metric

$$ds_{10D}^2 = e^{\frac{2}{3}\phi_0} \cos\left(\frac{\mu}{2}\right) \left[ds^2(\text{AdS}_4) + 4L^2 \left(\frac{1}{4} d\mu^2 + \sin^2\left(\frac{\mu}{2}\right) ds^2(S_1^3) + \frac{1}{4} \cos^2\left(\frac{\mu}{2}\right) ds^2(S_2^2) \right) \right], \quad (3.1.6)$$

which is supported by the NSNS and RR fields

$$F_4 = \frac{3}{L} \text{Vol}(\text{AdS}_4), \quad F_2 = -\frac{k}{2} \text{Vol}(S_2^2), \quad e^{2\Phi_0} = e^{2\phi_0} \cos^3\left(\frac{\mu}{2}\right), \quad e^{\frac{2}{3}\phi_0} = \frac{2L}{k}. \quad (3.1.7)$$

Note that this reduction also generates a Dilaton and a F_2 flux. The magnetic F_2 and F_4 fluxes signal the presence of D2 and D4-branes respectively, as was pointed out in [42]. The geometries studied in this chapter consist of the Abelian T-dual along the $U(1)$ fiber of S_1^3 and the NATD along the $SU(2)$ isometry of S_1^3 . These geometries are given below (up to a change of variables) in (3.2.10) and equations (3.2.20) and (3.2.22) respectively.

We start by reviewing the electrostatic formalism for 3D $\mathcal{N} = 4$ SCFTs, and how to obtain the desired geometries within that framework.

3.2 AdS₄ Type IIB Background

In [39], an infinite family of solutions to Type IIB Supergravity that are dual to $\mathcal{N} = 4$ super-conformal field theories were proposed. It was then shown in [46] that the same family of solutions can be obtained in the ‘electrostatic’ formalism. In order to match the conformality and global symmetries of the field theory, the Supergravity background geometry must contain an AdS₄ factor and a couple of two spheres S_1^2 and S_2^2 . The other NS and R background fields must preserve the isometries of the geometry. The Type IIB AdS₄ $\times$ $S^2 \times S^2$ has the following background

$$\begin{aligned} ds_{10,st}^2 &= f_1(\sigma, \eta) \left[ds^2(\text{AdS}_4) + f_2(\sigma, \eta) ds^2(S_1^2) + f_3(\sigma, \eta) ds^2(S_2^2) + f_4(\sigma, \eta) (d\sigma^2 + d\eta^2) \right], \\ e^{-2\Phi} &= f_5(\sigma, \eta), \quad B_2 = f_6(\sigma, \eta) \text{Vol}(S_1^2), \quad C_2 = f_7(\sigma, \eta) \text{Vol}(S_2^2), \quad \tilde{C}_4 = f_8(\sigma, \eta) \text{Vol}(\text{AdS}_4), \\ f_1 &= \frac{\pi}{2} \sqrt{\frac{\sigma^3 \partial_{\eta\sigma}^2 V}{\partial_{\sigma}(\sigma \partial_{\eta} V)}}, \quad f_2 = -\frac{\partial_{\eta} V \partial_{\sigma}(\sigma \partial_{\eta} V)}{\sigma \Lambda}, \quad f_3 = \frac{\partial_{\sigma}(\sigma \partial_{\eta} V)}{\sigma \partial_{\eta\sigma}^2 V}, \quad f_4 = -\frac{\partial_{\sigma}(\sigma \partial_{\eta} V)}{\sigma^2 \partial_{\eta} V}, \\ f_5 &= -16 \frac{\Lambda \partial_{\eta} V}{\partial_{\eta\sigma}^2 V}, \quad f_6 = \frac{\pi}{2} \left(\eta - \frac{\sigma \partial_{\eta} V \partial_{\eta}^2 V}{\Lambda} \right), \quad f_7 = -2\pi \left(\partial_{\sigma}(\sigma V) - \frac{\sigma \partial_{\eta} V \partial_{\eta}^2 V}{\partial_{\eta\sigma}^2 V} \right), \\ f_8 &= -\pi^2 \sigma^2 \left(3\partial_{\sigma} V + \frac{\sigma \partial_{\eta} V \partial_{\eta}^2 V}{\partial_{\sigma}(\sigma \partial_{\eta} V)} \right), \quad \Lambda = \partial_{\eta} V \partial_{\eta\sigma}^2 V + \sigma \left((\partial_{\eta\sigma}^2 V)^2 + (\partial_{\eta}^2 V)^2 \right), \end{aligned} \quad (3.2.1)$$

where σ and η are coordinates on a Riemann surface and

$$F_1 = 0, \quad H = dB_2, \quad F_3 = dC_2, \quad F_5 = d\tilde{C}_4 + *d\tilde{C}_4. \quad (3.2.2)$$

Notice that all functions $f_1, \dots, f_8$ can be determined in terms of a single function $V(\sigma, \eta)$, and partial derivatives. In order to be a solution of Type IIB Supergravity, V must satisfy

$$\partial_\sigma(\sigma^2 \partial_\sigma V(\sigma, \eta)) + \sigma^2 \partial_\eta^2 V(\sigma, \eta) = 0. \quad (3.2.3)$$

By using the redefinition

$$V(\sigma, \eta) = \frac{\partial_\eta W(\sigma, \eta)}{\sigma}, \quad (3.2.4)$$

equation (3.2.3) becomes the 2D Laplace equation

$$\partial_\sigma^2 W(\sigma, \eta) + \partial_\eta^2 W(\sigma, \eta) = 0. \quad (3.2.5)$$

Within this set-up, the boundary conditions ² that leads to a well defined dual CFT, imposed in [46] were analogous to an electrostatic problem in which there are two parallel conducting planes and a line of charge density given by $\mathcal{R}(\eta)$, which is usually called the ‘Rank function’. This Rank function, which must be piecewise linear in order to have quantized Page charges, completely characterises the brane configuration and therefore the field content of the dual quiver theory.

We are interested in solutions of the form (3.2.1) which do not necessarily satisfy the same boundary conditions, with the aim of finding solutions that would not be dual to the same quiver field theories. This solutions might have metric singularities or a finite volume internal space. Then, this will allow us to describe solutions that do not necessarily have a well defined holographic dual within the electrostatic formalism.

In the variables $z, \bar{z} = \sigma \pm i\eta$, any solution to the Laplace equation can be written as $W = f(z) + \bar{f}(\bar{z})$. For simplicity we will only consider the case in which $f(z)$ and $\bar{f}(\bar{z})$ are regular³, hence we can expand them in a Taylor series. After imposing a reality condition for W , namely $W = W^*$, and going back to the variables σ, η , the most general solution to equation (3.2.5) (ignoring boundary conditions) takes the form

$$W = \alpha_0 + \sum_{n=1}^{+\infty} \left[\alpha_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \sigma^{n-2k} \eta^{2k} + \beta_n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k \sigma^{n-2k-1} \eta^{2k+1} \right], \quad (3.2.6)$$

²These are explicitly given by

$$W(\sigma, \eta = 0) = 0, \quad W(\sigma, \eta = P) = 0, \quad \partial_\sigma W(\sigma = 0^+, \eta) - \partial_\sigma W(\sigma = 0^-, \eta) = -\mathcal{R}(\eta),$$

³We are also motivated to consider $f(z)$ to be regular because the solutions on [46] are all regular

from which we obtain the following general solution to equation (3.2.3)

$$V = \sum_{n=1}^{+\infty} \left[\alpha_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2k \binom{n}{2k} (-1)^k \sigma^{n-2k-1} \eta^{2k-1} \right. \\ \left. + \beta_n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2k+1) \binom{n}{2k+1} (-1)^k \sigma^{n-2k-2} \eta^{2k} \right]. \quad (3.2.7)$$

We will now show that different solutions appear by considering truncations of the Taylor series.

3.2.1 Abelian T-dual (ATD) Solution

We first consider the case in which all the coefficients α_n and β_n are set to zero, except for α_2 and α_3 . This then leads to the following potential

$$V_{\text{ATD}}(\sigma, \eta) = 2\alpha_2 \frac{\eta}{\sigma} - 6\alpha_3 \eta. \quad (3.2.8)$$

Let us now set $\alpha_2 = 6\alpha_3$, $\alpha_3 = M/12$ and use the change of coordinates

$$\sigma = 2 \cos^2 \left(\frac{\mu}{2} \right), \quad \eta = \frac{2}{\pi} r. \quad (3.2.9)$$

The range of the coordinates are $\mu \in [0, \pi]$ and we choose $r \in [0, \pi[$, where the points $r = 0$ and $r = \pi$ are identified. With this parametrization, the background metric reads

$$ds^2 = \pi \cos^2 \left(\frac{\mu}{2} \right) \left[ds^2(\text{AdS}_4) + \sin^2 \left(\frac{\mu}{2} \right) ds^2(S_1^2) \right. \\ \left. + \cos^2 \left(\frac{\mu}{2} \right) ds^2(S_2^2) + d\mu^2 + \frac{4}{\pi^2 \sin^2(\mu)} dr^2 \right], \quad (3.2.10)$$

and the rest of the background fields are given by

$$e^{-2\Phi} = 4M^2 \tan^2 \left(\frac{\mu}{2} \right), \\ B_2 = r \text{Vol}(S_1^2), \\ C_2 = 2Mr \text{Vol}(S_2^2), \\ \tilde{C}_4 = 6\pi Mr \text{Vol}(\text{AdS}_4). \quad (3.2.11)$$

The field strengths for the gauge forms are

$$H_3 = dr \wedge \text{Vol}(S_1^2), \\ F_3 = 2Mdr \wedge \text{Vol}(S_2^2), \\ F_5 = 6\pi Mdr \wedge \text{Vol}(\text{AdS}_4) - \frac{3}{4}\pi^2 M \sin^3(\mu) d\mu \wedge \text{Vol}(S_1^2) \wedge \text{Vol}(S_2^2). \quad (3.2.12)$$

By observation, one can easily see the presence of singularities at $\mu = \pi$ (from the D6 branes in Type IIA) and at $\mu = 0$ (from the NS5 branes), as discussed in [42], and hence the system is defined in the interval $\mu \in [0, \pi]$ (corresponding to $\sigma \in [0, 1]$). The dilaton diverges to positive infinity as $\mu \rightarrow 0$ and to negative infinity as $\mu \rightarrow \pi$. Hence, the above Supergravity description is not well defined in these limits.

This solution corresponds to the Abelian T-dual of the Type IIA solution obtained from a dimensional reduction of 11D Supergravity [42].

Page Charges

We now calculate the Page Charges, using the following definition

$$Q_{D_p/NS5} = \frac{1}{(2\pi)^{7-p}\alpha'} \int_{\Sigma_{8-p}} \hat{F}_{8-p}, \quad (3.2.13)$$

with $\alpha' = 1$, $2\kappa_{10}^2 T_{D_p} = (2\pi)^{7-p}$ and $\hat{F}_p = F_p \wedge e^{-B_2}$. The background of equation (3.2.1) has the following Page fluxes

$$\hat{F}_3 = F_3, \quad \hat{F}_5 = F_5 - B_2 \wedge F_3. \quad (3.2.14)$$

For the NS5 charge, we integrate in the submanifold $\mathcal{M}_1 = (r, S_1^2)$ for any value of μ

$$N_{NS5} = \frac{1}{(2\pi)^2} \int_{\mathcal{M}_1} H_3 = 1. \quad (3.2.15)$$

For the D5 charge, the integration is performed in the three-cycle $\mathcal{M}_2 = (r, S_2^2)$ for any value of μ

$$N_{D5} = \frac{1}{(2\pi)^2} \int_{\mathcal{M}_2} F_3 = 2M. \quad (3.2.16)$$

Finally, for the D3 charge, we integrate in the cycle $\mathcal{M}_3 = (\mu, S_1^2, S_2^2)$ for any value of r

$$N_{D3} = -\frac{1}{(2\pi)^4} \int_{\mathcal{M}_3} \hat{F}_5 = M, \quad (3.2.17)$$

where the negative charge suggests that the orientation of the cycle should be swapped in this case.

The full brane configuration for this geometry is described in figure 3.1(a), in which M D3-branes are stretched between NS5-branes located at $r = 0$ and $r = \pi$, with $2M$ D5-branes in each interval. The two NS5-branes represent the same brane, as the points $r = 0$ and $r = \pi$ are identified. The corresponding balanced linear quiver diagram is then given in figure 3.1(b).

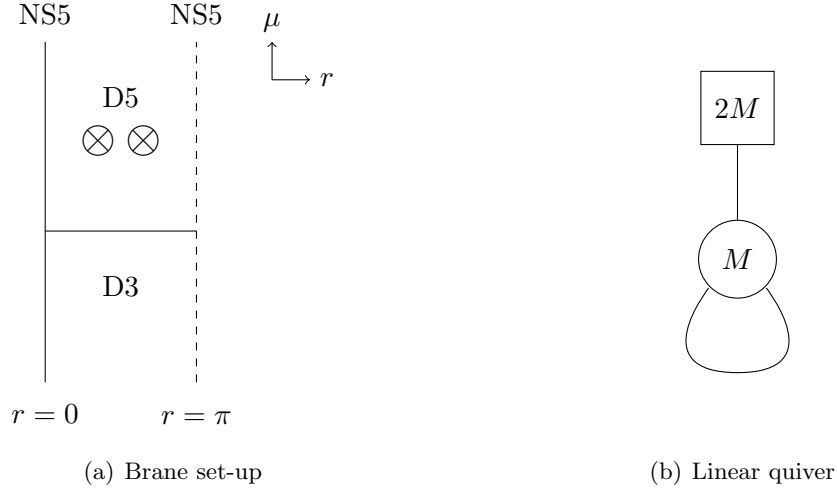


Figure 3.1: The Hanany-Witten (NS5, D3, D5) brane set-up for the ATD solution in (a), along with the corresponding balanced linear quiver diagram, (b). The brane set-up consists of M D3-branes stretched between NS5-branes located at $r = 0$ and $r = \pi$, with $2M$ D5-branes in each interval (taking $M = 1$). Since the points $r = 0$ and $r = \pi$ are identified, the two NS5-branes in the figure correspond to the same brane (represented using a dashed line).

3.2.2 Non-Abelian T-dual (NATD) Solution

Another well known solution can be obtained by considering the case in which only β_3 and β_4 are non-vanishing, namely

$$V_{\text{NATD}} = \beta_3 \left(3 \frac{\eta^2}{\sigma} - 3\sigma \right) + \beta_4 (4\sigma^2 - 12\eta^2). \quad (3.2.18)$$

After fixing $\beta_3 = 8\beta_4$ and $\beta_4 = M/96$, and using the change of coordinates

$$\sigma = 2 \cos^2 \left(\frac{\mu}{2} \right), \quad \eta = \frac{2}{\pi} r, \quad (3.2.19)$$

we obtain the following metric

$$ds^2 = \pi \cos \left(\frac{\mu}{2} \right) \left[ds^2(\text{AdS}_4) + \frac{4}{\pi^2} \frac{r^2}{\Delta} \sin^2 \left(\frac{\mu}{2} \right) ds^2(S_1^2) \right. \\ \left. + \cos^2 \left(\frac{\mu}{2} \right) ds^2(S_2^2) + d\mu^2 + \frac{4}{\pi^2 \sin^2(\mu)} dr^2 \right], \quad (3.2.20)$$

where $\Delta = \Delta(\mu, r)$ is given by

$$\Delta(\mu, r) = \sin^2 \left(\frac{\mu}{2} \right) \sin^2(\mu) + \frac{4}{\pi^2} r^2, \quad (3.2.21)$$

with the rest of the background fields given by

$$\begin{aligned}
e^{-2\Phi} &= M^2 \tan^2 \left(\frac{\mu}{2} \right) \Delta, \\
B_2 &= \frac{4}{\pi^2} \frac{r^3}{\Delta} \text{Vol}(S_1^2), \\
C_2 &= \frac{\pi}{4} M \left(\frac{4}{\pi^2} r^2 - \cos(\mu) (\cos^2(\mu) - 3) \right) \text{Vol}(S_2^2), \\
\tilde{C}_4 &= \frac{\pi^2}{8} M \left(\frac{24}{\pi^2} r^2 - (\cos(2\mu) - 4 \cos(\mu)) \right) \text{Vol}(\text{AdS}_4).
\end{aligned} \tag{3.2.22}$$

The gauge fields lead to the following field strengths

$$\begin{aligned}
H_3 &= \frac{r^2}{2\pi^2 \Delta^2} \left[8 \left(\frac{4}{\pi^2} r^2 + 3 \sin^2 \left(\frac{\mu}{2} \right) \sin^2(\mu) \right) dr \right. \\
&\quad \left. - r \left(\sin(\mu) + 4 \sin(2\mu) - 3 \sin(3\mu) \right) d\mu \right] \wedge \text{Vol}(S_1^2), \\
F_3 &= \frac{\pi}{2} M \left[\frac{4}{\pi^2} r dr - \frac{3}{2} \sin^3(\mu) d\mu \right] \wedge \text{Vol}(S_2^2),
\end{aligned} \tag{3.2.23}$$

together with the self-dual 5-form

$$\begin{aligned}
F_5 &= M \frac{\pi^2}{8} d \left(\frac{24}{\pi^2} r^2 - (\cos(2\mu) - 4 \cos(\mu)) \right) \wedge \text{Vol}(\text{AdS}_4) \\
&\quad - M \frac{r^2 \sin^2(\mu)}{\pi \Delta} \left(3r \sin(\mu) d\mu + 2 \sin^2 \left(\frac{\mu}{2} \right) dr \right) \wedge \text{Vol}(S_1^2) \wedge \text{Vol}(S_2^2).
\end{aligned} \tag{3.2.24}$$

This solution reproduces the non-Abelian T-dual of the Type IIA solution, in [42]. As in the ATD, we have $\mu \in [0, \pi]$ due to the presence of singularities. Once again, the supergravity description is no valid when $\mu \rightarrow 0$ and $\mu \rightarrow \pi$, as the dilaton diverges in these limits.

Page Charges

Now we compute the Page charges using equations (3.2.13). Note that unlike the ATD, in this case, the region of r is non-compact. However, as will be explained below, the charges below are calculated over the interval $r \in [n\pi, (n+1)\pi]$, with $n \in \mathbb{N}$.

Before computing the charges, it is convenient to perform the following large gauge transformation

$$B_2 \rightarrow B_2 - n\pi \text{Vol}(S_1^2). \tag{3.2.25}$$

When integrating the charges, it is convenient to choose $r \in [n\pi, (n+1)\pi]$ since this is the *minimal building block* of the brane set-up, i.e. is the smallest size interval in which we get integer charges for all the gauge forms.

For the NS5 charge, we integrate over the submanifold $\Sigma_1 = (r, S_1^2)$ for $\mu = 0$ and $\mu = \pi$

$$N_{NS5} = \frac{1}{(2\pi)^2} \int_{\Sigma_1|_{\mu=0,\pi}} H_3 = 1. \quad (3.2.26)$$

This suggests that the location of the NS5-branes correspond to the location of the singularities of the background. For the D5 charge, we integrate over the submanifold $\Sigma_2 = (r, S_2^2)$ for any value of μ

$$N_{D5} = \frac{1}{(2\pi)^2} \int_{\Sigma_2} F_3 = (2n+1)M. \quad (3.2.27)$$

Finally, for the D3 charge, we integrate over the cycle $\Sigma_3 = (\mu, S_1^2, S_2^2)$ for any value of r

$$N_{D3} = -\frac{1}{(2\pi)^4} \int_{\Sigma_3} \hat{F}_5 = Mn. \quad (3.2.28)$$

If instead we had considered the interval $r \in [n\pi, (n+2)\pi]$, the NS5 and D5 charges would have counted the branes between $[n\pi, (n+1)\pi]$ and $[(n+1)\pi, (n+2)\pi]$, thus not giving the smallest possible building block.

The full brane configuration of the entire geometry (for the case $M = 1$) is described in figure 3.2, in which Mn D3-branes are stretched between two NS5-branes located at $n\pi$ and $(n+1)\pi$, with $(2n+1)M$ D5-branes in each interval. The corresponding (overbalanced) linear quiver diagram is then given in figure 3.3.

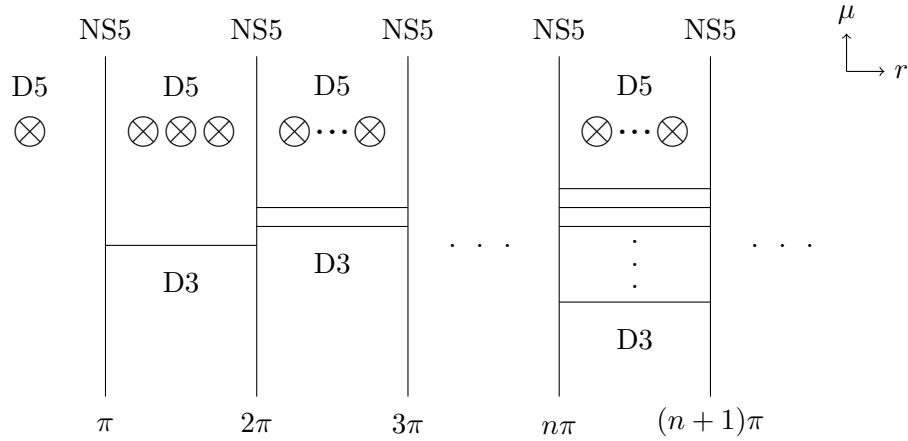


Figure 3.2: The Hanany-Witten (NS5, D3, D5) brane set-up for the NATD solution (taking $M = 1$), in which Mn D3-branes are stretched between two NS5 branes located at $n\pi$ and $(n+1)\pi$, with $(2n+1)M$ D5-branes in each interval.

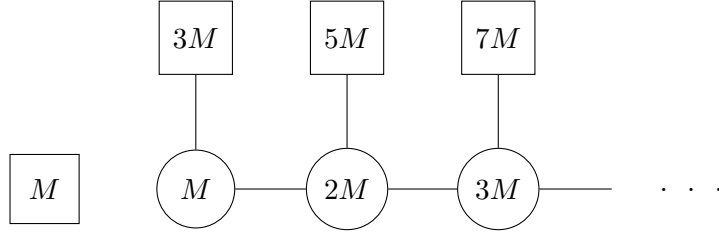


Figure 3.3: The corresponding overbalanced linear quiver diagram for the NATD geometry.

3.3 On the Completion of Geometries

3.3.1 NATD as a zoom-in

In [61], by using the electrostatic problem formalism, evidence was presented on how the non-Abelian T-dual of AdS_6 in massive Type IIA backgrounds could be thought of as a particular zoom-in of a more general solution, which possesses a well-defined holographic dual. Here, we give a similar argument for the NATD of AdS_4 in Type IIA Supergravity.

In [46], the infinite family of backgrounds dual to balanced linear quivers is derived from the following potential (we consider $\sigma > 0$ for simplicity)

$$V_{\text{LQ}}(\sigma, \eta) = \sum_{k=0}^{+\infty} \frac{a_k}{\sigma} \cos\left(\frac{k\pi\eta}{P}\right) e^{-\frac{k\pi\sigma}{P}}, \quad (3.3.1)$$

with a_k the coefficients of the Fourier expansion of the respective Rank function (which contains the ranks of colour and flavour groups). Expanding this potential in a Taylor series gives

$$\begin{aligned} V_{\text{LQ}}(\sigma, \eta) = & \frac{1}{\sigma} \sum_{k=0}^{+\infty} a_k - \frac{\pi}{P} \sum_{k=0}^{+\infty} k a_k + \frac{\pi^2}{2P^2} \left(-\frac{\eta^2}{\sigma} + \sigma\right) \sum_{k=0}^{+\infty} k^2 a_k, \\ & + \frac{\pi^3}{6P^3} (3\eta^2 - \sigma^2) \sum_{k=0}^{+\infty} k^3 a_k + \dots \end{aligned} \quad (3.3.2)$$

By defining

$$\beta_1 = \sum_{k=0}^{+\infty} a_k, \quad \beta_2 = -\frac{\pi}{P} \sum_{k=0}^{+\infty} k a_k, \quad \beta_3 = -\frac{1}{3} \frac{\pi^2}{2P^2} \sum_{k=0}^{+\infty} k^2 a_k, \quad \beta_4 = -\frac{1}{4} \frac{\pi^3}{6P^3} \sum_{k=0}^{+\infty} k^3 a_k, \quad (3.3.3)$$

one can check that the truncated expansion completely contains the potential for the NATD (β_1 does not appear on any background fields nor in the Rank function, and β_2 appears as a gauge transformation of C_2). As for the AdS_6 case, this suggest that the AdS_4 NATD corresponds to a zoom-in of a background for which we can completely characterise the

holographic dual. This means that even though the NATD holographic dual is not properly defined, it can be completed into a 3d $\mathcal{N} = 4$ linear quiver.

3.3.2 Comments on the ATD

The ATD background can be found via a similar zoom-in in the family of orthogonal functions to (3.3.1), namely

$$V_{\perp}(\sigma, \eta) = \sum_{k=0}^{+\infty} \frac{a_k}{\sigma} \sin\left(\frac{k\pi\eta}{P}\right) e^{-\frac{k\pi\sigma}{P}}. \quad (3.3.4)$$

Notice that, although this is a solution to the equation of motion, it does not satisfy the boundary conditions of the electrostatic problem. This means that it is not possible to use the same arguments to suggest that the ATD be completed into a more general geometry that has a well defined holographic dual. This is consistent with the fact that the ATD already has a precise holographic dual: the same one as its seed background.

3.4 Summary

In this chapter we used the electrostatic problem formalism developed in [46] to obtain two $\text{AdS}_4 \times S^2 \times S^2$ backgrounds in Type IIB Supergravity, derived from the polynomial solutions of the 2D Laplace equation defining the geometry, which do not satisfy the usual electrostatic problem boundary conditions. These solutions correspond to the Abelian and non-Abelian T-duals of the Type IIA background obtained from the dimensional reduction of $\text{AdS}_4 \times S^7$. We therefore re-derive the NATD solution using a different approach to [42], characterising the solutions by specific terms in the polynomial expansion of the potential. Hence, the dual Field Theory should be more easily determined via this approach. By computing the Page charges for each background we described the Hanany-Witten brane set-up, from where we recovered the previously known result that the NATD brane set-up is unbounded, making the dual field theory description obscure.

We then showed how the potential leading to the NATD solution emerges as a truncation of the full solution of the electrostatic problem, which leads to linear quiver field theories as holographic duals. This agrees with the picture that NATD solutions can be completed into a background that has a well-defined holographic dual. The ATD backgrounds are self-consistent on their own, i.e. since they describe the same dynamics as their seed, since they are dual to the same field theory.

Chapter 4

Compactification of 6D $\mathcal{N} = (1, 0)$ Quivers, 4D SCFTs and their Holographic dual Massive IIA Backgrounds

This chapter is dedicated to the study, via holography, of twisted compactifications of 6D non-Lagrangian CFTs down to 4D. This procedure consists on formulating the 6D theory on $\mathbb{R}^{3,1} \times \Sigma_2$, with Σ_2 a Riemann surface, and introducing a background gauge field for the R-symmetry, which allows to preserve some amount of SUSY. By doing so, an RG-flow across dimensions is triggered, and in the case we are interested in, the 6D theory flows to 4D IR fixed point.

These flows across dimensions have already appeared in the literature, where the flow between a SCFT_{D+1} and a SCFT_{d+1} (also with the possibility of ending in gapped systems) have been studied using their holographic duals, see for example [74, 91, 106–111], for early works with twisted compactifications from the holographic point of view. The topic progressed considerably after the paper [112]. This was followed by many works studying compactifications (twisted or with fluxes) from a purely QFT point of view. In the particular case of compactifications of 6D to 4D systems (preserving minimal SUSY in both dimensions), we find the works [113–120]. For a very nice summary of these developments from a field theoretical perspective, see [121].

We focus on the twisted compactification on a Riemann surface of an infinity family of 6D $\mathcal{N} = (1, 0)$ long linear quiver SCFTs, whose IR fixed point is an 4D $\mathcal{N} = 1$ SCFT. These flow is holographically described by a massive Type IIA background which interpolates between AdS_7 in the UV and AdS_5 in the IR. We start by constructing the interpolating

geometry at the level of a 7D gauged Supergravity, which is done in Section 4.1. In Section 4.2 we lift the 7D solution to massive Type IIA, and study its Page fluxes and charges. Then, Section 4.3 presents a proposal for the dual field theory, based on the brane configuration studied in the previous Section. We propose two quantities that measure the number of degrees of freedom along the flow.

4.1 Obtaining the Interpolating Geometry

We begin by considering 7D $SU(2)$ Topologically Massive gauged Supergravity. We propose an ansatz and derive the BPS equations from the SUSY variations. Finally, we outline the uplift of the 7D background to 10D Massive Type IIA, following the procedure used in [122].

4.1.1 7D $SU(2)$ Topologically Massive Gauged Supergravity

The Lagrangian, in string frame, of the bosonic fields of the 7D $SU(2)$ Topologically Massive Supergravity is given by,

$$\mathcal{L} = \sqrt{-g}e^{-2\Phi} \left[R - \frac{1}{8}F_{mn}^i F^{mni} + 4\partial^m \Phi \partial_m \Phi - \left(\frac{m^2}{2}e^{-4\Phi} - 4m e^{-2\Phi} - 4 \right) \right] - \frac{1}{2}e^{2\Phi} * G_4 \wedge G_4 + \frac{1}{4}F^i \wedge F^i \wedge B_3 - \frac{m}{2}G_4 \wedge B_3, \quad (4.1.1)$$

where $G_4 = dB_3$, $F = dA + iA \wedge A$, and A is a $su(2)$ valued gauge 1-form, i.e. $A = A^i T^i$. The SUSY variations of the fermionic fields of the theory are

$$\delta\lambda = \left[\Gamma^m \partial_m \Phi + \frac{i}{8}\Gamma^{mn} F_{mn}^i \sigma^i + \frac{1}{48}e^{2\Phi}\Gamma^{mnlr} G_{mnlr} - m e^{-2\Phi} + 1 \right] \epsilon, \quad (4.1.2)$$

$$\delta\Psi_m = \left[D_m - \frac{i}{2}A_m^i \sigma^i + \frac{i}{4}\Gamma^n F_{mn}^i \sigma^i + \frac{1}{96}e^{2\Phi}\Gamma_m{}^{nlrs} G_{nlrs} - \frac{m}{4}e^{-2\Phi}\Gamma_m \right] \epsilon, \quad (4.1.3)$$

where the covariant derivative is given by

$$D_m \epsilon = \left(\partial_m + \frac{1}{4}\omega_m{}^{ab}\Gamma_{ab} \right) \epsilon. \quad (4.1.4)$$

Here m, n are spacetime indices while a, b are tangent space ones, both ranging from 0, 1, ..., 6. Notice that we explicitly wrote the $su(2)$ generators in terms of the Pauli matrices $T^i = \frac{\sigma^i}{2}$, and also redefined $A \rightarrow -A$ and $F \rightarrow -F$ when compared with [122], since we want to derive the BPS equations using a method similar to the one showed in [123], where the author studied the same Supergravity theory but for vanishing topological mass ($m = 0$).

Background Ansatz

We are interested in compactifying the dual theory on a regular 2D Riemann surface. These are classified by their genus g . The sphere and the torus corresponds to the $g = 0$ and $g = 1$ surfaces respectively. The $g \geq 2$ surfaces correspond to identifications of the hyperbolic space H^2 . Therefore, locally, the metric on the Riemann surface is described by the metric on S^2 , $\mathbb{R}^2$ or H^2 . To describe holographically the compactification of the 6D theory on a Riemann surface, we consider the background metric to be a warped version of $\mathbb{R}^{3,1} \times \Sigma_k \times \mathbb{R}$

$$ds^2 = e^{2f(r)} dx_{3,1}^2 + e^{2h(r)} \left(d\theta^2 + \frac{1}{k} \sin^2(\sqrt{k}\theta) d\phi^2 \right) + dr^2, \quad (4.1.5)$$

where k is proportional to the curvature of Σ_k . Taking $k = -1, 0, 1$, gives $\Sigma_k = H^2, \mathbb{R}^2$ or S^2 , respectively. Because we are interested studying topological twisted compactifications we will only consider the cases $k = \pm 1$. The rest of the background fields are $B_3 = 0$, $\Phi = \Phi(r)$ and

$$A^1 = -k a(r) d\theta, \quad A^2 = \frac{1}{\sqrt{k}} a(r) \sin(\sqrt{k}\theta) d\phi, \quad A^3 = -\frac{1}{k} \cos(\sqrt{k}\theta) d\phi, \quad (4.1.6)$$

which leads to the field strength

$$\begin{aligned} F^1 &= -k \dot{a}(r) dr \wedge d\theta, \quad F^2 = \frac{1}{\sqrt{k}} \dot{a}(r) \sin(\sqrt{k}\theta) dr \wedge d\phi, \\ F^3 &= (1 - k a^2(r)) \frac{\sin(\sqrt{k}\theta)}{\sqrt{k}} d\theta \wedge d\phi. \end{aligned} \quad (4.1.7)$$

where $(\dot{}) = d/dr$. We choose to use the following vielbein basis

$$e^{\hat{m}} = e^{f(r)} dx^m, \quad e^{\hat{\theta}} = e^{h(r)} d\theta, \quad e^{\hat{\phi}} = \frac{1}{\sqrt{k}} e^{h(r)} \sin(\theta) d\phi, \quad e^{\hat{r}} = dr, \quad (4.1.8)$$

which lead to the spin connection

$$\begin{aligned} \omega^{\hat{m}\hat{r}} &= \dot{f}(r) e^{f(r)} dx^m, \quad \omega^{\hat{\theta}\hat{\phi}} = -\cos(\sqrt{k}\theta) d\phi, \\ \omega^{\hat{\theta}\hat{r}} &= \dot{h}(r) e^{h(r)} d\theta, \quad \omega^{\hat{\phi}\hat{r}} = \frac{1}{\sqrt{k}} \dot{h}(r) e^{h(r)} \sin(\sqrt{k}\theta) d\phi. \end{aligned} \quad (4.1.9)$$

In the vielbein basis, the gauge field, its field strength and the spin connection read

$$A^1 = -k a(r) e^{-h(r)} e^{\hat{\theta}}, \quad A^2 = a(r) e^{-h(r)} e^{\hat{\phi}}, \quad A^3 = \frac{1}{\sqrt{k}} e^{-h(r)} \cot(\sqrt{k}\theta) e^{\hat{\phi}}, \quad (4.1.10)$$

$$F^1 = k \dot{a}(r) e^{-h(r)} e^{\hat{\theta}} \wedge e^{\hat{r}}, \quad F^2 = -\dot{a}(r) e^{-h(r)} e^{\hat{\phi}} \wedge e^{\hat{r}}, \quad F^3 = (1 - k a^2(r)) e^{-2h(r)} e^{\hat{\theta}} \wedge e^{\hat{\phi}}, \quad (4.1.11)$$

$$\omega^{\hat{m}\hat{r}} = \dot{f}(r) e^{\hat{m}}, \quad \omega^{\hat{\theta}\hat{\phi}} = -\sqrt{k} e^{-h(r)} \cot(\sqrt{k}\theta) e^{\hat{\phi}}, \quad \omega^{\hat{\theta}\hat{r}} = \dot{h}(r) e^{\hat{\theta}}, \quad \omega^{\hat{\phi}\hat{r}} = \dot{h}(r) e^{\hat{\phi}}. \quad (4.1.12)$$

Note that the A^3 component of the $su(2)$ gauge field is proportional to the $\omega^{\hat{\theta}\hat{\phi}}$ component of the spin connection. This would play an important role below.

4.1.2 From SUSY Variations to BPS Equations

Substituting the above ansatz into the expressions for the SUSY variations, derives BPS equations as outlined below. We begin with the Dilatino variation and move onto the Gravitino variations.

Dilatino Variation

Substituting the ansatz into the Dilatino variation leads to

$$\delta\lambda = \left[\Gamma_{\hat{r}} \dot{\Phi}(r) + \frac{i}{4} \left(k \dot{a}(r) e^{-h(r)} \Gamma_{\hat{\theta}\hat{r}} \sigma^1 - \dot{a}(r) e^{-h(r)} \Gamma_{\hat{\phi}\hat{r}} \sigma^2 + \left(1 - k a^2(r) \right) e^{-2h(r)} \Gamma_{\hat{\theta}\hat{\phi}} \sigma^3 \right) - m e^{-2\Phi(r)} + 1 \right] \epsilon. \quad (4.1.13)$$

We now impose the projection

$$\Gamma_{\hat{\theta}\hat{\phi}} \epsilon = \frac{1}{k} \sigma^1 \sigma^2 \epsilon. \quad (4.1.14)$$

Then, since the Pauli matrices satisfy $\sigma^1 \sigma^2 \sigma^3 = i$, multiplying by $\Gamma_{\hat{r}}$, we obtain

$$\begin{aligned} \Gamma_{\hat{r}} \delta\lambda = & \left[\dot{\Phi}(r) - \frac{i}{4} \dot{a}(r) e^{-h(r)} \left(k \Gamma_{\hat{\theta}} \sigma^1 - \Gamma_{\hat{\phi}} \sigma^2 \right) \right. \\ & \left. + \left(1 - m e^{-2\Phi(r)} - \frac{1}{4k} \left(1 - k a^2(r) \right) e^{-2h(r)} \right) \Gamma_{\hat{r}} \right] \epsilon. \end{aligned} \quad (4.1.15)$$

Using the projection (4.1.14) and the fact that $k = \pm 1$ one can show that

$$\left(k \Gamma_{\hat{\theta}} \sigma^1 - \Gamma_{\hat{\phi}} \sigma^2 \right) \epsilon = 2k \Gamma_{\hat{\theta}} \sigma^1 \epsilon. \quad (4.1.16)$$

Replacing this in the Dilatino variation yields

$$\begin{aligned} \Gamma_{\hat{r}} \delta\lambda = & \left[\dot{\Phi}(r) - \frac{i}{2} k \dot{a}(r) e^{-h(r)} \Gamma_{\hat{\theta}} \sigma^1 \right. \\ & \left. + \left(1 - m e^{-2\Phi(r)} - \frac{1}{4k} \left(1 - k a^2(r) \right) e^{-2h(r)} \right) \Gamma_{\hat{r}} \right] \epsilon. \end{aligned} \quad (4.1.17)$$

Gravitino Variation

We now switch our focus to the Gravitino variations, focusing on the $\hat{m}$, $\hat{\theta}$, $\hat{\phi}$ and $\hat{r}$ components in turn. From the $\hat{m} = 0, 1, 2, 3$ components of the variation of the gravitino in tangent space indices, we have

$$\delta\psi_{\hat{m}} = \left(\frac{1}{2} \hat{f}(r) \Gamma_{\hat{m}\hat{r}} - \frac{m}{4} e^{-2\Phi(r)} \Gamma_{\hat{m}} \right) \epsilon, \quad (4.1.18)$$

and multiplying by $2\Gamma_{\hat{r}\hat{m}}$, gives (here there is no sum in $\hat{m}$)

$$2\Gamma_{\hat{r}\hat{m}}\delta\psi_{\hat{m}} = \left(f'(r) - \frac{m}{2}e^{-2\Phi(r)}\Gamma_{\hat{r}}\right)\epsilon. \quad (4.1.19)$$

From the $\hat{\theta}$ component, we have

$$\begin{aligned} \delta\psi_{\hat{\theta}} = & \left[\frac{1}{2}\dot{h}(r)\Gamma_{\hat{\theta}\hat{r}} + \frac{i}{2}k a(r) e^{-h(r)}\sigma^1 \right. \\ & \left. + \frac{i}{4} \left(k \dot{a}(r) e^{-h(r)}\Gamma_{\hat{r}}\sigma^1 + \left(1 - k a^2(r)\right) e^{-2h(r)}\Gamma_{\hat{\phi}}\sigma^3 \right) - \frac{m}{4}e^{-2\Phi(r)}\Gamma_{\hat{\theta}} \right] \epsilon. \end{aligned} \quad (4.1.20)$$

Multiplying by $2\Gamma_{\hat{r}\hat{\theta}}$ and using the projection (4.1.14) leads to

$$\begin{aligned} 2\Gamma_{\hat{r}\hat{\theta}}\delta\psi_{\hat{\theta}} = & \left[\dot{h}(r) + i k a(r) e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1 \right. \\ & \left. - \frac{i}{2}k \dot{a}(r) e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1 - \frac{1}{2k} \left(1 - k a^2(r)\right) e^{-2h(r)}\Gamma_{\hat{r}} - \frac{m}{2}e^{-2\Phi(r)}\Gamma_{\hat{r}} \right] \epsilon. \end{aligned} \quad (4.1.21)$$

From the $\hat{\phi}$ component we have

$$\begin{aligned} \delta\psi_{\hat{\phi}} = & \left[\frac{1}{2} \left(\dot{h}(r)\Gamma_{\hat{\phi}\hat{r}} - \sqrt{k}e^{-h(r)}\cot(\sqrt{k}\theta)\Gamma_{\hat{\theta}\hat{\phi}} \right) - \frac{i}{2} \left(a(r) e^{-h(r)}\sigma^2 + \frac{1}{\sqrt{k}}e^{-h(r)}\cot(\sqrt{k}\theta)\sigma^3 \right) \right. \\ & \left. + \frac{i}{4} \left(-\dot{a}(r)e^{-h(r)}\Gamma_{\hat{r}}\sigma^2 - \left(1 - k a^2(r)\right) e^{-2h(r)}\Gamma_{\hat{\theta}}\sigma^3 \right) - \frac{m}{4}e^{-2\Phi(r)}\Gamma_{\hat{\phi}} \right] \epsilon. \end{aligned} \quad (4.1.22)$$

Notice that, using (4.1.14), the second term, coming from the spin connection, and the fourth term, coming from the gauge field, cancel each other. This is the effect of the topological twist. We are left with

$$\begin{aligned} \delta\psi_{\hat{\phi}} = & \left[\frac{1}{2}\dot{h}(r)\Gamma_{\hat{\phi}\hat{r}} - \frac{i}{2}a(r) e^{-h(r)}\sigma^2 \right. \\ & \left. + \frac{i}{4} \left(-\dot{a}(r)e^{-h(r)}\Gamma_{\hat{r}}\sigma^2 - \left(1 - k a^2(r)\right) e^{-2h(r)}\Gamma_{\hat{\theta}}\sigma^3 \right) - \frac{m}{4}e^{-2\Phi(r)}\Gamma_{\hat{\phi}} \right] \epsilon. \end{aligned} \quad (4.1.23)$$

Multiplying by $2\Gamma_{\hat{r}\hat{\phi}}$, and using the projection (4.1.14) a second time, and due to (4.1.16), we have $k\Gamma_{\hat{\theta}}\sigma^1\epsilon = -\Gamma_{\hat{\phi}}\sigma^2\epsilon$, giving

$$\begin{aligned} 2\Gamma_{\hat{r}\hat{\phi}}\delta\psi_{\hat{\phi}} = & \left[\dot{h}(r) + i k a(r) e^{-h(r)}\Gamma_{\hat{r}\hat{\theta}}\sigma^1 - \frac{i}{2}k \dot{a}(r) e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1 \right. \\ & \left. - \frac{1}{2} \left(1 - k a^2(r)\right) e^{-2h(r)} - \frac{m}{2}e^{-2\Phi(r)}\Gamma_{\hat{r}} \right] \epsilon, \end{aligned} \quad (4.1.24)$$

from where we note that $\Gamma_{\hat{r}\hat{\theta}}\delta\psi_{\hat{\theta}} = \Gamma_{\hat{r}\hat{\phi}}\delta\psi_{\hat{\phi}}$.

