Prifysgol Abertawe
Swansea University

Thomas–Fermi Type Variational Problems With Low Regularity

by

Damiano Greco

Department of Mathematics

Swansea University

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Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Abstract

This thesis is dedicated to the study of two different Thomas–Fermi type variational problems under optimal assumptions.

The first problem concerns studying the existence and qualitative properties of the minimizers for a Thomas–Fermi type energy functional with non local repulsion involving a convolution with the Riesz kernel and an interaction with an external potential. Under mild assumptions, we establish uniqueness and qualitative properties such as positivity, regularity, and decay at infinity of the global minimizer.

The second problem concerns the study of optimizers of a Gagliardo–Nirenberg type inequality again involving a convolution with the Riesz kernel. Such a problem is well understood in connection with Keller–Segel models and appears in the study of Thomas–Fermi limit regimes for the Choquard equations with local repulsion and non local attraction. We establish optimal ranges of parameters for the validity of the inequality, discuss the existence and qualitative properties of the optimizers. We further prove that optimizers are either positive, smooth, and fully supported functions or continuous and compactly supported on a ball, or discontinuous and represented as a linear combination of the characteristic function of a ball and a nonconstant nonincreasing Hölder continuous function supported on the same ball.

Summary

We present here an overview of the content of this thesis. The main subjects are two Thomas–Fermi (TF) integral equations. The thesis is divided into four chapters. Chapter 1 collects some well known background results. Chapter 2 focuses on Brezis–Browder type results which will be extensively employed in the subsequent chapters. Chapters 3 and 4, focusing on Thomas–Fermi type variational problems, are the core of the thesis. For this reason, we underline below the main features of such problems.

Thomas–Fermi models have recently emerged as a new mathematical paradigm for analysing the collective behaviour of many–body systems, which is one of the most difficult problems in modern Science. In general, one of the primary goals of the analysis of non local problems is to develop a rigorous mathematical component of physical studies, which will result in a better understanding of the phenomenon. Thomas–Fermi models turn out to be particularly appropriate in the study of astrophysical, relativistic, and quantum mechanical phenomena. The main feature of TF–equations is the presence of a non local interaction term representing a long–range attraction/repulsion, a local nonlinear term representing short–range attraction/repulsion, and the absence of a (regularising) diffusion type term. As a matter of fact, in some cases, TF–equations arise as limit equations of a more general class of Choquard equations which are particularly suited for the study of self–gravitating Bose–Einstein condensates with repulsive short–range interactions. We refer to [90] and references therein for a survey on such class of equations. Furthermore, we refer the reader to Chapter 4 for a more in-depth introduction to this topic. Remarkably, even though Choquard equations typically admit fully supported and regular ground state solutions (they can be indeed seen as stationary Nonlinear Schrödinger (NLS) equations with an attractive long range interaction, represented by the non local term, coupled with a repulsive short range interaction, represented by the local nonlinearity), the corresponding limit TF–equation might admit only compactly

supported and discontinuous ground state solutions. This lack of regularity is expected and can be attributed to the absence of a diffusion term. The non local interaction terms taken into account in this thesis are convolutions with the well known Riesz kernel defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{d-\alpha}}, \quad \alpha \in (0,d),$$

where A_α is a suitable constant to be specified later, while the short range interaction terms are power type functions.

In Chapter 3, motivated by the Thomas–Fermi type model of charge screening in graphene [89], we study the existence and qualitative properties of the minimisers for the following energy functional:

$$\mathcal{E}_1^{TF}(\rho) := \frac{1}{q}\int_{\mathbb{R}^d} |\rho(x)|^q dx - \int_{\mathbb{R}^d} \rho(x)V(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} (I_\alpha * \rho)(x)\rho(x)dx, \qquad (\mathcal{E}_1^{TF})$$

where $I_\alpha * \rho$, as mentioned above, represents the convolution of ρ with the Riesz kernel. The graphene case corresponds to the special two–dimensional case $d = 2$, $q = \frac{3}{2}$ and $\alpha = 1$. The function $\rho(x)$ here has the meaning of a charge density of fermionic quasiparticles (electrons and holes) in a two–dimensional graphene layer. In general (and unlike in the classical Thomas–Fermi models of atoms and molecules), the density ρ is a sign–changing function, with $\rho > 0$ representing electrons and $\rho < 0$ representing holes. Moreover, because of the nature of the model, we minimise the energy ($\mathcal{E}_1^{TF}$) on its domain without any mass constraint.

The main results address the regularity of the unique minimiser for ($\mathcal{E}_1^{TF}$) and its asymptotic behaviour at infinity, which is proved to depend in a non trivial way on the parameters. Namely, we prove that if the potential V is sufficiently regular (e.g., continuous or Hölder continuous) then the (unique) minimiser for ($\mathcal{E}_1^{TF}$) inherits the same regularity. Furthermore, if $\alpha \in (0, 2)$, we identify three different polynomial decay regimes and two critical regimes where log–corrections arise. In order to obtain such asymptotic behaviour, assuming that V is fractional superharmonic, we link the (unique) minimiser for ($\mathcal{E}_1^{TF}$) with the (unique) non negative solution u to the fractional semilinear PDE

$$(-\Delta)^{\frac{\alpha}{2}} u + u^{\frac{1}{q-1}} = (-\Delta)^{\frac{\alpha}{2}} V \quad \text{in } \mathbb{R}^d, \qquad \text{(PDE)}$$

where by $(-\Delta)^{\frac{\alpha}{2}}$ we denote the fractional Laplace operator. The decay of solutions to the local prototype ($\alpha = 2$) of (PDE) has been widely studied in [105]. In particular, it has been proved the existence of two critical exponents, being respectively $q = \frac{2d-2}{d}$ and $q = 2$. The non local counterpart of $q = \frac{2d-2}{d}$ is $q = \frac{2d-\alpha}{d}$ which remains critical in the fractional framework as well. On the other hand, when passing from the local to the non local case, the critical exponent $q = 2$ moves to $q = \frac{2d+\alpha}{d+\alpha}$, which is a purely fractional critical exponent. Furthermore, interesting phenomena arise when analysing the decay rate. That is, in some regimes the local decay can be formally recovered by simply replacing $\alpha = 2$ in the non local decay, while in other regimes the behaviour is completely different. Moreover, we stress that the criticality of $q = \frac{2d+\alpha}{d+\alpha}$ is caused by the different behaviour of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ when acting on polynomial decaying functions. As a matter of fact, if we consider a function $v(x) \simeq |x|^{-d}$ and $q = \frac{2d+2}{d+2}$, the non linear term $v^{\frac{1}{q-1}}$ balances the behaviour of the linear term $-\Delta v$ at infinity. Note also that with this choice of q we have that $\frac{2(q-1)}{2-q} = d$, which is consistent with Figure 1. On the other hand, cf. Lemma 3.6.1, if $\alpha \in (0, 2)$ we have that

$$(-\Delta)^{\frac{\alpha}{2}} v(x) \simeq -\frac{\log |x|}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty.$$

Hence, if $q = \frac{2d+\alpha}{d+\alpha}$, the function v (and any polynomial behaving function) does not balance the behaviour of the non linear term $v^{\frac{1}{q-1}}$. Thus, some further work is needed to capture the sharp decay rate. See Figure 1 for a complete picture of the decay rate. In particular, we notice that in the fractional framework the solution is never compactly supported. This strongly depends on the non local nature of the fractional Laplacian. Indeed, if u is a sufficiently regular non negative solution of (PDE) and there exists x_0 such that $u(x_0) = 0$, its fractional Laplacian at x_0 (up to a positive constant) satisfies

$$0 \leq (-\Delta)^{\frac{\alpha}{2}} V(x_0) = (-\Delta)^{\frac{\alpha}{2}} u(x_0) = -\frac{1}{2} \int_{\mathbb{R}^d} \frac{u(x_0 + y) + u(x_0 - y)}{|y|^{d+\alpha}} dy < 0,$$

leading to a contradiction.

Before moving our attention to a Thomas–Fermi equation with non local attraction, we recall that the above repulsive model aims to generalise the one of charge screening in graphene in dimension 2. Nevertheless, several similar models of

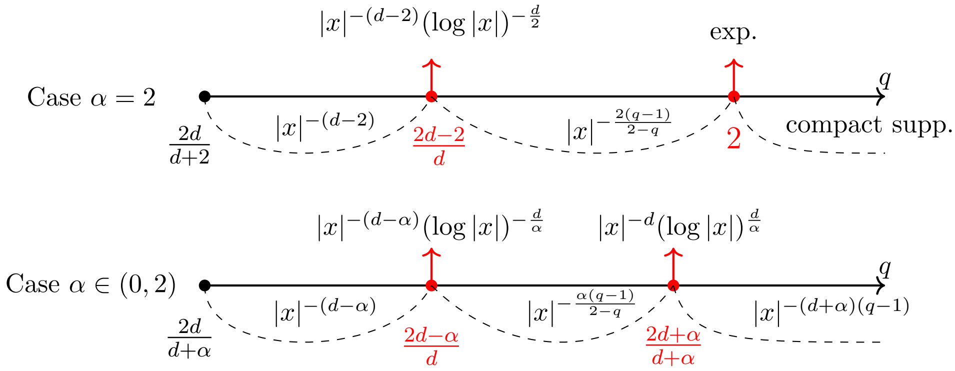

Figure 1: Comparison between the local and non local case.

Thomas–Fermi type have been studied in the literature. For example, in dimension 3, the energy introduced by Thomas and Fermi, and mathematically studied by Lieb and Simon (see [17, 76, 79]), corresponds to $q = \frac{5}{3}$ and $\alpha = 2$ and V being a Coulomb–type potential

$$V(x) = \sum_{i=1}^{k} \frac{Z_i}{|x - a_i|}, \tag{V_Z}$$

where $Z_i > 0$, $a_i \in \mathbb{R}^3$ and $k \in \mathbb{N}_{\geq 1}$. Here, the problem is to minimise ($\mathcal{E}_1^{TF}$) among all non negative functions with fixed mass M. In this case, ρ represents the electronic density and hence positivity and mass constraint arise naturally from the physical point of view. It turns out that the relation between the constant $Z := \sum_{i=1}^{k} Z_i$ and the total mass M plays a crucial role. Namely, if $M \leq Z$ there exists a unique minimiser for ($\mathcal{E}_1^{TF}$) with prescribed M–mass, while if $M > Z$ the infimum is not achieved.

In [17] the authors generalise the above model. Namely, the quantity $\frac{3}{5}\|\rho\|_{L^{\frac{5}{3}}(\mathbb{R}^d)}$ is replaced by a convex function J defined on $[0, +\infty)$ such that $J(0) = 0$, $J(r) > 0$ in $(0, +\infty)$ and $J(r) = +\infty$ if $r \leq 0$. Moreover, the potential (V_Z) is replaced with a generic function V. The approach used by P. Benilan and H. Brezis is fundamental, and was used for example in the papers [62, 97] where the authors replaced $J(\cdot)$ with $J(\cdot)w(x)$ for some $w : \mathbb{R}^d \to \mathbb{R}$ being a measurable function.

In Chapter 4, motivated by the study of ground state solutions for a Choquard type equation [82], we study critical points of the following energy functional $\mathcal{E}_2^{TF}$ defined by

$$\mathcal{E}_2^{TF}(\rho) := \frac{1}{2}\int_{\mathbb{R}^d} |\rho(x)|^2 dx + \frac{1}{q}\int_{\mathbb{R}^d} |\rho(x)|^q \, dx - \frac{1}{2p}\int_{\mathbb{R}^d} (I_\alpha * |\rho(x)|^p)|\rho(x)|^p dx. \tag{$\mathcal{E}_2^{TF}$}$$

To begin with, we notice that ($\mathcal{E}_2^{TF}$) is non convex and unbounded from below. Thus, we minimise the above functional subject to a constraint, i.e., a Pohožaev type constraint $\mathscr{P} = \{\rho \in (L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)) \setminus \{0\} : \mathcal{P}(\rho) = 0\}$ where $\mathcal{P}$ is defined by

$$\mathcal{P}(\rho) := \frac{d}{2}\int_{\mathbb{R}^d} |\rho|^2 dx + \frac{d}{q}\int_{\mathbb{R}^d} |\rho|^q dx - \frac{d+\alpha}{2p}\int_{\mathbb{R}^d} (I_\alpha * |\rho|^p)|\rho|^p dx.$$

For $p = 2$ this problem is well understood in connection with Keller–Segel models (see [29, 33, 35]) while the case $p \neq 2$ appears in the study of Thomas–Fermi limit regimes for the Choquard equations with local repulsion. Note also that, because of the lack of convexity of ($\mathcal{E}_2^{TF}$), uniqueness of a minimiser is a more challenging problem that has been solved only under strong restrictions on the parameters. In general, by introducing a constraint, if a minimiser for ($\mathcal{E}_2^{TF}$) exists it's not necessarily a critical point of ($\mathcal{E}_2^{TF}$) unless the Lagrange multiplier is zero, i.e., the constraint is a natural constraint. However, the standard methods for proving the last statement require some a priori regularity on the critical points of ($\mathcal{E}_2^{TF}$) (e.g., some local weak differentiability). Typically, in the Choquard case, the presence of a diffusion term allows us to achieve such regularity and prove that a Pohožaev type constraint is natural, see e.g., [82]. Nevertheless, this type of argument can not work because of the the lack of continuity of critical points in some regimes (for example if $p > 2$). To overcome this, we instead study optimisers of the Gagliardo–Nirenberg type inequality of the form

$$\mathcal{D}_\alpha(|\rho|^p, |\rho|^p) = \int_{\mathbb{R}^d} (I_\alpha * |\rho|^p)|\rho|^p dx \le \mathscr{C}_{d,\alpha,p,q}\Big(\int_{\mathbb{R}^d} |\rho|^2 dx\Big)^{p\theta}\Big(\int_{\mathbb{R}^d} |\rho|^q dx\Big)^{\frac{2p(1-\theta)}{q}}, \tag{GN}$$

where $0 < \alpha < d$, $\frac{2}{p} < \frac{2d}{d+\alpha} < \frac{q}{p}$ and $\theta \in (0,1)$. The optimal constant for (GN) is related to $\mathscr{C}_{d,\alpha}$ that is (up to the constant A_α) the best constant for the Hardy-Littlewood-Sobolev inequality. Namely, we have the trivial upper bound

$$\mathscr{C}_{d,\alpha,p,q} \le \mathscr{C}_{d,\alpha}.$$

Furthermore, we can see that $\mathscr{C}_{d,\alpha,p,q} < \mathscr{C}_{d,\alpha}$ for every $0 < \alpha < d$, $\frac{2}{p} < \frac{2d}{d+\alpha} < \frac{q}{p}$. Indeed, if the equality holds we would immediately obtain that every optimiser ρ_* for (GN) is an optimiser for the Hardy–Littlewood–Sobolev inequality. Therefore

[78, Theorem 4.3] implies that

$$\rho_*(x) = \frac{C}{(\gamma^2 + |x-a|^2)^{\frac{d+\alpha}{2p}}},$$

for some $C \in \mathbb{R} \setminus \{0\}$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}^d$. However, by writing the Euler–Lagrange equation corresponding to (GN), we deduce that such type of function can not be an optimiser for $\mathscr{C}_{d,\alpha,p,q}$.

After proving the existence of optimisers for (GN), because of the homogeneity and scaling invariance of the quotient associated to (GN), one can prove the existence of an optimiser that is also a ground state solution, i.e., it's a critical point of the energy functional ($\mathcal{E}_2^{TF}$) having minimal energy among all the functions satisfying the Pohožaev constraint.

The main contribution of Chapter 4 addresses regularity in the case $p > 2$. Namely, we prove that in the last regime, non negative radially non increasing ground state solutions are discontinuous and represented as a linear combination of the characteristic function of a ball and a nonconstant non increasing Hölder continuous function supported on the same ball.

Next, let us briefly describe the main differences and analogies between the two Thomas–Fermi problems introduced in previous lines.

Non negative optimisers for the energy functionals $\mathcal{E}_1^{TF}$, $\mathcal{E}_2^{TF}$ satisfy the following equations with repulsive (REP), respectively attractive (ATTR), non local interaction

$$\underbrace{\rho^{q-1}}_{\text{short range}} = V - \underbrace{I_\alpha * \rho}_{\text{non local repulsion}} \quad \text{in } \mathbb{R}^d \tag{REP}$$

$$\underbrace{\rho^{q-1} + \rho}_{\text{short range}} = \underbrace{(I_\alpha * \rho^p)\rho^{p-1}}_{\text{non local attraction}} \quad \text{in } \mathbb{R}^d. \tag{ATTR}$$

From the above equations, it's readily seen that in both cases the (regularising) role of the Riesz potential operator is a key feature to obtain regularity of optimisers. However, important differences arise. As a matter of fact, in the case (REP), under regularity assumptions on the external potential V, via a bootstrap argument one can improve the regularity of the Riesz potential until continuity of the optimiser is achieved. On the other hand, dealing with (ATTR) is more delicate from the point of view of gaining regularity. Indeed, dividing both sides of the equation by ρ^{p-1} we obtain the following

$$\rho^{2-p} + \rho^{q-p} = I_\alpha * \rho^p \quad \text{in } \{\rho > 0\}\,.$$

In particular, if $p > 2$, because of the blow up near the origin of the function $x \mapsto x^{2-p}$, it's easy to see that ρ must be compactly supported and discontinuous. Moreover, since in the same regime the function $f(x) = x^{2-p} + x^{q-p}$ is not bijective, we can not in principle reduce to study

$$\rho = f^{-1}(I_\alpha * \rho^p) \quad \text{in } \{\rho > 0\}\,.$$

For these reasons, a more careful argument is needed to recover continuity in the set $\{\rho > 0\}$. However, unlike (REP), if $p < 2$, corresponding to the case when the optimiser has full support, it is not difficult to obtain the decay rate.

Open problems

- Limit as $\alpha \to 0$ and $\alpha \to d$.

  Here, we intend to highlight how the Gagliardo–Nirenberg inequality (GN) can be seen as the non local intermediate inequality between two different Hölder inequalities. Indeed, because of the choice of the normalization constant A_α, in a suitable sense we have that $I_\alpha \to \delta_0$ as $\alpha \to 0$. Furthermore, since $\lim_{\alpha\to 0} \mathscr{C}_{d,\alpha,p,q} = 1$, at least formally, the limit of (GN) as α approaches zero is

$$\int_{\mathbb{R}^d} |\rho|^{2p} \le \left(\int_{\mathbb{R}^d} |\rho|^2\right)^{\frac{q-2p}{q-2}} \left(\int_{\mathbb{R}^d} |\rho|^q\right)^{\frac{2(p-1)}{q-2}}\,, \tag{LIM1}$$

  which corresponds to a standard interpolation inequality for the L^{2p} norm.

  Similarly, since as $\alpha \to d$ the constant $A_\alpha \to \infty$ and $\lim_{\alpha\to d} A_\alpha^{-1}\mathscr{C}_{d,\alpha,p,q} = 1$, we take into account a normalized (GN) inequality obtained by dividing by A_α. Again formally, cf. [101, Proposition 2.6], our limit inequality is now

$$\int_{\mathbb{R}^d} |\rho|^{p} \le \left(\int_{\mathbb{R}^d} |\rho|^2\right)^{\frac{q-p}{q-2}} \left(\int_{\mathbb{R}^d} |\rho|^q\right)^{\frac{p-2}{q-2}}\,, \tag{LIM2}$$

  that is another standard interpolation inequality for the L^p norm. It's well known that in (LIM1) and (LIM2) the equality is achieved by functions satisfying the following relation

$$|\rho|^2 = c|\rho|^q \quad \text{a.e. in } \mathbb{R}^d, \quad c > 0. \tag{LOC}$$

A natural question in this case would be whether a sequence of optimisers $(\rho_\alpha)_\alpha$ for (GN) converges as $\alpha \to 0$ (respectively $\alpha \to d$) to an optimiser for (LIM1) (respectively (LIM2)). Numerical simulations seem to suggest that a family $(\rho_\alpha)_\alpha$ of radially non increasing optimisers for (GN) converges to a multiple of a characteristic function of a certain ball, which clearly satisfies (LOC).

- Uniqueness of ground state solutions.

  Because of the lack of convexity of ($\mathcal{E}_2^{TF}$), proving uniqueness of ground state solutions is a challenging problem. Recently, in [35], uniqueness has been proved for the special case $p = 2$ and $\alpha \in (0, 2)$ when the model actually links to a Keller–Segel one. In an on–going project, uniqueness has been proved also in the case $\alpha = 2$, $q = 2p$ and $p > 2$. However, as far as we know, the other cases remain open.

As it has been stated in the beginning of this summary, the main content of the thesis is contained in three chapters, each one corresponding to a paper or a preprint as follows:

- Chapter 2: D. Greco, *Brezis–Browder type results and representation formulae for s–harmonic functions*, arXiv:2407.06442;
- Chapter 3: D. Greco, *Optimal decay and regularity for a Thomas–Fermi type variational problem*, Nonlinear Analysis, Volume 251, 2025;
- Chapter 4: D. Greco, Y. Huang, Z. Liu, V. Moroz, *Ground states of a non-local variational problem and Thomas–Fermi limit for the Choquard equation*, arXiv:2406.18472.

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Chapter 1

Preliminaries

In this chapter we begin by introducing the basic notations used in the whole of the thesis, and then we proceed by presenting some standard results known in the literature about the fractional Laplacian and its inverse operator, the Riesz potential. Although many results concerning these topics are generally well known, it's not always easy to find references for them. For this reason, in this thesis we try to be as self-contained as possible or provide specific references when necessary.

1.1 Notations

First, we start by recalling the standard notation for sets theory. Namely, for every $x \in \mathbb{R}^d$ and $R > 0$, by $B_R(x)$ we understand the set defined as

$$B_R(x) := \left\{ y \in \mathbb{R}^d : |x - y| < R \right\}.$$

We will often write $B_R := B_R(0)$. As it is customary we denote by ω_d the measure of the unit ball in $\mathbb{R}^d$. Let Ω be a subset of $\mathbb{R}^d$. By $|\Omega|$, Ω^c and $\overline{\Omega}$ we understand respectively the d–dimensional Lebesgue measure, the complementary and the closure of Ω. Moreover, we say that Ω is smooth (e.g., continuous) if for every $x \in \partial\Omega$ there exists a ball $B_R(x)$ such that the set $B_R(x) \cap \Omega$ is, up to an isometry, the hypograph of a smooth function.

In what follows we focus on the notation about functions and function spaces.

Let Ω be an open subset of $\mathbb{R}^d$. First of all, given $u : \Omega \to \mathbb{R}$, we define the positive and negative part function by $[u]_+ := \max\{u, 0\}$ and $[u]_- := -\min\{u, 0\}$.

Next, as it is customary we have the following:

- By $C_c^\infty(\Omega)$ we denote the space of infinitely many times differentiable functions having compact support contained in Ω;

- If $k \in \mathbb{N}$, the space $C^k(\Omega)$ (respectively $C_b^k(\Omega)$) represents the space of functions whose derivatives are continuous up to order k on Ω (respectively continuous and bounded);

- By $\mathcal{D}'(\Omega)$ we denote the dual space of $C_c^\infty(\Omega)$ endowed with the standard topology. Moreover, if $T \in \mathcal{D}'(\Omega)$, by the the notation $T \geq 0$ in $\mathcal{D}'(\Omega)$ we understand the following inequality

$$\langle T, \varphi \rangle \geq 0, \quad \forall \varphi \in C_c^\infty(\Omega),\, \varphi \geq 0, \tag{1.1.1}$$

  where by the symbol $\langle \cdot, \cdot \rangle$ we denote the action of T against the test function φ;

- If $\gamma \in (0,1]$ and $k \in \mathbb{N}$, we define $C^{k,\gamma}(\Omega)$ (respectively $C^{k,\gamma}_{loc}(\Omega)$) by

$$\begin{aligned} C^{0,\gamma}(\Omega) &:= \left\{ u \in C_b(\Omega) : \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma} < \infty \right\}, \\ C^{k,\gamma}(\Omega) &:= \left\{ u \in C_b^k(\Omega) : D^\beta u \in C^{0,\gamma}(\Omega) \text{ for all } |\beta| = k \right\},\ k \geq 1, \end{aligned} \tag{1.1.2}$$

  endowed with the norm

$$\|u\|_{C^{k,\gamma}(\Omega)} := \sum_{|\beta|\leq k} \|D^\beta u\|_{L^\infty(\Omega)} + \sum_{|\beta|=k} [D^\beta u]_{C^{0,\gamma}(\Omega)}.$$

  Similarly, if $\gamma \notin \mathbb{N}$ we define the space $C^\gamma(\Omega)$ (respectively $C^\gamma_{loc}(\Omega)$ by

$$C^\gamma(\Omega) := C^{[\gamma],\{\gamma\}}(\Omega), \tag{1.1.3}$$

  where $[\gamma] := \max\{n \in \mathbb{N}_0 : n < \gamma\}$ and $\{\gamma\} := \gamma - [\gamma]$;

- $L^p(\Omega)$ with $p \in [1,\infty]$ denotes the standard Lebesgue space endowed with the norm

$$\begin{aligned} \|u\|_{L^p(\Omega)} &:= \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}, \ p \in [1, \infty); \\ \|u\|_{L^\infty(\Omega)} &:= \operatorname*{ess\,sup}_\Omega |u|, \qquad p = \infty. \end{aligned} \tag{1.1.4}$$

- If $\Omega = \mathbb{R}^d$, $\mathscr{S}(\mathbb{R}^d)$ denotes the class of Schwartz functions while by $\mathscr{S}'(\mathbb{R}^d)$ we denote the space of tempered distributions.

Furthermore, by $\Gamma(\cdot), \Gamma(\cdot, \cdot)$ and ${}_2F_1(\cdot, \cdot, \cdot; \cdot)$ we denote respectively the Gamma function, the incomplete Gamma function and the Gaussian Hypergeometric function. Next, if $u \in L^1(\mathbb{R}^d)$ the Fourier transform is defined by the formula

$$\mathscr{F}[u](\xi) = \widehat{u}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x) dx,$$

where by $x \cdot \xi$ we denote the standard inner product in $\mathbb{R}^d$.

In order to conclude this section, we present some asymptotic notation useful in the study of decay estimates, cf. Chapter 3, Section 3.5.
Let $f, g : \mathbb{R}^d \to \mathbb{R}$. We write

- $f(x) \lesssim g(x)$ in Ω if there exists $C > 0$ such that $f(x) \leq Cg(x)$ for every $x \in \Omega$.
- $g(x) \simeq f(x)$ in Ω if both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ in Ω.

In particular, by writing $f(x) \lesssim g(x)$ as $|x| \to +\infty$ we mean that there exists $C > 0$ such that $f(x) \leq Cg(x)$ for every x sufficiently large and by writing $f(x) \simeq g(x)$ as $|x| \to +\infty$ we understand that $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ as $|x| \to +\infty$. Similarly we write $f(x) \sim g(x)$ as $|x| \to +\infty$ if

$$\lim_{|x| \to +\infty} \frac{f(x)}{g(x)} = 1.$$

1.2 Fractional Sobolev Spaces

In this section we discuss various definitions and properties of the fractional Sobolev spaces $\mathring{H}^{s,p}(\mathbb{R}^d)$. In particular, we mainly focus on the case $p = 2$ whose properties will play a crucial role for the whole of the thesis. As it is customary, to simplify the notation we will set $\mathring{H}^s(\mathbb{R}^d) := \mathring{H}^{s,2}(\mathbb{R}^d)$.

1.2.1 Case $p = 2$

Let $0 < s < \frac{d}{2}$. We define the homogeneous fractional Sobolev space $\mathring{H}^s(\mathbb{R}^d)$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|^2_{\mathring{H}^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (2\pi|\xi|)^{2s} |\widehat{u}(\xi)|^2 d\xi.$$

Equivalently, see [94, eq. 22] or [12, Definition 1.31], the last space can be also defined as

$$\mathring{H}^s(\mathbb{R}^d) := \left\{ u \in L^{\frac{2d}{d-2s}}(\mathbb{R}^d) : (2\pi|\xi|)^s \widehat{u}(\xi) \in L^2(\mathbb{R}^d) \right\}.$$

From its definition one can easily verify that $\mathring{H}^s(\mathbb{R}^d)$ is a Hilbert space, cf. [12, Proposition 1.34] having $C_c^\infty(\mathbb{R}^d)$ as a dense subspace. Furthermore, it is actually continuously embedded into $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.

Theorem 1.2.1. *Let $0 < s < \frac{d}{2}$. Then there exists a positive constant C such that*

$$\|u\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leq C \|u\|_{\mathring{H}^s(\mathbb{R}^d)} \quad \forall u \in \mathring{H}^s(\mathbb{R}^d).$$

We further recall that, by definition of the $\mathring{H}^s(\mathbb{R}^d)$–norm, we have that

$$\langle u, v \rangle_{\mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2s} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi, \tag{1.2.1}$$

the dual space to $\mathring{H}^s(\mathbb{R}^d)$ identifies with $\mathring{H}^{-s}(\mathbb{R}^d)$ and, by the Riesz representation theorem, for every $T \in \mathring{H}^{-s}(\mathbb{R}^d)$ there exists a unique element $U_T \in \mathring{H}^s(\mathbb{R}^d)$ such that

$$\langle T, \varphi \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \langle U_T, \varphi \rangle_{\mathring{H}^s(\mathbb{R}^d)} \quad \forall \varphi \in \mathring{H}^s(\mathbb{R}^d), \tag{1.2.2}$$

where by $\langle \cdot, \cdot \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)}$ we understand the duality between $\mathring{H}^{-s}(\mathbb{R}^d)$ and $\mathring{H}^s(\mathbb{R}^d)$. Moreover,

$$\|U_T\|_{\mathring{H}^s(\mathbb{R}^d)} = \|T\|_{\mathring{H}^{-s}(\mathbb{R}^d)}.$$

Note that, when dealing with real valued functions (as it is done in the whole of this thesis), the inner product defined in (1.2.2) is in fact real. Indeed, by assuming for example that $u, v \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} (2\pi|\xi|)^{2s}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi &= \int_{\mathbb{R}^d_+} (2\pi|\xi|)^{2s}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi + \int_{\mathbb{R}^d_-} (2\pi|\xi|)^{2s}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi \\ &= \int_{\mathbb{R}^d_+} (2\pi|\xi|)^{2s}\widehat{u}(\xi)\widehat{v}(-\xi)d\xi + \int_{\mathbb{R}^d_+} (2\pi|\xi|)^{2s}\widehat{u}(-\xi)\widehat{v}(\xi)d\xi \\ &= \int_{\mathbb{R}^d_+} (2\pi|\xi|)^{2s}(\widehat{u}(\xi)\widehat{v}(-\xi) + \overline{\widehat{u}(\xi)\widehat{v}(-\xi)})d\xi \\ &= 2\int_{\mathbb{R}^d_+} (2\pi|\xi|)^{2s}\,\mathrm{Re}(\widehat{u}(\xi)\widehat{v}(-\xi))d\xi,\end{aligned} \tag{1.2.3}$$

where we denoted by $\mathbb{R}^d_+$ (respectively $\mathbb{R}^d_-$) the upper (respectively lower) half space.

Next, we give an alternative useful formula for the $\|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$–norm which is valid for $0 < s < 1$ whose proof is based on a Fourier transform argument, see [12, Proposition 1.37] or [45, Proposition 3.4].

Lemma 1.2.1. *Let $0 < s < 1$. If $u \in \mathring{H}^s(\mathbb{R}^d)$ then*

$$\|u\|^2_{\mathring{H}^s(\mathbb{R}^d)} = \frac{C_{d,2s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy, \tag{1.2.4}$$

where $C_{d,2s}$ is defined by

$$C_{d,2s} := \left(\int_{\mathbb{R}^d} \frac{1-\cos(y_d)}{|y|^{d+2s}}dy\right)^{-1}.$$

In particular, by using Theorem 1.2.1 and Lemma 1.2.1 we deduce that for $0 < s < 1$ the space $\mathring{H}^s(\mathbb{R}^d)$ coincide with the space $\mathcal{D}^{s,2}(\mathbb{R}^d)$ defined by

$$\mathcal{D}^{s,2}(\mathbb{R}^d) := \left\{u \in L^{\frac{2d}{d-2s}}(\mathbb{R}^d) : \frac{C_{d,2s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy < \infty\right\}, \tag{1.2.5}$$

endowed with the norm defined by the right hand side of (1.2.4). Indeed, cf. [23, Theorem 3.1], $\mathcal{D}^{s,2}(\mathbb{R}^d)$ is a Hilbert space having $C_c^\infty(\mathbb{R}^d)$ as a dense subset. In particular, assume that $u \in \mathcal{D}^{s,2}(\mathbb{R}^d)$. Then, by density and Lemma 1.2.1 above, there exists a smooth sequence $(\varphi_n)_n$ converging to u in $\mathcal{D}^{s,2}(\mathbb{R}^d)$ which is also a cauchy sequence in $\mathring{H}^s(\mathbb{R}^d)$. Thus, by completeness we conclude that $(\varphi_n)_n$ converges to a function v belonging to $\mathring{H}^s(\mathbb{R}^d)$. However, by Theorem 1.2.1 we conclude that

$v = u$ and

$$\|u\|_{\mathring{H}^s(\mathbb{R}^d)} = \lim_{n\to+\infty} \|\varphi_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^d)} = \|v\|_{\mathcal{D}^{s,2}(\mathbb{R}^d)}.$$

In what follows, we show an important result which easily follows from the Gagliardo representation of the $\|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$–norm. We also refer to [92, page 3].

Lemma 1.2.2. *Let $0 < s < 1$ and $u \in \mathring{H}^s(\mathbb{R}^d)$. Then, $[u]_\pm \in \mathring{H}^s(\mathbb{R}^d)$. Moreover,*

$$\langle [u]_+, [u]_- \rangle_{\mathring{H}^s(\mathbb{R}^d)} \leq 0.$$

Proof. From (1.2.5) it's easy to see that $[u]_\pm$ belong to $\mathring{H}^s(\mathbb{R}^d)$ and $\|[u]_\pm\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \|u\|_{\mathring{H}^s(\mathbb{R}^d)}$. Moreover we have

$$\begin{aligned}\langle [u]_+, [u]_- \rangle_{\mathring{H}^s(\mathbb{R}^d)} &= \frac{C_{d,2s}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{([u(x)]_+ - [u(y)]_+)([u(x)]_- - [u(y)]_-)}{|x-y|^{d+2s}} dxdy \\ &= -C_{d,2s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{[u(x)]_+ [u(y)]_-}{|x-y|^{d+2s}} dxdy \leq 0,\end{aligned}$$

concluding the proof. □

Finally, in order to conclude this subsection, we define the Fractional Sobolev space on a proper subset of $\mathbb{R}^d$. Namely, let $\Omega \subset \mathbb{R}^d$ be an open and connected set and $0 < s < \frac{d}{2}$. We define the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$. That is

$$\overline{C_c^\infty(\Omega)}^{\|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}} =: \mathring{H}_0^s(\Omega). \tag{1.2.6}$$

Note that, for a general domain Ω, the space $\mathring{H}_0^s(\Omega)$ defined by (1.2.6) is different from

$$\widetilde{\mathring{H}}^s(\Omega) := \left\{ u \in \mathring{H}^s(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \right\}. \tag{1.2.7}$$

Nevertheless, our domains of interest will be either $\mathbb{R}^d$, or $\overline{B_R}^c$ with $R > 0$, and in these cases the two spaces coincide. We refer for example to [36, Theorem 3.19] for counterexamples.

Remark 1.2.1. To be precise, in [36, Lemma 3.15] or [84, Theorem 3.29], it has been proved that if Ω is a domain (even unbounded) with continuous boundary then

$$\widetilde{H}^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \right\} = H_0^s(\Omega),$$

where $H^s(\mathbb{R}^d)$ denotes the inhomogeneous fractional Sobolev space defined by

$$H^s(\mathbb{R}^d) := \left\{u \in L^2(\mathbb{R}^d) : (2\pi|\xi|)^s \hat{u}(\xi) \in L^2(\mathbb{R}^d)\right\}.$$

However, the result remains valid in the homogeneous case by a density argument. Indeed, every $u \in \widetilde{\mathring{H}^s}(\Omega)$ can be approximated by functions in $\widetilde{H^s}(\Omega)$ by simply multiplying it with a suitable smooth function φ_λ (see Lemma 2.3.2 for the precise statement).

1.2.2 Remarks on the case $p \neq 2$

Similarly to what has been done in the previous subsection, if $p > 1$ and $0 < s < \frac{d}{p}$, one can define the space $\mathring{H}^{s,p}(\mathbb{R}^d)$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{\mathring{H}^{s,p}(\mathbb{R}^d)} := \|\mathscr{F}^{-1}[(2\pi|\cdot|)^s \widehat{u}(\cdot)]\|_{L^p(\mathbb{R}^d)}$$

cf. [70, p. 90]. Moreover, $\mathring{H}^{s,p}(\mathbb{R}^d)$ continuously embeds into $L^{\frac{dp}{d-sp}}(\mathbb{R}^d)$. However, for $0 < s < 1$ and $p \neq 2$, unlike in (1.2.5) we have

$$\mathring{H}^{s,p}(\mathbb{R}^d) \neq \left\{u \in L^{\frac{dp}{d-sp}}(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dxdy < \infty\right\} =: \mathcal{D}^{s,p}(\mathbb{R}^d). \tag{1.2.8}$$

The right hand side of (1.2.8) is often denoted by $\mathring{W}^{s,p}(\mathbb{R}^d)$ in the literature. Indeed, if one wants to follow the proof of Lemma 1.2.1, it's clear that a fundamental fact is that the Fourier transform is an isometry from L^2 to L^2. However, this is false for $p \neq 2$. We refer to [23], [45, Remark 3.5] and references therein for a deep analysis on the topic.

1.3 Fractional Laplacian

Let $d \geq 2$ and $0 < s < \frac{d}{2}$. The fractional Laplacian $(-\Delta)^s$ of a smooth function on $\mathbb{R}^d$ is defined by means of the Fourier transform ([61, eq. (2.6)], [70, eq (1.1)] or [71, eq (25.2), p. 486])

$$\widehat{(-\Delta)^s u}(\xi) := (2\pi|\xi|)^{2s}\widehat{u}(\xi). \tag{1.3.1}$$

In what follows, we first focus on the pure fractional Laplacian, i.e., when s is smaller than one. We will recall some standard results such as regularity of such operator when acting on smooth functions as well as equivalent integral formulations.

1.3.1 Case $0 < s < 1$.

If $0 < s < 1$, the fractional Laplace operator arises in a rather natural way in the study of Lévy processes. The latter essentially generalizes the Brownian motion. In particular, in view of (1.3.3) below, the fractional Laplacian is the most canonical and well studied example of elliptic integro–differential operator. We refer the reader to [61, 98] and references therein for an extensive study of this topic.

If u is a smooth function rapidly decaying, we have a pointwise representation of the operator $(-\Delta)^s$ given by

$$(-\Delta)^s u(x) = C_{d,2s}\text{P.V.}\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}dy := C_{d,2s}\lim_{\varepsilon\to 0}\int_{\{|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{d+2s}}dy \tag{1.3.2}$$

or by the singular integral

$$(-\Delta)^s u(x) = \frac{C_{d,2s}}{2}\int_{\mathbb{R}^d} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{d+2s}}dy, \tag{1.3.3}$$

where

$$C_{d,2s} := \left(\int_{\mathbb{R}^d} \frac{1-\cos(y_d)}{|y|^{d+2s}}dy\right)^{-1}.$$

The above statements rely on Proposition 1.3.1, whose proof can be found in [61].

Proposition 1.3.1. *Let $u \in \mathscr{S}(\mathbb{R}^d)$. Then, the three expressions in* (1.3.1), (1.3.2) *and* (1.3.3) *coincide. Furthermore, $(-\Delta)^s u$ belongs to $C^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and*

$$|(-\Delta)^s u(x)| \lesssim |x|^{-(d+2s)} \quad \text{as } |x| \to +\infty. \tag{1.3.4}$$

1.3.2 Generic $0 < s < \frac{d}{2}$.

Assume again that u is a smooth function rapidly decaying and $0 < s < \frac{d}{2}$. Then, for every $m \in \mathbb{N}$ and $s \in (0, m)$ one can define the operator

$$L_{m,s}u(x) := \frac{C_{d,m,s}}{2}\int_{\mathbb{R}^d} \frac{\delta_m u(x,y)}{|y|^{d+2s}}dy, \quad x \in \mathbb{R}^d, \tag{1.3.5}$$

where $\delta_m u(x,y)$ is defined by

$$\delta_m u(x,y) := \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} u(x+ky) \quad \forall x,y \in \mathbb{R}^d, \tag{1.3.6}$$

for a suitable positive constant $C_{d,m,s}$, cf. [3, eq. (1.2)]. Note that, for $m=1$ we obtain $L_{1,s}u(x) = (-\Delta)^s u(x)$ where $(-\Delta)^s u$ is now understood via (1.3.3). In particular, one can prove that $L_{m,s}$ has $|\xi|^{2s}$ as a Fourier multiplier. Namely, for every $m \in \mathbb{N}$ and $s \in (0,m)$ it holds that

$$L_{m,s}\varphi = \mathscr{F}^{-1}\left[(2\pi|\cdot|)^{2s}\mathscr{F}[\varphi]\right] = (-\Delta)^s\varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d), \tag{1.3.7}$$

(see [3, Theorem 1.9]).
Next, we introduce a useful space. Namely, if $s \in \mathbb{R}$ we define $\mathcal{L}^1_{2s}(\mathbb{R}^d)$ by

$$\mathcal{L}^1_{2s}(\mathbb{R}^d) := \left\{ u \in L^1_{loc}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{|u(x)|}{(1+|x|)^{d+2s}} dx < +\infty \right\}. \tag{1.3.8}$$

It's easy to see (from Theorem 1.2.1 and Hölder inequality) that for $0 < s < \frac{d}{2}$ we have

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{(1+|x|)^{d+2s}} dx \leq \|u\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{dx}{(1+|x|)^{2d}} \right)^{\frac{d+2s}{2d}} < \infty,$$

which in turn implies

$$\mathring{H}^s(\mathbb{R}^d) \subset \mathcal{L}^1_{2s}(\mathbb{R}^d). \tag{1.3.9}$$

In general, by an analogous argument we see that $L^p(\mathbb{R}^d) \subset \mathcal{L}^1_{2s}(\mathbb{R}^d)$ for every $p \geq 1$ and $s > 0$.

Remark 1.3.1. We further recall that (1.3.4) holds in fact for every $s \in (0, \frac{d}{2})$, where now $(-\Delta)^s u$ is understood via (1.3.1) or equivalently via (1.3.5) (see e.g., [3, Lemma 3.9]).

Next, we state the following important result whose proof can be found in [1, Lemma 1.1].

Proposition 1.3.2. *Let $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d) \cap C^{2s+\varepsilon}_{loc}(\mathbb{R}^d)$, $s \in (0,m)$, $m \in \mathbb{N}$, and $\varepsilon > 0$ small. Then the operator $L_{m,s}u$ can be understood pointwise via the integral (1.3.3). Moreover, $L_{m,s}u = L_{n,s}u$ for every $n \in \mathbb{N}$ such that $n > m$, and $L_{m,s}u \in C^{0,\varepsilon}_{loc}(\mathbb{R}^d)$.*

Furthermore, in view of (1.3.7) and Remark 1.3.1, if u belongs to the space

$\mathcal{L}^1_{2s}(\mathbb{R}^d)$ defined by (1.3.8) we can define a distribution on every open set $\Omega \subset \mathbb{R}^d$ by

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}^d} u(x) L_{m,s}\varphi(x)dx = \int_{\mathbb{R}^d} u(x)(-\Delta)^s\varphi(x)dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.3.10}$$

Moreover, see e.g., [3, Lemma 2.5], if Ω is an open subset of $\mathbb{R}^d$ and $u \in C^{2s+\varepsilon}(\Omega) \cap \mathcal{L}^1_{2s}(\mathbb{R}^d)$ then $L_{m,s}u$ is well defined pointwisely in Ω and

$$\int_{\mathbb{R}^d} L_{m,s}u(x)\varphi(x)dx = \int_{\mathbb{R}^d} u(x)L_{m,s}\varphi(x)dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.3.11}$$

Remark 1.3.2. Here we point out the notion of weak solution for the fractional Poisson equation and we recall that any weak solution is a distributional solution as well. In particular, the function U_T defined by (1.2.2) is called the weak solution of the equation

$$(-\Delta)^s u = T \quad \text{in } \mathring{H}^{-s}(\mathbb{R}^d). \tag{1.3.12}$$

In what follows, we will often replace the notation U_T with the more explicit one $(-\Delta)^{-s}T$. Furthermore, as we have already anticipated, if U_T solves (1.3.12) then it also solves

$$(-\Delta)^s U_T = T \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{1.3.13}$$

where (1.3.13) is understood in the sense of (1.3.10). Indeed, by the Sobolev embedding $U_T \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$ (see again (1.3.9)). Furthermore, by combining (1.2.1) with (1.2.2) we have that for every $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\langle T, \varphi \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \langle U_T, \varphi \rangle_{\mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} U_T(x)(-\Delta)^s\varphi(x)dx,$$

where the last equality can be easily seen by a density argument as follows.
If $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$ are real valued, we clearly have that $\mathscr{F}^{-1}[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)] \in L^2(\mathbb{R}^d)$ and, by Plancharel Theorem, we infer

$$\begin{aligned} \langle \psi, \varphi \rangle_{\mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2s}\widehat{\psi}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi &= \int_{\mathbb{R}^d} \psi(x)\mathscr{F}^{-1}[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)](x)dx \\ &= \int_{\mathbb{R}^d} \psi(x)(-\Delta)^s\varphi(x)dx. \end{aligned} \tag{1.3.14}$$

Then, if we consider $(\psi_n)_n \subset \mathring{H}^s(\mathbb{R}^d)$ converging to U_T we have

$$\langle U_T, \varphi\rangle_{\mathring{H}^s(\mathbb{R}^d)} = \lim_{n\to+\infty} \langle \psi_n, \varphi\rangle_{\mathring{H}^s(\mathbb{R}^d)} = \lim_{n\to+\infty} \int_{\mathbb{R}^d} \psi_n(x)(-\Delta)^s\varphi(x)dx. \tag{1.3.15}$$

To conclude, by the Sobolev embedding (Theorem 1.2.1) and the fact that $(-\Delta)^s\varphi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \psi_n(x)(-\Delta)^s\varphi(x)dx = \int_{\mathbb{R}^d} U_T(x)(-\Delta)^s\varphi(x)dx,$$

concluding the proof.

1.4 Riesz potential

In this section, we focus on an important operator that is often understood as the inverse of the fractional Laplacian, i.e., the Riesz potential operator. However, this relation is valid only under some restrictions on the functions involved. In Lemma 1.4.4, 1.4.5 below we start exploring such a relation. We will carry on this analysis in Chapter 2.

1.4.1 Definition and regularity

Let $f \in L^1_{loc}(\mathbb{R}^d)$ and $0 < s < \frac{d}{2}$. We define the quanity

$$(I_{2s} * f)(x) := A_{2s} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2s}} dy, \tag{1.4.1}$$

where

$$A_{2s} := \frac{\Gamma(\frac{d-2s}{2})}{\pi^{\frac{d}{2}} 2^{2s}\Gamma(s)}. \tag{1.4.2}$$

The choice of the normalisation constant A_α ensures that the convolution kernel I_α satisfies the semigroup property $I_{\alpha+\beta} = I_\alpha * I_\beta$ for each $\alpha, \beta \in (0, d)$ such that $\alpha + \beta < d$ (see for example [103, eq. (6.6), p. 118]).

Formally, by (1.3.1) and the standard properties of Fourier transforms (cf. [12, Propositions 1.24, 1.29]), we have that

$$(-\Delta)^s(I_{2s} * f) = \mathscr{F}^{-1}[(2\pi|\cdot|)^{2s}\widehat{I}_{2s}(\cdot)\widehat{f}(\cdot)] = \mathscr{F}[\widehat{f}] = f,$$

i.e., the fractional Laplacian is formally the left inverse of the Riesz potential. Such relation will be better justified later. First, we will prove its validity in the distributional sense provided the function f is smooth with compact support. Next, already in Lemma 1.4.5 the same outcome is extended to L^p functions. Subsequently, in Chapter 2 we will further extend its validity to the case of non negative functions generating a linear and continuous functional on $\mathring{H}^s(\mathbb{R}^d)$, see e.g., Corollary 2.4.2.

Lemma 1.4.1. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ be a non negative function and $0 < s < \frac{d}{2}$. Then the Riesz potential defined as in (1.4.1) is finite a.e. if and only if $f \in \mathcal{L}^1_{-2s}(\mathbb{R}^d)$.*

Proof. Assume $I_{2s} * f < +\infty$ a.e. Then, for $x \neq 0$, there exists a positive constant C such that

$$|x-y|^{d-2s} \leq C(|x|^{d-2s} + |y|^{d-2s}) = C|x|^{d-2s}\left(1 + \left(\frac{|y|}{|x|}\right)^{d-2s}\right) := g(x,y)$$

and clearly,

$$g(x,y) \leq \begin{cases} C(1+|y|^{d-2s}) & \text{if } |x| \leq 1, \\ C|x|^{d-2s}(1+|y|^{d-2s}), & \text{if } |x| \geq 1 \end{cases} \tag{1.4.3}$$

which means that

$$\int_{\mathbb{R}^d} \frac{f(y)}{(1+|y|)^{d-2s}} dy \leq C \max\left\{1, |x|^{d-2s}\right\} (I_{2s} * f)(x) < +\infty.$$

Conversely, if $f \in \mathcal{L}^1_{-2s}(\mathbb{R}^d)$, then

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2s}} dy &= \int_{|y| \leq 2|x|} \frac{f(y)}{|x-y|^{d-2s}} dy + \int_{|y|>2|x|} \frac{f(y)}{|x-y|^{d-2s}} dy \\ &\leq \int_{|y| \leq 2|x|} \frac{f(y)}{|x-y|^{d-2s}} dy + 2^{d-2s} \int_{|y|>2|x|} \frac{f(y)}{|y|^{d-2s}} dy < +\infty \end{aligned}$$

□

Next, it's useful to recall some well known properties of the Riesz potential operator when it acts on L^q functions. The first coincides with [103, Theorem 1, Section 5].

Lemma 1.4.2. *Let $0 < 2s < d$, $1 < q < \frac{d}{2s}$ and $\frac{1}{t} = \frac{1}{q} - \frac{2s}{d}$. Then $I_{2s} * (\cdot)$ defined* *by*

$$
\begin{aligned}
I_{2s} * (\cdot) : L^q(\mathbb{R}^d) &\longrightarrow L^t(\mathbb{R}^d) \\
f &\longmapsto I_{2s} * f
\end{aligned}
$$

is a bounded linear operator. Namely there exists $C > 0$ independent of f such that

$$
\|I_{2s} * f\|_{L^t(\mathbb{R}^d)} \leq C \|f\|_{L^q(\mathbb{R}^d)} \quad \forall \rho \in L^q(\mathbb{R}^d). \tag{1.4.4}
$$

If $q \geq \frac{d}{2s}$ then $I_{2s} * (\cdot)$ is in general not well defined on the whole space $L^q(\mathbb{R}^d)$. Indeed, let us consider the space of function of bounded mean oscillation (BMO) as the subspace of $L^1_{loc}(\mathbb{R}^d)$ for which

$$
|f|_{BMO(\mathbb{R}^d)} := \sup_Q \int_Q |f(x) - f_Q| dx < +\infty, \tag{1.4.5}
$$

where the supremum is taken among all the cube $Q \subset \mathbb{R}^d$ and by ρ_Q we understand the mean integral defined by $\frac{1}{|Q|} \int_Q \rho(x) dx$. If $f \in L^{\frac{d}{2s}}(\mathbb{R}^d)$ and $I_{2s} * f$ is finite for almost every $x \in \mathbb{R}^d$ then $I_{2s} * f \in BMO(\mathbb{R}^d)$.

The following classic result gives regularity of the Riesz potential under some restrictions. See for example [48, Theorem 2] for a proof.

Lemma 1.4.3. *If $I_{2s} * f$ is almost everywhere finite on $\mathbb{R}^d$, $f \in L^q(\mathbb{R}^d)$ and $\frac{d}{q} < 2s < \frac{d}{q} + 1$ then $I_{2s} * f \in L^\infty(\mathbb{R}^d)$ and is Hölder continuous of order $2s - \frac{d}{q}$*

We refer also to [75, Theorem 3.1] for a local version providing regularity when $s \in (0, 1)$.

We finally mention that in the case $q \geq \frac{d}{2s}$, even if the Riesz potential is not well defined as a Lebesgue integral, the quantity $I_{2s} * f$ can be understood in the distributional sense via the formula

$$
\langle I_{2s} * f, \phi \rangle := \langle f, I_{2s} * f \rangle \quad \forall f \in \Phi,
$$

where Φ is defined as the subspace of Schwarz functions ortogonal to all the polynomials. See [100].

Now, we state the famous Hardy–Littlewood–Sobolev inequality which is a fundamental tool for all the subsequent chapters of the thesis. For a proof we refer to [78, Theorem 4.3].

Theorem 1.4.1. *(Hardy–Littlewood–Sobolev) Let $p, r > 1$ and $0 < \lambda < d$ with $\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$. Then there exists a sharp positive constant $C(\lambda, d, p)$ independent of f and g such that*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)h(y)}{|x-y|^\lambda} dxdy \right| \leq C(\lambda, d, p) \|f\|_{L^p(\mathbb{R}^d)} \|h\|_{L^r(\mathbb{R}^d)}. \tag{1.4.6}$$

Moreover, if $p = r = \frac{2d}{2d-\lambda}$ the sharp constant is given by

$$C(\lambda, d) := \pi^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(d - \frac{\lambda}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)} \right)^{-1+\frac{\lambda}{d}}$$

and (1.4.6) is an equality.

1.4.2 Relation with the fractional Laplacian

Lemma 1.4.4. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and $0 < s < \frac{d}{2}$. Then*

$$(-\Delta)^s (I_{2s} * f) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{1.4.7}$$

In particular, (1.4.7) holds also pointwisely.

Proof. In order to prove (1.4.7), we first recall that, cf. [103, Lemma 2, Chapter V],

$$\int_{\mathbb{R}^d} (I_{2s} * f)(x)\varphi(x)dx = \int_{\mathbb{R}^d} \widehat{f}(x)(2\pi|x|)^{-2s}\overline{\widehat{\varphi}(x)}dx \quad \text{for all } f, \varphi \in C_c^\infty(\mathbb{R}^d). \tag{1.4.8}$$

In particular, by Remark 1.3.1, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have that $(-\Delta)^s\varphi \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Finally, it remains to prove that

$$\int_{\mathbb{R}^d} (I_{2s} * f)(x)\mathscr{F}^{-1}\left[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)\right](x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d).$$

To this aim, since $(-\Delta)^s\varphi = \mathscr{F}^{-1}\left[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)\right] \in L^{\frac{2d}{d+2s}}(\mathbb{R}^d)$ (cf. Remark 1.3.1), we consider a sequence $(\varphi_n)_n \subset C_c^\infty(\mathbb{R}^d)$ converging to $(-\Delta)^s\varphi$ in $L^{\frac{2d}{d+2s}}(\mathbb{R}^d)$. In particular, by the Sobolev embedding we further conclude that $\{\varphi_n\}_n$ converges to $(-\Delta)^s\varphi$ in $\mathring{H}^{-s}(\mathbb{R}^d)$. Thus, by combining (1.4.4) with (1.4.8) we have that $I_{2s} * f \in$

$L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d}(I_{2s}*f)(x)\mathscr{F}^{-1}\left[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)\right](x)dx = \lim_{n\to+\infty}\int_{\mathbb{R}^d}(I_{2s}*f)(x)\varphi_n(x)dx$$
$$= \lim_{n\to+\infty}\int_{\mathbb{R}^d}\widehat{f}(x)(2\pi|x|)^{-2s}\overline{\widehat{\varphi}_n(x)}dx = \lim_{n\to+\infty}\langle f,\varphi_n\rangle_{\mathring{H}^{-s}(\mathbb{R}^d)}$$
$$= \langle f,\mathscr{F}^{-1}\left[(2\pi|\cdot|)^{2s}\widehat{\varphi}(\cdot)\right]\rangle_{\mathring{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\widehat{f}(x)\overline{\widehat{\varphi}(x)}dx = \int_{\mathbb{R}^d}f(x)\varphi(x)dx$$

where we have also used the definition of the inner product in $\mathring{H}^{-s}(\mathbb{R}^d)$ (see e.g., [12, definition 3.1], [78, Corollary 5.10]) and the fact that f and φ are real valued. Finally, from (1.4.7) and (1.3.11) we conclude the proof. □

In what follows, we prove a well known result extending the validity of Lemma 1.4.4 to a wider class of functions.

Lemma 1.4.5. *Let $0<s<\frac{d}{2}$, $0<p<\frac{d}{2s}$ and $f\in L^p(\mathbb{R}^d)$. Then,*

$$(-\Delta)^s(I_{2s}*f)=f \quad \textit{in } \mathcal{D}'(\mathbb{R}^d).$$

Proof. Let $\varphi\in C_c^\infty(\mathbb{R}^d)$ and $\psi\in C_c^\infty(\mathbb{R}^d)$. From Lemma 1.4.4 we have that

$$\int_{\mathbb{R}^d}(I_{2s}*\psi)(x)(-\Delta)^s\varphi(x)dx = \int_{\mathbb{R}^d}\psi(x)\varphi(x)dx. \tag{1.4.9}$$

From Remark 1.3.1 or [70, Lemma 1.2] we first recall that $(-\Delta)^s\varphi$ is a smooth function such that

$$|(-\Delta)^s\varphi(x)|\le C(1+|x|)^{-(d+2s)}, \tag{1.4.10}$$

which in particular implies $(-\Delta)^s\varphi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, by Theorem 1.4.1, for every $g\in L^p(\mathbb{R}^d)$

$$\left|\int_{\mathbb{R}^d}(I_{2s}*g)(x)(-\Delta)^s\varphi(x)dx\right| \le C\|\varphi\|_{L^r(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}, \tag{1.4.11}$$

where $\frac{1}{r}=2-\frac{1}{p}-\frac{d-2s}{d}$. Next, let $\{\psi_k\}_k\subset C_c^\infty(\mathbb{R}^d)$ be a sequence converging to f

in $L^p(\mathbb{R}^d)$. Therefore, (1.4.11) yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}(I_{2s}*\psi_k)(x)(-\Delta)^s\varphi(x)dx-\int_{\mathbb{R}^d}(I_{2s}*f)(x)(-\Delta)^s\varphi(x)dx\right| \\
&=\left|\int_{\mathbb{R}^d}(I_{2s}*(\psi_k-f))(x)(-\Delta)^s\varphi(x)dx\right|\leq C_\varphi\|\psi_k-f\|_{L^p(\mathbb{R}^d)}\to 0 \quad \text{as } k\to+\infty.
\end{aligned} \tag{1.4.12}$$

Taking into account (1.4.9), (1.4.12) and Hölder inequality

$$\begin{aligned}
\int_{\mathbb{R}^d}(I_{2s}*f)(x)(-\Delta)^s\varphi(x)dx &= \lim_{k\to\infty}\int_{\mathbb{R}^d}(I_{2s}*\psi_k)(x)(-\Delta)^s\varphi(x)dx \\
&= \lim_{k\to\infty}\int_{\mathbb{R}^d}\psi_k(x)\varphi(x)dx=\int_{\mathbb{R}^d}f(x)\varphi(x)dx
\end{aligned}$$

which concludes the proof. □

1.4.3 Local and nonlocal rearrangement inequalities

As a conclusion to this preliminary chapter, we recall some rearrangement inequalities involving local and nonlocal terms. Most of the following results are contained in [78, Chapter 3]. However, for the sake of completeness, we provide the basic results that will be heavily used in this thesis.

Let $f:\mathbb{R}^d\to\mathbb{R}$ be a measurable function such that

$$\left|\left\{x\in\mathbb{R}^d : |f(x)|>t\right\}\right|<\infty \quad \forall t>0. \tag{1.4.13}$$

Because of (1.4.13), we have the equality

$$|f(x)|=\int_0^\infty \chi_{\{|f|>t\}}(x)\,dt,$$

cf. [78, Theorem 1.13]. Then, given a measurable set A, we define A^* to be the open ball centred at the origin having the same measure of A. For the sake of clarity we also set $\chi_A^* := \chi_{A^*}$.

In view of the above definitions we further set f^* as follows

$$f^*(x):=\int_0^\infty \chi^*_{\{|f|>t\}}(x)\,dt. \tag{1.4.14}$$

The function f^* defined by (1.4.14) is radially non increasing and preserves the L^p norms, i.e.,

$$\|f^*\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.4.15}$$

for any $1 \leq p \leq \infty$.

Next, we state two results which will be used respectively in Chapter 3 and 4, see e.g., Corollary 3.2.3 and Section 4.2.

Theorem 1.4.2 ([78, Theorem 3.7])**.** *Let f, g, h be three non negative functions satisfying* (1.4.13)*. Then,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(x)dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(x)dxdy. \tag{1.4.16}$$

In this thesis, Theorem 1.4.2 will be applied to $g(x) = |x|^{-(d-2s)}$ and $f = h$, where the non local terms in (1.4.16) are associated to the norms $\|f\|_{\mathring{H}^{-s}(\mathbb{R}^d)}$ and $\|f^*\|_{\mathring{H}^{-s}(\mathbb{R}^d)}$, see e.g., Lemma 2.4.1. In particular, this shows that such norm increases under rearrangements.

Next, we recall rearrangements inequalities involving the Sobolev norm $\mathring{H}^s(\mathbb{R}^d)$, proving that in this case (1.4.16) has the opposite sign, i.e., rearrangements decrease the $\mathring{H}^s$ norm.

Theorem 1.4.3. *Let $0 < s \leq 1$, $0 < s < \frac{d}{2}$. If $f \in \mathring{H}^s(\mathbb{R}^d)$ then*

$$\|f^*\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \|f\|_{\mathring{H}^s(\mathbb{R}^d)}. \tag{1.4.17}$$

Proof. The proof follows the same lines of [78, Lemma 7.17] or [8, Proposition 2.1]. For the sake of completeness we provide a proof here.

If $s = 1$ we directly refer to [78, Lemma 7.17, (1)].

If $s \in (0, 1)$, by Lemma 1.2.1 and (1.2.5) it's enough to prove that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f^*(x) - f^*(y)|^2}{|x-y|^{d+2s}}dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2s}}dxdy. \tag{1.4.18}$$

To begin with, for $f \in \mathring{H}^s(\mathbb{R}^d)$, we define the function f_c as follows

$$f_c(x) := \min\left\{\max\left\{f(x) - c, 0\right\}, \frac{1}{c}\right\}, \quad c > 0.$$

In particular, from the inequality $|f_c(x) - f_c(y)| \leq |f(x) - f(y)|$ and monotone convergence we derive that

$$\lim_{c\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f_c(x)-f_c(y)|^2}{|x-y|^{d+2s}}dxdy=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}dxdy, \qquad (1.4.19)$$

and the same holds when f_c and f are replaced by $(f_c)^*$ and f^*. In view of (1.4.19) and the fact that $f_c \in L^2(\mathbb{R}^d)$, it's enough to prove (1.4.18) for functions in $L^2(\mathbb{R}^d)$. In this case, we split the kernel $|x-y|^{-(d+2s)}$ as follows

$$|x-y|^{-(d+2s)}=K_+(x-y)+K_-(x-y), \quad K_-(x-y):=(1+|x-y|^2)^{-\frac{d+2s}{2}}. \quad (1.4.20)$$

Note that K_+ is non negative and radially non increasing. From (1.4.20) we also split

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}dxdy=:I_1(f)+I_2(f). \qquad (1.4.21)$$

By non negativity of f, Fubini and Young's Theorem we derive

$$I_1(f)=2\int_{\mathbb{R}^d}|f(x)|^2\int_{\mathbb{R}^d}K_-(x-y)dy\,dx-2\int_{\mathbb{R}^d}f(x)(f*K_-)(x)dx. \qquad (1.4.22)$$

Indeed, since $K_- \in L^1(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$, Young's Theorem implies that $f*K_- \in L^2(\mathbb{R}^d)$ which in turn implies that $f(f*K_-)\in L^1(\mathbb{R}^d)$ proving that the second term in the right hand side of (1.4.22) is finite. Furthermore, the first term in the right hand side of (1.4.22) coincide (up to a constant) with the L^2 norm of f. Namely,

$$I_1(f)=2\|K_-\|_{L^1(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}+2\int_{\mathbb{R}^d}f(x)(f*K_-)(x)dx. \qquad (1.4.23)$$

Hence, by (1.4.15) and Theorem 1.4.2 we conclude that $I_1(f^*)\le I_1(f)$.

In order to finish the proof, it remains to prove that $I_2(f^*)\le I_2(f)$. To this aim, we consider the truncated kernel K_+^c defined as

$$K_+^c(x):=\min\{K_+(x),c\}.$$

Since $K_+^c \in L^1(\mathbb{R}^d)$, by arguing as before we derive that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f^*(x)-f^*(y)|^2 K_+^c(x-y)dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)-f(y)|^2 K_+^c(x-y)dxdy. \tag{1.4.24}$$

By monotone convergence, we can pass to the limit as $c \to +\infty$ in the inequality (1.4.24) to conclude the proof. □

Chapter 2

Brezis–Browder type results

This chapter is devoted to Brezis–Browder type results for homogeneous fractional Sobolev spaces $\mathring{H}^s(\mathbb{R}^d)$ and quantitative type estimates for s–harmonic functions. Such outcomes give sufficient conditions for a linear and continuous functional T defined on $\mathring{H}^s(\mathbb{R}^d)$ to admit (up to a constant) an integral representation of its norm in terms of the Coulomb type energy

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{T(x)T(y)}{|x-y|^{d-2s}} dxdy,$$

and for distributional solutions of $(-\Delta)^s u = T$ on $\mathbb{R}^d$ to be of the form

$$u(x) = \int_{\mathbb{R}^d} \frac{T(y)}{|x-y|^{d-2s}} dy + l, \quad l \in \mathbb{R}.$$

2.1 Introduction

Historically, if we denote by $W^{m,p}(\mathbb{R}^d)$ the standard Sobolev space where m is an integer number and p is a real number strictly larger than one, H. Brezis and F. Browder in [26] proved that if $T \in W^{-m,p'}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ and $u \in W^{m,p}(\mathbb{R}^d)$ are such that $T(x)u(x) \geq -|f(x)|$ a.e. for some integrable function f then $Tu \in L^1(\mathbb{R}^d)$ and

$$\langle T, u\rangle_{W^{-m,p'}(\mathbb{R}^d), W^{m,p}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)u(x)dx.$$

The first order case $m = 1$ was already studied in [25] where the same authors provide also counterexamples. We further refer to [42] for similar outcomes. The proof of the above results heavily relies on a truncation procedure introduced by L.

Hedberg (see [6, Theorem 3.4.1] and [67]) stating that for every $u \in W^{m,p}(\mathbb{R}^d)$ there exists a sequence $(u_n)_n$ such that

$$\begin{cases} u_n \in W^{m,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), & \text{supp}(u_n) \text{ is compact;} \\ |u_n(x)| \leq |u(x)| \text{ and } u_n(x)u(x) \geq 0 & \text{a.e. in } \mathbb{R}^d; \\ u_n \to u \text{ in } W^{m,p}(\mathbb{R}^d). \end{cases}$$

Such result has been extended by considering Ω a generic open subset of $\mathbb{R}^d$ and $T = \mu + h$ for some Radon measure μ and $h \in L^1_{loc}(\Omega)$. Namely, in [18] the authors proved that if $u(x) \geq 0$ and $h(x)u(x) \geq -|f(x)|$ a.e. in Ω for some $f \in L^1(\Omega)$ then $u \in L^1(\Omega, d\mu)$, $hu \in L^1(\Omega)$ and

$$\langle T, u \rangle_{W^{-m,p'}(\Omega), W_0^{m,p}(\Omega)} = \int_\Omega u(x)d\mu(x) + \int_\Omega h(x)u(x)dx.$$

We further mention that the case $m = 1$, $h = 0$ and Ω an open subset of $\mathbb{R}^d$ was studied in [25] and, already in [26] the case $m > 1$ was analysed under some restriction on T, see [26, Theorem 3]. In the latter the authors further required

$$\int_{\Omega \cap B_R} |T(x)|dx < \infty \quad \forall R > 0.$$

In Section 2.3 we prove Brezis–Browder type results in the framework of the fractional homogeneous Sobolev spaces $\mathring{H}^s(\mathbb{R}^d)$ for $d \geq 1$ and $0 < s < \frac{d}{2}$. In general, if $0 < s \leq 1$ such outcomes follow from the fact that the fractional Sobolev spaces $\mathring{H}^s(\mathbb{R}^d)$ are closed under truncations in such low order regime, cf. Proposition 2.3.1. As a consequence, in Lemma 2.3.1 we provide the fractional counterpart of [26, Theorem 1]. In the higher order case $1 < s < \frac{d}{2}$ we employ a different truncation argument. Indeed, by using a result proved in [94], we approximate any function in $\mathring{H}^s(\mathbb{R}^d)$ by multiplication against a suitable family of smooth functions. In general, such approximating sequence is not in $L^\infty(\mathbb{R}^d)$ as in the Hedbger's case. However, we overcome this issue by further requiring higher local integrability of the element T generating a linear functional on $\mathring{H}^s(\mathbb{R}^d)$. To be precise, if we require

$T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^{\bar{q}}_{loc}(\mathbb{R}^d)$ where $\bar{q} = \bar{q}(d,s)$ is defined by

$$\bar{q}(d,s) := \begin{cases} \frac{2d}{d+2s}, & \text{if } 1 < s < \frac{d}{2} \\ 1, & \text{if } 0 < s \leq 1, \end{cases} \tag{2.1.1}$$

we prove the following:

Theorem 2.1.1. *Let $d \geq 1$, $0 < s < \frac{d}{2}$ and $\bar{q}$ as in (2.1.1). Assume that $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^{\bar{q}}_{loc}(\mathbb{R}^d)$ and $u \in \mathring{H}^s(\mathbb{R}^d)$ are such that $Tu \geq -|f|$ for some $f \in L^1(\mathbb{R}^d)$. Then, $Tu \in L^1(\mathbb{R}^d)$ and*

$$\langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)u(x)dx.$$

As a byproduct, we give sufficient conditions under which an element $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ has an integral representation of its $\|\cdot\|_{\mathring{H}^{-s}(\mathbb{R}^d)}$ norm i.e.,

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (I_{2s} * T)(x)T(x)dx =: \mathcal{D}_{2s}(T,T). \tag{2.1.2}$$

As it will be pointed out in Section 2.4, the relation (2.1.2) holds when considering smooth functions T, or more generally when $\mathcal{D}_{2s}(|T|,|T|)$ is finite. From the Hardy–Littlewood–Sobolev inequality [78, Theorem 4.3], the last condition is satisfied for instance by every function in $L^{\frac{2d}{d+2s}}(\mathbb{R}^d)$.

The link with the Brezis–Browder type results proved in Section 2.3 can be seen as follows. Under some restrictions, the Riesz potential $I_{2s} * T$ belongs to $\mathring{H}^s(\mathbb{R}^d)$ and

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \langle T, I_{2s} * T \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)}, \tag{2.1.3}$$

see e.g., Corollary 2.4.1. Hence, in view of (2.1.3), asking whether (2.1.2) is valid or not reduces to ask if the action of the linear operator generated by T against the function $I_{2s} * T$ can be expressed as a Lebesgue integration. In Lemma 2.4.2 and Corollary 2.4.2, under different assumptions, we indeed derive that the $\mathring{H}^{-s}(\mathbb{R}^d)$–norm of T can be written as the Coulomb type energy $\mathcal{D}_{2s}(T,T)$.

One of the ingredients employed to obtain such outcomes involves interior regularity and quantitative type estimates for s–harmonic functions on balls in the spirit of [83, Lemma 4.2], see Lemma 2.2.1. These type of results are related to [61, Theorem 12.19] that provides regularity in the context of pseudodifferential operators.

As a consequence of the above mentioned quantitative estimates, we derive that s–harmonic functions are polynomials.

Theorem 2.1.2. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$. If $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$ solves*

$$(-\Delta)^s u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

then u is a polynomial of degree strictly smaller than $2s$.

We refer to Section 2.2 for the details. As a consequence we derive uniqueness of the Riesz potential as a distributional (and weak) solution to the fractional Poisson equation for all $0 < s < \frac{d}{2}$, cf. Corollary 2.4.1. This result can be seen as the high order version of [52, Corollary 1.4]. Furthermore, in Theorem 2.1.3 we deduce a representation formula for distributional solutions to the fractional Poisson equation in the spirit of [31, Theorem 2.4] where a similar result has been obtained in the case of the polyharmonic operator $(-\Delta)^k$, k being an integer number strictly larger than one.

Theorem 2.1.3. *Let $d \geq 1$, $0 < s < \frac{d}{2}$, $0 \leq T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ and $l \in \mathbb{R}$. The following are equivalent:*

(i) $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$ satisfies

$$(-\Delta)^s u = T \quad \mathcal{D}'(\mathbb{R}^d)$$

and

$$\liminf_{R\to\infty} \frac{1}{R^d} \int_{R<|x-y|<2R} |u(y) - l| dy < \infty \quad \textit{for a.e. } x \in \mathbb{R}^d; \tag{2.1.4}$$

(ii) u can be written as

$$u(x) = A_{2s} \int_{\mathbb{R}^d} \frac{T(y)}{|x-y|^{d-2s}} dy + l \quad \textit{for a.e. } x \in \mathbb{R}^d.$$

2.2 Quantitative estimates

We begin this section by proving interior regularity for s–harmonic functions on balls. This result links with the one in [61, Theorem 12.19]. Regardless, in the following Lemma we prove a quantitative version in the spirit of [52, Lemma 3.1] or [83, Lemma 4.2]. The following result can be refined by further assuming some Lebesgue integrability as in Corollary 2.2.1.

Lemma 2.2.1. *Let $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$ and $0 < s < \frac{d}{2}$. Assume that u solves*

$$(-\Delta)^s u = 0 \quad \text{in } \mathcal{D}'(B_{2R}). \tag{2.2.1}$$

Then $u \in C^\infty(\overline{B_R})$. Moreover, for every multi-index $n \in \mathbb{N}^d$,

$$\|D^n u\|_{L^\infty(\overline{B_R})} \le CR^{2s-|n|} \int_{\mathbb{R}^d} \frac{|u(x)|}{(R^{d+2s} + |x|^{d+2s})} dx$$

for some positive constant C independent of R.

Proof. Let $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x) = 1$ if $|x| \ge 2$, $\eta(x) = 0$ if $|x| < \frac{3}{2}$ and $0 \le \eta(x) \le 1$ for every $x \in \mathbb{R}$. Let now $\varphi \in C_c^\infty(\mathbb{R}^d)$ be supported in B_R. Then, if we denote by ψ the Riesz potential of φ defined by

$$\psi(x) := I_{2s} * \varphi(x) = A_{2s} \int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-2s}} dy, \tag{2.2.2}$$

it satisfies (see Lemma 1.4.4)

$$(-\Delta)^s \psi = \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{2.2.3}$$

In particular, $\psi \in C^\infty(\mathbb{R}^d) \cap \mathcal{L}^1_{2s}(\mathbb{R}^d)$. Now, let's fix $m \in \mathbb{N}$, $m > s$. Then, from [3, Lemma 1.5] we have that

$$\int_{\mathbb{R}^d} \psi(x) L_{m,s}\chi(x) dx = \int_{\mathbb{R}^d} L_{m,s}\psi(x)\chi(x) dx \quad \forall \chi \in C_c^\infty(\mathbb{R}^d). \tag{2.2.4}$$

In particular, by combining (2.2.3) with (2.2.4) we obtain that $L_{m,s}\psi(x) = \varphi(x)$ for every $x \in \mathbb{R}^d$. Next, let us consider the function $(1-\eta_R)\psi$ where $\eta_R(x) := \eta(|x|/R)$. Note that $(1-\eta_R)\psi \in C_c^\infty(\mathbb{R}^d)$ and it is supported in B_{2R}. Hence, we can test (2.2.1) against $(1-\eta_R)\psi$ obtaining

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} u(x) L_{m,s}((1-\eta_R)\psi)(x) dx = \int_{\mathbb{R}^d} u(x) L_{m,s}\psi(x) dx - \int_{\mathbb{R}^d} u(x) L_{m,s}(\eta_R\psi)(x) dx \\ & \int_{B_R} u(x)\varphi(x) dx - \int_{\mathbb{R}^d} u(x) L_{m,s}(\eta_R\psi)(x) dx. \end{aligned} \tag{2.2.5}$$

Then, by (1.3.5), (2.2.2), (2.2.5), assuming Fubini's theorem holds we obtain that there exists a positive constant $K_{d,m,s}$ such that

$$\begin{aligned} &K_{d,m,s}\int_{\mathbb{R}^d} u(x)L_{m,s}(\eta_R\psi)(x)dx \\ &= \int_{B_R}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1}{|y|^{d+2s}}\left(\sum_{k=-m}^{m}(-1)^k\binom{2m}{m-k}\frac{\eta_R(x+ky)}{|x+ky-z|^{d-2s}}\right)dy\,u(x)dx\right)\varphi(z)dz \\ &= K_{d,m,s}\int_{B_R}\left(\int_{\mathbb{R}^d} J_R(x,z)u(x)dx\right)\varphi(z)dz = K_{d,m,s}\int_{B_R} u(z)\varphi(z)dz, \end{aligned} \tag{2.2.6}$$

where $K_{d,m,s} := (2/(A_{2s}C_{d,m,s}))$ and

$$J_R(x,z) := \frac{A_{2s}C_{d,m,s}}{2}\int_{\mathbb{R}^d}\frac{1}{|y|^{d+2s}}\left(\sum_{k=-m}^{m}(-1)^k\binom{2m}{m-k}\frac{\eta_R(x+ky)}{|x+ky-z|^{d-2s}}\right)dy. \tag{2.2.7}$$

In particular, from (2.2.6) we deduce that for almost every $z\in B_R$

$$u(z) = \int_{\mathbb{R}^d} J_R(x,z)u(x)dx. \tag{2.2.8}$$

From now on, we focus on proving validity and estimates for (2.2.8). First of all we notice that the function $i_R(x,z)$ defined by

$$i_R(x,z) := \frac{\eta_R(x)}{|x-z|^{d-2s}} \tag{2.2.9}$$

satisfies the following the properties:

- If $|z|<R$, $|x|<\frac{3R}{2}$ then $i_R(x,z)=0$,
- if $|z|<R$ and $|x|\geq\frac{3R}{2}$ then $|x-z|\geq\frac{R}{2}$.

In particular, $i_R(x,z)\in C^\infty(\mathbb{R}^d\times\overline{B_R})$. Moreover, in view of the above analysis, the function $i_R(\cdot,z)\in C^\infty(\overline{B_R})\cap\mathcal{L}^1_{2s}(\mathbb{R}^d)$ and $J_R(\cdot,z) = A_{2s}L_{m,s}(i_R(\cdot,z))$ from which (2.2.7) is well defined.

Furthermore, by differentiating, for every $n\in\mathbb{N}^d$ multi-index the following inequalities hold

$$|D_z^n i_R(x,z)| \leq C(R+|x-z|)^{-(|n|+d-2s)}, \tag{2.2.10}$$

$$|D_x^{2m}D_z^n i_R(x,z)| \leq CR^{-2m}(R+|x-z|)^{-(|n|+d-2s)}, \tag{2.2.11}$$

where C is a positive constant not depending on R. For the convenience of reader we point out that, unless stated, all the constants from now on depend only on d, m, s and n. Hence, by (2.2.10), (2.2.11) and [3, Lemma 2.4] we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{|y|^{d+2s}} \left| \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} D_z^n i_R(x+ky,z) \right| dy \\
&= \int_{B_R} \frac{1}{|y|^{d+2s}} \left| \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} D_z^n i_R(x+ky,z) \right| dy \\
&\quad + \int_{B_R^c} \frac{1}{|y|^{d+2s}} \left| \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} D_z^n i_R(x+ky,z) \right| dy \\
&\leq \left\| D_x^{2m} D_z^n i_R(\cdot,z) \right\|_{L^\infty(\mathbb{R}^d)} \int_{B_R} |y|^{2m-(d+2s)} dy + C_m \| D_z^n i_R(\cdot,z) \|_{L^\infty(\mathbb{R}^d)} \int_{B_R^c} |y|^{-(d+2s)} dy \\
&\leq C R^{-(|n|+d)},
\end{aligned} \tag{2.2.12}$$

for some positive constant C. Thus, (2.2.12) implies that $J_R(x,\cdot) \in C^\infty(\overline{B_R})$ and

$$|D_z^n J_R(x,z)| \leq C R^{-(|n|+d)} \quad \forall (x,z) \in \mathbb{R}^d \times B_R. \tag{2.2.13}$$

Next, we further prove that for every $n \in \mathbb{N}^d$ multi-index there exists a positive constant C such that

$$|D_z^n J_R(x,z)| \leq \frac{C}{R^{|n|-2s}(R^{d+2s} + |x|^{d+2s})} \quad \forall (x,z) \in \mathbb{R}^d \times B_R. \tag{2.2.14}$$

Note that (2.2.14) combined with $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$ in particular implies that

$$\int_{\mathbb{R}^d} |J_R(x,z)||u(x)| dx \leq C R^{2s} \int_{\mathbb{R}^d} \frac{|u(x)|}{R^{d+2s} + |x|^{d+2s}} dx \quad \forall z \in B_R, \tag{2.2.15}$$

that is the right hand side of (2.2.8) is well defined and moreover

$$\int_{B_R} \left(\int_{\mathbb{R}^d} |J_R(x,z)||u(x)| dx \right) |\varphi(z)| dz < \infty. \tag{2.2.16}$$

Furthermore, (2.2.12) in particular implies that for every $x \in \mathbb{R}^d$

$$\int_{B_R} \left(\int_{\mathbb{R}^d} \frac{1}{|y|^{d+2s}} \left| \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} i_R(x+ky, z) \right| dy \right) |\varphi(z)| dz < \infty. \tag{2.2.17}$$

Taking into account (2.2.7), (2.2.16) and (2.2.17) we can apply Fubini's theorem twice to prove (2.2.6). In particular, (2.2.8) holds. Thus, in oder to complete the proof it remains to prove (2.2.14).

To this aim, we first notice that if $|x| \leq 4mR$, the inequality (2.2.14) is easily obtained from (2.2.12). Then we only need to consider the case $|x| > 4mR$.