Finally, repeating the process above for the $\hat{r}$ component of the gravitino variation, we have

$$\delta\psi_{\hat{r}} = \left[\partial_r - \frac{i}{2}k \dot{a}(r) e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1 - \frac{m}{4}e^{-2\Phi(r)}\Gamma_{\hat{r}} \right] \epsilon. \quad (4.1.25)$$

After imposing $\delta\lambda = 0$ and $\delta\psi_\mu = 0$, the fermionic variations (4.1.17), (4.1.19) and (4.1.21) lead to the following BPS equations

$$\dot{\Phi}(r)\epsilon - \frac{i}{2}k\dot{a}(r)e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1\epsilon + \left(1 - m e^{-2\Phi(r)} - \frac{1}{4k}\left(1 - k a^2(r)\right)e^{-2h(r)}\right)\Gamma_{\hat{r}}\epsilon = 0, \quad (4.1.26)$$

$$\dot{f}(r)\epsilon - \frac{m}{2}e^{-2\Phi(r)}\Gamma_{\hat{r}}\epsilon = 0, \quad (4.1.27)$$

$$\partial_r\epsilon = \frac{i}{2}k a'(r)e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1\epsilon + \frac{m}{4}e^{-2\Phi(r)}\Gamma_{\hat{r}}\epsilon, \quad (4.1.28)$$

together with

$$\begin{aligned} \dot{h}(r)\epsilon + i k a(r) e^{-h(r)}\Gamma_{\hat{r}\hat{\theta}}\sigma^1\epsilon - \frac{i}{2}k\dot{a}(r)e^{-h(r)}\Gamma_{\hat{\theta}}\sigma^1\epsilon \\ - \frac{1}{2k}\left(1 - k a^2(r)\right)e^{-2h(r)}\Gamma_{\hat{r}}\epsilon - \frac{m}{2}e^{-2\Phi(r)}\Gamma_{\hat{r}}\epsilon = 0, \end{aligned} \quad (4.1.29)$$

Rearranging the SUSY variations

Following the procedure given in [123], we now rearrange the SUSY variations to obtain the BPS equations. For this, we use the properties of the Γ matrices to simplify the equations obtained in the previous subsection. Let us start by rewriting the dilatino variation (4.1.17) as

$$\Gamma_{\hat{r}}\epsilon = \beta\epsilon + \tilde{\beta}i\Gamma_{\hat{\theta}}\sigma^1\epsilon, \quad (4.1.30)$$

with

$$\beta = \frac{-\dot{\Phi}(r)}{1 - m e^{-2\Phi(r)} - \frac{1}{4k}(1 - k a^2(r))e^{-2h(r)}}, \quad (4.1.31)$$

$$\tilde{\beta} = \frac{\frac{1}{2}k\dot{a}(r)e^{-h(r)}}{1 - m e^{-2\Phi(r)} - \frac{1}{4k}(1 - k a^2(r))e^{-2h(r)}}. \quad (4.1.32)$$

Since $(\Gamma_{\hat{r}})^2 = 1$, we can obtain a constraint between β and $\tilde{\beta}$ by squaring (4.1.30). This leads to

$$\epsilon = (\beta^2 + \tilde{\beta}^2)\epsilon \quad \Rightarrow \quad \beta^2 + \tilde{\beta}^2 = 1. \quad (4.1.33)$$

This also applies to (4.1.27), from where, after multiplying by $\Gamma_{\hat{r}}$, we obtain a first order differential equation for $f(r)$

$$f'(r) = \pm \frac{m}{2} e^{-2\Phi(r)}. \quad (4.1.34)$$

Now we go back to (4.1.27). Multiplying by $\Gamma_{\hat{r}}$ and using (4.1.30) and collecting terms with and without $i\Gamma_{\hat{\theta}}\sigma^1$, we get

$$\left(f'(r)\beta - \frac{m}{2}e^{-2\Phi(r)}\right)\epsilon + f'(r)\tilde{\beta}i\Gamma_{\hat{\theta}}\sigma^1 = 0, \quad (4.1.35)$$

from where we get two equations

$$\dot{f}(r)\beta = \frac{m}{2} e^{-2\Phi(r)}, \quad (4.1.36)$$

$$\dot{f}(r)\tilde{\beta} = 0. \quad (4.1.37)$$

From (4.1.34) and (4.1.36) we get $\beta = \pm 1$, and from (4.1.37) we get $\tilde{\beta} = 0$, which in (4.1.32) gives

$$\dot{a}(r) = 0, \quad (4.1.38)$$

and hence, (4.1.30) becomes

$$\Gamma_{\hat{r}}\epsilon = \beta\epsilon. \quad (4.1.39)$$

Now we turn to (4.1.29). By replacing (4.1.39) and $\dot{a}(r) = 0$, we get

$$\dot{h}(r)\epsilon - i\beta k a(r) e^{-h(r)} \Gamma_{\hat{\theta}} \sigma^1 \epsilon - \beta \frac{1}{2k} \left(1 - k a^2(r)\right) e^{-2h(r)} \epsilon - \beta \frac{m}{2} e^{-2\Phi(r)} \epsilon = 0. \quad (4.1.40)$$

Again, by collecting terms with and without $i\Gamma_{\hat{\theta}}\sigma^1$, we get

$$\left(\dot{h}(r) - \beta \frac{1}{2k} \left(1 - k a^2(r)\right) e^{-2h(r)} - \beta \frac{m}{2} e^{-2\Phi(r)} \right) \epsilon - i\beta k a(r) e^{-h(r)} \Gamma_{\hat{\theta}} \sigma^1 \epsilon = 0, \quad (4.1.41)$$

which leads to the equations

$$\dot{h}(r) - \beta \frac{1}{2k} \left(1 - k a^2(r)\right) e^{-2h(r)} - \beta \frac{m}{2} e^{-2\Phi(r)} = 0, \quad (4.1.42)$$

$$a(r) e^{-h(r)} = 0. \quad (4.1.43)$$

From the second one, we necessarily get $a(r) = 0$. After implementing this condition into (4.1.31) and (4.1.42), we finally derive the following BPS equations for $f(r)$ (which we got from (4.1.34)), $h(r)$ and $\Phi(r)$

$$\dot{f}(r) = \pm \frac{m}{2} e^{-2\Phi(r)}, \quad (4.1.44)$$

$$\dot{h}(r) = \pm \frac{1}{2} \left(\frac{1}{k} e^{-2h(r)} + m e^{-2\Phi(r)} \right), \quad (4.1.45)$$

$$\dot{\Phi}(r) = \pm \left(-1 + \frac{1}{4k} e^{-2h(r)} + m e^{-2\Phi(r)} \right). \quad (4.1.46)$$

From now on, we keep the $+$ branch, since the minus sign can be absorbed by $r \rightarrow -r$.

4.1.3 Numerical Solution of the BPS system

Here we give a detailed derivation of the numerical solutions that describe the flow from AdS_7 to $\text{AdS}_5 \times \Sigma_k$. Since we are looking to holographically describe the flow between a 6D SCFT and the 4D SCFT fixed point, the dual background should interpolate between

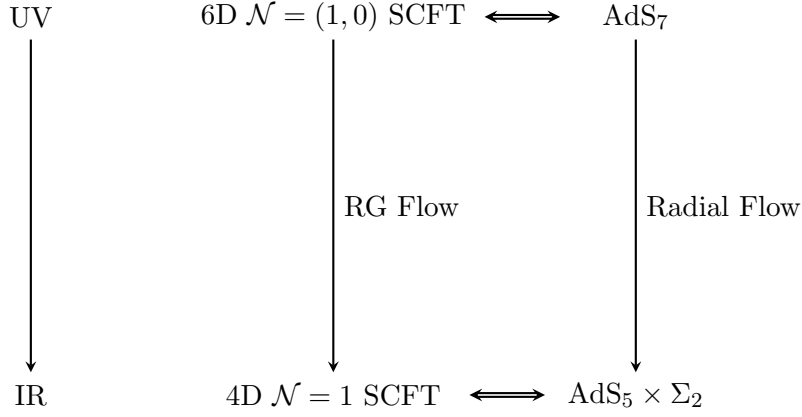


Figure 4.1: Prescription for the flow between the infinite family of 6D $\mathcal{N} = (1, 0)$ SCFTs and the 4D $\mathcal{N} = 1$ SCFT fixed point, and its holographic realisation.

AdS₇ for large values of the holographic coordinates (UV) and AdS₅ × Σ_k in the IR. This picture is described in Figure 4.1

The starting point is to find the IR fixed point, and then consider a linear perturbation around the it.

Infrared Fixed Point

We are looking for solutions to the BPS equations (4.1.44), (4.1.45) and (4.1.46) for $f(r)$, $h(r)$ and $\Phi(r)$ that have constant $h(r)$ and $\Phi(r)$, so that the geometry nicely factorises into AdS₅ × Σ_k . This will correspond to the IR fixed point, and we note that it exists only for $k = -1$. The solutions

$$f(r) = \frac{2}{3}r, \quad h(r) = \frac{1}{2} \ln \left(\frac{3}{4} \right), \quad \Phi(r) = \frac{1}{2} \ln \left(\frac{3m}{4} \right), \quad (4.1.47)$$

are exact solutions to the BPS equations, and the spacetime obtained from this corresponds to AdS₅ × H^2 .

Linear Perturbations

Here we consider linear deviations from the IR fixed point

$$f(r) = \frac{2}{3}r + \epsilon \bar{f}(r), \quad (4.1.48)$$

$$h(r) = \frac{1}{2} \ln \left(\frac{3}{4} \right) + \epsilon \bar{g}(r), \quad (4.1.49)$$

$$\Phi(r) = \frac{1}{2} \ln \left(\frac{3m}{4} \right) + \epsilon \bar{\Phi}(r). \quad (4.1.50)$$

Replacing in the BPS equations and keeping only first order terms in ϵ , we obtain linear equations for the linear perturbations

$$\dot{\bar{f}} = -\frac{4}{3}\bar{\Phi}, \quad \dot{\bar{h}} = \frac{4}{3}(\bar{h} - \bar{\Phi}), \quad \dot{\bar{\phi}} = \frac{1}{3}(2\bar{h} - 8\bar{\Phi}). \quad (4.1.51)$$

This systems has as solution

$$\bar{f}(r) = \frac{C_1}{3}(\sqrt{7} + 2)e^{-\frac{2}{3}(\sqrt{7}+1)r} - \frac{C_2}{3}(\sqrt{7} - 2)e^{\frac{2}{3}(\sqrt{7}-1)r}, \quad (4.1.52)$$

$$\bar{h}(r) = C_1 e^{-\frac{2}{3}(\sqrt{7}+1)r} + C_2 e^{\frac{2}{3}(\sqrt{7}-1)r}, \quad (4.1.53)$$

$$\bar{\Phi}(r) = \frac{C_1}{2}(\sqrt{7} + 3)e^{-\frac{2}{3}(\sqrt{7}+1)r} - \frac{C_2}{2}(\sqrt{7} - 3)e^{\frac{2}{3}(\sqrt{7}-1)r}. \quad (4.1.54)$$

We are interested in solutions to vanish for $r \rightarrow -\infty$, which is the location of the fixed point, hence we set $C_1 = 0$ and also without loss of generality we can set $C_2 = 1$.

Finally, to obtain the numerical solutions we use the perturbations around the fixed point (4.1.48)-(4.1.50) as initial conditions at $r = -8$. Figure 4.2 shows these numerical solutions.

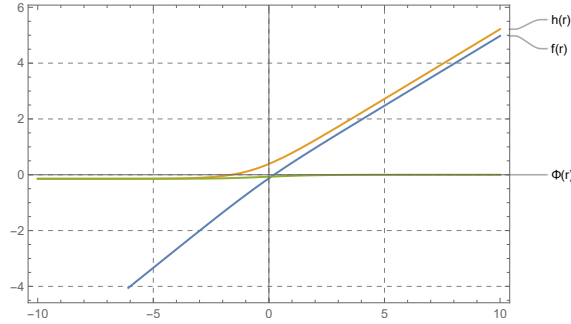


Figure 4.2: The plot for the case of the H_2 compactification ($k = -1$) of the functions $f(r), h(r), \Phi(r)$. The UV asymptotics at $r \rightarrow +\infty$ and the IR at $r \rightarrow -\infty$. We clearly see the attainment of both IR and UV fixed points.

4.1.4 Uplift to 10D Massive Type IIA

Einstein Frame and Normalizations

The procedure which we will follow to uplift our 7D topologically massive solution is given in [122], however, there the action we used for the 7D $\mathcal{N} = 2$ $SU(2)$ Gauged SUGRA is written in string frame, while in [122] is written in Einstein frame, hence we move our action to Einstein frame and then we compare normalizations and relevant constants. To move to Einstein frame we use the transformation

$$g_{\mu\nu}^{(E)} = e^{-\alpha\Phi(r)} g_{\mu\nu}^{(S)}, \quad (4.1.55)$$

in the action (4.1.1). Then, by comparing with [122] we see that $\alpha = \frac{4}{5}$, $m = 1$, $g = \sqrt{2}$, $h = \frac{1}{2}$ and that our Dilaton $\Phi(r)$ is related to the [122] one, which we call $\varphi(r)$, by

$$\varphi(r) = \frac{2\sqrt{10}}{5}\Phi(r). \quad (4.1.56)$$

Also, from the kinetic term for the gauge field, we see that in [122] the gauge field is normalised in a way in which the coupling constant appears in the definition of F rather than a coefficient of the kinetic term F^2 , while in (4.1.1) we consider the opposite. To match conventions we re-scale our gauge field and field strength as

$$A \rightarrow -\frac{1}{g}A, \quad F \rightarrow -\frac{1}{g}F. \quad (4.1.57)$$

With this, our 7D ansatz now reads

$$ds^2 = e^{-\frac{4}{5}\Phi(r)} \left(e^{2f(r)} ds_{3,1}^2 + e^{2h(r)} \left(d\theta^2 + \frac{1}{k} \sin^2(\sqrt{k}\theta) d\phi^2 \right) + dr^2 \right), \quad (4.1.58)$$

together with the gauge field

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = \frac{1}{g} \frac{1}{k} \cos(\sqrt{k}\theta) d\phi, \quad (4.1.59)$$

which leads to

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = -\frac{1}{g} \text{Vol}(\Sigma_k^2). \quad (4.1.60)$$

Uplift of 7D Topologically Massive Gauged SUGRA

The massive Type IIA ansatz is given by

$$ds^2 = \frac{1}{\ell} X(r)^{-\frac{1}{2}} e^{2A(\rho)} ds_7^2 + X(r)^{\frac{5}{2}} ds_{M_3}^2, \quad (4.1.61)$$

with $\ell = 4$, ds_7^2 a solution of the 7D $\mathcal{N} = 2$ Gauged Supergravity presented in the previous section, and

$$ds_{M_3}^2 = d\rho^2 + \frac{1 - x(\rho)^2}{16\omega} e^{2A(\rho)} Ds_{S^2}^2, \quad (4.1.62)$$

where $X(r)$ is given by the 7D dilaton

$$X(r) = e^{\frac{2}{5}\Phi(r)}, \quad (4.1.63)$$

and

$$\omega = X(r)^5 (1 - x(\rho)^2) + x(\rho)^2. \quad (4.1.64)$$

The covariantized metric $Ds_{S^2}^2$ on the sphere is constructed as follows. First we consider the normal unitary vector to the sphere

$$y^i = (\cos(\phi_2) \sin(\theta_2), \sin(\phi_2) \sin(\theta_2), \cos(\theta_2)), \quad (4.1.65)$$

then, the covariant line element is given by

$$Ds_{S^2}^2 = Dy^i Dy^i, \quad Dy^i = dy^i + g \epsilon^{ijk} y^k A^j, \quad (4.1.66)$$

where A^i is the component along the i th Pauli matrix of the 7D gauge field. This spacetime is supported by the 10D Dilaton Ψ

$$e^{2\Psi} = \frac{X(r)^{\frac{5}{2}}}{\omega} e^{2\psi(\rho)}, \quad (4.1.67)$$

and the background forms

$$B_2 = \left[\frac{1}{16} e^{2A(\rho)} \frac{x(\rho) \sqrt{1-x(\rho)^2}}{\omega} \text{Vol}_2 - \frac{1}{2} e^{A(\rho)} d\rho \wedge \left(a - \frac{1}{g} y^i A^i \right) \right], \quad (4.1.68)$$

$$F_2 = -q (\text{Vol}_2 + g y^i F^i) + \frac{1}{16\omega} F_0 e^{2A(\rho)} x(\rho) \sqrt{1-x(\rho)^2} \text{Vol}_2, \quad (4.1.69)$$

$$F_4 = \frac{\ell}{4} \left[-g \frac{q}{16\omega} e^{2A(\rho)} x(\rho) \sqrt{1-x(\rho)^2} y^i F^i \wedge \text{Vol}_2 - g \frac{q}{4} e^{A(\rho)} d\rho \wedge \epsilon^{ijk} F^i y^j Dy^k \right] + G_4 \text{ terms}, \quad (4.1.70)$$

where Vol_2 is the volume element of the covariantised sphere, $a = \frac{1}{2} \cos(\theta_2) d\phi_2$ and F_0 is the Ramond mass (which is constant). Note that we have not written explicitly the terms proportional to the 7D 4-form, since we will not use them.

This configuration is a solution to the equations of motion

$$\frac{1}{4} R + \nabla^2 \Psi - (\nabla \Psi)^2 - \frac{1}{8} H_3^2 = 0, \quad (4.1.71)$$

$$dF_p + H_3 \wedge *F_{p-2} = 0, \quad (4.1.72)$$

$$d(e^{-2\Psi} * H_3) - (F_0 * F_2 + F_2 \wedge *F_4 + F_4 \wedge F_4) = 0, \quad (4.1.73)$$

$$R_{MN} + 2\nabla_M \nabla_N \Psi - \frac{1}{2} (H_3^2)_{MN} - \frac{1}{4} e^{2\Psi} \sum_p (F_p^2)_{MN}, \quad (4.1.74)$$

with $p = 2, 4, 6, 8, 10$, and

$$(F_p^2)_{MN} = \frac{1}{(p-1)!} F_M^{N_1 \dots N_{p-1}} F_{MN_1 \dots N_{p-1}}, \quad (4.1.75)$$

and similarly for $(H_3^2)_{MN}$, provided $\psi(\rho)$, $x(\rho)$ and $A(\rho)$ satisfy

$$\frac{d}{d\rho}\psi(\rho) = \frac{1}{4} \frac{e^{-A(\rho)}}{\sqrt{1-x(\rho)^2}} \left(12x(\rho) + (2x(\rho)^2 - 5) F_0 e^{A(\rho)+\psi(\rho)} \right), \quad (4.1.76)$$

$$\frac{d}{d\rho}x(\rho) = -\frac{1}{2} e^{-A(\rho)} \sqrt{1-x(\rho)^2} \left(4 + x(\rho) F_0 e^{A(\rho)+\psi(\rho)} \right), \quad (4.1.77)$$

$$\frac{d}{d\rho}A(\rho) = \frac{1}{4} \frac{e^{-A(\rho)}}{\sqrt{1-x(\rho)^2}} \left(4x(\rho) - F_0 e^{A(\rho)+\psi(\rho)} \right). \quad (4.1.78)$$

Finally, it can be shown (as in [124]) that the 10D BPS equations can be solved in terms of just one function, $\alpha(z)$, provided (A, x, ψ) are of the following form

$$A(\rho) = \frac{1}{2} \ln \left(8\pi\sqrt{2} \sqrt{-\frac{\alpha(z)}{\alpha''(z)}} \right), \quad (4.1.79)$$

$$\psi(\rho) = \frac{1}{4} \ln \left(e^{4\psi_0} \frac{\left(-\frac{\alpha(z)}{\alpha''(z)} \right)^3}{\left(\alpha'(z)^2 - 2\alpha(z)\alpha''(z) \right)^2} \right), \quad (4.1.80)$$

$$x(\rho) = \sqrt{1 + \frac{2\alpha(z)\alpha''(z)}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)}}, \quad (4.1.81)$$

where the coordinate z is related to ρ via the following change of coordinates

$$d\rho = \sqrt{\pi\sqrt{2}} \left(-\frac{\alpha''(z)}{\alpha(z)} \right)^{\frac{1}{4}} dz. \quad (4.1.82)$$

This is a solution of the 10D BPS equations provided $\alpha(z)$ satisfy

$$\alpha'''(z) = 2^{-\frac{1}{4}} \sqrt{\pi} e^{\psi_0} F_0. \quad (4.1.83)$$

In the next section, we present the metric and background fields in terms of this new variable.

We now move to the study of the explicit lift of the 7D $su(2)$ topologically massive gauged Supergravity.

4.2 Supergravity Backgrounds

We start this section by describing an infinite family of supergravity solutions. This is a family of Massive Type IIA backgrounds, preserving four supersymmetries ($\mathcal{N}=1$ in four dimensional notation). These are written in terms of coordinates, parameters and functions,

$$\text{Coordinates: } (t, x_1, x_2, x_3, r, \theta_1, \phi_1, z, \theta_2, \phi_2). \quad \text{Parameters: } (\Psi_0, k). \quad (4.2.1)$$

$$\text{Functions: } \alpha(z), f(r), h(r), X(r) = e^{\frac{2}{5}\Phi(r)}, \omega(r, z) = \left(\frac{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)X(r)^5}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)} \right).$$

The equations constraining these functions are written below. In terms of these coordinates and functions, the string-frame spacetime metric reads

$$\begin{aligned}
ds_{st}^2 = & 2\pi\sqrt{2}\sqrt{-\frac{\alpha(z)}{\alpha''(z)}}X(r)^{-\frac{1}{2}}e^{-\frac{4\Phi(r)}{5}}\left[e^{2f(r)}dx_{3,1}^2\right. \\
& \left.+dr^2 + e^{2h(r)}\left(d\theta_1^2 + \frac{1}{k}\sin^2(\sqrt{k}\theta_1)d\phi_1^2\right)\right] \\
& + X(r)^{5/2}\left[\pi\sqrt{2}\sqrt{-\frac{\alpha''(z)}{\alpha(z)}}dz^2\right. \\
& \left.+ \frac{\sqrt{2}\pi}{\omega(r,z)}\frac{\sqrt{-\alpha^3(z)\alpha''(z)}}{2\alpha(z)\alpha''(z) - \alpha'^2}\left(d\theta_2^2 + \sin^2(\theta_2)\left(d\phi_2 - \frac{1}{k}\cos(\sqrt{k}\theta_1)d\phi_1\right)^2\right)\right].
\end{aligned} \tag{4.2.2}$$

The Neveu-Schwarz (B_2, Ψ) and Ramond (F_0, F_2, F_4) background fields are,

$$\begin{aligned}
B_2 = & \left(\frac{\pi}{\omega(r,z)}\frac{\alpha(z)\alpha'(z)}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)}\sin(\theta_2)d\theta_2 - \pi\cos(\theta_2)dz\right) \wedge \left(d\phi_2 - \frac{1}{k}\cos(\sqrt{k}\theta_1)d\phi_1\right), \\
e^{4\Psi(r,z)} = & \frac{X^5(r)}{\omega^2(r,z)}\left(\frac{-\alpha(z)}{\alpha''(z)}\right)^3\left(\frac{e^{2\Psi_0}}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)}\right)^2, \\
F_0 = & 2^{\frac{1}{4}}\frac{e^{-\Psi_0}}{\sqrt{\pi}}\alpha'''(z), \\
F_2 = & 2^{\frac{1}{4}}\sqrt{\pi}e^{-\Psi_0}\alpha''(z)\left[\cos(\theta_2)\text{Vol}(\Sigma_k) - \text{Vol}(S_c^2)\right] + F_0\frac{\pi}{\omega(r,z)}\frac{\alpha(z)\alpha'(z)}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)}\text{Vol}(S_c^2), \\
F_4 = & \left(\frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}e^{-\Psi_0}}{\omega(r,z)}\right)\left(\frac{\alpha(z)\alpha'(z)\alpha''(z)}{\alpha'(z)^2 - 2\alpha(z)\alpha''(z)}\right)\cos(\theta_2)\text{Vol}(\Sigma_k) \wedge \text{Vol}(S^2) \\
& + 2^{\frac{1}{4}}\pi^{\frac{3}{2}}e^{-\Psi_0}\alpha''(z)\sin^2(\theta_2)dz \wedge d\phi_2 \wedge \text{Vol}(\Sigma_k).
\end{aligned} \tag{4.2.3}$$

We have defined the volume elements,

$$\begin{aligned}
\text{Vol}(S^2) = & \sin(\theta_2)d\theta_2 \wedge d\phi_2, \quad \text{Vol}(S_c^2) = \sin(\theta_2)d\theta_2 \wedge \left(d\phi_2 - \frac{1}{k}\cos(\sqrt{k}\theta_1)d\phi_1\right), \\
\text{Vol}(\Sigma_k) = & \frac{\sin(\sqrt{k}\theta_1)}{\sqrt{k}}d\theta_1 \wedge d\phi_1.
\end{aligned} \tag{4.2.4}$$

The functions $f(r), h(r), \Phi(r)$ are the ones obtained in the previous section. The BPS equation for $\alpha(z)$ is already contained in (4.2.3). In fact, the mass-parameter of massive Type IIA F_0 , that should be constant by pieces for an interpretation in terms of localised D8 branes dictates that $\alpha'''(z)$ must be piece-wise constant. It is in the many possible choices for a piece-wise constant F_0 that the family of backgrounds is originated. More on this in Section 4.2.1 below.

The configuration in (4.2.2)-(4.2.4), is a solution to the equations of motion of Massive

IIA. In string frame these read,

$$\begin{aligned} \frac{1}{4}R + \nabla^2\Psi - (\nabla\Psi)^2 - \frac{1}{8}H_3^2 &= 0, \\ dF_p + H_3 \wedge *F_{p-2} &= 0, \quad d(e^{-2\Psi} * H_3) - (F_0 * F_2 + F_2 \wedge *F_4 + F_4 \wedge F_4) = 0, \\ R_{MN} + 2\nabla_M \nabla_N \Psi - \frac{1}{2}(H_3^2)_{MN} - \frac{1}{4}e^{2\Psi} \sum_p (F_p^2)_{MN} &= 0. \end{aligned} \quad (4.2.5)$$

In eq. (4.2.5) $p = 2, 4, 6, 8, 10$, and

$$(F_p^2)_{MN} = \frac{1}{(p-1)!} F_M^{N_1 \dots N_{p-1}} F_{N N_1 \dots N_{p-1}}, \quad (H_3^2)_{MN} = \frac{1}{2} H_M^{N_1 N_2} H_{N N_1 N_2}. \quad (4.2.6)$$

Regarding the volume form $\text{Vol}(\Sigma_k)$ in (4.2.4), for the allowed values for the parameter k , namely $k = (1, -1)$, Σ_k is describing a two-sphere or a hyperbolic plane. The case $k = 0$, corresponding to a torus, is slightly more subtle and will be briefly addressed below.

This concludes the presentation of the backgrounds. To interpret these in terms of branes, we calculate the Page charges (quantised and gauge-variant) associated with this family of SUSY solutions.

4.2.1 Page fluxes and Charges

The Page fluxes $\hat{F}_p$, defined as a polyform $\hat{F} = e^{-B_2} \wedge F$ are quantised. This implies a certain form for some of the functions in the background, as we discuss below.

The Page fluxes are gauge-variant. They do change under a large gauge transformation of B_2 . We use this in our favour, performing a particular transformation that makes explicit the quantised charges present and the role of the function $\alpha(z)$. To write the Page fluxes in a concise fashion, it proves useful to define a one form Θ_1 and its exterior derivative,

$$\Theta_1 = -\cos(\theta_2) \left(d\phi_2 - \frac{1}{k} \cos(\sqrt{k}\theta_1) d\phi_1 \right), \quad d\Theta_1 = \text{Vol}(S_c^2) - \cos(\theta_2) \text{Vol}(\Sigma_k). \quad (4.2.7)$$

It is also convenient to change the B_2 -field by a large gauge transformation (this has no effect on $H_3 = dB_2$). Below, we explain the purpose of such transformation,

$$\begin{aligned} B_{2,new} &= B_{2,old} - \pi d \left[(z - \Delta) \Theta_1 \right], \quad \Delta \text{ is a constant,} \\ B_{2,new} &= \left(\frac{\pi}{\omega(r, z)} \frac{\alpha(z) \alpha'(z)}{\alpha'(z)^2 - 2\alpha(z) \alpha''(z)} \text{Vol}(S_c^2) - \pi(z - \Delta) d\Theta_1 \right), \\ H_3 &= \pi \left[-\frac{1}{\omega^2} \frac{\alpha(z) \alpha'(z)}{\alpha'(z)^2 - 2\alpha(z) \alpha''(z)} \frac{\partial \omega}{\partial X} X'(r) dr \right. \\ &\quad \left. + \frac{d}{dz} \left(\frac{1}{\omega} \frac{\alpha(z) \alpha'(z)}{\alpha'(z)^2 - 2\alpha(z) \alpha''(z)} \right) dz \right] \wedge \text{Vol}(S_c^2) \\ &\quad - \pi \left(\frac{1}{\omega} \frac{\alpha(z) \alpha'(z)}{\alpha'(z)^2 - 2\alpha(z) \alpha''(z)} \right) \sin(\theta_2) d\theta_2 \wedge \text{Vol}(\Sigma_k) \\ &\quad - \pi dz \wedge \text{Vol}(S_c^2) + \pi \cos(\theta_2) dz \wedge \text{Vol}(\Sigma_k) \end{aligned} \quad (4.2.8)$$

With this new- B_2 we compute the Page flux $\hat{F}_2$ and obtain,

$$\hat{F}_2 = F_2 - B_2 F_0 = -2^{\frac{1}{4}} \sqrt{\pi} e^{-\psi_0} (\alpha'' - (z - \Delta) \alpha''') d\Theta_1. \quad (4.2.9)$$

Similarly we calculate the $\hat{F}_4$ Page flux,

$$\begin{aligned} \hat{F}_4 &= F_4 - B_2 \wedge F_2 + \frac{1}{2} B_2 \wedge B_2 F_0, \\ \hat{F}_4 &= 2^{\frac{1}{4}} \pi^{\frac{3}{2}} e^{-\Psi_0} \alpha''(z) \sin^2 \theta_2 dz \wedge d\phi_2 \wedge \text{Vol}(\Sigma_k) \\ &\quad + 2^{1/4} \pi^{3/2} e^{-\Psi_0} (z - \Delta) (2\alpha''(z) - \alpha'''(z)(z - \Delta)) \cos \theta_2 \text{Vol}(S_c^2) \wedge \text{Vol}(\Sigma_k). \end{aligned} \quad (4.2.10)$$

Finally, for F_0 we have the same as in (4.2.3), $\hat{F}_0 = 2^{\frac{1}{4}} \frac{e^{-\Psi_0}}{\sqrt{\pi}} \alpha'''(z)$. We now calculate the charges associated with these Page fluxes, and impose their quantisation.

Page Charges

To calculate the charges, we need to compute the integrals (as in the rest of the paper we set $g_s = \alpha' = 1$),

$$Q_{NS5} = \frac{1}{4\pi^2} \int_{M_3} H_3, \quad Q_{D_p} = \frac{1}{(2\pi)^{7-p}} \int_{M_{8-p}} \hat{F}_{8-p}, \quad p = 4, 6, 8. \quad (4.2.11)$$

These integrals need to be defined over suitable cycles, some of which contain the sub-manifold Σ_k . Then, it is useful to first calculate the volume of the two-manifold Σ_k defined in eq.(4.2.4). As we stated above, in the cases $k = (1, 0, -1)$ the space described is a two-sphere, a torus or a hyperbolic plane. Its volume is calculated using Gauss-Bonnet's theorem and the fact that Σ_k has curvature $R = 2k$. For a genus g Riemann surface, we have

$$\int_{\Sigma_k} d^2x \sqrt{g} R = 8\pi(1 - g) \quad \longrightarrow \quad \int_{\Sigma_k} d^2x \sqrt{g} = \frac{4\pi}{k}(1 - g).$$

For the sphere, $k = 1$ and $g = 0$, while for the hyperbolic plane, $k = -1$ with $g > 1$. The case of the torus needs some care, as we have $k = 0, g = 1$ (and the volume is 4π). We will not discuss the case of T^2 in what follows (except for the purpose of making an intuitive argument). This allows us to write (for S^2, H_2)

$$\int_{\Sigma_k} d^2x \sqrt{g} = 4\pi|g - 1|. \quad (4.2.12)$$

After these preliminaries, we calculate the Page charges. Inspecting the H_3 -field in eq.(4.2.8) we find two possible three-cycles on which the integral can be performed. These three-cycles are

$$\mathcal{M}_1 = (\theta_1, \phi_1, z)|_{\theta_2=0}, \quad \text{and} \quad \mathcal{M}_2 = (\theta_2, \phi_2, z)|_{r \rightarrow +\infty}. \quad (4.2.13)$$

We find that there are two sets of NS-five branes. Their total numbers being,

$$\begin{aligned} N_{NS5}^{(1)} &= \frac{1}{4\pi^2} \int_{\mathcal{M}_1} H_3 = \frac{1}{4\pi} \text{Vol}(\Sigma_k) P = |g - 1| P, \\ N_{NS5}^{(2)} &= \frac{1}{4\pi^2} \int_{\mathcal{M}_2} H_3 = P. \end{aligned} \quad (4.2.14)$$

To calculate $N_{NS5}^{(2)}$ a boundary condition $\alpha(z=0) = \alpha(z=P) = 0$ has been imposed, more on this below. We have also taken the orientation of the manifolds such that all the charges are positive. Importantly, we have set the range of the z -coordinate to be $z \in [0, P]$, with P an integer.

For the charges of D6-branes, we have a pair of two-cycles on which to integrate the Page flux of eq.(4.2.9),

$$\mathcal{M}_3 = (\theta_1, \phi_1)|_{\theta_2=0}, \quad \text{and} \quad \mathcal{M}_4 = (\theta_2, \phi_2). \quad (4.2.15)$$

Performing the integrals, we find,

$$\begin{aligned} N_{D6}^{(1)} &= \frac{1}{2\pi} \int_{\mathcal{M}_3} \hat{F}_2 = 2^{\frac{5}{4}} \sqrt{\pi} e^{-\Psi_0} |g-1| (\alpha'' - (z-\Delta)\alpha'''), \\ N_{D6}^{(2)} &= \frac{1}{2\pi} \int_{\mathcal{M}_4} \hat{F}_2 = 2^{\frac{5}{4}} \sqrt{\pi} e^{-\Psi_0} (\alpha'' - (z-\Delta)\alpha'''). \end{aligned} \quad (4.2.16)$$

For the D8 branes we use eq.(4.2.3) and find,

$$N_{D8} = 2\pi \int dz F'_0 = 2^{\frac{5}{4}} \sqrt{\pi} e^{-\Psi_0} \int dz \alpha^{(4)}. \quad (4.2.17)$$

Finally, for the D4 branes, the four-cycle is $\mathcal{M}_5 = (\theta_2, \phi_2, \theta_1, \phi_1)$, at constant values of r, z . Calculating

$$Q_{D4} = \frac{1}{8\pi^3} \int_{\mathcal{M}_5} \hat{F}_4 = 0.$$

In other words, there is no charge of D4 branes in the system¹.

Inspecting the charges in eqs.(4.2.14), (4.2.16) and (4.2.17), suggests to set $e^{\Psi_0} = 2^{\frac{5}{4}} \sqrt{\pi}$. Imposing quantisation, it is clear from eqs. (4.2.16) that the function $\alpha''(z)$ must be a linear function with integer coefficients N_l . In fact, we can divide the range of the z -coordinate in intervals of unit size. In each interval, $\alpha''(z)$ should be a linear function. The charge of D8 branes in eq.(4.2.17) suggests that $\alpha''(z)$ should be piecewise linear and continuous (with integer coefficients), the third derivative piecewise constant (integer and generically discontinuous), whilst the fourth-derivative a sum of delta functions with integer coefficients. In other words, if we choose for $\alpha''(z)$,

$$\alpha''(z) = \begin{cases} N_1 z & , 0 \leq z < 1 \\ N_1 + (N_2 - N_1)(z - 1) & , 1 \leq z < 2 \\ \vdots & \\ N_l + (N_{l+1} - N_l)(z - l) & , l \leq z < l + 1 \\ \vdots & \\ N_{P-1}(P - z) & , (P - 1) \leq z < P. \end{cases} \quad (4.2.18)$$

¹One might wonder about computing Q_{D4} integrating $\hat{F}_4$ over the manifold $\mathcal{M}_6 = [z, \theta_1, \phi_1, \phi_2]_{\theta_2=\frac{\pi}{2}}$. We believe this is not a well defined four-cycle.

This implies,

$$\alpha'''(z) = \begin{cases} N_1 & , 0 \leq z < 1 \\ N_2 - N_1 & , 1 \leq z < 2 \\ \vdots & \\ N_{l+1} - N_l & , l \leq z < l+1 \\ \vdots & \\ -N_{P-1} & , (P-1) \leq z < P, \end{cases} \quad (4.2.19)$$

and

$$\alpha^{(4)}(z) = \sum_{l=1}^{P-1} (2N_l - N_{l+1} - N_{l-1}) \delta(z - l). \quad (4.2.20)$$

At this point we choose a convenient coefficient Δ in the large gauge transformation of B_2 — see eq.(4.2.8) . In fact, choosing $\Delta = l$ for $z \in [l, l+1]$ we have in each interval

$$(\alpha'' - (z - l)\alpha''') = N_l.$$

In summary, we have two kinds of NS-five branes, their total number is given in eqs.(4.2.14). In the interval $z \in [l, l+1]$ we have two types of D6 brane charges and one kind of D8 charge given by (in each interval),

$$\begin{aligned} N_{D6}^{(1)}[l, l+1] &= |g-1|N_l, \quad N_{D6}^{(2)}[l, l+1] = N_l, \\ N_{D8}[l, l+1] &= 2N_l - N_{l+1} - N_{l-1}. \end{aligned} \quad (4.2.21)$$

Though it does not feature in the the calculation of Page charges, the function $\alpha(z)$ is obtained after two integrations of eq.(4.2.18). The integration constants must be chosen such that $\alpha(z)$ is a piecewise continuous cubic function with continuous derivative $\alpha'(z)$. To avoid singular behaviours (not associated with the presence of localised D8 branes), it must satisfy $\alpha(0) = \alpha(P) = 0$.

This is a good point to discuss the physical effect of the large gauge transformation on the B_2 -field— see eq.(4.2.8). Being a gauge transformation, it does not affect the Physics of our system, but it makes the counting of charges more transparent. Had we not performed it and calculated Page charges with the $B_{2,old}$, we would have obtained a combination between charges of D6 branes induced on the D8 branes and those of 'actual' D6 branes. The large gauge transformation separates these, making the counting clearer.

To better understand these systems, it is a good (and intuitive) guide to go back to the case of the torus. In this case, we are compactifying a six dimensional $\mathcal{N} = (1, 0)$ SCFT on T^2 without any flux that breaks SUSY. In other words, we would end with an $\mathcal{N} = 2$ four dimensional SCFT. For more on this perspective see, for example [125]. What follows in the next paragraph is an *intuitive* argument.

The compactification on T^2 (which sets $k = 0$) should be handled with some care. For example, the 'twisting' in the one-form Θ_1 is absent, leading to $\Theta_1 = -\cos\theta_2 d\phi_2$. Also, in the BPS equations (4.1.44), (4.1.45) and (4.1.46), the terms $\frac{e^{-2h}}{k}$ are absent. This gives $2\Phi(r) = 0$ with $X(r) = 1$ (for the parameter $m = 1$) and $2f(r) = 2h(r) = r$. This leads to a background metric—see eq.(4.2.2)—of the form $\text{AdS}_7 \times S^2 \times R_z$. For this case the genus is $g = 1$ and the charges in eqs.(4.2.14) and (4.2.21) indicate the presence of only one type of NS-five branes and D6 branes, together with D8 branes. The reader will recognise that this is the compactification on T^2 of the backgrounds in [62–66]. Intuitively, this is what our background in (4.2.2)-(4.2.4) is describing for large values of the r -coordinate. Of course, since we compactify on a curved two manifold (either S^2 or H_2), the twisting needs to be performed even at high energies (large values of the r -coordinate). The system preserves four supercharges all along the flow. The twisted compactification of the six dimensional system of NS-D6-D8 induces new sets of NS and D6 branes. We identify these new branes as those whose charge comes with the $(g - 1)$ -factor in front—see (4.2.14) and (4.2.16).

Let us be more precise about the compactifications on S^2 and H_2 .

4.2.2 The Cases of H_2 and S^2 Compactifications

For $k = -1$, the IR fixed point and numerical solution of this system was studied in Section 4.1.3. Let us gain some understanding by discussing asymptotic solutions. For large values of the coordinate r we find an asymptotic solution,

$$e^{2h(r)} \sim e^r, \quad e^{2f(r)} \sim e^{2f_0+r}, \quad e^{2\Phi} \sim \frac{km}{k - e^{-r}} \sim m. \quad (4.2.22)$$

The fixed point found before, is probably the same solution found in [126], after conventions are matched. We quote the solution here for completeness

$$e^{2f(r)} = e^{\frac{2}{3}r}, \quad e^{2h(r)} = \frac{3}{4}, \quad e^{2\Phi(r)} = \frac{3m}{4}. \quad (4.2.23)$$

As is usual, the r -coordinate plays the role of energy-coordinate. The solution in (4.2.22) asymptotes to $\widehat{\text{AdS}}_7 \times S_c^2 \times R_z$ when replaced in (4.2.2)-(4.2.3), describing approximately a six dimensional SCFT (the 6d SCFT is formulated on $\mathbb{R}^{3,1} \times H_2$ and the $\widehat{\text{AdS}}_7$ is written as a foliation over this six-space). On the other end, when the fixed point solution of (4.2.23) is replaced in the family of backgrounds of Massive Type IIA of equations (4.2.2)-(4.2.3), the space time takes the form $\text{AdS}_5 \times \mathcal{M}_5$. These describe the dual to a family of four dimensional $\mathcal{N} = 1$ SCFTs. The numerical solution connecting the large- r asymptotics in (4.2.22) with the fixed-point exact solution (4.2.23) was described in Section 4.1.3.

The numerical study of the case of the two-sphere ($k = 1$) leads to badly singular behaviour as we decrease the r -coordinate. We believe that a background more elaborated

than that in equations (4.2.2)-(4.2.3) could resolve this singular behaviour, but leave this for a future investigation. In what follows we focus our attention on the backgrounds describing the twisted compactification on H_2 (preserving 4 SUSYs) of a six dimensional $\mathcal{N} = (1, 0)$ 6D SCFT.

In the next section we attempt to be more detailed about the dual field theory interpretation of the numerical solution described by the flow between the UV and IR fixed point.

We close this section emphasising that we have presented a large family of backgrounds in (4.2.2)-(4.2.3). Each member of this family, labelled by each possible $\alpha(z)$, is (in the IR fix point) a Massive IIA background with an AdS_5 factor preserving four supercharges.

4.3 A Proposal for the Dual Field Theory

In this section we give a proposal for the brane system and the field theory dual to the background in (4.2.2)-(4.2.3). We will be particularly interested on the fixed point solution of equation (4.2.23), that gives a family of backgrounds with an AdS_5 factor preserving four Poincare supercharges.

As we advanced in the previous section, for large values of the r -coordinate, the system is dual to a QFT that is to a good approximation six-dimensional. In fact, using the solution of eq.(4.2.22) in the background of eqs.(4.2.2)-(4.2.3), we find that the space asymptotes to $\text{AdS}_7 \times S^2 \times R_z$. The six dimensional SCFT is defined on a spacetime of the form $\mathbb{R}^{3,1} \times H_2$. This forces the twisting of the SCFT (to partially preserve SUSY). The different fibrations in the Massive IIA background (encoded by the one form Θ_1) are the effects of the twist in the holographic perspective. Following the formalism developed in the references [62–66], we can *roughly and intuitively* think that the field theory is encoded in a long linear quiver and Hanany-Witten set up of D6-D8-NS5 branes depicted in Figure 4.3

The numbers $N_1, N_2, \dots, N_P$ and $F_1, \dots, F_P$ must satisfy the relation $F_i = 2N_i - N_{i-1} - N_{i+1}$ in order for the six dimensional field theory to be free of gauge anomalies. This condition is ensured by the function $\alpha''(z)$ chosen in eq.(4.2.18). Indeed, note that from the fourth-derivative $\alpha^{(4)}(z)$ we derive the relation between flavours and colours required, see eq.(4.2.20). The function $\alpha(z)$, continuous and cubic by pieces is chosen such that the z -coordinate begins and ends smoothly. This is imposed by the conditions $\alpha(0) = \alpha(P) = 0$. There is one UV conformal point for each choice of $\alpha(z)$. Of course, this is just a *very rough* picture for the 4D SCFTs.

Indeed, these UV conformal points are deformed (either by VEVs or by relevant operators). The dimension of these operators can be read from the near- AdS_7 expansion of

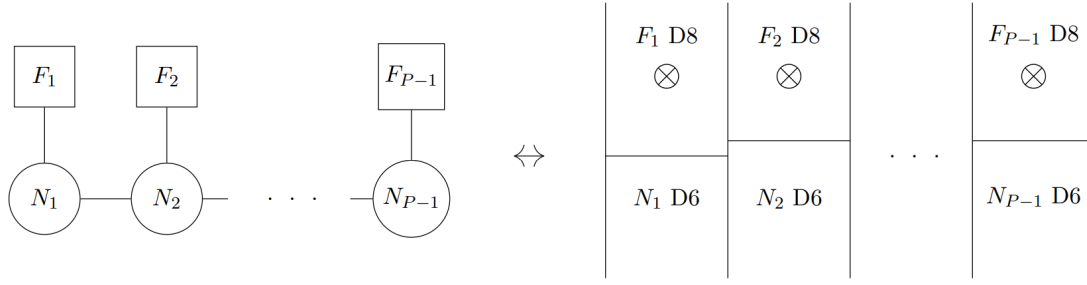


Figure 4.3: We show a *rough* approximation to the UV field theory. This would be just the torus reduction of a family of $\mathcal{N} = (1, 0)$ six dimensional SCFTs. As we explain below, this picture is deformed by other brane stacks appearing.

the metric. These deformations (analogously, the presence of the fibrations) topologically twist the 6d CFT and trigger a RG flow, that ends in a CFT_4 . As we discussed around eqs. (4.2.14), (4.2.21), a new set of NS and D6-branes appears due to the twisted compactification. These new branes lead to the presence of new gauge groups, not present in the original linear quiver without twisting (it is in this sense that the discussion for the T^2 compactification, whilst not rigorous is useful).

Following this line of arguments, our system is represented by the Hanany-Witten set up of Figure 4.4.

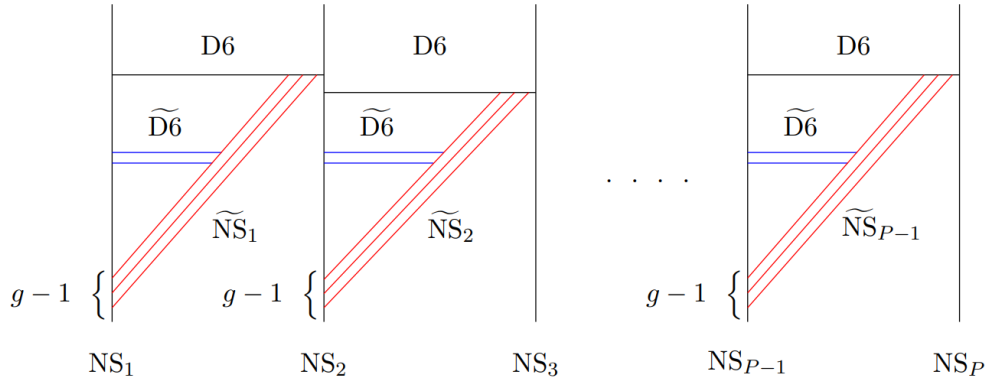


Figure 4.4: We see the new set of branes—denoted with a tilde—that are induced by the twisting. These new branes give place to new gauge groups and new matter fields represented by strings extending between different stacks of branes.

As expressed in eq.(4.2.14), a new set of NS5 branes appears due to the twisting. These new $(g - 1) \times P$ NS5 branes share the $\mathbb{R}^{3,1}$ directions with the other branes and extend along the coordinates (θ_2, ϕ_2) . They are orthogonal to the original NS branes that extend along $(\mathbb{R}^{3,1}, \theta_1, \phi_1)$. This leads to the generation of new gauge groups. In fact, the D6

branes originally extended along $(\mathbb{R}^{3,1}, \theta_1, \phi_1, z)$ can now extend between the two stacks of NS5 branes. This generates a new stack of D6 branes extended along $(\mathbb{R}^{3,1}, \theta_2, \phi_2, z)$. We associate with these the new gauge groups. As calculated in eq.(4.2.21), there are $(g-1) \times N_l$ of them, for each interval. There are also D8 sources, playing the role of flavour groups. The system must cancel all gauge anomalies. We emphasise the $(g-1) \times P \times P$ -plication of the original quiver. This will reflect in quantities like the free energy, that we calculate in Section 4.3.2. It should be interesting to carefully quantise the open strings between these different stacks of branes, to have a more concrete handle on the low energy QFT.