Assume that $|x| > 4mR$ and $|z| < R$. Since I_{2s} is the fundamental solution for $(-\Delta)^s$ and the function $x \mapsto \frac{A_{2s}}{|x-z|^{d-2s}} \in C^\infty(\mathbb{R}^d) \cap \mathcal{L}^1_{2s}(\mathbb{R}^d)$ if $|x| > 4mR$ and $|z| < R$, by applying again [3, Lemma 1.5] we infer that for every $(x, z) \in B^c_{4mR} \times B_R$

$$\frac{A_{2s} C_{d,m,s}}{2} \int_{\mathbb{R}^d} \frac{1}{|y|^{d+2s}} \left(\sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} \frac{1}{|x+ky-z|^{d-2s}} \right) dy = 0. \tag{2.2.18}$$

Hence, by combining (2.2.7), (2.2.18) with the fact that $\eta_R(x) = 1$ if $|x| > 4mR$, we deduce the equality

$$\begin{aligned} J_R(x,z) &= \frac{A_{2s} C_{d,m,s}}{2} \int_{\mathbb{R}^d} \frac{1}{|y|^{d+2s}} \left(\sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} \frac{(\eta_R(x+ky)-1)}{|x+ky-z|^{d-2s}} \right) dy \\ &= \frac{A_{2s} C_{d,m,s}}{2} \int_{\mathbb{R}^d} |\bar{y}-z|^{-(d-2s)} \underbrace{\left(\sum_{k=-m, k\neq 0}^{m} (-1)^k \binom{2m}{m-k} \frac{(\eta_R(\bar{y})-1)}{|x+k\bar{y}|^{(d+2s)}} \right)}_{=:h_R(x,\bar{y})} d\bar{y}. \end{aligned} \tag{2.2.19}$$

Furthermore, for fixed $|x| > 4mR$, we see that if $|\bar{y}| \geq 2R$ then $h_R(x, \bar{y}) = 0$ while if $|\bar{y}| < 2R$ we still have $|x + k\bar{y}| > 2mR$. We have therefore proved that,

$$h_R(x, \cdot) \in C^\infty_c(\mathbb{R}^d) \quad \forall |x| > 4mR.$$

Furthermore, by differentiating, we have the following estimate

$$|D^n_{\bar{y}} h_R(x, \bar{y})| \leq C R^{-|n|} |x|^{-(d+2s)} \quad \forall |x| > 4mR, \ n \in \mathbb{N}^d, \tag{2.2.20}$$

for some positive constant C. Hence, putting together (2.2.19) with (2.2.20) yields

$$|D^n_z J_R(x,z)| \leq \frac{C}{R^{|n|}|x|^{d+2s}} \int_{B_{2R}} \frac{d\bar{y}}{|\bar{y}-z|^{d-2s}} \leq \frac{\tilde{C}}{R^{|n|-2s}|x|^{d+2s}}. \tag{2.2.21}$$

In particular, from (2.2.21), if $|x| > 4mR$ we deduce (2.2.14). Finally, since $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$, from (2.2.8) and (2.2.14) we can differentiate under the sign of the integral obtaining that for every $n \in \mathbb{N}^d$ there exists $C > 0$ such that

$$|D^n u(z)| \leq \frac{C}{R^{|n|-2s}} \int_{\mathbb{R}^d} \frac{|u(x)|}{R^{d+2s}+|x|^{d+2s}} dx \quad \forall |z| < R, \tag{2.2.22}$$

concluding the proof. □

Corollary 2.2.1. *Let $k \in \mathbb{N}$. Let $u = \sum_{i=1}^k u_i$, where $u_i \in L^{p_i}(\mathbb{R}^d)$ and $p_i \in [1,\infty)$. If u solves*

$$(-\Delta)^s u = 0 \quad \text{in } \mathcal{D}'(B_{2R}), \tag{2.2.23}$$

then $u \in C^\infty(\overline{B_R})$ and moreover, for every multi-index $n \in \mathbb{N}^d$ we have

$$\|D^n u\|_{L^\infty(\overline{B_R})} \leq C \sum_{i=1}^k R^{-|n|-\frac{d}{p_i}} \|u\|_{L^{p_i}(\mathbb{R}^d)},$$

for some positive constant C independent of R.

Proof. Clearly $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$. Hence, Lemma 2.2.1 yields

$$\|D^n u\|_{L^\infty(B_R)} \leq C R^{2s-|n|} \int_{\mathbb{R}^d} \frac{|u(x)|}{(R^{d+2s}+|x|^{d+2s})} dx. \tag{2.2.24}$$

Then, if we denote by ω_{d-1} the surface area of the unitary ball in $\mathbb{R}^d$ and p_i' the Hölder conjugate of p_i, we obtain

$$\int_{\mathbb{R}^d} \frac{dx}{(R^{d+2s}+|x|^{d+2s})^{p_i'}} = \omega_{d-1} R^{d-(d+2s)p_i'} \int_0^\infty \frac{s^{d-1}}{(1+s^{d+2s})^{p_i'}} ds. \tag{2.2.25}$$

Next, by combining (2.2.24), (2.2.25) and Hölder inequality, for every multi–index $n \in \mathbb{N}^d$ there exists a positive constant C independent of R such that

$$\|D^n u\|_{L^\infty(B_R)} \leq C \sum_{i=1}^k R^{-|n|-\frac{d}{p_i}} \|u\|_{L^{p_i}(\mathbb{R}^d)} \tag{2.2.26}$$

which concludes the proof. □

Proof of Theorem 2.1.2.

Proof. From Lemma 2.2.1 we obtain that u is smooth and for every multi-index $n \in \mathbb{N}^d$,

$$\|D^n u\|_{L^\infty(\overline{B_R})} \le C R^{2s-|n|} \int_{\mathbb{R}^d} \frac{|u(x)|}{(R^{d+2s} + |x|^{d+2s})} dx,$$

where C is a constant not depending on R. By sending R to infinity (and using again that $u \in \mathcal{L}^1_{2s}(\mathbb{R}^d)$) we obtain that $D^n u \equiv 0$ for every $n \in \mathbb{N}^d$ such that $|n| \ge 2s$. □

2.3 Brezis–Browder type results

In this section we focus on proving some Brezis–Browder type results. To this aim, we first recall the notion of *regular distribution*.

Regular distributions

Let $0 < s < \frac{d}{2}$. We recall that by $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ we understand a distribution such that

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^d} T(x)\varphi(x)dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d), \tag{2.3.1}$$

and there exists a positive constant C independent of T such that

$$|\langle T, \varphi \rangle| \le C \|\varphi\|_{\mathring{H}^s(\mathbb{R}^d)} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d). \tag{2.3.2}$$

From (2.3.2), we get that T can be identified as the unique continuous extension with respect to the $\mathring{H}^s(\mathbb{R}^d)$–norm of the linear functional defined by (2.3.1). An element T satisfying (2.3.1) is called a *regular distribution*. Note that, if $T \in L^{\frac{2d}{d+2s}}(\mathbb{R}^d)$, $u \in \mathring{H}^s(\mathbb{R}^d)$ and again by $\langle \cdot, \cdot \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)}$ we denote the duality evaluation, by density of $C_c^\infty(\mathbb{R}^d)$ and the Sobolev embedding we infer

$$\lim_{n\to+\infty} \langle T, \varphi_n \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \lim_{n\to+\infty} \int_{\mathbb{R}^d} T(x)\varphi_n(x)dx = \int_{\mathbb{R}^d} T(x)u(x)dx. \tag{2.3.3}$$

Since the left hand side of (2.3.3) converges to $\langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)}$ we conclude that the action of T on $\mathring{H}^s(\mathbb{R}^d)$ can be represented as an integral.

Next, we prove the following result:

Lemma 2.3.1. *Let* $0 < s < \frac{d}{2}$, $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^q_{loc}(\mathbb{R}^d)$ *and* $q \geq \frac{2d}{d+2s}$. *Then*

$$\langle T, \psi \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)\psi(x)dx \quad \forall \psi \in \mathring{H}^s(\mathbb{R}^d) \cap L^{q'}_c(\mathbb{R}^d).$$

Proof. Let's fix $\psi \in \mathring{H}^s(\mathbb{R}^d) \cap L^{q'}_c(\mathbb{R}^d)$. In particular, cf. [12, Proposition 1.55], $\psi \in H^s(\mathbb{R}^d)$. Then, if we set $K := \mathrm{supp}(\psi)$, there exists a compact set $K' \supset K$ and a sequence $(\psi_n)_n \subset C^\infty_c(K')$ converging to ψ in $\mathring{H}^s(\mathbb{R}^d)$ (see e.g., [36, p. 191, eq. (11)]). Hence, by the Sobolev embedding and the fact that $q' \leq \frac{2d}{d-2s}$ we obtain that $\psi_n \to \psi$ in $L^{q'}(K')$. To conclude, by (2.3.1),

$$\lim_{n\to+\infty} \langle T, \psi_n \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \lim_{n\to+\infty} \int_{K'} T(x)\psi_n(x)dx = \int_{\mathbb{R}^d} T(x)\psi(x)dx. \tag{2.3.4}$$

Since the left hand side of (2.3.4) converges to $\langle T, \psi \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)}$ we conclude. □

Remark 2.3.1. In view of Lemma 2.3.1, if $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, where such intersection is understood in the sense of (2.3.1)–(2.3.2), and $q \geq \frac{2d}{d+2s}$ then T generates a linear and continuous operator on $\mathring{H}^s(\mathbb{R}^d) + L^{q'}(\mathbb{R}^d)$. Indeed, let us fix $u \in \mathring{H}^s(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$. By Lemma 2.3.2, there exists a sequence $u_\lambda \in \mathring{H}^s(\mathbb{R}^d) \cap L^{q'}_c(\mathbb{R}^d)$ such that

$$\begin{aligned} &\|u_\lambda - u\|_{\mathring{H}^s(\mathbb{R}^d)} \to 0 \quad \text{as } \lambda \to +\infty; \\ &\|u_\lambda - u\|_{L^{q'}(\mathbb{R}^d)} \to 0 \quad \text{as } \lambda \to +\infty. \end{aligned} \tag{2.3.5}$$

In particular, by combining (2.3.5) with Lemma 2.3.1, for every $u \in \mathring{H}^s(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$

$$\begin{aligned} \langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \lim_{\lambda\to+\infty} \langle T, u_\lambda \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} &= \lim_{\lambda\to+\infty} \int_{\mathbb{R}^d} T(x)u_\lambda(x)dx \\ &= \int_{\mathbb{R}^d} T(x)u(x)dx. \end{aligned}$$

Thus, the operator $T : \mathring{H}^s(\mathbb{R}^d) + L^{q'}(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\langle T, u \rangle := \langle T, u_1 \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} + \int_{\mathbb{R}^d} T(x)u_2(x)dx \quad u = u_1 + u_2,$$

is linear and continuous.

Now, similarly to what has been proved in [26], we give sufficient conditions for an element $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ and $u \in \mathring{H}^s(\mathbb{R}^d)$ to have integrable product. As already mentioned, in the general case $0 < s < \frac{d}{2}$ we further require $T \in L_{loc}^{\frac{2d}{d+2s}}(\mathbb{R}^d)$ even though such assumption is actually not needed in the case $0 < s \leq 1$ (see Corollary 2.3.1).

2.3.1 Full regime: $0 < s < \frac{d}{2}$

First of all let's recall the following technical result from [94, Lemma 5]. In the whole of this subsection, unless specified, we will assume $0 < s < \frac{d}{2}$.

Lemma 2.3.2. *Let $u \in \mathring{H}^s(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. If we define $\varphi_\lambda(x) := \varphi(\lambda^{-1}x)$ with $\lambda > 0$, we have that $u\varphi_\lambda \in H^s(\mathbb{R}^d)$. Moreover, if $\varphi = 1$ in a neighbourhood of the origin then $u\varphi_\lambda \to u \quad$ in $\mathring{H}^s(\mathbb{R}^d)$ as $\lambda \to \infty$.*

Theorem 2.3.1. *Let $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L_{loc}^{\frac{2d}{d+2s}}(\mathbb{R}^d)$. Assume $u \in \mathring{H}^s(\mathbb{R}^d)$ is such that $Tu \geq -|f|$ for some $f \in L^1(\mathbb{R}^d)$. Then, $Tu \in L^1(\mathbb{R}^d)$ and*

$$\langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)u(x)dx.$$

Proof. Let's define

$$u_\lambda(x) := u(x)\varphi(\lambda^{-1}x), \quad \lambda > 0,$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, and $\varphi = 1$ in B_1. Lemma 2.3.2 implies that $u_\lambda \to u$ in $\mathring{H}^s(\mathbb{R}^d)$. Moreover, $u_\lambda \in \mathring{H}^s_c(\mathbb{R}^d)$ and, combining Lemma 2.3.1 and 2.3.2 yields

$$\langle T, u_\lambda \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)u_\lambda(x)dx \to \langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)} \quad \text{as } \lambda \to \infty.$$

Therefore, by Fatou's Lemma,

$$\int_{\mathbb{R}^d} T(x)u(x)dx \leq \liminf_{\lambda \to +\infty} \int_{\mathbb{R}^d} T(x)u_\lambda(x)dx = \langle T, u \rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)}. \tag{2.3.6}$$

By combining (2.3.6) with the inequality $Tu \geq -|f| \in L^1(\mathbb{R}^d)$ we also get $Tu \in L^1(\mathbb{R}^d)$. Finally, since $|Tu_\lambda| \leq |Tu| \in L^1(\mathbb{R}^d)$, by dominated convergence we deduce that $Tu_\lambda \to Tu$ in $L^1(\mathbb{R}^d)$, concluding the proof. □

2.3.2 Low order regime: $0 < s \leq 1$

In the previous subsection it has been requested for a distribution in $\mathring{H}^{-s}(\mathbb{R}^d)$ to be more than locally integrable. Here, if $s \in (0,1]$ we prove an analogous result to Theorem 2.3.1 by only requiring T to be a regular distribution. The proof is indeed based on Proposition 2.3.1 where, unlike Lemma 2.3.2, we provide a bounded approximating sequence for elements of $\mathring{H}^s(\mathbb{R}^d)$.

Proposition 2.3.1. *Let $0 < s \leq 1$ and $u \in \mathring{H}^s(\mathbb{R}^d)$ be a non negative function. Then, there exists a sequence $(u_n)_n \subset \mathring{H}^s(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d)$ of non negative functions such that*

(i) $0 \leq u_n(x) \leq u(x)$*;*

(ii) $\|u_n - u\|_{\mathring{H}^s(\mathbb{R}^d)} \to 0$ *as* $n \to \infty$*.*

Proof. Let $(\varphi_n)_n \subset C_c^\infty(\mathbb{R}^d)$ such that $\varphi_n \to u$ in $\mathring{H}^s(\mathbb{R}^d)$. We define

$$u_n := \min\{[\varphi_n]_+, u\}. \tag{2.3.7}$$

Note that (2.3.7) can be rewritten in the more useful form as

$$u_n := u - [[\varphi_n]_+ - u]_- \tag{2.3.8}$$

and, by construction, $0 \leq u_n \leq u$. Recalling that $\|[\cdot]_\pm\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$ for every $s \in (0,1]$ (see Lemma 1.2.2) we conclude that $u_n \in \mathring{H}^s(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d)$.

Next, we prove that u_n converges to u in $\mathring{H}^s(\mathbb{R}^d)$. First of all we recall again that $\||\cdot|\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$. Then, by the parallelogram law we infer

$$\|[\varphi_n]_+\|^2_{\mathring{H}^s(\mathbb{R}^d)} + \|[\varphi_n]_-\|^2_{\mathring{H}^s(\mathbb{R}^d)} = \frac{\||\varphi_n|\|^2_{\mathring{H}^s(\mathbb{R}^d)} + \|\varphi_n\|^2_{\mathring{H}^s(\mathbb{R}^d)}}{2}. \tag{2.3.9}$$

In particular, we further obtain the inequality

$$\|[\varphi_n]_+\|^2_{\mathring{H}^s(\mathbb{R}^d)} + \|[\varphi_n]_-\|^2_{\mathring{H}^s(\mathbb{R}^d)} \leq \|\varphi_n\|^2_{\mathring{H}^s(\mathbb{R}^d)}. \tag{2.3.10}$$

Now, by definition of φ_n and the fact that $u \geq 0$, up to a subsequence, we have

$$[\varphi_n]_+ \rightharpoonup u \quad \text{in } \mathring{H}^s(\mathbb{R}^d),$$

which combined with the weak lower semicontinuity of $\|\cdot\|_{\mathring{H}^s(\mathbb{R}^d)}$ yields

$$\begin{aligned} \|u\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \liminf_{n\to+\infty} \|[\varphi_n]_+\|_{\mathring{H}^s(\mathbb{R}^d)} &\leq \limsup_{n\to+\infty} \|[\varphi_n]_+\|_{\mathring{H}^s(\mathbb{R}^d)} \\ &\leq \limsup_{n\to+\infty} \|\varphi_n\|_{\mathring{H}^s(\mathbb{R}^d)} = \|u\|_{\mathring{H}^s(\mathbb{R}^d)}, \end{aligned} \tag{2.3.11}$$

i.e., $\|[\varphi_n]_+\|_{\mathring{H}^s(\mathbb{R}^d)} \to \|u\|_{\mathring{H}^s(\mathbb{R}^d)}$. Then, by (2.3.10) we also derive $\|[\varphi_n]_-\|_{\mathring{H}^s(\mathbb{R}^d)} \to 0$ which in turn implies that $\|[\varphi_n]_+ - u\|_{\mathring{H}^s(\mathbb{R}^d)} \to 0$. Then, by (2.3.8) we get

$$\|u_n - u\|_{\mathring{H}^s(\mathbb{R}^d)} = \|[\varphi_n]_+ - u]_-\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \|[\varphi_n]_+ - u\|_{\mathring{H}^s(\mathbb{R}^d)} \to 0,$$

concluding the proof. □

Now, we prove a similar result to Theorem 2.3.1 but using Proposition 2.3.1 rather than Lemma 2.3.2.

Corollary 2.3.1. *Let $0 < s \leq 1$, $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$, and $u \in \mathring{H}^s(\mathbb{R}^d)$. If $Tu \geq -|f|$ for some f belonging to $L^1(\mathbb{R}^d)$ then $Tu \in L^1(\mathbb{R}^d)$ and*

$$\langle T, u\rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} T(x)u(x)dx. \tag{2.3.12}$$

Proof. Let $u \in \mathring{H}^s(\mathbb{R}^d)$. Then, $[u]_\pm \in \mathring{H}^s(\mathbb{R}^d)$ and, by Proposition 2.3.1 there exist two sequences $(u_n^\pm)_n \subset \mathring{H}^s(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d)$ converging to $[u]_\pm$ in $\mathring{H}^s(\mathbb{R}^d)$ such that $0 \leq u_n^\pm(x) \leq [u(x)]_\pm$ a.e. Then, the function $\bar{u}_n := u_n^+ - u_n^-$ converges to u in $\mathring{H}^s(\mathbb{R}^d)$. Furthermore, by a direct computation it's easy to see that $|\bar{u}_n(x)| \leq |u(x)|$ and $\bar{u}_n(x)u(x) \geq 0$ a.e. Next, by approximating $\bar{u}_n \in \mathring{H}^s(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d)$ via convolution with mollifiers $(\rho_k)_k$ (see e.g. [23, Lemma A.1]), we notice that for every fixed n

$$\begin{aligned} \langle T, \bar{u}_n\rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)} &= \lim_{k\to+\infty} \int_{\mathbb{R}^d} T(x)(\bar{u}_n * \rho_k)(x)dx \\ &= \lim_{k\to+\infty} \int_{\mathrm{supp}(\bar{u}_n)} (T * \rho_k)(x)\bar{u}_n(x)dx = \int_{\mathbb{R}^d} T(x)\bar{u}_n(x)dx. \end{aligned}$$

Then, by arguing as in the proof of Theorem 2.3.1 we conclude that

$$\lim_{n\to+\infty} \langle T, \bar{u}_n\rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)} = \lim_{n\to+\infty} \int_{\mathbb{R}^d} T(x)\bar{u}_n(x)dx = \int_{\mathbb{R}^d} T(x)u(x)dx.$$

The thesis follows since the left hand side converges to $\langle T, u\rangle_{\mathring{H}^{-s}(\mathbb{R}^d), \mathring{H}^{-s}(\mathbb{R}^d)}$. □

Proof of Theorem 2.1.1.

The proof follows by combining Theorem 2.3.1 with Corollary 2.3.1. □

Remark 2.3.2. Note that, as a direct application of Theorem 2.1.1 we can conclude that the Riesz potential operator is well defined for every $0 \leq T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^{\bar{q}}_{loc}(\mathbb{R}^d)$. To see this, let's take any non negative (and non identically zero) smooth function φ with compact support. Then, $I_{2s} * \varphi \in \mathring{H}^s(\mathbb{R}^d)$. Moreover, by Lemma 3.5.1 we conclude that

$$(I_{2s} * \varphi)(x) \geq \frac{C}{(1+|x|)^{d-2s}} \quad \forall x \in \mathbb{R}^d, \tag{2.3.13}$$

for some positive constant C. Then, by Theorem 2.1.1 we infer that $T(I_{2s} * \varphi) \in L^1(\mathbb{R}^d)$. Thus, by combining (2.3.13) with [73, eq. (1.3.10)] we conclude that $I_{2s} * T$ is finite almost everywhere. Note that such result can be improved in the higher order case $1 < s < \frac{d}{2}$ in view of Lemma 2.4.2.

2.4 Applications and final remarks

This section is devoted to some applications of the Brezis–Browder results contained in Section 2.3 and of the quantitative estimates of Section 2.2. In particular, as we have already mentioned, an application concerns the question of whether a given *regular distribution* $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ admits an integral representation of $\|T\|_{\mathring{H}^{-s}(\mathbb{R}^d)}$ given by

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(T, T), \tag{2.4.1}$$

where we recall that by $\mathcal{D}_{2s}(T, T)$ we denote the right hand side of (2.1.2). To begin with, we prove the following result.

Lemma 2.4.1. *Let $0 < s < \frac{d}{2}$ and $T \in L^1_{loc}(\mathbb{R}^d)$ satisfying*

$$\mathcal{D}_{2s}(|T|, |T|) < \infty. \tag{2.4.2}$$

Then T generates a linear and continuous operator on $\mathring{H}^s(\mathbb{R}^d)$, $U_T = I_{2s} * T$ a.e. and (2.4.1) holds.

Proof. By combining (2.4.2) with the semigroup property of the Riesz kernel, the Fubini's Theorem yields $I_s * T \in L^2(\mathbb{R}^d)$ and $\mathcal{D}_{2s}(T, T) = \|I_s * T\|^2_{L^2(\mathbb{R}^d)}$. Furthermore, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, it's easy to see that $I_s * ((-\Delta)^{\frac{s}{2}}\varphi) = \varphi$ holds pointwisely. Indeed, let's define the function $\psi := I_s * ((-\Delta)^{\frac{s}{2}}\varphi) - \varphi$. Clearly,

$$(-\Delta)^{\frac{s}{2}}\psi = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

On the other hand, since $\psi \in C^\infty(\mathbb{R}^d) \cap \mathcal{L}_s^1(\mathbb{R}^d)$, by Theorem 2.1.2 we deduce that $\psi = 0$. Then, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\left|\int_{\mathbb{R}^d} T(x)\varphi(x)dx\right| = \left|\int_{\mathbb{R}^d} T(x)(I_s * (-\Delta)^{\frac{s}{2}}\varphi)(x)dx\right| = \left|\int_{\mathbb{R}^d} (I_s * T)(x)(-\Delta)^{\frac{s}{2}}\varphi(x)dx\right|$$
$$\leq \|I_s * T\|_{L^2(\mathbb{R}^d)}\|(-\Delta)^{\frac{s}{2}}\varphi\|_{L^2(\mathbb{R}^d)} = \|I_s * T\|_{L^2(\mathbb{R}^d)}\|\varphi\|_{\mathring{H}^s(\mathbb{R}^d)}, \tag{2.4.3}$$

proving that $T \in \mathring{H}^{-s}(\mathbb{R}^d)$. Note that we can change the order of integration in the second equality of (2.4.3) since Fubini's Theorem and (2.4.2) yield

$$A_s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|T(x)||(-\Delta)^{\frac{s}{2}}\varphi(x)|}{|x-y|^{d-s}}dxdy \leq \sqrt{\mathcal{D}_{2s}(|T|,|T|)}\ \|\varphi\|_{\mathring{H}^s(\mathbb{R}^d)} < \infty. \tag{2.4.4}$$

Then, since $I_s * T \in L^2(\mathbb{R}^d)$ and $I_{2s} * T$ is well defined a.e. because of (2.4.2), we have that $I_{2s} * T = I_s * (I_s * T)$ defines an element of $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$, cf. Lemma 1.4.2. In particular, by Corollary 2.1.2 and Fubini's Theorem we obtain

$$\int_{\mathbb{R}^d} T(x)\varphi(x)dx = \int_{\mathbb{R}^d} T(x)(I_{2s} * (-\Delta)^s\varphi)(x)dx = \int_{\mathbb{R}^d} (I_{2s} * T)(x)(-\Delta)^s\varphi(x)dx,$$

i.e., $I_{2s} * T$ solves

$$(-\Delta)^s(I_{2s} * T) = T \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Hence, since $T \in \mathring{H}^{-s}(\mathbb{R}^d)$ we have that $U_T - I_{2s} * T \in L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$ is s–harmonic, where we recall that U_T is defined by (1.2.2) and satisfies (1.3.2) as well. Again by Corollary 2.1.2 we infer that $U_T = I_{2s} * T$ for a.e. $x \in \mathbb{R}^d$. As a consequence, $I_{2s} * T \in \mathring{H}^s(\mathbb{R}^d)$ and

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \|U_T\|^2_{\mathring{H}^s(\mathbb{R}^d)} = \|I_{2s} * T\|^2_{\mathring{H}^s(\mathbb{R}^d)} = \|I_s * T\|^2_{L^2(\mathbb{R}^d)}.$$

The desired equality (2.4.1) follows by noting again that

$$\mathcal{D}_{2s}(T,T) = \|I_s * T\|^2_{L^2(\mathbb{R}^d)}. \tag{2.4.5}$$

□

Lemma 2.4.2 and Remark 2.4.1 below complement Lemma 2.4.1 essentially proving that, for a non negative element T generating a linear and continuous operator $T : \mathring{H}^s(\mathbb{R}^d) \to \mathbb{R}$, the relation (2.4.1) holds if and only if (2.4.2) holds. However, see Example 2.4.1, it's easy to find (sign changing) elements $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ such that (2.4.2) does not hold.

Example 2.4.1. Let's fix $d = 2$, $s = \frac{3}{4}$. Then, it's well known that

$$\mathcal{F}[\chi_{B_1}](x) = |x|^{-1} J_1(|x|) =: T(x), \tag{2.4.6}$$

where J_1 denoted the Bessel function of the first kind (see [61, eq. (4.22)] and [61, Lemma 4.5]). In particular, from (2.4.6) we deduce that T generates a linear and continuous functional on $\mathring{H}^s(\mathbb{R}^d)$, cf. [12, Proposition 1.36]. On the other hand, if $r > 0$ is large enough

$$J_1(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{3\pi}{4}\right) + O(r^{-\frac{3}{2}}), \tag{2.4.7}$$

(see e.g., [61, Lemma 4.3]). In particular, combining (2.4.6) with (2.4.7) yields for $R > 0$ sufficiently large that

$$\int_{B_R^c} \frac{|T(x)|}{\sqrt{|x|}} dx \gtrsim \int_R^\infty \frac{|J_1(r)|}{\sqrt{r}} dr \gtrsim \int_R^\infty \left| \frac{1}{r} \cos\left(r - \frac{3\pi}{4}\right) \right| dr = \infty,$$

that is $|T| \notin \mathcal{L}^1_{-2s}(\mathbb{R}^d)$ or equivalently $I_{2s} * |T| = \infty$ a.e. (see [73, eq. (1.3.10)]). As a consequence (2.4.2) does not hold.

In what follows we recall a result which is essentially contained in [70, Lemma 1.1].

Lemma 2.4.2. *Let $0 < s < \frac{d}{2}$. If $0 \le T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ then $U_T = I_{2s} * T$ where U_T is the potential defined by (1.2.2), and*

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(T,T).$$

Proof. Let $\lambda \in \mathbb{N}$. We consider the sequence $T_\lambda(x) := T(x)\varphi_\lambda(x)$ where φ_λ is defined in Proposition 2.3.2. Then, by combining (2.3.2) wtih [94, Lemma 5, eq. (64)], we infer that there exists $\tilde{C} > 0$ independent of λ such that

$$\left| \int_{\mathbb{R}^d} T_\lambda(x)\psi(x)dx \right| \leq C \, \|\varphi_\lambda \psi\|_{\mathring{H}^s(\mathbb{R}^d)} \leq \widetilde{C} \|\psi\|_{\mathring{H}^s(\mathbb{R}^d)} \quad \forall \psi \in C_c^\infty(\mathbb{R}^d), \tag{2.4.8}$$

i.e., $T_\lambda \in \mathring{H}^{-s}(\mathbb{R}^d)$ and $\|T_\lambda\|_{\mathring{H}^{-s}(\mathbb{R}^d)}$ is uniformly bounded in λ. In particular, up to a subsequence, we have that $T_\lambda \rightharpoonup T \quad \text{in } \mathring{H}^{-s}(\mathbb{R}^d)$. Furthermore, by arguing as in [70, Lemma 1.1., eq. (1.43)] we infer that $\mathcal{D}_{2s}(T_\lambda, T_\lambda) < \infty$. Finally, by Lemma 2.4.1 we also get that

$$\|T_\lambda\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(T_\lambda, T_\lambda). \tag{2.4.9}$$

Then, by Mazur's Lemma, there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a nonnegative sequence $a_{k,\lambda} \in [0, 1]$ with $k \in [\lambda, N(\lambda)]$, such that

$$\sum_{k=\lambda}^{N(\lambda)} a_{k,\lambda} = 1, \quad f_\lambda := \sum_{k=\lambda}^{N(\lambda)} a_{k,\lambda} \, T_k \to T \quad \text{in } \mathring{H}^{-s}(\mathbb{R}^d). \tag{2.4.10}$$

Note that by combining Lemma 2.4.1 with (2.4.9) we have

$$\|f_\lambda\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(f_\lambda, f_\lambda).$$

Then, if we further assume that $0 \leq \varphi \leq 1$ is radially non increasing we have that $T_k \leq T_j$ if $k \geq j$. Thus, by (2.4.10) we get

$$0 \leq T_\lambda \leq f_\lambda \leq T_{N(\lambda)} \leq T, \tag{2.4.11}$$

In particular, by (2.4.10), (2.4.11), Fatou's Lemma and the fact that $T_\lambda \to T$ a.e. we infer that

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \liminf_{\lambda \to +\infty} \mathcal{D}_{2s}(f_\lambda, f_\lambda) \geq \liminf_{\lambda + \infty} \mathcal{D}_{2s}(T_\lambda, T_\lambda) \geq \mathcal{D}_{2s}(T, T). \tag{2.4.12}$$

Thus, by putting together (2.4.11) with Lemma 2.4.1 we infer

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(T, T).$$

and $I_{2s} * T = U_T$. □

Remark 2.4.1. Note that, from (2.4.8) and (2.4.9) the conclusion of the proof of Lemma 2.4.2 can be essentially derived from [49]. Indeed, since T_λ converges to T a.e. and $\mathcal{D}_{2s}(T_\lambda, T_\lambda)$ is uniformly bounded, we directly conclude that $\mathcal{D}_{2s}(T, T) < \infty$, see e.g., [49, Lemma 3.12-3.13]. Then, the proof of Lemma 2.4.2 follows again by Lemma 2.4.1.

Next, we present a representation theorem in the spirit of [31, Theorem 2.4] where the authors find representation formulas in the case of the polyharmonic operator $(-\Delta)^k$, $k \in \mathbb{N}$, as well as related Liouville theorems.

Proof of Theorem 2.1.3.
By combining [70, Lemma 1.1] with Lemma 2.4.1 we deduce that $I_{2s} * T \in \mathring{H}^s(\mathbb{R}^d)$ and solves

$$(-\Delta)^s(I_{2s} * T) = T \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Hence, by Corollary 2.1.2 we infer that any s–harmonic function u can be written as

$$u(x) = (I_{2s} * T)(x) + P(x), \quad P(x) = \sum_{i=0}^{\deg(\mathrm{P})} P_i(x), \tag{2.4.13}$$

where P_i is homogeneous polynomial of degree i, $P_{\deg(\mathrm{P})} \not\equiv 0$ and by deg(P) we denote the degree of P. Next, we claim that there exists a positive constant C such that

$$\frac{1}{R^d} \int_{R<|x-y|<2R} |P(y)| dy \geq C R^{\deg(\mathrm{P})} + o(R^{\deg(\mathrm{P})}) \quad \text{as } R \to +\infty. \tag{2.4.14}$$

Proof of (2.4.14). For every $i = 0, \ldots, \deg(P)$, by the substitution $z := y/(2R)$ we obtain

$$\begin{aligned}\frac{1}{R^d} \int_{|y-x|<2R} |P_i(y)| dy = 2^d \int_{|2Rz-x|<2R} |P_i(2Rz)| dz = R^i 2^d \int_{|2Rz-x|<2R} |P_i(z)| dz \\ \geq R^i 2^{d+i} \int_{|z|<1-\frac{|x|}{2R}} |P_i(z)| dz,\end{aligned} \tag{2.4.15}$$

where we used that by definition $P_i(2Rz) = (2R)^i P_i(z)$. Similarly,

$$\frac{1}{R^d}\int_{|y-x|<R} |P_i(y)|dy \leq R^i \int_{|z|<1+\frac{|x|}{R}} |P_i(z)|dz. \tag{2.4.16}$$

In particular, for every $i = 0, \ldots, \deg(P) - 1$ we have that

$$\frac{1}{R^d}\int_{|y-x|<R} |P_i(y)|dy = o(R^{\deg}) \quad \text{as } R \to +\infty.$$

Then, by combining (2.4.15) with (2.4.16) we derive

$$\begin{aligned}
&\frac{1}{R^d}\int_{R<|x-y|<2R} |P(y)|dy = \frac{1}{R^d}\int_{|x-y|<2R} |P(y)|dy - \frac{1}{R^d}\int_{|x-y|<R} |P(y)|dy \\
&\geq R^{\deg(P)}\left(2^{d+\deg(P)}\int_{|z|<1-\frac{|x|}{2R}} |P_{\deg(P)}(z)|dz - \int_{|z|<1+\frac{|x|}{R}} |P_{\deg(P)}(z)|dz\right) + o(R^{\deg(P)}) \\
&= R^{deg(P)}\left((2^{d+\deg(P)} - 1)\int_{|z|<1} |P_{\deg(P)}(z)|dz\right) + o(R^{\deg(P)}),
\end{aligned} \tag{2.4.17}$$

concluding the proof of (2.4.14). Assume now by contradiction that u satisfies (2.1.4) and $P \not\equiv l$. Then, by (2.4.14) we conclude that deg(P) ≥ 1 and there exists a positive constant C such that

$$\begin{aligned}
&\frac{1}{R^d}\int_{R<|y-x|<2R} |u(y) - l|dy \\
&\quad \geq \frac{1}{R^d}\int_{R<|y-x|<2R} |P(y) - l|dy - \frac{1}{R^d}\int_{R<|y-x|<2R} (I_{2s} * T)(y)dy \\
&\quad \geq CR^{\deg(\mathrm{P})} + o(R^{\deg(\mathrm{P})}) \quad \text{as } R \to +\infty,
\end{aligned} \tag{2.4.18}$$

where we have also employed the Sobolev embedding to conclude that

$$\lim_{R\to+\infty} \frac{1}{R^d}\int_{R<|y-x|<2R} (I_{2s} * T)(y)dy = 0.$$

The inequality (2.4.18) contradicts (2.1.4) completing the proof. □

To conclude the chapter, we formulate two corollaries concerning the Riesz potential operator. Nevertheless, instead of restricting to non negative functions we impose some standard integrability assumptions in order to have the Riesz potential well defined, cf. [103, Chapter V, Theorem 1].

Corollary 2.4.1. *Let* $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $0 < s < \frac{d}{2}$ *and* $1 < q < \frac{d}{2s}$. *Then,* $U_T = I_{2s} * T$, *where* U_T *is the potential defined by* (1.2.2).

Proof. Since $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, by (1.2.2) and (1.3.2) there exists an element $U_T \in \mathring{H}^s(\mathbb{R}^d)$ such that

$$(-\Delta)^s U_T = T \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{2.4.19}$$

On the other hand, since $p < \frac{d}{2s}$ the Riesz potential $I_{2s} * T$ is well defined as a Lebesgue integral and solves (2.4.19) as well. Note that, by the Sobolev embedding, $U_T \in L^{2_*}(\mathbb{R}^d)$ and $I_{2s} * T \in L^{q_*}(\mathbb{R}^d)$ where, for the convenience of the reader, we set $q_* := \frac{qd}{d-2sq}$ and $2_* = \frac{2d}{d-2s}$. Hence, $U_T - I_{2s} * T \in \mathcal{L}^1_s(\mathbb{R}^d)$ is s–harmonic, i.e.,

$$(-\Delta)^s (U_T - I_{2s} * T) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Then, by applying Corollary 2.2.1 to $u := U_T - I_{2s} * T \in L^{2_*}(\mathbb{R}^d) + L^{q_*}(\mathbb{R}^d)$ and by sending R to infinity we derive that $U_T = I_{2s} * T$. □

Corollary 2.4.2. *Let* $T \in \mathring{H}^{-s}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \cap L^{\bar{q}}_{loc}(\mathbb{R}^d)$, $0 < s \leq 1$, $1 < p < \frac{d}{2s}$ *and* $\bar{q}$ *as in* (2.1.1). *If* $T(I_{2s} * T) \geq -|f|$ *for some* $f \in L^1(\mathbb{R}^d)$ *then,*

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \mathcal{D}_{2s}(T, T).$$

Proof. By Corollary 2.4.1 we conclude that $I_{2s} * T$ is well defined as a Lebesgue integral and coincides with the potential U_T defined by (1.2.2). In particular it also belongs to $\mathring{H}^s(\mathbb{R}^d)$. Then, the condition $T(I_{2s} * T) \geq -|f|$ ensures the validity of Theorem 2.4.2. Namely,

$$\|T\|^2_{\mathring{H}^{-s}(\mathbb{R}^d)} = \langle T, I_{2s} * T \rangle_{\mathring{H}^{-s}(\mathbb{R}^d),\, \mathring{H}^s(\mathbb{R}^d)} = \mathcal{D}_{2s}(T, T). \tag{2.4.20}$$

□

Chapter 3

Repulsive non local interaction

In this chapter, we study existence and qualitative properties such as uniqueness, regularity and decay estimates of minimizers for a Thomas–Fermi type energy functional with non local repulsion.

In the whole of Chapters 3 and 4, for the convenience of the reader we set $2s := \alpha$, $\alpha \in (0, d)$.

3.1 Introduction

We analyse minimizers of a Thomas–Fermi type energy of the form

$$\mathcal{E}_\alpha^{TF}(\rho) := \frac{1}{q}\int_{\mathbb{R}^d} |\rho(x)|^q dx - \int_{\mathbb{R}^d} \rho(x)V(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} (I_\alpha * \rho)(x)\rho(x)dx \tag{3.1.1}$$

where $d \geq 2$, $0 < \alpha < d$, $\rho : \mathbb{R}^d \to \mathbb{R}$ is a measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is a potential and $I_\alpha * \rho$ is defined by (1.4.1).

Our study is motivated by the Thomas–Fermi type model of charge screening in graphene [89] (see also [69, 83]) that corresponds to the special two–dimensional case $q = \frac{3}{2}$ and $\alpha = 1$,

$$\mathcal{E}_1^{TF}(\rho) = \frac{2}{3}\int_{\mathbb{R}^2} |\rho(x)|^{\frac{3}{2}} dx - \int_{\mathbb{R}^2} \rho(x)V(x)dx + \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|}dxdy. \tag{3.1.2}$$

The function $\rho(x)$ here has a meaning of a charge density of fermionic quasiparticles (electrons and holes) in a two–dimensional graphene layer. In general (and unlike in the classical Thomas–Fermi models of atoms and molecules [78]), the density ρ is a sign changing function, with $\rho > 0$ representing electrons and $\rho < 0$ representing holes. The first term in (3.1.2) is an approximation of the kinetic energy of a uniform gas of non–interacting particles. The middle term is the interaction with an external potential V, and the last quantity is a non–local Coulombic interaction between quasi–particles. In [89], the authors prove the existence, uniqueness, and decay of the minimizer of the energy (3.1.2). In particular, they establish a *log–correction* to the decay rates of the minimizer, previously identified by M. Katsnelson in [69].

The main goal of this chapter is to deduce existence, uniqueness and qualitative properties of the minimizer in the range

$$\frac{2d}{d+\alpha} < q < \infty, \tag{3.1.3}$$

where $\frac{2d}{d+\alpha}$ is a critical exponent with respect to the Hardy–Littlewood–Sobolev inequality (see [78, Theorem 4.3, p. 106]). Our main result is Theorem 3.1.7 where we establish five different asymptotic regimes for the minimizer, depending on the values of d, α and q. Two of the regimes are entirely new and not visible in the "local" case $\alpha = 2$, when the non–local term is the standard Newtonian potential.

Before presenting our results in details, let us emphasise the main differences between our case and the classical three dimensional TF–theory of atoms and molecules [14, 17, 76, 79] (see also [62, 97] for generalisations of the mentioned three dimensional models).

Unlike in [14, 17, 76, 79], we are looking at

- global minimizers without a mass constraint since there is no a priori reason for the density function $\rho(x)$ to be in $L^1(\mathbb{R}^d)$;
- potentially sign changing profiles $\rho(x)$, since electrons and holes could co–exist in a graphene layer;
- general range of $\alpha \in (0, d)$, as for istance in (3.1.2) $\alpha = 1$.

In particular, we mention that several difficulties arise if sign changing profiles $\rho(x)$ are allowed and at the same time no further integrability is required. Namely, in all of the above models, the density $\rho(x)$ belongs to $L^q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for some $q > \frac{2d}{d+\alpha}$. As a consequence, by the Hardy–Littlewood–Sobolev inequality [78, Theorem 4.3], the non local Coulomb type term is well defined as a Lebesgue integral. Namely,

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\rho(x)\rho(y)}{|x-y|^{d-\alpha}}dxdy\right| \le \mathscr{C}_{d,\alpha}\|\rho\|^2_{L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)} \le \mathscr{C}_{d,\alpha}\|\rho\|^{2\theta}_{L^q(\mathbb{R}^d)}\|\rho\|^{2(1-\theta)}_{L^1(\mathbb{R}^d)},$$

where $\mathscr{C}_{d,\alpha}$ is a sharp constant and $\theta \in (0,1)$. However, as it has been already underlined above and in [89], requiring a priori the density to be in $L^1(\mathbb{R}^d)$ is not natural in the graphene case. To overcome this issue, we consider a more general energy functional which will coincide with the one in (3.1.1) under suitable restrictions. To this aim, we first introduce the Banach space $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ which can be defined as the space of tempered distributions such that $\hat{\rho} \in L^1_{loc}(\mathbb{R}^d)$ and

$$\|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} := \int_{\mathbb{R}^d}|\xi|^{-\alpha}|\widehat{\rho}(\xi)|^2 d\xi < \infty,$$

where $\hat{\rho}$ stands for the Fourier transform of ρ (see [12, Definition 1.31]). it is easy to see (cf. [103, Lemma 2, Section 5] and [78, Corollary 5.10]) that for functions ρ belonging to $L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)$ we indeed have the equality

$$\mathcal{D}_\alpha(\rho,\rho) = A_\alpha \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\rho(y)\rho(x)}{|x-y|^{d-\alpha}}dxdy = \|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}.$$

However, in general $\mathcal{D}_\alpha$ can not be extended as an integral to the whole of $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ without any fast decay or sign restriction. We refer to Chatper 2 and references therein for a more detailed analysis of the topic. As a consequence, the term $\mathcal{D}_\alpha$ is replaced by $\|\cdot\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}$ in the sequel.

To be precise, we consider the energy functional

$$\mathcal{E}^{TF}_\alpha(\rho) := \frac{1}{q}\int_{\mathbb{R}^d}|\rho(x)|^q dx - \langle\rho, V\rangle + \frac{1}{2}\|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} \tag{3.1.4}$$

on the domain $\mathcal{H}_\alpha$ defined as follows

$$\mathcal{H}_\alpha := L^q(\mathbb{R}^d) \cap \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d), \tag{3.1.5}$$

and endowed with the norm $\|\cdot\|_{\mathcal{H}_\alpha} := \|\cdot\|_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} + \|\cdot\|_{L^q(\mathbb{R}^d)}$. In (3.1.4) the function V belongs to the dual space $(\mathcal{H}_\alpha)'$ and $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{H}_\alpha$ and $(\mathcal{H}_\alpha)'$. We recall that $(\mathcal{H}_\alpha)'$ identifies with $L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. More precisely, for every function $V = V_1 + V_2 \in L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, $V_1 \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and $V_2 \in L^{q'}(\mathbb{R}^d)$ we define

$$\langle \rho, V \rangle := \langle \rho, V_1 \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d), \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \rho(x) V_2(x) dx, \tag{3.1.6}$$

where again by $\langle \cdot, \cdot \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d), \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ we identify the duality between $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. [1]

The first result of Section 3.2 establishes existence and uniqueness of a minimizer for $\mathcal{E}_\alpha^{TF}$.

Theorem 3.1.1. *Assume $q > \frac{2d}{d+\alpha}$. Then, for every $V \in L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, the functional $\mathcal{E}_\alpha^{TF}$ admits a unique minimizer $\rho_V \in \mathcal{H}_\alpha$ that satisfies*

$$\int_{\mathbb{R}^d} sign(\rho_V(x)) |\rho_V(x)|^{q-1} \varphi(x) dx - \langle \varphi, V \rangle + \langle \rho_V, \varphi \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = 0 \quad \forall \varphi \in \mathcal{H}_\alpha. \tag{3.1.7}$$

An equivalent way to read (3.1.7) is by the following relation

$$\text{sign}(\rho_V)|\rho_V|^{q-1} = V - (-\Delta)^{-\frac{\alpha}{2}} \rho_V \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{3.1.8}$$

where by $(-\Delta)^{-\frac{\alpha}{2}} \rho_V$ we understand the unique element of $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ weakly solving

$$(-\Delta)^{\frac{\alpha}{2}} u = \rho_V \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d).$$

See Section 1 for all the details.

If, e.g., $\rho_V \in L^q(\mathbb{R}^d)$ with $q < \frac{d}{\alpha}$, the Riesz potential of ρ_V is well defined (see [103, Theorem 1, Sect. 5]) and identifies with $(-\Delta)^{-\frac{\alpha}{2}} \rho_V$ (cf. Chapter 2, Corollary 2.4.1 and [70, Lemma 1.8]). Moreover, without any restriction on q, if $\rho_V \geq 0$, the Riesz potential $I_\alpha * \rho_V$ can be identified again with the operator $(-\Delta)^{-\frac{\alpha}{2}} \rho_V$ (see again [58, Example 2.2.1] for the case $\alpha \in (0, 2]$ and [70, Lemma 1.1] for the general case). Thus, in the above cases, (3.1.8) reads also pointwisely as

$$u_V(x) = V(x) - A_\alpha \int_{\mathbb{R}^d} \frac{\rho_V(y)}{|x-y|^{d-\alpha}} dy \quad \text{a.e. in } \mathbb{R}^d, \tag{3.1.9}$$

where u_V is defined by

$$u_V := \text{sign}(\rho_V)|\rho_V|^{q-1}. \tag{3.1.10}$$

[1] We further recall that the space $L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ is a Banach space endowed with the norm

$$\|V\|_{L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \inf \left\{ \|V_1\|_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \|V_2\|_{L^{q'}(\mathbb{R}^d)} : V = V_1 + V_2, V_1 \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d), V_2 \in L^{q'}(\mathbb{R}^d) \right\}.$$

Such function u_V plays an important role for the analysis of the sign and decay at infinity of the minimizer. As a matter of fact, if $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ we prove that the Euler–Lagrange equation (3.1.7) is equivalent to the fractional semilinear PDE

$$(-\Delta)^{\frac{\alpha}{2}} u + \operatorname{sign}(u)|u|^{\frac{1}{q-1}} = (-\Delta)^{\frac{\alpha}{2}} V \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \tag{3.1.11}$$

in the sense that u_V weakly solves it. We further prove that (3.1.11) satisfies a weak comparison principle provided $\alpha \in (0, 2]$. See for example [24, Lemma 2, (27), p. 8] for some general results in the case $\alpha = 2$.

Section 3.3 is devoted to the study of the regularity of the minimizer. The way in which we obtain the regularity is based on a classical bootstrap argument similar to the one in [33, Theorem 8]. However, some differences occur. The main one is the presence of the function V which clearly might be an obstruction for the regularity of the minimizer. Indeed, from (3.1.8), it is clear that in general the best regularity that u_V can achieve is the same regularity as V. Such regularity is achieved for example in Corollary 3.3.4. We further summarize below the basic regularity result for the minimizer ρ_V. We stress that such regularity can be improved and specialised according to the regularity assumptions we impose on V. In particular, if we denote by $C_b(\mathbb{R}^d)$ the space of bounded continuous function the following holds:

Theorem 3.1.2. *Assume $q > \frac{2d}{d+\alpha}$. If $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ then $\rho_V \in C(\mathbb{R}^d)$.*

In Section 3.4 we focus on the relation between unconstrained minimizers and minimizers of $\mathcal{E}_\alpha^{TF}$ subject to constraint to non–negative functions, defined by

$$\mathcal{H}_\alpha^+ := \{f \in \mathcal{H}_\alpha : f \geq 0\}. \tag{3.1.12}$$

Then, we establish existence and uniqueness of a minimizer in $\mathcal{H}_\alpha^+$. Namely, denoted by $[x]_+ := \max\{0, x\}$, we prove the following:

Theorem 3.1.3. *Assume $q > \frac{2d}{d+\alpha}$. Then, for every $V \in L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ the function $\mathcal{E}_\alpha^{TF}$ admits a unique minimizer ρ_V^+ belonging to $\mathcal{H}_\alpha^+$. Moreover, the minimizer ρ_V^+ satisfies*

$$\int_{\mathbb{R}^d} (\rho_V^+)^{q-1}\varphi - \langle \varphi, V\rangle + \langle \rho_V^+, \varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} \geq 0 \quad \forall \varphi \in \mathcal{H}_\alpha^+, \tag{3.1.13}$$

and

$$\rho_V^+ = \left[V - (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+\right]_+^{\frac{1}{q-1}} \quad \textit{in } \mathcal{D}'(\mathbb{R}^d).$$

Next, we discuss the relation between the non–negative and free minimizer. We also prove that the equality $[\rho_V]_+ = \rho_V^+$ holds only in the trivial case in which $\rho_V \geq 0$ (see Theorem 3.4.1) while we obtain the inequality $[\rho_V]_+ \geq \rho_V^+$ provided the potential V is regular and decays sufficiently fast (see e.g., Remark 3.4.4). We further highlight that, from now on, most of the statements will be restricted the case $0 < \alpha \leq 2$. This is related to the fact that such results heavily relies on a comparison principles for semilinear PDEs involving the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. Such comparison principles are known not to be valid (in general) in the higher order regime $2 < \alpha < d$, see for example [4] for a discussion on the topic.

We start with the following:

Theorem 3.1.4. *Assume $d \geq 2$, $0 < \alpha \leq 2$. Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be compactly supported and not identically zero. Then,*

$$[\rho_V]_+ \geq \rho_V^+ \quad \textit{in } \mathbb{R}^d, \tag{3.1.14}$$

the inequality being strict in a set of positive measure.

In Section 3.5 we study decay properties of ρ_V in the fractional framework $\alpha \in (0, 2)$. First of all we study positivity of ρ_V. If $q \leq 2$, the total power of u in (3.1.11) is bigger than 1 and equation (3.1.11) is superlinear. If $q > 2$ then equation (3.1.11) is sublinear which implies that studying the positivity of ρ_V is more delicate and strongly relates to the non–local nature of the fractional Laplacian (see Proposition 3.5.2 and Lemma 3.5.10). Indeed, in the local case $\alpha = 2$ the support of a non–negative solution of (3.1.11) could be compact (see [96, Theorem 1.1.2] and [46, Corollary 1.10, Remark 1.5]). As a consequence, if $\alpha \in (0, 2)$ we can summarise the information about the sign of ρ_V as follows:

Theorem 3.1.5. *Let $\alpha \in (0, 2)$. Assume that $V \in \left(\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)\right) \setminus \{0\}$. Then, the following possibilities hold:*

(i) If $\lim_{|x| \to +\infty} |x|^{d-\alpha} V(x) = 0$ then ρ_V is sign changing;

(ii) If $(-\Delta)^{\frac{\alpha}{2}} V \geq 0$ then $\rho_V \geq 0$. Furthermore, If $V \in C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$ then $\rho_V > 0$.

We refer the reader to Proposition 3.2.2, 3.4.1 and 3.5.2 for the precise statements. See also Remark 3.4.3 for examples of potentials satisfying (i) or (ii).

Next, we outline the decay of the minimizer ρ_V in the fractional framework. However, for the convenience of the reader, we state the result by requiring that V satisfies the stronger decay assumption (3.1.15) and $C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$ regularity even though such assumptions are not sharp and can be weakened in some regimes of q (see eq. (1.1.3) for the definition of the the latter space and Remarks 3.5.1, 3.5.4). The same considerations apply to Theorem 3.1.8.

Before formulating Theorem 3.1.7 we first recall its local counterpart, i.e., the case $\alpha = 2$ that was studied in [105]. Namely, assuming some regularity on ρ_V and that $-\Delta V$ is non negative and compactly supported we have the following:

(i) If $\frac{2d}{d+2} < q < \frac{2d-2}{d}$ then

$$\lim_{|x|\to+\infty} |x|^{\frac{d-2}{q-1}}\rho_V(x) \in (0,\infty);$$

(ii) If $q = \frac{2d-2}{d}$ then

$$\lim_{|x|\to+\infty} |x|^{d}(\log|x|)^{\frac{d}{2}}\rho_V(x) \in (0,\infty);$$

(iii) If $\frac{2d-2}{d} < q < 2$ then

$$\lim_{|x|\to+\infty} |x|^{\frac{2}{2-q}}\rho_V(x) \in (0,\infty);$$

(iv) If $q = 2$ and we let $x = |x|\varepsilon$ for some fixed ε then

$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}}e^{|x|}\rho_V(x) = L(\varepsilon) \in (0,\infty);$$

(v) If $q > 2$ then ρ_V has compact support.

The statement of such local result has been adapted to the notation of this thesis. Thus, we refer the reader to [105] for the original statement of the result.

As far as concerns the fractional framework, the case $d = 2$, $q = \frac{3}{2}$ and $\alpha = 1$ has been recently studied in [89, Theorem 2.2, Proposition 4.8] resulting in a particular istance of Theorem 3.1.7, case (ii). In particular, the authors proved the following:

Theorem 3.1.6 ([89, Theorem 2.2]). *Let $d = 2$, $\alpha = 1$, $q = \frac{3}{2}$ and V_Z be defined in (3.1.16). Then, the minimizer ρ_{V_z} is Hölder continuous, radially symmetric non–increasing and satisfies*

$$0 < \rho_{V_z} \leq V_Z \quad \text{in } \mathbb{R}^2$$

and

$$0 < \liminf_{|x|\to+\infty} |x|^2(\log|x|)^2 \rho_{V_z}(x) \leq \limsup_{|x|\to+\infty} |x|^2(\log|x|)^2 \rho_{V_z}(x) < +\infty.$$

In particular, $\rho_{V_z} \in L^1(\mathbb{R}^2)$ and $\|\rho_{V_z}\|_{L^1(\mathbb{R}^2)} = Z$.

Following that, we state the general non local counterpart of the local result described above as well as examine the difference and similarities between the two outcomes.

Theorem 3.1.7. *Assume $d \geq 2$, $0 < \alpha < 2$ and $q > \frac{2d}{d+\alpha}$. If V is a non–identically zero element of $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$ and $(-\Delta)^{\frac{\alpha}{2}} V$ is a non–negative function such that*

$$\limsup_{|x|\to+\infty} |x|^{d+\alpha}(-\Delta)^{\frac{\alpha}{2}} V(x) < +\infty, \tag{3.1.15}$$

then $\rho_V \in L^1(\mathbb{R}^d)$. Moreover, the following cases occur:

(*i*) *If $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ then*

$$0 < \liminf_{|x|\to+\infty} |x|^{\frac{d-\alpha}{q-1}} \rho_V(x) \leq \limsup_{|x|\to+\infty} |x|^{\frac{d-\alpha}{q-1}} \rho_V(x) < +\infty;$$

(*ii*) *If $q = \frac{2d-\alpha}{d}$ and, either $1 < \alpha < 2$ and $d > \alpha + 1$ or $q = \frac{3}{2}$, then*

$$0 < \liminf_{|x|\to+\infty} |x|^d(\log|x|)^{\frac{d}{\alpha}} \rho_V(x) \leq \limsup_{|x|\to+\infty} |x|^d(\log|x|)^{\frac{d}{\alpha}} \rho_V(x) < +\infty;$$

(*iii*) *If $\frac{2d-\alpha}{d} < q < \frac{2d+\alpha}{d+\alpha}$ then*

$$0 < \liminf_{|x|\to+\infty} |x|^{\frac{\alpha}{2-q}} \rho_V(x) \leq \limsup_{|x|\to+\infty} |x|^{\frac{\alpha}{2-q}} \rho_V(x) < +\infty;$$

(*iv*) *If $q = \frac{2d+\alpha}{d+\alpha}$ then*

$$0 < \liminf_{|x|\to+\infty} |x|^{d+\alpha}(\log|x|)^{-\frac{d+\alpha}{\alpha}} \rho_V(x) \leq \limsup_{|x|\to+\infty} |x|^{d+\alpha}(\log|x|)^{-\frac{d+\alpha}{\alpha}} \rho_V(x) < +\infty;$$

(*v*) *If* $q > \frac{2d+\alpha}{d+\alpha}$ *then*

$$0 < \liminf_{|x|\to+\infty} |x|^{d+\alpha}\rho_V(x) \leq \limsup_{|x|\to+\infty} |x|^{d+\alpha}\rho_V(x) < +\infty.$$

In particular, if $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ *then*

$$\lim_{|x|\to+\infty} |x|^{\frac{d-\alpha}{q-1}}\rho_V(x) = \left[A_\alpha\left(\left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)} - \|\rho_V\|_{L^1(\mathbb{R}^d)}\right)\right]^{\frac{1}{q-1}} > 0.$$

First of all, unlike the cases (i)–(ii)–(iii) of Theorem 3.1.7 where the decay of the local framework can be formally recovered by substituting $\alpha = 2$, we underline how the critical threshold $q = 2$ for the local case has been replaced by $q = \frac{2d+\alpha}{d+\alpha}$ at which the decay is not exponential, resulting in a purely fractional phenomenon. We also mention that such criticality occurs in correspondence with the value of q where the decay exponent $\frac{\alpha}{2-q}$ reaches the value $d+\alpha$. This relates to the different behaviour, w.r.t. to the classical Laplace operator, of the fractional Laplacian when acting on polynomial decaying functions (see Lemma 3.6.1, case (iii)). Finally, if $q > \frac{2d+\alpha}{d+\alpha}$, differently from the local case where the solution has compact support, we have the fastest decay possible as in [53], resulting again in a fractional phenomenon. Furthermore, we point out that the additional restrictions on the parameters in case (ii) of Theorem 3.1.7 are technical but not structural. In particular, we conjecture that they can be removed.

Before showing some applications of Theorem 3.1.7, we briefly underline how the proof develops. We intend to construct positive super and subsolutions of (3.1.11) in exterior domains of the form $\Omega = \overline{B_R}^c$. The scenario is the following:

- If $q \in \left(\frac{2d}{d+\alpha}, \frac{2d+\alpha}{d+\alpha}\right) \setminus \left\{\frac{2d-\alpha}{d}\right\}$, we employ polynomial–decaying functions as barriers. Namely, we use a linear combination of the functions f and g where $f(x) = (1+|x|^2)^{-\beta}$, for some suitable $\beta > 0$, and g is defined as the optimizer for the following problem

  $$\mathcal{I} := \inf_{u\in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \left\{\|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} : u \geq \chi_{B_1}\right\}$$

  (see Lemma 3.5.3, eq. (3.5.21)). Decay estimates for such functions were derived in [55, equation (4.4)] and [22, Proposition 3.6];

- If $q \in \left\{\frac{2d-\alpha}{d}, \frac{2d+\alpha}{d+\alpha}\right\}$, polynomial–behaving barriers are not enough to obtain sharp estimates. Indeed, we apply [55, Theorem 1.1] to deduce asymptotic decay for functions with logarithmic behaviour (see Lemma 3.6.2 and 3.6.3);
- If $q \in \left(\frac{2d+\alpha}{d+\alpha}, \infty\right) \setminus \{2\}$, by applying again [55, Theorem 1.1], we employ polynomial–behaving functions with fast decay (see Lemma 3.6.1);
- If $q = 2$, the conclusion is essentially a consequence of [53, Lemmas 4.2–4.3].

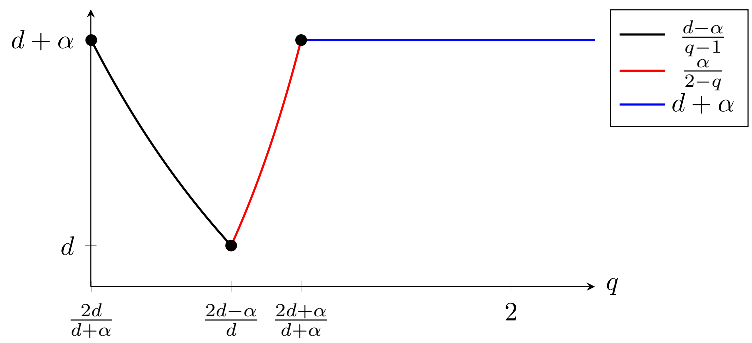

Figure 3.1: Plot of the decay's exponent in the range $q \in \left(\frac{2d}{d+\alpha}, \infty\right) \setminus \left\{\frac{2d-\alpha}{d}, \frac{2d+\alpha}{d+\alpha}\right\}$. Recall that log–corrections appear at the critical values $\frac{2d-\alpha}{d}$ and $\frac{2d+\alpha}{d+\alpha}$.

An important class of potentials satisfying the assumptions of Theorem 3.1.7 is the following:

$$V_Z(x) := Z A_\alpha \left(1 + |x|^2\right)^{-\frac{d-\alpha}{2}}, \tag{3.1.16}$$

where A_α is the Riesz constant and Z is a positive constant. Then, we can prove that

(i) If $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ then

$$\|\rho_{V_Z}\|_{L^1(\mathbb{R}^d)} < Z;$$

(ii) If $q > \frac{2d-\alpha}{d}$ then

$$\|\rho_{V_Z}\|_{L^1(\mathbb{R}^d)} = Z.$$

For the precise statements we refer to Corollaries 3.5.1 and 3.5.2.

Finally, we study asymptotic decay of sign changing minimizers where we do not expect the existence of a universal lower bound. Namely, we obtain a similar version of Theorem 3.1.7 with upper bounds only. Note that, in view of Remark 3.5.3, case (ii) of Theorem 3.1.8 requires less restrictions than case (ii) of Theorem 3.1.7.

Theorem 3.1.8. *Assume $d \geq 2$, $0 < \alpha < 2$ and $q > \frac{2d}{d+\alpha}$. Assume that V belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$ and*

$$\limsup_{|x|\to+\infty} |x|^{d+\alpha}|(-\Delta)^{\frac{\alpha}{2}}V(x)| < +\infty. \tag{3.1.17}$$

If $|\rho_V| \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then the following possibilities hold:

(i) If $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ then

$$\limsup_{|x|\to+\infty} |x|^{\frac{d-\alpha}{q-1}}|\rho_V(x)| < +\infty;$$

(ii) If $q = \frac{2d-\alpha}{d}$ and, either $1 < \alpha < 2$ or $q = \frac{3}{2}$, then

$$\limsup_{|x|\to+\infty} |x|^{d}(\log|x|)^{\frac{d}{\alpha}}|\rho_V(x)| < +\infty;$$

(iii) If $\frac{2d-\alpha}{d} < q < \frac{2d+\alpha}{d+\alpha}$ then

$$\limsup_{|x|\to+\infty} |x|^{\frac{\alpha}{2-q}}|\rho_V(x)| < +\infty;$$

(iv) If $q = \frac{2d+\alpha}{d+\alpha}$ then

$$\limsup_{|x|\to+\infty} |x|^{d+\alpha}(\log|x|)^{-\frac{d+\alpha}{\alpha}}|\rho_V(x)| < +\infty;$$

(v) If $q > \frac{2d+\alpha}{d+\alpha}$ then

$$\limsup_{|x|\to+\infty} |x|^{d+\alpha}|\rho_V(x)| < +\infty.$$

The proof is essentially based on two facts:

- similarly to the proof of Theorem 3.1.4, we employ a Kato–type inequality of the form

$$(-\Delta)^{\frac{\alpha}{2}}|u| \leq \operatorname{sign}(u)(-\Delta)^{\frac{\alpha}{2}}u \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

  which allows us to conclude that the function $|u_V|$ is a non–negative weak subsolution of the semilinear PDE

$$(-\Delta)^{\frac{\alpha}{2}}u + \operatorname{sign}(u)|u|^{\frac{1}{q-1}} = |(-\Delta)^{\frac{\alpha}{2}}V| \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d); \tag{3.1.18}$$

- we employ the same barriers used for the proof of Theorem 3.1.7 as supersolutions of (3.1.18) in exterior domains of the form $\Omega = \overline{B_R}^c$.

3.2 Existence of a minimizer

In the following section, we prove existence and uniqueness of a minimizer for the Thomas–Fermi energy $\mathcal{E}_\alpha^{TF}$. Furthermore, we derive the fractional semilinear PDE (3.1.7) which will play a key role for the entire chapter and especially for the study of asymptotic decay of the minimizer (see Sect. 3.5). We also recall that the basic assumptions throughout all the chapter are

$$d \geq 2, \quad 0 < \alpha < d, \quad \frac{2d}{d+\alpha} < q < \infty,$$

and

$$V \in \left(L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)\right) \setminus \{0\}$$

where as usual $q' = \frac{q}{q-1}$. We also recall that $\mathcal{H}_\alpha := L^q(\mathbb{R}^d) \cap \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$, the energy functional $\mathcal{E}_\alpha^{TF}$ is defined by

$$\mathcal{E}_\alpha^{TF}(\rho) := \frac{1}{q}\int_{\mathbb{R}^d} |\rho(x)|^q dx - \langle \rho, V\rangle + \frac{1}{2}\,\|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} \quad \forall \rho \in \mathcal{H}_\alpha,$$

and by ρ_V we denote a minimizer (unique by Theorem 3.1.1) of $\mathcal{E}_\alpha^{TF}$ in $\mathcal{H}_\alpha$ emphasising the dependence on the potential V. Note that we don't consider $V \equiv 0$ since in this case we trivially have $\rho_V \equiv 0$. Further restrictions and notations will be explicitly written. We refer to [89, Proposition 3.1] for a special case of Theorem 3.1.1.

3.2.1 Proof of Theorem 3.1.1.

Proof. By definition of $\mathcal{E}_\alpha^{TF}$, if $V = V_1 + V_2 \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) + L^{q'}(\mathbb{R}^d)$, then

$$\mathcal{E}_\alpha^{TF}(\rho) \geq \frac{1}{q}\|\rho\|^q_{L^q(\mathbb{R}^d)} - \|V_2\|_{L^{q'}(\mathbb{R}^d)}\|\rho\|_{L^q(\mathbb{R}^d)} + \frac{1}{2}\|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} - \|V_1\|_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\|\rho\|_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)},$$

that is clearly bounded from below in $\mathcal{H}_\alpha$. Namely, $\inf_{\mathcal{H}_\alpha} \mathcal{E}^{TF}_\alpha(\rho) > -\infty$ Moreover, If $\{\rho_n\}_n \subset \mathcal{H}_\alpha$ is a minimizing sequence, then

$$\sup_{n\in\mathbb{N}} \left(\|\rho_n\|_{L^q(\mathbb{R}^d)} + \|\rho_n\|_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} \right) < +\infty.$$

In particular, up to subsequence we can assume that

$$\begin{aligned} \rho_n &\rightharpoonup \rho_V \quad \text{in } L^q(\mathbb{R}^d), \\ \rho_n &\rightharpoonup F \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d). \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d} \rho_n(x)\varphi(x)dx \to \int_{\mathbb{R}^d} \rho_V(x)\varphi(x)dx \quad \forall \varphi \in L^{q'}(\mathbb{R}^d),$$
$$\langle \rho_n, \varphi \rangle \to \langle F, \varphi \rangle \quad \forall \varphi \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d),$$

from which

$$\int_{\mathbb{R}^d} \rho_V(x)\varphi(x)dx = \langle F, \varphi \rangle \quad \forall \varphi \in L^{q'}(\mathbb{R}^d) \cap \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d).$$

In particular, the distribution F is $\mathcal{D}(\mathbb{R}^d)$–regular and we can indentify $F = \rho_V$ with an element of $L^q(\mathbb{R}^d) \cap \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ such that $\rho_n \rightharpoonup \rho_V$ in $L^q(\mathbb{R}^d) \cap \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Finally, from the weak lower semicontinuity of $\|\cdot\|_{L^q(\mathbb{R}^d)}$ and $\|\cdot\|_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}$, and the weak continuity of the linear operator $\cdot \mapsto \langle \cdot, V \rangle$. we obtain

$$\mathcal{E}^{TF}_\alpha(\rho_V) \le \liminf_{n\to+\infty} \mathcal{E}^{TF}_\alpha(\rho_n).$$

Uniqueness is a consequence of the strict convexity of $\mathcal{E}^{TF}_\alpha$. Furthermore, since ρ_V is a critical point for $\mathcal{E}^{TF}_\alpha$, by differentiating the L^q and $\mathring{H}^{-\frac{\alpha}{2}}$ norm we infer

$$\int_{\mathbb{R}^d} \operatorname{sign}(\rho_V(x))|\rho_V(x)|^{q-1}\varphi(x)dx - \langle \varphi, V \rangle + \langle \rho_V, \varphi \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = 0 \quad \forall \varphi \in \mathcal{H}_\alpha,$$

concluding the proof.

□

Remark 3.2.1. Note that Theorem 3.1.1 in particular implies that the map

$$\begin{aligned} L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) &\longrightarrow L^q(\mathbb{R}^d) \cap \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \\ V &\longmapsto \rho_V \end{aligned}$$

is a bijection.

3.2.2 Equivalent PDE

As we have pointed out in the introduction, by testing the Euler–Lagrange equation (3.1.7) against $\varphi \in C_c^\infty(\mathbb{R}^d)$ we derive that

$$\operatorname{sign}(\rho_V)|\rho_V|^{q-1} = V - (-\Delta)^{-\frac{\alpha}{2}}\rho_V \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{3.2.1}$$

where we recall that $(-\Delta)^{-\frac{\alpha}{2}}\rho_V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ is the unique solution of (1.3.12).

Let $\rho_V \in \mathcal{H}_\alpha$ be the unique minimizer of $\mathcal{E}_\alpha^{TF}$. We introduce the function

$$u_V := \operatorname{sign}(\rho_V)|\rho_V|^{q-1}. \tag{3.2.2}$$

Then, writing the Euler–Lagrange equation (3.1.7) in terms of u_V yields

$$\int_{\mathbb{R}^d} u_V(x)\varphi(x)dx - \langle \varphi, V\rangle + \langle \operatorname{sign}(u_V)|u_V|^{\frac{1}{q-1}}, \varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = 0 \quad \forall \varphi \in \mathcal{H}_\alpha. \tag{3.2.3}$$

As a consequence of (3.2.3), in Proposition 3.2.1 we deduce that the function u_V weakly solves the semilinear PDE defined in (3.1.11).

Proposition 3.2.1 (Equivalent PDE)**.** *Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and u_V defined by* (3.2.2)*. Then the function u_V belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and weakly solves*

$$(-\Delta)^{\frac{\alpha}{2}}u + \mathit{sign}(u)|u|^{\frac{1}{q-1}} = (-\Delta)^{\frac{\alpha}{2}}V \quad \mathit{in}\ \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d). \tag{3.2.4}$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$. Then, we recall that the quantity $\varphi := (-\Delta)^{\frac{\alpha}{2}}\psi$ belongs to $C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and

$$|(-\Delta)^{\frac{\alpha}{2}}\psi(x)| \lesssim |x|^{-(d+\alpha)} \quad \text{as } |x| \to +\infty.$$

We can therefore test (3.2.3) against ψ to get

$$\int_{\mathbb{R}^d} u_V(x)(-\Delta)^{\frac{\alpha}{2}}\psi(x)dx - \int_{\mathbb{R}^d} V(x)(-\Delta)^{\frac{\alpha}{2}}\psi(x)dx + \langle \rho_V, (-\Delta)^{\frac{\alpha}{2}}\psi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = 0. \tag{3.2.5}$$

Note that the inclusion $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \subset \mathcal{L}^1_\alpha(\mathbb{R}^d)$ implies

$$\int_{\mathbb{R}^d} |V(x)(-\Delta)^{\frac{\alpha}{2}}\psi(x)|dx < +\infty.$$

Furthermore, by combining (1.2.2) with the fact that $T \mapsto (-\Delta)^{-\frac{\alpha}{2}}T$ gives an isometry from $\mathring{H}^{-\frac{\alpha}{2}}$ to $\mathring{H}^{\frac{\alpha}{2}}$ we obtain

$$\langle \rho_V, (-\Delta)^{\frac{\alpha}{2}}\psi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = \langle (-\Delta)^{-\frac{\alpha}{2}}\rho_V, \psi\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \langle \rho_V, \psi\rangle = \int_{\mathbb{R}^d} \rho_V(x)\psi(x)dx, \tag{3.2.6}$$

where by $\langle\cdot,\cdot\rangle$ we understand the duality between $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. By combining (3.2.5) with (3.2.6) we obtain that u_V solves the equation

$$(-\Delta)^{\frac{\alpha}{2}}u_V + \operatorname{sign}(u_V)|u_V|^{\frac{1}{q-1}} = (-\Delta)^{\frac{\alpha}{2}}V \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{3.2.7}$$

where $\operatorname{sign}(u_V)|u_V|^{\frac{1}{q-1}} = \rho_V \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ and $(-\Delta)^{\frac{\alpha}{2}}V \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Then, from (3.2.7) we infer that $(-\Delta)^{\frac{\alpha}{2}}u_V \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ and (cf. Corollary 2.1.2) u_V identifies with U_T, with $T := (-\Delta)^{\frac{\alpha}{2}}V - \rho_V$. In particular, $u_V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and weakly solves (3.2.7). □

In what follows, assuming $\alpha \in (0, 2]$, we prove that the PDE defined by (3.2.4) satisfies a weak comparison principle (Lemma 3.2.1) which will allow us to deduce information about the minimizer such as asymptotic behaviour and non negativity (under some suitable assumptions). Note that, avoiding the cases where $\alpha > 2$ is crucial for the following reasons. First, given any element $u \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, it's not in general true that the quantity $[u]_+$ (respectively $[u]_-$) belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ unless $\alpha \in (0, 3)$ (see [92, p.3], [1, p. 12, 13] and references therein). On the other hand, if $\alpha \in (2, 3)$, it's not in general true that $\langle [u]_+, [u]_-\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \le 0$ which is fundamental for the proofs of Proposition 3.2.2 and Lemma 3.2.1 below.

Proposition 3.2.2 (Non negativity). *Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, $\alpha \in (0, 2]$. If $(-\Delta)^{\frac{\alpha}{2}}V \ge 0$, then $u_V \ge 0$.*

Proof. Let $\alpha \in (0, 2)$. From the decomposition $u_V = [u_V]_+ - [u_V]_-$ and (3.2.4), we have

$$(-\Delta)^{\frac{\alpha}{2}}([u_V]_+ - [u_V]_-) + \mathrm{sign}(u_V)|u_V|^{\frac{1}{q-1}} = (-\Delta)^{\frac{\alpha}{2}} V \geq 0 \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d). \quad (3.2.8)$$

Consequently, testing (3.2.8) by $[u_V]_-$ (which belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$) we have

$$\langle(-\Delta)^{\frac{\alpha}{2}}([u_V]_+ - [u_V]_-), [u_V]_-\rangle + \int_{\mathbb{R}^d} \mathrm{sign}(u_V)|u_V|^{\frac{1}{q-1}}[u_V(x)]_- \geq 0, \quad (3.2.9)$$

where the above integral representation is due to the obvious inclusions

$$[u_V]_- \in L^{q'}(\mathbb{R}^d), \quad \mathrm{sign}(u_V)|u_V|^{\frac{1}{q-1}} \in L^q(\mathbb{R}^d).$$

Therefore, by combining (3.2.9) with $\langle [u_V]_+, [u_V]_- \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \leq 0$, we obtain

$$\|[u_V]_-\|_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \leq 0,$$

which implies that $[u_V]_- = 0$. The case $\alpha = 2$ easily follows as well by combining the above argument with the equality

$$\langle [u_V]_+, [u_V]_- \rangle_{\mathring{H}^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \nabla[u_V]_+ \nabla[u_V]_- = 0.$$

□

Next, we prove that the PDE (3.1.11) satisfies a weak comparison principle on smooth domains Ω. However, such result will be applied on domains of the form $\Omega = \overline{B_R}^c$, $R > 0$ (see for example Lemma 3.5.4), or $\Omega = \mathbb{R}^d$ (Theorem 3.1.4).

Lemma 3.2.1 (Comparison Principle)**.** *Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, $\alpha \in (0,2]$. Let $u, v \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$, $sign(u)|u|^{\frac{1}{q-1}}, sign(v)|v|^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Assume that u, v are respectively super and subsolution of (3.2.4) in a smooth domain $\Omega \subset \mathbb{R}^d$. Namely*

$$(-\Delta)^{\frac{\alpha}{2}} u + sign(u)|u|^{\frac{1}{q-1}} \geq (-\Delta)^{\frac{\alpha}{2}} V \qquad in\ \mathcal{D}'(\Omega), \quad (3.2.10)$$

$$(-\Delta)^{\frac{\alpha}{2}} v + sign(v)|v|^{\frac{1}{q-1}} \leq (-\Delta)^{\frac{\alpha}{2}} V \qquad in\ \mathcal{D}'(\Omega). \quad (3.2.11)$$

If $\mathbb{R}^d \setminus \Omega \neq \emptyset$ we further require that $u \geq v$ in $\mathbb{R}^d \setminus \Omega$. Then $u \geq v$ in $\mathbb{R}^d$.

Proof. Assume that $\alpha \in (0,2)$. Then, by subtracting (3.2.10) to (3.2.11) we obtain

$$(-\Delta)^{\frac{\alpha}{2}}(v-u) + (\mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}}) \leq 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.2.12}$$

By density of $C_c^\infty(\Omega)$ in $\mathring{H}_0^{\frac{\alpha}{2}}(\Omega)$ and (3.2.12) we conclude that

$$\langle v-u, \varphi\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \langle \mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}}, \varphi\rangle \leq 0 \ \forall \varphi \in \mathring{H}_0^{\frac{\alpha}{2}}(\Omega), \varphi \geq 0, \tag{3.2.13}$$

where by $\langle\cdot,\cdot\rangle$ we denoted the duality between $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Note that (1.2.2) implies that $[v-u]_+ \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Moreover, since $u \geq v$ in $\mathbb{R}^d \setminus \Omega$ and $\partial\Omega$ is regular, the function $[v \ -u]_+$ (which by definition belongs to $\widetilde{\mathring{H}^{\frac{\alpha}{2}}}(\Omega)$) can be approximated by smooth functions supported in Ω. Namely, $[v-u]_+ \in \mathring{H}_0^{\frac{\alpha}{2}}(\Omega)$ (see Remark 1.2.1). In particular we can test (3.2.13) against $[v-u]_+$. Note also that, since $\mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $[v-u]_+ \in \mathring{H}_0^{\frac{\alpha}{2}}(\Omega) \cap L^{q'}(\mathbb{R}^d)$, we have the equality

$$\begin{aligned}\langle \mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}}, [v-u]_+\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d), \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \\ = \int_{\mathbb{R}^d} (\mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}})[v-u]_+.\end{aligned} \tag{3.2.14}$$

Thus by combining (3.2.13), (3.2.14) with (1.2.2) we infer

$$\begin{aligned}0 &\geq \langle v-u, [v-u]_+\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \int_{\mathbb{R}^d} (\mathrm{sign}(v)|v|^{\frac{1}{q-1}} - \mathrm{sign}(u)|u|^{\frac{1}{q-1}})[v-u]_+ \geq \\ &\geq \langle [v-u]_+, [v-u]_+\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \|[v-u]_+\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)},\end{aligned}$$

from which $[v-u]_+ = 0$. This concludes the proof for the case $\alpha \in (0,2)$.

The case $\alpha = 2$ follows by the same argument. □

Corollary 3.2.1. *Let $\alpha \in (0,2]$. For every $f \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ there exists a unique $u_f \in L^{q'}(\mathbb{R}^d) \cap \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ weakly solving*

$$(-\Delta)^{\frac{\alpha}{2}}u + \mathit{sign}(u)|u|^{\frac{1}{q-1}} = f \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d). \tag{3.2.15}$$

Proof. Let u be a solution of (3.2.15). From $f \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ we further conclude that $\mathrm{sign}(u)|u|^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Assume now that there exist two solutions u_1, u_2 of (3.2.15). Then, by applying Lemma 3.2.1 to $\Omega = \mathbb{R}^d$ we conclude that $u_1 = u_2$.

Furthermore, let us consider the minimizer ρ_V for $\mathcal{E}^{TF}_\alpha$ with $V := (-\Delta)^{-\frac{\alpha}{2}} f$. By Proposition 3.2.1 and the definition of V we derive that u_V solves (3.2.15) as well. Thus, $u_V = \bar{u}$ by the uniqueness just proved. □

Corollary 3.2.2. *Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$, $\alpha \in (0,2]$. If $V^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$, $V \geq 0$ then*

$$u_V \leq V \quad \text{in } \mathbb{R}^d.$$

Proof. Since V is non negative it clearly satisfies

$$(-\Delta)^{\frac{\alpha}{2}} V + V^{\frac{1}{q-1}} \geq (-\Delta)^{\frac{\alpha}{2}} V \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Then, Lemma 3.2.1 implies

$$u_V \leq V \quad \text{in } \mathbb{R}^d.$$

□

Next we prove that the minimizer ρ_V is non negative and radially non increasing provided $(-\Delta)^{\frac{\alpha}{2}} V$ is non negative and radially non increasing as well. In particular Corollary 3.2.3 below generalises [89, Corollary 4.4] to every $\alpha \in (0,2]$ without requiring the additional decay assumption $(-\Delta)^{\frac{\alpha}{2}} V \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)$.

Corollary 3.2.3. *Let $\alpha \in (0,2]$ and $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. If $(-\Delta)^{\frac{\alpha}{2}} V$ is a non negative, locally integrable, and radially non increasing function then u_V is the unique minimizer for the following problem*

$$\mathcal{J} := \inf_{u \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)} J(u), \tag{3.2.16}$$

where J is defined by

$$J(u) := \frac{1}{2} \|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \frac{1}{q'} \int_{\mathbb{R}^d} |u(x)|^{q'} dx - \langle u, V \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}. \tag{3.2.17}$$

Furthermore, u_V is non negative and radially non increasing.