A natural question that arises is the following: can we find a Lagrangian description for the 4d QFT? If so, we could propose a (low energy) duality between a 4d Lagrangian QFT and a 6d SCFT on a geometry of the form $\mathbb{R}^{3,1} \times \Sigma_k$. This is perhaps too much to ask. We should be content with a 4d quiver (even when strongly coupled) that captures aspects of the compactified 6d theory. In fact, the above analysis leads us to propose a quiver, that consist on $P^2(g-1)$ copies of the six-dimensional 'mother' theory. Of course, the system is now four dimensional and preserves $\mathcal{N} = 1$ SUSY. In the usual notation (lines with arrows indicating chiral multiplets, circles denoting vector multiplets, boxes indicating flavour symmetries), we propose the quiver in Figure 4.5.

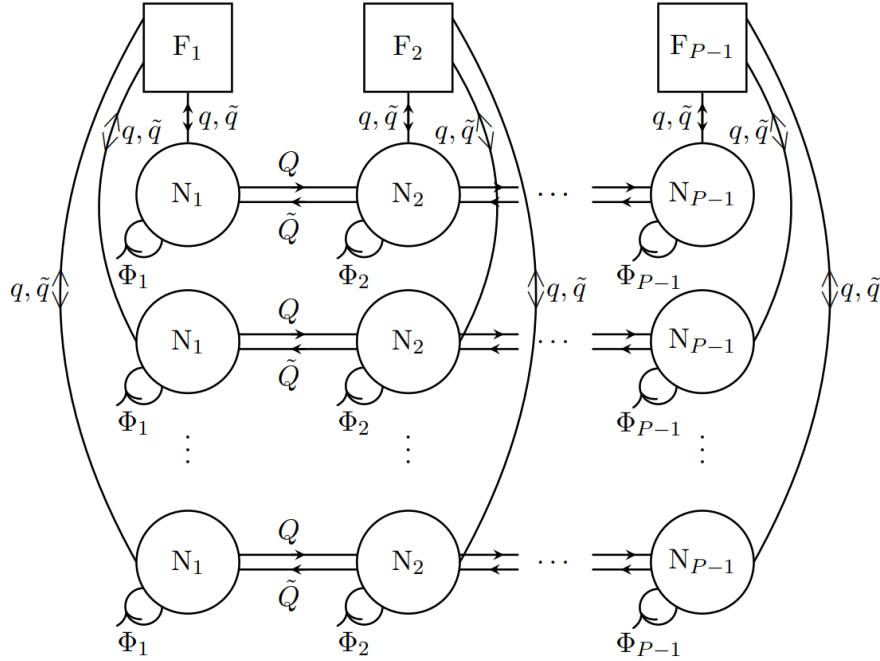


Figure 4.5: The proposed quiver field theory. The adjoint fields Φ_i get a mass, the bi-fundamentals $(Q, \bar{Q})$, the fundamentals $(q, \bar{q})$ and the vector multiplets represented by the circles. All the fields are in four dimensional $\mathcal{N} = 1$ notation.

In the next Section, we analyse this QFT proposal by computing the beta functions

and R-symmetry anomalies from the field theory side for each of the gauge groups. We also calculate a_{CFT} and c_{CFT} , the a, c -central charges of the quiver. Also a suitable superpotential is proposed, this indicates an interaction between different rows of the quiver in Figure 4.5.

Below, we compare these field theory calculations with the holographic ones using the background of eq.(4.2.2)-(4.2.3). In particular, we will study the free energy (we will refer to it as holographic central charge) and its properties along the flow. And then, to gain some intuition, we discuss explicitly two illustrative examples.

4.3.1 Analysis of the 4D QFT

Referring to the four dimensional $\mathcal{N} = 1$ quiver in Figure 4.5, we assign R-charges and anomalous dimensions for each of the fields. For the adjoint scalars Φ_i , the vector multiplets W_i , the bifundamentals between gauge nodes $(Q, \tilde{Q})$ and the fundamentals $(q, \bar{q})$ we propose

$$\text{The R-charges: } R[\Phi_i] = 1, \quad R[W_i] = 1, \quad R[Q] = R[\tilde{Q}] = R[q] = R[\bar{q}] = \frac{1}{2}. \quad (4.3.1)$$

$$\text{The anomalous dimensions: } \gamma_{\Phi_i} = 1, \quad \gamma_Q = \gamma_{\tilde{Q}} = \gamma_q = \gamma_{\bar{q}} = -\frac{1}{2}. \quad (4.3.2)$$

Notice that the dimension of any combination of fields $\mathcal{O}$ satisfy

$$\dim(\mathcal{O}) = \frac{3}{2}R_{\mathcal{O}}. \quad (4.3.3)$$

As should occur at conformal points. In particular the following superpotential terms are present,

$$\mathcal{W} \sim \mu_i \Phi_i \Phi_i + q_i \bar{q}_j q_j \bar{q}_i + \text{all other combinations}. \quad (4.3.4)$$

The R-charge of each of these possible terms is $R[\mathcal{W}] = 2$ and their dimension $\dim[\mathcal{W}] = 3$. The mass term for all the adjoint scalars decouples them from the IR dynamics. They do not participate in the quantities computed below. The quartic term, on the other hand, generates interactions between all the different $P^2(g-1)$ horizontal lines of the quiver, by closing loops using the flavour groups. As one can start seeing, the behaviour of these quivers is somewhat reminiscent of those in [127, 128]. Let us calculate the beta functions and the R-symmetry anomaly for each gauge group of the quiver in Figure 4.5.

Beta functions

For any particular node we use the NSVZ beta function $\beta \sim 3N_c - N_{fi}(1 - \gamma_i)$, we find

$$\beta_i \sim 3N_i - (N_{i+1} + N_{i-1} + F_i)(1 - (-\frac{1}{2})) = \frac{3}{2} \left[2N_i - F_i - N_{i+1} - N_{i-1} \right] = 0. \quad (4.3.5)$$

As we start with a balanced quiver in the UV (the six dimensional quiver must cancel gauge anomalies), the beta function in four dimensions vanishes for each gauge group. It

is nice to see how the six dimensional anomaly transmuted into the four dimensional beta function condition for conformality.

R-symmetry anomaly

We calculate using the expression $\Delta\Theta = T(R_i)R(f_i)$. We use that $T(adj_i) = 2N_i$ and $T(fund) = 1$. We find

$$\Delta\Theta_i = 2N_i \times 1 + (F_i + N_{i+1} + N_{i-1}) \times 2 \times \left(-\frac{1}{2}\right) = 2N_i - F_i - N_{i+1} - N_{i-1} = 0. \quad (4.3.6)$$

The comments written below eq.(4.3.5) also apply here. Both these calculations add to the proposal that the quiver in Figure 4.5 describes the dynamics of our AdS_5 fixed point.

Other interesting quantities are the central charges a and c .

Central charges

They are defined as,

$$a = \frac{3}{32\pi} [3\text{Tr}R^3 - \text{Tr}R], \quad c = \frac{1}{32\pi} [9\text{Tr}R^3 - 5\text{Tr}R]. \quad (4.3.7)$$

Calculating explicitly for the quiver in Figure 4.5, we find

$$\begin{aligned} \text{Tr}R &= P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1) \times 1 + N_j F_j \times \left(-\frac{1}{2}\right) \times 2 + \sum_{j=1}^{P-2} N_j N_{j+1} \times \left(-\frac{1}{2}\right) \times 2 \right] \\ &= P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1 - N_j F_j) - \sum_{j=1}^{P-2} N_j N_{j+1} \right]. \\ \text{Tr}R^3 &= P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1) \times 1^3 + N_j F_j \times \left(-\frac{1}{2}\right)^3 \times 2 + \sum_{j=1}^{P-2} N_j N_{j+1} \times \left(-\frac{1}{2}\right)^3 \times 2 \right] \\ &= P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1 - \frac{1}{4} N_j F_j) - \frac{1}{4} \sum_{j=1}^{P-2} N_j N_{j+1} \right]. \end{aligned} \quad (4.3.8)$$

Using eqs.(4.3.7) we have,

$$a = \frac{3}{16\pi} P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1) + \frac{1}{8} N_j F_j + \frac{1}{8} \sum_{j=1}^{P-2} N_j N_{j+1} \right], \quad (4.3.9)$$

$$c = \frac{1}{8\pi} P^2(g-1) \left[\sum_{j=1}^{P-1} (N_j^2 - 1) + \frac{11}{16} N_j F_j + \frac{11}{16} \sum_{j=1}^{P-2} N_j N_{j+1} \right]. \quad (4.3.10)$$

4.3.2 Free Energy and Holographic Central Charge

One meaningful observable of these SCFTs is the free energy. For the case of SCFTs in diverse dimensions this was calculated holographically. See for example the papers [33, 45,

46, 52, 59, 61–66]. These calculations were checked against field theoretical computations (typically a global anomaly coefficient or a localisation calculation in matrix models). These checks of the AdS/CFT correspondence are valid in the regime of parameters for which the supergravity background is a reliable representation of the QFT dynamics. This is typically for long linear quivers $P \rightarrow \infty$ and for large rank of the gauge nodes $N_i \rightarrow \infty$.

The calculations done with the supergravity backgrounds are either computing a regularised on-shell action for a putative reduced supergravity or alternatively the calculation of the Newton constant associated with a reduced theory of gravity in lower dimensions.

In particular, it was shown in [129, 130] that for any generic holographic background dual to a QFT in $d + 1$ spacetime dimensions, with metric and dilaton given by,

$$ds^2 = a(r, y^i) (dx_{d,1}^2 + b(r)dr^2) + g_{ij}dy^i dy^j, \quad \Psi(r, y^i), \quad (4.3.11)$$

we can define quantities $V_{int}, \hat{H}$ according to

$$V_{int} = \int dy^i \sqrt{e^{-4\Psi} a(r, y^i)^d \det[g_{int}]}, \quad H = V_{int}^2. \quad (4.3.12)$$

From these we define the holographic central charge (or free energy),

$$c_{hol} = d^d \frac{b(r)^{\frac{d}{2}} H^{\frac{2d+1}{2}}}{G_N^{(10)} (H')^d}, \quad (4.3.13)$$

where $G_N^{(10)} = 8\pi^6$ is in our conventions, the ten-dimensional Newton constant.

For the case of our backgrounds in eq.(4.2.2), we compare with eq.(4.3.11) and read,

$$a(r, z) = 2\beta\pi\sqrt{2}\sqrt{-\frac{\alpha(z)}{\alpha''(z)}} X(r)^{-\frac{1}{2}} e^{-\frac{4\Phi(r)}{5} + 2f(r)}, \quad b(r) = e^{-2f(r)}, \quad d = 3. \quad (4.3.14)$$

The metric of the internal space has determinant,

$$\det(g_{int}) = \frac{16\sqrt{2}\pi^5}{\omega^2} e^{4h(r) + \Phi(r)} \frac{\sqrt{-\alpha(z)^7 \alpha''(z)}}{(\alpha'(z)^2 - 2\alpha(z)\alpha''(z))^2} \frac{\sin^2(\sqrt{k}\theta_1)}{k} \sin^2(\theta_2). \quad (4.3.15)$$

This leads, after using eqs.(4.3.12)-(4.3.13), to

$$H = \hat{\mathcal{N}}^2 e^{6f(r) + 4h(r) - 4\Phi(r)}, \quad \hat{\mathcal{N}} = 16\sqrt{2}\pi^4 e^{-2\Psi_0} \int [-\alpha(z)\alpha''(z)] dz \text{Vol}(\Sigma_k) \text{Vol}(S^2). \\ c_{hol} = \frac{27\hat{\mathcal{N}}}{8G_N^{(10)}} \frac{e^{2h(r) - 2\Phi(r)}}{(3\dot{f} + 2\dot{h} - 2\dot{\Phi})^3}. \quad (4.3.16)$$

After using the BPS equations in (4.1.44), (4.1.45) and (4.1.46), the holographic central charge reads

$$c_{hol} = \frac{27\hat{\mathcal{N}}}{G_N^{(10)}} \frac{e^{2h(r) - 2\Phi(r)}}{\left(4 + \frac{1}{k}e^{-2h(r)} + m e^{-2\Phi(r)}\right)^3}. \quad (4.3.17)$$

Since this computes the central charge of the four dimensional SCFTs— we have set $d = 3$ in eq.(4.3.14)— we evaluate this quantity at the IR fixed point for $k = -1$, the hyperbolic plane twisted compactification of the 6d SCFTs (setting $m = 1$), and find

$$c_{hol,IR} = \frac{27\hat{\mathcal{N}}}{64 G_N^{(10)}}. \quad (4.3.18)$$

Let us analyse the results in eqs.(4.3.17)-(4.3.18). First, notice that the factor $\hat{\mathcal{N}}$ in eq.(4.3.16), has a part proportional to $\left(-\text{Vol}(S^2) \int dz \alpha'' \alpha \right)$. We recognise this as information coming from the UV, six dimensional SCFT. Indeed, when calculating the free energy of a six dimensional $(1, 0)$ SCFT, see equation (2.14) in the paper [66] or equation (4.10) in [62]. This is coming from the UV-part of the flow.

Note that $\hat{\mathcal{N}}$ in eq.(4.3.16), also contains a factor of $\text{Vol}(\Sigma_k) = 4\pi|(g-1)|$. This is in line with our proposal that the number of gauge groups is multiplied by $(g-1)$ and is represented by the D6 branes induced by the twisting, see our eq.(4.2.21).

4.3.3 Comparison Between Field Theory and Holographic Results

Let us evaluate a_{CFT} and c_{CFT} in (4.3.9) and (4.3.10), and compare them with the holographic central charge in eq.(4.3.18) for two examples. Whilst these examples are not generic, they capture many aspects of the dynamics of the QFTs.

Example 1

Referring to the quiver in Figure 4.5, we study the field theory with $N_j = jN$, $F_j = NP\delta_{j,P-1}$, which leads to a $SU(PN)$ flavour group. We find that (4.3.9) and (4.3.10) give

$$\begin{aligned} a_{CFT} &= \frac{3}{128\pi}(g-1)(P-1)P^2 [N^2P(3P-2) - 8], \\ c_{CFT} &= \frac{1}{128\pi}(g-1)(P-1)P^2 [N^2P(9P-10) - 16]. \end{aligned} \quad (4.3.19)$$

For large values of N and P , which is the holographic limit, both a and c anomalies are the same

$$a_{CFT} = \frac{9}{128\pi}P^5N^2(g-1) \left(1 + O\left(\frac{1}{P}\right) \right), \quad c_{CFT} = a_{CFT} \quad (4.3.20)$$

Holographically, this quiver corresponds to a function $\alpha(z)$ given by,

$$\alpha(z) = \begin{cases} \frac{N}{6}(1-P^2)z + \frac{N}{6}z^3 & , 0 \leq z \leq (P-1) \\ -\frac{1}{6}(2P^2-3P+1)(P-z) + \frac{N}{6}(P-1)(P-z)^3 & , (P-1) \leq z \leq P. \end{cases} \quad (4.3.21)$$

This is a function that vanishes at $z = 0$ and $z = P$, is continuous at $z = (P - 1)$ and the derivative $\alpha'(z)$, is continuous at the same point. The second derivative is

$$\alpha''(z) = \begin{cases} Nz & , 0 \leq z \leq (P - 1) \\ N(P - 1)(P - z) & , (P - 1) \leq z \leq P. \end{cases} \quad (4.3.22)$$

Using $\alpha''(z)$ above and eq.(4.2.21) gives the brane charges reproducing the quiver we wish to study.

We now use the expressions in eqs. (4.3.16)-(4.3.18), the fact that $e^{2\Psi_0} = 4\sqrt{2}\pi$, $G_N = 8\pi^6$ and find,

$$\hat{\mathcal{N}} = \frac{64}{45}\pi^5 N^2 P^5 (g - 1), \quad c_{hol,ex1} = \frac{3}{40\pi} N^2 P^5 (g - 1) \left(1 + O\left(\frac{1}{P^2}\right)\right). \quad (4.3.23)$$

We find a scaling with the length of the quiver and the number of nodes, that is reminiscent of what occurs for this kind of quiver for the case of six dimensional SCFTs.

The comparison between eqs.(4.3.20) and (4.3.23) suggests that in the QFT side there must be other fields that whilst not contributing to the beta function and R-anomaly, it makes a contribution to the a and c central charges. The contribution of these fields, that we denote as $\mathcal{X}$ should modify the result in eq.(4.3.20) (at leading order in P) according to,

$$a_{CFT} = \frac{9}{128\pi} N^2 P^5 (1 + \delta_{\mathcal{X}}), \quad 15\delta_{\mathcal{X}} = 1. \quad (4.3.24)$$

Example 2

The quiver of example 2 consists of a line of $(P - 1) - SU(N)$ gauge groups, closed in the beginning and in the end by $SU(N)$ -flavour groups, which means $N_j = N$ and $F_j = N(\delta_{j,1} + \delta_{j,P-1})$. In the holographic limit, (4.3.9) and (4.3.10) take the form

$$a_{CFT} = \frac{27}{128\pi} P^3 N^2 (g - 1) \left(1 + O\left(\frac{1}{P}\right)\right), \quad c_{CFT} = a_{CFT}. \quad (4.3.25)$$

On the gravity side, this system is analysed starting from the function $\alpha(z)$,

$$\alpha(z) = \begin{cases} \frac{N}{2}(1 - P)z + \frac{N}{6}z^3 & , 0 \leq z \leq 1 \\ \frac{N}{6} - \frac{PN}{2}z + \frac{N}{2}z^2 & , 1 \leq z \leq (P - 1) \\ -\frac{N}{2}(P - 1)(P - z) + \frac{N}{6}(P - z)^3 & , (P - 1) \leq z \leq P. \end{cases} \quad (4.3.26)$$

This is a function that vanishes at $z = 0$ and $z = P$, is continuous at $z = 1$ and at $z = (P - 1)$

and the derivative $\alpha'(z)$ is continuous at the same two points. The second derivative is

$$\alpha''(z) = \begin{cases} Nz & , 0 \leq z \leq 1 \\ N & , 1 \leq z \leq (P-1) \\ N(P-z) & , (P-1) \leq z \leq P. \end{cases} \quad (4.3.27)$$

This gives brane charges in agreement with the quiver we want to study, namely $N_j = N$ and $F_J = N(\delta_{J,1} + \delta_{J,P-1})$. Using eqs. (4.3.16)-(4.3.18), we calculate,

$$c_{hol,ex2} = \frac{9}{32\pi} N^2 P^3 (g-1) \left(1 + O\left(\frac{1}{P^2}\right) \right). \quad (4.3.28)$$

Again, this 4d SCFT has a scaling characteristic of a six dimensional quiver of the same type.

As before, the comparison between eqs.(4.3.25) and (4.3.28) suggests that in the QFT side there must be other fields (not contributing to the beta function and R-anomaly), but adding a contribution to the a and c central charges. At leading order in P

$$a = \frac{27}{128\pi} N^2 P^3 (1 + \delta_{\mathcal{X}}), \quad 3\delta_{\mathcal{X}} = 1. \quad (4.3.29)$$

This suggests that in our definition of the QFT quiver, new fields should enter adding these small contributions to the a central charge. Indeed, the presence of 'flip fields' that couple to irrelevant operators of the baryonic type is common in field theories of the class S_k . These fields are gauge singlets and decouple at low energies. They do not contribute to the beta functions or R-symmetry anomaly, but do change the result of a, c . It would be nice to see that their contribution can be $\delta_{\mathcal{X}}$ given in eqs.(4.3.24)-(4.3.29).

This result, together with the one in the first example and their respective comparison with the field theoretical calculation in eqs.(4.3.20)-(4.3.25), suggest that the four dimensional quiver is deconstructing a six dimensional CFT.

Let us close this section with three important comments.

- First, it may be the case that other fields that do not contribute to the beta functions and the R-symmetry anomaly, but still contribute to the a, c central charges, are present in our quiver of Figure 4.5. To decide if they should be there, we should compute the integral of the anomaly polynomial I_8 on the compactification manifold Σ_- . We will get a hint at those fields, by comparing the material of Section 4.3.2 with that in Section 4.3.1.
- Second, the scaling with the number of nodes P of the a -central charge (the free energy) is two powers higher than usual. In fact, it actually scales as a six dimensional CFT. The quiver we write is *deconstructing* two dimensions, in the sense of [131].

What is interesting about this case is that this is occurring due to the twisting, whilst in [131] it is an effect of going to a particular point in the moduli space, expanding, etc.

- Finally, as explained in Section 4.3.1, a mass term for Φ_i in eq.(4.3.4) appears. This generates a new set of $(P - 1) - U(1)$'s. These are all anomalous, except for one. The R-symmetry we have taken is the one after this mixing has taken place. Importantly, this makes the beautiful result of Bobev and Cricigno [132], not applicable here. In fact, the authors of [132] assume the absence of mixings. In this sense, the coefficient in their eq.(3.6) for the a central charge, is not enforced on us².

Energy Dependence of the Holographic Central Charge

Let us now interpret the r -dependence of our free energy/holographic central charge in eq.(4.3.17). Inspecting this quantity, we find at the IR fixed point the result in eq.(4.3.18). We associate with this the free energy of a family of four dimensional SCFTs preserving four Poincare supercharges. The family of SCFTs in 4d is labelled by the different functions $\alpha(z)$ that we choose as input, as in the examples above. On the other end, at high energies, we find using eq.(4.2.22), that the quantity in eq.(4.3.17) diverges. A plot along the numerical solution found in Section 4.1.3 shows this.

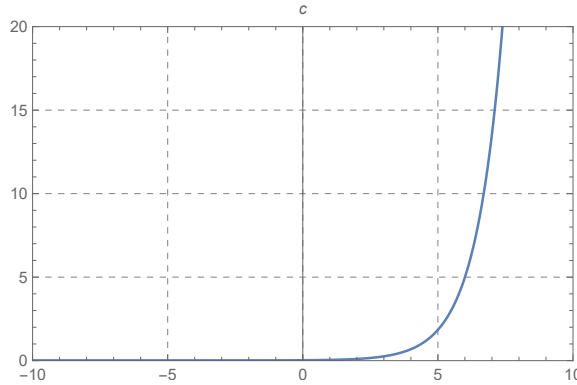


Figure 4.6: Central Charge calculated for the numerical solutions of Figure 4.2. Here we see that the central charge is divergent in the UV.

The divergence of this monotonically increasing (towards the UV) free energy is understandable. Massive fields that originated in the wrapping of both D6's and NS-five branes on the surface $\Sigma_k = H_2$ generically have a mass inversely proportional to the size of Σ_k and are frozen at low energies (of course, there are also massless fields present in the 4d SCFT). When flowing towards the UV these massive fields become active. The number of these

²Special thanks to Nikolay Bobev for a discussion on this.

Kaluza-Klein modes grows with energy and leads to the divergence of c_{hol} . Somewhat, the quantity defined in eqs.(4.3.14)-(4.3.18) for a *four* dimensional QFT (we have set $d = 3$) is not able to recognise that the system is approximately approaching a six dimensional SCFT.

This same arguments were expressed by the authors of [133]. In that paper the authors define a quantity $c_{monotonic}$ using a calculation of Entanglement Entropy. We believe that $c_{monotonic}$ in [133] and our eq.(4.3.17) should be related by a proportionality factor when studied in our family of backgrounds.

This calls for a quantity that is actually able to detect both fixed points (this was also one of the objectives in [133]). A quantity called c_{flow} in [130, 134] can play this role. It should be emphasised that this quantity is monotonic, but decreases towards the UV fixed point. We study this quantity below.

4.3.4 Flow Central Charge

Let us now define a quantity that is capable of detecting both the UV and IR fixed points of our flow across dimensions. With this in mind, we write a generic holographic metric dual to a QFT defined on anisotropic spacetime—see [130, 134] for details,

$$ds^2 = -a_0 dt_0^2 + \sum_{i=1}^{d-p} a_i dx_i^2 + a_{d-p+1} ds_p^2 + \prod_{i=1}^{d-p} (a_i a_{d-p+1}^p)^{\frac{1}{d}} \tilde{b} dr^2 + h_{ab} dy^a dy^b. \quad (4.3.30)$$

where t and x_i are coordinates of the macroscopic space, p is the dimension of the compactification manifold and ds_p^2 its line element. The metric and the coordinates of the internal manifold are denoted by h_{ab} and y^a respectively. The dilaton is $\Psi(r, y^a)$. We define the quantities,

$$G_{ij} dX^i dX^j = \sum_{i=1}^{d-p} a_i dx_i^2 + a_{d-p+1} ds_p^2 + h_{ab} dy^a dy^b, \quad (4.3.31)$$

$$\tilde{H} = \left(\int \prod_{a=1}^{8-d} dX^a \sqrt{e^{-4\Psi} \det(G_{ij})} \right)^2, \quad c_{flow} = d^d \frac{\tilde{b}^{\frac{d}{2}} \hat{H}^{\frac{2d+1}{2}}}{G_N^{(10)} (\hat{H}')^d}.$$

Specialising these quantities in our metric of eq.(4.2.2), we have

$$d = 5, \quad p = 2, \quad a_0 = a_1 = a_2 = a_3 = 2\pi\sqrt{2} \sqrt{-\frac{\alpha(z)}{\alpha''(z)}} X(r)^{-\frac{1}{2}} e^{-\frac{4\Phi(r)}{5} + 2f(r)},$$

$$a_4 = a_0 e^{2h(r) - 2f(r)}, \quad \tilde{b} = e^{-\frac{6}{5}f(r) - \frac{4}{5}h(r)},$$

$$\tilde{H} = \hat{\mathcal{N}}^2 e^{6f(r) + 4h(r) - 4\Phi(r)}, \quad \hat{\mathcal{N}} = 16\sqrt{2}\pi^4 e^{-2\Psi_0} \int (-\alpha(z)\alpha''(z)) dz \text{Vol}(\Sigma_k) \text{Vol}(S^2).$$

Using eq.(4.3.31) we compute,

$$c_{flow} = \frac{27\hat{\mathcal{N}}}{32G_N^{(10)}} \frac{e^{-2\Phi(r)}}{(3\dot{f} + 2\dot{h} - 2\dot{\Phi})^5}, \quad (4.3.32)$$

and as before we use the BPS equations (4.1.44), (4.1.45) and (4.1.46) to find

$$c_{flow} = \frac{27\hat{\mathcal{N}}}{G_N^{(10)}} \frac{e^{-2\Phi(r)}}{\left(4 + \frac{1}{k}e^{-2h(r)} + m e^{-2\Phi(r)}\right)^5}. \quad (4.3.33)$$

Evaluating in the IR fixed point of eq.(4.2.23) and in the asymptotic UV-expansion of eq.(4.2.22), leads to³

$$c_{flow,IR} = \frac{9\hat{\mathcal{N}}}{256 G_N^{(10)}}, \quad c_{flow,UV} = \frac{27\hat{\mathcal{N}}}{5^5 G_N^{(10)}}. \quad (4.3.34)$$

We can evaluate this quantity numerically on the flow solutions discussed in Section 4.1.3. The plot in Figure 4.7 shows this.

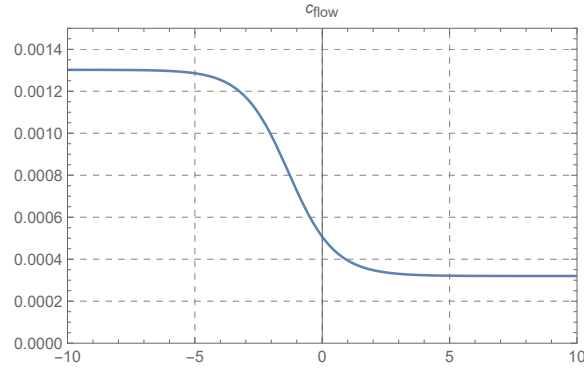


Figure 4.7: Flow central charge calculated for the numerical solutions. We clearly see the attainment of both IR and UV fixed points.

This plot is very similar to the analogous quantity calculated from the Entanglement Entropy in reference [133]. The discussion about the factors in $\hat{\mathcal{N}}$ is similar to that in the previous subsection. The qualitative lessons for the QFT along the flow are similar. In summary, we have defined a quantity for an anisotropic QFT realising a flow across dimensions. It should be a nice problem to tackle this same problem from a purely field theoretical perspective.

4.4 Summary

Here, we constructed a 7D gauged Supergravity solution that interpolates between AdS_7 in the UV and $\text{AdS}_5 \times \Sigma_2$ in the IR. We then embedded this 7D configuration into an

³Note that already in the IR there is a difference when comparing $c_{flow,IR}$ with $c_{hol,IR}$ in (4.3.18), which is not enough to match the field theory computation. The UV central charge could in principle be computed from the anomaly polynomial as in [62], however a more careful analysis is needed in order to take into the account the topological twist. See for example [116].

infinity family of Massive Type IIA backgrounds. These describe holographically the flow between a family of six-dimensional $\mathcal{N} = (1, 0)$ SCFTs and four dimensional $\mathcal{N} = 1$ SCFTs. The charges of the brane system are discussed. New sets of branes appear induced by the twisted-compactification and are carefully discussed.

Then we proposed a family of dual four dimensional SCFTs of the linear quiver type. The form and the numbers characterising the quiver are inherited from the six dimensional description. The 4d QFTs are proposed to reach a conformal point at low energies compared to the inverse size of the compactification manifold. These 4d SCFTs are holographically described by an infinite family of Massive Type IIA backgrounds with an AdS_5 factor.

We use the quiver field theory description to calculate field theory observables. Interestingly our field theoretical calculation of the a and c central charges in comparison with the holographic one, suggests that other fields should intervene in the dynamics, aside from those in the proposed quiver. We present and calculate two quantities that measure the number of degrees of freedom along the flow. One of these quantities coincides with the holographic central charge for the 4d SCFTs, and diverges in the UV. The second quantity detects both fixed points, it is monotonous, its value being bigger in the IR than in the UV.

Chapter 5

Twisted Circle Compactification of $\mathcal{N} = 4$ SYM and its Holographic Dual

The aim of this chapter is to analyse a mechanism of SUSY preservation. When formulating a SUSY QFT on a circle with SUSY breaking boundary conditions, introducing a background gauge field for the abelian subgroup of the R-symmetry group of the theory, it is possible to preserve some amount of SUSY, by means of a twist between the $U(1)$ isometry along the circle, and the $U(1)$ subgroups of the R-symmetry.

In particular, we focus on 4D maximally supersymmetric $SU(N)$ Yang-Mills theories, but the analysis extends to different dimensions. We start by reviewing compactifications of the above mentioned theory on a spatial circle with anti-periodic boundary conditions and how this leads to the breaking of Bose-Fermi degeneracy. This, together with its holographic description, is done in Section 5.1. In Section 5.2, we consider 4D $\mathcal{N} = 1$ SYM as a toy model, to show how introducing a constant background gauge field for the R-symmetry can restore SUSY. We then move to the 4D $\mathcal{N} = 4$ SYM and study SUSY restoration, low energy spectrum and vacua. In Section 5.3, we study its holographic description and explain how these describe the compactified field theory, and examine various probes of these backgrounds revealing aspects of the dual field theory. Then, in Section 5.4, we propose that it is possible to extend the SUSY preserving mechanism to maximally SUSY Yang-Mills in even dimensions by looking at their holographic duals, and point out the difficulties to apply this method in odd dimensions.

5.1 Preliminary: $\mathcal{N} = 4$ SYM on $\mathbb{R}^{2,1} \times S^1$

Formulating a QFT with fermions on a manifold with compact directions requires specifying the spin structure. When the compact dimension is a circle, one needs to impose periodic or anti-periodic boundary conditions for the fermions on the circle. Choosing anti-periodic boundary conditions makes all fermions massive at the classical level. Indeed, consider a free fermion in $\mathbb{R}^{2,1} \times S^1$ (here, the 4D coordinates x^μ are split into $x^\mu = (x^i, \phi)$)

$$\mathcal{L} = i\lambda\sigma^\mu\partial_\mu\bar{\lambda}. \quad (5.1.1)$$

imposing anti-periodic boundary conditions as

$$\lambda(x^j, \phi) = \sum_{n \in \mathbb{Z}} e^{\frac{i}{R}(n+\frac{1}{2})\phi} \lambda^{(n)}(x^j), \quad (5.1.2)$$

leads to a mass term in the Lagrangian

$$\mathcal{L} = \sum_n i\lambda^{(n)}\sigma^i\partial_i\bar{\lambda}^{(n)} - \frac{1}{R}\left(n + \frac{1}{2}\right)\lambda^{(n)}\sigma^3\bar{\lambda}^{(n)}. \quad (5.1.3)$$

In the context of supersymmetric (SUSY) theories, and in particular SUSY Yang-Mills (SYM) theories, choosing anti-periodic boundary conditions breaks SUSY by making all fermion modes massive at the classical level and spoiling Bose-Fermi degeneracy. Any massless scalar superpartners of the fermions then get masses through radiative corrections beginning at one-loop in perturbation theory. At energies much lower than the scale set by the size of the circle, the effective theory is pure Yang-Mills theory in one dimension less.

Holographically, the SUSY breaking spin-structure on the circle is realised by a cigar-like geometry: shrinking of the circle coordinate along which the boundary field theory is compactified. Canonical examples [75] of such backgrounds are double Wick rotations of non-extremal Dp-brane geometries, dual to maximally SUSY Yang-Mills in $(p+1)$ dimensions compactified on a circle with anti-periodic (thermal) boundary conditions for the fermions. Take for example $p = 3$, which is dual to 4D $\mathcal{N} = 4$ SYM with anti-periodic boundary conditions on along the circle

$$\begin{aligned} ds^2 &= \frac{r^2}{\ell^2} \left(dx_{2,1}^2 + \tilde{f}(r) d\phi^2 \right) + \frac{dr^2}{\frac{r^2}{\ell^2} \tilde{f}(r)} + \ell^2 ds^2(S^5), \\ F_5 &= -\frac{4}{\ell} (1 + \star) \text{Vol}(\text{AdS}_5), \end{aligned} \quad (5.1.4)$$

with

$$\tilde{f}(r) = 1 - \frac{\mu}{r^2}. \quad (5.1.5)$$

Although the Killing spinors on this geometry vanish for any $\mu \neq 0$, it will be instructive to do the following exercise: asymptotically, the geometry looks like pure $\text{AdS}_5 \times S^5$ with the

coordinate ϕ identified. We can then solve the Killing spinor on pure $\text{AdS}_5 \times S^5$ and impose anti-periodic boundary conditions along ϕ to see what is the effect of the spin-structure on the Killing spinor. The Killing spinor is then

$$\epsilon_{\text{AdS}_5} = \sqrt{\frac{r}{\ell}} \epsilon_+ + \sqrt{\frac{r}{\ell}} \left(\frac{\ell}{r} + \frac{1}{\ell} x^\mu \Gamma_\mu \right) \epsilon_- . \quad (5.1.6)$$

When imposing anti-periodic boundary conditions along ϕ we note that the term proportional to ϵ_- depends explicitly on ϕ in a non-periodic way, so we must set $\epsilon_- = 0$. On the other hand, the term proportional to ϵ_+ is constant, so the same argument as before applies. We learn that choosing the anti-periodic spin-structure along ϕ makes the Killing spinor completely vanish. This is the holographic counterpart of the breaking of the Bose-Fermi degeneracy by the effective mass terms.

Below, we consider the scenario described above, but with a twist, in theories with extended supersymmetry in various dimensions. In particular, we will discuss circle compactifications with SUSY breaking boundary conditions for fermions, simultaneously switching on a constant abelian background gauge field for a global R-symmetry along the compact dimension, in such a way that the low energy effective theory consists of a classically massless SUSY Yang-Mills multiplet preserving at least four supercharges. Quantum effects in the low energy theory can then generate a mass gap, which can be explored quantitatively via a holographic description.

A key motivation for this study is to understand the field theoretic implication of the (holographic) supersymmetric soliton supergravity backgrounds found first in [135] (see [136] for a precursor), and generalisations thereof [77, 78, 137–140]. What makes these backgrounds particularly interesting is that, like holographic duals of SUSY field theories with SUSY-breaking spin structure they contain a cigar-like geometry. Crucially however, the Killing spinors in [77, 78, 135, 137–140] are non-vanishing while satisfying anti-periodic boundary conditions around the circle. Moreover, a further puzzling feature of the supersymmetric soliton backgrounds of [135] is that unlike other known gravity duals of SUSY field theories with a mass gap [74, 141], there are no additional Type II fields turned on that could correspond to field theory condensates accompanying a nontrivial vacuum structure (however, at the level of the 5D gauged SUGRA, there is a gauge field turned on, signalling the presence of a current in the dual field theory).

5.2 $\mathcal{N} = 4$ SYM on $\mathbb{R}^{2,1} \times S^1$ with background gauge field

In this section we explain how coupling a SUSY theory to a constant background gauge field for the maximal $U(1)$ subgroups of the R-symmetry, allows to preserve some amount of SUSY when imposing SUSY breaking boundary conditions on a spatial circle. We first

look at the illustrative example $\mathcal{N} = 1$ SYM in four dimensions and subsequently move to the maximally SUSY theory, which is the main objective of this note.

5.2.1 Toy example: $\mathcal{N} = 1$ SYM on $\mathbb{R}^{2,1} \times S^1$

Consider the $\mathcal{N} = 1$ SYM theory in 4D with some gauge group G . The $\mathcal{N} = 1$ vector multiplet consists of the gauge field A_μ and its superpartner gaugino, a Weyl fermion λ transforming in the adjoint representation. The Lagrangian of the theory is

$$S_{\mathcal{N}=1} = \frac{1}{g_{\text{YM}}^2} \int d^4x \text{Tr} \left(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i\lambda\sigma^\mu D_\mu \bar{\lambda} \right), \quad (5.2.1)$$

where we define the gauge covariant derivative as $D_\mu \equiv \partial_\mu - i[A_\mu, \cdot]$. We now compactify the theory on a circle of radius R along the $x^3 \equiv \phi$ direction, identifying points under the shift,

$$\phi \sim \phi + 2\pi R, \quad (5.2.2)$$

with anti-periodic boundary conditions for λ and periodic for A_μ . This yields mode expansions for the fields around the circle,

$$A_\mu(x^j, \phi) = \sum_{n \in \mathbb{Z}} e^{i\frac{n}{R}\phi} \left(A_i^{(n)}(x^j), \Theta^{(n)}(x^j) \right), \quad \lambda(x^j, \phi) = \sum_{n \in \mathbb{Z}} e^{i\frac{n}{R}(n+\frac{1}{2})\phi} \lambda^{(n)}(x^j), \quad (5.2.3)$$

where $\Theta^{(n)}$ are the Kaluza-Klein (KK) modes of A_ϕ , appearing as scalars in the 3D effective theory, and $i, j = t, x^1, x^2$, the (2+1) Minkowski coordinates. The fermion modes thus acquire effective masses

$$m_{n,\lambda} = \left| n + \frac{1}{2} \right| R^{-1}, \quad n \in \mathbb{Z}. \quad (5.2.4)$$

Reducing on the circle, the 3D effective theory has a massless gauge field $A_i^{(0)}$ and a classically massless 3D scalar $\Theta^{(0)}$, with gauge and Yukawa interactions respectively to fermion modes which are all massive, thus breaking SUSY.

The splitting of the bosonic and fermionic perturbative spectrum can potentially be undone by deforming the Lagrangian with a constant background $U(1)$ gauge field $\mathcal{A}_\mu$ with only non-vanishing component along ϕ , and which couples only to the fermionic states. This can of course be viewed as a background field for the $U(1)_R$ symmetry of the theory, which is anomalous quantum mechanically. Introducing the background gauge potential is achieved by the replacement

$$D_\mu \rightarrow \mathcal{D}_\mu = D_\mu - iq\mathcal{A}_\mu, \quad \mathcal{A}_\mu = Q\delta_\mu^\phi, \quad (5.2.5)$$

where q is the R-charge of λ , and the holonomy of the background gauge field around the S^1 is,

$$\Phi_{\mathcal{R}} \equiv \oint_{S^1} d\phi \mathcal{A}_\phi = 2\pi R Q. \quad (5.2.6)$$

For the gaugino, $q = +1$, and hence the shifted covariant derivative leads to shifted masses for the fermion modes in the 3D effective theory,

$$m_{n,\lambda} \rightarrow \left| \frac{n}{R} + \frac{1}{2R} - Q \right|. \quad (5.2.7)$$

The background field induces a spectral flow, and by setting $Q = 1/2R$ we restore the Bose-Fermi degeneracy of the perturbative spectrum. In this situation the Wilson line around the S^1 is $\Phi_{\mathcal{R}} = \pi$. The massless perturbative modes correspond to the four supercharge, $\mathcal{N} = 2$ SYM multiplet in 3D. Of course, in this case the full quantum theory has an anomaly for the R-symmetry which one must worry about. In particular, the effect of the background gauge field can be absorbed in the field redefinition

$$\tilde{\lambda} = e^{-iq\mathcal{A}_\phi\phi}\lambda, \quad (5.2.8)$$

where choosing anti-periodic boundary conditions for λ leads to periodic boundary conditions for $\tilde{\lambda}$, but this is then accompanied by a position (ϕ -)dependent θ -term in the 4D Lagrangian due to the anomaly. In a theory with a non-anomalous R-symmetry, we can expect an operation analogous to the above to lead to a supersymmetric effective 3D description.

5.2.2 $\mathcal{N} = 4$ SYM on $\mathbb{R}^{2,1} \times S^1$

Let us now consider the $SU(N)$ $\mathcal{N} = 4$ SYM theory in (3+1)-dimensions. The field content consists of a gauge field A_μ , four Weyl fermions $\{\lambda_a\}$ ($a = 1, \dots, 4$) and six real scalars $\{X^I\}$ ($I = 1, \dots, 6$), all transforming in the adjoint representation of $SU(N)$. The theory is invariant under the $SO(6)$ R-symmetry, under which the scalars transform in the vector representation or **6** of $SO(6)$, while the fermions transform in the spinor representation i.e. in the fundamental representation **4** of $SU(4)$.

The $SO(6) \sim SU(4)$ R-symmetry has $SO(2)^3 \sim U(1)^3$ as its maximal abelian subgroup. We pick a basis in which the action of each $SO(2)$ corresponds to a rotation on a specific pair of scalars, which can accordingly be grouped into 3 complex fields,

$$Y_1 = X^1 + iX^2, \quad Y_2 = X^3 + iX^4, \quad Y_3 = X^5 + iX^6. \quad (5.2.9)$$

Each $U(1)$ then acts on one of the complex fields $\{Y_a\}$ as a phase rotation. Assembling the fields and their complex conjugates into the 6-vector,

$$\vec{Y} = (Y_1, Y_1^*, Y_2, Y_2^*, Y_3, Y_3^*), \quad (5.2.10)$$

the generators in the vector representation of the three $U(1)$ subgroups are given by¹

$$\mathcal{R}_1^6 = \text{diag}(1, -1, 0, 0, 0, 0), \quad \mathcal{R}_2^6 = \text{diag}(0, 0, 1, -1, 0, 0), \quad \mathcal{R}_3^6 = \text{diag}(0, 0, 0, 0, 1, -1), \quad (5.2.11)$$

while the generators in the spinor representation are²

$$\mathcal{R}_1^4 = \frac{1}{2} \text{diag}(1, -1, -1, 1), \quad \mathcal{R}_2^4 = \frac{1}{2} \text{diag}(-1, 1, -1, 1), \quad \mathcal{R}_3^4 = \frac{1}{2} \text{diag}(-1, -1, 1, 1). \quad (5.2.12)$$

We take the theory to be compactified on a circle as before with SUSY breaking boundary conditions, so we have the same mode expansions for the gauge field components $A_i(x^j, \phi)$ and the effective 3D scalar $\Theta(x^j, \phi)$ as in (5.2.3). In addition, the four fermions $\{\lambda_a\}$ and six scalars $\{X^I\}$ have Fourier expansions with half-integer and integer modings, respectively:

$$X^I = \sum_n e^{i \frac{n}{R} \phi} X^{I(n)}(x^i), \quad \lambda_a = \sum_n e^{i \frac{n}{R} (n + \frac{1}{2}) \phi} \lambda_a^{(n)}(x^i). \quad (5.2.13)$$

We now couple the theory to a background gauge field $\mathcal{A}_\mu$ for the diagonal combination of the three $U(1)$ generators discussed above, with non-vanishing component along the compact ϕ coordinate, accordingly modifying the gauge-covariant derivatives as,

$$D_\mu \rightarrow \mathcal{D}_\mu = D_\mu - i \mathcal{R}_{\text{diag}} \mathcal{A}_\mu, \quad \mathcal{A}_\mu = Q \delta_\mu^\phi. \quad (5.2.14)$$

Here $\mathcal{R}_{\text{diag}}$ is the generator of the diagonal $U(1)$ symmetry and the generators are normalised so the complex scalars have unit charge under phase rotations,

$$\mathcal{R}_{\text{diag}}^6 = \sum_{n=1}^3 \mathcal{R}_n^6 = \text{diag}(1, -1, 1, -1, 1, -1), \quad (5.2.15)$$

while for the fermions λ (the charge of $\bar{\lambda}$ has the opposite sign)

$$\mathcal{R}_{\text{diag}}^4 = \sum_{n=1}^3 \mathcal{R}_n^4 = \frac{1}{2} \text{diag}(-1, -1, -1, 3). \quad (5.2.16)$$

The shifted covariant derivatives for each of the fields now imply the following set of effective masses for the charged fields, namely the 3D fermions,

$$m_{n, \lambda_4} = \left| \frac{n}{R} + \frac{1}{2R} - \frac{3}{2}Q \right|, \quad m_{n, \lambda_{\tilde{a}}} = \left| \frac{n}{R} + \frac{1}{2R} + \frac{1}{2}Q \right|, \quad \tilde{a} = 1, 2, 3. \quad (5.2.17)$$

and scalars,

$$m_{n, Y_{\tilde{a}}} = \left| \frac{n}{R} - Q \right|, \quad \tilde{a} = 1, 2, 3, \quad n \in \mathbb{Z}. \quad (5.2.18)$$

¹In the real scalar frame, the infinitesimal transformation generated by, for example, $\mathcal{R}_1^6$ is $\delta X^1 = -\theta X^2$, $\delta X^2 = \theta X^1$, with θ the infinitesimal parameter of the transformation.

²Under $\mathcal{R}_1^4$, $\lambda_1 \rightarrow e^{i \frac{\theta}{2}} \lambda_1$, with θ the same parameter for the scalar transformation. The factor of $1/2$ indicates the spinorial character under the R-symmetry.