Proof. The unique minimizer u in $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$ for the convex energy functional J is characterised by

$$\langle u, \varphi \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \operatorname{sign}(u(x)) |u(x)|^{\frac{1}{q-1}} \varphi(x) dx - \langle \varphi, V \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = 0 \tag{3.2.18}$$

for every $\varphi \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$. Hence, by Proposition 3.2.1, we conclude that $u = u_V$. In particular, by Proposition 3.2.2, the function u_V is the unique minimizer of J on the set $\mathcal{P}$ defined as follows

$$\mathcal{P} := \left\{ u \geq 0 : u \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d) \right\}.$$

Thus, by combining (1.2.2) with Corollary 2.3.1 we have that $u\,(-\Delta)^{\frac{\alpha}{2}} V \in L^1(\mathbb{R}^d)$ and

$$\langle u, V \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \langle (-\Delta)^{\frac{\alpha}{2}} V, u \rangle = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(x) u(x) dx \quad \forall u \in \mathcal{P}.$$

This implies that the functional J restricted on $\mathcal{P}$ takes the form of

$$J(u) = \frac{1}{2} \|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \frac{1}{q'} \int_{\mathbb{R}^d} |u(x)|^{q'} dx - \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(x) u(x) dx \quad \forall u \in \mathcal{P}. \tag{3.2.19}$$

We now claim that $u_V^* = u_V$, where by u_V^* we denote the radially symmetric rearrangement of u_V. It holds that (see [7, Theorem 9.2] or [8, Proposition 2.1] and [78, Lemma 7.17, eq (4) p.81])

$$\|u_V^*\|_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \leq \|u_V\|_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}, \quad \|u_V^*\|_{L^{q'}(\mathbb{R}^d)} = \|u_V\|_{L^{q'}(\mathbb{R}^d)}, \tag{3.2.20}$$

and

$$\int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(x) u_V(x) dx \leq \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} V(x) u_V^*(x) dx, \tag{3.2.21}$$

where (3.2.21) follows by radial symmetry and monotonicity of $(-\Delta)^{\frac{\alpha}{2}} V$. Then, putting together (3.2.20) and (3.2.21) we deduce that $u_V^* \in \mathcal{P}$ and

$$J(u_V^*) \leq J(u_V) = \mathcal{J}.$$

Hence, u_V^* is also a minimizer and $u_V^* = u_V$ (by uniqueness). □

3.3 Regularity

This section is entirely devoted to prove regularity of the minimizer ρ_V. The idea is to take advantage of the Euler–Lagrange equation

$$\operatorname{sign}(\rho_V)|\rho_V|^{q-1} = V - (-\Delta)^{-\frac{\alpha}{2}} \rho_V \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{3.3.1}$$

where we recall that by $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$ we denote the unique element of $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ whose Laplacian is ρ_V in the sense of (1.3.12). Indeed, in view of (3.3.1), it's readily seen that the regularity of $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$ implies regularity of ρ_V (assuming some regularity of the potential V). Thus, in the first two subsections (see 3.3.1 and 3.3.2 below), we focus our attention on the case $q \geq \frac{d}{\alpha}$ where in general the Riesz potential $I_\alpha * \rho_V$ is not well defined as a Lebesgue integral and can not be identified with the operator $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$.

In the last subsection, we study regularity of the minimizer in the subcritical regime $q < \frac{d}{\alpha}$ (Corollaries 3.3.3 and 3.3.4), where (3.3.1) reads as

$$\operatorname{sign}(\rho_V)|\rho_V|^{q-1} = V(x) - A_\alpha \int_{\mathbb{R}^d} \frac{\rho_V(y)}{|x-y|^{d-\alpha}} dy \quad \text{a.e. in } \mathbb{R}^d, \tag{3.3.2}$$

(see again Corollary 2.4.1). We further point out that, from now on, the basic assumption on V is to be a non zero element of $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ while other assumptions will be explicitly written when necessary.

Remark 3.3.1. Before presenting the regularity results of ρ_V on the subcritical regime, we emphasise the fact that Corollaries 3.3.1 and 3.3.2 can be improved by simply requiring more regularity (e.g., Hölder regularity) of the potential V in the spirit of Corollary 3.3.4.

3.3.1 Supercritical cases: $q > \frac{d}{\alpha}$, $\alpha - \frac{d}{q} \notin \mathbb{N}$

In this regime, the Euler–Lagrange equation reads as (3.3.1) and, by a classical localisation argument that, we can easily see (Proposition 3.3.1) that the function $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$ is regular.

Proposition 3.3.1. *Let $0 < \alpha < d$, β be a positive number and $u \in \mathcal{L}^1_\alpha(\mathbb{R}^d)$ be a function solving*

$$(-\Delta)^{\frac{\alpha}{2}} u = f \quad \text{in } \mathcal{D}'(B_R),$$

for some $R > 0$. The following hold:

(i) If $f \in L^q(B_R)$ for some $q > \frac{d}{\alpha}$, $\alpha - \frac{d}{q} \notin \mathbb{N}$ then $u \in C^{\alpha - \frac{d}{q}}(\overline{B_{R/2}})$;

(ii) If $u \in L^\infty(\mathbb{R}^d)$, $f \in C^\beta(B_R)$ with β, $\beta + \alpha \notin \mathbb{N}$ and $0 < \alpha < 2$ then $u \in C^{\beta+\alpha}(\overline{B_{R/2}})$.

Proof. We begin by proving case (i). To this aim, let us define the function

$$u_R(x) := A_\alpha \int_{\mathbb{R}^d} \frac{f(y)\chi_{B_R}(y)}{|x-y|^{d-\alpha}} dy.$$

Then, $u - u_R \in \mathcal{L}^1_\alpha$ is α–harmonic in B_R. In particular, from Lemma 2.2.1, the latter function belongs to $C^\infty(\overline{B_{R/2}})$. We mention also the general result [61, Theorem 12.19] providing regularity in a framework of pseudodifferential operators and [102, Proposition 2.2]. Moreover, since $f \in L^q(B_R)$, the function $f\chi_{B_R}$ has compact support and belongs to $L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Then, since $\alpha - \frac{d}{q}$ is not an integer, we infer $u_R \in C^{[\beta],\{\beta\}}(\mathbb{R}^d)$ with $\beta = \alpha - \frac{d}{q}$ (cf. [2, Remark 5.18] and references therein). This concludes the proof of (i).
Proof of (ii). We refer to [99, Theorem 1.1–(b), Corollary 3.5] or [33, eq. (3.23)–(3.25)]. □

In view of Proposition 3.3.1, in Corollary 3.3.1 we prove the regularity of ρ_V when $\alpha - \frac{d}{q}$ is not an integer.

Corollary 3.3.1. *Assume* $0 < \alpha < d$, $q > \frac{d}{\alpha}$ *and* $\alpha - \frac{d}{q} \notin \mathbb{N}$. *If* $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *then the minimizer* $\rho_V \in C(\mathbb{R}^d)$.

Proof. By Proposition 3.3.1 we obtain continuity of $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$. Then, if $V \in C(\mathbb{R}^d)$, the Euler–Lagrange equation (3.3.1) implies continuity of ρ_V. □

Next, we prove the regularity of the minimizer ρ_V in the cases $\alpha - \frac{d}{q}$ being an integer. This completes the study of regularity in the regime $q \geq \frac{d}{\alpha}$.

3.3.2 Critical cases: $\alpha - \frac{d}{q} \in \mathbb{N}$

In this cases we can reduce the analysis of the regularity to the supercritical case already analysed above. Namely, we prove the following:

Corollary 3.3.2. *Assume* $0 < \alpha < d$ *and* $\alpha - \frac{d}{q} \in \mathbb{N}$. *If* $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *then* $\rho_V \in C(\mathbb{R}^d)$.

Proof. By the Sobolev embedding, V and $(-\Delta)^{-\frac{\alpha}{2}}\rho_V$ belong to $L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$. In particular, from the Euler–Lagrange equation (3.3.1) we obtain that $\rho_V \in L^q(\mathbb{R}^d) \cap L^{\frac{2d}{d-\alpha}(q-1)}(\mathbb{R}^d)$. Then, $\rho_V \in L^{q+\varepsilon}(\mathbb{R}^d)$ for every $\varepsilon > 0$ sufficiently small. Consequently, from Proposition 3.3.1 we infer that $(-\Delta)^{-\frac{\alpha}{2}}\rho_V \in C^{\alpha-\frac{d}{q+\varepsilon}}_{loc}(\mathbb{R}^d)$ for every

$\varepsilon > 0$ small such that $\alpha - \frac{d}{q+\varepsilon}$ is not an integer. Then, by continuity of V and the Euler–Lagrange equation (3.3.1), we deduce the continuity of ρ_V. □

3.3.3 Subcritical case: $q < \frac{d}{\alpha}$

In this regime, the Riesz potential $I_\alpha * \rho_V$ is well defined and equation (3.3.1) reads as

$$\mathrm{sign}(\rho_V)|\rho_V|^{q-1} = V - I_\alpha * \rho_V \quad \text{a.e. in } \mathbb{R}^d. \tag{3.3.3}$$

We begin the subsection by proving Hölder continuity of the Riesz potential provided that the potential V belongs to $L^p(\mathbb{R}^d)$ for some p sufficiently large.

Lemma 3.3.1. *Let $\rho_V \in \mathcal{H}_\alpha$ be the minimizer and $q < \frac{d}{\alpha}$. Then, there exists $p > \frac{2d}{d-\alpha}$ such that if $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ then $I_\alpha * \rho_V$ belongs to $C^{0,\gamma}(\mathbb{R}^d)$ for some $\gamma \in (0,1]$.*

Proof. By (1.4.4), $I_\alpha * \rho_V \in L^{t_1}(\mathbb{R}^d)$ where $\frac{1}{t_1} = \frac{1}{q} - \frac{\alpha}{d}$. Assume that $V \in L^{t_1}(\mathbb{R}^d)$. Then $\rho_V \in L^{t_1(q-1)}(\mathbb{R}^d)$. Moreover, we set $s_1 := t_1(q-1) > q := s_0$ and we note that $t_1 > \frac{2d}{d-\alpha}$.

Let us start now from s_1 above defined. We split the analysis into three cases:

(i) $s_1 > \frac{d}{\alpha}$,

(ii) $s_1 = \frac{d}{\alpha}$,

(iii) $s_1 < \frac{d}{\alpha}$.

If (i) holds, we have that $\rho_V \in L^q(\mathbb{R}^d) \cap L^{s_1}(\mathbb{R}^d)$ from which Lemma 1.4.3 implies Hölder continuity of $I_\alpha * \rho_V$.
If (ii) holds, we have that $\rho_V \in L^q(\mathbb{R}^d) \cap L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ which in turn implies that $\rho_V \in L^{\frac{d}{\alpha}-\varepsilon}(\mathbb{R}^d)$ for every $\varepsilon > 0$ small. Hence, (3.3.3) implies that $I_\alpha * \rho_V \in L^{t_\varepsilon}(\mathbb{R}^d)$ where t_ε is defined by

$$t_\varepsilon := \frac{d(d-\alpha\varepsilon)}{\alpha^2\varepsilon}.$$

Thus, if we assume that $V \in L^{t_\varepsilon}(\mathbb{R}^d)$ we obtain $\rho_V \in L^{(q-1)t_\varepsilon}(\mathbb{R}^d)$. Consequently, if we choose ε small enough (i.e. $0 < \varepsilon < \frac{d(q-1)}{\alpha q}$) it holds that $(q-1)t_\varepsilon > \frac{d}{\alpha}$. Again from Lemma 1.4.3 the desired Hölder regularity holds.

If (iii) holds the idea is to iterate the process. In general, we consider a recursive sequence defined as follows:

$$\begin{cases} s_0 := q, \\ s_k := \frac{ds_k(q-1)}{d-\alpha s_{k-1}}, \quad \text{if } s_{k-1} < \frac{d}{\alpha}. \end{cases} \tag{3.3.4}$$

Next, we claim the above sequence consists of a finite number of elements, i.e., there exists $\bar{k} \in \mathbb{N}$ such that $s_{\bar{k}-1} < \frac{d}{\alpha}$ and $s_{\bar{k}} \geq \frac{d}{\alpha}$. The proof of the claim goes by contradiction. Assume that from (3.3.4) we construct a sequence $\{s_k\}_k$ such that $s_k < \frac{d}{\alpha}$ for every $k \in \mathbb{N}$. Then, we notice that the function $f : (0, \frac{d}{\alpha}) \to \mathbb{R}$ defined by

$$f(x) := \frac{dx(q-1)}{d - \alpha x}$$

is increasing. As a consequence, the inequality $s_{k+1} > s_k$ is equivalent to the inequality $f(s_k) > f(s_{k-1})$ that is in turn true by induction since $s_1 > s_0 = q$. Therefore, the sequence $\{s_k\}_k$ is increasing and there exists $L := \lim_{k\to+\infty} s_k$.

Case 1. If $L = \frac{d}{\alpha}$ then

$$\frac{d}{\alpha} = \lim_{k\to+\infty} \frac{ds_k}{d-\alpha s_k}(q-1) = +\infty$$

which is a contradiction.

Case 2. If $0 < L < \frac{d}{\alpha}$, passing to the limit on the equation

$$s_{k+1} = \frac{ds_k}{d-\alpha s_k}(q-1),$$

we obtain the following relation

$$L = \frac{d(2-q)}{\alpha}. \tag{3.3.5}$$

On the other hand, since the sequence $\{s_k\}_k$ is increasing and $s_0 = q$, we conclude that L must be strictly bigger than q. However, the assumption $q > \frac{2d}{d+\alpha}$ ensures that the right hand side of (3.3.5) is smaller than q leading to a contradiction.

In view of the above analysis, if we further assume that $V \in \bigcap_{k=1}^{\bar{k}} L^{t_k}(\mathbb{R}^d)$, $\frac{1}{t_k} = \frac{1}{s_{k-1}} - \frac{\alpha}{d}$, by iterating $\bar{k}$–times the argument in the first lines of this proof we conclude that $\rho_V \in L^{s_{\bar{k}}}(\mathbb{R}^d)$, where $s_{\bar{k}} \geq \frac{d}{\alpha}$. We also recall that the sequence $\{t_k\}_k$,

$k = 1, \dots, \bar{k}$ is increasing and $t_1 > \frac{2d}{d-\alpha}$. Hence, by the Sobolev embedding it is enough to require $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{t_{\bar{k}}}(\mathbb{R}^d)$. Now, if $s_{\bar{k}} > \frac{d}{\alpha}$ from Lemma 1.4.3 we derive the Hölder regularity. If $s_{\bar{k}} = \frac{d}{\alpha}$ by arguing as in (ii) we conclude that there exists t_ε such that if $V \in L^{t_\varepsilon}(\mathbb{R}^d)$ then $I_\alpha * \rho_V$ is Hölder continuous. In particular, if $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p := \max\{t_{\bar{k}}, t_\varepsilon\}$, we conclude again.

Moreover, $I_\alpha * \rho_V \in L^\infty(\mathbb{R}^d)$. Indeed, by the previous argument, $\rho_V \in L^q(\mathbb{R}^d) \cap L^t(\mathbb{R}^d)$ for some $q < \frac{d}{\alpha} < t$. Thus,

$$\begin{aligned} |(I_\alpha * \rho_V)(x)| &= A_\alpha \left| \int_{\mathbb{R}^d} \frac{\rho_V(x-y)}{|y|^{d-\alpha}} dy \right| \\ &\le A_\alpha \left(\|\rho_V\|_{L^q(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d \setminus B_1} \frac{dy}{|y|^{(d-\alpha)q'}} \right)^{\frac{1}{q'}} + \|\rho_V\|_{L^t(\mathbb{R}^d)} \left(\int_{B_1} \frac{dy}{|y|^{(d-\alpha)t'}} \right)^{\frac{1}{t'}} \right) \\ &< \infty, \end{aligned} \tag{3.3.6}$$

from which there exists $\gamma \in (0,1]$ such that $\|I_\alpha * \rho_V\|_{C^{0,\gamma}(\mathbb{R}^d)} < \infty$. Furthermore, from the inclusion $I_\alpha * \rho_V \in L^{\frac{dq}{d-\alpha q}}(\mathbb{R}^d) \cap C^{0,\gamma}(\mathbb{R}^d)$ we conclude that $I_\alpha * \rho_V$ vanishes at infinity. □

Next, to simplify the notation, in Corollaries 3.3.3, 3.3.4 below we restrict the statements to the case $0 < \alpha < 2$. The other cases can be obtained by an iterations of the same arguments in the spirit of [33, Theorem 10].

Corollary 3.3.3. *Let $0 < \alpha < 2$ and $q < \frac{d}{\alpha}$. Let $\rho_V \in \mathcal{H}_\alpha$ be the global minimizer. If $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ then $\rho_V \in C_b(\mathbb{R}^d)$ and the following possibilities hold:*

(i) If $\alpha \le 1$ then $I_\alpha * \rho_V \in C^{0,\gamma}(\mathbb{R}^d)$ for every $\gamma < \alpha$;

(ii) If $\alpha > 1$ then $I_\alpha * \rho_V \in C^{1,\gamma}(\mathbb{R}^d)$ for every $\gamma < \alpha - 1$.

In particular, $\rho_V(x) \to 0$ as $|x| \to +\infty$ provided $V(x) \to 0$ as $|x| \to +\infty$.

Proof. Assuming V is also bounded continuous we obtain that $V \in L^s(\mathbb{R}^d)$ for every $s \ge \frac{2d}{d-\alpha}$. Thus, by Lemma 3.3.1, $I_\alpha * \rho_V$ is Hölder continuous of some exponent γ. Then, from (3.3.3) we obtain $\rho_V \in C_b(\mathbb{R}^d)$. This precisely implies the desired regularity of $I_\alpha * \rho_V$ (see e.g. [102, Proposition 2.9] or [75, Theorem 3.1]). Furthermore, since $I_\alpha * \rho_V$ goes to zero, the same holds for ρ_V provided V goes to zero. □

In what follows, by applying Corollary 3.3.3 and Proposition 3.3.1, we show that assuming some additional regularity of the potential V yields more regularity of ρ_V. In particular, Corollary 3.3.4 proves that if V is bounded locally Hölder continuous of exponent $\gamma \in (0,1]$, and $q < q_*$ where q_* is defined in (3.3.7), then ρ_V is locally Hölder continuous of exponent $\min\left\{\frac{\gamma}{q-1}, \gamma\right\}$, which is in general the best regularity achievable. We also notice that if V is locally Lipschitz continuous then the exponent q_* coincides with the critical exponent already mentioned in [33, Theorem 8].

Let $0 < \alpha < d$, $0 < \gamma \le 1$ and $q > \frac{2d}{d+\alpha}$. We define the quantities q_*, γ_* as follows

$$q_* := \begin{cases} \frac{2\gamma-\alpha}{\gamma-\alpha}, & \text{if } \alpha < \gamma, \\ +\infty, & \text{if } \alpha \ge \gamma, \end{cases} \tag{3.3.7}$$

$$\gamma_* := \min\left\{\frac{\gamma}{q-1}, \gamma\right\}. \tag{3.3.8}$$

Taking into account the above notations, we formulate the main regularity result of the section requiring low regularity of the potential V.

Corollary 3.3.4 (Hölder regularity)**.** *Let $0 < \alpha < 2$. Assume $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C^{0,\gamma}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for some $\gamma \in (0,1]$. If $q < \frac{d}{\alpha}$ then ρ_V is bounded and locally Hölder continuous. Moreover, if we further assume $q < q_*$, then $\rho_V \in C^{0,\gamma_*}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ where q_*, γ_* are defined in* (3.3.7)-(3.3.8).

Proof. The first part of the statement is a direct consequence of Lemma 3.3.1 and Corollary 3.3.3. For the second part, we follow the proof of [33, Theorem 8].
Case $\alpha < 1$. By Corollary 3.3.3 we have $I_\alpha * \rho_V \in C^{0,\beta}(\mathbb{R}^d)$ for every $\beta \in (0,\alpha)$. Let us start by assuming $q \le 2$. Then, we define the quantity

$$\beta_n := \beta + (n-1)\alpha \in (1-\alpha, 1), \tag{3.3.9}$$

where $n \in \mathbb{N}$ is such that $\frac{1}{n+1} \le \alpha \le \frac{1}{n}$. Clearly we can assume that $\alpha \le \gamma$. Indeed, from (3.3.3), if $\gamma < \alpha$ we immediately get $\mathrm{sign}(\rho_V)|\rho_V|^{q-1} \in C^{0,\gamma}_{loc}(\mathbb{R}^d)$.

Assume that $\alpha \le \gamma$. By the choice of n and the definition of β_n provided in (3.3.9), it holds that $\beta_n < 1$ and $\beta_n + \alpha > 1$. Thus, starting from β and applying $(n-1)$–times Proposition 3.3.1 we get $I_\alpha * \rho_V \in C^{0,\beta_n}_{loc}(\mathbb{R}^d)$. Hence, (3.3.3) yields $\mathrm{sign}(\rho_V)|\rho_V|^{q-1} \in C^{0,\beta_n}_{loc}(\mathbb{R}^d)$. Again, if $\gamma \le \beta_n$ we conclude. If not, another iteration of Proposition 3.3.1 leads to $I_\alpha * \rho_V \in C^{[\beta_n+\alpha],\{\beta_n+\alpha\}}_{loc}(\mathbb{R}^d) = C^{1,\beta_n+\alpha-1}_{loc}(\mathbb{R}^d)$. The desired regularity is therefore obtained by (3.3.3).

Assume now $2 < q < q_*$. As before, $I_\alpha * \rho_V \in C^{0,\beta}(\mathbb{R}^d)$ for every $\beta \in (0, \alpha)$. Hence if $\alpha > \gamma$ we obtain the thesis. If not, we argue as follows. Let us define the quantities α_∞, α_n by

$$\alpha_n := \sum_{j=0}^{n} \frac{\alpha}{(q-1)^j}, \quad \alpha_\infty := \sum_{j=0}^{\infty} \frac{\alpha}{(q-1)^j} = \frac{\alpha(q-1)}{q-2}.$$

Note also that $\alpha_\infty > \gamma$ since $q < q^*$. In particular, there exists $\bar{n} \in \mathbb{N}$ such that $\alpha_{\bar{n}-1} \le \gamma$ and $\alpha_{\bar{n}} > \gamma$. Thus, after $\bar{n}-1$ iterations of Proposition 3.3.1 we obtain that $I_\alpha * \rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, \alpha_{\bar{n}-1})$. As a consequence, (3.3.3) yields $\rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, \frac{\alpha_{\bar{n}-1}}{q-1})$. In particular, since by assumption $\alpha_{\bar{n}} = \alpha + \frac{\alpha_{\bar{n}-1}}{q-1} > \gamma$, another iteration of Proposition 3.3.1 implies that there exists $\beta \notin \mathbb{N}$ strictly larger than γ such that $I_\alpha * \rho_V \in C^{[\beta],\{\beta\}}_{loc}(\mathbb{R}^d)$. Thus, $I_\alpha * \rho_V \in C^{0,\gamma}_{loc}(\mathbb{R}^d)$ and, from (3.3.3), we conclude the proof of this case.

Case $\alpha = 1$. By Corollary 3.3.3 we get $I_\alpha * \rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, 1)$. Hence, if $\gamma < 1$ the thesis follows. On the other hand, If $\gamma = 1$ and $q \le 2$, the Euler–Lagrange equation (3.3.3) implies that $\rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, 1)$. Thus, Proposition 3.3.1 implies $I_\alpha * \rho_V \in C^{[\beta+\alpha],\{\beta+\alpha\}}_{loc}(\mathbb{R}^d) = C^{1,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, 1)$. In particular, $I_\alpha * \rho_V \in C^{0,1}_{loc}(\mathbb{R}^d)$ and, by (3.3.3), the same holds for ρ_V. If $\gamma = 1$ and $q > 2$, from (3.3.3) we deduce that $\rho_V \in C^{0,\frac{\beta}{q-1}}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, 1)$. In particular, Proposition 3.3.1 yields $I_\alpha * \rho_V \in C^{\left[\frac{\beta}{q-1}+\alpha\right],\left\{\frac{\beta}{q-1}+\alpha\right\}}_{loc}(\mathbb{R}^d) = C^{1,\frac{\beta}{q-1}}_{loc}(\mathbb{R}^d)$. The conclusion follows again by (3.3.3).

Case $\alpha > 1$. This case follows again by Proposition 3.3.1 and by using the same arguments as in the previous cases. □

Remark 3.3.2. If $q \ge q_*$ and $\alpha < 1$ we can still get information about the Hölder exponent of $I_\alpha * \rho_V$. Namely $I_\alpha * \rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $0 < \beta < \frac{\alpha(q-1)}{q-2}$. In particular, since $\frac{\alpha(q-1)}{q-2} > \alpha$, we improved the Hölder exponent of $I_\alpha * \rho_V$ from α to $\frac{\alpha(q-1)}{q-2}$. This fact will be important for Proposition 3.5.2, where such regularity is enough to compute the fractional Laplacian via the singular integral (1.3.3).

3.4 Non negative minimizer

We begin the section by minimizing the same energy functional $\mathcal{E}_\alpha^{TF}$ over the cone of non negative functions. To be precise, we minimize the energy $\mathcal{E}_\alpha^{TF}$ over the set

$$\mathcal{H}_\alpha^+ := \{\rho \in \mathcal{H}_\alpha : \rho \geq 0\},$$

where we recall that $\mathcal{H}_\alpha$ is defined by (3.1.5). Note also that if $V \leq 0$ in the whole of $\mathbb{R}^d$ we clearly obtain the inequality

$$\mathcal{E}_\alpha^{TF}(\rho) \geq 0 \quad \forall \rho \in \mathcal{H}_\alpha^+,$$

and hence $\rho_V^+ = 0$ is the unique trivial minimizer of $\mathcal{E}_\alpha^{TF}$ in $\mathcal{H}_\alpha^+$. Then, for the rest of the section we will always assume V to be an element of $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) + L^{q'}(\mathbb{R}^d)$ which is positive on a set of positive Lebesgue measure. Further restrictions will be emphasised when needed.

Next, in Theorem 3.1.4 and Theorem 3.4.1, we study the relation between the free and constrained minimizer

3.4.1 Proof of Theorem 3.1.3.

Proof. Once noticed that $\mathcal{H}_\alpha^+$ is weakly closed in $\mathcal{H}_\alpha$, the proofs of the existence and uniqueness follow the same lines as in Theorem 3.1.1. In particular, (3.1.13) follows. Moreover, by arguing as in [78, Theorem 11.13] or [29, Proposition 3.6], we further obtain the Euler–Lagrange equation.

Let ρ_V^+ be the minimizer. Then, we have

$$\langle (\mathcal{E}_\alpha^{TF})'(\rho_V^+), \varphi\rangle = \int_{\mathbb{R}^d} (\rho_V^+(x))^{q-1}\varphi(x)dx - \langle \varphi, V\rangle + \langle \rho_V^+, \varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}} \geq 0 \quad \forall \varphi \in \mathcal{H}_\alpha^+, \tag{3.4.1}$$

where by $(\mathcal{E}_\alpha^{TF})'$ we understand the Frechét derivative of $\mathcal{E}_\alpha^{TF}$ and by $\langle \cdot, \cdot \rangle$ we denote the evaluation of the linear functional $(\mathcal{E}_\alpha^{TF})'(\rho_V^+)$ against the test function φ. Furthermore, inequality (3.4.1) implies

$$\int_{\mathbb{R}^d} \left((\rho_V^+(x))^{q-1} - V(x) + (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+)(x) \right) \varphi(x)dx \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \varphi \geq 0 \tag{3.4.2}$$

that in turn proves

$$(\rho_V^+)^{q-1} \geq V - (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+ \quad a.e. \text{ in } \mathbb{R}^d. \tag{3.4.3}$$

Let $\psi := \rho_V^+ \varphi$ where $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $\rho_\varepsilon := \rho_V^+ - \varepsilon\psi$ where $0 < \varepsilon < ||\varphi||^{-1}_{L^\infty(\mathbb{R}^d)}$. Such kind of construction has been used for example in [10, Lemma 2]. Note that since $q > \frac{2d}{d+\alpha}$ and $\psi \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d) \subset \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$, we have that ψ and ρ_ε belong to $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. By the choice on ε and minimality of ρ_V^+, we have $\rho_\varepsilon \in \mathcal{H}_\alpha^+$ and

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \to 0^+} \frac{\mathcal{E}_\alpha^{TF}(\rho_\varepsilon) - \mathcal{E}_\alpha^{TF}(\rho_V^+)}{\varepsilon} \\ &= \int_{\mathbb{R}^d} \left((\rho_V^+(x))^{q-1} - V(x) + (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+(x) \right) \rho_V^+(x)\varphi(x)dx. \end{aligned} \tag{3.4.4}$$

Now, replacing ψ with $-\psi$ we obtain also the opposite inequality which implies that

$$\int_{\mathbb{R}^d} \left((\rho_V^+(x))^{q-1} - V(x) + (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+(x) \right) \rho_V^+(x)\varphi(x)dx = 0. \tag{3.4.5}$$

Thus, since φ is arbitrary we obtain

$$\rho_V^+ \left((\rho_V^+)^{q-1} - V + (-\Delta)^{-\frac{\alpha}{2}}\rho_V^+ \right) = 0 \quad a.e. \text{ in } \mathbb{R}^d. \tag{3.4.6}$$

Relation (3.1.3) follows combining (3.4.3) with (3.4.6). □

Theorem 3.4.1. *Assume $q > \frac{2d}{d+\alpha}$. Let V be a function in $L^{q'}(\mathbb{R}^d) + \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Let ρ_V, ρ_V^+ be respectively the minimizer of $\mathcal{E}_\alpha^{TF}$ in $\mathcal{H}_\alpha$ and in $\mathcal{H}_\alpha^+$. Then, $[\rho_V]_+ = \rho_V^+$ if and only if $\rho_V \geq 0$.*

Proof. If ρ_V is non negative the conclusion is trivial since ρ_V is a minimizer in the larger set $\mathcal{H}_\alpha$ and by uniqueness it has to coincide with ρ_V^+. On the contrary, assume $[\rho_V]_+ = \rho_V^+$. By definition ρ_V and ρ_V^+ satisfy respectively

$$\int_{\mathbb{R}^d} \text{sign}(\rho_V(x))|\rho_V(x)|^{q-1}\varphi(x)dx - \langle \varphi, V \rangle + \langle \rho_V, \varphi \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} = 0 \quad \forall \varphi \in \mathcal{H}_\alpha, \tag{3.4.7}$$

$$\int_{\mathbb{R}^d} (\rho_V^+)^{q-1}(x)\varphi(x)dx - \langle \varphi, V \rangle + \langle \rho_V^+, \varphi \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)} \geq 0 \quad \forall \varphi \in \mathcal{H}_\alpha^+. \tag{3.4.8}$$

In view of (3.4.7), we rewrite the quantity $\langle \varphi, V \rangle$ and we replace it into equation

(3.4.8) to obtain

$$\int_{\mathbb{R}^d}[\rho_V]_+^{q-1}\varphi+\langle[\rho_V]_+,\varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}-\langle\rho_V,\varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}-\int_{\mathbb{R}^d}\mathrm{sign}(\rho_V)|\rho_V|^{q-1}\varphi\geq 0\ \ \forall\varphi\in\mathcal{H}_\alpha^+,$$

which implies that

$$\int_{\mathbb{R}^d}[\rho_V]_-^{q-1}\varphi+\langle[\rho_V]_-,\varphi\rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}\geq 0\quad\forall\varphi\in\mathcal{H}_\alpha^+. \tag{3.4.9}$$

Hence, if $[\rho_V]_+=\rho_V^+$, the function $[\rho_V]_-$ belongs to $\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ and satisfies (3.4.9) that is the Euler–Lagrange equation related to the convex functional

$$\mathcal{F}(\rho):=\frac{1}{q}\int_{\mathbb{R}^d}|\rho(x)|^q+\frac{1}{2}\|\rho\|^2_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)}.$$

Clearly, the unique minimizer of $\mathcal{F}$ in $\mathcal{H}_\alpha^+$ is the zero function and so $[\rho_V]_-$ (which is the minimizer since it satisfies (3.4.9)) must be the zero function. □

Similarly to what has been in the previous section, we can prove regularity of ρ_V^+. We formulate the following:

Corollary 3.4.1. *Let $\rho_V^+\in\mathcal{H}_\alpha^+$ be the non negative minimizer. If $V\in\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)\cap C_b(\mathbb{R}^d)$ then $\rho_V^+\in C(\mathbb{R}^d)$.*

Proof. First of all let's notice that $\cdot\mapsto[\cdot]_+$ is a Lipschitz continuous functions. Thus, if $q<\frac{d}{\alpha}$ the proof follows the same argument as in Corollary 3.3.3 while if $q\geq\frac{d}{\alpha}$, the continuity is a direct consequence of Corollaries 3.3.1 and 3.3.2. □

Remark 3.4.1. In view of the argument provided in Corollary 3.4.1, it's easy to see that we can prove that the analogous version of Corollary 3.3.4 for the constrained minimizer ρ_V^+.

Next, we state a more general version of Lemma 3.2.1 that will be used for the proof of Theorem 3.1.4.

Lemma 3.4.1. *Let $u,v\in\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, $\alpha\in(0,2]$, and $f:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}_+$ such that $t\mapsto f(x,t)$ is non decreasing for a.e. $x\in\mathbb{R}^d$. Assume that $f(x,u(x)),f(x,v(x))\in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)$ and that there exists a smooth domain $\Omega\subset\mathbb{R}^d$ where*

$$(-\Delta)^{\frac{\alpha}{2}}u+f(x,u)\geq 0\quad \textit{in }\mathcal{D}'(\Omega), \tag{3.4.10}$$

$$(-\Delta)^{\frac{\alpha}{2}} v + f(x, v) \leq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.4.11}$$

If $\mathbb{R}^d \setminus \Omega \neq \emptyset$ we further require $u \geq v$ in $\mathbb{R}^d \setminus \Omega$. Then, $u \geq v$ in $\mathbb{R}^d$.

Proof. The proof is similar to the one of Lemma 3.2.1. □

Remark 3.4.2. Note that, Lemma 3.4.1 applies also when we add $g \in \mathcal{D}'(\mathbb{R}^d)$ to the right hand side of (3.4.10) and of (3.4.11).

Proposition 3.4.1. *Let $V \in \big(\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)\big) \setminus \{0\}$, $\alpha \in (0, d)$. Assume that*

$$\lim_{|x| \to +\infty} |x|^{d-\alpha} V(x) = 0. \tag{3.4.12}$$

Then ρ_V is sign changing.

Proof. Assume now that $\rho_V \geq 0$. From Corollary 2.4.2, 3.3.1, 3.3.2 and 3.3.3 we infer that ρ_V is a continuous solution of

$$\rho_V^{q-1} = V - I_\alpha * \rho_V \quad \text{in } \mathbb{R}^d. \tag{3.4.13}$$

Moreover, if $|x|$ is sufficiently large, there exist $C, R > 0$ independent on x such that

$$\begin{aligned} (I_\alpha * \rho_V)(x) = A_\alpha \int_{\mathbb{R}^d} \frac{\rho_V(y)}{|x-y|^{d-\alpha}} dy &\geq \frac{A_\alpha}{(2|x|)^{d-\alpha}} \int_{B(x,2|x|)} \rho_V(y) dy \\ &\geq \frac{C}{|x|^{d-\alpha}} \int_{B_R} \rho_V(y) dy. \end{aligned} \tag{3.4.14}$$

In view of (3.4.14) we obtain

$$\liminf_{|x| \to +\infty} |x|^{d-\alpha} (I_\alpha * \rho_V)(x) > 0. \tag{3.4.15}$$

Hence by, (3.4.12), (3.4.13) and (3.4.14)

$$0 \leq \limsup_{|x| \to +\infty} |x|^{d-\alpha} \rho_V^{q-1}(x) = \lim_{|x| \to +\infty} |x|^{d-\alpha} V(x) - \limsup_{|x| \to +\infty} |x|^{d-\alpha} (I_\alpha * \rho_V(x)) < 0, \tag{3.4.16}$$

that is a contradiction. The same contradiction arises assuming $\rho_V \leq 0$. □

Remark 3.4.3. In this remark we provide non trivial examples of potentials satisfying or not satisfying the conditions (i) and (ii) of Theorem 3.1.5. First of all, besides smooth functions with compact support, if V is C^2, non negative and radially non increasing of the form

$$V(|x|) := \begin{cases} |x|^{-\gamma}, & \text{if } |x| \geq 1 \\ v(|x|), & \text{if } |x| < 1, \end{cases}$$

with $v \in C^2(\overline{B_1})$ and $\gamma > d - \alpha$, the assumptions of Theorem 3.1.5–(i) are satisfied. See e.g., Lemma 3.6.1. As far as concerns (ii), to find examples of bounded superharmonic functions belonging to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, it's enough to consider a function V of the form $V := I_\alpha * T$ for some $0 \leq T \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $p < \frac{d}{\alpha} < q$. Indeed, by arguing as in Corollary 2.4.1 and Lemma 3.3.1 we have that $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. Moreover, by construction

$$(-\Delta)^{\frac{\alpha}{2}} V = T \geq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

Finally, let us consider a sign changing function $T \in L^q_c(\mathbb{R}^d)$, for some $q > \frac{d}{\alpha}$. Then, as before, we consider the function $V := I_\alpha * T \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. By construction, we have that $(-\Delta)^{\frac{\alpha}{2}} V = T$ is sign changing, i.e., V does not satisfy the assumption of (ii). Furthermore, if $\|[T]_+\|_{L^1(\mathbb{R}^d)} \neq \|[T]_-\|_{L^1(\mathbb{R}^d)}$, by Lemma 3.5.1 we infer

$$(I_\alpha * T)(x) = (I_\alpha * [T]_+)(x) - (I_\alpha * [T]_-)(x) = \frac{\|[T]_+\|_{L^1(\mathbb{R}^d)} - \|[T]_-\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right),$$

i.e., (i) is not satisfied neither.

In what follows we are going to prove Theorem 3.1.4 by further assuming V being a continuous function with compact support. To this aim, we will employ a fractional Kato–type inequality which essentially comes from [8, Theorem 3.3]. For completeness we provide a proof in Lemma 3.4.2. The proof of the local case $\alpha = 2$ follows by a similar argument (see also [24, Lemma A.1]). Note also that, the condition on V being compactly supported in particular implies

$$\lim_{|x| \to +\infty} |x|^{d-\alpha} V(x) = 0, \tag{3.4.17}$$

from which ρ_V is sign changing (Proposition 3.4.1) and the statement of Theorem 3.1.4 is not trivial. (Note that if ρ_V is non negative then Theorem 3.1.4 is trivial since $\rho_V = \rho_V^+$ by uniqueness.)

Lemma 3.4.2 (Fractional Kato inequality)**.** *Let $\alpha \in (0,2)$. Let $u \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}^d)$ such that*

$$\langle u, \psi \rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\psi(x)dx \quad \forall \psi \in \mathring{H}_c^{\frac{\alpha}{2}}(\mathbb{R}^d). \tag{3.4.18}$$

Then,

$$(-\Delta)^{\frac{\alpha}{2}}|u| \leq f\, sign(u) \quad in\ \mathcal{D}'(\mathbb{R}^d).$$

Proof. Let $0 \leq \varphi \in C_c^\infty(\mathbb{R}^d)$ and $\delta > 0$. We define the function

$$w_\delta(x) := \frac{u(x)}{u_\delta(x)}\varphi(x),$$

where $u_\delta(x) = \sqrt{u(x)^2 + \delta^2}$. First, we claim that $w_\delta \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Since by the Sobolev embedding $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$, we deduce that

$$\int_{\mathbb{R}^d} |w_\delta|^{\frac{2d}{d-\alpha}} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}^{\frac{2d}{d-\alpha}} \int_{\mathbb{R}^d} \left|\frac{u}{u_\delta}\right|^{\frac{2d}{d-\alpha}} \leq \left(\frac{\|\varphi\|_{L^\infty(\mathbb{R}^d)}}{\delta}\right)^{\frac{2d}{d-\alpha}} \int_{\mathbb{R}^d} |u|^{\frac{2d}{d-\alpha}}. \tag{3.4.19}$$

Moreover,

$$\begin{aligned} w_\delta(x) - w_\delta(y) &= \frac{u(x)}{u_\delta(x)}\varphi(x) - \frac{u(y)}{u_\delta(y)}\varphi(y) \\ &= [u(x) - u(y)]\frac{\varphi(x)}{u_\delta(x)} \\ &\quad + \left[\frac{1}{u_\delta(x)} - \frac{1}{u_\delta(y)}\right]\varphi(x)u(y) + [\varphi(x) - \varphi(y)]\frac{u(y)}{u_\delta(y)}. \end{aligned} \tag{3.4.20}$$

By (3.4.20) we have that

$$\begin{aligned} |w_\delta(x) - w_\delta(y)|^2 \leq & \frac{4}{\delta^2}|u(x) - u(y)|^2 \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \\ & +4\left|\frac{u(y)}{u_\delta(y)}\right|^2 \|\varphi\|^2_{L^\infty(\mathbb{R}^d)}|u_\delta(x) - u_\delta(y)|^2 + 4|\varphi(x) - \varphi(y)|^2 \\ \leq & \frac{4}{\delta^2}|u(x) - u(y)|^2 \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} + \frac{4}{\delta^2}||u(x)| - |u(y)||^2 \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \\ & + 4|\varphi(x) - \varphi(y)|^2 \end{aligned}$$

which in turn implies that

$$\begin{aligned}\|w_\delta\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} &\le \frac{4}{\delta^2}\|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\|\varphi\|^2_{L^\infty(\mathbb{R}^d)} + \frac{4}{\delta^2}\||u|\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\|\varphi\|^2_{L^\infty(\mathbb{R}^d)} + 4\|\varphi\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\\ &\le \frac{8}{\delta^2}\|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\|\varphi\|^2_{L^\infty(\mathbb{R}^d)} + 4\|\varphi\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}.\end{aligned} \tag{3.4.21}$$

From (3.4.21), we can test (3.4.18) by w_δ obtaining that

$$\frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(w_\delta(x)-w_\delta(y))}{|x-y|^{d+\alpha}}dxdy = \int_{\mathbb{R}^d} f w_\delta.$$

Moreover,

$$(u(x)-u(y))(w_\delta(x)-w_\delta(y)) \ge \frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y)) \tag{3.4.22}$$

and, by combining the definition of w_δ with Hölder inequality and Lemma 1.2.2, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|}{u_\delta(x)}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}}dxdy \le \frac{2}{C_{d,\alpha}}\|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}\|\varphi\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}. \tag{3.4.23}$$

In view of (3.4.23) and of the fact that $\frac{u(x)}{u_\delta(x)} \to 1$ a.e. as $\delta \to 0$, by Fatou's Lemma we obtain

$$\begin{aligned}&\liminf_{\delta\to 0}\left[\frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(w_\delta(x)-w_\delta(y))}{|x-y|^{d+\alpha}}dxdy\right]\\ &\ge \frac{C_{d,\alpha}}{2}\liminf_{\delta\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|}{u_\delta(x)}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}}dxdy\\ &\ge \frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}}dxdy\end{aligned} \tag{3.4.24}$$

Finally, since $|fw_\delta| \le |f\varphi| \in L^1(\mathbb{R}^d)$ and $fw_\delta \to f\text{sign}(u)\varphi$ a.e., dominated convergence theorem implies that $fw_\delta \to f\text{sign}(u)\varphi$ in $L^1(\mathbb{R}^d)$. We have therefore proved that for every $\varphi \in C_c^\infty(\mathbb{R}^d),\ \varphi \ge 0$.

$$\frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}}dxdy \le \int_{\mathbb{R}^d} f\text{sign}(u)\varphi.$$

□

3.4.2 Proof of Theorem 3.1.4.

Proof. By Corollaries 3.3.1, 3.3.2, 3.3.3 and 3.4.1, the functions ρ_V and ρ_V^+ are continuous. Moreover, by Proposition 3.4.1, we have that $[\rho_V]_+ \not\equiv \rho_V^+$ and they solve

$$\begin{aligned} \rho_V^+(x) &= [V(x) - (-\Delta)^{-\frac{\alpha}{2}} \rho_V^+(x)]_+^{\frac{1}{q-1}} \quad && \forall x \in \mathbb{R}^d, \\ \operatorname{sign}(\rho_V(x))|\rho_V(x)|^{q-1} &= V(x) - (-\Delta)^{-\frac{\alpha}{2}} \rho_V(x) && \forall x \in \mathbb{R}^d. \end{aligned} \tag{3.4.25}$$

Now, to simplify the notation, we introduce again the functions $U_{\rho_V} = (-\Delta)^{-\frac{\alpha}{2}} \rho_V$ and $U_{\rho_V^+} = (-\Delta)^{-\frac{\alpha}{2}} \rho_V^+$.

By applying the positive part function $[\cdot]_+$ to the second equation in (3.4.25) and multiplying by a non negative smooth function with compact support φ we infer

$$\int_{\mathbb{R}^d} \rho_V(x)\varphi(x)dx \leq \int_{\mathbb{R}^d} [\rho_V(x)]_+ \varphi(x)dx = \int_{\mathbb{R}^d} [V(x) - U_{\rho_V}(x)]_+^{\frac{1}{q-1}} \varphi(x)dx.$$

Similarly,

$$\int_{\mathbb{R}^d} \rho_V^+(x)\varphi(x)dx = \int_{\mathbb{R}^d} [V(x) - \varphi_V^+(x)]^{\frac{1}{q-1}} dx.$$

Consequently, from (1.3.13) we obtain

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} U_{\rho_V} &\leq [V - U_{\rho_V}]_+^{\frac{1}{q-1}} \quad && \text{in } \mathcal{D}'(\mathbb{R}^d), \\ (-\Delta)^{\frac{\alpha}{2}} U_{\rho_V^+} &= [V - U_{\rho_V^+}]_+^{\frac{1}{q-1}} && \text{in } \mathcal{D}'(\mathbb{R}^d). \end{aligned} \tag{3.4.26}$$

Furthermore, since $\rho_V \in L^q(\mathbb{R}^d)$ and $q > \frac{2d}{d+\alpha}$, we can apply Lemma 2.3.1 to conclude that

$$\langle \rho_V, \psi \rangle_{\mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d), \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \rho_V(x)\psi(x)dx \quad \forall \psi \in \mathring{H}_c^{\frac{\alpha}{2}}(\mathbb{R}^d). \tag{3.4.27}$$

Thus, by combining Lemma 3.4.2 with (3.4.26) we deduce that

$$(-\Delta)^{\frac{\alpha}{2}} [U_{\rho_V}]_+ \leq \chi_{\{U_{\rho_V} \geq 0\}}\, \rho_V \leq [V - [U_{\rho_V}]_+]_+^{\frac{1}{q-1}} \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{3.4.28}$$

On the other hand, the case $\alpha = 2$ is classical and follows the same lines. Note also that the assumption on the support of V combined with its continuity ensures that

$$\begin{aligned} [V - [U_{\rho_V}]_+]_+^{\frac{1}{q-1}} &\in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d), \\ [V - U_{\rho_V^+}]_+^{\frac{1}{q-1}} &\in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d). \end{aligned} \tag{3.4.29}$$

Moreover, from (3.4.25) and non negativity of $U_{\rho_V^+}$, we deduce that ρ_V^+ has compact support. Thus, by (1.2.2) and the above analysis, the functions $[U_{\rho_V}]_+$, $U_{\rho_V^+}$ belong to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and satisfy

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}}[U_{\rho_V}]_+ &\leq [V-[U_{\rho_V}]_+]_+^{\frac{1}{q-1}} \quad \text{in } \mathcal{D}'(\mathbb{R}^d),\\ (-\Delta)^{\frac{\alpha}{2}}U_{\rho_V^+} &= [V-U_{\rho_V^+}]_+^{\frac{1}{q-1}} \qquad \text{in } \mathcal{D}'(\mathbb{R}^d). \end{aligned} \tag{3.4.30}$$

Then, Lemma 3.2.1 applied to $\Omega=\mathbb{R}^d$ and $f(x,u)=-[V(x)-u]_+^{\frac{1}{q-1}}$ implies the inequality

$$U_{\rho_V^+} \geq [U_{\rho_V}]_+ \quad \text{in } \mathbb{R}^d. \tag{3.4.31}$$

Hence, from (3.4.25), inside the open set $\left\{\rho_V^+>0\right\}$ we have

$$\begin{aligned} \operatorname{sign}(\rho_V)|\rho_V|^{q-1}-(\rho_V^+)^{q-1} &= U_{\rho_V^+}-U_{\rho_V}\\ &\geq U_{\rho_V^+}-[U_{\rho_V}]_+ \geq 0, \end{aligned}$$

proving the desired inequality.

To conclude, it remains to prove the existence of a set of positive Lebesgue measure where the inequality is strict. To this aim we argue by contradiction.

Assume that $[\rho_V]_+=\rho_V^+$ in the whole of $\mathbb{R}^d$. Then, by Theorem 3.4.1 we deduce that ρ_V is non negative. However, since V has compact support, Proposition 3.4.1 implies that ρ_V must be sign changing leading to a contradiction. □

Remark 3.4.4. Note that the assumptions on V to prove the inequality (3.1.14) can be relaxed. Indeed, the above proof remains the same as long as (3.4.29) is valid and this is true for example if $[V]_+\in L^{\frac{2d}{(d+\alpha)(q-1)}}(\mathbb{R}^d)$.

As an immediate Corollary we state the following result:

Corollary 3.4.2. *Assume that the same conditions of Theorem 3.1.4 hold. If V is non negative function supported in compact set K of $\mathbb{R}^d$, then*

$$supp(\rho_V^+) \subseteq supp([\rho_V]_+) \subseteq K.$$

In particular, $\rho_V \leq 0$ in K^c.

Proof. By Corollary 3.2.2 and Theorem 3.1.4

$$0 \le \rho_V^+ \le [\rho_V]_+ \le V^{\frac{1}{q-1}}$$

from which the thesis follows. □

Next, we move our attention on the study of the asymptotic behaviour of the minimizer.

3.5 Decay estimates

The main goal of the following section is to study the asymptotic behaviour of the minimizer ρ_V extending the results of [89] concerning the special case $d = 2$, $\alpha = 1$ and $q = \frac{3}{2}$. In order to take advantage of the elliptic formulation (3.1.11), we focus on the case $\alpha \in (0, 2)$. We further recall that, in the local case $\alpha = 2$, the asymptotic behaviour of solutions for (3.1.11) has been widely studied in [96, Theorem 1.1.2] and [105] where log–corrections in the decay in the spirit of Theorem 3.1.7 have been proved. Furthermore, except for Theorem 3.1.8 where we study decay properties of sign changing minimizers, we will always assume that $(-\Delta)^{\frac{\alpha}{2}} V \ge 0$ (which ensures non negativity of the minimizer by Proposition 3.2.2).

For simplicity's sake, we summarise below the basic assumptions of this subsection. We denote by $(\mathcal{A})$ the following set of conditions

$$(\mathcal{A}) = \begin{cases} d \in \mathbb{N} \cap [2, \infty); \\ \alpha \in (0, 2); \\ q \in \left(\frac{2d}{d+\alpha}, \infty\right); \\ V \in \left(\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right) \setminus \{0\}\,; \\ (-\Delta)^{\frac{\alpha}{2}} V \ge 0. \end{cases}$$

We further recall that in this context, given two non negative functions f and g, by writing $f(x) \lesssim g(x)$ as $|x| \to +\infty$ we mean that there exists $C > 0$ such that $f(x) \le C g(x)$ for every x sufficiently large and similarly by writing $f(x) \simeq g(x)$ as $|x| \to +\infty$ we understand that $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ as $|x| \to +\infty$.

Remark 3.5.1. We further notice that, the regularity assumptions on V imposed above can be weakened according to Proposition 3.5.1.

We begin by studying the sublinear regime corresponding to $q < 2$.

3.5.1 Sublinear case: $q < 2$

Proposition 3.5.1. *Let $\alpha \in (0,2)$. Let $V \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, V vanishing at infinity and such that $0 \leq (-\Delta)^{\frac{\alpha}{2}} V \in L^p(\mathbb{R}^d)$ for some $p > \frac{d}{\alpha}$. If $q \leq 2$ then,*

$$0 < u_V(x) < V(x) \quad for\ all\ x \in \mathbb{R}^d. \tag{3.5.1}$$

Proof. The proof is similar to [83, Lemma 7.1]. From Proposition 3.3.1 we derive continuity of V. This, combined with the fact that V vanishes at infinity in particular implies boundedness of V as well. Thus, from regularity and superharmonicity of V we derive regularity and non negativity of u_V (see Corollaries 3.3.1, 3.3.2 and 3.3.3 for regularity and Proposition 3.2.2 for non negativity). Moreover, since $I_\alpha * \rho_V$ is clearly non negative, the Euler–Lagrange equation (3.1.9) implies that the function u_V is dominated by the bounded function V. In particular, since V vanishes at infinity the same happens to u_V. Next, if we define $c := \|u_V\|_{L^\infty(\mathbb{R}^d)}^{\frac{2-q}{q-1}}$, the function u_V solves

$$((-\Delta)^{\frac{\alpha}{2}} + c)u_V = u_V(c - u_V^{\frac{2-q}{q-1}}) + (-\Delta)^{\frac{\alpha}{2}} V \geq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{3.5.2}$$

It is known that the operator $(-\Delta)^{\frac{\alpha}{2}} + c$ has a positive and radially non increasing Green function $\mathcal{K}_c$ behaving as follows

$$\mathcal{K}_c(r) \lesssim \begin{cases} r^{-d+\alpha}, & \text{as } r \to 0, \\ r^{-d-\alpha}, & \text{as } r \to +\infty \end{cases} \tag{3.5.3}$$

(see [53, Lemma 4.2] or [57, Lemma C.1] for more details). From (3.5.3) we deduce that $\mathcal{K}_c \in L^1(\mathbb{R}^d) \cap L^t(\mathbb{R}^d)$ for every $t < \frac{d}{d-\alpha}$.

Let g be the right hand side of (3.5.2). Then g takes the form of

$$g = g_1 + g_2,$$

where

$$g_1 := u_V(c - u_V^{\frac{2-q}{q-1}}), \quad g_2 := (-\Delta)^{\frac{\alpha}{2}} V.$$

As a consenquence, $g_1 \in L^{q'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $g_2 \in L^p(\mathbb{R}^d)$, $p > \frac{d}{\alpha}$. Thus, $\mathcal{K}_c * g$ defines a bounded uniformly continuous function which solves by construction

$$((-\Delta)^{\frac{\alpha}{2}} + c)(\mathcal{K}_c * g) = g \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{3.5.4}$$

cf. [83, Lemma 7.1, eq. (7.16)]. Note also that by the properties of convolutions we have that both $\mathcal{K}_c * g_1$ and $\mathcal{K}_c * g_2$ vanish at infinity, i.e., $\mathcal{K}_c * g$ vanishes as well. Then, by combining (3.5.2) with (3.5.4) we infer that u_V can be written as

$$u_V = \mathcal{K}_c * g + \bar{u}, \tag{3.5.5}$$

where $\bar{u}$ solves the eigenvalue problem

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u} = -c\bar{u} \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad c = \|u_V\|_{L^\infty(\mathbb{R}^d)}^{\frac{2-q}{q-1}} > 0. \tag{3.5.6}$$

Next, we prove that $\bar{u} \equiv 0$. First of all, from (3.5.5) we conclude that $\bar{u} \in L^\infty(\mathbb{R}^d)$ and $\bar{u} \to 0$ as $|x| \to \infty$. Then, by Proposition 3.3.1 we conclude that $\bar{u} \in C^\infty(\mathbb{R}^d)$ and so (3.5.6) is satisfied pointwisely. Namely,

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u}(x) = \frac{C_{d,2s}}{2} \int_{\mathbb{R}^d} \frac{2\bar{u}(x) - \bar{u}(x+y) - \bar{u}(x-y)}{|y|^{d+2s}} dy = -c\bar{u}(x) \quad \forall x \in \mathbb{R}^d. \tag{3.5.7}$$

Then, we consider three possibilities:

(*i*) $\bar{u} \geq 0$;

(*ii*) $\bar{u} \leq 0$;

(*iii*) $\bar{u}$ is sign changing.

If (*i*) holds, since $\bar{u} \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\bar{u} \to 0$ as $|x| \to +\infty$, we deduce that either $\bar{u} \equiv 0$ or there exists a positive global maximum at $\bar{x}$. If this is the case, from (3.5.7) we get

$$0 < (-\Delta)^{\frac{\alpha}{2}} \bar{u}(\bar{x}) = -c\bar{u}(\bar{x}) < 0,$$

contradiction. Hence, $\bar{u} \equiv 0$.
If (*ii*) holds, either $\bar{u} \equiv 0$ or there exists a negative global minimum at $\bar{x}$. Then, again by (3.5.7) we obtain

$$0 < -c\bar{u}(\bar{x}) = (-\Delta)^{\frac{\alpha}{2}} \bar{u}(\bar{x}) < 0,$$

contradiction. Hence also in this case we have that $\bar{u} \equiv 0$.
Finally, if (iii) holds we get that there exist a negative global minimum and a positive global maximum. This leads to a contradiction by arguing as in the previous cases. We have therefore proved that $\bar{u} \equiv 0$ and so, cf. (3.5.5), u_V satisfies

$$u_V(x) = \int_{\mathbb{R}^d} \mathcal{K}_c(|x-y|)g(y)dy \quad \forall x \in \mathbb{R}^d. \tag{3.5.8}$$

From (3.5.8), non negativity of g and positivity of $\mathcal{K}_c$ we derive positivity of u_V. $\square$

Lemma 3.5.1. *Let $f \in L^1(\mathbb{R}^d)$ be non negative and φ be a radially non increasing function defined on $\mathbb{R}^d$. If $f(x) \leq \varphi(|x|)$ for every $x \in \mathbb{R}^d$ and*

$$\lim_{|x|\to+\infty} \varphi(|x|)|x|^d = 0,$$

then

$$(I_\alpha * f)(x) = \frac{A_\alpha \|f\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty.$$

Proof. Let $0 \neq x \in \mathbb{R}^d$. We split $\mathbb{R}^d$ by the following sets $B := \{y : |y-x| < |x|/2\}$, $A := \{y \notin B : |y| \leq |x|\}$, $C := \{y \in B : |y| > |x|\}$. Then,

$$\left|\int_{A\cup C} f(y)\left(\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\right)dy\right| \leq \int_{A\cup C} f(y)\left|\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\right|dy.$$

If $y \in A$, from $|x|/2 \leq |y-x| \leq 2|x|$ and the Mean Value theorem, there exists $c \in (0,1]$ such that

$$\begin{aligned}\left|\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\right| &\leq |\nabla g((1-c)x + c(x-y))||y| \\ &\leq \frac{(d-\alpha)|y|}{|x-cy|^{d-\alpha+1}} \leq \frac{(d-\alpha)(2^{d-\alpha+1})|y|}{|x|^{d-\alpha+1}}.\end{aligned} \tag{3.5.9}$$

where $g(z) := \frac{1}{|z|^{d-\alpha}}$. Thus, if $y \in A$

$$\left|\int_A f(y)\left(\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\right)dy\right| \leq \frac{(d-\alpha)2^{d-\alpha+1}}{|x|^{d-\alpha+1}} \int_A f(y)|y|dy =: I_1.$$

On the other hand, if $y \in C$, since $|y-x| > |x|/2$,

$$\left|\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\right| \leq \frac{1}{|x|^{d-\alpha}}$$

from which,

$$\left|\int_C f(y)\left(\frac{1}{|x-y|^{d-\alpha}}-\frac{1}{|x|^{d-\alpha}}\right)dy\right| \leq \frac{1}{|x|^{d-\alpha}}\int_C f(y)dy =: I_3.$$

Moreover

$$\left|\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\alpha}}-\frac{||f||_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}}\right| \leq I_1+\underbrace{\int_B \frac{f(y)}{|x-y|^{d-\alpha}}dy}_{=:I_2}+I_3.$$

Now we can estimate

$$I_1 \leq \frac{\omega_d(d-\alpha)2^{d-\alpha+1}}{|x|^{d-\alpha+1}}\int_{r=0}^{|x|}\varphi(r)r^d dr$$

and

$$\int_{r=0}^{|x|}\varphi(r)r^d dr = o(|x|) \quad \text{as } |x| \to +\infty.$$

Thus,

$$I_1 = o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty.$$

As regards I_2, by monotonicity of φ we have

$$\begin{aligned} I_2 &\leq \int_{|y-x|\leq\frac{|x|}{2}}\frac{\varphi(|y|)}{|x-y|^{d-\alpha}}dy \leq \varphi\left(\frac{|x|}{2}\right)\int_{|y-x|\leq\frac{|x|}{2}}\frac{dy}{|x-y|^{d-\alpha}} \\ &= \omega_d\,\varphi\left(\frac{|x|}{2}\right)\int_{r=0}^{|x|}\frac{dr}{r^{1-\alpha}} = \begin{cases}\frac{\omega_d}{\alpha}\varphi\left(\frac{|x|}{2}\right)|x|^\alpha & \text{if } \alpha\neq 1,\\ \omega_d\varphi\left(\frac{|x|}{2}\right)\log(|x|) & \text{if } \alpha = 1\end{cases} \longrightarrow 0, \end{aligned} \tag{3.5.10}$$

where by ω_d we denoted the surface area of the unitary ball in $\mathbb{R}^d$. Finally, since $f \in L^1(\mathbb{R}^d)$ also I_3 goes to zero which concludes the proof. □

Remark 3.5.2. Note that Lemma 3.5.1 it's not in general valid for sign changing functions where the decay may be faster. To see this, it's enough to consider a function f such that $|f|$ satisfies the assumptions of Lemma 3.5.1 and

$$\int_{\mathbb{R}^d} f(x)dx = \|[f]_+\|_{L^1(\mathbb{R}^d)} - \|[f]_-\|_{L^1(\mathbb{R}^d)} = 0. \tag{3.5.11}$$

Then, by applying Lemma 3.5.1 to $[f]_+$ and $[f]_-$ we obtain that

$$\begin{aligned}(I_\alpha * [f]_+)(x) &= \frac{A_\alpha \,\|[f]_+\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty,\\ (I_\alpha * [f]_-)(x) &= \frac{A_\alpha \,\|[f]_-\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty.\end{aligned} \tag{3.5.12}$$

Consequently, from (3.5.11) and (3.5.12)

$$(I_\alpha * f)(x) = (I_\alpha * [f]_+)(x) - (I_\alpha * [f]_-)(x) = o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty.$$

Now we state two useful results proved in [55]. We also refer to [5] for the definitions and properties of Hypergeometric and Gamma functions that will be used in the sequel.

Theorem 3.5.1. *Assume $d \geq 2$, $\alpha \in (0,2)$. Then, for every radial function $u \in C^2(\mathbb{R}^d) \cap \mathcal{L}^1_\alpha(\mathbb{R}^d)$ the following equality holds*

$$(-\Delta)^{\frac{\alpha}{2}} u(r) = c_{d,\alpha} r^{-\alpha} \int_{\tau=1}^{\infty} \Big[u(r) - u(r\tau) + (u(r) - u(r/\tau))\,\tau^{-d+\alpha}\Big] \tau(\tau^2-1)^{-1-\alpha} H(\tau) d\tau, \tag{3.5.13}$$

where $c_{d,\alpha}$ is a positive constant and $H(\tau)$ is a positive continuous function such that $H(\tau) \sim \tau^\alpha$ as τ goes to infinity.

Next, we refer again to [55, eq. (4.4)] for Lemma 3.5.2 below, providing the behaviour at infinity of the fractional Laplacian of polynomial behaving functions. We further refer to Lemma 3.6.1 for a similar result providing the decay of the fractional Laplacian a power–type function for a wider range of exponents.

Lemma 3.5.2. *Let $d \geq 2$, $0 < \alpha < 2$ and β be a positive number. Let $g_\beta : \mathbb{R}^d \to \mathbb{R}$ be the function defined by $g_\beta(x) := (1 + |x|^2)^{-\frac{\beta}{2}}$. If $d - \alpha < \beta < d$ then*

$$(-\Delta)^{\frac{\alpha}{2}} g_\beta(x) \simeq -(1 + |x|^2)^{-\frac{\alpha+\beta}{2}} \quad \textit{as } |x| \to +\infty.$$

Before investigating the first regime corresponding to the interval $\left(\frac{2d}{d+\alpha}, \frac{2d-\alpha}{d}\right)$ we recall some well known results.

If $\alpha \in (0,2)$ we define the following quantity

$$\mathcal{I} := \inf_{u \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} \left\{ \|u\|^2_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} : u \geq \chi_{B_1} \right\}. \tag{3.5.14}$$

It can proved that the infimum defined in (3.5.14) is achieved by a positive, continuous, and radially non increasing function $\bar{u}$. Moreover, $\bar{u}$ weakly solves

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}\bar{u} = 0 & \text{in } \overline{B_1}^c, \\ \bar{u} = 1 & \text{in } B_1, \end{cases} \tag{3.5.15}$$

and

$$\bar{u}(x) \simeq \frac{1}{|x|^{d-\alpha}} \quad \text{as } |x| \to +\infty. \tag{3.5.16}$$

For the proof we refer for example to [22, Propositions 3.5, 3.6].

We have now all the ingredients to deduce the desired asymptotic decay in the range $q \in \left(\frac{2d}{d+\alpha}, \frac{2d+\alpha}{d+\alpha}\right) \setminus \left\{\frac{2d-\alpha}{d}\right\}$.

Lemma 3.5.3. *Assume that* $(\mathcal{A})$ *holds and* $q < \frac{2d-\alpha}{d}$. *If there exists a non negative, radially non increasing function* $g : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\lim_{|x|\to+\infty} g(|x|)|x|^d = 0, \tag{3.5.17}$$

and

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \leq g(|x|), \tag{3.5.18}$$

then,

$$u_V(x) \simeq \frac{1}{|x|^{d-\alpha}} \quad \textit{as } |x| \to +\infty.$$

Proof. By (3.5.17)–(3.5.18) and $(\mathcal{A})$, we have that $(-\Delta)^{\frac{\alpha}{2}}V \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In particular, $(I_\alpha * (-\Delta)^{\frac{\alpha}{2}}V)(x) = V(x)$ for every $x \in \mathbb{R}^d$ and, by Lemma 3.5.1,

$$V(x) = \frac{\left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5.19}$$

Moreover, in view of Proposition 3.5.1 we obtain positivity of u_V. In addition, by combining the poinwise Euler–Lagrange equation (3.1.9) with (3.5.19) there exists $R > 0$ large enough and a positive constant C depending on R such that

$$u_V(x) \leq C(1+|x|)^{-(d-\alpha)} \quad \text{in } \overline{B_R}^c. \tag{3.5.20}$$

Let δ be a positive real number satisfying $d - \alpha < \delta < d$, and $\lambda > 0$. We define the function

$$v_\lambda^\delta(x) := \lambda \left(\bar{u}(|x|) + (1+|x|^2)^{-\frac{\delta}{2}}\right), \tag{3.5.21}$$

where $\bar{u}$ is the solution of (3.5.15). Note that the inequality $q < \frac{2d-\alpha}{d}$ implies that $v_\lambda^\delta(x) \in L^{q'}(\mathbb{R}^d)$ and $(v_\lambda^\delta)^{\frac{1}{q-1}} \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d) \subset \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Moreover, by putting together [52, Corollary 1.4] with Lemma 3.5.2 we infer $v_\lambda^\delta \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Furthermore, there exist $R > 0$ large enough, c_1, c_2 and c_3 positive constants such that for all λ sufficiently small

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda^\delta + (v_\lambda^\delta)^{\frac{1}{q-1}} - (-\Delta)^{\frac{\alpha}{2}} V \le -c_1\lambda |x|^{-(\delta+\alpha)} + c_2 \lambda^{\frac{1}{q-1}} |x|^{-\frac{d-\alpha}{q-1}} \le 0 \quad \text{in } \overline{B_R}^c, \tag{3.5.22}$$

provided δ satisfies $d-\alpha < \delta < (d-\alpha)/(q-1) - \alpha$. Such δ exists again by the restriction $q < \frac{2d-\alpha}{d}$. Hence, by positivity of u_V (Proposition 3.5.1) we can find $\lambda_0 > 0$ small such that (3.5.22) holds and

$$\sup_{x\in\overline{B_R}} v_{\lambda_0}^\delta(x) \le \inf_{x\in\overline{B_R}} u(x). \tag{3.5.23}$$

Finally, choosing $\lambda \le \lambda_0$ satisfying (3.5.22) yields

$$v_\lambda^\delta \le u_V \quad \text{in } \overline{B_R}. \tag{3.5.24}$$

By Lemma 3.2.1 we infer

$$v_{\lambda_0}^\delta \le u_V \quad \text{in } \mathbb{R}^d, \tag{3.5.25}$$

that combined with (3.5.20) yields the thesis. □

Lemma 3.5.4. *Assume that* ($\mathcal{A}$) *holds and* $\frac{2d-\alpha}{d} < q < \frac{2d+\alpha}{d+\alpha}$. *If*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^{\frac{\alpha}{2-q}}} \quad \text{as } |x| \to +\infty, \tag{3.5.26}$$

then,

$$u_V(x) \simeq \frac{1}{|x|^{\frac{\alpha(q-1)}{2-q}}} \quad \text{as } |x| \to +\infty.$$

Proof. Let λ be a positive constant and $v_\lambda(x) := \lambda(1+|x|^2)^{-\frac{\alpha(q-1)}{2(2-q)}}$. From Lemma

3.5.2 and [61, Proposition 2.15], $(-\Delta)^{\frac{\alpha}{2}} v_\lambda$ is a continuous function such that

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda(x) \simeq -|x|^{-\frac{\alpha(q-1)}{(2-q)}-\alpha} \in L^{\frac{2d}{d+\alpha}}(\overline{B_1}^c).$$

Again by [52, Corollary 1.4] we therefore have that $v_\lambda \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$. From Lemma 3.5.2, there exist $R_1 > 0$ large, $c_1, c_3 < 0$, $c_2 > 0$ such that for all $\lambda > 0$ large enough

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda + b v_\lambda^{\frac{1}{q-1}} - (-\Delta)^{\frac{\alpha}{2}} V \geq \left(\lambda c_1 + \lambda^{\frac{1}{q-1}} c_2 + c_3\right) |x|^{-\frac{\alpha}{2-q}} \geq 0 \quad \text{in } \overline{B_{R_1}}^c. \tag{3.5.27}$$

Taking into account boundedness of u_V, there exists λ_1 large such that (3.5.27) holds and

$$\inf_{x \in \overline{B_{R_1}}} v_{\lambda_1}(x) \geq \sup_{x \in \overline{B_{R_1}}} u_V(x).$$

In view of Lemma 3.2.1 applied to $\Omega = \overline{B_{R_1}}^c$, we get

$$u_V(x) \leq v_{\lambda_1}(x) \quad \text{in } \mathbb{R}^d.$$

Similarly, again from Lemma 3.5.2, there exist $R_2 > 0$ large such that for all $\lambda > 0$ sufficiently small

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda + b v_\lambda^{\frac{1}{q-1}} - (-\Delta)^{\frac{\alpha}{2}} V \leq 0 \quad \text{in } \overline{B_{R_2}}^c. \tag{3.5.28}$$

Then, by Proposition 3.5.1 we can choose $\lambda_2 > 0$ small such that (3.5.28) is valid and

$$\sup_{x \in \overline{B_{R_2}}} v_{\lambda_1}(x) \leq \inf_{x \in \overline{B_{R_2}}} u_V(x).$$

From Lemma 3.2.1 we deduce that

$$u_V(x) \geq v_{\lambda_2}(x) \quad \text{in } \mathbb{R}^d.$$

This concludes the proof of Lemma 3.5.4. □

Before investigating the range $\frac{2d+\alpha}{d+\alpha} < q < 2$ we study the critical cases $q_1 = \frac{2d-\alpha}{d}$ and $q_2 = \frac{2d+\alpha}{d+\alpha}$. In Lemma 3.5.5 and 3.5.6, we derive the decay of the minimizer ρ_V at the critical values $q = \frac{2d-\alpha}{d}$ and $q = \frac{2d+\alpha}{d+\alpha}$. For the particular case $d = 2$, $\alpha = 1$

and $q = \frac{3}{2}$ we refer to [89, Proposition 4.8].

Lemma 3.5.5. *Assume that* ($\mathcal{A}$) *holds and* $q = \frac{2d-\alpha}{d}$. *Assume that either* $\alpha \in (1,2)$ *and* $d > \alpha + 1$ *or* $q = \frac{3}{2}$. *If*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^d (\log |x|)^{\frac{d}{\alpha}}} \quad \text{as } |x| \to +\infty, \tag{3.5.29}$$

then

$$u_V(x) \simeq \frac{1}{|x|^{d-\alpha} (\log |x|)^{\frac{d-\alpha}{\alpha}}} \quad \text{as } |x| \to +\infty. \tag{3.5.30}$$

Proof. Let $\lambda > 0$. We define $v_\lambda(x) := \lambda U(|x|)$ where $U(|x|)$ is the barrier function defined in Lemma 3.6.2, and $b := \frac{d-\alpha}{\alpha}$. First of all we notice that $(-\Delta)^{\frac{\alpha}{2}} v_\lambda \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \subset \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. In particular, $v_\lambda = I_\alpha * (-\Delta)^{\frac{\alpha}{2}} v_\lambda \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$. Moreover, it's easy to see that $v_\lambda \in L^{q'}(\mathbb{R}^d)$. Now, we argue like in the proof of Lemma 3.5.4. Consequently, we can find $R > 0$ sufficiently large such that for all $\lambda > 0$ large enough

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda + v_\lambda^{\frac{d}{d-\alpha}} - (-\Delta)^{\frac{\alpha}{2}} V \geq 0 \quad \text{in } \overline{B_R}^c,$$

and

$$\lambda \inf_{x \in \overline{B_R}} U(|x|) \geq \sup_{x \in \overline{B_R}} u(x). \tag{3.5.31}$$

Thus, from Lemma 3.2.1 applied to $\Omega = \overline{B_R}^c$ we obtain

$$u(x) \leq v_{\lambda_1}(x) \quad \text{in } \mathbb{R}^d. \tag{3.5.32}$$

On the other hand, there exist $R_0 > 0$ large, such that for all $\lambda > 0$ sufficiently small we have

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda + b v_\lambda^{\frac{d}{d-\alpha}} - (-\Delta)^{\frac{\alpha}{2}} V \leq 0 \quad \text{in } \overline{B_{R_0}}^c$$

and

$$\sup_{x \in \overline{B_{R_0}}} v_\lambda(x) \leq \inf_{x \in \overline{B_{R_0}}} u(x). \tag{3.5.33}$$

Again from Lemma 3.2.1 we have

$$u(x) \geq v_\lambda(x) \quad \text{in } \mathbb{R}^d. \tag{3.5.34}$$

By putting together inequality (3.5.32) with (3.5.34) we get (3.5.30). □

Remark 3.5.3. Note that, in view of Remark 3.6.1, we still obtain a lower bound (respectively upper bound) on the decay rate of u_V if $\alpha \in (1,2)$ (respectively $d > \alpha + 1$). Namely, under the assumptions $(\mathcal{A})$ and $q = \frac{2d-\alpha}{d}$ the following occur:

(i) If either $\alpha \in (1,2)$ and $d \geq 2$, or $q = \frac{3}{2}$ then

$$u_V(x) \lesssim \frac{1}{|x|^{d-\alpha}(\log|x|)^{\frac{d-\alpha}{\alpha}}} \quad as\ |x| \to +\infty; \tag{3.5.35}$$

(ii) If either $\alpha \in (0,2)$ and $d > \alpha + 1$, or $q = \frac{3}{2}$ then

$$u_V(x) \gtrsim \frac{1}{|x|^{d-\alpha}(\log|x|)^{\frac{d-\alpha}{\alpha}}} \quad as\ |x| \to +\infty. \tag{3.5.36}$$

Up to our knowledge, the following asymptotic result is new and was never observed before.

Lemma 3.5.6. *Assume that $(\mathcal{A})$ holds and $q = \frac{2d+\alpha}{d+\alpha}$. If*

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{(\log(e|x|))^{\frac{d+\alpha}{\alpha}}}{|x|^{d+\alpha}} \quad as\ |x| \to +\infty, \tag{3.5.37}$$

then

$$u_V(x) \simeq \frac{(\log|x|)^{\frac{d}{\alpha}}}{|x|^d} \quad as\ |x| \to +\infty.$$

Proof. The proof uses Lemma 3.6.3 and follows the same lines of Lemma 3.5.5. □

We now investigate the range $\frac{2d+\alpha}{d+\alpha} < q < 2$.

Lemma 3.5.7. *Assume $\gamma > d$. Let $U \in C^2(\mathbb{R}^d)$ be a decreasing function of the form*

$$U(|x|) := \begin{cases} |x|^{-\gamma} & if\ |x| \geq 1, \\ v(|x|) & if\ |x| < 1, \end{cases} \tag{3.5.38}$$

where $v \in C^2(\overline{B_1})$ is positive and radially decreasing. Then,

$$(-\Delta)^{\frac{\alpha}{2}}U(x) \simeq -\frac{1}{|x|^{d+\alpha}} \quad as\ |x| \to +\infty.$$

Proof. As usual we use Theorem 3.5.1 and we split the integral on the right hand side of (3.5.13) (up to the constant $c_{d,\alpha}$) into three parts I_1^2, I_2^r and I_r^∞ respectively

$$I_1^2 := \int_{\tau=1}^2 (\cdot),\ I_2^r := \int_{\tau=2}^r (\cdot),\ I_r^\infty := \int_{\tau=r}^\infty (\cdot). \tag{3.5.39}$$

First of all let's notice that for every $\tau \in [1, r]$

$$\begin{aligned} \phi(\tau, r) &= U(r) - U(r\tau) + \left(U(r) - U(r/\tau)\right)\tau^{-d+\alpha} \\ &= r^{-\gamma}(1 - \tau^{-\gamma} - \tau^{-d+\alpha+\gamma} + \tau^{-d+\alpha}). \end{aligned} \tag{3.5.40}$$

Next, we prove that

$$\phi(\tau, r) \le 0 \quad \forall \tau \in [1, r]. \tag{3.5.41}$$

By (3.5.40), (3.5.41) is equivalent to

$$\tau^{-\gamma} \le \tau^{-d+\alpha},$$

that is true by the assumption on γ. Thus,

$$\begin{aligned} I_2^r &= r^{-\gamma-\alpha} \int_{\tau=2}^r \left(1 - \tau^{-\gamma} + \tau^{-d+\alpha}(1 - \tau^\gamma)\right)(\tau - 1)^{-1-\alpha} F(\tau) d\tau \\ &\simeq r^{-\gamma-\alpha} \int_{\tau=2}^r \left(1 - \tau^{-\gamma} + \tau^{-d+\alpha}(1 - \tau^\gamma)\right)\tau^{-1-\alpha} d\tau \\ &= -\frac{r^{-d-\alpha}}{\gamma - d} + o(r^{-d-\alpha}) \quad \text{as } r \to +\infty, \end{aligned} \tag{3.5.42}$$

where by F we denoted the function

$$F(\tau) := \tau(\tau + 1)^{-1-\alpha} H(\tau),$$

and H is defined in [55, Theorem 1.1]. Furthermore, by Taylor's formula, there exists $\xi_\tau \in [1, \tau]$ such that

$$I_1^2 = r^{-\gamma-\alpha} \int_{\tau=1}^2 \frac{\phi''(1)}{2}(\tau - 1)^{-1-\alpha} F(\tau) d\tau + r^{-\gamma-\alpha} \int_{\tau=1}^2 \frac{\phi'''(\xi_\tau)}{6}(\tau - 1)^{2-\alpha} F(\tau) d\tau. \tag{3.5.43}$$

Next, from the fact that

$$\sup_{t\in[1,2]} |\phi'''(t)| < \infty,$$

(3.5.43) implies

$$I_1^2 = o(r^{-d-\alpha}) \quad \text{as } r \to +\infty. \tag{3.5.44}$$

Moreover, by definition of U,

$$\begin{aligned} I_r^\infty &= \\ &= r^{-\gamma-\alpha} \int_{\tau=r}^{\infty} (1 - \tau^{-\gamma} + \tau^{-d+\alpha})(\tau-1)^{-1-\alpha} F(\tau) d\tau - r^{-\alpha} \int_{\tau=r}^{\infty} U(r/\tau)(\tau-1)^{-1-\alpha} F(\tau) d\tau \\ &= -r^{-\alpha} \int_{\tau=r}^{\infty} U(r/\tau)(\tau-1)^{-1-\alpha} F(\tau) d\tau + o(r^{-d-\alpha}) \quad \text{as } r \to +\infty. \end{aligned} \tag{3.5.45}$$

As a consequence, since U is decreasing and $(\tau-1)^{-1-\alpha}F(\tau)$ is a positive function such that $(\tau-1)^{-1-\alpha}F(\tau) \simeq \tau^{-1-\alpha}$ as $\tau \to +\infty$, we deduce the two sided estimate

$$-U(0) r^{-\alpha} \int_{\tau=r}^{\infty} \tau^{-d-1} d\tau + o(r^{-d-\alpha}) \lesssim I_r^\infty \lesssim -U(1) r^{-\alpha} \int_{\tau=r}^{\infty} \tau^{-d-1} d\tau + o(r^{-d-\alpha}). \tag{3.5.46}$$

The above two sided bound yields

$$I_r^\infty \simeq -r^{-d-\alpha} + o(r^{-d-\alpha}) \quad \text{as } r \to +\infty. \tag{3.5.47}$$

Putting together (3.5.42), (3.5.44) and (3.5.47) we derive the thesis.

□

As a consequence, we formulate the following result. For the sake of simplicity, we shorten the proof since it follows the same lines of the proof of Lemma 3.5.5 or 3.5.6.

Lemma 3.5.8. *Assume that* ($\mathcal{A}$) *holds and* $\frac{2d+\alpha}{d+\alpha} < q < 2$. *If*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \textit{as } |x| \to +\infty, \tag{3.5.48}$$

then

$$u_V(x) \simeq \frac{1}{|x|^{(d+\alpha)(q-1)}} \quad \textit{as } |x| \to +\infty.$$

Proof. First of all we notice that the function $U(x)$ defined in Lemma 3.5.7, with $\gamma := (d+\alpha)(q-1)$, belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$. Then, arguing as we did for example in the proof of Lemma 3.5.5 we obtain the thesis. □

3.5.2 Linear case: $q = 2$

As we have pointed out in Proposition 3.5.1, it's easy to deduce positivity of the minimizer ρ_V. Moreover, using some well known results we can prove the precise asymptotic decay of ρ_V.