We observe that the antiperiodic fermionic field λ_4 can now have a zero mode precisely when

$$Q = \frac{1}{3R}, \quad (5.2.19)$$

and can be identified as the gaugino for the 3D gauge multiplet. At this value it also follows immediately that the spectra of modes of the three complex scalars $\{Y_{\tilde{a}}\}$ and their fermionic partners $\{\lambda_{\tilde{a}}\}$ are *degenerate* and *massive*.

So the perturbative spectrum is supersymmetric. Unlike the toy example discussed previously, the R-symmetry is non-anomalous. The holonomy at the supersymmetric point is

$$\Phi_{\mathcal{R}} = \oint_{S^1} d\phi \mathcal{A}_{\phi} = 2\pi R Q = \frac{2\pi}{3}. \quad (5.2.20)$$

The effect of the background gauge field can also be packaged as ϕ -dependent phases for each of the fields,

$$\eta = e^{-i\frac{3}{2}Q\phi}\lambda_4, \quad \eta_{\tilde{a}} = e^{\frac{i}{2}Q\phi}\lambda_{\tilde{a}}, \quad Z_{\tilde{a}} = e^{-iQ\phi}Y_{\tilde{a}}. \quad (5.2.21)$$

With anti-periodic boundary conditions for all the fermions and periodic for $Y_{\tilde{a}}$ the mode expansions for the redefined fields at the supersymmetric point are,

$$\begin{aligned} \eta &= \sum_n e^{i\frac{n}{R}\phi} \eta^{(n)}(x^i), \\ \eta_{\tilde{a}} &= \sum_n e^{i\frac{(n+\frac{2}{3})}{R}\phi} \eta_{\tilde{a}}^{(n)}(x^i), \\ Z_{\tilde{a}} &= \sum_n e^{i\frac{(n-\frac{1}{3})}{R}\phi} Z_{\tilde{a}}^{(n)}(x^i). \end{aligned} \quad (5.2.22)$$

In this picture we have one periodic fermion, identified as the gaugino, and the rest of the fermions and the scalars have non-trivial winding around the circle, such that we only have periodicity under $\phi \rightarrow \phi + 6\pi i$ shifts.

5.2.3 4D $\mathcal{N} = 1$ picture

It is useful to revisit the argument above in the 4D $\mathcal{N} = 1$ language which is natural in this setting. The $\mathcal{N} = 4$ theory can be viewed as an $\mathcal{N} = 1$ vector multiplet coupled to three adjoint chiral multiplets $\{Y_{\tilde{a}}\}$ where we denote the chiral multiplets with the same symbol as their lowest scalar component, along with the superpotential,

$$W = \frac{1}{g_{\text{YM}}^2} \text{Tr} (Y_1[Y_2, Y_3]). \quad (5.2.23)$$

In this language, the $\mathcal{N} = 1$ theory has a manifest $U(1)_R$ symmetry under which the gaugino λ_4 has charge +1, and the superpotential transforms with charge +2. The two other abelian R-symmetries of $\mathcal{N} = 4$ appear as additional independent global $U(1)$ symmetries which

leave W invariant, corresponding to simultaneous and opposite phase rotations of a pair of the three chiral multiplets. If we choose $U(1)_R$ to act democratically on the three multiplets, they each must have R -charge $+\frac{2}{3}$ in order for W to have R -charge 2, and the corresponding fermions have R -charge $-\frac{1}{3}$. Notice that these charge assignments are, up to an overall rescaling of $\frac{2}{3}$, precisely the assignments in (5.2.15) and (5.2.16).

Introducing a background gauge field, $\hat{\mathcal{A}}_\mu = \frac{1}{2R} \delta_\mu^\phi$, for the R-symmetry kills the half-integer moding of the gaugino λ_4 , yields the same spectra³ and phase redefinitions for the chiral multiplets as discussed above.

Note that we can reach the same supersymmetric point by compactifying 4D $\mathcal{N} = 4$ SYM on the S^1 with SUSY *preserving* boundary conditions and subsequently turning on a background gauge field for a global $U(1)$ global symmetry which is *not* an R-symmetry. Consider the global $U(1)$ which acts on the chiral multiplets as

$$Y_{1,2} \rightarrow Y_{1,2} e^{2i\alpha/3}, \quad Y_3 \rightarrow Y_3 e^{-4i\alpha/3} \quad (5.2.24)$$

leaving the superpotential invariant. Picking a background gauge field $\hat{\mathcal{A}}_\mu = \delta_\mu^\phi/2R$ for this $U(1)$ yields the same theory we obtained above with degenerate masses for all matter fields and their harmonics.

Of the four sets of SUSY transformations, the one that is manifest in the $\mathcal{N} = 1$ picture respects the natural pairing that follows from the boundary conditions (5.2.22). Indeed, the SUSY transformations are compatible with the boundary conditions following from the phase redefinitions (5.2.22). This can be seen explicitly from the boundary term obtained from the $\mathcal{N} = 1$ SUSY variation⁴:

$$\begin{aligned} \delta_\epsilon S &= \int_{\mathbb{R}^{2,1} \times S^1} d^4x \quad \partial_\mu \text{Tr} \left(D^\mu (Z^{\tilde{a}})^\dagger \epsilon \eta_{\tilde{a}} - D_\nu (Z^{\tilde{a}})^\dagger \epsilon \sigma^\nu \bar{\sigma}^\mu \eta_{\tilde{a}} \right) \\ &= \int_{\mathbb{R}^{2,1}} d^3x \text{Tr} \left(D^\mu (Z^{\tilde{a}})^\dagger \epsilon \eta_{\tilde{a}} - D_\nu (Z^{\tilde{a}})^\dagger \epsilon \sigma^\nu \bar{\sigma}^\mu \eta_{\tilde{a}} \right) \Big|_{\phi=0}^{\phi=2\pi} = 0. \end{aligned} \quad (5.2.25)$$

5.2.4 Connection to refined index

On $S^3 \times S^1$, the procedure described above mirrors the definition of the refined Witten index [142] with complexified chemical potentials (ω_1, ω_2) which can be thought of as complex structure parameters on $S^1 \times S^3$. The refined index for $\mathcal{N} = 4$ SYM is only periodic under shifts of $\omega_{1,2} \rightarrow \omega_{1,2} + 6\pi i$ and lives on a three-sheeted cover of the space of complex

³The rescaled charge assignment means that $\hat{\mathcal{A}}_\mu$ and $\mathcal{A}_\mu$ (defined in (5.2.19)), are related as $\mathcal{A}_\phi = \frac{2}{3} \hat{\mathcal{A}}_\phi$ so the holonomy of the background gauge field, and consequently, the spectrum is indeed the same as encountered previously.

⁴Note that for anti-periodic boundary conditions *without* the background gauge field, this boundary term would not vanish, signalling the breaking of SUSY.

structures. In the notation of [142], we are setting $\omega_2 \rightarrow 0$ and $\omega_1 \rightarrow 2\pi i$ which can be viewed as a Cardy-like limit on the second sheet, so that the partition function takes the form

$$Z_{S^3 \times S^1} = \text{Tr} e^{-\{Q, Q^\dagger\} + i\pi \mathcal{R}}, \quad (5.2.26)$$

where fermions are antiperiodic around the S^1 and $\mathcal{R}$ the R -charge of the states⁵.

5.2.5 Effective 3D $\mathcal{N} = 2$ SYM with real masses

Masses obtained by turning on background gauge fields for global $U(1)$'s correspond to real mass deformations of 3D $\mathcal{N} = 2$ theories [142, 143]. The lightest KK harmonics in the setup described above fill out the massless $\mathcal{N} = 2$ vector multiplet coupled to three adjoint chiral multiplets with identical real masses

$$m_1 = m_2 = m_3 = \frac{1}{3R}. \quad (5.2.27)$$

The higher harmonics are however important and their presence has a nontrivial effect on the IR physics as argued below.

The real scalar $\Theta^{(0)}$ in the 3D gauge multiplet is massless and can have a VEV classically. We can always use gauge transformations to diagonalise $\Theta^{(0)}$ so that,

$$\Theta^{(0)} = \text{diag}(a_1, a_2, \dots, a_N), \quad (5.2.28)$$

where the $\{a_i\}$ are compact scalars, with identification under shifts⁶ by multiples of $1/R$,

$$a_i \sim a_i + \frac{1}{R}, \quad 2\pi R \sum_{i=1}^N a_i = 0 \bmod 2\pi, \quad (5.2.29)$$

as such a shift can be absorbed in a relabelling of the Kaluza-Klein harmonics from the compact direction [144]. A VEV for $\Theta^{(0)}$ thus breaks $SU(N)$ to $U(1)^{N-1}$, assuming all $Y_{\tilde{a}} = 0$, and there is a classical Coulomb branch moduli space.

Gapped vacuum: $SU(N)_N$ $\mathcal{N} = 2$ SYM

Near the origin of the classical Coulomb branch where $\Theta^{(0)} = 0$, the $SU(N)$ gauge symmetry is unbroken and fluctuations of all chiral multiplets are massive. We might then expect that the low energy theory is effectively pure $\mathcal{N} = 2$ SYM in 3D which has a non-perturbatively generated superpotential [143, 144] and potentially runaway vacua in the

⁵A chiral multiplet with R -charge $\mathcal{R}_b$ contains a fermion with charge $\mathcal{R}_f = \mathcal{R}_b - 1$. This shift cancels the half-integer moding of the fermions, precisely at the special value of the complexified chemical potential in (5.2.26).

⁶The normalisation of the shift is fixed by the definition $\Theta^{(0)} = \oint_{S^1} A_\phi d\phi / 2\pi R$.

decoupling limit for massive matter. However, as argued in [142], the physics is subtler. The twist with the R-symmetry yields a spectrum for the fermions in the chiral multiplets, $m_{n,\lambda_{\tilde{a}}} = |n + \frac{2}{3}|\frac{1}{R}|$, $n \in \mathbb{Z}$, which is not vector-like. Integrating out all the fermion KK harmonics produces a Chern-Simons term with a level given formally by an alternating infinite sum. When regulated, the sum leaves behind a residual Chern-Simons interaction at level N [142, 145] for the massless $\mathcal{N} = 2$ vector multiplet. The vacuum is trivially gapped.

At the perturbative level, the $\mathcal{N} = 2$ vector multiplet (which includes the scalar $\Theta^{(0)}$) acquires a mass $\sim g_{\text{YM}}^2 N$, lifting the Coulomb branch. As discussed below, we expect N such vacua owing to spontaneous breaking of the $\mathbb{Z}_N$ symmetry for the theory on $\mathbb{R}^{2,1} \times S^1$.

Higgs branches and massless vacua

The theory potentially has additional interesting vacua, visible classically, where the gauge group is Higgsed i.e. the unbroken gauge group has rank $r < N - 1$. Such branches appear as a result of the scalars in the chiral multiplets acquiring non-commuting VEVs. At the roots of such branches, new light states appear for particular values of the 3D scalar $\Theta^{(0)}$. This follows upon examining the form of the 3D scalar potential for $\Theta^{(0)}$ and the chiral multiplets,

$$V(\Theta^{(0)}, Y_{\tilde{a}}) = \frac{1}{g_{\text{YM}}^2} \sum_{\tilde{a}=1}^3 \sum_n \text{Tr} \left| [\Theta^0, Y_{\tilde{a}}^{(n)}] + \frac{1}{R} \left(\frac{1}{3} - n \right) Y_{\tilde{a}}^{(n)} \right|^2 + \frac{1}{g_{\text{YM}}^2} \oint_{S^1} \text{Tr} \left(|[Y_1, Y_2]|^2 + |[Y_1, Y_3]|^2 + |[Y_2, Y_3]|^2 \right), \quad (5.2.30)$$

where, in the first term, we have made explicit the sum over the KK modes of the chiral multiplet scalars, $Y_{\tilde{a}} = \sum_n Y_{\tilde{a}}^{(n)} e^{in\phi/R}$.

Case $N = 3$: Keeping in mind that eigenvalues of $2\pi R \Theta^{(0)}$ must lie in the interval $[-\pi, \pi]$, we point out that something interesting happens for $SU(3)$ gauge group, when the VEV takes the form,

$$2\pi R \Theta^{(0)} = \text{diag} \left(\frac{2\pi}{3}, 0, -\frac{2\pi}{3} \right) \equiv \frac{2\pi}{3} J_3, \quad (5.2.31)$$

where $J_3 = \text{diag}(1, 0, -1)$. For this choice, there is a precise cancellation of the real mass for some of the fluctuations with $n = 0$ or 1 in $Y_{\tilde{a}}^{(n)}$. Each of the three $Y_{\tilde{a}}$ yield three massless chiral multiplets at this point. In fact, it is fairly easy to show that the scalar potential vanishes with the following VEVs for both the $n = 0$ and $n = 1$ harmonics of the

scalar fields,

$$Y_{\tilde{a}}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ Y_{\tilde{a},21} & 0 & 0 \\ 0 & Y_{\tilde{a},32} & 0 \end{pmatrix} \quad Y_{\tilde{a}}^{(1)} = \begin{pmatrix} 0 & 0 & Y_{\tilde{a},13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.2.32)$$

when the VEVs are constrained to satisfy

$$Y_{2,ij} = \gamma Y_{1,ij} \quad Y_{3,ij} = \tilde{\gamma} Y_{1,ij}, \quad \gamma, \tilde{\gamma} \in \mathbb{C}, \quad (5.2.33)$$

and the bifundamental labels $(ij) = (21), (32)$ or (13) , matching the non-zero entries in (5.2.36). In the $SU(3)$ theory, generic non-zero VEVs of this type completely Higgs the gauge group. At the origin of this Higgs branch $SU(3)$ is broken to $U(1) \times U(1)$. The matrix of VEVs $Y_{\tilde{a}} = Y_{\tilde{a}}^{(0)} + e^{i\phi} Y_{\tilde{a}}^{(1)}$ has eigenvalues proportional to the cube roots of unity,

$$\text{spec}(Y) = (Y_{13}Y_{32}Y_{21})^{1/3} e^{i\phi/3R} \{1, \omega, \omega^2\}, \quad Y^3 = \mathbf{1}_{3 \times 3} (Y_{13}Y_{32}Y_{21}) e^{i\phi/R}. \quad (5.2.34)$$

Here we have omitted the subscript $\tilde{a}$ for clarity. Relatedly, single trace gauge-invariant operators with non-vanishing classical VEVs are at least cubic in the $\{Y_{\tilde{a}}\}$, e.g. $\text{Tr } Y_{\tilde{a}}^3$, $\text{Tr } Y_1 Y_2 Y_3$ etc. In general, traces of monomials of $Y_{\tilde{a}}$ are non-vanishing only when the number of terms is a multiple of 3. This indicates that states which are not left invariant by the simultaneous phase rotations and shift of the ϕ coordinate

$$Y_{\tilde{a}} \rightarrow e^{-2\pi i/3} Y_{\tilde{a}}, \quad \phi \rightarrow \phi + 2\pi R, \quad (5.2.35)$$

are projected out.

A natural interpretation of the VEVs, (5.2.32) and (5.2.34), as three dimensional matrices on the Higgs branch, is in terms of the locations of whole D-branes, which can fractionate at the origin of the Higgs branch [146].

General N : There are two ways of generalising the $N = 3$ example.

First, consider the theory with $N = 3k$, $k \in \mathbb{Z}$, where the $Y_{\tilde{a},ij}$ are now $k \times k$ blocks and

$$2\pi R \Theta^{(0)} = \frac{2\pi}{3} J_3 \otimes \mathbf{1}_{k \times k}. \quad (5.2.36)$$

At the root of this Higgs branch ($\mathbf{H}_1$), the massless states constitute a 3D $\mathcal{N} = 2$, $S(U(k) \times U(k) \times U(k))$ quiver gauge theory with 3 sets of massless bifundamental chiral multiplets $Y_{\tilde{a},ij}$, ($\tilde{a} = 1, 2, 3$), for the pair of nodes (ij) in the quiver.

Exploring the Higgs branch with bifundamental VEVs proportional to the identity, $Y_{\tilde{a},ij} = y_{\tilde{a}} \mathbf{1}_{k \times k}$, only the diagonal $SU(k)$ $\mathcal{N} = 2$ gauge multiplet is unbroken, and we find precisely 3 massless chiral multiplets transforming in the adjoint representation in the low

energy theory. Since KK harmonics of the massless fields on the Higgs branch have integer modings no CS terms will be generated, and we expect this theory to then flow to the 3D $\mathcal{N} = 8$ SCFT in the IR. When N is not a multiple of 3, i.e. when $N = 3k + 1$ or $3k + 2$, we can still consider classical vacua which consist of k blocks of the kind discussed above, with the remaining entries set to zero. Such vacua would have an additional unbroken $U(1)$ or $U(2)$ factor which we expect to get gapped due to induced Chern-Simons terms.

There is a second generalisation ($\mathbf{H}_2$), of the Higgs vacuum seen above for the $SU(3)$ gauge group. Consider again the case with $N = 3k$, but with the real scalar VEV of the form

$$2\pi R\Theta^{(0)} = \frac{2\pi}{3k} \text{diag} \left(\frac{3k-1}{2}, \frac{3k-3}{2}, \dots, -\frac{3k-1}{2} \right). \quad (5.2.37)$$

For $N = 3k + 1$ or $N = 3k + 2$ we add one or two vanishing entries, respectively, to (5.2.37). In general we find $3k$ massless perturbative modes for each member of a chiral multiplet $Y_{\tilde{a}}$. $2k$ of these arise from off-diagonal entries $\sim (Y_{\tilde{a}}^{(0)})_{k+i,i}$ for $1 \leq i \leq 2k$. The remaining massless fluctuations are those of the $n = 1$ harmonic $\sim (Y_{\tilde{a}}^{(1)})_{i,i+2k}$ with $1 \leq i \leq k$.

Matrices with these entries non-vanishing, naturally split into k three-dimensional irreducible blocks which each have eigenvalues proportional to the cube roots of unity. These correspond to k whole D-branes exploring the Higgs branch.

5.2.6 Polyakov loop

The Wilson loop of the $SU(N)$ gauge field around the S^1 is a natural gauge-invariant observable in the compactified theory. We will refer to this as the ‘‘Polyakov loop’’, in a slight abuse of terminology since the Polyakov loop usually denotes the Wilson loop around the thermal circle⁷:

$$P = \frac{1}{N} \text{Tr } U, \quad U = \exp \left(i \oint_{S^1} A_{\phi} d\phi \right), \quad (5.2.38)$$

with expectation value,

$$\langle P \rangle = \frac{1}{N} \langle \text{Tr } e^{2\pi i R \Theta^{(0)}} \rangle. \quad (5.2.39)$$

The Polyakov loop is an order parameter for $\mathbb{Z}_N$ center symmetry of the theory on $\mathbb{R}^{1,2} \times S^1$. In the classically trivial vacuum where $\Theta^{(0)} = 0$, the Polyakov loop is non-vanishing and spontaneous breaking of centre symmetry then implies N distinct quantum vacua.

When $N = 3k$, we can consider vacua where the VEVs consist of k copies of the three

⁷In the Euclidean thermal setting on $\mathbb{R}^3 \times S^1$, the Polyakov loop will have a non-zero expectation value which cannot be computed analytically, although it can be computed holographically [75]. Consistent weak coupling [147] and even zero coupling calculations [148] can be performed on $S^3 \times S^1$ at large- N revealing competing phases with vanishing and non-vanishing expectation values of the Polyakov loop.

dimensional representations, (5.2.36), and then the Polyakov loop in this Higgs phase is:

$$\langle P \rangle_{\mathbf{H}_1} = 0, \quad N = 3k, \quad k \in \mathbb{Z}^+. \quad (5.2.40)$$

But since the eigenvalues of the holonomy matrix U lie clumped at the cube roots of unity, $\frac{1}{N} \langle \text{Tr} U^3 \rangle_{\mathbf{H}_1} = 1$. When $N = 3k + 1$ or $3k + 2$, the vacuum with k three dimensional blocks has additional unbroken gauge symmetries and $\langle P \rangle_{\mathbf{H}_1} = \mathcal{O}(1/N)$.

More interesting is the classical vacuum $\mathbf{H}_2$ with the arrangement (5.2.37). For $N = 3k$, the Polyakov loop and its multiply wound versions vanish identically in this vacuum,

$$\frac{1}{N} \langle \text{Tr} U^m \rangle_{\mathbf{H}_2} = 0, \quad 1 \leq m < 3k. \quad (5.2.41)$$

For $N = 3k + 1$ and $3k + 2$, there will be non-vanishing pieces $\sim 1/N$. In the large- N limit, this vacuum is $\mathbb{Z}_N$ invariant.

5.3 Holographic Description

The holographic dual of $\mathcal{N} = 4$ SYM compactified on a circle with SUSY breaking spin structure along with a background gauge field was studied in [135]. The key observation of [135] was that two competing non-singular, asymptotically $\text{AdS}_5 \times S^5$ dual backgrounds exist, one containing a cigar-like geometry (AdS soliton) with a shrinking circle and another where the circle always remains finite (quotiented Poincaré-AdS). Both backgrounds consist of an S^5 fibration over the circle direction. Importantly, the two geometries become degenerate when the asymptotic value of the background gauge field is tuned so that some amount of SUSY is preserved. As we will review below, it is the nontrivial fibration of the S^5 that allows to preserve some amount of SUSY.

5.3.1 SUSY AdS_5 Soliton and Poincaré-AdS quotient

AdS soliton: The supersymmetric AdS_5 soliton background in type IIB supergravity is specified by the following form for the metric and five-form flux⁸

$$\begin{aligned} ds^2 &= \frac{r^2}{\ell^2} (dx_{2,1}^2 + f(r)d\phi^2) + \frac{dr^2}{\frac{r^2}{\ell^2}f(r)} + \ell^2 \sum_{i=1}^3 [d\mu_i^2 + \mu_i^2 (d\phi_i + Q^3 \ell^4 \zeta(r) d\phi)^2], \\ F_5 &= G_5 + \star G_5, \\ G_5 &= -\frac{4}{\ell} \text{Vol}(\mathcal{M}_5) + Q^3 \ell^4 \sum_{i=1}^3 d(\mu_i^2) \wedge (d\phi_i + Q^3 \ell^4 \zeta(r) d\phi) \wedge dt \wedge dx_1 \wedge dx_2 \end{aligned} \quad (5.3.1)$$

⁸Our parametrisation of the solution differs from that of [135].

where μ_i satisfy $\sum_i \mu_i^2 = 1$, $\text{Vol}(\mathcal{M}_5) = \frac{r^3}{\ell^3} dt \wedge dx_1 \wedge dx_2 \wedge d\phi \wedge dr$ and

$$f(r) = 1 - \frac{Q^6 \ell^{12}}{r^6}, \quad \zeta(r) = \frac{1}{r^2} - \frac{1}{Q^2 \ell^4}, \quad \ell = (4\pi g_s N)^{1/4}. \quad (5.3.2)$$

Here ℓ is the AdS radius and that of the S^5 , and as usual there are N units of five-form flux through the S^5 . Q is a dimensionful constant with mass dimension $+1$. In this parametrisation, the three $U(1)$ abelian subgroups of the $SO(6)$ symmetry are manifest in the metric via the angular coordinates ϕ_i . The fibration does not introduce extra brane charges. This solution preserves four real supersymmetries.

The function $f(r)$ has one positive root at $r_0 = Q\ell^2$. In the limit $r \rightarrow r_0$, the ϕ cycle shrinks to zero size. In order for it to shrink smoothly the circumference $2\pi R$ of the circle is fixed in terms of Q ,

$$2\pi R = \frac{4\pi}{f'(r_0)} \frac{\ell^2}{r_0^2} \implies Q = \frac{1}{3R}. \quad (5.3.3)$$

This is precisely the field theory relation (5.2.19) between the magnitude of the background gauge field and the radius of the S^1 .

AdS quotient: For a given Q , there is another background with identical asymptopia, which can be viewed as a smooth quotient of $\text{AdS}_5 \times S^5$ in the presence of a Wilson line in the ϕ direction,

$$ds^2 = ds_{\text{AdS}_5}^2 + \ell^2 \sum_{i=1}^3 [d\mu_i^2 + \mu_i^2 (d\phi_i - Qd\phi)^2]. \quad (5.3.4)$$

The metric and five-form flux have the general form of (5.3.1) but with $f(r) = 1$ and $\zeta(r) = -1/Q^2 \ell^4$. The ten dimensional metric is a quotient of $\text{AdS}_5 \times S^5$ with the identifications,

$$(\phi, \tilde{\phi}_i) \sim (\phi + 2\pi R, \tilde{\phi}_i - 2\pi/3), \quad \tilde{\phi}_i \equiv \phi_i - Q\phi, \quad (5.3.5)$$

where the shift of the $\tilde{\phi}_i$ coordinates is by the background Wilson line $\Phi = 2\pi/3$ along the S^1 . Note that this quotienting precisely mirrors the dual field theory description (5.2.35) in the Higgs branch (5.2.32).

Therefore, both backgrounds above in the limit, $r \rightarrow +\infty$, are asymptotically $\text{AdS}_5 \times S^5$, with two key differences. First, the coordinate ϕ is compact with period $2\pi R$. The second difference is that in both cases there is a non-vanishing fibration of the S^5 over the field theory S^1 :

$$ds^2(S^5)|_{r \rightarrow \infty} = \ell^2 \sum_{i=1}^3 [d\mu_i^2 + \mu_i^2 (d\phi_i - Qd\phi)^2]. \quad (5.3.6)$$

This fibration cannot be undone by a gauge transformation of ϕ_i since the 1-form $\mathcal{A} = Qd\phi$ has a non-trivial holonomy around the S^1 . This is precisely the gauge field used to cancel

the SUSY breaking spin structure in the boundary field theory, namely $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,2} \times S^1$. Each of the three fibrations corresponds to deforming the boundary theory by three equal background gauge fields $\mathcal{A} = Qd\phi$ for the three $U(1)$ subgroups of the R-symmetry. The holonomy $\Phi_{\mathcal{R}}$ of this gauge field has to be tuned to $2\pi/3$ to preserve four supercharges. We briefly review how this works below.

5.3.2 SUSY Preservation in Supergravity

How the supersymmetric soliton preserves SUSY was already explained in [135] and [78]. We now schematically review the mechanism for the solution presented here. In type IIB supergravity with only the F_5 flux turned on, the condition for vanishing SUSY variations of the gravitino leads to the Killing spinor equation for the SUSY transformation parameter ϵ ,

$$\delta\psi_M = D_M\epsilon + \frac{i}{16}\not{F}_5\Gamma_M\epsilon = 0. \quad (5.3.7)$$

Here $\{x^M\}_{M=0,1\dots 9}$ are spacetime coordinates and $\not{F}_p = \frac{1}{p!}F_{a_1\dots a_p}\Gamma^{a_1\dots a_p}$ ($a = 0, \dots, 9$ are tangent space indices). In pure $\text{AdS}_5 \times S^5$ the solution is [149],

$$\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_{\text{AdS}_5} \otimes \epsilon_{S^5}, \quad (5.3.8)$$

where ϵ_{AdS_5} and ϵ_{S^5} are the Killing spinors on AdS_5 and S^5 respectively. The AdS_5 spinor is,

$$\epsilon_{\text{AdS}_5} = \sqrt{\frac{r}{\ell}}\epsilon_+ + \sqrt{\frac{r}{\ell}}\left(\frac{\ell}{r} + \frac{1}{\ell}x^\mu\Gamma_\mu\right)\epsilon_-, \quad (5.3.9)$$

where $\epsilon_\pm$ are constant spinors satisfying $\Gamma_{\hat{r}}\epsilon_\pm = \pm\epsilon_\pm$, so that each of them have two independent complex constants. Here, ϵ_+ corresponds to field theory Poincaré supercharges, while ϵ_- to super-conformal transformations. On the other hand the Killing spinor on the S^5 is of the form

$$\epsilon_{S^5} = e^{\frac{i}{2}(\phi_1+\phi_2+\phi_3)}\epsilon_1(\theta_1, \theta_2) + \sum_{i=1}^3 e^{-i\phi_i}e^{\frac{i}{2}(\phi_1+\phi_2+\phi_3)}\epsilon_{i+1}(\theta_1, \theta_2) \quad (5.3.10)$$

where $\epsilon_1, \dots, \epsilon_4$ are the four independent complex spinors and $\theta_{1,2}$ are the polar angles on the S^5 .

Periodically identifying the spatial coordinate $x^3 = \phi$ with $\phi \sim \phi + 2\pi R$, with R as in (5.2.2), and imposing anti-periodic boundary conditions for the Killing spinor along ϕ completely breaks SUSY, so there are no non-vanishing solutions.

We can now see how SUSY is preserved by including the constant part of the fibering function $\zeta(r)$ and setting $f(r) = 1$ in (5.3.2), whilst considering the identification $\phi \sim \phi + 2\pi R$. This is equivalent to coupling the theory on $\mathbb{R}^{2,1} \times S^1$ to the background gauge

field $\mathcal{A}$ and picking an ungapped IR phase (corresponding to the Poincaré-AdS quotient geometry). The inclusion of $\mathcal{A}$ makes the spinor charged under shifts of the ϕ coordinate which appear as phase shifts. One can obtain the resulting spinor by the replacement $\phi_i \rightarrow \phi_i - Q\phi$ in (5.3.10)

$$\epsilon_{S^5} = e^{-i\frac{3Q}{2}\phi} e^{\frac{i}{2}(\phi_1+\phi_2+\phi_3)} \epsilon_1(\theta_1, \theta_2) + e^{-i\frac{Q}{2}\phi} \sum_{i=1}^3 e^{-i\phi_i} e^{\frac{i}{2}(\phi_1+\phi_2+\phi_3)} \epsilon_{i+1}(\theta_1, \theta_2), \quad (5.3.11)$$

with $Q = 1/3R$. The coefficients of the spinors $\epsilon_2, \dots, \epsilon_4$ still do not satisfy anti-periodic boundary conditions around the ϕ , so these must vanish. Similarly the piece of the AdS_5 spinor that depends on ϵ_- in (5.3.9), contains the term $\phi \Gamma_\phi$ which does not respect boundary conditions under shifts of ϕ . The Killing spinor which does satisfy the boundary conditions is of the form

$$\epsilon = e^{-i\frac{3Q}{2}\phi} e^{\frac{i}{2}(\phi_1+\phi_2+\phi_3)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_+ \otimes \epsilon_1(\theta_1, \theta_2), \quad (5.3.12)$$

wherein we count four real independent spinors, hence the background preserves four supercharges, matching the field theoretic expectation from weak coupling.

In order to obtain the spinor in the supersymmetric soliton background which is dual to the gapped phase of $\mathcal{N} = 4$ SYM with a background gauge field on a circle, we modify the previous expression to have an r -dependant spinor in the AdS part $\epsilon_+ \rightarrow \epsilon_+(r)$. This solution also preserves four supercharges.

The fact that we can preserve SUSY by only introducing the constant part of the fibering gauge field is related to the fact that the mechanism by which the AdS-soliton preserves SUSY is very similar to a twisting procedure, as was explained in [78]. There is a twist between the R-symmetry with the spin structure on the shrinking circle ϕ , realised by the non-vanishing fibration of the S^5 over the S^1 , which allows for the existence of anti-periodic spinors. The r -dependent part of the fibration and the function $f(r)$ allow the space to end smoothly at $r = r_0$, and the one-form $Q^3 \ell^4 \zeta(r) d\phi$ to vanish smoothly at the end of the space.

5.3.3 Polyakov loop

The two supersymmetric backgrounds, namely the AdS_5 soliton and the (smooth) quotient of Poincaré-AdS, are distinguished by two main features. The first difference is that the ϕ -circle shrinks smoothly in the soliton background, whereas it remains finite sized for all r in the Poincaré-AdS quotient. The second, related difference is that the non-trivial fibration function $\zeta(r)$ in the AdS-soliton background leads to a normalisable mode for the 5D effective gauge field [135] which implies a non-zero, constant expectation value for the

ϕ -component of the R -current,

$$\text{AdS soliton :} \quad \langle J_\phi \rangle = \frac{\lambda}{27R^3}, \quad (5.3.13)$$

$$\text{Poincaré AdS quotient :} \quad \langle J_\phi \rangle = 0,$$

where $\lambda = g_{\text{YM}}^2 N \gg 1$ is the 't Hooft coupling of $\mathcal{N} = 4$ SYM. Both backgrounds yield vanishing expectation value for the QFT stress tensor. This is natural from the point of view of the dual field theory. Unlike ordinary masses, real masses through the background gauge field deformation naturally act as a source for the zero mode of the ϕ component of the $U(1)_R$ current, so we generically expect

$$\frac{1}{2\pi R} \left\langle \int_{S^1} d\phi J_R^\phi \right\rangle \neq 0. \quad (5.3.14)$$

The shrinking cycle in the AdS-soliton geometry means that we should be able to deduce a non-zero expectation value for the Polyakov-Maldacena loop [22], by wrapping a Euclidean string worldsheet on the cigar geometry [75]. We can always choose the worldsheet coordinates (τ, σ) such that the string embedding is,

$$r(\sigma) = \sigma, \quad \phi(\tau) = \tau, \quad (5.3.15)$$

with all other coordinates set to constant. The induced metric on the worldsheet is then

$$ds_{\text{ind}}^2 = \frac{dr^2}{\frac{r^2}{\ell^2} f(r)} + \left(\frac{r^2}{\ell^2} f(r) + \frac{r_0^6}{\ell^2} \zeta(r)^2 \right) d\phi^2, \quad (5.3.16)$$

and the on-shell worldsheet action (setting $\alpha' = 1$),

$$S_{\text{F1}} = \frac{1}{2\pi} \int_{r_0}^{+\infty} dr \int_0^{2\pi R} d\phi \sqrt{1 + \frac{r_0^6}{r^2 f(r)} \zeta(r)^2}. \quad (5.3.17)$$

Using (5.3.2), $r_0 = Q\ell^2$, $Q = 1/3R$ and the change of variables $u = r/r_0$, we find

$$S_{\text{F1}} = \sqrt{\lambda} QR \int_1^{+\infty} du \sqrt{1 + \frac{(u^2 - 1)^2}{(u^6 - 1)}}. \quad (5.3.18)$$

The integral is regular at $u = 1$, but divergent when $u \rightarrow +\infty$. As is standard in holography, the action of extended probes must be regulated by adding suitable boundary terms [150]. In effect, we regulate the integral by cutting it off at $u = \Lambda$, then subtracting the leading divergence, and taking the limit $\Lambda \rightarrow \infty$,

$$S_{\text{F1}}^{(\text{reg})} = \frac{\sqrt{\lambda}}{3} \lim_{\Lambda \rightarrow \infty} \left(\int_1^\Lambda du \sqrt{1 + \frac{(u^2 - 1)^2}{(u^6 - 1)}} - \Lambda \right) \approx -0.243 \sqrt{\lambda}, \quad (5.3.19)$$

valid at large 'tHooft coupling λ . The Polyakov-Maldacena loop is then $P = \exp(-S_{\text{F1}}^{\text{(reg)}})$. The numerical value of the holographic Polyakov loop is not so important. The main observation is that it is non-zero and therefore the $\mathbb{Z}_N$ symmetry associated to large gauge transformations around the ϕ circle is broken⁹. This further implies that there should be N vacua following identical arguments to those presented in the thermal context in [75]. The domain walls between adjacent vacua correspond to D1-branes along a noncompact spatial direction in the field theory. Such a D1-brane will want to minimise its tension and sit at the tip of the cigar where its tension is $T_{\text{D1}} = 2NQ^2/\sqrt{\lambda}$.

It is natural to identify this phase as the IR description of the classical vacuum of the compactified theory with unbroken $SU(N)$ gauge symmetry and a level N Chern-Simons term that trivially gaps the theory. This is consistent with the bulk picture where no additional fields corresponding to gauge theory condensates are turned on, apart from the zero mode of the R -current.

For the AdS quotient geometry, the ϕ -circle is non-shrinking and there is no finite action static string worldsheet configuration wrapping the circle and extended in the radial AdS direction. The Polyakov loop must therefore vanish. From the gauge theory viewpoint, the candidate for this vacuum is at the root of the Higgs branch $\mathbf{H}_2$, characterised by the VEVs (5.2.37), where the expectation value of the Polyakov loop is always vanishing in the large- N limit (5.2.41).

5.3.4 Coulomb branch probes

We expect that the Coulomb branch of $\mathcal{N} = 4$ SYM is completely lifted in the gauge theory due to real masses for the chiral multiplets, since this deformation only preserves four supercharges.

D3-brane in the AdS-soliton: To verify this we introduce a probe D3-brane extended in the field theory directions $x^\mu = (t, x^1, x^2, \phi)$. The induced metric on the D3-brane is

$$ds_{\text{D3}}^2 = \frac{r^2}{\ell^2} dx_{2,1}^2 + \left(\frac{r^2}{\ell^2} f(r) + \frac{r_0^6}{\ell^2} \zeta(r)^2 \right) d\phi^2, \quad \sqrt{-\det g_{\text{D3}}} = \frac{r^4}{\ell^4} \sqrt{f(r) + \frac{r_0^6}{r^2} \zeta(r)^2} \quad (5.3.20)$$

while the pullback of the C_4 potential is

$$C_4^* = - \left(\frac{r^4}{\ell^4} - \frac{r_0^4}{\ell^4} \right) dt \wedge dx_1 \wedge dx_2 \wedge d\phi. \quad (5.3.21)$$

Then, the action of the D3-brane probe at fixed radial coordinate has the form,

$$S_{\text{D3}} = T_{\text{D3}} \int d^4x \frac{r^4}{\ell^4} \left(\sqrt{\left(1 - \frac{r_0^2}{r^2}\right) \left(1 + \frac{2r_0^2}{r^2}\right)} - \left(1 - \frac{r_0^4}{r^4}\right) \right) \geq 0. \quad (5.3.22)$$

⁹Note that spontaneous breaking of this $\mathbb{Z}_N$ is not relevant for determining whether the effective 3D theory confines or not.

In the AdS_5 soliton geometry, the probe D3-branes are therefore attracted to the origin at $r = r_0$, and the Coulomb branch is lifted.

D3-brane in the AdS quotient: One may naively expect the D3-brane probe potential to vanish in the Poincaré AdS quotient geometry. However, this turns out not to be the case since the effect of the twisting is always present and the probe action in this background is

$$S_{\text{D3}} = T_{\text{D3}} \int d^4x \frac{r^4}{\ell^4} \left(\sqrt{1 + \frac{r_0^2}{r^2}} - 1 \right) \geq 0. \quad (5.3.23)$$

Probe D3-branes of the type considered above have a potential that drags them to the origin.

D3'-brane probe with moduli space

Now let us consider a D3-brane probe extended in $x^\mu = (t, x^1, x^2, \phi)$ but such that $\phi_i = Q\phi$, that is, the D3-brane wraps the three $U(1)$ subgroups of the R-symmetry directions. This is a particularly interesting configuration, as it effectively undoes the twisting procedure for both the soliton and quotiented Poincaré AdS backgrounds. The untwisting is immediately obvious in the latter case, but less trivial for the soliton background. The metric and four-form potential pulled back onto this D3-brane worldvolume,

$$ds_{\text{D3}'}^2 = \frac{r^2}{\ell^2} dx_{3,1}^2, \quad C_4^* = -\frac{r^4}{\ell^4} dt \wedge dx_1 \wedge dx_2 \wedge d\phi, \quad (5.3.24)$$

exhibit perfect cancellation so that there is a probe moduli space.

From our discussion of the dual field theory in section 5.2.5, we naturally expect such probe moduli spaces. Consider a classical vacuum of the $SU(N)$ theory where the gauge symmetry is broken to $SU(N-3) \times U(1)^3$, where the VEVs for $U(1)^3$ follow the arrangement (5.2.31) at the root of the Higgs branch. In the large- N limit, the vacuum is “mostly gapped” and the dynamics of the Higgs branch is seen in the probe limit alone. Analogous interpretation applies for the probe moduli space in the AdS quotient geometry.

5.4 Extension to general Dp-brane theories

It is possible to extend the twisted circle compactification procedure which preserves SUSY to different dimensions, particularly even dimensions. Consider maximally supersymmetric Yang-Mills theory in $2(l+1)$ dimensions, formulated on $\mathbb{R}^{1,2l} \times S^1$ with SUSY breaking boundary conditions for the fermions. The R-symmetry group of the theory is $SO(8-2l)$ with maximal abelian group $U(1)^\mathfrak{r}$, where $\mathfrak{r} = 4-l$, the rank of the R-symmetry group. In such cases we expect the background gauge field holonomy to be $\Phi_{\mathcal{R}} = 2\pi/\mathfrak{r}$. We limit ourselves to the holographic descriptions of corresponding field theories.

5.4.1 Maximally SUSY Yang-Mills in (1+1)

The background configuration of a supersymmetric soliton in the background produced by a large number of D1-branes, does not appear in previous works, to the best of our knowledge. We present this below via the twisting procedure,

$$\begin{aligned}
ds^2 &= \frac{r^3}{\ell^3} (-dt^2 + f(r)d\phi^2) + \frac{dr^2}{\frac{r^3}{\ell^3}f(r)} + \frac{\ell^3}{r} \left[\sum_{i=1}^4 d\mu_i^2 + \mu_i^2 (d\phi_i + Q^2 \ell^3 \zeta(r) d\phi)^2 \right], \\
F_3 &= \frac{6r^5}{\ell^6} dr \wedge dt \wedge d\phi - Q^2 \ell^3 \sum_{i=1}^4 d(\mu_i^2) \wedge (d\phi_i + Q^2 \ell^3 \zeta(r) d\phi) \wedge dt, \\
\Phi &= \frac{1}{2} \log \left(\frac{\ell^6}{r^6} \right),
\end{aligned} \tag{5.4.1}$$

where Φ is the dilaton, and μ_i satisfy $\sum_i \mu_i^2 = 1$,

$$f(r) = 1 - \frac{Q^4 \ell^{12}}{r^8}, \quad \zeta(r) = \frac{1}{r^2} - \frac{1}{Q \ell^3}, \tag{5.4.2}$$

and $\ell = (32\pi^2 N)^{1/6}$ such that there are N units of $F_7 = -\star F_3$ flux through the S^7 . This solution preserves four supercharges. As in the $\text{AdS}_5 \times S^5$ case, we have used a parametrisation of the S^7 that makes manifest the $U(1)^4$ maximal abelian subgroup via the coordinates ϕ_i . At $r_0 = \sqrt{Q \ell^3}$ the cycle ϕ shrinks to zero size. The period of ϕ is

$$2\pi R = \frac{\pi}{2Q} \Rightarrow R = \frac{1}{4Q}, \tag{5.4.3}$$

with R the radius of the S^1 at $r \rightarrow +\infty$. Asymptotically the fibration of the S^7 on the shrinking cycle does not vanish and cannot be undone by a gauge transformation. It corresponds to the introduction of a background gauge field for the diagonal $U(1)$ of the $U(1)^4$ Cartan subalgebra of the $SO(8)$ R-symmetry in the boundary field theory. The holonomy of this background field is $\Phi_{\mathcal{R}} = \pi/2$. We defer the discussion of the dual (0+1) dimensional effective four-supercharge theory to future work.

5.4.2 Maximally SUSY Yang-Mills in (5+1)

The background of the supersymmetric soliton in the D5-brane background was presented in [78] and will be reviewed on the next chapter of this thesis. Here we present the configuration in a set of variables more suited to the discussion in this paper, and where

the maximal abelian subgroup of the $SO(4)$ R-symmetry is manifest:

$$\begin{aligned}
ds^2 &= \frac{r}{\ell} (dx_{4,1}^2 + f(r)d\phi^2) + \frac{dr^2}{\ell f(r)} + r\ell \left[\sum_{i=1}^2 d\mu_i^2 + \mu_i^2 (d\phi_i + Q^4 \ell^5 \zeta(r) d\phi) \right], \\
F_7 &= -\frac{2r}{\ell^2} dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge d\phi \wedge dr \\
&\quad + Q^4 \ell^5 \sum_{i=1}^2 d(\mu_i)^2 \wedge (d\phi_i + Q^4 \ell^5 \zeta(r) d\phi) \wedge dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \\
\Phi &= \frac{1}{2} \log \left(\frac{r^2}{\ell^2} \right).
\end{aligned} \tag{5.4.4}$$

Here $\ell = \sqrt{N}$ such that there are N units of $F_3 = -\star F_7$ through the S^3 and

$$f(r) = 1 - \frac{Q^8 \ell^{12}}{r^4}, \quad \zeta(r) = \frac{1}{r^2} - \frac{1}{Q^4 \ell^6}. \tag{5.4.5}$$

As discussed in [78] the period of the shrinking cycle ϕ is independent of Q ,

$$2\pi R = \pi \ell. \tag{5.4.6}$$

This is traceable to the linear term in $rf(r)$ and the corresponding linear dilaton growth for large r , and that the effective theory on the D5-brane is UV-completed by Little String Theory. The value of Q is fixed by the requirement that we have non-vanishing Killing spinors satisfying the antiperiodic boundary conditions around the ϕ -circle.

The dual field theory is maximally SUSY Yang-Mills in 6D with $SU(N)$ gauge group, compactified on a circle with a SUSY preserving background gauge field. Curiously, and differently to the other examples, this solution actually preserves eight supercharges. This is because the dual field theory in the IR is five-dimensional, and minimal SUSY in 5D has eight Poincaré supercharges.

5.4.3 Odd Dimensional Maximally SUSY Yang-Mills and Their Duals

So far, we have only focussed attention on maximally SUSY gauge theories in even dimensions. Holographic duals to these theories, (5.3.1), (5.4.1) and (5.4.4) realising the twist, can be obtained by truncating the maximal $SO(8-2l)$ gauged SUGRA in $(2l+3)$ dimensions down to the minimal $U(1)$ gauged SUGRA¹⁰ (comprising of Einstein-Maxwell theory and possibly a dilaton, originating from the 10D dilaton), where the $U(1)$ gauge field in lower dimensions corresponds to the fibration of the sphere over the shrinking S^1 .

Let us turn to odd dimensional field theories. Compactifying the 5D gauge theory on a circle yields an additional compact scalar in the 4D effective theory. This scalar pairs

¹⁰This does not apply for the supersymmetric soliton in the D1-brane background, which was found by directly solving the 10D supergravity equations.

with the real scalar from the 5D minimal SUSY vector multiplet, and naturally sits in the $\mathcal{N} = 2$ vector multiplet in 4D. The four real scalars sit in the $\mathcal{N} = 2$ adjoint hypermultiplet. A background gauge field along the fifth compact dimension for a linear combination of $U(1)_J$ and $U(1)_R$ can be chosen to leave a massless $\mathcal{N} = 2$ vector multiplet in 4D while rendering the hypermultiplets massive in a way that continues to preserve SUSY. A similar story can play out in the compactification from 3D to 2D (but with four supercharges).

However, on the gauged supergravity side, in $2l + 2$ dimensions it may not be possible to consistently truncate the maximal gauged SUGRA to the Einstein-Maxwell-dilaton sector. In other words, it may not be possible to consider a supersymmetric soliton in the background of a large number of D2- or D4-branes with only the $U(1)$ gauge field turned on, corresponding to fibering the internal sphere over the S^1 . In such a situation, if one chooses to set higher degree p -form fields to zero, then scalars corresponding to breathing modes of the sphere cannot be set to the minimum of the potential, and vice versa. From this, we learn that when compactifying the theory on a circle with a SUSY breaking spin structure, in addition to a background gauge field for the diagonal $U(1)$ R-symmetry used to preserve SUSY, there must be extra operators or VEVs turned on, dual to the extra non-vanishing fields in the bulk.

5.5 Summary

Motivated by the gravity backgrounds found in [135], in this chapter we investigated the compactification of $\mathcal{N} = 4$ SYM on a circle with SUSY breaking boundary conditions, including a constant background gauge field for the maximal abelian subgroup of the R -symmetry. We understood how this procedure allows to preserve four supersymmetries, and how the IR physics of the 3D effective $\mathcal{N} = 2$ SYM is in accord with expectation from the holographic dual description in terms of the supersymmetric AdS soliton and the quotiented $\text{AdS}_5 \times S^5$ geometries. We also pointed out that it should be possible to extend this mechanism to maximally SUSY theories in other dimensions. For even dimensions, the corresponding dual gravity backgrounds can be readily obtained. For odd dimensional field theories however, we expect the corresponding dual geometries to be more complicated.

Chapter 6

Twisted Circle Compactification of Maximally SUSY Yang-Mills in 6D

This chapter is dedicated to the study of the supersymmetric soliton in the D5-brane background presented in Section 5.4.2. The dual field theory corresponds to a twisted circle compactification of 6D maximally SUSY Yang-Mills. The UV regime of this theory is defined in terms of a non-local theory, since the theory is UV completed in to $\mathcal{N} = (1, 1)$ Little String Theory.

We start Section 6.1 by presenting a more general version of the supersymmetric soliton discussed before. We also discuss how the twisted circle compactification works in this example, and finish the section by discussing aspects of dual field theory. Then, Section 6.2 is dedicated to the study of observables in this background. Wilson loops, 't Hooft loops and Entanglement Entropy are calculated, together with a quantity that gives an estimate of the number of degrees of freedom as a function of the energy (density of states). Masses for spin-two glueballs are calculated. Finally, in Section 6.3 we compare the twisted circle compactification presented here, with a the twisted compactification on a Riemann surface, highlighting their differences and similarities.

We note that this chapter contains very detailed derivations of the computation of different observables. Section 6.2.2 starts with a generic treatment of holographic Wilson loops, which provides the expression of the Length and Energy of the loop, as well as conditions to have regular behaviour and a stability criterion, in terms of the embedding functions. The main results are equations (6.2.35) and (6.2.34). Then Section 6.2.3 and Section 6.2.5 deal with similar aspects for 't Hooft loops and Entanglement Entropy. Then Section 6.2.6 deals with spin-two fluctuations. After a very detailed derivation of all the needed equations, the main results follow from (6.2.105) onwards.

6.1 The D5-brane Supersymmetric Soliton

In this section we present a Type IIB background describing a stack of D5 branes that, in a SUSY preserving fashion, wraps a circle direction, leading to a smooth manifold over the whole space.

To describe this system we use coordinates $[t, x_1, x_2, x_3, x_4, \varphi, r, \theta, \phi, \psi]$. We define the left invariant forms of $SU(2)$, in terms of the the coordinates $[\theta, \varphi, \psi]$ that parameterise a three sphere,

$$\begin{aligned}\omega_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, & \omega_2 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi, & \omega_3 &= d\psi + \cos \theta d\phi, \\ \text{Vol}(S^3) &= \frac{1}{8} \omega_1 \wedge \omega_2 \wedge \omega_3 = \frac{1}{8} \sin \theta d\theta \wedge d\phi \wedge d\psi.\end{aligned}\tag{6.1.1}$$

The range of the coordinates in the three sphere is $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. We set the constants $\alpha' = 1, g_s = 1$. In string frame the background reads,

$$\begin{aligned}ds_{st}^2 &= r \left[dx_{4,1}^2 + f_s(r) d\varphi^2 + \frac{N dr^2}{r^2 f_s(r)} + \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right) \right], \\ F_3 &= 2N \text{Vol}(S^3) + \sqrt{\frac{N}{2}} Q d(\zeta(r) \omega_3 \wedge d\varphi), \quad C_2 = \frac{N}{4} \psi \sin \theta d\theta \wedge d\phi + \sqrt{\frac{N}{2}} Q \zeta(r) \omega_3 \wedge d\varphi, \\ \Phi &= \log(r),\end{aligned}\tag{6.1.2}$$

the electric RR 7-form is

$$\begin{aligned}F_7 &= -\star F_3 = -dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge \left[2r dr \wedge d\varphi + Q \sqrt{\frac{N}{2}} \sin \theta d\theta \wedge d\phi \right] \\ C_6 &= dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge \left[r^2 d\varphi - Q \sqrt{\frac{N}{2}} \cos \theta d\phi \right].\end{aligned}\tag{6.1.3}$$

The background is defined in terms of the functions $f_s(r)$, $\zeta(r)$ and three parameters $(N, r_\pm)$ or (M, Q, m) ,

$$\begin{aligned}f_s(r) &= 1 - \frac{m}{r^2} - \frac{2Q^2}{r^4} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, & \zeta(r) &= \frac{1}{r^2} - \frac{1}{r_+^2}, \\ 2r_\pm^2 &= m \pm \sqrt{m^2 + 8Q^2}.\end{aligned}\tag{6.1.4}$$

To calculate the number of D5-branes, we integrate F_3 on the manifold $\Sigma_3 = [\theta, \phi, \psi]$. Using that

$$(2\pi)^{7-p} g_s \alpha'^{\frac{7-p}{2}} N_{Dp} = \int_{\Sigma_{8-p}} F_{8-p},\tag{6.1.5}$$

and setting $g_s = \alpha' = 1$ we find,

$$N_{D5} = \frac{1}{4\pi^2} \int_{\Sigma_3} F_3 = N.\tag{6.1.6}$$

6.1.1 Supersymmetry Analysis

This background is a solution to the equations of motion of Type IIB Supergravity

$$\begin{aligned}
\nabla^2 \Phi - \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{4} R - \frac{1}{8} |H_3|^2 &= 0, \\
d(e^{-2\Phi} \star H_3) &= -F_5 \wedge F_3 - F_1 \wedge F_7, \\
dF_5 - H_3 \wedge F_3 &= 0, \\
dF_7 - H_3 \wedge F_5 &= 0, \\
dF_9 - H_3 \wedge F_7 &= 0, \\
R_{\mu\nu} &= -2\nabla_\mu \nabla_\nu \Phi + \frac{1}{2} |H_3|_{\mu\nu}^2 + \frac{e^{2\Phi}}{2} \left(|F_1|_{\mu\nu}^2 + |F_3|_{\mu\nu}^2 + \frac{1}{2} |F_5|_{\mu\nu}^2 - \frac{1}{2} g_{\mu\nu} (|F_1|^2 + |F_3|^2) \right)
\end{aligned} \tag{6.1.7}$$

where the R-R fluxes are given in terms of the C_0 , C_2 and C_4 potentials by

$$F_1 = dC_0, \quad F_3 = dC_2 - C_0 \wedge H_3, \quad F_5 = dC_4 - C_2 \wedge H_3. \tag{6.1.8}$$

Also, $|F_p|_{\mu\nu}^2 = F_{\mu\nu_1 \dots \nu_{p-1}} F^{\nu_1 \dots \nu_{p-1}} / (p-1)!$, similarly for H_3 , and

$$F_7 = -\star F_3, \quad F_9 = \star F_1. \tag{6.1.9}$$

The equations of motion are complemented by the Bianchi identities

$$dF_1 = 0, \quad dF_3 - H_3 \wedge F_1 = 0. \tag{6.1.10}$$

Solutions of the purely bosonic sector of Type IIB has all the fermionic partners, the dilatino λ and gravitino Ψ_μ , set to zero. If we are interested in finding SUSY solutions, we need to be consistent with the fact that we turn off the fermions by asking for the SUSY variations of these fields to vanish. In string frame, the SUSY variations of the fermionic fields are [151],

$$\delta\lambda = \frac{1}{2} \left(\Gamma^\mu \partial_\mu \Phi + \frac{1}{2 \cdot 3!} H_{\mu\nu\lambda} \Gamma^{\mu\nu\lambda} \sigma^3 - e^\Phi \left(F_\mu \Gamma^\mu (i\sigma_2) + \frac{1}{2 \cdot 3!} F_{\mu\nu\lambda} \Gamma^{\mu\nu\lambda} \sigma^1 \right) \right) \epsilon, \tag{6.1.11}$$

$$\begin{aligned}
\delta\Psi_\mu &= \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \epsilon + \frac{1}{4 \cdot 2!} H_{\mu\nu\lambda} \Gamma^{\nu\lambda} \sigma^3 \epsilon \\
&\quad + \frac{e^\Phi}{8} \left(F_\nu \Gamma^\nu (i\sigma_2) + \frac{1}{3!} F_{\nu\lambda\rho} \Gamma^{\nu\lambda\rho} \sigma^1 + \frac{1}{2 \cdot 5!} F_{\nu\lambda\rho\sigma\tau} \Gamma^{\nu\lambda\rho\sigma\tau} (i\sigma_2) \right) \Gamma_\mu \epsilon.
\end{aligned} \tag{6.1.12}$$

Here ω_μ^{ab} is the spin connection of the 10D background, where the a, b indexes are flat space ones, and σ^1 , σ^2 and σ^3 are Pauli matrices. Also

$$\Gamma^{\mu_1 \dots \mu_p} = \Gamma^{[\mu_1 \dots \mu_p]} \tag{6.1.13}$$

Here ϵ is a 64 component spinor,

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad (6.1.14)$$

where both 32-component parts are left-handed.

SUSY Variations

For the parameter $m = 0$, it can be shown by operating with the SUSY variations in equations (6.1.11) and (6.1.12), that there are eight preserved spinors. Let us perform explicitly the computation of the Killing spinors by directly solving (6.1.11) and (6.1.12). For simplicity, let us start with the S-dual frame of (6.1.2) with $m = Q = 0$, that is, with the pure NS5-brane background. Then, we identify one of the field theory directions and impose anti-periodic boundary conditions for the Killing spinor, which leads to a vanishing one. However, as we reviewed on the previous chapter, introducing a background gauge field makes the Killing spinor charged under the circle, which allows to preserve some SUSY. Finally, we discuss the solution with non-vanishing parameters.

In what follows we use the following representation of the Dirac matrices

$$\begin{aligned} \Gamma^0 &= i\sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^1 &= \sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^2 &= -\mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^3 &= -\mathbb{I}_2 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^4 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^5 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3, \\ \Gamma^6 &= -\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^3, \\ \Gamma^7 &= -\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^2 \otimes \sigma^3, \\ \Gamma^8 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^1, \\ \Gamma^9 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma^2, \end{aligned} \quad (6.1.15)$$

where $(\sigma^1, \sigma^2, \sigma^3)$ are Pauli matrices and $\mathbb{I}_2$ is the 2×2 identity matrix.

The NS5-brane background configuration is given by

$$\begin{aligned} ds_{st}^2 &= dx_{5,1}^2 + \frac{Ndr^2}{r^2} + \frac{N}{4} (\omega_1^2 + \omega_2^2 + \omega_3^2), \\ H_3 &= 2N \text{Vol}(S^3), \\ \Phi &= -\log(r). \end{aligned} \quad (6.1.16)$$

Recalling that

$$\omega_1^2 + \omega_2^2 = d\theta^2 + \sin^2(\theta)d\phi^2, \quad \omega_3 = d\psi + \cos(\theta)d\phi, \quad (6.1.17)$$

it is convenient to choose the following vielbein basis

$$\begin{aligned} e^0 &= dt, \quad e^m = dx^m, \quad e^6 = \frac{\sqrt{N}}{r\sqrt{f_s(r)}} dr, \\ e^7 &= \frac{\sqrt{N}}{2} d\theta, \quad e^8 = \frac{\sqrt{N}}{2} \sin(\theta) d\phi, \quad e^9 = \frac{\sqrt{N}}{2} \omega_3, \end{aligned} \quad (6.1.18)$$

where $m = 1, \dots, 5$. From (6.1.11) and (6.1.12) we find the SUSY spinor to be

$$\epsilon_1 = \begin{pmatrix} c_1 e^{i\frac{\psi}{2}} + c_2 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ c_1 e^{i\frac{\psi}{2}} - c_2 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_3 e^{i\frac{\psi}{2}} + c_4 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ c_3 e^{i\frac{\psi}{2}} - c_4 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ c_5 e^{i\frac{\psi}{2}} + c_6 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ c_5 e^{i\frac{\psi}{2}} - c_6 e^{-i\frac{\psi}{2}} \\ c_7 e^{i\frac{\psi}{2}} + c_8 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ c_7 e^{i\frac{\psi}{2}} - c_8 e^{-i\frac{\psi}{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_9 e^{\frac{i}{2}(\theta+\phi)} + c_{10} e^{\frac{i}{2}(\theta-\phi)} \\ c_9 e^{\frac{i}{2}(-\theta+\phi)} - c_{10} e^{\frac{i}{2}(-\theta-\phi)} \\ 0 \\ 0 \\ c_{11} e^{\frac{i}{2}(\theta+\phi)} + c_{12} e^{\frac{i}{2}(\theta-\phi)} \\ c_{11} e^{\frac{i}{2}(-\theta+\phi)} - c_{12} e^{\frac{i}{2}(-\theta-\phi)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_{13} e^{\frac{i}{2}(\theta+\phi)} + c_{14} e^{\frac{i}{2}(\theta-\phi)} \\ c_{13} e^{\frac{i}{2}(-\theta+\phi)} - c_{14} e^{\frac{i}{2}(-\theta-\phi)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_{15} e^{\frac{i}{2}(\theta+\phi)} + c_{16} e^{\frac{i}{2}(\theta-\phi)} \\ c_{15} e^{\frac{i}{2}(-\theta+\phi)} - c_{16} e^{\frac{i}{2}(-\theta-\phi)} \\ 0 \end{pmatrix}, \quad (6.1.19)$$

which has 16 independent solutions, as expected.

If we take one of the field theory directions to be compact $x^5 \rightarrow \bar{\varphi}$ (with $\bar{\varphi} \sim \bar{\varphi} + 2\pi$) and take anti-periodic boundary conditions for the spinor in $\bar{\varphi}$, we see that none of the spinor components satisfy this boundary conditions, so that SUSY is completely broken.

There is a way of preserving some amount of SUSY: in the above parametrisation of the 3-sphere there is an explicit $U(1)$ isometry along ψ . In the previous chapter we discussed

that background gauge field that we introduce to preserve SUSY, should be one for the maximal abelian subgroup of the R -symmetry of the theory. For the D5-brane (or the NS5-brane) the R -symmetry group is $SO(4)$ which is realised by the 3-sphere in the geometry, which has two $U(1)$ subgroups, as seen from (5.4.4). In the parametrisation used in this chapter however, only the $U(1)$ direction ψ is related to the maximal abelian subgroup. Group theoretically, we are just using a different basis of the generators, under which the simultaneous action of the two $U(1)$ subgroups is realised by only one generator.

By replacing

$$\psi \rightarrow \psi + \alpha \bar{\varphi}, \quad (6.1.20)$$

the spinor components that include ψ will be shifted to $e^{\pm i \frac{\psi}{2}} \rightarrow e^{\pm i \frac{1}{2}(\psi + \alpha \bar{\varphi})}$. Note that this is not a coordinate transformation: we are allowing α to take any possible values, not necessarily the one that respects the periodicity of ψ . This "shift" by $\bar{\varphi}$ corresponds to the insertion of a Wilson Loop in the boundary theory.

Imposing anti-periodic boundary conditions in $\bar{\varphi}$ fixes $\alpha = \pm 1$. After boundary condi-

tions are imposed the spinor reads

$$\epsilon_1 = \begin{pmatrix} c_1 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} + c_2 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ c_1 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} - c_2 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_3 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} + c_4 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ c_3 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} - c_4 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ 0 \\ 0 \\ c_5 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} + c_6 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ c_5 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} - c_6 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ c_7 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} + c_8 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ c_7 e^{\frac{i}{2}(\psi \pm \bar{\varphi})} - c_8 e^{-\frac{i}{2}(\psi \pm \bar{\varphi})} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_2 = 0. \quad (6.1.21)$$

In this way, by charging the spinor under the compact field theory directions, we manage to preserve half of the supersymmetries. SUSY in the case where the coordinate $\bar{\varphi}$ shrinks to zero works in the exact same way. We now move to the study of the SUSY variations

on (6.1.2). As before we work on the S-dual frame

$$\begin{aligned}
ds_{st}^2 &= dx_{5,1}^2 + f_s(r)d\varphi^2 + \frac{Ndr^2}{r^2 f_s(r)} + \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right), \\
H_3 &= 2N \text{Vol}(S^3) + \sqrt{\frac{N}{2}} Q d(\zeta(r) \omega_3 \wedge d\varphi), \\
\Phi &= -\log(r),
\end{aligned} \tag{6.1.22}$$

We choose the vielbein basis

$$\begin{aligned}
e^0 &= dt, \quad e^l = dx^l, \quad e^5 = \sqrt{f_s(r)} d\varphi, \quad e^6 = \frac{\sqrt{N}}{r \sqrt{f_s(r)}} dr, \\
e^7 &= \frac{\sqrt{N}}{2} d\theta, \quad e^8 = \frac{\sqrt{N}}{2} \sin(\theta) d\phi, \quad e^9 = \frac{\sqrt{N}}{2} \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right),
\end{aligned} \tag{6.1.23}$$

where now $l = 1, \dots, 4$. The background in eq.(6.1.2) is BPS when $m = 0$. The most general solution of the Killing spinor equation is

$$\epsilon_1 = \frac{\Omega(r)}{r} \begin{pmatrix} \epsilon_1^{(1)} \\ \epsilon_1^{(2)} \\ \epsilon_1^{(3)} \\ \epsilon_1^{(4)} \\ \epsilon_1^{(5)} \end{pmatrix}, \quad \epsilon_2 = \vec{0}, \tag{6.1.24}$$

where $\epsilon_1^{(i)}$ are 8 component spinors given by

$$\epsilon_1^{(1)} = \begin{pmatrix} c_1 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_2 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \\ 0 \\ 0 \\ c_1 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_2 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \\ 0 \\ -\frac{r^2}{\sqrt{2}Q} \left(1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_1 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_2 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ \frac{r^2}{\sqrt{2}Q} \left(1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_1 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_2 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ 0 \end{pmatrix} \tag{6.1.25}$$

$$\epsilon_1^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{\Omega^2(r)} \left(c_3 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_4 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ \frac{1}{\Omega^2(r)} \left(c_3 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_4 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ 0 \\ -\frac{r^2}{\sqrt{2}Q\Omega^2(r)} \left(-1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_3 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_4 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ 0 \\ 0 \\ \frac{r^2}{\sqrt{2}Q\Omega^2(r)} \left(-1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_3 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_4 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \end{pmatrix} \quad (6.1.26)$$

$$\epsilon_1^{(3)} = \begin{pmatrix} 0 \\ \frac{1}{\Omega^2(r)} \left(c_5 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_6 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ \frac{1}{\Omega^2(r)} \left(c_5 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_6 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ 0 \\ -\frac{r^2}{\sqrt{2}Q\Omega^2(r)} \left(-1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_5 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} + c_6 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \\ 0 \\ 0 \\ \frac{r^2}{\sqrt{2}Q\Omega^2(r)} \left(-1 + \sqrt{1 - \frac{2Q^2}{r^4}} \right) \left(c_5 e^{\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} - c_6 e^{-\frac{i}{2} \left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi \right)} \right) \end{pmatrix} \quad (6.1.27)$$

$$\epsilon_1^{(4)} = \begin{pmatrix} c_7 e^{\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} + c_8 e^{-\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} \\ 0 \\ 0 \\ c_7 e^{\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} - c_8 e^{-\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} \\ 0 \\ -\frac{r^2}{\sqrt{2}Q} \left(1 + \sqrt{1 - \frac{2Q^2}{r^4}}\right) \left(c_7 e^{\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} + c_8 e^{-\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} \right) \\ \frac{r^2}{\sqrt{2}Q} \left(1 + \sqrt{1 - \frac{2Q^2}{r^4}}\right) \left(c_7 e^{\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} - c_8 e^{-\frac{i}{2}\left(\psi + \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2}\varphi\right)} \right) \\ 0 \end{pmatrix} \quad (6.1.28)$$

where c_i are integration constants and

$$\Omega(r) = \left(\frac{1 - \sqrt{1 - \frac{2Q^2}{r^4}}}{1 + \sqrt{1 - \frac{2Q^2}{r^4}}} \right)^{\frac{1}{4}}. \quad (6.1.29)$$

In order to see the connection with the spinor for the NS5-brane on a circle, first recall that when $m = 0$ we have $r_+^2 = \sqrt{2}|Q|$, so that

$$\frac{i}{2} \frac{2\sqrt{2}Q}{\sqrt{N}r_+^2} \varphi = i \text{sign}(Q) \frac{\varphi}{\sqrt{N}}, \quad (6.1.30)$$

also recall that $\varphi \sim \varphi + \sqrt{N}\pi$. Defining $\bar{\varphi} = \frac{2}{\sqrt{N}}\varphi$, we recover what we had before, that is, the dependence of the spinor on $\bar{\varphi}$ is of the form

$$\epsilon \sim e^{\frac{i}{2}(\psi \pm \bar{\varphi})}, \quad (6.1.31)$$

so that the spinor is anti-periodic in $\bar{\varphi}$.

6.1.2 Comments on the Dual Field Theory

We interpret the background in eq.(6.1.2) as a stack of N D5 branes. On these five branes, the φ -direction is compactified and fibered over the external three sphere –this is the effect of the function $\zeta(r)$. This fibration is such that some amount of SUSY is preserved and the space time ends at $r = r_+$ in a smooth fashion (we explain this in Section 6.1.1). The smoothness-condition imposes the periodicity $\varphi \sim \varphi + L_\varphi$ to be

$$L_\varphi = \frac{2\pi\sqrt{N}r_+^2}{(r_+^2 - r_-^2)} = \sqrt{N}\pi \left(1 + \frac{m}{\sqrt{m^2 + 8Q^2}} \right). \quad (6.1.32)$$

The Ricci scalar for the background of eq.(6.1.2) is,

$$R = -\frac{12}{Nr} + \frac{4m}{Nr^3} - \frac{8Q^2}{Nr^5}, \quad (6.1.33)$$

this is bounded as r varies in $[r_+, \infty)$.

The field theory holographically dual to this background is, when $m = 0$, a SUSY QFT in $(4 + 1)$ -dimensions when observed at low energies (the region close to $r \sim r_+$). At higher energies the φ -direction decompactifies and the QFT is UV completed by the theory on D5 branes $(5 + 1)$ dimensional Super Yang-Mills and then, at even higher energies, a Little String Theory (after S-duality). In fact, whilst the Ricci scalar and other curvature invariants are bounded in the region of large radial coordinate r , the dilaton grows unbounded. An S-duality to the NS-five brane frame is needed. The effect of the fibration is to introduce a Wilson line (in the QFT) along the compactified direction φ .

In summary, at low energies, the field theory is $(4 + 1)$ dimensional, apparently gapped and confining (we substantiate these properties in the forthcoming sections). At higher energies, the theory recovers its $(5 + 1)$ dimensional character and is UV completed by the theory on NS-five branes, a Little String theory (LST).

We can rescale $e^\Phi \rightarrow e^{\Phi(r)} e^{\Phi_0}$ and $F_3 \rightarrow e^{-\Phi_0} F_3$, which rescales the Newton constant but keeps the same equations of motion. We can choose $e^{-\Phi_0}$ to be a large integer number. Hence, the range of the radial coordinate for which $e^\Phi < 1$ can be made large (at the same time, the number of five branes made large and quantised).

The scales at which these different description take over are: $E \sim r_+$, for the scale that compactifies to $(4 + 1)$ dimensions. The theory becomes six dimensional when $f_s(r_*) \sim 1$. Then, when $e^\Phi \sim 1$ is the scale at which the S-duality is needed and the NS-five brane description takes over.

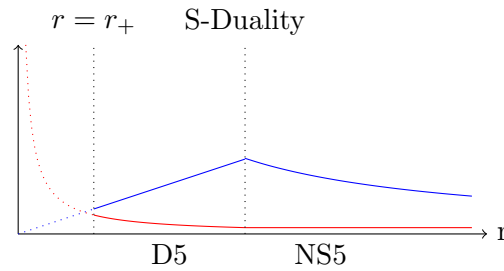


Figure 6.1: Behaviour of the string coupling constant (blue) and the Ricci scalar (red) as a function of the holographic coordinate. The two scales of the theory are depicted: at $r = r_+$ the compactification circle is effectively of zero size and the theory is $(4+1)$ -dimensional. When the string coupling constant becomes of order one, we S-dualise and the theory is described in terms of the NS5-brane.

In what follows, we calculate various observables associated with the QFT. These display behaviours that give account of the IR properties of the field theory (like confinement of quarks, screening of monopoles) and also some of the high energy properties, like the presence of a minimal length, characteristic of the above mentioned non-local UV-completion. We also propose a definition of an effective gauge coupling, we discuss how to calculate the Entanglement Entropy in a strip region, and describe the calculation and spectrum of spin-two glueballs in the QFT.

6.2 Observables

In this section we calculate various observables using the holographic background in eq.(6.1.2), by probing it with various available objects in Type IIB. This produces values characterising the strongly coupled QFT that we describe above. We start by finding a gauge coupling of the $(4 + 1)$ QFT.

6.2.1 Gauge Coupling

To study the background using Dp-branes probes, it is first useful to set our conventions for the Dirac-Born-Infeld-Wess-Zumino (DBIWZ) action, describing the dynamics of branes in our background.

Consider a ten dimensional manifold M_{10} equipped with a metric tensor $G_{\mu\nu}$, Neveu-Schwarz two form $B_{\mu\nu}$, dilaton Φ and Ramond potentials encoded in the poly-form $\mathcal{C}$. In this space, there is an embedded manifold Σ of dimension $(p + 1)$ with $(9 - p)$ space-like normal vectors. This embedded manifold hosts a Dp-brane. We denote the coordinates on the Dp-brane as X^M with $M = 0, 1, \dots, p$. and the induced metric g_{MN} . The action of a single Dp-brane is the Dirac-Born-Infeld-Wess-Zumino action given by

$$S_{\text{Dp,DBI}}[g_{MN}, \mathcal{F}_{MN}] = T_p \int_{\Sigma} d^{p+1}x e^{-\Phi} \sqrt{-\det(g_{MN} + \mathcal{F}_{MN})}. \quad (6.2.1)$$

$$S_{\text{Dp,WZ}}[\mathcal{C}, \mathcal{F}_{MN}] = -T_p \int_{\Sigma} \mathcal{C} \wedge e^{-\mathcal{F}_{MN}}. \quad (6.2.2)$$

Here $\mathcal{F}_{MN} = B_{MN} + 2\pi\alpha' F_{MN}$, where B_{MN} is the pull-back of the background Neveu-Schwarz two-form on Σ , and F_{MN} is an Abelian gauge field strength defined on the brane. The tension of the Dp brane is $T_p = \frac{1}{(2\pi)^{7-p}}$, in our chosen units ($g_s = \alpha' = 1$). We can perform a small field F_{MN} expansion of the action in eq.(6.2.1), equivalent to a small- α' expansion. We obtain an effective action for the Dp-brane,

$$S_{\text{Dp,DBI}} = T_p \int d^{p+1}x e^{-\Phi} \sqrt{-\det g_{MN}} \left[1 - \frac{1}{4} (2\pi)^2 F^{BC} F_{BC} + \mathcal{O}(F^3) \right] \quad (6.2.3)$$

where $F^{BC} = g^{BM}g^{CN}F_{MN}$. The Wess-Zumino part of the action in eq.(6.2.2) contains a finite number of terms. Specialising for $B_{MN} = 0$ we have for any D_p brane probe

$$S_{Dp,WZ} = -T_p \int_{\Sigma} C_{p+1} - 2\pi C_{p-1} \wedge F_2 + \frac{(2\pi)^2}{2} C_{p-3} \wedge F_2^2 - \frac{(2\pi)^3}{6} C_{p-5} \wedge F_2^3 + \frac{(2\pi)^4}{24} C_{p-7} \wedge F_2^4. \quad (6.2.4)$$

To calculate the gauge coupling of the QFT we are interested in, we probe the back-ground with a D5-brane that extends over $[t, x_1, x_2, x_3, x_4, \varphi]$. We switch on an Abelian gauge field on the brane and calculate the coefficient in front of the gauge field kinetic term (for a small-field expansion). This coefficient in general depends on the radial coordinate, indicating that the gauge coupling is energy dependent.

The induced metric on the D5 brane is

$$ds_{D5,ind}^2 = r dx_{4,1}^2 + r (f_s(r) + 2Q^2\zeta(r)^2) d\varphi^2, \quad (6.2.5)$$

with determinant

$$e^{-\Phi} \sqrt{-\det \mathbf{g}_{D5,ind}} = r^2 (f_s(r) + 2Q^2\zeta(r)^2)^{1/2}. \quad (6.2.6)$$

The Dirac-Born-Infeld action is

$$S_{DBI} = T_5 \int d^6x e^{-\Phi} \sqrt{-\det (g_{MN} + 2\pi F_{MN})}, \quad (6.2.7)$$

with $\mathbf{g}_{D5,ind} = g_{MN} dx^M dx^N$. When expanded in the weak field limit this leads to,

$$S_{DBI} = T_{D5} r^2 (f_s(r) + 2Q^2\zeta(r)^2)^{1/2} \int d^5x d\varphi \left(1 - \frac{1}{4} (2\pi)^2 g^{MA} g^{NB} F_{AB} F_{MN} \right). \quad (6.2.8)$$

Turning on a gauge field F_{mn} on "macroscopic" field theory directions (here $m, n = t, x_1, \dots, x_4$), we obtain

$$\begin{aligned} S_{BI} &= T_{D5} r^2 (f_s(r) + 2Q^2\zeta(r)^2)^{1/2} \int d^5x d\varphi \left(1 - \frac{1}{4} (2\pi)^2 2g^{ma} g^{nb} F_{ab} F_{mn} \right), \\ &= T_{D5} L_{\varphi} r^2 (f_s(r) + 2Q^2\zeta(r)^2)^{1/2} \int d^5x \left(1 + \frac{1}{2r^2} (2\pi)^2 \eta^{ma} \eta^{nb} F_{ab} F_{mn} \right) \end{aligned} \quad (6.2.9)$$

We have performed the integration over the φ -direction (represented by the factor L_{φ} above), to construct the effective (4+1)d gauge coupling. We read the gauge coupling,

$$\frac{1}{4g_{YM}^2} = 2\pi^2 T_5 L_{\varphi} (f_s(r) + 2Q^2\zeta(r)^2)^{1/2}. \quad (6.2.10)$$

The asymptotic values of the coupling are

$$\frac{1}{g_{YM}^2} = \begin{cases} 8\pi^2 T_5 L_{\varphi} \sqrt{1 + \frac{2Q^2}{r_+^4}} & , \quad r \rightarrow \infty \\ 0 & , \quad r \rightarrow r_+ \end{cases}. \quad (6.2.11)$$

We interpret this result as follows: at high energies (large values of r), the gauge coupling becomes a large constant value. This value is characteristic of the LST. It is the constant coupling of the LST. Note that the constant part of the fibration (the term $\frac{2Q^2}{r_+^2}$) enters the coupling above. On the other hand, for low energies, when $r \sim r_+$, we find a gauge coupling that grows unbounded. This suggest that the QFT confines at low energies.

Let us observe that the Wess-Zumino term for the probe D5 branes is,

$$S_{WZ} = -T_{D5} \int \left(C_6 - C_4 \wedge F_2 + \frac{1}{2} C_2 \wedge F_2 \wedge F_2 - \frac{1}{6} C_0 F_2 \wedge F_2 \wedge F_2 \right) = -T_{D5} \int C_6. \quad (6.2.12)$$

Using the expression for C_6 in eq.(6.1.3), we find

$$\begin{aligned} S_{WZ} &= -r^2 T_{D5} \int dt \wedge d\varphi \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \\ &= -r^2 T_{D5} L_\varphi \int d^5x. \end{aligned} \quad (6.2.13)$$

Comparing the expression above with the "tension" term in eq.(6.2.9), we observe that (even in the case $m = 0$) this probe is not SUSY preserving. The probe is effectively attracted towards r_+ (the tension is bigger than the charge).

We close this section with some comments motivated by the above material. Note that the probe we used is not SUSY. This is in line with the analysis performed in chapter 5: the twisted circle compactification lifts the Coulomb branch of the theory. Moreover, it was shown that for 4D $\mathcal{N} = 4$ theory, integrating out the massive chiral multiplets leads to a Chern-Simons term in the lower energy effective theory, which is consistent with the fact that the background is dual to a confining theory. It would be interesting to study if there is also an induced Chern-Simons theory in 5D field theory dual to the supersymmetric soliton in the D5-brane background.

Also, note that the naive dimensional analysis, indicating that five dimensional QFTs should have weakly coupled IR dynamics is not working here. As mentioned, a VEV for an R -symmetry current might be changing this. Note the same would occur if working with a Witten-like compactification on a circle (breaking SUSY) for Dp branes with $p \geq 5$.

We go back to the point made below eq.(6.2.11) that the QFT confines. To substantiate it, we compute the Wilson loop.

6.2.2 Wilson Loops

We start summarising general results pertaining holographic Wilson loops [22, 152]. We follow the treatment of [153, 154]. This generic treatment is also useful for the study of other probes that reduce to an 'effective string' in the background. Hence it will apply to 't Hooft loops, Entanglement Entropy, as we discuss below.

Consider a generic holographic background of the form

$$ds^2 = -g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{rr}dr^2 + g_{ij}d\theta^i d\theta^j . \quad (6.2.14)$$

We assume that g_{tt} , g_{xx} , g_{rr} depend only on the radial coordinate r . As usual, we propose a string embedding (parametrised in terms of (τ, σ) the worldsheet coordinates), which leads to a Nambu-Goto action for the F1-string of the form,

$$t = \tau, \quad x = x(\sigma), \quad r = r(\sigma). \\ S_{NG} = T_{F1} \int d\tau d\sigma \sqrt{g_{tt}(r)g_{xx}(r)x'^2 + g_{tt}(r)g_{rr}(r)r'^2}. \quad (6.2.15)$$

From this action, the equations for the string moving in the generic background reduce to (see [154] for a detailed derivation)

$$\frac{dr}{d\sigma} = \pm \frac{dx}{d\sigma} V_{eff}(r) . \quad (6.2.16)$$

We defined the effective potential

$$V_{eff}(r) = \frac{F(r)}{CG(r)} \sqrt{F^2(r) - C^2}, \quad F^2(r) = g_{tt}g_{xx}, \quad G^2(r) = g_{tt}g_{rr}. \quad (6.2.17)$$

The constant $C = \frac{F^2 x'}{\sqrt{F^2 x'^2 + G^2 r'^2}}$ is obtained from one of the equations of motion. In the simpler case in which we take $x(\sigma) = \sigma$ we find eq.(6.2.16) from the conserved Hamiltonian. In that case $C = F(r_0)$, being r_0 the turning point of the string satisfying $r'(\sigma) = 0$ (these are called *U-shaped* embeddings). We set $C = F(r_0)$ in what follows. We enumerate below a set of properties of the U-shaped embeddings.

1. This formalism applies to an open string whose end points are at $r \rightarrow \infty$, where we add a Dp -brane. Dirichlet boundary conditions for the string at $r \rightarrow \infty$ require that $V_{eff}|_{r \rightarrow \infty} \sim \infty$.
2. We compute the separation between the two ends of the string on the Dp -brane, which can be thought as the separation between a quark-antiquark pair. The energy of the pair of quarks calculated from the Nambu-Goto action needs regularisation, implemented by subtracting the mass of two non-dynamical strings extended along the whole range of the radial coordinate $[r_+, \infty)$.

The separation and energy are given as functions of r_0 (the distance from the origin of the radial coordinate, r_+ , to the position of the turning point of the string). The expressions for these quantities are,

$$L_{QQ}(r_0) = 2 \int_{r_0}^{+\infty} \frac{dz}{V_{eff}(z)}, \quad (6.2.18)$$

$$E_{QQ}(r_0) = F(r_0) L_{QQ}(r_0) + 2 \int_{r_0}^{+\infty} dz \frac{G(z)}{F(z)} \sqrt{F(z)^2 - F(r_0)^2} - 2 \int_{r_+}^{+\infty} dz G(z) . \quad (6.2.19)$$

3. To obtain a finite contribution coming from the upper limit of the $Q\bar{Q}$ pair separation in (6.2.18), a further restriction on the behaviour of the effective potential at infinity is needed. See [154] for a derivation,

$$V_{eff}|_{r \rightarrow +\infty} \sim r^\beta, \quad \text{with } \beta > 1. \quad (6.2.20)$$

4. When expanded close to the end of the space, which is at $r = r_+$ in the case of the background in (6.1.2), we find $V_{eff} \sim (r - r_+)^\gamma$. If $1 \leq \gamma$, the separation between the pair becomes infinite—see [154]. Otherwise (if $\gamma < 1$) we have screening behaviour.
5. There is an analytic relation between E_{QQ} and L_{QQ} which is

$$\frac{dE_{QQ}}{dr_0} = F(r_0) \frac{dL_{QQ}}{dr_0} \quad \longrightarrow \quad \frac{dE_{QQ}}{dL_{QQ}} = F(r_0). \quad (6.2.21)$$

We have inverted the relation (6.2.18) as $r_0 = r_0(L_{QQ})$.

6. For generic backgrounds, the evaluation of the integral in eq.(6.2.18) need not be simple nor have an expression in terms of elementary functions. Nevertheless, the quantity

$$\hat{L}_{QQ}(r_0) = \pi \frac{G}{F'} \Big|_{r_0}, \quad (6.2.22)$$

provides a reasonable approximation to eq.(6.2.18). We check this approximate expression for different observables studied below.

7. Following [155, 156] we define

$$Z(r_0) = \frac{d}{dr_0} \hat{L}_{QQ}(r_0) = \pi \frac{d}{dr_0} \left(\frac{G(r_0)}{F'(r_0)} \right). \quad (6.2.23)$$

The stability of the U-shaped string embedding in eq.(6.2.15) is guaranteed if $Z(r_0) < 0$ [155, 156]. This is valid for any observable that can be reduced to an effective string action of the form (6.2.15)¹

8. In calculations like those in eqs.(6.2.18)-(6.2.19) we can introduce a cutoff r_{UV} to regulate divergences coming from the upper limit in the integrals. We define a quantity analog to (6.2.18),

$$L_{QQ}(r_0, r_{UV}) = 2 \int_{r_0}^{r_{UV}} \frac{dz}{V_{eff}(z)}. \quad (6.2.24)$$

Following [155, 156] we calculate

$$L_a = \lim_{r_0 \rightarrow \infty} \lim_{r_{UV} \rightarrow \infty} L_{QQ}(r_0, r_{UV}) \quad \text{and} \quad L_b = \lim_{r_{UV} \rightarrow \infty} \lim_{r_0 \rightarrow r_{UV}} L_{QQ}(r_0, r_{UV}). \quad (6.2.25)$$

¹It was shown in [155]-[157] that embeddings for which $Z(r_0) \geq 0$ do not satisfy two physically well motivated criteria [158]: The force between the quark and the antiquark is always attractive $\frac{dE_{QQ}}{dL_{QQ}} > 0$. It is also a non-increasing function of the separation $\frac{d^2 E_{QQ}}{dL_{QQ}^2} \leq 0$. The proposal in [155, 156] is that the two criteria above are equivalent to the stability of the U-shaped embedding or conversely $Z(r_0) < 0$.

Whilst $L_b = 0$ by definition, it is sometimes the case that L_a is nonzero. If in this case the U-shaped configuration is unstable ($Z(r_0) > 0$), there exist 'short configurations' (that appear very close to the cutoff). These short configurations are energetically favoured and induce a phase transition in the observable calculated [155], [156], [159].

The existence of these short configurations that appear when introducing the UV-cutoff is not an artefact of the cutoff. They indicate that the string embedding we proposed in eq.(6.2.15) is not capturing the dynamically favoured configuration. In the study of the 't Hooft loop and the Entanglement Entropy we encounter these short configurations, that cure the problem of the instability of the embedding and introduce a phase transition in the observable. We take the instability of the configuration (cured by the introduction of a cutoff) as an indication that the dynamics of the LST is driving the observable.

Let us now apply this general treatment to our background in (6.1.2). To calculate the Wilson loop for non-dynamical quarks in the fundamental representation, we follow, [22, 152] and embed a fundamental string in the configuration,

$$t = \tau, \quad x = \sigma, \quad r = r(\sigma). \quad (6.2.26)$$

The Nambu-Goto action is

$$S_{NG} = T_{F1} L_\tau \int d\sigma \sqrt{F^2 + G^2 r'^2}, \quad F = r, \quad G = \frac{\sqrt{N}}{\sqrt{f_s(r)}}. \quad (6.2.27)$$

The effective potential is

$$V_{eff}(r) = \frac{1}{\sqrt{N}} \frac{r}{r_0} \sqrt{f_s(r) (r^2 - r_0^2)} \quad (6.2.28)$$

The approximated separation of the quark-anti-quark pair is

$$\hat{L}_{QQ}(r_0) = \frac{\sqrt{N} \pi}{\sqrt{f_s(r_0)}}. \quad (6.2.29)$$

This expression for $\hat{L}_{QQ}$ (approximating the separation between the quark pair) becomes divergent in $r_0 \sim r_+$. This suggest a confining behaviour, as the pair can be separated to arbitrarily large distances. Below we confirm this when we study the dependence of the energy of the pair, where we find a linear behaviour with the separation. On the other hand, for $r_0 \rightarrow \infty$, when studying the high energy regime, we find that there is a minimal separation $\hat{L}_{min} = \pi\sqrt{N}$ between the pair. This indicates a non-local behaviour of the QFT, with a minimal length, associated with the size of little strings.

The stability of this embedding is determined by the function

$$Z(r_0) = \frac{d}{dr_0} \hat{L}(r_0) = -\frac{\sqrt{N} \pi}{2 [f_s(r_0)]^{3/2}} f'_s(r_0), \quad (6.2.30)$$

$$= -\frac{\sqrt{N} \pi r_0 (r_0^2(r_+^2 + r_-^2) - 2r_+^2 r_-^2)}{[(r_0^2 - r_+^2)(r_0^2 - r_-^2)]^{3/2}}, \quad (6.2.31)$$

which is always negative for $r_0 > r_+$. Consequently the proposed configuration for the Wilson loop is stable. We can analyse numerically the expressions for the length and the energy of the Wilson loop, see eqs (6.2.18) and (6.2.19). We have,

$$L_{QQ} = 2r_0\sqrt{N} \int_{r_0}^{\infty} \frac{r \, dr}{\sqrt{(r^2 - r_0^2)(r^2 - r_+^2)(r^2 - r_-^2)}} \quad (6.2.32)$$

$$E_{QQ} = r_0 L_{QQ}(r_0) + 2\sqrt{N} \left[\int_{r_0}^{\infty} \sqrt{\frac{r^2(r^2 - r_0^2)}{(r^2 - r_+^2)(r^2 - r_-^2)}} dr - \int_{r_+}^{\infty} \frac{r^2 \, dr}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} \right].$$

The integrals can be performed analytically and expressed in terms of Elliptic integrals of the first kind, which are give by

$$\mathbf{K}(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2 \theta}}, \quad \mathbf{E}(x) = \int_0^{\frac{\pi}{2}} \sqrt{1 - x \sin^2 \theta} d\theta. \quad (6.2.33)$$

The explicit expressions for the separation and energy of the quark-antiquark pair, which read

$$L_{QQ}(r_0) = 2\sqrt{N} \frac{r_0/r_+}{\sqrt{r_0^2/r_+^2 - 1}} \mathbf{K} \left(\frac{1 - r_-^2/r_+^2}{1 - r_0^2/r_+^2} \right), \quad (6.2.34)$$

$$E_{QQ}(r_0) = 2r_+\sqrt{N} \left[\frac{r_0^2/r_+^2}{\sqrt{r_0^2/r_+^2 - 1}} \mathbf{K} \left(\frac{1 - r_-^2/r_+^2}{1 - r_0^2/r_+^2} \right) - \mathbf{E} \left(\frac{1 - r_-^2/r_+^2}{1 - r_0^2/r_+^2} \right) \sqrt{r_0^2/r_+^2 - 1} + C(r_-/r_+) \right], \quad (6.2.35)$$

where

$$C(r_-/r_+) = \mathbf{E}(r_-^2/r_+^2) + \lambda_- \mathbf{K}(1 - r_-^2/r_+^2) + i\lambda_- \mathbf{K}(r_-^2/r_+^2) - (1 - r_-^2/r_+^2) \mathbf{K}(r_-^2/r_+^2). \quad (6.2.36)$$

$$\lambda_-^2 = -\frac{r_-^2}{r_+^2}.$$

We plot these results in Figure 6.2. The various panels of Figure 6.2 show some conventional and other less conventional behaviours. First, note that the expression for $\hat{L}_{QQ}$ in eq.(6.2.22) very well approximates the exact expression in eq.(6.2.34). As is usual, the concavity of the curve $E_{QQ}(L_{QQ})$ is 'downwards', indicating that the Nambu-Goto string configuration of (6.2.26) is stable, as confirmed by the $Z(r_0)$ in (6.2.31). Note also that for large separations between the quark pair L_{QQ} , the energy grows linearly (signalling confinement). What is less conventional is that there is a minimal separation, given by the Little String Theory scale. This indicates that the far UV of the QFT dynamics is not field theoretical (but has the dynamics of the LST). The plot of the strings profiles confirms this. Indeed, strings that barely explore the bulk (with $\frac{r_0}{r_+}$ large) show a minimal fixed

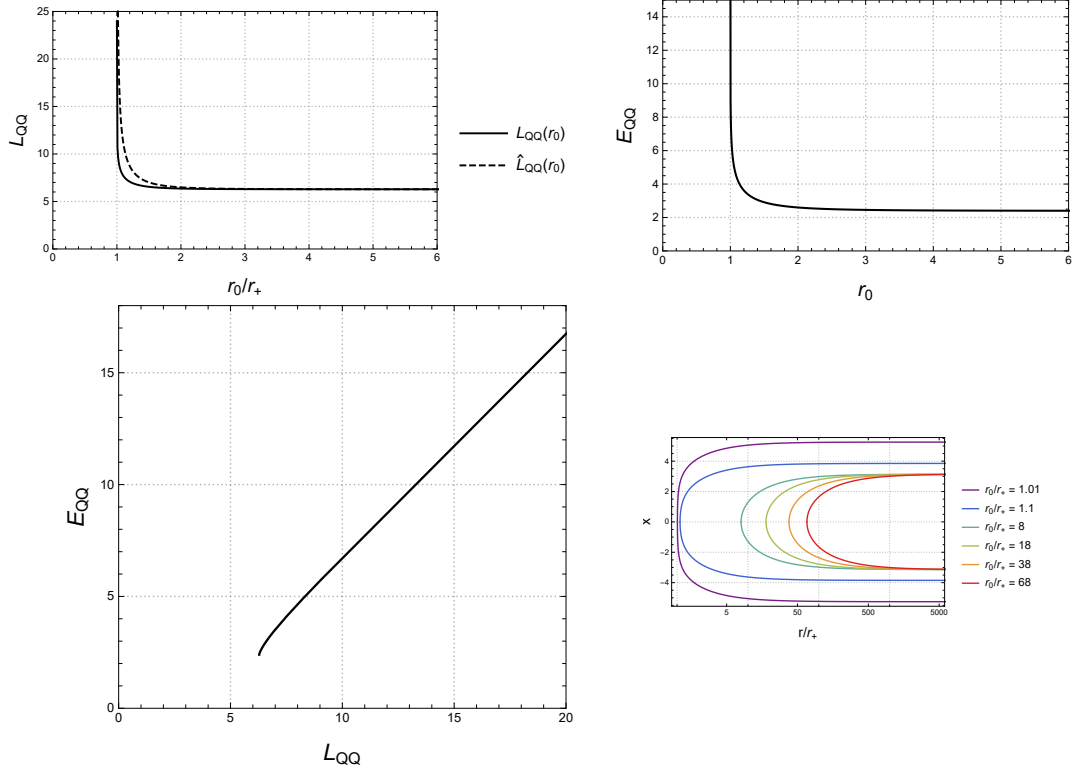


Figure 6.2: Upper left: Plot comparing the exact expression for the quark separation L_{QQ} in eq.(6.2.34) with the approximate one $\hat{L}_{QQ}$ in (6.2.22). The minimal separation as r_0 grows large hints at a LST behaviour. Upper right: Plot of $E_{QQ}(r_0)$ with r_0 in units of r_+ . The plot is made in the BPS limit with $m = 0$. Bottom left: Parametric plot of $E_{QQ}(L_{QQ})$ in the BPS bound $m = 0$, $r_+ = N = 1$. Bottom right: the profiles of different strings as they explore the bulk. The longer the separation L_{QQ} , the more the string approaches $\frac{r_0}{r_+} \sim 1$. This is usual of the backgrounds dual to a confining QFT behaviour.

separation between the quark pair. On the other hand, the strings that explore deeper into the bulk display a bigger quark separation and carry higher energy.