Lemma 3.5.9. *Assume* $(\mathcal{A})$ *holds. If* $q = 2$ *and*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty$$

then

$$\rho_V(x) \simeq \frac{1}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty.$$

Proof. Since by assumption $(-\Delta)^{\frac{\alpha}{2}} V \geq 0$, the function $u_V (= \rho_V)$ weakly solves

$$(-\Delta)^{\frac{\alpha}{2}} u_V + u_V \geq 0 \quad \text{in } \overline{B_1}^c.$$

Now, from [53, Lemma 4.2] there exists a positive, continuous function v belonging to $H^\alpha(\mathbb{R}^d)$ weakly and pointwisely solving the equation

$$(-\Delta)^{\frac{\alpha}{2}} v + v = 0 \quad \text{in } \overline{B_1}^c,$$

and $v(x) \gtrsim \frac{1}{|x|^{d+\alpha}}$ as $|x| \to +\infty$. In particular, from positivity of u_V, for all $\lambda > 0$ small enough the function $v_\lambda(x) = \lambda v(x)$ satisfies

$$(-\Delta)^{\frac{\alpha}{2}} v_\lambda + v_\lambda = 0 \quad \text{in } \overline{B_1}^c \tag{3.5.49}$$

and $v_\lambda \leq u_V$ in B_1. Then, by Lemma 3.2.1 we obtain that

$$\frac{1}{|x|^{d+\alpha}} \lesssim v_\lambda(x) \leq u_V(x) \quad \text{as } |x| \to +\infty. \tag{3.5.50}$$

On the other hand, taking into account the decay assumptions on $(-\Delta)^{\frac{\alpha}{2}} V$, the function u_V weakly solves

$$(-\Delta)^{\frac{\alpha}{2}} u_V + u_V \leq \frac{C}{(1+|x|^2)^{\frac{d+\alpha}{2}}} \quad \text{in } \overline{B_R}^c \tag{3.5.51}$$

for some positive constant C, and $R > 0$ sufficiently large. Then, by [59, Lemma

A.1] there exists $v \in H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ such that $v = u_V$ in B_R and weakly sastisfying

$$(-\Delta)^{\frac{\alpha}{2}} v + v = \frac{C}{(1+|x|^2)^{\frac{d+\alpha}{2}}} \quad \text{in } \overline{B_R}^c. \tag{3.5.52}$$

By combining (3.5.51) with (3.5.52) we obtain that the function $U_V := u_V - v$ weakly solves

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_V + U_V \leq 0 & \text{in } \overline{B_R}^c, \\ U_V = 0 & \text{in } \overline{B_R}. \end{cases} \tag{3.5.53}$$

Then, by the comparison principle (see e.g. [59, Lemma A.3]) we conclude that $U_V \leq 0$ in $\mathbb{R}^d$. Finally, (see [59, Lemma 3.5 and A.1]) we recall that v satisfies

$$\limsup_{|x|\to+\infty} |x|^{d+\alpha} v(x) < \infty \tag{3.5.54}$$

The thesis follows by putting together (3.5.50) with (3.5.54) □

3.5.3 Superlinear case: $q > 2$

In the following subsection, we first show how to recover positivity of u_V (and so of ρ_V) when $q \in (2, +\infty)$. This result is strongly related to the non local nature of the fractional Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$. Indeed, such result fails in the local case of the Laplacian $-\Delta$ (see [46, Corollary 1.10, Remark 1.5]). We refer also to [53, Theorem 1.3] for a similar argument.

Proposition 3.5.2. *Let $\alpha \in (0, 2)$ and $V \in \left(\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right) \setminus \{0\}$ be such that $(-\Delta)^{\frac{\alpha}{2}} V \geq 0$. Then, the same conclusion of Proposition 3.5.1 holds.*

Proof. The idea is to obtain enough regularity in order to use the singular integral representation (1.3.3) for the fractional Laplacian. Notice that non negativity of ρ_V implies that $I_\alpha * \rho_V$ is always a convergent integral (see again Corollary 2.4.2). In what follows, we prove that u_V defined by (3.2.2), belongs to $C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$ for some $\varepsilon > 0$ (see (1.1.3) for the definition of the latter space). In particular, from such regularity, u_V is proved to be not only a weak (and so distributional) solution of (3.1.11) but also a pointwise solution of the latter. First of all we notice that, from non negativity of ρ_V and boundedness of V, the Euler–Lagrange equation (3.3.2) implies boundedness of ρ_V and $I_\alpha * \rho_V$. Hence, by [102, Proposition 2, 9] we have the following

(i) If $\alpha \leq 1$ then $I_\alpha * \rho_V \in C^{0,\beta}(\mathbb{R}^d)$ for every $\beta < \alpha$;

(ii) If $\alpha > 1$ then $I_\alpha * \rho_V \in C^{1,\beta}(\mathbb{R}^d)$ for every $\beta < \alpha - 1$.

Assume first that $q > 2$.

Case $0 < \alpha < 1$. In this case, by arguing as in Corollary 3.3.4 and Remark 3.3.2, we can improve (i) up to $I_\alpha * \rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, \beta_0)$, where β_0 is defined as

$$\beta_0 := \min\left\{1, \frac{\alpha(q-1)}{q-2}\right\}.$$

In particular, since $\frac{\alpha(q-1)}{q-2} > \alpha$ and V is regular, the Euler–Lagrange equation (3.1.9) implies that $u_V \in C^{0,\gamma}_{loc}(\mathbb{R}^d)$ for some $\alpha < \gamma < 1$. Thus, by [61, Proposition 2.15], formula (1.3.3) holds true and $(-\Delta)^{\frac{\alpha}{2}} u_V \in C(\mathbb{R}^d)$.

Case $1 < \alpha < 2$. By [102, Proposition 2, 9] we obtain that $I_\alpha * \rho_V \in C^{1,\beta}(\mathbb{R}^d)$ for every $\beta \in (0, \alpha - 1)$. Thus, from (3.1.9) and regularity of V we have that $u_V \in C^{1,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta \in (0, \alpha - 1)$. Now we consider three subcases.

(i) If $\alpha + \frac{1}{q-1} < 2$, we first apply Proposition 3.3.1 obtaining that $I_\alpha * \rho_V \in C^{1,\beta_1}_{loc}(\mathbb{R}^d)$ with $\beta_1 := \alpha + \frac{1}{q-1} - 1$. Then, putting together (3.1.9) with regularity of V, we derive that $u_V \in C^{1,\min\{\beta_1, \alpha-1+\varepsilon\}}_{loc}(\mathbb{R}^d)$. Since $\beta_1 > \alpha - 1$, [61, Proposition 2.15] again implies that formula (1.3.3) holds.

(ii) If $\alpha + \frac{1}{q-1} = 2$, we can choose $0 < \delta < \frac{1}{q-1}$ obtaining (arguing as in the above case) that $I_\alpha * \rho_V \in C^{1,\gamma^*}(\mathbb{R}^d)$, where $\gamma^* := \alpha + \frac{1}{q-1} - 1 - \delta$. Then $u_V \in C^{1,\min\{\gamma^*, \alpha-1+\varepsilon\}}_{loc}(\mathbb{R}^d)$.

(iii) If $2 < \alpha + \frac{1}{q-1} < 3$, arguing as in case ($i$), we deduce that $I_\alpha * \rho_V \in C^{2,\beta_2}_{loc}(\mathbb{R}^d)$ where $\beta_2 := \alpha + \frac{1}{q-1} - 2$. Then, taking into account (3.1.9) and regularity of V, we conclude again validity of (1.3.3) for $(-\Delta)^{\frac{\alpha}{2}} u_V$.

Note that since $q > 2$ and $\alpha \in (1, 2)$, cases (i), (ii) or (iii) are the only admissible ones.

Case $\alpha = 1$. This case follows by a similar argument.

Assume now that $q \leq 2$. If $0 < \alpha \leq 1$ we have that $I_\alpha * \rho_V \in C^{0,\beta}(\mathbb{R}^d)$ for every $\beta < \alpha$. In particular, from (3.1.9) and the regularity of V we derive that $\rho_V \in C^{0,\beta}_{loc}(\mathbb{R}^d)$ for every $\beta < \alpha$. Then, Proposition 3.3.1 yields $I_\alpha * \rho_V \in C^{[\beta+\alpha],\{\beta+\alpha\}}_{loc}(\mathbb{R}^d)$ provided $\beta + \alpha \notin \mathbb{N}$. By combining such regularity with the regularity of V and

(3.1.9), we conclude that there exists $\varepsilon > 0$ small enough such that $u_V \in C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d)$. The proof of the case $1 < \alpha < 2$ can be performed in an analogous way.

As a consequence of the above analysis, all of $u_V, I_\alpha * \rho_V$ and V belong to $C^{\alpha+\varepsilon}_{loc}(\mathbb{R}^d) \cap \mathcal{L}^1_\alpha(\mathbb{R}^d)$. Assume now by contradiction that there exists x_0 such that $u_V(x_0) = 0$ and consider a ball B around x_0. Then, by [54, Corollary 2.2.29] applied to $f := (-\Delta)^{\frac{\alpha}{2}} V - u_V^{\frac{1}{q-1}} \in L^1_{loc}(\mathbb{R}^d)$, (and continuity of the functions involved) we have that

$$(-\Delta)^{\frac{\alpha}{2}} u_V(x) = -u_V^{\frac{1}{q-1}}(x) + (-\Delta)^{\frac{\alpha}{2}} V(x) \quad \forall x \in B, \tag{3.5.55}$$

where in (3.5.55) the operator $(-\Delta)^{\frac{\alpha}{2}}$ is understood in the sense of (1.3.3). In particular, from non negativity of u_V, we have that

$$(-\Delta)^{\frac{\alpha}{2}} u_V(x_0) = -\frac{C_{d,\alpha}}{2} \int_{\mathbb{R}^d} \frac{u_V(x_0+y) + u_V(x_0-y)}{|y|^{d+\alpha}} dy < 0. \tag{3.5.56}$$

On the other hand, by definition of x_0, (3.5.55), (3.5.56) and non negativity of $(-\Delta)^{\frac{\alpha}{2}} V$, we obtain

$$0 \le (-\Delta)^{\frac{\alpha}{2}} V(x_0) < 0, \tag{3.5.57}$$

leading to a contradiction. We have therefore proved that u_V (and so ρ_V) is positive. This concludes the proof. □

Lemma 3.5.10. *Assume that* ($\mathcal{A}$) *holds and* $q > 2$. *If*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty$$

then

$$u_V(x) \simeq \frac{1}{|x|^{(d+\alpha)(q-1)}} \quad \text{as } |x| \to +\infty. \tag{3.5.58}$$

Proof. Let $\gamma := (d+\alpha)(q-1)$ and U be the function defined by

$$U(x) := \begin{cases} v(|x|) & \text{if } |x| \le 1 \\ |x|^{-\gamma}, & \text{if } |x| > 1, \end{cases} \tag{3.5.59}$$

where $v \in C^2(\overline{B_1})$ is such that U is $C^2(\mathbb{R}^d)$ and radially decreasing. By Lemma 3.5.7, there exists $c > 0$ and $\overline{R} > 0$, such that

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \le -\frac{c}{|x|^{d+\alpha}} \quad \text{if } |x| \ge \overline{R}. \tag{3.5.60}$$

Next, we claim that there exists $R > 0$ large, $\lambda = \lambda(R) > 0$ such that

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v_{\lambda(R),R} + (v_{\lambda(R),R})^{\frac{1}{q-1}} \leq 0 & \text{if } |x| > 1, \\ v_{\lambda(R),R}(x) \leq u_V(x) & \text{if } |x| \leq 1. \end{cases} \tag{3.5.61}$$

where $v_{\lambda,R}$ is defined by

$$v_{\lambda,R}(x) := \lambda U(Rx), \tag{3.5.62}$$

and U is defined in (3.5.59). It's well known (cf. [61, Lemma 2.6]) that

$$(-\Delta)^{\frac{\alpha}{2}} v_{\lambda,R}(x) = \lambda R^{\alpha}((-\Delta)^{\frac{\alpha}{2}} U)(Rx). \tag{3.5.63}$$

Then, if $R \geq \max\{\overline{R}, 1\}$, by combining (3.5.60) with (3.5.63) we infer

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} v_{\lambda,R}(x) + (v_{\lambda,R}(x))^{\frac{1}{q-1}} &\leq \left(-\frac{c\lambda}{R^d} + \frac{\lambda^{\frac{1}{q-1}}}{R^{d+\alpha}} \right) \frac{1}{|x|^{d+\alpha}} \quad \text{in } \overline{B_1}^c \\ &\leq 0 \quad \text{in } \overline{B_1}^c, \end{aligned} \tag{3.5.64}$$

provided $\lambda \geq (c^{-1}R^{-\alpha})^{\frac{q-1}{q-2}}$. In particular, if we consider $\lambda = \lambda(R) := (c^{-1}R^{-\alpha})^{\frac{q-1}{q-2}}$ then the inequality (3.5.64) holds true. Furthermore, if $|x| \leq 1$, by monotonicity of U and Proposition 3.5.2 we have that

$$v_{\lambda(R),R}(x) = \lambda(R)U(Rx) \leq (c^{-1}R^{-\alpha})^{\frac{q-1}{q-2}} U(0) \leq \inf_{x \in \overline{B_1}} u_V(x),$$

provided R is large enough. Putting together (3.5.64) with the above inequality we obtain (3.5.61). Next, it's easy to verify that for all $R > 0$ the function $v_{\lambda(R),R}$ belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$ and $v_{\lambda(R),R}^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Then, by applying Lemma 3.2.1 to $\Omega := \mathbb{R}^d$, we obtain that

$$u_V(x) \geq v_{\lambda(R),R}(x) \quad \forall x \in \mathbb{R}^d, \tag{3.5.65}$$

provided R is large enough. Next, we prove the upper bound. To this aim we employ again the function $v_{\lambda,R}$ defined in (3.5.62) and we claim that there exists

$R > 0$ small, $\lambda = \lambda(R) > 0$ and $\tilde{R} = \tilde{R}(R)$ such that

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v_{\lambda(R),R} + (v_{\lambda(R),R})^{\frac{1}{q-1}} \geq (-\Delta)^{\frac{\alpha}{2}} V & \text{if } |x| > \tilde{R}(R), \\ v_{\lambda(R),R}(x) \geq u_V(x) & \text{if } |x| \leq \tilde{R}(R). \end{cases} \tag{3.5.66}$$

By Lemma 3.5.7, there exists $c_1 > 0$ and $R_1 > 0$, such that

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \geq -\frac{c_1}{|x|^{d+\alpha}} \quad \text{if } |x| \geq R_1. \tag{3.5.67}$$

Furthermore, by the decay assumption on $(-\Delta)^{\frac{\alpha}{2}} V$ there exists $c_2 > 0$ and $R_2 > 0$ such that

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \leq \frac{c_2}{|x|^{d+\alpha}} \quad \text{if } |x| \geq R_2. \tag{3.5.68}$$

Hence, by combining (3.5.63), (3.5.67) with (3.5.68), if $|x| > \max\left\{\frac{1}{R}, \frac{R_1}{R}, R_2\right\}$ we conclude that

$$(-\Delta)^{\frac{\alpha}{2}} v_{\lambda,R} + (v_{\lambda,R})^{\frac{1}{q-1}} - (-\Delta)^{\frac{\alpha}{2}} V \geq \frac{1}{|x|^{d+\alpha}} \left(-\frac{\lambda c_1}{R^d} + \frac{\lambda^{\frac{1}{q-1}}}{R^{d+\alpha}} - c_2 \right). \tag{3.5.69}$$

By an explicit computation, if we choose $\lambda = \lambda(R) := (c_1^{-1} R^{-\frac{\alpha}{2}})^{\frac{q-1}{q-2}}$, there exists $\bar{R}$ small such that for every $R \leq \bar{R}$ the right hand side of (3.5.69) is positive. In particular, we can assume that R is small enough such that

$$\max\left\{\frac{1}{R}, \frac{R_1}{R}, R_2\right\} = R^{-1} \max\{R_1, 1\}.$$

Furthermore, if $|x| \leq R^{-1} \max\{R_1, 1\}$, monotonicity of U yields

$$v_{\lambda(R),R}(x) = \lambda(R) U(Rx) \geq (c_1^{-1} R^{-\frac{\alpha}{2}})^{\frac{q-1}{q-2}} (\max\{R_1, 1\})^{-(d+\alpha)(q-1)} \geq \|u_V\|_{L^\infty(\mathbb{R}^d)}, \tag{3.5.70}$$

provided R is small enough. We have therefore proved that (3.5.66) holds with $\tilde{R}(R) = R^{-1} \max\{R_1, 1\}$ and R sufficiently small. Hence, again by applying Lemma 3.2.1 we conclude that there exists $R > 0$ small such that

$$u_V(x) \leq v_{\lambda(R),R}(x) \quad \forall x \in \mathbb{R}^d. \tag{3.5.71}$$

By combining (3.5.65) with (3.5.71) we derive (3.5.58). □

3.5.4 Proof of Theorems 3.1.7 and 3.1.8.

Remark 3.5.4. For the sake of clarity, we highlights below all the decay assumptions on $(-\Delta)^{\frac{\alpha}{2}}V$ required to prove Lemmas 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.8, 3.5.9 and 3.5.10:

- If $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ we require (3.5.17) and (3.5.18);
- If $q = \frac{2d-\alpha}{d}$ and, either $\alpha \in (1,2)$ and $d > \alpha + 1$ or $q = \frac{3}{2}$, we require

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{1}{|x|^d(\log|x|)^{\frac{d}{\alpha}}} \quad \text{as } |x| \to +\infty;$$

- If $\frac{2d-\alpha}{d} < q < \frac{2d+\alpha}{d+\alpha}$ we require

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{1}{|x|^{\frac{\alpha}{2-q}}} \quad \text{as } |x| \to +\infty;$$

- If $q = \frac{2d+\alpha}{d+\alpha}$ we require

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{(\log|x|)^{\frac{d+\alpha}{\alpha}}}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty;$$

- If $q > \frac{2d+\alpha}{d+\alpha}$ we require

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty.$$

In particular, if

$$(-\Delta)^{\frac{\alpha}{2}}V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty, \tag{3.5.72}$$

all of the above assumptions are satisfied and so, for simplicity, in the statement of Theorems 3.1.7, 3.1.8 we require (3.5.72) instead of distinguishing between the different regimes of q.

Proof of Theorem 3.1.7.

Proof. By combining (3.1.15) with Lemma 3.5.1 we deduce that $V \in L^\infty(\mathbb{R}^d)$. Furthermore, by combining the results of Lemmas 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.8, 3.5.9

and 3.5.10 we obtain the desired decay estimates $(i), (ii), (iii), (iv)$ and (v). It remains only to prove that

$$\lim_{|x|\to+\infty} |x|^{\frac{d-\alpha}{q-1}} \rho_V(x) = \left[A_\alpha \left(\left\|(-\Delta)^{\frac{\alpha}{2}} V\right\|_{L^1(\mathbb{R}^d)} - \|\rho_V\|_{L^1(\mathbb{R}^d)}\right)\right]^{\frac{1}{q-1}} \tag{3.5.73}$$

provided $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$. To this aim, by the decay assumption (3.5.72), Lemma 3.5.1 implies that

$$V(x) = \frac{A_\alpha \left\|(-\Delta)^{\frac{\alpha}{2}} V\right\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5.74}$$

Moreover, (i) ensures that in such regimes of q the minimizer ρ_V satisfies the assumption of Lemma 3.5.1. In particular,

$$(I_\alpha * \rho_V)(x) = \frac{A_\alpha \|\rho_V\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha}} + o\left(\frac{1}{|x|^{d-\alpha}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5.75}$$

Combining (3.5.74) and (3.5.75) with the Euler–Lagrange equation (3.4.13) we obtain the following asymptotic expansion for ρ_V

$$\rho_V(x) = \frac{\left[A_\alpha \left(\left\|(-\Delta)^{\frac{\alpha}{2}} V\right\|_{L^1(\mathbb{R}^d)} - \|\rho_V\|_{L^1(\mathbb{R}^d)}\right)\right]^{\frac{1}{q-1}}}{|x|^{\frac{d-\alpha}{q-1}}} + o\left(\frac{1}{|x|^{\frac{d-\alpha}{q-1}}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5.76}$$

In order to conlude it's enough to notice that, by positivity of ρ_V and validity of (i), the coefficient of $|x|^{-\frac{d-\alpha}{q-1}}$ in the right hand side of (3.5.76) must be positive, concluding the proof. □

Next, we apply Theorem 3.1.7 to estimate the L^1–norm of the minimizer. As before, the decay assumption on $(-\Delta)^{\frac{\alpha}{2}} V$ in Corollary 3.5.1 can be weakened accordingly to the range of q.

Corollary 3.5.1. *Assume that* ($\mathcal{A}$) *holds. If*

$$(-\Delta)^{\frac{\alpha}{2}} V(x) \lesssim \frac{1}{|x|^{d+\alpha}} \quad \textit{as } |x| \to +\infty$$

then the following possibilities hold:

(*i*) *If* $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ *then*

$$\|\rho_V\|_{L^1(\mathbb{R}^d)} < \left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)};$$

(*ii*) *If* $q = \frac{2d-\alpha}{d}$ *and, either* $1 < \alpha < 2$ *or* $q = \frac{3}{2}$, *then*

$$\|\rho_V\|_{L^1(\mathbb{R}^d)} = \left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)};$$

(*iii*) *If* $q > \frac{2d-\alpha}{d}$ *then*

$$\|\rho_V\|_{L^1(\mathbb{R}^d)} = \left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)}.$$

Proof. Proceding as in the proof of Theorem 3.1.7, we obtain

$$\rho_V(x) = \frac{\left[A_\alpha\left(\left\|(-\Delta)^{\frac{\alpha}{2}}V\right\|_{L^1(\mathbb{R}^d)} - \|\rho_V\|_{L^1(\mathbb{R}^d)}\right)\right]^{\frac{1}{q-1}}}{|x|^{\frac{d-\alpha}{q-1}}} + o\left(\frac{1}{|x|^{\frac{d-\alpha}{q-1}}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5.77}$$

Thus, under the assumptions of Lemma 3.5.3, again from positivity of ρ_V we conclude validity of (*i*). On the other hand, Theorem 3.1.7–(*ii*)–(*iii*)–(*iv*)–(*v*) and Remarks 3.6.1–3.5.3 tell us that ρ_V decays faster than $|x|^{-\frac{d-\alpha}{q-1}}$ at infinity and so the coefficient of $|x|^{-\frac{d-\alpha}{q-1}}$ in the right hand side of (3.5.77) is zero. This completes the proof. □

In what follows, before studying the case when the minimizer is sign changing, we apply Corollary 3.5.1 to an important family of potentials V.

Corollary 3.5.2. *Let* $d \geq 2$, $0 < \alpha < d$ *and* V_Z *be the function defined by*

$$V_Z(x) := \frac{ZA_\alpha}{(1+|x|^2)^{\frac{d-\alpha}{2}}}, \tag{3.5.78}$$

where Z *is positive constant and* A_α *defined in* (1.4.2)*. The following statements hold:*

(*i*) *If* $\frac{2d}{d+\alpha} < q < \frac{2d-\alpha}{d}$ *then*

$$0 < \|\rho_{V_Z}\|_{L^1(\mathbb{R}^d)} < Z;$$

(*ii*) *If* $q = \frac{2d-\alpha}{d}$ *and, either* $1 < \alpha < 2$ *or* $q = \frac{3}{2}$, *then*

$$\|\rho_{V_Z}\|_{L^1(\mathbb{R}^d)} = Z;$$

(iii) If $q > \frac{2d-\alpha}{d}$ *then*

$$\|\rho_{V_Z}\|_{L^1(\mathbb{R}^d)} = Z.$$

Proof. By [41, Theorem 1], (see also [43, Theorem 1.1]) we deduce that V_Z defined by (3.5.78) belongs to $\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and solves

$$(-\Delta)^{\frac{\alpha}{2}} V_Z(x) = C(d,\alpha,Z) V_Z^{\frac{d+\alpha}{d-\alpha}}(x),$$

where $C(d,\alpha,Z)$ is a suitable normalisation costant. In particular, V_Z satisfies all the conditions of Corollary 3.5.1. The thesis follows combining

$$A_\alpha \left\|(-\Delta)^{\frac{\alpha}{2}} V_Z\right\|_{L^1(\mathbb{R}^d)} = \lim_{|x|\to+\infty} |x|^{d-\alpha} V_Z(x) = Z A_\alpha,$$

with Corollary 3.5.1. □

Next, we deduce asymptotic upper bounds for the absolute value of the minimizer ($|\rho_V|$) without requiring any sign restriction. In particular, notice that the upper bounds are the same as the one proved in Theorem 3.1.7.

Proof of Theorem 3.1.8.

Proof. By the decay assumption (3.1.17) on $(-\Delta)^{\frac{\alpha}{2}} V$ and the regularity of V, we conclude that $V = I_\alpha * ((-\Delta)^{\frac{\alpha}{2}} V)$. In particular $V(x) \lesssim (1+|x|)^{\alpha-d}$ (cf. Lemma 3.5.1). Moreover, by Lemma 3.4.2 applied to $f := (-\Delta)^{\frac{\alpha}{2}} V - \operatorname{sign}(u_V)|u_V|^{\frac{1}{q-1}}$, we infer that

$$(-\Delta)^{\frac{\alpha}{2}} |u_V| + |u_V|^{\frac{1}{q-1}} \le |(-\Delta)^{\frac{\alpha}{2}} V| \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{3.5.79}$$

Indeed, by combining the regularity of V with (3.1.17) we have $(-\Delta)^{\frac{\alpha}{2}} V \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d) \subset \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$. Thus, by combining (3.1.11) with Lemma 2.3.1 and (2.3.3), the function u_V satisfies

$$\langle u_V, \psi\rangle_{\mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} ((-\Delta)^{\frac{\alpha}{2}} V(x) - \rho_V(x))\psi(x)dx \quad \forall \psi \in \mathring{H}_c^{\frac{\alpha}{2}}(\mathbb{R}^d).$$

Moreover, since by assumption $|\rho_V| = |u_V|^{\frac{1}{q-1}} \in \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ and $|u_V| \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ (see (1.2.2)), the function $|u_V|$ weakly solves (by density)

$$(-\Delta)^{\frac{\alpha}{2}} |u_V| + |u_V|^{\frac{1}{q-1}} \le |(-\Delta)^{\frac{\alpha}{2}} V| \quad \text{in } \mathring{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d). \tag{3.5.80}$$

Notice that the PDE obtained by taking the equality in (3.5.80) satisfies a weak comparison principle in the spirit of Lemma 3.2.1. Then, if $q \neq 2$ it's enough to use the same upper barries provided in Lemmas 3.5.3, 3.5.4, 3.5.6, 3.5.8, 3.5.10 and Remark 3.5.3. On the other hand, If $q = 2$ we can argue as in Lemma 3.5.9. □

Remark 3.5.5. As we have already noticed for the proof of Theorem 3.1.7, the decay assumption (3.1.17) can be replaced with the ones of Lemmas 3.5.3, 3.5.4, 3.5.5, 3.5.6 and 3.5.8 accordingly to the range of q.

3.6 Appendix

In this appendix we first provide the asymptotic decay for the fractional Laplacian of power–type functions for any negative exponent and of power–type function with logarithmic corrections.

Lemma 3.6.1. *Let $d \geq 2$ and $0 < \alpha < 2$, $\gamma > 0$ and $\gamma \neq d - \alpha$. Let $U \in C^2(\mathbb{R}^d)$ be a decreasing function of the form*

$$U(|x|) := \begin{cases} |x|^{-\gamma} & \text{if } |x| \geq 1, \\ v(|x|) & \text{if } |x| < 1, \end{cases} \tag{3.6.1}$$

where $v \in C^2(\overline{B_1})$ is positive and radially decreasing. Then the following possibilities hold:

(*i*) *If $0 < \gamma < d - \alpha$ then*

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \simeq \frac{1}{|x|^{\gamma+\alpha}} \quad \text{as } |x| \to +\infty;$$

(*ii*) *If $d - \alpha < \gamma < d$ then*

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \simeq -\frac{1}{|x|^{\gamma+\alpha}} \quad \text{as } |x| \to +\infty;$$

(*iii*) *If $\gamma = d$ then*

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \simeq -\frac{\log(|x|)}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty;$$

(iv) If $\gamma > d$ then

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \simeq -\frac{1}{|x|^{d+\alpha}} \quad \textit{as } |x| \to +\infty.$$

In particular, if $\gamma > \frac{d-\alpha}{2}$ then $U \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$.

Proof. If $\gamma > d$ we directly refer to Lemma 3.5.7. If not, we apply Theorem 3.5.1 and we split the integral on the right hand side of (3.5.13) (up to the constant $c_{d,\alpha}$) into three terms I_1^2, I_2^r and I_r^∞ respectively

$$I_1^2 := \int_{\tau=1}^{2} (\cdot), \;\; I_2^r := \int_{\tau=2}^{r} (\cdot), \;\; I_r^\infty := \int_{\tau=r}^{\infty} (\cdot). \tag{3.6.2}$$

First of all let's notice that for every $\tau \in [1, r]$

$$\begin{aligned} \phi(\tau, r) &= U(r) - U(r\tau) + (U(r) - U(r/\tau))\, \tau^{-d+\alpha} \\ &= r^{-\gamma}(1 - \tau^{-\gamma} - \tau^{-d+\alpha+\gamma} + \tau^{-d+\alpha}). \end{aligned} \tag{3.6.3}$$

From (3.6.3) we see that

(1) If $\gamma > d - \alpha$ then $\phi(\tau, r) \leq 0 \quad \forall \tau \in [1, r]$;

(2) If $\gamma < d - \alpha$ then $\phi(\tau, r) \geq 0 \quad \forall \tau \in [1, r]$.

Thus, by arguing as in the proof of Lemma 3.5.7, if r is sufficiently large we obtain the following possibilities:

$$\begin{aligned} I_2^r &= r^{-\gamma-\alpha} \int_{\tau=2}^{r} \left(1 - \tau^{-\gamma} + \tau^{-d+\alpha}(1 - \tau^{\gamma})\right) (\tau - 1)^{-1-\alpha} F(\tau) d\tau \\ &\simeq r^{-\gamma-\alpha} \int_{\tau=2}^{r} \left(1 - \tau^{-\gamma} + \tau^{-d+\alpha}(1 - \tau^{\gamma})\right) \tau^{-1-\alpha} d\tau \\ &= \begin{cases} r^{-\gamma-\alpha} \left[\frac{2^{-\alpha}}{\alpha} - \frac{2^{-\gamma-\alpha}}{\gamma+\alpha} + \frac{2^{\gamma-d}}{\gamma-d} + \frac{2^{-d}}{d}\right] + o(r^{-\gamma-\alpha}), & \text{if } \gamma < d \\ -r^{-d-\alpha} \log(r) + o(r^{-d-\alpha} \log(r)), & \text{if } \gamma = d. \end{cases} \end{aligned} \tag{3.6.4}$$

Furthermore, if we denote by F the function

$$F(\tau) := \tau(\tau + 1)^{-1-\alpha} H(\tau),$$

and H is defined in [55, Theorem 1.1], there exists $\xi_\tau \in [1, \tau]$ such that

$$I_1^2 = r^{-\gamma-\alpha} \left[\int_{\tau=1}^{2} \frac{\phi''(1)}{2} (\tau-1)^{-1-\alpha} F(\tau) d\tau + \int_{\tau=1}^{2} \frac{\phi'''(\xi_\tau)}{6} (\tau-1)^{2-\alpha} F(\tau) d\tau \right]. \tag{3.6.5}$$

In particular,

$$I_1^2 = \begin{cases} C_\gamma r^{-\gamma-\alpha}, & \text{if } \gamma < d \\ o(r^{-d-\alpha} \log(r)), & \text{if } \gamma = d, \end{cases} \tag{3.6.6}$$

where C_γ is the coefficient of $r^{-\gamma-\alpha}$ in (3.6.5). Note that we stressed the dependence on γ since from (1)–(2) we have that C_γ is negative (respectively positive) if $\gamma > d-\alpha$ (respectively $\gamma < d - \alpha$). Next, similarly to the proof of Lemma 3.5.7 we obtain that

$$I_r^\infty = \begin{cases} o(r^{-\gamma-\alpha}), & \text{if } \gamma < d \\ o(r^{-d-\alpha} \log(r)), & \text{if } \gamma = d \end{cases} \tag{3.6.7}$$

Putting together (3.6.4), (3.6.6) and (3.6.7) we derive the asymptotic decay.

Finally, if $\gamma > \frac{d-\alpha}{2}$, from the decay rates just proved and continuity of $(-\Delta)^{\frac{\alpha}{2}} U$ (cf. [61, Proposition 2.15]), it's straightforward to see that $(-\Delta)^{\frac{\alpha}{2}} U \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)$. Thus, $U \in \mathring{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$ concluding the proof.

□

Next, we provide the proof of two technical results that are used for proving Lemma 3.5.5 and 3.5.6.

Lemma 3.6.2. *Assume that either $b = 1$, $\alpha \in (0, 2)$ and $d > \alpha$, or $b > 1$, $\alpha \in (1, 2)$ and $d > \alpha + 1$. Let $U \in C^2(\mathbb{R}^d)$ be a decreasing function of the form*

$$U(|x|) := \begin{cases} |x|^{-(d-\alpha)} (\log(e|x|))^{-b} & \text{if } |x| \geq 1, \\ v(|x|) & \text{if } |x| < 1, \end{cases}$$

where $v \in C^2(\overline{B_1})$ is positive and radially decreasing. Then,

$$(-\Delta)^{\frac{\alpha}{2}} U(|x|) \simeq -\frac{1}{|x|^d (\log|x|)^{b+1}} \quad \text{as } |x| \to +\infty. \tag{3.6.8}$$

Proof. We start proving the lower bound. We split the right hand side of (3.5.13) in

Theorem 3.5.1 (up to the constant $c_{d,\alpha}$) into three parts I_1^2, I_2^r and I_r^∞ respectively,

$$I_1^2 := \int_{\tau=1}^2 (\cdot),\ I_2^r := \int_{\tau=2}^r (\cdot),\ I_r^\infty := \int_{\tau=r}^\infty (\cdot). \tag{3.6.9}$$

For the convenience of the reader we also define

$$\Phi(\tau, r) := U(r) - U(r\tau) + (U(r) - U(r/\tau))\,\tau^{-d+\alpha}, \tag{3.6.10}$$

and we recall that for the entire proof r is assumed to be large enough (at least $r > 2$). We perform the proof by the following steps:

(i) We prove that there exists $M > 0$ independent on τ, r such that

$$r^{d-\alpha}\Phi(\tau, r) \geq -\frac{M\log(e^{b+2}r)}{(\log(er))^{b+2}}(\tau-1)^2 \quad \forall \tau \in [1,2]; \tag{3.6.11}$$

(ii) We prove that

$$\Phi(\tau, r) \leq 0 \quad \forall \tau \in [1,2]; \tag{3.6.12}$$

(iii) We prove that

$$I_2^r \simeq -\frac{1}{r^d(\log(r))^{b+1}}, \quad I_r^\infty = o\left(\frac{1}{r^d(\log(r))^{b+1}}\right) \quad \text{as } r \to +\infty. \tag{3.6.13}$$

Let's start with (i), whose proof is similar to the one of [89, Lemma 4.7].

If we set $A := \log(er)$ and $\tau := e^x$, the inequality (3.6.11) becomes equivalent to

$$\frac{M}{A^{b+1}}(e^x-1)^2 + \left[\frac{1}{A^b} - \frac{1}{(A-x)^b} + e^{(-d+\alpha)x}\left(\frac{1}{A^b} - \frac{1}{(A+x)^b}\right)\right]e^{-(1+\alpha)x} \geq 0. \tag{3.6.14}$$

Inequality (3.6.14) is trivially verified for $x = 0$.

Assume $x \neq 0$. In this case (3.6.14) holds if there exists $M > 0$ such that

$$M \geq \frac{A^{b+1}}{x^2}\left[\frac{1}{(A-x)^b} - \frac{1}{A^b} + e^{x(\alpha-d)}\left(\frac{1}{(A+x)^b} - \frac{1}{A^b}\right)\right]e^{-x(1+\alpha)} =: F(x, A). \tag{3.6.15}$$

Since by definition $x \in [0, \log(2)]$, we only need to focus on x small. As a matter of fact, assume $x \geq \varepsilon > 0$. Then,

$$\begin{aligned}|F(x,A)| &\leq \frac{A}{\varepsilon^2}\left|\left[\frac{1}{(1-\frac{x}{A})^b} - 1 + e^{-x(d-\alpha)}\left(\frac{1}{(1+\frac{x}{A})^b} - 1\right)\right]\right| \\ &\leq \frac{A}{\varepsilon^2}\left(\frac{1}{(1-\frac{x}{A})^b} - \frac{1}{(1+\frac{x}{A})^b}\right) \leq \frac{\log(2)}{\varepsilon^2}\left[\frac{A}{x}\left(\frac{1}{(1-\frac{x}{A})^b} - \frac{1}{(1+\frac{x}{A})^b}\right)\right] \\ &=: G\left(\frac{x}{A}\right).\end{aligned}$$

Now, it's convenient to set $t := x/A$. Then, if $A \leq L$ for some $L > 0$ we clearly obtain boundedness of F. It remains to show what happens when A goes to infinity (or equivalently when t goes to zero). In the latter case we have that

$$\lim_{t\to 0} G(t) = \frac{2b\log(2)}{\varepsilon^2} < +\infty. \tag{3.6.16}$$

Consequently, to obtain boundedness of F we only need to focus on x going to zero. To this aim, we rewrite F as

$$F(x,A) = F_1(x,A) + F_2(x,A),$$

where

$$\begin{aligned}F_1(x,A) &:= \frac{A}{x^2}\left(\frac{1}{(1-\frac{x}{A})^b} + \frac{1}{(1+\frac{x}{A})^b} - 2\right)e^{-x(d-\alpha)}, \\ F_2(x,A) &:= \frac{A}{x^2}(e^{-x(d+1)} - 1)\left(\frac{1}{(1+\frac{x}{A})^b} - 1\right).\end{aligned}$$

Since $A \geq 1$, we have

$$|F_1(x,A)| \leq \left(\frac{A}{x}\right)^2\left|\frac{1}{(1-\frac{x}{A})^b} + \frac{1}{(1+\frac{x}{A})^b} - 2\right| = G_1\left(\frac{x}{A}\right).$$

Let's set again $t := x/A$. Then, if $t \geq \varepsilon > 0$ the function F_1 defined above is clearly bounded. On the other hand,

$$\lim_{t\to 0} G_1(t) = b(b+1) < +\infty,$$

proving that F_1 is bounded. Next, we notice that the function F_2 can be written as the product

$$F_2(x,A) = G_2(x) \cdot G_3\left(\frac{x}{A}\right)$$

where

$$G_2(x) := \frac{e^{-x(d+1)} - 1}{x},$$

and

$$G_3\left(\frac{x}{A}\right) := \frac{A}{x}\left(\frac{1}{(1+\frac{x}{A})^b} - 1\right).$$

Thus, it's enough to deduce boundedness as x goes to zero and A goes to infinity. In this case, it's clear that G_2 is bounded as x goes to zero and, by setting again $t = x/A$, we obtain

$$\lim_{t\to 0} G_3(t) = -b,$$

infering boundedness of G_3 as well. In view of the above arguments, there exists a positive costant M satisfying (3.6.11).

Next, we prove (ii) for sufficiently large r, or equivalently

$$r^{d-\alpha}\Phi(\tau,r) = \frac{1}{\log(er)^b} - \frac{1}{\log(er/\tau)^b} + \left(\frac{1}{\log(er)^b} - \frac{1}{\log(er\tau)^b}\right)\tau^{-d+\alpha} \leq 0 \quad \forall \tau \in [1,2]. \tag{3.6.17}$$

By simple computations, (3.6.17) is equivalent to

$$\left(\frac{\log(er)^b - \log(er\tau)^b}{\log(er\tau)^b - \log(er)^b}\right)\frac{\log(er\tau)^b}{\log(er/\tau)^b} \geq \tau^{-d+\alpha} \quad \forall \tau \in [1,2]. \tag{3.6.18}$$

If we set $s := \frac{\log(\tau)}{\log(er)}$, the left hand side of (3.6.18) becomes

$$h(s) := \left(\frac{1-(1-s)^b}{(1+s)^b - 1}\right)\left(\frac{1+s}{1-s}\right)^b = 1 + (b+1)s + o(s) \quad \text{as } s \to 0^+. \tag{3.6.19}$$

In particular, there exists $\bar{s} > 0$ such that $h(s) \geq 1$ provided $s \leq \bar{s}$. Therefore, if we choose r large enough such that

$$\frac{\log(2)}{\log(er)} \leq \bar{s},$$

we infer

$$\frac{\log(\tau)}{\log(er)} \leq \frac{\log(2)}{\log(er)} \leq \bar{s} \quad \forall \tau \in [1,2]. \tag{3.6.20}$$

The inequality (3.6.20) implies

$$\left(\frac{\log(er)^b - \log(er\tau)^b}{\log(er\tau)^b - \log(er)^b}\right) \frac{\log(er\tau)^b}{\log(er/\tau)^b} \geq 1 \geq \tau^{-d+\alpha} \quad \forall \tau \in [1,2],$$

from which (3.6.12) follows.