Another interesting observable is the 't Hooft loop, describing the potential between a pair of external magnetic monopoles. We study this next.

6.2.3 't Hooft Loops

If the theory confines electric charges, we expect magnetic monopoles to be screened. The 't Hooft loop can be calculated by proposing an object with magnetic charge that effectively appears like a string for the QFT. The ends of this magnetic string appear as a pair of monopoles of opposite charge. This object is studied by probing the background with a D5-brane that extends in the directions $[t, x, \varphi, \theta, \phi, \psi]$ and $r = r(x)$. The induced metric is

$$ds_{D5,ind}^2 = r \left\{ -dt^2 + \left(1 + \frac{Nr'^2}{r^2 f_s(r)} \right) dx^2 + f_s(r) d\varphi^2 + \frac{N}{4} \left[\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q\zeta(r) d\varphi \right)^2 \right] \right\}. \quad (6.2.37)$$

When we integrate over the directions $\varphi, \theta, \phi, \psi$ the effective probe is $(1+1)$ dimensional and carries magnetic charge. The effective action takes the form of (6.2.15). It reads,

$$S_{eff} = T_{D5} L_\varphi 16\pi^2 \left(\frac{N}{4} \right)^{3/2} \int dt \int dx \sqrt{r^4 f_s(r) + Nr^2 r'^2}. \quad (6.2.38)$$

From eq.(6.2.38) we identify the functions

$$F = r^2 \sqrt{f_s(r)}, \quad G = \sqrt{N} r. \quad (6.2.39)$$

This object has vanishing tension when it approaches $r = r_+$, as $F(r) = \sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}$. As a consequence of this, at low energies the monopole anti-monopole pair can be separated at no energy expense, In contraposition with the Wilson loop, this indicates a screening behaviour.

The effective potential defined in (6.2.17) is for this probe,

$$V_{eff}(r) = \frac{r}{\sqrt{N} r_0^2} \sqrt{\frac{f_s(r)}{f_s(r_0)} (r^4 f_s(r) - r_0^4 f_s(r_0))}. \quad (6.2.40)$$

The approximated separation for the monopole pair $\hat{L}_{MM}(r_0)$ is given by

$$\hat{L}_{MM}(r_0) = 2\pi\sqrt{N} \frac{\sqrt{f_s(r_0)}}{(4f_s(r_0) + r f'_s(r_0))} = \pi\sqrt{N} \frac{\sqrt{(r_0^2 - r_+^2)(r_0^2 - r_-^2)}}{2r_0^2 - r_+^2 - r_-^2}. \quad (6.2.41)$$

This approximate separation for the monopole pair displays the opposite behaviour to that of the Wilson loop (quark pair). In fact, there is a maximal separation, achieved when $r_0 \rightarrow \infty$ and the minimal (vanishing) separation is achieved when $r_0 \sim r_+$.

The function $Z(r_0)$ associated to the embedding is

$$Z(r_0) = \frac{d}{dr_0} \left(\hat{L}_{MM}(r_0) \right) = \pi \sqrt{N} r_0 \frac{(r_+^2 - r_-^2)^2}{(2r_0^2 - r_+^2 - r_-^2)^2 \sqrt{(r_0^2 - r_+^2)(r_0^2 - r_-^2)}}. \quad (6.2.42)$$

This is always positive, indicating that this embedding of the D5 brane is unstable. This instability suggest a transition between a connected embedding, like the one studied above, and a disconnected one. This indicates that the pair of monopoles can be arbitrarily separated at no energy cost. We conclude that the 't Hooft loop displays a screening type of behaviour.

We can write the expressions in eqs. (6.2.18),(6.2.19). These read,

$$\begin{aligned} L_{MM} &= 2\sqrt{N(r_0^2 - r_+^2)(r_0^2 - r_-^2)} \\ &\quad \times \int_{r_0}^{\infty} \frac{r dr}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)} [(r^2 - r_+^2)(r^2 - r_-^2) - (r_0^2 - r_+^2)(r_0^2 - r_-^2)]} \\ E_{MM} &= \sqrt{(r_0^2 - r_+^2)(r_0^2 - r_-^2)} L_{MM}(r_0) \\ &\quad + 2\sqrt{N} \left[\int_{r_0}^{\infty} \frac{r \sqrt{(r^2 - r_+^2)(r^2 - r_-^2) - (r_0^2 - r_+^2)(r_0^2 - r_-^2)}}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} dr - \int_{r_+}^{\infty} r dr \right]. \end{aligned} \quad (6.2.43)$$

We analyse these integrals by plotting $L_{MM}(r_0)$, comparing it with $\hat{L}_{MM}$ in eq.(6.2.41), $E_{MM}(r_0)$ and the parametric plot $E_{MM}(L_{MM})$. These are displayed in Figure 6.3.

The Wilson and 't Hooft loops indicate that the background of eq.(6.1.2) is dual to a $(4+1)$ dimensional QFT that confines electric charges and screens magnetic ones. An interesting quantity to study is the density of states (the number of degrees of freedom) in terms of the energy (the radial coordinate). Below, we calculate an observable that quantifies this.

6.2.4 Density of states, free energy or holographic central charge

In this section we study a quantity that becomes particularly important and well defined in conformal field theories (our QFT is not conformal). In the conformal case, this quantity coincides with the free energy or central charge of the CFT (indicating the number of degrees of freedom in the system), which was studied in Section 4.3.2. We rewrite the formulas mentioned there for convenience. This quantity has been generalised from calculations in backgrounds with an AdS-factor (dual to CFTs) to generic backgrounds and

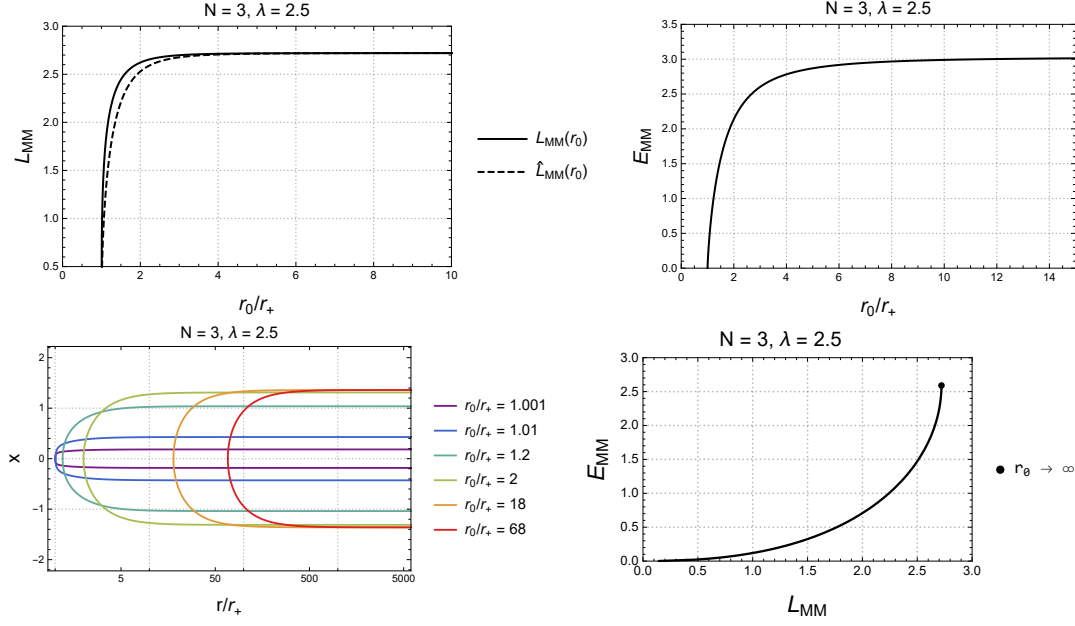


Figure 6.3: The integrals of the monopole-antimonopole t' Hooft loop (6.2.43) can be written in terms of two parameters $\lambda = -r_-^2/r_+^2$ and N . All the plots are considering $\lambda = 2.5$ and $N = 3$. Upper left: Plot comparing the exact expression for the monopole separation (6.2.43) with the approximate expression $\hat{L}_{MM}$ in (6.2.41). Upper right: The energy of the t'Hooft loop as a function of the r_0 in units of r_+ . Bottom left: Different profiles of the macroscopic strings as a function of the turn-around point r_0 . Bottom Left: Plot of the function $E_{MM}(L_{MM})$. The upwards concavity ascertains the prediction that the embedding is unstable.

dilaton, like our case in the background of eq.(6.1.2). See [129, 130]. For backgrounds and dilaton of the form,

$$ds^2 = \alpha(r, y^i) (dx_{d,1}^2 + \beta(r) dr^2) + g_{ij}(r, y^i) dy^i dy^j, \quad \Phi = \Phi(r, y^i). \quad (6.2.44)$$

Here y^i stands for all the internal coordinates. We define the following quantites

$$V_{\text{int}} = \int dy^i \sqrt{e^{-4\Phi} \det g_{ij} \alpha(r, y^i)^d}, \quad H = V_{\text{int}}^2. \quad (6.2.45)$$

The holographic central charge is defined as

$$c_{\text{hol}} = d^d \frac{\beta(r)^{d/2} H^{(2d+1)/2}}{G_N (H')^d}. \quad (6.2.46)$$

For the background in eq.(6.1.2) we have $y^i \in [\varphi, \theta, \phi, \psi]$ with

$$g_{ij} dy^i dy^j = r f_s(r) d\varphi^2 + \frac{Nr}{4} \left[\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right]. \quad (6.2.47)$$

The functions in eqs.(6.2.44)-(6.2.45) are

$$\alpha(r, y^i) = r, \quad \beta(r) = \frac{N}{r^2 f_s(r)}, \quad d = 4, \quad e^{-4\Phi} = \frac{1}{r^4}. \quad (6.2.48)$$

These lead to $H = \mathcal{N}^2 r^4 f_s(r)$ with $\mathcal{N} = 2\pi^2 \sqrt{N^3} L_\phi$, and the holographic central charge is

$$c_{\text{hol}} = \frac{256 N^2 \mathcal{N}^2 r^2 f_s(r)^{5/2}}{G_N [f_s(r) + \frac{r}{4} f'_s(r)]^4} = \frac{16 N^2 \mathcal{N}}{G_N} \frac{[(r^2 - r_+^2)(r^2 - r_-^2)]^{5/2}}{(2r^2 - r_+^2 - r_-^2)^4}. \quad (6.2.49)$$

This quantity vanishes for $r = r_+$, suggesting a gapped system, in agreement with the confining behaviour induced from the results of the Wilson and 't Hooft loops. On the other hand, for large values of r (at high energies), the quantity diverges quadratically. This is in agreement with the non-field theoretical UV-completion provided by the LST.

There is another quantity, c_{flow} , studied in Section 4.3.4, which in the case of CFTs, captures the UV and IR fixed points of the flow across dimensions. This considers the system as a six dimensional non-isotropic QFT. The behaviour of c_{flow} is qualitatively similar to the one described in eq.(6.2.49), but the divergence at high energies is faster.

6.2.5 Entanglement Entropy

A very interesting observable in all QFTs is the Entanglement Entropy (EE). This can be holographically computed following [160, 161]. The method is to find a minimal area eight-surface (Σ_8 , a codimension-two surface to which we refer below as RT surface) such that the boundary of the surface coincides with the two entangled regions. We focus on the case in which one of the regions is a strip of size L_{EE} and the other region is the complement. The EE between these regions is given in [160–162], minimising the quantity

$$S_{EE} = \frac{1}{4G_N} \int_{\Sigma_8} d^8 \sigma \sqrt{e^{-4\Phi} \det[g_{\Sigma_8}]}. \quad (6.2.50)$$

For the case of a strip-region of size L_{EE} and for a confining field theory, it was suggested in [162] that the Entanglement Entropy $S_{EE}(L_{EE})$ presents a phase transition as we vary L_{EE} . This proposal was critically analysed in [159] and more recently in [163]. In these papers, it was found that for gravity backgrounds dual to confining QFTs with non-local UV completion, the phase transition conjectured in [162] is absent. If a UV-cutoff is introduced (avoiding the non-local UV completion), the phase transition is recovered.

The QFT holographically described by the background in eq.(6.1.2), fits the above description. The EE on a strip is calculated by computing the area of an eight-manifold

Σ_8 that is parametrised by,

$$\begin{aligned} \Sigma_8 &= [x_1, x_2, x_3, x_4, \varphi, \theta, \phi, \psi], \quad r(x_1). \\ ds_{\Sigma_8, st}^2 &= r \left[dx_{2,3,4}^2 + dx_1^2 \left(1 + \frac{N}{r^2 f_s(r)} r'(x_1)^2 \right) + f_s(r) d\varphi^2 + \right. \\ &\quad \left. \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right) \right], \end{aligned} \quad (6.2.51)$$

and then calculating (for an entangling region of size L),

$$\begin{aligned} S_{EE} &= \frac{1}{G_N} \int d^8 x \sqrt{e^{-4\Phi} \det g_{\Sigma_8, st}} \\ &= \frac{16\pi^2 N^{3/2} L_{x_2} L_{x_3} L_{x_4} L_\varphi}{8G_N} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \sqrt{F^2(r) + G^2(r) r'^2}. \end{aligned} \quad (6.2.52)$$

$$F(r) = r^2 \sqrt{f_s(r)}, \quad G(r) = \sqrt{N} r.$$

The expression above fits in the generic effective action in (6.2.15), hence we can write integral expressions for the separation and EE following the formulas (6.2.18), (6.2.19). Indeed, we can write an approximate expression for the separation following (6.2.22), and also evaluate the function $Z(r_0)$ in (6.2.23), indicating stability or instability of the embedding.

At this point it is useful to observe that the functions $F(r)$ and $G(r)$ that appear in the study of the EE, eq.(6.2.52), are the same ones that appear in the study of the 't Hooft loop, in eq.(6.2.39). It follows from this that the approximate expression for the separation is a monotonic function given in eq.(6.2.41). This indicates the absence of phase transitions. The function $Z_{EE}(r_0)$ coincides with that in eq.(6.2.42). The plots are identical to those in Figure 6.3.

6.2.6 Spin-two Glueballs

We consider glueball-like excitations in the confining $(4+1)$ dimensional QFT holographically described by the background in eq.(6.1.2). The glueball excitations are studied by performing a fluctuation of some background field, either Ramond or Neveu-Schwarz in the Type II action. The nonlinear character of the equations of motion, implies that this fluctuation sources excitations for other fields in the background. The dynamics is described by linear second order coupled differential equations for the fluctuations. This is characteristically very hard to analyse (diagonalise, resolve, find the normal modes and spectra, etc)².

²A different approach is to study the equation of motion of a probe scalar in the background. It was recently observed in [164], that a new logarithmic branch of solutions appears in background of the form (6.1.2). Whilst we do not refer to these interesting probe scalars as "glueballs", their study is interesting.

We begin this section by reviewing some general formulas for spin-2 fluctuations, previously obtained in [165]. We adapt those result to have the macroscopic space to be $(p+1)$ -dimensional. We work in the *Einstein frame*, since we are studying particle-like excitations. In this frame, the equations of motion of Type IIB Supergravity only with F_3 flux are given by

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{g_s e^\Phi}{4} \left(F_{\mu\lambda\rho} F_\nu{}^{\lambda\rho} - \frac{1}{12} g_{\mu\nu} F_{\lambda\rho\sigma} F^{\lambda\rho\sigma} \right), \quad (6.2.53)$$

$$\nabla^2\Phi = \frac{g_s e^\Phi}{12} F_{\mu\nu\lambda} F^{\mu\nu\lambda}, \quad (6.2.54)$$

$$\partial_\mu \left(\sqrt{g} e^\Phi F^{\mu\nu\lambda} \right) = 0, \quad (6.2.55)$$

$$\partial_{[\mu} F_{\nu\lambda\rho]} = 0. \quad (6.2.56)$$

In what follows we will consider a spacetime of the following form

$$\begin{aligned} ds_{10D}^2 &= e^{2A(r)} ds^2(\mathcal{M}_{p+1}) + g_{ij}(y^k) dy^i dy^j, \\ F_3 &= \frac{1}{3!} f_{ijk}(y) dy^i \wedge dy^j \wedge dy^k, \\ \Phi &= \Phi(y), \end{aligned} \quad (6.2.57)$$

where $\mathcal{M}_{p+1}$ is a space of constant curvature. We will split the 10D indexes in the following way

$$\begin{aligned} a, b &= 0, 1, \dots, p \\ i, j &= y^1, \dots, y^{9-p}. \end{aligned} \quad (6.2.58)$$

We consider metric fluctuations of the form

$$\delta g_{\mu\nu} = e^{2A} \bar{h}_{\mu\nu}, \quad \bar{h}_{\mu\nu} = \begin{pmatrix} h_{ab}(x)\psi(y) & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.2.59)$$

i.e. metric fluctuations parallel to the macroscopic space $\mathcal{M}$. We work in the transverse-traceless gauge

$$h^a{}_a = 0, \quad \nabla_a h^a{}_b = 0. \quad (6.2.60)$$

Note that this conditions are expressed in the macroscopic space, but due to the form of the ansatz, they also can be written as a 10D condition

$$h^\mu{}_\mu = 0, \quad \nabla_\mu h^\mu{}_\nu = 0. \quad (6.2.61)$$

As showed in [165], with this choice of fluctuation, it is consistent to take

$$\delta F_{\mu\nu\lambda} = 0, \quad \delta\Phi = 0, \quad (6.2.62)$$

since any fluctuation of those field will not source the fluctuation (6.2.59).

First, we study the variation of the energy-stress tensor. For the configurations of interest the later is given by

$$T_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\partial\Phi)^2 \right) + \frac{g_s e^\Phi}{4} \left(F_{\mu\lambda\rho} F_\nu{}^{\lambda\rho} - \frac{1}{3!} g_{\mu\nu} F_{\lambda\rho\sigma} F^{\lambda\rho\sigma} \right). \quad (6.2.63)$$

Within the configurations of the form (6.2.57)

$$T_c^c = (p+1) \left(\frac{1}{4} (\partial\Phi)^2 + \frac{g_s e^\Phi}{24} F_{\lambda\rho\sigma} F^{\lambda\rho\sigma} \right), \quad (6.2.64)$$

so that with perturbations of the form (6.2.59) we have

$$\delta T_{\mu\nu} = \frac{1}{p+1} T_c^c \delta g_{\mu\nu}. \quad (6.2.65)$$

For $(\mu, \nu) = (a, b)$ Einstein's equations give

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}, \quad (6.2.66)$$

and tracing over a, b we get

$$R^a{}_a - \frac{(p+1)}{2} R = T^a{}_a. \quad (6.2.67)$$

From where we can write the $(\mu, \nu) = (a, b)$ components of (6.2.65) as

$$\delta T_{ab} = \frac{1}{p+1} R_c^c \delta g_{ab} - \frac{1}{2} R \delta g_{ab}. \quad (6.2.68)$$

We now turn to the variation of Einstein's equations. From the Ricci tensor variation we have

$$\delta R_{\mu\nu} = \frac{1}{2} \left(\nabla_\lambda \nabla_\mu \delta g^\lambda{}_\nu + \nabla_\lambda \nabla_\nu \delta g^\lambda{}_\mu - \nabla^2 \delta g_{\mu\nu} \right). \quad (6.2.69)$$

Note that since $\delta g_{\mu\nu}$ is transverse

$$[\nabla_\lambda, \nabla_\mu] \delta g^\lambda{}_\nu = R^\lambda{}_{\sigma\lambda\mu} \delta g^\sigma{}_\nu - R^\sigma{}_{\nu\lambda\mu} \delta g^\lambda{}_\sigma, \quad (6.2.70)$$

is equivalent to

$$\nabla_\lambda \nabla_\mu \delta g^\lambda{}_\nu = R_{\sigma\mu} \delta g^\sigma{}_\nu - R^\sigma{}_{\nu\lambda\mu} \delta g^\lambda{}_\sigma. \quad (6.2.71)$$

Then we have

$$\delta R_{\mu\nu} = \frac{1}{2} \left(R_{\sigma\mu} \delta g^\sigma{}_\nu - R^\sigma{}_{\nu\lambda\mu} \delta g^\lambda{}_\sigma + R_{\sigma\nu} \delta g^\sigma{}_\mu - R^\sigma{}_{\mu\lambda\nu} \delta g^\lambda{}_\sigma - \nabla^2 \delta g_{\mu\nu} \right). \quad (6.2.72)$$

In what follows, we rewrite the background metric as

$$ds^2 = e^{2A} (ds^2(\mathcal{M}_{p+1})) + e^{-2A} ds_{\text{Int}}^2 = e^{2A} \hat{ds}^2, \quad (6.2.73)$$

so that it takes the form of a Weyl rescaling of the metric $\hat{ds}^2$ of a product space. Since we are dealing with a Weyl rescaling we can use

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \hat{\Gamma}^\lambda_{\mu\nu} + \left(\delta^\lambda_\nu \partial_\mu A + \delta^\lambda_\mu \partial_\nu A - \hat{g}_{\mu\nu} \partial^\lambda A \right), \\ R^\mu_{\nu\lambda\rho} &= \hat{R}^\mu_{\nu\lambda\rho} - \delta^\mu_\lambda \hat{g}_{\nu\rho} (\partial A)^2 + \delta^\mu_\rho \hat{g}_{\nu\lambda} (\partial A)^2 + \dots, \\ R_{\mu\nu} &= \hat{R}_{\mu\nu} - \hat{g}_{\mu\nu} \left(\hat{\square} A - 8(\partial A)^2 \right) + \dots\end{aligned}\tag{6.2.74}$$

Here, the "...” terms will not play a role in our discussion. Then we have

$$\nabla_\lambda \delta g_{\mu\nu} = e^{2A} \left(\hat{\nabla}_\lambda \bar{h}_{\mu\nu} - \bar{h}_{\lambda\nu} \partial_\mu A - \bar{h}_{\lambda\mu} \partial_\nu A \right),\tag{6.2.75}$$

and after a bit of algebra

$$\nabla^\lambda \nabla_\lambda \delta g_{\mu\nu} = \hat{\square} \bar{h}_{\mu\nu} + 8 \partial^\sigma A \hat{\nabla}_\sigma \bar{h}_{\mu\nu} - 2 \bar{h}_{\mu\nu} (\partial A)^2.\tag{6.2.76}$$

We also use

$$\begin{aligned}R_{\sigma\mu} \delta g^\sigma_\nu &= \hat{R}_{\sigma\mu} \bar{h}^\sigma_\nu - \bar{h}_{\mu\nu} \left(\hat{\square} A - 8(\partial A)^2 \right), \\ R^\sigma_{\nu\lambda\mu} \delta g^\lambda_\sigma &= \hat{R}^\sigma_{\nu\lambda\mu} \bar{h}^\lambda_\sigma + \bar{h}_{\mu\nu} (\partial A)^2.\end{aligned}\tag{6.2.77}$$

Finally, (6.2.72) reads

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2} \left(\hat{R}_{\sigma\mu} \bar{h}^\sigma_\nu + \hat{R}_{\sigma\nu} \bar{h}^\sigma_\mu - 2 \bar{h}_{\mu\nu} \left(\hat{\square} A - 8(\partial A)^2 \right) \right. \\ &\quad \left. - 2 \hat{R}_{\sigma\nu\lambda\mu} \bar{h}^{\lambda\sigma} - \hat{\square} \bar{h}_{\mu\nu} - 8 \partial^\sigma A \hat{\nabla}_\sigma \bar{h}_{\mu\nu} \right),\end{aligned}\tag{6.2.78}$$

Now we use the fact that $\hat{ds}^2$ is a product space and that the macroscopic space is of constant curvature

$$\begin{aligned}\hat{R}_{abcd} &= k (\hat{g}_{ac} \hat{g}_{bd} - \hat{g}_{ad} \hat{g}_{bc}), \\ R_{ab} &= kp \hat{g}_{ab},\end{aligned}\tag{6.2.79}$$

then

$$\delta R_{ab} = \frac{1}{2} \left(2k(p+1) \bar{h}_{ab} - 2 \bar{h}_{ab} \left(\hat{\square} A - 8(\partial A)^2 \right) - \hat{\square} \bar{h}_{ab} - 8 \partial^\sigma A \hat{\nabla}_\sigma \bar{h}_{ab} \right).\tag{6.2.80}$$

Also, note that

$$\delta(g_{\mu\nu} R) = \delta g_{\mu\nu} R + g_{\mu\nu} \delta g^{\lambda\rho} R_{\lambda\rho} + g_{\mu\nu} g^{\lambda\rho} \delta R_{\lambda\rho}.\tag{6.2.81}$$

The second and third term do not contribute since $\delta g_{\mu\nu}$ is traceless. Then, specialising to $(\mu, \nu) = (a, b)$

$$\delta(g_{ab} R) = \delta g_{ab} R.\tag{6.2.82}$$

Putting everything together, from the variation of Einstein's equations

$$\delta R_{ab} - \frac{1}{2} R \delta g_{ab} = \delta T_{ab},\tag{6.2.83}$$

we have

$$\frac{1}{2} \left(2k(p+1)\bar{h}_{ab} - 2\bar{h}_{ab} \left(\hat{\square}A - 8(\partial A)^2 \right) - \hat{\square}\bar{h}_{ab} - 8\partial^\sigma A \hat{\nabla}_\sigma \bar{h}_{ab} \right) = \frac{1}{p+1} R^c_c \delta g_{ab}. \quad (6.2.84)$$

Finally, by noting that in our ansatz

$$R^c_c = e^{-2A} (p+1) \left(kp - \hat{\square}A - 8(\partial A)^2 \right), \quad (6.2.85)$$

we can write (6.2.84) as

$$2k\bar{h}_{ab} - \hat{\square}\bar{h}_{ab} - 8\partial^\sigma A \hat{\nabla}_\sigma \bar{h}_{ab} = 0. \quad (6.2.86)$$

Writing $\bar{h}_{ab}$ in terms of h_{ab} and ψ leads to

$$2kh_{ab}\psi - \psi\hat{\square}_x h_{ab} - h_{ab}\hat{\square}_y \psi - 8h_{ab}\partial^i A \hat{\nabla}_i \psi = 0. \quad (6.2.87)$$

Suppose that h_{ab} satisfies

$$\hat{\square}_x h_{ab} - \mu h_{ab} = 0, \quad (6.2.88)$$

then (6.2.87)

$$(2k - \mu)h_{ab}\psi - h_{ab}\hat{\square}_y \psi - 8h_{ab}\partial^i A \hat{\nabla}_i \psi = 0, \quad (6.2.89)$$

from where (here we define $M^2 = \mu - 2k$)

$$\hat{\square}_y \psi + 8\hat{g}^{ij}\partial_i A \partial_j \psi + M^2 \psi = 0 \quad (6.2.90)$$

Change of Variables

We now review the two changes of variables used in [165]. First we rewrite (6.2.90) in terms of $g_{ij} = e^{2A}\hat{g}_{ij}$

$$e^{(1-p)A} \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} e^{(p+1)A} g^{ij} \partial_j \psi \right) + M^2 \psi = 0. \quad (6.2.91)$$

From here, a well defined Sturm-Liouville norm

$$||\psi||^2 = \int d^{9-p}y \sqrt{\det(g_{ij})} e^{(p-1)A} |\psi|^2, \quad (6.2.92)$$

leads to the a positiveness condition for M^2

$$\begin{aligned} M^2 ||\psi||^2 &= - \int d^{9-p}y \psi \partial_i \left(\sqrt{\det(g_{ij})} e^{(p+1)A} g^{ij} \partial_j \psi \right) \\ &= \int d^{9-p}y \sqrt{\det(g_{ij})} e^{(p+1)A} (\partial \psi)^2 \geq 0, \end{aligned} \quad (6.2.93)$$

provided

$$\int d^{9-p}y \partial_i \left(\psi \sqrt{\det(g_{ij})} e^{(p+1)A} g^{ij} \partial_j \psi \right) = 0, \quad (6.2.94)$$

which is satisfied by imposing Dirichlet or Neumann boundary conditions.

Another useful change of variables is $\psi = e^{-4A}\Psi$, which leads to

$$\hat{\square}_y \Psi + \Psi e^{4A} \hat{\square}_y e^{-4A} - 32\Psi \hat{g}^{ij} \partial_i A \partial_j A + M^2 \Psi = 0. \quad (6.2.95)$$

Noting that

$$-e^{4A} \hat{\square}_y e^{-4A} + 32\hat{g}^{ij} \partial_i A \partial_j A = e^{-4A} \hat{\square}_y e^{4A}, \quad (6.2.96)$$

we can write (6.2.95) in a Schroedinger like form

$$-\hat{\square}_y \Psi + V(y) \Psi = M^2 \Psi, \quad (6.2.97)$$

where the effective potential is given by

$$V(y) = e^{-4A} \hat{\square}_y e^{4A}. \quad (6.2.98)$$

Note that in terms of Ψ and $\hat{g}_{ij}$, (6.2.93) takes the form (here we use $\sqrt{\det(g_{ij})} = e^{(9-p)A} \sqrt{\det(\hat{g}_{ij})}$ and $|\psi|^2 = e^{-8A} |\Psi|^2$)

$$||\psi||^2 = \int d^{9-p}y \sqrt{\det(\hat{g}_{ij})} |\Psi|^2, \quad (6.2.99)$$

Schroedinger Like Equation

Note that although (6.2.97) has the form of a Schroedinger equation, the Laplace operator can still contain first derivatives of Ψ . Here we show how to obtain an equation that only contains second derivatives of the function. This is achieved by using a change of variables and a tortoise coordinate for the radial direction.

In what follows we consider internal spaces of the form

$$ds_{\text{Int}}^2 = f(r) d\Omega_d^2 + \frac{dr^2}{g(r)} + ds^2(\tilde{M}_{8-p-d}), \quad (6.2.100)$$

where Ω_d^2 is a d -dimensional compact space and $\tilde{M}_{8-p-d}$ is a $(8-p-d)$ -dimensional space, possibly including fibrations over Ω_d . We will denote the unfibered version as M_{8-p-d} , and we assume that the fibration is such that $\text{Vol}(\tilde{M}_{8-p-d}) = \text{Vol}(M_{8-p-d})$. Also, we assume $A = A(r)$.

We will consider metric fluctuations that only depend on Ω_d and r , that is, we take the S-wave M_{8-p-d} . Then

$$\hat{\square}_y \Psi = \frac{1}{f(r)^{\frac{d}{2}} g(r)^{-\frac{1}{2}}} \frac{d}{dr} \left(f(r)^{\frac{d}{2}} g(r)^{\frac{1}{2}} \frac{d\Psi}{dr} \right) + \frac{1}{f(r)} \nabla_{\Omega}^2 \Psi, \quad (6.2.101)$$

with ∇_{Ω}^2 the Laplace operator in Ω_d . By using the tortoise coordinate $d\rho = dr/\sqrt{g(r)}$ and using the change of variables

$$\Psi = f(r)^{-\frac{d}{4}} \Theta(\rho) Y_l(\Omega), \quad (6.2.102)$$

with $Y_l(\Omega)$ the eigenfunction of ∇_Ω^2 , i.e. $\nabla_\Omega^2 Y_l = -l(l+d-1)$, we can write (6.2.97) as

$$-\frac{d^2\Theta}{d\rho^2} + \tilde{V}(\rho)\Theta(\rho) = M^2\Theta(\rho), \quad (6.2.103)$$

where the effective potential $\tilde{V}$ is given by

$$\tilde{V}(\rho) = \left(V(r) + \frac{l(l+d-1)}{f(r)} - \frac{g(r)^{\frac{1}{2}}}{f(r)^{\frac{d}{4}}} \frac{d}{dr} \left(f(r)^{\frac{d}{2}} g(r)^{\frac{1}{2}} \frac{d}{dr} \left(f(r)^{-\frac{d}{4}} \right) \right) \right) \Big|_{r=r(\rho)}. \quad (6.2.104)$$

We now proceed to apply this mechanism to the background in (6.1.2).

Glueball Spectra on the Supersymmetric Soliton

In Einstein frame, the background metric of eq.(6.1.2) reads,

$$ds_E^2 = \sqrt{r} \left[dx_{1,4}^2 + f_s(r) d\varphi^2 + \frac{N dr^2}{r^2 f_s(r)} + \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right) \right], \quad (6.2.105)$$

which can be written in the form (6.2.73) as

$$ds^2 = e^{2A(\vec{y})} \left(ds^2(\mathbb{R}^{4,1}) + \hat{g}_{ab}(\vec{y}) dy^a dy^b \right). \quad (6.2.106)$$

From here, we identify the warp factors and internal metric

$$e^{4A(\vec{y})} = r, \quad (6.2.107)$$

$$\hat{g}_{ab} dy^a dy^b = f_s(r) d\varphi^2 + \frac{N dr^2}{r^2 f_s(r)} + \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right).$$

This gives $\sqrt{\det \hat{g}_{ab}} = \frac{N^2}{8r} \sin(\theta)$. In what follows we focus on glueballs that depend on the radial coordinate r and the angular coordinate φ as

$$\Psi(r, \varphi) = e^{i \frac{2\pi}{L\varphi} n \varphi} \Psi(r), \quad n \in \mathbb{Z} \quad (6.2.108)$$

but are not excited in the $[\theta, \phi, \psi]$ directions. The potential in (6.2.98) is,

$$V(r) = \frac{1}{N} \partial_r (r f_s(r)), \quad (6.2.109)$$

and the Schroedinger-like equation in (6.2.97) reads

$$-\frac{r}{N} \frac{d}{dr} \left(r f_s(r) \frac{d\Psi}{dr} \right) + \left(V(r) + \frac{n^2}{f_s(r)} \left(\frac{2\pi}{L\varphi} \right)^2 \right) \Psi(r) = M^2 \Psi(r). \quad (6.2.110)$$

We can move (6.2.110) to more manageable forms. Moving to the tortoise coordinate ρ by

$$r = \cosh \left(\frac{\rho}{\sqrt{N}} \right) \sqrt{r_+^2 - r_-^2 \tanh^2 \left(\frac{\rho}{\sqrt{N}} \right)}, \quad (6.2.111)$$

which maps $r \in [r_+, +\infty[$ to $\rho \in [0, +\infty[$. Using the change of variables³ $\Psi(r) = f_s(r)^{-1/4}\Theta(\rho)$, leads to the equation (see (6.2.103) and (6.2.104))

$$-\frac{d^2\Theta}{d\rho^2} + \tilde{V}(\rho)\Theta = M^2\Theta, \quad (6.2.112)$$

$$\tilde{V}(\rho) = \left(V(r) + \frac{n^2}{f_s(r)} \left(\frac{2\pi}{L_\varphi} \right)^2 - \frac{r}{N} f_s^{1/4} \frac{d}{dr} \left[r f_s \frac{df_s^{-1/4}}{dr} \right] \right) \Big|_{r=r(\rho)}.$$

Explicitly, the effective potential reads

$$\tilde{V}(\rho) = \frac{1}{N} - \frac{1}{N} \frac{1}{\sinh^2\left(\frac{2\rho}{\sqrt{N}}\right)} + \frac{n^2}{N} \coth^2\left(\frac{\rho}{\sqrt{N}}\right) \left(1 - \frac{r_-^2}{r_+^2} \tanh^2\left(\frac{\rho}{\sqrt{N}}\right) \right)^2, \quad (6.2.113)$$

which has the following asymptotic expansions

$$\begin{aligned} \tilde{V}(\rho \rightarrow 0) &= \left(n^2 - \frac{1}{4} \right) \frac{1}{\rho^2} + \frac{4}{3N} + n^2 \frac{2(r_+^2 - 3r_-^2)}{3Nr_+^2} + \mathcal{O}(\rho^2), \\ \tilde{V}(\rho \rightarrow +\infty) &= \frac{1}{N} - n^2 \frac{2r_-^2}{Nr_+^2} + \mathcal{O}\left(e^{-2N^{-1/2}\rho}\right). \end{aligned} \quad (6.2.114)$$

Note that the equation only depends on the parameters $r_\pm$ through the ratio $\lambda = -\frac{r_-^2}{r_+^2}$. The potential is unbounded from below when the leading term of the effective potential close to $\rho \rightarrow 0$ is negative. Requiring a potential that is bounded below leads to a bound in the angular momentum along the circle $n^2 \geq 1$. The only excitation for which the effective potential is unbounded from below is the S-wave. In the later case, the potential near the origin can be approximated by $V \sim -1/4\rho^2$, which, as highlighted in [166], needs to be studied carefully.

The previous bound is not enough to guarantee a discrete spectrum of excitations. For $0 \leq \lambda \leq 1$ the effective potential has a runaway behaviour (the minimum is at infinity), while for $\lambda > 1$ the potential has a minimum. We have a transition between a spectrum with discrete states ($\lambda > 1$) or without discrete states ($\lambda \leq 1$). Note that $\lambda = 1$ is precisely the SUSY point.

In what follows, we study the discrete spectrum of glueballs for $\lambda > 1$. Also, note that the potential (6.2.113) is invariant under $n \rightarrow -n$. Without loss of generality, we consider $n \geq 1$ (we include the $n \leq -1$ cases when writing the full solution, see below). Using the change of variables

$$z = \cosh\left(\frac{\rho}{\sqrt{N}}\right), \quad (6.2.115)$$

³Note that in the tortoise coordinate $\sqrt{\det(\hat{g}_{ij})} = \sqrt{f_s(\rho)}\text{Vol}(S^3)$ and therefore the Sturm-Liouville norm, is

$$\|\tilde{\psi}\|^2 = \int d\varphi d\rho \text{Vol}(S^3) \sqrt{f_s(\rho)} \left(f_s(r)^{-1/4} \Theta(\rho) \right)^2 = \int d\varphi d\rho \text{Vol}(S^3) |\Theta(\rho)|^2.$$

Therefore, when imposing boundary conditions, it is enough to require $\Theta(\rho)$ to be normalisable.

which maps the region $\rho \in [0, \infty[$ to $z \in [1, +\infty[$, we can write the profile of the metric perturbation $e^{2A}\tilde{\psi} = e^{-2A}f_s^{-1/4}\Theta(z)e^{i\frac{2\pi}{L\varphi}n\varphi} = \mathcal{H}(z)e^{i\frac{2\pi}{L\varphi}n\varphi}$ as

$$\begin{aligned} \mathcal{H} &= (z^2 - 1)^{-\frac{n}{2}} (z^2 + \lambda(z^2 - 1))^{\frac{1}{4}} \\ &\times \left[C_1 z^{n\lambda} {}_2F_1(a_+, a_-; c_-; z^2) + C_2 z^{-n\lambda} {}_2F_1(b_+, b_-; c_+; z^2) \right], \end{aligned} \quad (6.2.116)$$

where

$$\begin{aligned} a_{\pm} &= \frac{1}{2} \left(1 - n(\lambda + 1) \pm \sqrt{1 - M^2 N + n^2(1 + \lambda)^2} \right), \\ b_{\pm} &= \frac{1}{2} \left(1 + n(\lambda - 1) \pm \sqrt{1 - M^2 N + n^2(1 + \lambda)^2} \right), \\ c_{\pm} &= 1 \pm \lambda n. \end{aligned} \quad (6.2.117)$$

In what follows we consider $0 \leq M^2 N < 1 + n^2(1 + \lambda)^2$. We show below that this condition leads to a discrete spectrum of states. On the other hand, $M^2 N > 1 + n^2(1 + \lambda)^2$ leads to a continuous spectrum. We explain the origin of the continuous spectrum in the following subsection.

To impose boundary conditions at $z = 1$ and $z \rightarrow +\infty$ we use the following Kumar identities for the hypergeometric functions

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} (1-z)^{c-a-b} {}_2F_1(a, b; a+b-c+1; 1-z), \end{aligned} \quad (6.2.118)$$

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(b-z)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{z}\right) \\ &+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, b-c+1; -a+b+1; \frac{1}{z}\right). \end{aligned} \quad (6.2.119)$$

These are convenient since they allow us to center the Hypergeometric functions at zero (recall that ${}_2F_1(a, b; c; 0) = 1$) when expanding around the $z = 1$ and $z \rightarrow +\infty$. Using (6.2.118) and expanding (6.2.116) around $z \rightarrow 1$ we have

$$\begin{aligned} \lim_{z \rightarrow 1} \mathcal{H} &= (z^2 - 1)^{\frac{n}{2}} \Gamma(-n) \left(C_1 \frac{\Gamma(c_-)}{\Gamma(a_+)\Gamma(a_-)} + C_2 \frac{\Gamma(c_+)}{\Gamma(b_+)\Gamma(b_-)} \right) \\ &+ (z^2 - 1)^{-\frac{n}{2}} \Gamma(n) \left(C_1 \frac{\Gamma(c_-)}{\Gamma(c_- - a_+)\Gamma(c_- - a_-)} + C_2 \frac{\Gamma(c_+)}{\Gamma(c_+ - b_+)\Gamma(c_+ - b_-)} \right), \end{aligned} \quad (6.2.120)$$

while at $z \rightarrow +\infty$ the expansion reads⁴

$$\begin{aligned} \lim_{z \rightarrow +\infty} \mathcal{H} = & z^{-\frac{1}{2} - \sqrt{1 - M^2 N + n^2(1+\lambda)^2}} \left(C_1 \frac{\Gamma(a_- - a_+) \Gamma(c_-)}{\Gamma(a_-) \Gamma(c_- - a_+)} + C_2 \frac{\Gamma(b_+ - b_-) \Gamma(c_+)}{\Gamma(b_+) \Gamma(c_+ - b_-)} \right) \\ & + z^{-\frac{1}{2} + \sqrt{1 - M^2 N + n^2(1+\lambda)^2}} \left(C_1 \frac{\Gamma(a_+ - a_-) \Gamma(c_-)}{\Gamma(a_+) \Gamma(c_- - a_-)} + C_2 \frac{\Gamma(b_- - b_+) \Gamma(c_+)}{\Gamma(b_-) \Gamma(c_+ - b_+)} \right). \end{aligned} \quad (6.2.121)$$

We require the leading term in both cases to vanish. This is, we impose regularity condition at the origin ($z = 1$) and the normalisability of the excitations (fast decay at $z \rightarrow +\infty$). Let us start with (6.2.121). In order to make the leading term vanish we impose

$$C_2 = -C_1 \frac{\Gamma(a_+ - a_-) \Gamma(c_-)}{\Gamma(a_+) \Gamma(c_- - a_-)} \frac{\Gamma(b_-) \Gamma(c_+ - b_+)}{\Gamma(b_- - b_+) \Gamma(c_+)}. \quad (6.2.122)$$

We now move to (6.2.120). It might seem that there is an unavoidable divergence due to the factor of $\Gamma(-n)$ but we will show that this divergence can be cancelled by a precise choice of M^2 which also sets the leading term to zero.

Since we are considering $n \geq 1$, the leading term in (6.2.120) is the second one. The coefficient of the leading term in (6.2.120) is

$$\text{coeff}_{\text{leading}} = \frac{\Gamma(c_-)}{\Gamma(c_- - a_-)} \left(\frac{1}{\Gamma(c_- - a_+)} - \frac{\Gamma(a_+ - a_-)}{\Gamma(a_+)} \frac{\Gamma(b_-)}{\Gamma(b_- - b_+)} \frac{1}{\Gamma(c_+ - b_-)} \right), \quad (6.2.123)$$

We can set the coefficient to zero by imposing $c_- - a_- = -p$, with $p \in \mathbb{N}$, which leads to a quantisation condition for the glueball mass provided p satisfies the following bound

$$p \leq \frac{1}{2} (n(\lambda - 1) - 1) = p_{\text{max}}. \quad (6.2.124)$$

From this is clear that for $0 \leq \lambda \leq 1$ does not lead to a discrete spectrum: in that case is impossible to satisfy $p > 0$ and $p < p_{\text{max}}$ simultaneously. The same holds for $n = 0$.

The spectrum for the glueball masses reads (notice that this includes the $n \leq -1$ cases)

$$M^2 = \frac{1}{N} (1 + n^2(\lambda + 1)^2 - 4(p_{\text{max}} - p)^2). \quad (6.2.125)$$

On the other hand, noting that $c_- - a_n = a_+ + n$ we have that $a_+ = -n - p$, so that the factor of $1/\Gamma(a_+)$ cancels the divergence in $\Gamma(-n)$ in (6.2.120). Furthermore, with this choice for M^2 , from (6.2.122) we see that $C_2 = 0$. The solution satisfying the boundary conditions is

$$e^{2A} \tilde{\psi} = \sum_{p=0}^{\lfloor p_{\text{max}} \rfloor} \sum_{n \neq 0} C_{p,n} e^{i \frac{2\pi n}{L\varphi} \varphi} \frac{z^{|n|\lambda} (z^2 + \lambda(z^2 - 1))^{\frac{1}{4}}}{(z^2 - 1)^{\frac{|n|}{2}}} {}_2F_1(-|n| - p, 1 + p - |n|\lambda; 1 - |n|\lambda; z^2) \quad (6.2.126)$$

⁴From here we see that for $M^2 N > 1 + n^2(1 + \lambda)^2$ both branches are regular at infinity. There is no need to implement a boundary condition at infinity and hence, no quantisation condition for M^2 .

In Figure 6.4 we plot the radial profiles for this functions. The figure also shows the potential in eq.(6.2.112) and the masses of the first few glueballs.

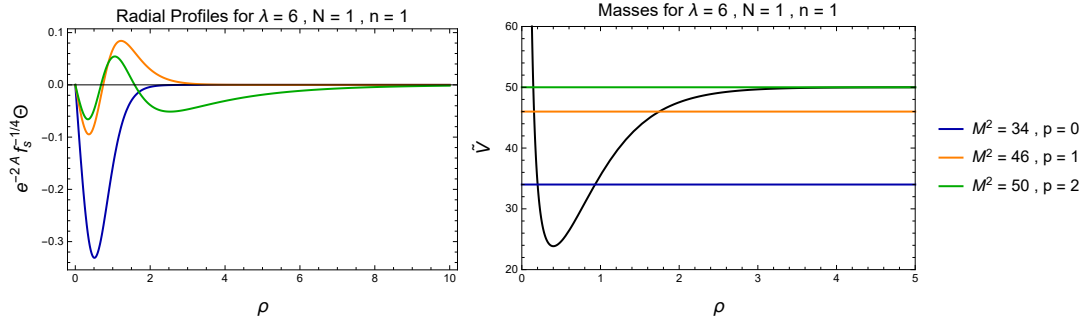


Figure 6.4: Glueball profile for a particular configuration $\lambda = 6, N = 1, n = 1$. Left: Radial profile for the three allowed values of $p = 0, 1, 2$. Right: In black the Schroedinger effective potential (6.2.113) and in colors the value of the glueball masses associated to the excitations $p = 0, 1, 2$.

On the Continuous Spectrum

Here we discuss the origin of the continuous spectrum that appears for the spin-2 glueballs discussed in the previous section. Let us begin by computing the spin-2 glueball spectrum on the pure D5-brane background, which in Einstein frame reads

$$ds_E^2 = \sqrt{r} \left(dx_{1,5}^2 + N \frac{dr^2}{r^2} + N ds^2(S^3) \right). \quad (6.2.127)$$

As before, we rewrite the metric as

$$ds_E^2 = e^{2A} (dx_{1,5}^2 + ds_y^2), \quad ds_y^2 = N \frac{dr^2}{r^2} + N ds^2(S^3), \quad e^{2A} = \sqrt{r}. \quad (6.2.128)$$

Before proceeding with the computation of the glueball solution, let us first go into the tortoise coordinate by $r = e^\rho$, which maps $r \in [0, +\infty[$ to $\rho \in \mathbb{R}$. The metric in the transverse space then reads

$$ds_y^2 = N d\rho^2 + N ds^2(S^3), \quad e^{2A(\rho)} = e^{\frac{\rho}{2}}. \quad (6.2.129)$$

Using (6.2.98) we have $V(\rho) = 1/N$. Separating variables $\Psi(y) = \Theta(\rho) Y_{\ell m}$, where $Y_{\ell m}$ are the spherical harmonics on the 3-sphere satisfying $\nabla_{S^3}^2 Y_{\ell m} = -\ell(\ell + 2) Y_{\ell m}$, then, eq.(6.2.97) takes the form

$$-\frac{d^2 \Theta}{d\rho^2} + [\ell(\ell + 2) + 1] \Theta(\rho) = M^2 N \Theta(\rho) \quad (6.2.130)$$

This is a Schroedinger-like equation with a constant potential $V_{eff} = \ell(\ell + 2) + 1$. Defining $E = M^2 N - \ell(\ell + 2) - 1$, the solutions of the problem depend on the sign of E (recall that M^2 is always positive) and are

$$\Theta(\rho) = \begin{cases} c_1 e^{i\sqrt{E}\rho} + c_2 e^{-i\sqrt{E}\rho} & E > 0 \\ c_1 e^{\sqrt{-E}\rho} + c_2 e^{-\sqrt{-E}\rho} & E < 0 \end{cases} \quad (6.2.131)$$

For $E < 0$, imposing regularity at $\rho \rightarrow \pm\infty$ sets $c_1 = c_2 = 0$ so that solution is not relevant. The only solution is the plane wave and we have a continuum of states.

The continuous spectrum for the spin-two excitations of the D5 brane suggests a natural interpretation for the spectrum we have found in above. In fact, we have there a spectrum that (for the parameter $\lambda > 1$) starts being discrete and then turns continuous. On the other hand, for $1 \leq \lambda$ the spectrum is solely continuous.

The way to think about this is the following. The size of the compactification circle φ is according to (6.1.32),

$$\frac{L_\varphi}{\pi\sqrt{N}} = 1 + \frac{m}{\sqrt{m^2 + 8Q^2}}. \quad (6.2.132)$$

In this expression the quantity $\pi\sqrt{N}$ is the size characteristic of the LST—see for example, the comment below (6.2.29).

For $m > 0$ we have $L_\varphi > L_{LST}$, for $m < 0$ we have $L_\varphi < L_{LST}$. The case $m = 0$ implies $\lambda = -\frac{r_-^2}{r_+^2} = 1$, which is the SUSY situation.

In the case $L_\varphi < L_{LST}$ there are some field theoretical discrete states. The energy of these states is less than the energy for which little strings take over the dynamics. Hence, the discrete set of states is followed by a continuous spectrum. This continuum is identified with the one encountered in the flat D5 brane case.

On the other hand, for $L_\varphi > L_{LST}$ the little string theory with its continuous spectrum controls the dynamics at all energies (for this particular spin-two observable).

We propose that this behaviour (a discrete of states, followed by a continuous spectrum) is characteristic of spin-two fluctuations like the ones considered here, see (6.2.59), (6.2.108) in holographic duals to QFTs constructed based on compactified D5 branes.

This feature should be absent in duals obtained compactifying Dp -branes ($p < 5$). For example, the analog glueball in a dual to field theoretical system written in [138] is purely discrete.

6.3 Comparison of the SUSY preservation mechanisms

In this Section we compare the SUSY preserving mechanism described in Section 6.1 with the partial topological twist. Although conceptually different, we highlight their similarities as can be seen from the holographic perspective

Let us now make a comparative study between the background in eq.(6.1.2), a dual to a $(4+1)$ confining QFT preserving eight supercharges and another background written in [74]. This second background is proposed as a dual to $(3+1)$ dimensional confining QFT with four preserved supercharges. The description above suggest that there are few commonalities between these solutions. To see some common features, let us quote the two metrics (a similar analysis can be made for the Dilaton and Ramond fields). For the background in eq.(6.1.2) we have

$$ds_{st}^2 = r \left[dx_{1,4}^2 + f_s(r) d\varphi^2 + \frac{N dr^2}{r^2 f_s(r)} + \frac{N}{4} \left(\omega_1^2 + \omega_2^2 + \left(\omega_3 - \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi \right)^2 \right) \right],$$

$$f_s(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \quad \zeta(r) = \frac{1}{r^2} - \frac{1}{r_+^2}. \quad (6.3.1)$$

For the background in [74]. we have

$$ds_{st}^2 = N e^{\Phi(r)} \left[\frac{1}{N} dx_{1,3}^2 + e^{2h(r)} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 \right. \\ \left. + \frac{1}{4} (\omega_1 + a(r) d\theta)^2 + \frac{1}{4} (\omega_2 - a(r) \sin \theta d\varphi)^2 + \frac{1}{4} (\omega_3 - \cos \theta d\varphi)^2 \right], \quad (6.3.2)$$

$$a(r) = \frac{2r}{\sinh(2r)}, \quad e^{2h(r)} = r \coth(2r) - \frac{1}{4} (a(r)^2 - 1), \quad e^{-4\Phi(r)} = \frac{4e^{-4\Phi_0 + 2h(r)}}{\sinh^2(2r)}.$$

The points we wish to emphasise are the following:

- The solution in eq.(6.3.2) is SUSY thanks to a twisting procedure, namely a mix between the Lorentz group and the $SO(4)$ R-symmetry group of the D5 brane theory. See [167, 168] for a detailed account of the twisting in this QFT. The twisting is reflected by the fibration of the $S^3(\omega_i)$ on the $S^2(\theta, \varphi)$. This fibre does not vanish at large values of the radial coordinate (the far UV of the QFT). The function $a(r)$ takes the form in eq.(6.3.2) to allow a smooth ending of the space at $r \sim 0$. Close to the end of the space, the dilaton presents a minimum and the (r, S^2) shrinks as flat space. The $SU(2)$ -valued one form performing the fibre $A = -a(r) d\theta T^1 + a(r) \sin \theta d\varphi T^2 + \cos \theta d\varphi T^3$ has zero curvature at the origin of the space.
- The background in eq.(6.3.1) preserves SUSY due to a mechanism nicely explained recently in [135], [137], see also Section 6.1.1. Whilst this is not a twisting procedure,

it shares some features with it. In fact, the fibre of the S^3 on the S^1 does not vanish for large values of the radial coordinate—by virtue of the $\frac{1}{r_+^2}$ term in the function $\zeta(r)$. This allows for spinors with anti-periodic boundary conditions to exist in this configuration. The rest of the one form performing the fibre (the part dependent on r) is what allows the configuration to have a smooth closure near $r \sim r_+$. In fact, the dilaton presents a minimum and the sub-space (r, φ) shrinks as flat space. The $U(1)$ -valued one form performing the fibre $A = \sqrt{\frac{8}{N}} Q \zeta(r) d\varphi$, vanishes at the end of the space $r = r_+$.

In other words, whilst both dual QFTs are different, some common features are shared. This is clear using the holographic (gravitational) description. The common characteristics of metrics and background fields are more apparent in that language.

6.4 Summary

In this chapter, we studied a more general version supersymmetric soliton in the D5-brane background. The background is SUSY for some choice of parameters, where it reduces to the holographic dual of the circle twisted compactification of maximally supersymmetric YM in 6D. In the IR, the QFT is $(4+1)$ dimensions. By computing different observables, we claim that the theory is confining, gapped and has a spectrum of spin-two glueballs that is either continuous or first discrete and then continuous (like the hydrogen atom) as the energy increases.

We then compared SUSY preservation via topological twist with the insertion of the Wilson line presented here: the key similarities being the non-vanishing of the fibre at infinity, and the radially dependent part making the IR regular.

Chapter 7

The Supersymmetric Soliton on the I-brane Background

On this final chapter, we focus on the 2D effective theory that arises at the intersection of two stacks of D5-branes on a line, known as I-brane theory [79]. Dynamical features of these theories imply that, as the coupling is increased the field theory turns $(2 + 1)$ -dimensional and doubles the amount of SUSY preserved.

The string background, holographic dual to this strongly coupled QFT is well understood at large values of the radial coordinate, where it can be written as an intersection of NS5-branes. Good control over the string σ -model in such background has been developed [79, 169]. Nevertheless, this background is singular in the IR (at small values of a suitable radial coordinate). In order to obtain a regular background where observables can be trusted in the IR regime, we include a supersymmetric soliton in the geometry.

In Section 7.1, we review general aspects of the theory on the D5-D5 intersection, and show how at strong coupling, the theory becomes 3D [79]. Then, Section 7.2, we present the supersymmetric soliton on the I-brane background, and analyse the dual field theory. In Section 7.3 we compute various characteristic observables. The IR confining behaviour is determined and the influence of the high energy LST dynamics on observables like Wilson and 't Hooft loops, and Entanglement Entropy is discussed. Some aspects of the string σ -model are discussed in Section 7.4.

7.1 I-brane Dynamics

Consider a stack of N_A D5-branes, stretched in the $(x^0, x^1, x^2, x^3, x^4, x^5)$ directions, intersecting a stack of N_B D5-branes spanning the $(x^0, x^1, x^6, x^7, x^8, x^9)$ directions [79]. The intersection between the two stacks is $(1 + 1)$ -dimensional consisting on the (x^0, x^1) direc-

tions. The configuration is depicted in Table 7.1. This configuration is invariant under $ISO(1,1) \times SO(4) \times SO(4)$ and it preserves 8 Supercharges.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D5	-	-	-	-	-	-	.	.	.	.
D5'	-	-	.	.	.	.	-	-	-	-

Table 7.1: Brane configuration of the D5-D5 intersection.

The effective theory on this brane configuration consists of the vector multiplets on each of the stacks, together with matter excitations arising from fundamental strings stretched between the two stacks of D5-branes. The later corresponds to $N_A N_B$ complex, Weyl fermions, which transform in the representation $(N_A, \bar{N}_B)$ of the $SU(N_A)$ and $SU(N_B)$ gauge groups.

As discussed in [79] the bifundamental Weyl fermions, which we take to be left-moving, lead to gravitational anomalies. The anomaly is resolved by the coupling between the fermions coming from the intersection with the 6D field living on the bulk of both D5-branes [170]. This anomaly inflow can be understood by, first, moving into spherical coordinates in the $\mathbb{R}_A^4 = (x^3, x^4, x^5, x^6)$ space as (r_A, S_A^3) , and similarly for $\mathbb{R}_B^4$ we use (r_B, S_B^3) . Reducing on S_A^3 and S_B^3 results in an effective 3D theory. The third direction, which we call y , consists of a combination of r_A and r_B , such that $y > 0$ corresponds to r_A , while $y < 0$ is r_B .

The 3D effective theory has a domain wall, the intersection, located at $y = 0$ ($r_A = r_B = 0$). At zero coupling, the theory on the intersection consists of the Weyl fermion, while on the upper half of the plane ($y > 0$) we have a $SU(N_A)$ gauge theory, similarly for $y < 0$ we have $SU(N_B)$ gauge theory. As we will show later, these gauge theories also have Chern-Simons terms coming from the fact that the D5-branes are wrapped on 3-spheres with non-vanishing R-R 3-form flux.

As the gauge coupling is increased, due to the interaction between the bulk (3D) gauge field, the chiral fermions are displaced from the intersection. When the defect moves to infinity, the system becomes gapped. Also, both Chern-Simons theories at different sides of the intersection combine nicely: due to level-rank duality, the $SU(N_A)_{N_B}$ and $SU(N_B)_{N_A}$ are the same. Effectively, the domain wall disappears and the theory becomes 3D. Moreover, the enhancement of the Poincare symmetry also leads to a SUSY enhancement, at strong coupling, the theory preserves 16 supercharges. For more details see [79].

It might seem odd that the theory gains an extra direction in the strong coupling regime. However recall that the theory was never purely 2D: the three dimensional degrees of freedom coming from the bulk gauge fields where necessary to cancel the fermion anomaly,

so that the dynamics of the theory was never 2D.

The appearance of an extra dimension at strong coupling is nicely captured by the Supergravity dual. We review this next.

7.1.1 Supergravity Dual

In order to construct the Supergravity background corresponding to the backreaction of the brane configuration described in Table 7.1, we use the harmonic superposition rule [171]. For this, first, we consider the Supergravity configuration for each stack independently. In terms of the harmonic factors (here we use spherical coordinates in both $\mathbb{R}_{A,B}^4$ as explained in the previous section, and also work in units in which $\alpha' = g_s = 1$)

$$H_A(r_B) = 1 + \frac{N_A}{r_B^2}, \quad H_B(r_A) = 1 + \frac{N_B}{r_A^2}, \quad (7.1.1)$$

where $H_{A,B}$ is the harmonic factor if the A, B stack respectively, the background configurations for each stack is

$$\begin{aligned} ds_{10D}^2 &= H_A^{-\frac{1}{2}} dx_{1,1}^2 + H_A^{-\frac{1}{2}} (dr_A^2 + r_A^2 ds^2(S_A^3)) + H_A^{\frac{1}{2}} (dr_B^2 + r_B^2 ds^2(S_B^3)), \\ F_3 &= 2N_A \text{Vol}(S_B^3), \\ \Phi &= -\frac{1}{2} \log(H_A(r_B)), \end{aligned} \quad (7.1.2)$$

for the stack A , and

$$\begin{aligned} ds_{10D}^2 &= H_B^{-\frac{1}{2}} dx_{1,1}^2 + H_B^{\frac{1}{2}} (dr_A^2 + r_A^2 ds^2(S_A^3)) + H_B^{-\frac{1}{2}} (dr_B^2 + r_B^2 ds^2(S_B^3)), \\ F_3 &= 2N_B \text{Vol}(S_A^3), \\ \Phi &= -\frac{1}{2} \log(H_B(r_A)), \end{aligned} \quad (7.1.3)$$

for the stack B . Recall that, in any Dp-brane metric, the factor of $H^{-1/2}$ means that the brane is parallel to those directions, while the factor of $H^{1/2}$ means that those directions are transverse to the brane. In order to construct the geometry of the intersection, we multiply the warping factors of the same coordinates, and we add the fluxes and Dilatons. This is the harmonic superposition rule, which results in

$$\begin{aligned} ds_{10D}^2 &= H_A^{-\frac{1}{2}} H_B^{-\frac{1}{2}} dx_{1,1}^2 + H_A^{-\frac{1}{2}} H_B^{\frac{1}{2}} (dr_A^2 + r_A^2 ds^2(S_A^3)) + H_A^{\frac{1}{2}} H_B^{-\frac{1}{2}} (dr_B^2 + r_B^2 ds^2(S_B^3)), \\ F_3 &= 2N_A \text{Vol}(S_B^3) + 2N_B \text{Vol}(S_A^3), \\ \Phi &= -\frac{1}{2} \log(H_A(r_B) H_B(r_A)). \end{aligned} \quad (7.1.4)$$

In order to obtain the geometry dual to the strong coupling regime of the field theory, we need to take the decoupling limit [19]. After the change of coordinates $r_A = e^{\frac{\rho_A}{\sqrt{N_B}}}$ and

$r_B = e^{\frac{\rho_B}{\sqrt{N_A}}}$ we find

$$\begin{aligned} ds_{10D}^2 &= e^{\frac{\rho_A}{\sqrt{N_B}} + \frac{\rho_B}{\sqrt{N_A}}} (dx_{1,1}^2 + d\rho_A^2 + N_B ds^2(S_A^3) + d\rho_B^2 + N_A ds^2(S_B^3)), \\ F_3 &= 2N_A \text{Vol}(S_B^3) + 2N_B \text{Vol}(S_A^3), \\ \Phi &= e^{\frac{\rho_A}{\sqrt{N_B}} + \frac{\rho_B}{\sqrt{N_A}}}. \end{aligned} \quad (7.1.5)$$

We seem to have two radial coordinates (ρ_A, ρ_B) , but this is artefact of the coordinate system. In fact, taking the linear combination

$$\begin{aligned} \sqrt{\frac{1}{N_A} + \frac{1}{N_B}} \rho_A &= \sqrt{\frac{1}{N_B}} \rho - \sqrt{\frac{1}{N_A}} x_2, \\ \sqrt{\frac{1}{N_A} + \frac{1}{N_B}} \rho_B &= \sqrt{\frac{1}{N_A}} \rho + \sqrt{\frac{1}{N_B}} x_2, \end{aligned} \quad (7.1.6)$$

we obtain

$$\begin{aligned} ds_{10D}^2 &= e^{\sqrt{\frac{1}{N_A} + \frac{1}{N_B}} \rho} (dx_{1,1}^2 + dx_2^2 + d\rho^2 + N_B ds^2(S_A^3) + N_A ds^2(S_B^3)), \\ F_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3), \\ \Phi &= \sqrt{\frac{1}{N_A} + \frac{1}{N_B}} \rho. \end{aligned} \quad (7.1.7)$$

Since $\rho_A, \rho_B \in \mathbb{R}$, the coordinates ρ, x_2 also are valued in $\mathbb{R}$. Note that neither the 3-form flux or the Dilaton depend on x_2 , so it is an isometry of the whole background configuration. Moreover, there is no difference between the Minkowski coordinates (t, x_1) and x_2 : they are on equal footing. For that reason, it is then natural to write instead

$$dx_{1,1}^2 + dx_2^2 \rightarrow dx_{2,1}^2, \quad (7.1.8)$$

which is precisely the enhancement of the dimensionality in the strong coupling regime of the I-brane theory discussed in the previous section. Finally, introducing the parameter $Q = \sqrt{\frac{1}{N_A} + \frac{1}{N_B}}$ and using $r = e^{Q\rho}$, we can write the Type IIB background configuration as (this solution was already found in [172, 173])

$$\begin{aligned} ds_{10D}^2 &= r \left(dx_{2,1}^2 + \frac{dr^2}{Q^2 r^2} + N_B ds^2(S_A^3) + N_A ds^2(S_B^3) \right), \\ F_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3), \\ \Phi &= \log(r). \end{aligned} \quad (7.1.9)$$

As mentioned before, the enhancement from $SO(1, 1) \rightarrow SO(2, 1)$ also leads to an enhancement of the amount of Killing spinors of the geometry from 8 to 16.

The effective string coupling constant and the Ricci scalar of the geometry are given by

$$g_s = e^\Phi = r, \quad R = -\frac{12Q^2}{r}. \quad (7.1.10)$$

Here the behaviour of this observables is similar to the one studied in the previous chapter¹ First, the effective string coupling constant grows large as we increase the energy (represented by the holographic coordinate r), so that perturbative String Theory is not the correct description in this regime. As in the D5-brane case, we need to S-dualise and give a description in terms of an intersection of NS5-branes. The system is then dual to two Little String Theories (LST) that intersect along (t, x_1, x_2) each one wrapping the spheres S_a^3 and S_B^3 . We further elaborate on this background in Section 7.4.

On the other hand, when $r \rightarrow 0$, while the string coupling constant is finite, the Ricci scalar diverges, so that the supergravity approximation is not correct anymore.

We are interested in resolving the singular behaviour at $r = 0$ of the background in (7.1.9), making the solution in terms of D5-branes trustable in the "IR-regime". The goal is to write a trustable dual description for a strongly coupled QFT (that is UV-completed by a LST). We are also interested in preserving some amount of SUSY for stability purposes. In order to achieve this goal, we introduce a supersymmetric soliton in the I-brane geometry.

7.2 SUSY Soliton in the I-brane Background

We write below a solution to the equations of motion of Type IIB supergravity that resolves the singular behaviour by compactifying the coordinate $x_2 = \varphi$, and including a SUSY soliton, which realises a twisted circle compactification of the dual field theory. This solution reads,

$$\begin{aligned}
 ds_{10D}^2 &= r \left[dx_{1,1}^2 + f_s(r) d\varphi^2 + \frac{dr^2}{Q^2 r^2 f_s(r)} + \frac{N_B}{4} \left(\hat{\omega}_1^2 + \hat{\omega}_2^2 + \left(\hat{\omega}_3 - \sqrt{\frac{8}{N_B}} Q_A \zeta(r) d\varphi \right)^2 \right) \right. \\
 &\quad \left. + \frac{N_A}{4} \left(\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \left(\tilde{\omega}_3 - \sqrt{\frac{8}{N_A}} Q_B \zeta(r) d\varphi \right)^2 \right) \right], \\
 F_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3) + 2Q_A \sqrt{\frac{N_B}{8}} d(\zeta(r) \hat{\omega}^3 \wedge d\varphi) \\
 &\quad + 2Q_B \sqrt{\frac{N_A}{8}} d(\zeta(r) \tilde{\omega}^3 \wedge d\varphi), \\
 \Phi &= \log(r)
 \end{aligned} \tag{7.2.1}$$

¹This similarities will appear all throughout this chapter, and it is due to the fact that both geometries are constructed using only D5-branes.

Here (N_A, Q_A, N_B, Q_B) are parameters. The functions $f_s(r), \zeta(r)$ are given by

$$\begin{aligned} f_s(r) &= 1 - \frac{m}{r^2} - 2 \frac{(Q_A^2 + Q_B^2)}{r^4} \equiv \frac{1}{r^4} (r^2 - r_+^2)(r^2 - r_-^2), \\ \zeta(r) &= \frac{1}{r^2} - \frac{1}{r_+^2}, \\ r_\pm^2 &= \frac{1}{2} \left(m \pm \sqrt{m^2 + 8(Q_A^2 + Q_B^2)} \right) \end{aligned} \quad (7.2.2)$$

If the parameter $m = 0$ and $N_A Q_B^2 = N_B Q_A^2$, the background preserves four supercharges. For the SUSY study and the details of the construction of the background in eqs.(7.2.1)-(7.2.2).

Note that the circle parametrised by the angle φ shrinks smoothly at $r = r_+$ if we choose its periodicity to be

$$\varphi \sim \varphi + L_\varphi, \quad L_\varphi = \frac{4\pi}{Q r_+ f'_s(r_+)} = \frac{\pi}{Q} \left(1 + \frac{m}{\sqrt{m^2 + 8(Q_A^2 + Q_B^2)}} \right). \quad (7.2.3)$$

In the BPS limit, the Ricci scalar associated with the geometry in eq.(7.2.1) is

$$R = -\frac{12Q^2}{r} + \frac{4mQ^2}{r^3} - \frac{8Q^2(Q_A^2 + Q_B^2)}{r^5}. \quad (7.2.4)$$

That is bounded for all the range of the radial coordinate $[r_+, \infty)$.

Notice that in the case $Q_A = Q_B = m = 0$, the background in eq.(7.2.1) becomes that in eq.(7.1.9). On the other hand, in the limit $r \rightarrow \infty$ since $\zeta(r)$ is non-vanishing, the background (7.2.1) is the same as (7.1.9) up to a large gauge transformation that cancels the fibration. In some of the observables we discuss below, we can perform a regularisation that takes away the effects on the observable that come from the background (7.1.9) from the same observable computed in the background (7.2.1). Also, in what follows, for any object, $\xi_{A,B}$, (like a D-brane) extended along the spheres we use the notation $\xi_A \equiv \hat{\xi}$ and $\xi_B \equiv \tilde{\xi}$.

To calculate the D5-brane charge associated with the Ramond Field F_3 in eq.(7.2.1), we integrate on both 3-spheres S_A^3 and S_B^3 . Let us call the A-stack of branes to be the one extended along the coordinates $[t, x, \varphi, \theta_A, \phi_A, \psi_A]$ and analogously for the B-stack.

To calculate the number of branes in the A-stack, we need to integrate F_3 over S_B^3 -as this is orthogonal to the A-brane stack. Analogously, the number of branes in the B-stack will be obtained by integrating F_3 over S_A^3 . We find

$$\begin{aligned} N_{D5}^A &= \frac{1}{(2\pi)^2} \int_{S_B^3} F_3 = N_A, \\ N_{D5}^B &= \frac{1}{(2\pi)^2} \int_{S_A^3} F_3 = N_B. \end{aligned} \quad (7.2.5)$$

7.2.1 Analysis of the Background and its Dual Field Theory

The Ricci scalar in eq.(7.2.4) is finite for $r > r_+$ and the string coupling (proportional to e^Φ) is bounded below some value $r < r_*$. The value of r_* is determined by observing that the dilaton in eq.(7.2.1) can be changed by $\Phi = \Phi_0 + \log r$ at the cost of rescaling the Ramond form $F_3 \rightarrow e^{-\Phi_0} F_3$. These scalings make the string coupling $g_s = e^\Phi = r e^{\Phi_0}$. The value $r_* \sim e^{-\Phi_0}$ (for which $g_s \sim 1$), can be made arbitrarily large by suitably choosing the parameter Φ_0 . Notice that a chosen large and negative Φ_0 , makes the charges of the D5 branes larger.

We then conclude that the background is trustable in a large region of the radial coordinate $[r_+, r_*)$. Consequently, in a large regime of energies the dual field theory is strongly coupled. At very high energies, when the string coupling becomes large we should S-dualise arriving to a configuration of intersecting NS5-branes with a SUSY Soliton. The high energy behaviour of the field theory is UV-completed in terms of a Little String Theory.

We now discuss the flowing to lower energies. In this case, the coordinate φ is compactified, shrinks to zero size and one ends with a QFT with less SUSY, smaller R-symmetry and effectively in $(1+1)$ dimensions. This is due to the twisted circle compactification realised by the supersymmetric soliton. Interestingly, note that the configuration is only SUSY if the precise tuning between the background gauge fields parameters $N_B Q_A^2 = N_B Q_B^2$. The effective theory is then the twisted reduction of the Yang-Mills-Chern-Simons $SU(N_A)_{N_B}$ to $(1+1)$, preserving four supercharges.

On the other hand, at large values of the radial coordinate, the SUSY soliton on the I-brane background in (7.2.1), asymptotes to the "pure" I-brane background in (7.1.9), except from the constant fibration of the S^1 over both three spheres, which is realised by the constant part of the function $\zeta(r)$, as in the previous SUSY solitons studied in this thesis.

7.3 Observables

In what follows, we start the study of this interesting field theory. We do so by calculating observables of the two-dimensional QFT using probes of the background in eqs. (7.2.1)-(7.2.2). These probes inform us about gauge couplings, theta-angles, symmetry breaking, confinement, etc. We present a quantity that indicates the number of degrees of freedom (density of states in terms of the energy).

7.3.1 Gauge Coupling

Calculation in the I-brane Background

Let us start by computing the gauge coupling in the background of eq.(7.1.9). For this, we follow the procedure explained in Section 6.2.1. We consider a probe D5-brane extended in the same directions as the A-stack of D5-branes of the background, that is, the brane is extended along the coordinates $(t, x_1, x_2, \theta_A, \phi_A, \psi_A)$. By expanding the Born-Infeld action, we find a Maxwell term, with coupling

$$\frac{1}{g_{YM,A}^2} = (2\pi)^4 T_{\hat{D}_5} N_B^{\frac{3}{2}}. \quad (7.3.1)$$

Similarly, the Wess-Zumino term gives,

$$\begin{aligned} S_{D5,WZ} &= -2\pi^2 T_{\hat{D}_5} \int C_2 \wedge F_2 \wedge F_2 \\ &= - \int_{S_A^3} F_3 \int_{t,x_1,x_2} A_1 \wedge F_2 \\ &= -N_B \int_{t,x_1,x_2} A_1 \wedge F_2. \end{aligned} \quad (7.3.2)$$

We have performed an integration by parts, used the quantisation condition in eq.(7.2.5) and set $T_5 = \frac{1}{(2\pi)^2}$ (in our units). There is a similarly symmetric calculation for a D5-brane probe along $[t, x, \varphi, \theta_B, \phi_B, \psi_B]$. In agreement with the field theory picture discussed above, we find two gauge groups with Yang-Mills Chern-Simons dynamics, $SU(N_A)_{N_B} \times SU(N_B)_{N_A}$, with fixed gauge couplings. This is exactly in agreement with the field theory expectations [79], that we summarised in the previous section.

Calculation in the SUSY Soliton Background

Let us now study the case for which the $(2+1)$ QFT is compactified using a twisting procedure along the φ -direction, so that we are dealing with a $(1+1)$ dimensional QFT. We work with the background in eqs.(7.2.1)-(7.2.2), and follow Section 6.2.1. We identify the Yang-Mills coupling for this probe D5-brane. In this case, there is an extra contribution to the gauge coupling coming from the fibration of S_φ^1 over S_B^3

$$\frac{1}{g_{YM,A}^2} = T_5 (2\pi)^2 \text{Vol}(S^3) N_B^{\frac{3}{2}} L_\varphi \sqrt{f_s(r) + 2Q_B^2 \zeta^2(r)} \quad (7.3.3)$$

In the BPS limit $N_A Q_B^2 = N_B Q_A^2$ and $m = 0$ on the parameters appearing in eqs.(7.2.2)-(7.2.3), the gauge coupling has the limiting values

$$\frac{1}{g_{YM,A}^2} = \begin{cases} T_5 (2\pi)^2 \text{Vol}(S^3) N_B^{\frac{3}{2}} L_\varphi \frac{N_A + 2N_B}{N_A + N_B} & , \quad r \rightarrow \infty \\ 0 & , \quad r \rightarrow r_+ \end{cases} \quad (7.3.4)$$

In other words, the gauge coupling grows very large at low energies and asymptotes to a constant value for high energies. As discussed, at very high energies the field theory is best

described in terms of a Little String Theory (LST). This calculation above refers to the gauge coupling $g_{YM,A}$. Other interactions in the QFT may become large at high energies, in such a way that the field theory is strongly coupled in the UV. This is in agreement with the background in eq.(7.2.1) being weakly curved for all values of the radial coordinate $[r_+, \infty)$.

Note that for large values of the holographic coordinate, the gauge coupling almost reduces to the one in (7.3.1). The difference in the factor of $(N_A + 2N_B)/(N_A + N_B)$ in (7.3.4), is due to the fact that the fibration does not vanish at infinity. In fact, it is possible to trace the contribution of the fibration to the factor of 2 which makes fraction not equal to one.

Had we studied a D5 probe extended along $[t, x_1, \varphi, \theta_B, \phi_B, \psi_B]$, with an electric field F_{tx} switched on the brane, the result would be,

$$\frac{1}{g_{YM,B}^2} = 8\pi^4 T_{\hat{D}_5} N_A^{\frac{3}{2}} \sqrt{f(r) + 2Q_A^2 \zeta(r)^2}. \quad (7.3.5)$$

7.3.2 Theta Angle

From the viewpoint of the $(2+1)$ dimensional QFT dual to (7.1.9), we can consider dimensionally reducing the Chern-Simons term obtained in eq.(7.3.2). We then obtain a theta-term proportional to $N_B \oint_{\varphi} A_{\varphi}$ for the QFT on the A-stack.

For the $(1+1)$ viewpoint, additional probes calculate the Θ -angle of each gauge group. Let us use a D3-brane probe, extended along $[t, x_1, \theta_A, \phi_A]$ with an electric field F_{tx_1} switched on. We study the Wess-Zumino term following eq.(6.2.4) and using the two-form potential pulled-back on this D3-brane probe

$$C_2|_{D_3} = -\frac{N_B}{4} \psi_A \sin \theta_A d\theta_A \wedge d\phi_A. \quad (7.3.6)$$

Replacing this C_2 in eq.(6.2.4) and using that $F_2 = F_{tx} dt \wedge dx$ we find the Wess-Zumino term for this probe is (note that $C_4 = 0$ in the background),

$$S_{WZ,D3} = -T_3 2\pi \int C_2 \wedge F_2 = 2T_3 \pi^2 N_B \psi_A \int F_{tx_1} dt \wedge dx_1. \quad (7.3.7)$$

The Θ -angle associated with the gauge group should be identified according to ,

$$S_{WZ,D3} = \frac{\Theta_A}{4\pi^2} \int dt dx_1 F_{tx_1}, \quad \longrightarrow \quad \Theta_A = T_3 8\pi^4 N_B = \frac{\psi_A N_B}{2}. \quad (7.3.8)$$

We have used $16\pi^4 T_3 = 1$ (in our units). Notice that the periodic identification $\Theta_A \sim \Theta_A + 2k\pi$ implies that the angle ψ_A gets quantised to the values

$$\Delta\psi_A = \frac{4k\pi}{N_B}, \quad \text{with } k = 0, 1, 2, \dots, N_B - 1. \quad (7.3.9)$$

For $k = N_B$ we have $\Delta\psi_A = 4\pi$, covering the full circle.

Had we considered the D3-brane probe extended along $[t, x_1, \theta_B, \phi_B]$ with all other coordinates fixed, we would have found (the calculation is exactly symmetrical), $\Theta_B = \frac{\psi_B N_A}{2}$.

Another way of understanding the R-symmetry breaking would be to consider euclidean D1-branes wrapping $[\theta_A, \phi_A]$. The Wess-Zumino action contributes to the partition function via

$$Z_{D1} \sim e^{\frac{i}{2\pi} \int C_2}. \quad (7.3.10)$$

This contribution should not depend on a (large) gauge transformation parameter ϵ_A , that appears as we change $\psi_A \rightarrow \psi_A + \epsilon_A$. We enforce

$$\left(\frac{i}{2\pi}\right) \left(\frac{-N_B \epsilon_A}{4}\right) \int \sin \theta_A d\theta_A d\phi_A = 2ik\pi, \text{ leads to } \epsilon_A = \frac{4k\pi}{N_B}. \quad (7.3.11)$$

These results indicate that the background's continuous isometries transforming $\psi_{A,B} \rightarrow \psi_{A,B} + \epsilon_{A,B}$, with $(\epsilon_{A,B})$ being constants, are actually broken. In fact, the allowed changes are $\Delta\psi_{A,B} = \frac{4k\pi}{N_{B,A}}$, with $k = 0, 1, 2, \dots, N_{B,A} - 1$, which should be interpreted as the breaking of the two field theory global symmetries $U(1)_{A,B}$ into discrete subgroups.

The argument used to derive eqs.(7.3.9)-(7.3.11) is not airtight. It uses a two manifold that has a boundary and at the same time a gauge choice is made for the potential C_2 in eq.(7.3.6). It would be nice to have an argument for R-symmetry breaking that is explicitly gauge invariant. In the coming section we present a different holographic perspective on the $U(1)_{A,B}$ breaking with this property.

7.3.3 $U(1)_{A,B}$ Symmetry Breaking Pattern

Some supersymmetric field theories exhibit a classical $U(1)$ R-symmetry that is quantum mechanically broken to a discrete subgroup. The symmetry breaking can be understood diagrammatically or in terms of instantons. The supergravity dual to the given field theory should encode this, but the mechanism should not involve instantons (as that are very suppressed in supergravity). The fact that the Ramond potentials are not gauge invariant under the $U(1)$ R-symmetry is key. The breaking of the global R-symmetry in the field theory manifest as spontaneous breaking in supergravity. The vector field in the bulk, dual of the R-symmetry current acquires a mass. We find this below for our background of eq.(7.2.1). The argument that follows is gauge invariant at all steps. All along this section, we set $m = 0$ and focus only in the BPS case.

The $U(1)$ symmetry of the metric is perturbed, the Lagrangian for this fluctuation is described by the usual $F_{\mu\nu}^2$ -term. For the perturbation to be consistent, the Ramond fields

must be also perturbed, this contributes to the mass term for the fluctuation. The massive gauge field is understood as symmetry breaking in supergravity.

In the previous section, we hinted at a breaking of the isometries represented by the Killing vectors $\partial_{\psi_{A,B}}$. To better understand the breaking of the field theory global $U(1)$ symmetries associated with the translations in $\psi_{A,B}$ we proceed as explained in [174, 175]. In the holographic background, we gauge the isometry by replacing, both in the metric and in the Ramond potential of eq.(7.2.1)

$$d\psi_{A,B} \rightarrow d\psi_{A,B} + A_{A,B}, \quad \psi_{A,B} \rightarrow \psi_{A,B} + \epsilon_{A,B}. \quad (7.3.12)$$

Such that a change $\psi_j \rightarrow \psi_j + \epsilon_j$ is compensated by $A_j \rightarrow A_j + d\epsilon_j$. We then study the Lagrangian for the gauge fields $A_{A,B}$, by replacing these changes in the string frame Lagrangian. After these replacements the metric and the R-R 3-form read

$$ds^2 = ds_{(0)}^2 + \frac{N_B r}{2} \left(\hat{\omega}_3 - \sqrt{\frac{8}{N_B}} Q_A \xi(r) d\varphi \right) A_{A\mu} dx^\mu + \frac{N_B r}{4} A_{A\mu} A_{A\nu} dx^\mu dx^\nu \\ + \frac{N_A r}{2} \left(\tilde{\omega}_3 - \sqrt{\frac{8}{N_A}} Q_B \xi(r) d\varphi \right) A_{B\mu} dx^\mu + \frac{N_A r}{4} A_{B\mu} A_{B\nu} dx^\mu dx^\nu \quad (7.3.13)$$

$$F_3 = F_3^{(0)} - 2d\epsilon_A \wedge \left(-Q_A \sqrt{\frac{N_B}{8}} \xi'(r) dr \wedge d\varphi + \frac{N_B}{8} \text{Vol}(S_A^2) \right) \\ - 2d\epsilon_B \wedge \left(-Q_B \sqrt{\frac{N_A}{8}} \xi'(r) dr \wedge d\varphi + \frac{N_A}{8} \text{Vol}(S_B^2) \right) \quad (7.3.14)$$

where $ds_{(0)}^2$ and $F_3^{(0)}$ denotes the metric and the 3-form of the configuration before gauging the $U(1)$ symmetries and $\text{Vol}(S_{A,B}^2) = \sin(\theta_{A,B}) d\theta_{A,B} \wedge d\phi_{A,B}$. Now we want to obtain an effective lagrangian for $A_{A,B}$ and $\epsilon_{A,B}$. In order to do this, we consider how the Ricci scalar and the kinetic term of F_3 change under the gauging of the symmetry. Explicitly, the Ricci scalar transforms as

$$R = R^{(0)} - \frac{1}{4} \frac{N_B r}{4} F_A^2 - \frac{1}{4} \frac{N_A r}{4} F_B^2, \quad (7.3.15)$$

where $F_{A,B}^2 = F_{A,B\mu\nu} F_{A,B}^{\mu\nu}$ and $F_{A,B\mu\nu}$ is the field strength of $A_{A,B\mu}$, while the kinetic term of the R-R 3-form reads

$$\frac{1}{12} F_{\mu\nu\lambda} F^{\mu\nu\lambda} = \frac{1}{12} F_{\mu\nu\lambda}^{(0)} F^{\mu\nu\lambda}_{(0)} + \frac{1}{2r^2} \left(\frac{Q_A^2 N_B r^2}{8} \xi'(r)^2 + 1 \right) (A_A - d\epsilon_A)^2 \\ + \frac{1}{2r^2} \left(\frac{Q_B^2 N_A r^2}{8} \xi'(r)^2 + 1 \right) (A_B - d\epsilon_B)^2 \\ + \frac{Q_A Q_B \sqrt{N_A N_B}}{8} (A_A - d\epsilon_A) \cdot (A_B - d\epsilon_B). \quad (7.3.16)$$

These perturbations around an isometry direction are usually known to be consistent. Finally, replacing these expressions in the Type IIB action (in string frame), leads to the

following effective Lagrangian

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} \frac{N_B}{4r} F_A^2 - \frac{1}{4} \frac{N_A}{4r} F_B^2 - \frac{1}{2r^2} \left(\frac{Q_A^2 N_B r^2}{8} \xi'(r)^2 + 1 \right) (A_A - d\epsilon_A)^2 \\ & - \frac{1}{2r^2} \left(\frac{Q_B^2 N_A r^2}{8} \xi'(r)^2 + 1 \right) (A_B - d\epsilon_B)^2 \\ & - \frac{Q_A Q_B \sqrt{N_A N_B}}{8} (A_A - d\epsilon_A) \cdot (A_B - d\epsilon_B)\end{aligned}\tag{7.3.17}$$

It is interesting to note that the kinetic term for the gauge field $A_{(A,B)}$ comes only from the Ricci scalar. The information of $\epsilon_{A,B}$ on the other hand, comes only from F_3^2 . Defining

$$W_{(A,B)} = A_{A,B} - d\epsilon_{A,B},\tag{7.3.18}$$

and imposing the BPS condition $N_A Q_B^2 = N_B Q_A^2$, the Lagrangian for the perturbation reads,

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} \frac{N_B}{4r} F_A^2 - \frac{1}{4} \frac{2N_A}{4r} F_B^2 + \frac{1}{2r^2} (W_\mu^{(A)} W^{(A)\mu} + W_\mu^{(B)} W^{(B)\mu}) \\ & + \frac{1}{2} \frac{4Q_A^2 N_B}{8r^6} (W_\mu^{(A)} + W_\mu^{(B)})^2.\end{aligned}\tag{7.3.19}$$

We interpret this result as follows. There are two $U(1)$ global symmetries in the QFT, holographically they are represented by the invariance of the metric under changes in ψ_A and ψ_B . These global symmetries are broken to discrete groups $\mathbb{Z}_{N_B}$ and $\mathbb{Z}_{N_A}$, as indicated by eq.(7.3.9). This breaking is an effect of the lack of invariance of the gauge potential C_2 . The breaking of the global symmetries is addressed in this section without appeal to the Ramond potentials, by observing that gauging the metric isometries $\partial_{\psi_{A,B}}$ leads to a breaking of the gauge symmetry, by a mass term. These mass terms are dependent on the radial coordinate.

Contrary to what happens for the duals to $\mathcal{N} = 1$ SYM, the metric in eq.(7.2.1) does not break these discrete isometries to $\mathbb{Z}_2$. In other words, there is not a radial-regime in the metric that explicitly breaks the isometry $\partial_{\psi_{A,B}}$. We interpret this result as the (anomalous) breaking of the two $U(1)_{A,B}$ in the QFT not being followed by a further spontaneous breaking. One might argue that VEVs are not allowed in a two dimensional QFT, hence no further breaking can take place by the formation of a condensate. This argument is not completely rigorous, as our QFT is two dimensional in the far IR, but get UV completed around the confining scale to a higher dimensional QFT.

We study now a different observable that gives an approximate idea of the number of degrees of freedom as a function of the energy (a density of states).

7.3.4 Holographic Central Charge

In this Section, we compute the holographic and flow central charges, defined in Section 4.3.2 and Section 4.3.4 respectively, to study the number of degrees of freedom on the QFT.

Let us first compute the quantities in eqs. (4.3.11), (4.3.12) and (4.3.13) for the solution in eq.(7.1.9), dual to a $(2 + 1)$ strong coupling regime of the I-brane theory. We find,

$$\begin{aligned} d = 2, \quad a(r, y^i) = r, \quad b(r) = \frac{1}{Q^2 r^2}, \quad V_{int} = \mathcal{N} r^2, \quad \mathcal{N} = (2\pi^2)^2 N_A^{\frac{3}{2}} N_B^{\frac{3}{2}}, \\ H = \mathcal{N}^2 r^4, \quad c_{hol} = \frac{\mathcal{N}}{4G_N Q^2} r^2. \end{aligned} \quad (7.3.20)$$

This quantity diverges at large energies, hinting at a UV completion in terms of a system in higher dimensions (a LST). Also, it vanishes for $r = 0$, indicating a gapped system. Notice nevertheless that the calculation should not be trusted close to $r = 0$, as the background (7.1.9) is singular there.

Let us now calculate the number of degrees of freedom for the background in eq.(7.2.1), as a dual to a $(1 + 1)$ -dimensional QFT. We find

$$\begin{aligned} d = 1, \quad a(r, y^i) = r, \quad b(r) = \frac{1}{Q^2 r^2 f_s(r)}, \\ V_{int} = \mathcal{N} r^2 \sqrt{f_s(r)}, \quad H = \mathcal{N}^2 r^4 f_s(r), \quad \mathcal{N} = (2\pi^2)^2 N_A^{\frac{3}{2}} N_B^{\frac{3}{2}} L_\varphi, \\ c_{hol} = \frac{\mathcal{N}}{4G_N Q} \frac{f_s(r) r^2}{(f_s(r) + \frac{r}{4} f'_s(r))}. \end{aligned} \quad (7.3.21)$$

At high energies the number of degrees of freedom grows unbounded (as r^2), signalling the UV completion in terms of a decompactified QFT in higher dimensions. At very low energies the number of degrees of freedom vanish, as $f_s(r_+) = 0$. In this case, the calculation is trustable, hence the gapped character of the system is clear. The reader might wonder about the difference in the factor of $1/Q$ in (7.3.20) compared to (7.3.21). This difference is resolved by recalling that $\mathcal{N}$ in (7.3.21) contains a factor of L_φ , and as seen from (7.2.3), it contains a factor of $1/Q$.

A related calculation can be done that encodes the fact that we can think our field theory as a three dimensional QFT with anisotropies (or a QFT with a flow across dimensions). This corresponds to the flow central charge defined in Section 4.3.4. Following that section, we have

$$\begin{aligned} d = 2, \quad a_0 = a_1 = r, \quad a_2 = r f_s(r), \quad \tilde{b} = \frac{1}{Q^2 r^2 f_s^{3/2}}, \\ \tilde{H} = \mathcal{N}^2 r^4 f_s(r), \quad \mathcal{N} = (2\pi^2)^2 N_A^{\frac{3}{2}} N_B^{\frac{3}{2}} L_\varphi, \\ c_{flow} = \frac{\mathcal{N}}{4Q^2 G_N} \frac{r^2 f_s}{(f_s(r) + \frac{r}{4} f'_s(r))^2} \end{aligned} \quad (7.3.22)$$

The result in eq.(7.3.22) has a similar interpretation. At low energies we have no degrees of freedom, at high energies an unbounded growth in the degrees of freedom indicates the UV completion. Note that the growth at high energies for the flow-central-charge—eq.(7.3.22), is slower than it is for the case in which we consider the QFT to be two dimensional, see eq. (7.3.21). This same feature occurs when considering flows between conformal points in different dimensions.

These results suggest that our QFT generates a mass gap at low energies. The behaviour of the gauge couplings at low energies, see eq.(7.3.4) suggest that the QFT is also confining. To ascertain the confining behaviour we calculate Maldacena-Wilson and 't Hooft loops, that provide order parameters for confinement. We also investigate the Entanglement Entropy (EE), which gives information about the interplay between a confining IR and the non-local UV dynamics of the QFT.

7.3.5 Wilson Loops

We start calculating the Maldacena-Wilson loops, following the generic treatment discussed in Section 6.2.2, in order to test the above proposal that at low energies the QFT presents a mass gap and confines. Let us apply the expressions in eqs.(6.2.14)-(6.2.22) to SUSY Soliton in eq.(7.2.1). When considering the rectangular embedding $t = \tau$, $x_1 = \sigma$ and $r = r(\sigma)$ the relevant functions are

$$F(r) = r, \quad G(r) = \frac{1}{Q\sqrt{f_s(r)}}, \quad (7.3.23)$$

which lead to the effective potential

$$V_{eff}(r) = Q \frac{r}{r_0} \sqrt{f_s(r)(r^2 - r_0^2)}. \quad (7.3.24)$$

Note that the effective potential for the Wilson for the rectangular embedding of the Wilson loop in the I-brane SUSY Soliton (7.3.24) is exactly the same as the one for the SUSY Soliton in the D5-brane background (6.2.28) studied in Section 6.2.2, up to the replacement $Q \rightarrow 1/\sqrt{N}$. This is due to the fact that $1/Q$ for the I-branes and $\sqrt{N}$ for the D5-branes correspond to the characteristic scale of these backgrounds constructed with D5-branes.

In spite of the field theory analysed here and that in Chapter 6 being qualitatively different, since the in both backgrounds the Wilson Loop computation reduces to the same 1D effective system, the analytic results for the energy and separation of the loop, together with the stability analysis studied in Section 6.2.2 also apply for this background. Put it differently, there is no difference between the rectangular Wilson loops computed in these two backgrounds constructed using D5-branes. Hence, we observe a confining behaviour, together with a minimal separation of the quark-antiquark pair (at large values of r_0), that

is associated with the scale of the LST. Plots for the quark-anti-quark energy and separation are thus the same as in Figure 6.2.

7.3.6 't Hooft Loops

As in Section 6.2.3, we consider an object with magnetic charge that effectively appears like a string for the (1+1) dimensional QFT. We propose to calculate the 't Hooft loop by studying the effective magnetic string obtained when extending a D5-brane along the directions $[t, x_1, \varphi, \theta_A, \phi_A, \psi_A]$, with $r(x_1)$. There is an analog magnetic string for the second gauge group, for the D5-brane in the configuration $[t, x_1, \varphi, \theta_B, \phi_B, \psi_B]$, with $r(x_1)$. We do not discuss the latter object, as its result is analog to the one obtained below. Note that this object becomes string-like when we consider the size of the S_φ^1 to be small enough to not admit excitations.

The induced metric for the above D5-brane is,

$$ds^2 = r \left\{ -dt^2 + \left(1 + \frac{1}{Q^2 r^2 f_s(r)} r'^2 \right) dx_1^2 + (f_s(r) + 2Q_B^2 \zeta(r)^2) d\varphi^2 + \frac{N_B}{4} \left[\hat{\omega}_1^2 + \hat{\omega}_2^2 + \left(\hat{\omega}_3 - \sqrt{\frac{8}{N_B}} Q_A \zeta(r) d\varphi \right)^2 \right] \right\}. \quad (7.3.25)$$

Then we calculate,

$$e^{-\Phi} \sqrt{-\det g_{MN}} = \left(\frac{N_B}{4} \right)^{3/2} r^2 \sin(\theta_A) \sqrt{\left(1 + \frac{r'^2}{Q^2 r^2 f_s(r)} \right) (f_s(r) + 2Q_B^2 \zeta(r)^2)}. \quad (7.3.26)$$

The action for this D5-brane is,

$$S_{\hat{D}_5} [r] = 2\pi^2 T_{D_5} L_\varphi N_B^{\frac{3}{2}} \int dt dx_1 \sqrt{(f_s(r) + 2Q_B^2 \zeta(r)^2) r^4 + \frac{r^2}{Q^2} \left(1 + \frac{2Q_B^2 \zeta(r)^2}{f_s(r)} \right) r'^2}. \quad (7.3.27)$$

Comparing with (6.2.15)-(6.2.17) we identify

$$F^2 = (f_s(r) + 2Q_B^2 \zeta(r)^2) r^4 \quad G^2 = \frac{1}{Q^2 r^2 f_s(r)} F^2, \quad T_{eff} = 2\pi^2 T_{D_5} L_\varphi L_t N_B^{\frac{3}{2}}, \quad (7.3.28)$$

where T_{eff} is the effective tension of the magnetic string. Following eq.(6.2.17), the effective potential

$$V_{eff}(r) = Q \frac{r^3}{C} \sqrt{f_s(r) \left(f_s(r) + 2Q_B^2 \zeta(r)^2 - \frac{C^2}{r^4} \right)}. \quad (7.3.29)$$

The constant $C = F(r_0) = \sqrt{(f_s(r_0) + 2Q_B^2 \zeta^2(r_0)) r_0^2}$. In the asymptotic region the effective potential in eq.(7.3.29) scales as $V_{eff}(r \rightarrow \infty) \sim r^3$, satisfying the condition in eq.(6.2.20).

The intuitive criteria discussed in the previous section applied for this case indicate a screening behaviour. In fact $F(r_+) = 0$ and $V_{eff} \sim (r - r_+)^{1/2}$, signalling a vanishing tension of the magnetic QCD-string in the IR, and a finite maximal separation, after which the monopoles are screened (more about this is discussed below).

We perform a similar analysis using the *approximate* expression for the separation between the monopole-anti-monopole pair $\hat{L}_{MM}$. Replacing in eq.(6.2.22) the functions in eq.(7.3.28), we find an involved expression that asymptotes as,

$$\hat{L}_{MM}(r_0 \rightarrow \infty) \sim \frac{\pi}{2Q}, \quad \hat{L}_{MM}(r_0 \rightarrow r_+) \sim \frac{\pi\sqrt{2}}{Q} \sqrt{\frac{r_+(r_0 - r_+)}{r_+^2 - r_-^2}}. \quad (7.3.30)$$

These asymptotic behaviours indicate that at high energies in the field theory, the pair is separated by a maximum distance characteristic of the UV completion (Little String Theory). In this sense, the magnetic string behaves oppositely to the electric one used to compute the Wilson loop (which shows a minimum separation). On the other hand, at low energies the separation decreases to zero (again, oppositely to the electric string case). We calculate $Z(r_0)$ in eq.(6.2.23) for this configuration and find that is positive in all the range. This indicates the instability of the U-shaped embeddings. A more general embedding that the one proposed here should drive the dynamics. Instead of finding this more complicated embedding, below we introduce a UV-cutoff. New short configurations appear close to a cutoff, that dominate the dynamics and produce a phase transition (to a deconfining behaviour).

More formally, we write expressions for the separation between the monopole-anti-monopole pair as those in eqs.(6.2.18)-(6.2.19). In terms of the elliptic integral of the first kind

$$\mathbf{F}(y|x) = \int_0^y \frac{d\theta}{\sqrt{1 - x \sin^2 \theta}}, \quad (7.3.31)$$

and the definition for $\mathbf{K}(\mathbf{x})$ in eq.(6.2.33), working in the BPS limit and defining

$$\xi = \frac{r}{r_+}, \quad \xi_0 = \frac{r_0}{r_+}, \quad \eta^2 = \frac{N_A}{N_B}, \quad \lambda_-^2 = -\frac{r_-^2}{r_+^2} \quad (7.3.32)$$

we find

$$\begin{aligned} L_{MM}^{BPS}(\xi_0) = & \frac{2}{e_B} \sqrt{\frac{2(\xi_0^2 - 1)(\eta^2 + 2)\xi_0^2 + \eta^2}{(\eta^2 + 1)(\xi_0^2(\eta^2 + 2) - 1)}} \\ & \times \left[\mathbf{F} \left(\arcsin \sqrt{\frac{2(-1 + (2 + \eta^2)\xi_0^2)}{(2\xi_0^2 + \eta^2(1 + \xi_0^2))}} \middle| \frac{(1 + \xi_0^2)(2\xi_0^2 + \eta^2(1 + \xi_0^2))}{-4 + 4(2 + \eta^2)\xi_0^2} \right) \right. \\ & \left. + i\mathbf{K} \left(1 - \frac{(1 + \xi_0^2)(2\xi_0^2 + \eta^2(1 + \xi_0^2))}{-4 + 4(2 + \eta^2)\xi_0^2} \right) \right]. \end{aligned}$$

For the energy of the t' Hooft loop we have a similar expression to the one obtained when computing the energy of the Wilson loop

$$E_{MM}(r_0) = F(r_0) L_{MM}(r_0) + I_2 + I_3, \quad (7.3.33)$$

where

$$I_2 = \frac{4r_+^2}{\sqrt{\eta^2 + 1}} \int_{\xi_0}^{+\infty} d\xi \sqrt{\frac{\xi^2 (\xi^2 - \xi_0^2) (\eta^2 (\xi^2 + \xi_0^2) + 2 (\xi^2 + \lambda_0^2 - 1))}{\xi^4 - 1}}, \quad (7.3.34)$$

$$I_3 = -2 \int_{r_+}^{+\infty} dz G(z) = -\frac{4r_+^2}{\sqrt{(1 + \eta^2)}} \int_1^\infty d\xi \xi \sqrt{\frac{2\xi^2 + \eta^2 (\xi^2 + 1)}{(1 + \xi^2)}}. \quad (7.3.35)$$

These integrals are quite involved and present technical difficulties to be performed in terms of known functions. Therefore we compute them numerically up to a large value of the upper limit $\tilde{\xi}_{\max}$. Since $\tilde{\xi}_{\max}$ is finite the integrals are convergent and we can write them together in terms of a single integral which depends on $\tilde{\xi}_{\max}$

$$\begin{aligned} E_{MM}(r_0) = F(r_0) L_{MM}(r_0) &- \left[\frac{4r_+^2}{\sqrt{1 + \eta^2}} \int_1^{\xi_0} \xi \sqrt{\frac{2\xi^2 + \eta^2 (\xi^2 + 1)}{(1 + \xi^2)}} \right. \\ &- \lim_{\tilde{\xi}_{\max} \rightarrow \infty} \int_{\xi_0}^{\tilde{\xi}_{\max}} d\xi \left(\sqrt{\frac{\xi^2 (\xi^2 - \lambda_0^2) (\eta^2 (\xi^2 + \xi_0^2) + 2 (\xi^2 + \xi_0^2 - 1))}{\xi^4 - 1}} \right. \\ &\quad \left. \left. - \xi \sqrt{\frac{2\xi^2 + \eta^2 (\xi^2 + 1)}{(1 + \xi^2)}} \right) \right]. \end{aligned} \quad (7.3.36)$$

We verify that the integral converges to a limiting value for $\tilde{\xi}_{\max}$ large enough and much bigger than $\xi_0 = r_0/r_+$.

In Figure 7.1 we compare this exact expression with the approximating function whose asymptotics we write in eq.(7.3.30). As stated above, opposite to the Wilson loop case, we encounter a maximal separation for the monopole pair. This maximal separation is associated with the Little String Theory scale. We also plot the string profiles as they enter the bulk. We observe a different behaviour to that found in the case of the Wilson loop—compare with the lower right panel of Figure 6.2.

For magnetic strings that barely explore the bulk, the separation between the monopole pair is large (equal to the LST scale). As we decrease the separation between the monopole pair, the magnetic string dives into $r_0 \rightarrow r_+$. These unconventional behaviours, together with the instability of the string embedding—note that $E_{MM}(L_{MM})$ has upwards concavity (and $Z(r_0)$ is positive), indicate the presence of a second 'disconnected' configuration for which the pair of monopoles separate without energy expense, which is indicative of screening. The transition to a disconnected configuration is dynamically favoured.

To avoid the 't Hooft loop to be driven by the UV (LST) dynamics and to realise explicitly these short configurations, we introduce a hard cutoff in the radial direction and recalculate things. Doing so, the behaviour changes qualitatively.

In fact, the separation between the pair of monopoles does not show a maximum value, instead we find a 'double valued' behaviour as displayed in the left panel of Figure 7.2. This leads to a phase transition in the curve $E_{MM}(L_{MM})$. Note that the curve now has the correct 'downwards' concavity, indicating that the configuration is stable. This phase transition is the physical manifestation of the magnetic string suddenly changing into two disconnected magnetic strings that move without energy expense. This is deconfinement for the pair monopole anti-monopole. We are finding confinement for the quark-anti-quark pair and screening for a pair of monopoles. These behaviours are consistent with the (electric) confining behaviour of the dual QFT.

The introduction of the UV cutoff might seem unsatisfactory. Here we use it as a device to show that the correct five brane embedding must be more elaborated than the one we proposed above. It is also used to avoid the LST overtaking the dynamics. The effect of the cutoff is clear considering the integral needed to calculate $L(r_0)$. This integral vanishes for $r_0 \rightarrow r_{MAX}$. This produces a double-valued $L(r_0)$ and a consequent phase transition.

For the Entanglement Entropy a very similar behaviour occurs. We study this next.

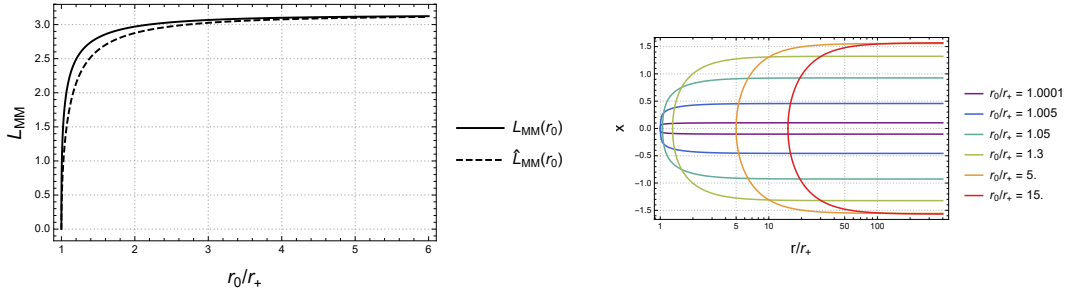


Figure 7.1: Left: Plot comparing the exact expression for the monopole separation L_{MM} in eq.(7.3.33) with the approximate one $\hat{L}_{MM}$ in eq.(7.3.30), both in the BPS limit. Right: the profiles of different strings as they enter the bulk. Strings with small separation of the monopole pair penetrate deeper into the bulk. There is a maximum separation for the pair of monopoles, associated with the Little String Theory scale.

7.3.7 Entanglement Entropy

In this section, we follow the procedure explained in Section 6.2.5, that is, we have to find a minimal area codimension-two surface that minimises (6.2.50). As discussed before, there are various eight-surfaces that minimise S_{EE} in eq.(6.2.50). Due to this, in some cases there is a phase transition between different extremal surfaces. It was suggested in

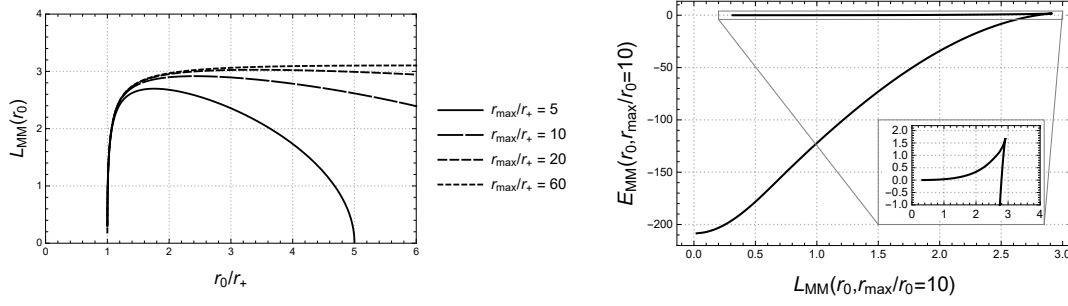


Figure 7.2: Left: after introducing a cutoff at r_{MAX} , the figure shows the double-valued character of L_{MM} . Removing the cutoff recovers the LST behaviour (on the left panel of Figure 7.1). Right: with the UV cutoff at $r_{MAX} = 20$, we see that E_{MM} as a function of L_{MM} has the 'upwards' concavity (indicating stability). The presence of the phase transition to a disconnected configuration is observed.

[162] that a criterion for confinement is the presence of a phase transition in the EE. This proposal was critically analysed in [159, 163]. It was found in [159], that for the case of field theories that confine, but have a non-local high energy behaviour, the phase transition in the EE is absent. The point is subtle, as introducing a UV cutoff or UV-completing the QFT to avoid the non-locality, recovers the phase transition.

It is in this way that the EE can serve as an order parameter for confinement, but also as a tool to diagnose non-locality in the UV-behaviour of the QFT (when used together with a confining Wilson loop). In [159] it was found that introducing a UV-cutoff implies the existence of new configurations realising the phase transition (and resolving a stability issue with the original eight-surface). Below, we perform an analysis of these features in our background of eq.(7.2.1).

We calculate the Entanglement Entropy on a strip by computing the area of an eight-surface $[x, \varphi, \theta_A, \phi_A, \psi_A, \theta_B, \phi_B, \psi_B]$ with $r = r(x)$ in the background of eq.(7.2.1). The induced metric on the RT eight-surface, its determinant and the Entanglement Entropy are,

$$ds_{st}^2 = r \left\{ \left(1 + \frac{r'^2}{Q^2 r^2 f_s(r)} \right) dx^2 + f_s(r) d\varphi^2 + \frac{N_B}{4} \left[\hat{\omega}_1^2 + \hat{\omega}_2^2 + \left(\hat{\omega}_3 - \sqrt{\frac{8}{N_B}} Q_A \zeta(r) d\varphi \right)^2 \right] \right. \\ \left. + \frac{N_A}{4} \left[\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \left(\tilde{\omega}_3 - \sqrt{\frac{8}{N_A}} Q_B \zeta(r) d\varphi \right)^2 \right] \right\}, \quad (7.3.37)$$

then

$$\begin{aligned}\sqrt{e^{-4\Phi} \det[g_8]} &= \left(\frac{N_A N_B}{16} \right)^{\frac{3}{2}} \sqrt{r^4 f_s(r) + \frac{r^2 r'^2}{Q^2} \sin(\theta_A) \sin(\theta_B)} \\ S_{EE} &= \frac{1}{4G_N} \int d^8x \sqrt{e^{-4\Phi} \det[g_8]} = \frac{\mathcal{N}}{G_N} \int_{-L/2}^{L/2} dx \sqrt{r^4 f_s(r) + \frac{r^2 r'^2}{Q^2}}, \\ \mathcal{N} &= \pi^4 L_\varphi (N_A N_B)^{\frac{3}{2}}.\end{aligned}\tag{7.3.38}$$

From eqs.(6.2.15)-(6.2.17), this implies

$$F(r) = r^2 \sqrt{f_s(r)}, \quad G(r) = \frac{r}{Q},\tag{7.3.39}$$

which lead to the effective potential

$$V_{eff} = \frac{Q}{r_0^2 \sqrt{f_s(r_0)}} \sqrt{r^2 f_s(r) (r^4 f_s(r) - r_0^4 f_s(r_0))}.\tag{7.3.40}$$

To minimise the S_{EE} above, we follow the usual conserved Hamiltonian treatment. The Entanglement Entropy needs to be regularised by the area of two eight-surfaces that hang straight from infinity. Then, computing the regulated area for a surface that turns around at r_0 , we find for the length of the interval and the Entanglement Entropy,

$$L = 2 \frac{r_0^2}{Q} \sqrt{f_s(r_0)} \int_{r_0}^{\infty} \frac{dr}{\sqrt{r^2 f_s(r) (r^4 f_s(r) - r_0^4 f_s(r_0))}},\tag{7.3.41}$$

$$S_{EE} = \frac{\mathcal{N}}{G_N} \left[\int_{r_0}^{\infty} \sqrt{\frac{r^6 f_s(r)}{r^4 f_s(r) - r_0^4 f_s(r_0)}} dr - \int_{r_+}^{\infty} r dr \right].\tag{7.3.42}$$

As in eq.(6.2.22), we can write a simple expression that approximates L_{EE} in eq.(7.3.41)—see [159],

$$\hat{L}_{EE} = \frac{\pi G(r_0)}{F'(r_0)} = 2\pi \frac{H(r) \sqrt{\beta(r)}}{H'(r)} \Big|_{r_0}, \quad \text{with } H(r) = \mathcal{N}^2 r^4 f_s(r), \quad \beta(r) = \frac{1}{Q^2 r^2 f_s(r)}.\tag{7.3.43}$$

Using eq.(7.3.43) we find

$$\hat{L}_{EE} = \left(\frac{\pi}{Q} \right) \frac{\sqrt{(r_0^2 - r_+^2)(r_0^2 - r_-^2)}}{(2r_0^2 - r_+^2 - r_-^2)}.\tag{7.3.44}$$

This function is monotonous, going from a vanishing value at $r_0 = r_+$ to a constant value at $r_0 \rightarrow \infty$. This behaviour prevents the possibility of phase transitions, which require that for a given L_{EE} there are two possible values of r_0 . In fact, the conditions for the presence of a phase transition (see section 2.4 of the work [159]) are not satisfied, in particular equations (2.26)-(2.29) of [159] imply $j = 2$ preventing a phase transition. The absence of a phase

transition in a confining model was interpreted in [159] as an effect of the non-locality of the completion of the QFT, in this case, by a LST.

Since $Z(r_0)$ defined in eq.(6.2.23) gives

$$Z(r_0) = \frac{\sqrt{8}\pi(r_+^2 - r_-^2)^2}{r_0 f_s(r_0)(r_+^2 + r_-^2 - 2r_0^2)^2} > 0, \quad (7.3.45)$$

the proposed embedding is unstable. Upon the introduction of a cutoff, new surfaces appear as found in [159], [176],[177]. These cure the instability problem of the embedding and give place to the phase transition, in agreement with confinement.

The treatment in the papers [159], [176], [177] applies to our background, even when the IR dynamics is different, the UV dynamics is similarly driven by a Little String Theory.

Before discussing the presence (or absence) of phase transitions we write the analytic expressions for the values of the separation between the two entangled regions $L_{EE}(r_0)$ and the Entanglement Entropy $S_{EE}(r_0)$. Using the definitions in eqs.(6.2.33),(7.3.31) and recalling that $\lambda_-^2 = -\frac{r_-^2}{r_+^2}$, we find

$$\begin{aligned} L_{EE}(r_0) = \frac{2}{Q} \sqrt{\frac{r_0^2/r_+^2 - 1}{r_0^2/r_+^2 + \lambda_-^2}} & \left[i\mathbf{K} \left(\frac{(r_0^2/r_+^2 - 1)^2}{(r_0^2/r_+^2 + \lambda_-^2)^2} \right) \right. \\ & \left. + \mathbf{F} \left(\arcsin \sqrt{\frac{r_0^2/r_+^2 - r_-^2/r_+^2}{1 - r_-^2/r_+^2}} \middle| \frac{(1 - r_-^2/r_+^2)(-r_-^2/r_+^2 + 2r_0^2/r_+^2 - 1)}{(r_0^2/r_+^2 - r_-^2/r_+^2)^2} \right) \right] \end{aligned} \quad (7.3.46)$$

and

$$S_{EE}^{BPS}(r_0) = \frac{\mathcal{N}}{G_N} r_+^2 \left[\frac{r_+^2}{2r_0^2} (-r_0^4/r_+^4 \mathbf{E}(r_+^4/r_0^4) - \mathbf{K}(r_+^4/r_0^4) + r_0^4/r_+^4 \mathbf{K}(r_+^4/r_0^4)) + \frac{1}{2} \right]. \quad (7.3.47)$$

To analyse these expressions, it is useful to show some plots. First, we check that the approximate $\hat{L}_{EE}(r_0)$ in eq.(7.3.44) approximates well the analytic expression in eq.(7.3.46), see the left panel of Figure 7.3. We also plot $S_{EE}(r_0)$, see the right panel of Figure 7.3. The plot of S_{EE} in terms of L_{EE} in the left panel of Figure 7.4, shows an upwards concavity indicating that the configuration is unstable. This follows the prediction of [159] that indicates that new configurations should appear as we introduce a UV-cutoff in the geometry. The profiles of the effective strings shown in the right panel of Figure 7.4, display a behaviour similar to the one we encountered in the study of 't Hooft loops (and opposite to that of the Wilson loop), again suggesting the need for a phase transition.

In analogy with the case of the 't Hooft loop, if we introduce a UV-cutoff, the separation between the two entangled regions becomes multiple-valued, as shown in Figure 7.5 (right panel). This is at the root of the phase transition. The plot of $S_{EE}(L_{EE})$ shows the correct concavity and the presence of a transition to the disconnected configuration is clearly displayed. See Figure 7.5

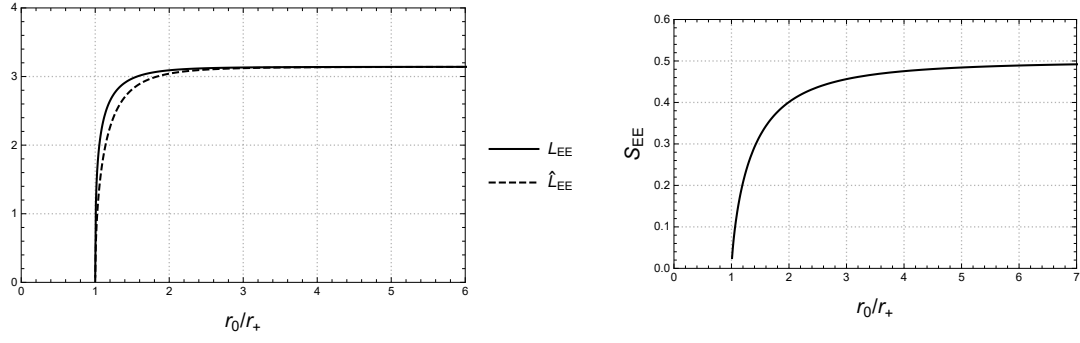


Figure 7.3: Left: Plot comparing the exact expression for the separation of the entangled regions L_{EE} in eq.(7.3.46) with the approximate one $\hat{L}_{EE}$ in eq.(7.3.44). Right: The plot of $S_{EE}(r_0)$.

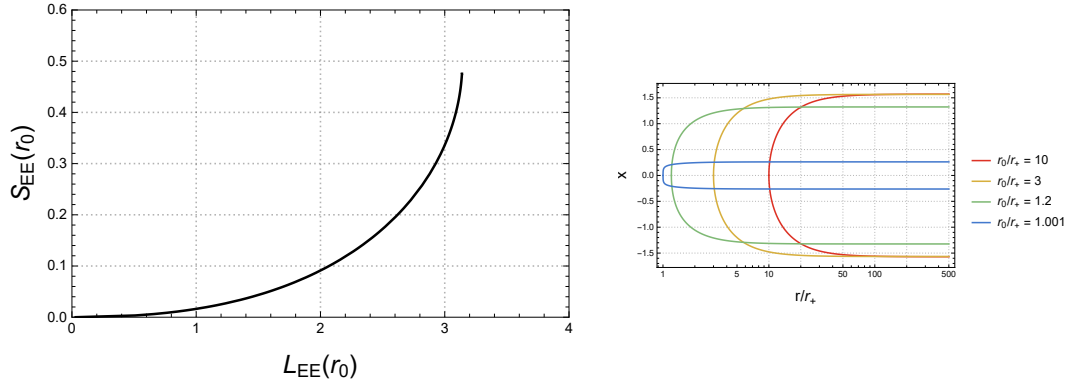


Figure 7.4: Left: The plot of $S_{EE}(L_{EE})$ showing the 'downwards' concavity (instability). Right: The profile of the effective strings used in computing the EE. Note the analogous behaviour with the 't Hooft loops strings.

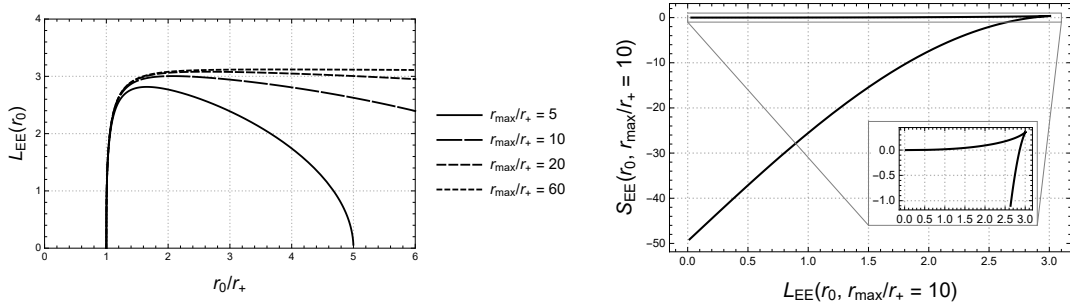


Figure 7.5: Left: Plot of L_{EE} in eq.(7.3.46) imposing the presence of a UV-cutoff. Right: The plot of $S_{EE}(L_{EE})$. Notice that the concavity has changed, indicating the stability of the configuration. The phase transition becomes apparent thanks to the cutoff, avoiding the non-local behaviour

Following the findings of [159], we state that if a field theoretical UV completion to our system (that is completed by a LST) were found, the phase transitions for the 't Hooft loops and the EE would become apparent. In this sense the UV-cutoff captures the correct dynamics.

7.4 Sigma Model

In this section we review some results [79, 169], concerning the string σ -model on the background in eq.(7.1.9). To properly study the string action we move to the S-dual frame, and work in terms of NS5-branes. The background reads (here we used $r = e^{Q\rho}$)

$$\begin{aligned} ds_{10D}^2 &= -dt^2 + dx_1^2 + dx_2^2 + d\rho^2 + N_B ds^2(S_A^3) + N_A ds^2(S_B^3), \\ H_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3), \\ \Phi &= -\sqrt{\frac{1}{N_A} + \frac{1}{N_B}} \rho. \end{aligned} \quad (7.4.1)$$

The background is a product space of the form

$$\mathbb{R}^{2,1} \times \mathbb{R}_\rho \times S_{N_A}^3 \times S_{N_B}^3, \quad (7.4.2)$$

where $\mathbb{R}_\rho$ denotes the direction (with the linear dilaton), and the subscript on the S^3 denotes the square of their radius. The metric on the spheres together with the H_3 flux on each of them allows us to write the σ -model on them as a WZW model on $SU(2)$. This is due to the fact the S^3 is a group manifold. Naively, the contribution of the ρ coordinate to the string action is (for clarity we reinstate the α' -factor)

$$S_\rho = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left(h^{ab} \partial_a \rho \partial_b \rho - \alpha' \sqrt{\frac{1}{N_A} + \frac{1}{N_B}} R^{(2)} \rho \right), \quad (7.4.3)$$

where $R^{(2)}$ is the world-sheet Ricci scalar. However when $\rho \rightarrow -\infty$ the theory becomes non-perturbative, since $g_s \sim e^{-\rho}$. In order to avoid the strong coupling region, it is necessary to add the tachyon operator $e^{2b\rho}$ to the action, so that

$$S_\rho = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left(h^{ab} \partial_a \rho \partial_b \rho - \alpha' Q R^{(2)} \rho + \Lambda e^{2b\rho} \right). \quad (7.4.4)$$

Here, b is related to the background charge $Q = \sqrt{\frac{1}{N_A} + \frac{1}{N_B}}$ as

$$Q = b + \frac{1}{b},$$

such that it does not have a strong coupling region. Thus the contribution of the ρ direction to the σ -model corresponds to a Liouville field. The complete σ -model on this geometry is

$$U(1)^3 \times \text{Liouville} \times SU(2)_{N_A} \text{WZW} \times SU(2)_{N_B} \text{WZW} \quad (7.4.5)$$

where the subscripts denote the WZW level. One can check that this is indeed a good σ -model by computing the central charge. Here we use

$$c(U(1)) = 1, \quad c(\text{Liouville}) = 1 + 6Q^2, \quad c(SU(2)_k) = \frac{3(k-2)}{k}. \quad (7.4.6)$$

In total we have

$$c_{\text{total}} = 3 + 1 + 6 \left(\frac{1}{N_A} + \frac{1}{N_B} \right) + \frac{3(N_A - 2)}{N_A} + \frac{3(N_B - 2)}{N_B} = 10. \quad (7.4.7)$$

To this we should add the ten free fermions, that contribute to the central charge $c_{\text{ferm}} = 5$. The central charge of the SUSY system $c_{\text{SUSY}} = 15$, is then cancelled by the $b - c$ and $\beta - \gamma$ ghosts.

We now study the interesting case $Q_A = Q_B = 0$, but with $m > 0$. The configuration of interest is obtained by S-dualising eq.(7.2.1),

$$\begin{aligned} ds_{10D}^2 &= -dt^2 + dx^2 + f_s(r)d\varphi^2 + \frac{dr^2}{Q^2 r^2 f_s(r)} + N_B ds^2(S_A^3) + N_A ds^2(S_B^3), \\ H_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3), \\ \Phi &= -\log(r). \end{aligned} \quad (7.4.8)$$

Recall that here $\varphi \sim \varphi + \pi/Q^2$, see eq.(7.2.3), with $Q^2 = \frac{1}{N_A} + \frac{1}{N_B}$, as above. Before proceeding to the σ -model analysis, it is convenient to perform some changes of variables. First, we want to rewrite the background (7.4.8) in such a way that it reduces to (7.4.1) when $m = 0$. For this use

$$r = e^{Q\rho}, \quad \tilde{m} = \frac{m}{4Q^2}, \quad (7.4.9)$$

which leads to

$$\begin{aligned} ds_{10D}^2 &= -dt^2 + dx^2 + 4Q^2 (1 - \tilde{m}e^{-2Q\rho}) d\varphi^2 + \frac{d\rho^2}{1 - \tilde{m}e^{-2Q\rho}} + N_B ds^2(S_A^3) + N_A ds^2(S_B^3), \\ H_3 &= N_B \text{Vol}(S_A^3) + N_A \text{Vol}(S_B^3), \\ \Phi &= -Q\rho. \end{aligned} \quad (7.4.10)$$

We change coordinates as,

$$\tanh^2(\lambda) = 1 - \tilde{m}e^{-2Q\rho}, \quad \varphi = \frac{\phi}{2Q^2}, \quad (7.4.11)$$

which puts the geometry in the usual cigar form (note that ϕ has period 2π)

$$\begin{aligned} ds_{10D}^2 &= -dt^2 + dx^2 + \frac{1}{Q^2} (\tanh^2(\lambda)d\phi^2 + d\lambda^2) + N_B ds^2(S_A^3) + N_A ds^2(S_B^3), \\ H_3 &= 2N_B \text{Vol}(S_A^3) + 2N_A \text{Vol}(S_B^3), \\ \Phi &= -\log(\cosh(\lambda)) - \frac{1}{2} \log(\tilde{m}). \end{aligned} \quad (7.4.12)$$

As explained in [178], for more details see Section 2 of [169], the above backgrounds leads to an exact σ -model

$$U(1)^2 \times \frac{SL(2, \mathbb{R})_k}{U(1)} \times SU(2)_{N_B} \text{WZW} \times SU(2)_{N_A} \text{WZW}, \quad (7.4.13)$$

where $k^{-1} = Q^2 = \frac{1}{N_A} + \frac{1}{N_B}$. To check that this is also a good string σ -model we use

$$c(SL(2, \mathbb{R})_k) = \frac{3(k+2)}{k}, \quad c(G/H) = c(G) - c(H), \quad (7.4.14)$$

so that

$$c\left(\frac{SL(2, \mathbb{R})_k}{U(1)}\right) = 2 + \frac{6}{k} = 2 + \frac{6}{N_A} + \frac{6}{N_B}, \quad (7.4.15)$$

from where is easy to see that $c_{\text{total}} = 10$. We leave for future research the study of the σ -model in the background with m, Q_A, Q_B arbitrary.

7.5 Summary

The I-brane QFT, defined as the $(1+1)$ field theory on the intersection of two stacks of D5-branes, has the remarkable behaviour that as the coupling is increased, the system gains one more dimension and enhances its SUSY (with a peculiar SUSY algebra in $(2+1)$ dimensions [179]). The background dual to this strongly coupled QFT was written in [79]. This presents a singular behaviour for large values of the radial coordinate, where the dilaton diverges and string coupling effects cannot be neglected. This is solved by performing an S-duality and working with the NS5-branes system. The background of eq.(7.1.9) is also singular for small values of the radial coordinate, $r \rightarrow 0$, as indicated by eq.(7.1.10). This ill-defined IR behaviour is amended, while preserving some amount of SUSY, by introducing a SUSY soliton in the geometry, see eq.(7.2.1). Our simple and explicit solution describes the holographic dual to the circle twisted compactification of the $(2+1)$ QFT, preserving four supercharges and ending the flow with a confining and gapped behaviour.

We holographically studied different aspects of this peculiar QFT. Maldacena-Wilson loops, 't Hooft loops, Entanglement entropy were discussed in dedicated sections, with emphasis on the effects of the UV-completion in terms of LST. R-symmetry and its breaking, a suitably defined gauge coupling and a quantity measuring the number of degrees of freedom as a function of the radial coordinate (the energy) were presented and discussed.

Chapter 8

Conclusion

In this thesis, we used two different String Theory frameworks to study different aspects of the strong coupling regime of Super-Yang-Mills and SQCD theories via strong-weak dualities.

First, by constructing a brane configuration that has as effective theory on its world-volume $\mathcal{N} = 1$ SQCD Chern-Simons theories, in Chapter 2, using Brane Dynamics and Giveon-Kutasov duality, we proposed the magnetic dual of these theories to explore the IR dynamics of the electric theory. When $\frac{N_f}{2} \geq \frac{N_c}{2} - k$ the IR dynamics is described by either a TQFT for large quark masses, or a Grassmannian and a Wess-Zumino (WZ) term for small masses. We also consider non-supersymmetric mass deformations and RG flows in the vicinity of the SUSY point and find agreement between the IR of the electric and its magnetic dual. When $\frac{N_f}{2} < \frac{N_c}{2} - k$ supersymmetry is broken and the IR dynamics is described by a TQFT accompanied by a Goldstino. We also discuss SQCD theories based on SO/USp gauge groups.

It will be interesting to exploit the techniques used in Chapter 2 in other cases. In particular to obtain a better understanding of the WZ term in other field theories. In addition, our technique suggests that 4D Seiberg duality can be understood even when $N_f < N_c$, namely that the Affleck-Dine-Seiberg runaway superpotential can be obtained from a magnetic dual.

In the following chapters, we used the Gauge/Gravity Duality to address open questions and analyse various observables of different field theories and Supergravity configurations. In Chapter 3, we attempted to answer the question of what is the field theory dual of a background obtained after a NATD transformation. For this, we used the electrostatic formalism for the holographic dual of 3D $\mathcal{N} = 4$ SCFTs to argue that the NATD of the reduction to Type IIA of $AdS_4 \times S^7$ can be completed into a background with a well defined holographic dual. A natural question to ask is whether this completion can occur in other

dimensions.

Then, in Chapter 4 we constructed an infinite family of massive Type IIA solutions that interpolate between AdS_7 in the UV and AdS_5 in the IR. These holographically describe the compactification of a infinite family of 6D $\mathcal{N} = (1, 0)$ linear quiver SCFTs on a Riemann surface. The analysis of the branes involved suggests a four dimensional linear quiver QFT, that deconstructs the theory in six dimensions. We proposed a dual field theory and compared the a and c central charges with the field theory computation, and also proposed a quantity that measures the number of degrees of freedom along the flow.

Although the a and c central charges computed in the proposed field theory match the scaling with the gauge group rank when compared with the holographic computation, the overall coefficient is not the same. This means that the field theory proposal is not completely correct. It is then important to study the string quantisation in the Hanany-Witten set up that arises as a result of the twisted compactification. The massless modes would give firm clues about the four dimensional quiver field theory.

Moreover, it should be interesting to attempt this kind of compactifications with systems in different dimensions. The systematic study of examples might indicate dimension-dependent characteristics.

In Chapter 5, we considered a compactification of 4D $\mathcal{N} = 4$ SYM, with $SU(N)$ gauge group, on a circle with anti-periodic boundary conditions for the fermions. We couple the theory to a constant background gauge field along the circle for an abelian subgroup of the R -symmetry which allows to preserve four supersymmetries. The 3D effective theory exhibits gapped and ungapped phases, which we argue are holographically dual, respectively, to a supersymmetric soliton in $\text{AdS}_5 \times S^5$, and a particular quotient of $\text{AdS}_5 \times S^5$. The gapped phase corresponds to an IR 3D $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons theory at level N , while the ungapped phase is naturally identified with the root of a Higgs branch in the 3D theory. We discuss the extension of the twisting procedure to maximally SUSY Yang-Mills theories in different dimensions, obtaining the relevant duals for 2D and 6D, and comment on the odd dimensional cases.

We argued that for 4D $\mathcal{N} = 1$ SYM, it is possible to preserve all the SUSY through the twisting procedure. This is questionable when the R -symmetry is anomalous. It would be interesting to try to find supersymmetric solitons on backgrounds dual to theories preserving four Poincaré supercharges, and non-anomalous R -symmetry, for example $\text{AdS}_5 \times T^{1,1}$, and check whether the resulting background also preserves four SUSY. Also, we discussed a potential difficulty in obtaining supersymmetric solitons in Dp -brane backgrounds for p even. A more careful analysis of the field theory for these cases may open a route to obtaining such solitons. New families of Type II backgrounds, based on uplifts of the AdS_5 soliton of [135], dual to field theories preserving four or more supercharges were studied

in [180, 181].

Afterwards, the study of the Supergravity background dual to the twisted circle compactification of maximally supersymmetry YM in 6D was done in Chapter 6. Similarly, Chapter 7 deals with the study of the supersymmetric soliton on the I-brane background. Both of these chapters are quite similar: in both cases, the soliton allows to resolve a singularity at the origin of the spacetime in a smooth way. The high energy regime of both of these field theories is that of a Little String Theory. We study various observables (Wilson and 't Hooft loops, Entanglement entropy, density of degrees of freedom and the spectrum of spin-two glueballs, among others). Although the theories are qualitatively different, certain observables behave in exactly the same way, which is due to the fact that both theories are built out of D5-branes. The Penrose limit on these geometries can also be analysed, obtaining two different pp-wave geometries. We can match the string oscillators and masses to spin-chains in (2+1)D and (1+1)D. These are the results of the work [182], which is a collaboration of the author of this thesis, but they are not discussed in the bulk of this thesis.

It was suggested that the D5-brane SUSY soliton preserved a $SU(2)$ G-structure. It would be interesting to understand whether the other SUSY solitons presented in this theses also preserve some structure. This could provide more evidence to support the claim that it is not possible to consider a supersymmetric soliton in a D2 or D4-brane background without exciting more fields.

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