Next, we investigate point (iii). To this aim, we start by estimating the quantity I_2^r. Assume first $b > 1$. In the sequel, we denote by F the function

$$F(\tau) := \tau(\tau+1)^{-1-\alpha} H(\tau),$$

where $H(\tau)$ is the function defined in Theorem 3.5.1, eq. (3.5.13). We further recall that $F(\tau)$ is continuous, positive and bounded on the whole of $\mathbb{R}^d$. Then, if we further set $\bar{H}(\tau) := (\tau-1)^{-1-\alpha} F(\tau)$ we have

$$\begin{aligned} I_2^r &= \\ &= r^{-d} \int_{\tau=2}^{r} \left[\frac{1}{(\log(er))^b} - \frac{1}{(\log(er/\tau))^b} + \tau^{-d+\alpha} \left(\frac{1}{(\log(er))^b} - \frac{1}{(\log(er\tau))^b} \right) \right] \bar{H}(\tau) d\tau \\ &\geq r^{-d} \int_{\tau=2}^{r} \left[\frac{1}{(\log(er))^b} - \frac{1}{(\log(er/\tau))^b} \right] \bar{H}(\tau) d\tau \\ &\gtrsim r^{-d} \int_{\tau=2}^{r} \left[\frac{1}{(\log(er))^b} - \frac{1}{(\log(er/\tau))^b} \right] \tau^{-1-\alpha} d\tau = r^{-d} \int_{\tau=2}^{r} [g(1) - g(\tau)]\, \tau^{-1-\alpha} d\tau. \end{aligned} \tag{3.6.21}$$

Hence, by Lagrange Theorem,

$$\begin{aligned} |g(\tau) - g(1)| &\leq \sup_{t\in[1,\tau]} |g'(t)|(\tau-1) = \sup_{t\in[1,\tau]} \left(\frac{b}{t(\log(er) - \log(t))^{b+1}} \right) \cdot (\tau - 1) \\ &\leq \frac{b(\tau-1)}{(\log(er) - \log(\tau))^{b+1}}. \end{aligned} \tag{3.6.22}$$

Then, putting together (3.6.21) with (3.6.22)

$$I_2^r \gtrsim -r^{-d} \int_{\tau=2}^{r} \frac{\tau^{-\alpha}}{(\log(er) - \log(\tau))^{b+1}} d\tau = -\frac{r^{-d}}{(\log(er))^{b+1}} \int_{\tau=2}^{r} \frac{\tau^{-\alpha}}{\left(1 - \frac{\log(\tau)}{\log(er)}\right)^{b+1}} d\tau$$

$$- \frac{r^{-d}}{(\log(er))^{b+1}} \int_{x=\log(2)}^{\log(r)} \frac{e^{(-\alpha+1)x}}{\left(1 - \frac{x}{\log(er)}\right)^{b+1}} dx,$$

where $\tau = e^x$. Then, we claim that

$$\lim_{r\to+\infty}\int_{x=\log(2)}^{\log(r)} \frac{e^{(-\alpha+1)x}}{\left(1-\frac{x}{\log(er)}\right)^{b+1}}dx = \int_{\log(2)}^{\infty} e^{(-\alpha+1)x}dx = \frac{2^{1-\alpha}}{\alpha-1}. \tag{3.6.23}$$

provided $\alpha > 1$. To this aim, we first prove that there exists $C > 0$ such that, for all r sufficiently large we have

$$\frac{e^{(-\alpha+1)x}}{\left(1-\frac{x}{\log(er)}\right)^{b+1}}\chi_{[\log(2),\log(r))}(x) \le Ce^{(-\alpha+1+\varepsilon)x}\chi_{[\log(2),\infty)}(x) \quad \forall x \in \mathbb{R}^d, \tag{3.6.24}$$

where $\varepsilon > 0$ is small enough. Indeed, if $0 < x < \log(2)$ or $x \ge \log(r)$ (3.6.24) is trivial. It remains only to consider the case $x \in [\log(2), \log(r))$. In this case we split the analysis in two subcases:

Case $\frac{\log(er)}{2} \le x < \log(r)$. In this regime, (3.6.24) is equivalent to the inequality

$$e^{-\varepsilon x} \le C\left(1-\frac{x}{\log(er)}\right)^{b+1} \quad \forall x \in \left[\frac{\log(er)}{2}, \log(r)\right). \tag{3.6.25}$$

Because of monotonicity in the x–variable of left hand side and right hand side of (3.6.25) we infer

$$\begin{aligned} e^{-\varepsilon x} &\le e^{-\frac{\varepsilon}{2}}r^{-\frac{\varepsilon}{2}} \\ &\le C\left(\frac{1}{\log(er)}\right)^{b+1} \le C\left(1-\frac{x}{\log(er)}\right)^{b+1} \quad \forall x \in \left[\frac{\log(er)}{2}, \log(r)\right), \end{aligned} \tag{3.6.26}$$

provided r is large enough.

Case $\log(2) \le x < \frac{\log(er)}{2}$. In this range, again by monotonicity we obtain that

$$e^{-\varepsilon x} \le 1 \le C\left(\frac{1}{2}\right)^{b+1} \le C\left(1-\frac{x}{\log(er)}\right)^{b+1} \quad \forall x \in \left[\log(2), \frac{\log(er)}{2}\right), \tag{3.6.27}$$

provided $C \ge 2^{b+1}$. Hence, by combining (3.6.26) with (3.6.27) we obtain that if $C \ge 2^{b+1}$, for all r sufficiently large (3.6.24) holds. Thus, since the right hand side of (3.6.24) belongs to $L^1(\mathbb{R}^d)$ and the left hand side of (3.6.24) converges to $e^{(-\alpha+1)x}$ for almost every $x \in \mathbb{R}^d$, Dominated convergence's theorem yields (3.6.23).

Then, by taking into account (3.6.23) we infer that

$$I_2^r \gtrsim -\frac{1}{r^d(\log(r))^{b+1}} \quad \text{as } r \to +\infty. \tag{3.6.28}$$

In what follows we shall prove the converse inequality of (3.6.28). If we define the functions $f_1(\tau, r) := \frac{1}{(\log(er/\tau))^b}$ and $f_2(\tau, r) := \frac{1}{(\log(er\tau))^b}$, by Lagrange Theorem, there exist $t_{\tau,r}, \bar{t}_{\tau,r} \in (1, \tau)$ such that

$$\begin{aligned} r^{d-\alpha}\Phi(\tau, r) &= \frac{1}{(\log(er))^b} - \frac{1}{(\log(er/\tau))^b} + \tau^{-d+\alpha}\left(\frac{1}{(\log(er))^b} - \frac{1}{(\log(er\tau))^b}\right) \\ &= f_1(1, r) - f_1(\tau, r) + \tau^{-d+\alpha}\left(f_2(1, r) - f_2(\tau, r)\right) \\ &= -\frac{b(\tau-1)}{t_{\tau,r}(\log(er) - \log(t_{\tau,r}))^{b+1}} + \tau^{-d+\alpha}\frac{b(\tau-1)}{\bar{t}_{\tau,r}(\log(er) + \log(\bar{t}_{\tau,r}))^{b+1}} \\ &\le -\frac{b(\tau-1)}{\tau(\log(er))^{b+1}} + \tau^{-d+\alpha}\frac{b(\tau-1)}{(\log(er))^{b+1}}. \end{aligned} \tag{3.6.29}$$

Then, if $d > \alpha + 1$, from (3.6.29) we obtain

$$\begin{aligned} & r^{d-\alpha}\int_{\tau=1}^{r} \Phi(\tau, r)(\tau-1)^{-1-\alpha}F(\tau)d\tau \le \\ & \le -\frac{b}{(\log(er))^{b+1}}\int_{\tau=1}^{r}(\tau-1)^{-\alpha}\tau^{-1}F(\tau)d\tau + \frac{b}{(\log(er))^{b+1}}\int_{\tau=1}^{r}(\tau-1)^{-\alpha}F(\tau)\tau^{-d+\alpha}d\tau \\ & = -\frac{b}{(\log(er))^{b+1}}\left[\int_{\tau=1}^{r}(\tau-1)^{-\alpha}(\tau^{-1} - \tau^{-d+\alpha})F(\tau)d\tau\right]. \end{aligned}$$

Since by assumption $\alpha < 2$, $d > \alpha + 1$ we have that

$$\lim_{\tau \to 1^+} \frac{\tau^{-1} - \tau^{-d+\alpha}}{\tau - 1} = d - \alpha - 1 > 0$$

and

$$\int_{\tau=1}^{1+\varepsilon}(\tau-1)^{-\alpha}(\tau^{-1} - \tau^{-d+\alpha})F(\tau)d\tau < +\infty, \quad \varepsilon > 0.$$

Finally, (from boundedness and positivity of F) we get

$$0 < \int_{\tau=1}^{r}(\tau-1)^{-\alpha}(\tau^{-1} - \tau^{-d+\alpha})F(\tau)d\tau \le \int_{\tau=1}^{\infty}(\tau-1)^{-\alpha}(\tau^{-1} - \tau^{-d+\alpha})F(\tau)d\tau < \infty.$$

We have therefore proved that

$$I_2^r \lesssim -\frac{1}{r^d(\log(r))^{b+1}} \quad \text{as } r \to +\infty \tag{3.6.30}$$

provided $d > \alpha + 1$. Then, by combining (3.6.28) with (3.6.30) we obtain

$$I_2^r \simeq -\frac{1}{r^d(\log(r))^{b+1}} \quad \text{as } r \to +\infty. \tag{3.6.31}$$

Now, we further claim that

$$I_r^\infty = o\left(\frac{1}{r^d(\log(r))^{b+1}}\right) \quad \text{as } r \to +\infty \tag{3.6.32}$$

which will complete the proof of $(i) - (ii) - (iii)$. Indeed,

$$\begin{aligned} r^d|I_r^\infty| &\lesssim r^{d-\alpha}\int_{\tau=r}^{+\infty} |\Phi(\tau,r)|\tau^{-1-\alpha}d\tau \leq \\ &\leq r^{d-\alpha}U(r)\int_{\tau=r}^{+\infty}\tau^{-1-\alpha}d\tau + r^{d-\alpha}\int_{\tau=r}^{+\infty}U(r\tau)\tau^{-1-\alpha}d\tau \\ &+ r^{d-\alpha}\int_{\tau=r}^{+\infty}(v(r/\tau)-U(r))\tau^{-1-d}d\tau \\ &\lesssim r^{d-\alpha}U(r)\int_{\tau=r}^{+\infty}\tau^{-1-\alpha}d\tau + r^{d-\alpha}\int_{\tau=r}^{+\infty}\tau^{-1-d}d\tau \lesssim r^{-\alpha} \quad \text{as } r \to +\infty. \end{aligned}$$

Validity of (3.6.8) is therefore obtained since, by combining $(i) - (ii) - (iii)$,

$$\begin{aligned} -\frac{1}{r^d(\log(r))^{b+1}} + o\left(\frac{1}{r^d(\log(r))^{b+1}}\right) &\lesssim \\ \lesssim I_1^2 + I_2^r + I_r^\infty &\lesssim -\frac{1}{r^d(\log(r))^{b+1}} + o\left(\frac{1}{r^d(\log(r))^{b+1}}\right). \end{aligned} \tag{3.6.33}$$

Assume now $b = 1$. In this case, the function $\phi(\tau, r) \leq 0$ for every $\tau \in [1, r]$ and moreover it can be directly computed the limit

$$\lim_{r\to+\infty}(\log(r))^2\int_{\tau=2}^{r}\phi(\tau,r)\tau^{-1-\alpha}d\tau = \frac{2^{-d}}{d^2} + \frac{2^{-d}\log(2)}{d} - \frac{2^{-\alpha}(1+\alpha\log(2))}{\alpha^2}. \tag{3.6.34}$$

See [89, Lemma 4.7] for an explicit computation in the case $d = 2$ and $\alpha = 1$. From (3.6.34) we obtain the thesis. □

Remark 3.6.1. Note that, if $b \geq 1$, $\alpha \in (1,2)$, $d \geq 2$ we obtained the following lower bound on the decay given by

$$(-\Delta)^{\frac{\alpha}{2}} U(|x|) \gtrsim -\frac{1}{|x|^d (\log |x|)^{b+1}} \quad \text{as } |x| \to +\infty,$$

without the further restriction $d > \alpha + 1$ stated in Lemma 3.6.2. Similarly, if $b \geq 1$, $\alpha \in (0,2)$ and $d > \alpha + 1$ we have

$$(-\Delta)^{\frac{\alpha}{2}} U(|x|) \lesssim -\frac{1}{|x|^d (\log |x|)^{b+1}} \quad \text{as } |x| \to +\infty,$$

without the restriction $\alpha \in (1,2)$ stated in Lemma 3.6.2. These facts will play a role (see Remark 3.5.3) in Corollary 3.5.1 where only an upper bound on the decay rate of the minimizer ρ_V is needed.

The following result prove the asymptotic decay for another class of functions with logarithmic correction. We recall again that the purpose of Lemma 3.6.3 is to find asymptotic decay of the minimizer ρ_V when $q = \frac{2d+\alpha}{d+\alpha}$ (see Lemma 3.5.6).

Lemma 3.6.3. *Let $U \in C^2(\mathbb{R}^d)$ be a decreasing function of the form*

$$U(|x|) := \begin{cases} |x|^{-d} (\log(e|x|))^{\frac{d}{\alpha}} & \text{if } |x| \geq 1, \\ v(|x|) & \text{if } |x| < 1, \end{cases} \tag{3.6.35}$$

where $v \in C^2(\overline{B_1})$ is positive and radially decreasing. Then,

$$(-\Delta)^{\frac{\alpha}{2}} U(x) \simeq -\frac{(\log |x|)^{\frac{d+\alpha}{\alpha}}}{|x|^{d+\alpha}} \quad \text{as } |x| \to +\infty.$$

Proof. Similarly to what has been done in Lemma 3.6.2, Theorem 3.5.1 plays a key role. As a matter of fact, in view of [55, Theorem 1.1], we can write the following

$$(-\Delta)^{\frac{\alpha}{2}} U(r) = c_{d,\alpha} r^{-\alpha} \int_{\tau=1}^{+\infty} \Phi(\tau, r) G(\tau) d\tau, \tag{3.6.36}$$

where G can be written in terms of the Gaussian hypergeometric function

$$G(\tau) = \frac{(2\pi)^{\frac{d}{2}}}{\Gamma(d/2)} \tau^{-1-\alpha} {}_2F_1(a, b, c, \tau^{-2}),$$

with parameters

$$a := \frac{d+\alpha}{2}, \quad b := 1 + \frac{\alpha}{2}, \quad c := \frac{d}{2}.$$

As we have already done for the proof of Lemma 3.6.2, we define

$$\Phi(\tau, r) := U(r) - U(r\tau) + (U(r) - U(r/\tau))\, \tau^{-d+\alpha},$$

and we split the integral (3.6.36) into three parts I_1^2, I_2^r and I_r^∞ respectively

$$I_1^2 := \int_{\tau=1}^{2} (\cdot), \;\; I_2^r := \int_{\tau=2}^{r} (\cdot), \;\; I_r^\infty := \int_{\tau=r}^{\infty} (\cdot). \tag{3.6.37}$$

Furthermore, by [93, page 159, eq. (9.03)–(9.04)] we have

$${}_2F_1(a, b, c, \tau^{-2}) = 1 + \frac{ab}{\tau^2} + o(\tau^{-3}) \quad \text{as } \tau \to +\infty.$$

Then, for $\tau \geq 2$ we decompose the function ${}_2F_1(a, b, c, \tau^{-2})$ as

$${}_2F_1(a, b, c, \tau^{-2}) = 1 + W(\tau), \tag{3.6.38}$$

where $W(\tau)$ is a continuous function such that

$$W(\tau) = o(\tau^{-1}) \quad \text{as } \tau \to +\infty. \tag{3.6.39}$$

Taking into account (3.6.38), the quantity I_2^r in turn splits as

$$\begin{aligned} I_2^r &= \frac{(2\pi)^{\frac{d}{2}}}{\Gamma(d/2)} r^{-\alpha} \int_{\tau=2}^{r} \Phi(\tau, r)\tau^{-1-\alpha} d\tau + \frac{(2\pi)^{\frac{d}{2}}}{\Gamma(d/2)} r^{-\alpha} \int_{\tau=2}^{r} \Phi(\tau, r)\tau^{-1-\alpha} W(\tau)\, d\tau \\ &=: I_{2,1}^r + I_{2,2}^r. \end{aligned} \tag{3.6.40}$$

By performing an explicit computation, if we denote again by $\Gamma(\cdot, \cdot)$ the incomplete Gamma function (see e.g., [93, page 110, eq. (2.02)] for the asymptotic properties of the latter function) we have that

$$\begin{aligned}
&\int_{\tau=2}^{r} \Phi(\tau,r)\tau^{-1-\alpha}d\tau = \\
&(d+\alpha)^{-\frac{d+\alpha}{\alpha}} r^{-d} \log(er)^{d+\alpha} \left(-\Gamma\left(\frac{d+\alpha}{\alpha}, (d+\alpha)(\log(2er)\right) + \Gamma\left(\frac{d+\alpha}{\alpha}, (d+\alpha)(1+2\log(r)\right)\right) \\
&-\frac{\alpha}{d+\alpha} r^{-d}(-1+\log(er/2))^{\frac{d+\alpha}{\alpha}}) + \frac{2^{-d}r^{-2d}(-2^d+r^d)}{d}\log(er)^{\frac{d}{\alpha}} + \frac{2r^{-d}(2^{-\alpha}-r^{-\alpha})}{\alpha}\log(er)^{\frac{d}{\alpha}} \\
&= -\frac{\alpha}{d+\alpha} r^{-d}(-1+\log(er/2))^{\frac{d+\alpha}{\alpha}}) + o(r^{-d}\log(er)^{\frac{d+\alpha}{\alpha}}) \quad \text{as } r \to +\infty.
\end{aligned} \tag{3.6.41}$$

Hence, from (3.6.41) we conclude the following

$$\lim_{r\to+\infty} \frac{r^d}{\log(er)^{\frac{d+\alpha}{\alpha}}} \int_{\tau=2}^{r} \Phi(\tau,r)\tau^{-1-\alpha}d\tau = -\frac{\alpha}{d+\alpha}. \tag{3.6.42}$$

Furthermore, we claim that

$$I_{2,2}^r = o\left(\frac{\log(er)^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) \quad \text{as } r \to +\infty. \tag{3.6.43}$$

To derive (3.6.43), by simple estimates

$$\begin{aligned}
r^\alpha |I_{2,2}^r| \le{}& r^{-d}(\log(er))^{\frac{d}{\alpha}} \int_{\tau=2}^{\infty} \tau^{-1-\alpha}|W(\tau)|d\tau + 2^{\frac{d}{\alpha}} r^{-d}(\log(er))^{\frac{d}{\alpha}} \int_{\tau=2}^{\infty} \tau^{-1-\alpha}|W(\tau)|d\tau \\
&+ r^{-d}(\log(er))^{\frac{d}{\alpha}} \int_{\tau=2}^{\infty} \tau^{-1-d}|W(\tau)|d\tau + Cr^{-d}(\log(er))^{\frac{d}{\alpha}} \int_{\tau=2}^{\infty} \tau^{-1}|W(\tau)|d\tau \\
&\lesssim r^{-d}(\log(er))^{\frac{d}{\alpha}} = o\left(\frac{(\log(er))^{\frac{d+\alpha}{\alpha}}}{r^d}\right) \quad \text{as } r \to +\infty,
\end{aligned} \tag{3.6.44}$$

where we used continuity of W and (3.6.39). This proves the claim.

Next we focus on $\tau \in [1,2]$. Here, it is useful to write G in the form

$$G(\tau) = (\tau-1)^{-1-\alpha}F(\tau)$$

where $F(\tau)$ is continuous, bounded and never zero in $[1,2]$. In this way, by Taylor's formula there exists $\xi_\tau \in (1,\tau)$ such that

$$r^\alpha I_{1,2} = \int_{\tau=1}^{2} \Phi(\tau,r)(\tau-1)^{-1-\alpha}F(\tau)d\tau$$
$$= \int_{\tau=1}^{2} \frac{\partial^2}{\partial\tau^2}\Phi(\tau,r)_{|\tau=1}(\tau-1)^{1-\alpha}F(\tau)d\tau + \int_{\tau=1}^{2} \frac{\partial^3}{\partial\tau^3}\Phi(\tau,r)_{|\tau=\xi_\tau}(\tau-1)^{2-\alpha}F(\tau)d\tau. \tag{3.6.45}$$

To conclude, by an explicit computation,

$$\sup_{\tau\in[1,2]} \left|\frac{\partial^3}{\partial\tau^3}\Phi(\tau,r)\right| \lesssim \frac{(\log(er))^{\frac{d}{\alpha}}}{r^d} \quad \text{as } r\to+\infty, \tag{3.6.46}$$

and

$$\left|\frac{\partial^2}{\partial\tau^2}\Phi(\tau,r)_{|\tau=1}\right| \lesssim \frac{(\log(er))^{\frac{d}{\alpha}}}{r^d} \quad \text{as } r\to+\infty. \tag{3.6.47}$$

From (3.6.46) and (3.6.47) we derive that

$$I_{1,2} = o\left(\frac{(\log(er))^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) \quad \text{as } r\to+\infty. \tag{3.6.48}$$

Assume now $\tau \in [r,+\infty[$. In this case,

$$\Phi(\tau,r) = r^{-d}(\log(er))^{\frac{d}{\alpha}} + \tau^{-d+\alpha}\Big(r^{-d}(\log(er))^{\frac{d}{\alpha}} - v(r)\Big) - (r\tau)^{-d}(\log(er\tau))^{\frac{d}{\alpha}}$$
$$\le r^{-d}(\log(er))^{\frac{d}{\alpha}} + \underbrace{F(r,\tau)}_{<0} < U(r). \tag{3.6.49}$$

Inequality (3.6.49) leads to

$$(-\Delta)^{\frac{\alpha}{2}}U(r) \lesssim -\frac{(\log(er))^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}} + o\left(\frac{(\log(er))^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) \quad \text{as } r\to+\infty.$$

In order to conclude, let's also notice that

$$|I_r^\infty| \lesssim r^{-\alpha}\int_{\tau=r}^{\infty} |\Phi(\tau,r)|\tau^{-1-\alpha}d\tau \lesssim \frac{\log(er))^{\frac{d}{\alpha}}}{r^{d+2\alpha}} + \frac{1}{r^{d+\alpha}} + \frac{(\log(er))^{\frac{d}{\alpha}}}{r^{2d+\alpha}}. \tag{3.6.50}$$

Finally, by combining (3.6.42), (3.6.43), (3.6.48) and (3.6.50) we deduce

$$(-\Delta)^{\frac{\alpha}{2}}U(r) \simeq -\frac{(\log(er))^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}} \quad \text{as } r \to +\infty.$$

This concludes the proof. □

Chapter 4

Attractive non local interaction

In this chapter we study a Thomas–Fermi type variational problem with non local attraction. We start by introducing the model and its physical origin.

4.1 Introduction

Our starting point is the Choquard type equation

$$-\Delta w + \varepsilon w + |w|^{q-2}w = (I_\alpha * |w|^p)|w|^{p-2}w \quad \text{in } \mathbb{R}^d, \tag{P_ε}$$

where $d \geq 3$, $p > 1$, $q > 2$ and $\varepsilon \geq 0$. When $d = 3$, $\alpha = 2$, $p = 2$ and $q = 4$ in (P_ε), under the name of Gross–Pitaevskii–Poisson (GPP) equation, was proposed in cosmology as a model to describe the Cold Dark Matter (CDM) made of axions or bosons in the form of self–gravitating the Bose–Einstein Condensate (BEC) at absolute zero temperatures [20, 21, 38, 51, 106]. The non local convolution term on the right hand side of (P_ε) represents the Newtonian gravitational attraction between bosonic particles. The local term $|w|^{q-2}w$ accounts for the repulsive short–range quantum force self–interaction between bosons. Similar models appear in the literature under the names Ultralight Axion Dark Matter, and Fuzzy Dark Matter, see [21, 74] for a history survey. More generally, (P_ε) can be seen as a Hartree type nonlinear Schrödinger equation with an attractive long range interaction, represented by the non local Coulomb term, and repulsive short range interactions, represented by the local nonlinearity. While for the most of the relevant physical applications $p = 2$, the values $p \neq 2$ appear in several relativistic models of the density functional theory [13, 15, 16].

By a *ground state* of (P_ε) we understand a weak solution $w \in H^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ which has a minimal energy given by

$$\mathcal{I}_\varepsilon(w) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^d} |w|^2 dx + \frac{1}{q}\int_{\mathbb{R}^d} |w|^q dx - \frac{1}{2p}\int_{\mathbb{R}^d} (I_\alpha * |w|^p)|w|^p dx \tag{4.1.1}$$

amongst all nontrivial finite energy solutions of (P_ε). The following was proved in [82, Theorem 1.1], under optimal or near optimal assumptions on the parameters.

Theorem 4.1.1. *Let $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ and $q > 2$, or $p \geq \frac{d+\alpha}{d-2}$ and $q > \frac{2dp}{d+\alpha}$. Then for each $\varepsilon > 0$, (P_ε) admits a positive, radially monotone decreasing ground state solution $w_\varepsilon \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$. Moreover, there exists $C_\varepsilon > 0$ such that*

- *if $p > 2$,*
$$\lim_{|x|\to\infty} w_\varepsilon(x)|x|^{\frac{d-1}{2}} e^{\sqrt{\varepsilon}|x|} = C_\varepsilon;$$

- *if $p = 2$,*
$$\lim_{|x|\to\infty} w_\varepsilon(x)|x|^{\frac{d-1}{2}} \exp\left(\int_{\rho_\varepsilon}^{|x|} \sqrt{\varepsilon - \frac{A_\alpha \|w_\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{s^{d-\alpha}}}\, ds\right) = C_\varepsilon,$$

  where
$$\rho_\varepsilon := \left(\frac{A_\alpha \|w_\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\varepsilon}\right)^{\frac{1}{d-\alpha}};$$

- *if $p < 2$,*
$$\lim_{x\to\infty} w_\varepsilon(x)|x|^{\frac{d-\alpha}{2-p}} = \left(\varepsilon^{-1} A_\alpha \|w_\varepsilon\|^p_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{2-p}}.$$

In addition to the existence of ground states for every fixed $\varepsilon > 0$, in [82] the authors have identified and studied several limit regimes for ground states of (P_ε), as $\varepsilon \to 0$ or $\varepsilon \to \infty$. One of the relevant limit regimes is associated with the rescaling

$$u(x) := \varepsilon^{-\frac{1}{q-2}} w\big(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\big), \tag{4.1.2}$$

that converts (P_ε) to the equation

$$-\varepsilon^\nu \Delta u + u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{a.e. in } \mathbb{R}^d, \tag{4.1.3}$$

where $\nu := \frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}$. The *Thomas–Fermi limit regime* for the Choquard equation (P_ε) is the scenario when $\varepsilon \to 0$ and $\nu > 0$, or $\varepsilon \to \infty$ and $\nu < 0$. In this regime, ε^ν approaches zero and the *formal* limit equation for Eq. (4.1.3) is the *Thomas–Fermi* type integral equation

$$u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{a.e. in } \mathbb{R}^d. \tag{TF}$$

When $p = 2$ and $\alpha = 2$, equations equivalent to (*TF*) are well-known in astrophysical literature, cf. [37, p.92]. Mathematical analysis of (*TF*) with $p = 2$ and $\alpha = 2$ goes back to [10, 80]. More recently (*TF*) with $p = 2$ and general $\alpha \in (0, d)$ was studied in [29, 33] (existence of solutions), [30, 34, 35] (uniqueness), in connection with Keller-Segel models. See also [82, Theorem 2.6] which proves the existence of a ground state for (*TF*) with $p = 2$ for the optimal range $q > \frac{4d}{d+\alpha}$, extending some of the existence results in [29, 33].

In [82, Theorems 2.7 and 3.2], for the special case $p = 2$ and $\alpha = 2$ the authors established the convergence of the rescaled ground states u_ε of (4.1.3) to the ground state of (*TF*) in the Thomas–Fermi regimes, thus justifying the formal analysis of the rescaling (4.1.2). Recall that for $p = 2$ the limit ground state of (*TF*) is compactly supported on a ball, so that the rescaled ground states u_ε develop a steep "boundary layer" near the boundary of the support of the limit ground state. This phenomenon is well-known in astrophysics, where the radius of support of the limit ground state provides approximate radius of the astrophysical object. In the context of self–gravitating Bose–Einstein Condensate models, the Thomas–Fermi limit regime (under the name of *Thomas–Fermi approximations*) was used as the key tool in the astrophysical studies of the GPP-equation ($\alpha = 2$, $p = 2$, $q = 4$) in [20, 38, 106].

The main goal of this work is to study the existence and qualitative properties of ground states for (*TF*) for $p \neq 2$ and $\alpha \in (0, d)$, under optimal assumptions on the parameters. We are going to prove that:

- if $p < 2$ ground states for (*TF*) are positive smooth functions supported on $\mathbb{R}^d$,
- if $p = 2$ it is well–known [29, 33] that ground states for (*TF*) are continuous and compactly supported on a ball,

- for $p > 2$ ground states for (*TF*) are discontinuous and represented as a linear combination of the characteristic function of a ball, and a nonconstant nonincreasing Hölder continuous function supported on the same ball.

We also establish some qualitative properties of the ground states, including decay at infinity if $p < 2$, or regularity near the boundary of the support if $p \geq 2$. This information becomes crucial in the proofs of convergence of the rescaled ground states of (P_ε) to the limit profiles of (*TF*).

4.1.1 Existence of a solution of (P_ε).

Before presenting the other results contained in this chapter, for the convenience of the reader we provide a shorten proof (without including all the details) of the existence of ground state solutions for (P_ε). Essential tool to control the non local term in (4.1.1) is the Hardy–Littlewood–Sobolev inequality, see again Theorem 1.4.1. Namely

$$\int_{\mathbb{R}^d} (I_\alpha * |u|^p)|u|^p \, dx \leq \mathscr{C}_{d,\alpha} \|u\|^{2p}_{L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)} \quad \forall u \in L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d),$$

which is valid for any $p \geq 1$; and the Sobolev inequality

$$\|u\|^2_{\mathring{H}^1(\mathbb{R}^d)} = \|\nabla u\|^2_{L^2(\mathbb{R}^d)} \geq S^* \|u\|^2_{L^{2^*}(\mathbb{R}^d)} \quad \forall u \in \mathring{H}^1(\mathbb{R}^d),$$

where $2^* = \frac{2d}{d-2}$ is the critical Sobolev exponent HLS and Sobolev inequalities can be used to control the non local term. In particular the following possibilities hold:

- if $\frac{d+\alpha}{d} \leq p \leq \frac{d+\alpha}{d-2}$ then $L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \cap L^{2^*}(\mathbb{R}^d)$
- if $p \geq \frac{d+\alpha}{d}$ and $q \geq L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)$ then $L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$.

As a consequence, if $\frac{d+\alpha}{d} \leq p \leq \frac{d+\alpha}{d-2}$ and $q > 2$ or $p > \frac{d+\alpha}{d-2}$ and $q > \frac{2dp}{d+\alpha}$ then the energy functional $\mathcal{I}_\varepsilon$

$$\mathcal{I}_\varepsilon(w) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |w|^2 dx + \frac{1}{q} \int_{\mathbb{R}^d} |w|^q dx - \frac{1}{2p} \int_{\mathbb{R}^d} (I_\alpha * |w|^p)|w|^p dx$$

is well defined on the space

$$\mathcal{H}_q := H^1(\mathbb{R}^d) \cap L^{q^*}(\mathbb{R}^d), \quad q^* := \max\{q, 2^*\}.$$

The space $\mathcal{H}_q$ endowed with the norm

$$\|\cdot\|_{\mathcal{H}_q} := \|\cdot\|_{H^1(\mathbb{R}^d)} + (q^* - 2^*)\|\cdot\|_{L^{q^*}(\mathbb{R}^d)}$$

is a Banach space and $\mathcal{I}_\varepsilon \in C^1(\mathcal{H}_q, \mathbb{R})$. Furthermore, see e.g., Lemma 4.6.1,

$$\langle \mathcal{I}_\varepsilon'(u), \varphi\rangle_{\mathcal{H}_q} = \int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi \, dx + \varepsilon \int_{\mathbb{R}^d} u\varphi \, dx - \int_{\mathbb{R}^d} (I_\alpha * |u|^p)|u|^{p-2} u\varphi \, dx \tag{4.1.4}$$

$$+ \int_{\mathbb{R}^d} |u|^{q-2} u\varphi \, dx = 0, \quad \forall \varphi \in \mathcal{H}_q, \tag{4.1.5}$$

proving that critical points for $\mathcal{I}_\varepsilon$ correspond to weak solutions of (P_ε). In particular, by testing (4.1.4) against u, weak solutions $u \in \mathcal{H}_q$ of (P_ε) satisfy the Nehari identity

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \varepsilon \int_{\mathbb{R}^d} |u|^2 \, dx + \int_{\mathbb{R}^d} |u|^q \, dx - \int_{\mathbb{R}^d} (I_\alpha * |u|^p)|u|^p \, dx = 0.$$

We further remark that by definition $\mathcal{H}_q \hookrightarrow L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)$ and $\mathcal{H}_q = H^1(\mathbb{R}^d)$ if $2 < q \leq 2^*$.

Next, we show how to construct a groundstate of (P_ε) by minimizing over the Pohožaev manifold of (P_ε). Let us set

$$\mathscr{P}_\varepsilon := \{u \in \mathcal{H}_q \setminus \{0\} : \mathcal{P}_\varepsilon(u) = 0\},$$

where $\mathcal{P}_\varepsilon : \mathcal{H}_q \to \mathbb{R}$ is defined by

$$\mathcal{P}_\varepsilon(u) := \frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{\varepsilon d}{2} \int_{\mathbb{R}^d} |u|^2 \, dx + \frac{d}{q} \int_{\mathbb{R}^d} |u|^q \, dx - \frac{d+\alpha}{2p} \int_{\mathbb{R}^d} (I_\alpha * |u|^p) \, |u|^p \, dx.$$

For each $u \in \mathcal{H}_q \setminus \{0\}$, set

$$u_t(x) := u\left(\frac{x}{t}\right).$$

Then,

$$\begin{aligned} f_u(t) &:= \mathcal{I}_\varepsilon(u_t) \\ &= \frac{t^{d-2}}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\varepsilon t^d}{2} \int_{\mathbb{R}^d} |u|^2 dx + \frac{t^d}{q} \int_{\mathbb{R}^d} |u|^q dx - \frac{t^{d+\alpha}}{2p} \int_{\mathbb{R}^d} (I_\alpha * |u|^p) \, |u|^p dx. \end{aligned} \tag{4.1.6}$$

It's also easy to see that there exists a unique $t_u > 0$ such that

$$f_u(t_u) = \max\{f_u(t) : t > 0\}, \quad f_u'(t_u)t_u = 0,$$

which in turn implies that $u(x/t_u) \in \mathscr{P}_\varepsilon$. Therefore, $\mathscr{P}_\varepsilon \neq \emptyset$. Next, we define $M : \mathcal{H}_q \to \mathbb{R}$ as follows

$$M(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{q}\|u\|^q_{L^q(\mathbb{R}^d)}.$$

Moreover, it's readily seen that $M(u) = 0$ if and only if $u = 0$. Furthermore, from the definition of M and the norm $\|\cdot\|_{\mathcal{H}_q}$, one can also prove that

$$2^{-\frac{q}{2}}\|u\|^q_{\mathcal{H}_q} \leq M(u) \leq C\|u\|^2_{\mathcal{H}_q}$$

if either $M(u) \leq 1$ or $\|u\|_{\mathcal{H}_q} \leq 1$, where $C > 0$ is independent of u.

Next, we state the following technical lemma proved in [82, Lemma 4.2]. In this thesis we provide only a partial proof.

Lemma 4.1.1. *Assume that either* $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ *and* $q > 2$*, or* $p \geq \frac{d+\alpha}{d-2}$ *and* $q > \frac{2dp}{d+\alpha}$*. Then there exists* $C > 0$ *such that for all* $u \in \mathcal{H}_q$*,*

$$\int_{\mathbb{R}^d} (I_\alpha * |u|^p)\,|u|^p\,dx \leq C\max\left\{M(u)^{\frac{d+\alpha}{d}}, M(u)^{\frac{d+\alpha}{d-2}}\right\}.$$

STEP 1: $0 \notin \mathscr{P}_\varepsilon$.

Proof of Step 1: If $u \in \mathscr{P}_\varepsilon$, by using the HLS and Sobolev inequalities,

$$\begin{aligned} 0 = \mathcal{P}_\varepsilon(u) &\geq \min\left(\frac{d-2}{2}, \frac{\varepsilon d}{2}, \frac{d}{q}\right) M(u) - C\|u\|^{\frac{2d}{d+\alpha}}_{L^{2p}(\mathbb{R}^d)} \\ &\geq \min\left(\frac{d-2}{2}, \frac{\varepsilon d}{2}, \frac{d}{q}\right) M(u) - C(M(u))^{\frac{d+\alpha}{d}}, \end{aligned} \tag{4.1.7}$$

yielding that there exists $C > 0$ such that $M(u) \geq C$ for all $u \in \mathscr{P}_\varepsilon$.

STEP 2: $c_\varepsilon = \inf_{u\in\mathscr{P}_\varepsilon} \mathcal{I}_\varepsilon(u) > 0$.

Proof of Step 2: For each $u \in \mathscr{P}_\varepsilon$, we have

$$\mathcal{I}_\varepsilon(u) = \frac{1}{d}\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \frac{\alpha}{2dp}\int_{\mathbb{R}^d}(I_\alpha * |u|^p)|u|^p\,dx,$$

therefore $c_\varepsilon \geq 0$. If $c_\varepsilon = 0$, then there exists a sequence $(u_n)_n \subset \mathscr{P}_\varepsilon$ such that $\mathcal{I}_\varepsilon(u_n) \to 0$, which means that

$$\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} \to 0, \int_{\mathbb{R}^d} (I_\alpha * |u_n|^p)|u_n|^p \, dx \to 0.$$

Recall that $\mathcal{P}_\varepsilon(u_n) = 0$. Then we conclude that $\|u_n\|^2_{L^2(\mathbb{R}^d)} \to 0$ and $\|u_n\|^q_{L^q(\mathbb{R}^d)} \to 0$. This implies that $\|u_n\|^2_{L^2(\mathbb{R}^d)} \to 0$, which contradicts to $0 \notin \partial\mathscr{P}_\varepsilon$.
STEP 3: There exists $u_0 \in \mathscr{P}_\varepsilon$ such that $\mathcal{I}_\varepsilon(u_0) = c_\varepsilon$.
Proof of Step 3: Since c_ε is well defined, there exists a sequence $(u_n)_n \subset \mathscr{P}_\varepsilon$ such that $\mathcal{I}_\varepsilon(u_n) \to c_\varepsilon$. Then, we infer that $(\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)})_n$ and $(\int_{\mathbb{R}^d} (I_\alpha * |u_n|^p)|u_n|^p \, dx)_n$ are bounded. Note that $\mathcal{P}_\varepsilon(u_n) = 0$. Then we see that $(\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)})_n$ and $(\|u_n\|^q_{L^q(\mathbb{R}^d)})_n$ are bounded, and therefore $(u_n)_n$ is bounded in $\mathcal{H}_q$.

Let u_n^* be the Schwartz spherical rearrangement of $|u_n|$. Then $u_n^* \in \mathcal{H}_{q,rad}$, the subspace of $\mathcal{H}_q$ which consists of all spherically symmetric functions in $\mathcal{H}_q$, and

$$\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} \geq \|\nabla u_n^*\|^2_{L^2(\mathbb{R}^d)}, \|u_n\|^2_{L^2(\mathbb{R}^d)} = \|u_n^*\|^2_{L^2(\mathbb{R}^d)}, \|u_n\|^q_{L^q(\mathbb{R}^d)} = \|u_n^*\|^q_{L^q(\mathbb{R}^d)},$$

$$\int_{\mathbb{R}^d} (I_\alpha * |u_n|^p)|u_n|^p \, dx \leq \int_{\mathbb{R}^d} (I_\alpha * |u_n^*|^p)|u_n^*|^p \, dx.$$

Next, for each u_n^*, there exists a unique $t_n \in (0,1)$ such that $v_n := u_n^*(t_n x) \in \mathscr{P}_\varepsilon$. Therefore we obtain that

$$\mathcal{I}_\varepsilon(u_n) \geq \mathcal{I}_\varepsilon(u_n(t_n x)) \geq \mathcal{I}_\varepsilon(v_n) \geq c_\varepsilon,$$

which in turn implies that $(v_n)_n$ is also a minimizing sequence for c_ε. We have therefore proved that $\mathcal{I}_\varepsilon(v_n) \to c_\varepsilon$. Furthermore, the sequence $(v_n)_n \subset \mathcal{H}_{q,rad}$ is bounded. Then there exists $v \in \mathcal{H}_{q,rad}$ such that $v_n \rightharpoonup v$ weakly in $\mathcal{H}_q$ and $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^d$, by the local compactness of the embedding $\mathcal{H}_q \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^d)$ on bounded domains. Using Strauss's L^s-bounds with $s = 2$ and $s = q^*$, we conclude that

$$v_n(|x|) \leq U(x) := C \min \left\{ |x|^{-d/2}, |x|^{-d/q_*} \right\}.$$

Since $U \in L^s(\mathbb{R}^d)$ for $s \in (2, q^*)$, by the Lebesgue dominated convergence we conclude that for $s \in (2, q^*)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |v_n|^s \, dx = \int_{\mathbb{R}^d} |v|^s \, dx.$$

Note that $q^* > \frac{2dp}{d+\alpha}$ and hence we can always choose $s > \frac{2dp}{d+\alpha} > p$ such that $(v_n)_n$ is bounded in $L^s(\mathbb{R}^d)$. Then by the non local Brezis–Lieb Lemma [85, Proposition 4.7] we further conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}(I_\alpha * |v_n|^p)|v_n|^p\,dx = \int_{\mathbb{R}^d}(I_\alpha * |v|^p)|v|^p\,dx.$$

This means that $v \neq 0$, since by Lemma 4.2 the sequence $\{M(v_n)\}_n$ has a positive lower bound. Then there exists a unique $t_0 > 0$ such that $v(x/t_0) \in \mathscr{P}_\varepsilon$. By the weakly lower semi-continuity of the norm, we see that

$$\mathcal{I}_\varepsilon(v_n) \geq \mathcal{I}_\varepsilon(v_n(x/t_0)) \geq \mathcal{I}_\varepsilon(v(x/t_0)) \geq c_\varepsilon,$$

which implies that $\mathcal{I}_\varepsilon(v(x/t_0)) = c_\varepsilon$. For the next step we fix $u_0(x) := v(x/t_0))$.
STEP 4: $\mathcal{P}'_\varepsilon(u_0) \neq 0$, where u_0 is obtained in Step 3.
Proof of Step 4: Arguing by contradiction, we assume that $\mathcal{P}'_\varepsilon(u_0) = 0$. Then u_0 is a weak solution of the following equation,

$$-(d-2)u + \varepsilon du - (d+\alpha)(I_\alpha * |u|^p)|u|^{p-2}u + d|u|^{q-2}u = 0 \text{ in } \mathbb{R}^d.$$

In particular, u_0 (by applying [82, Corollary 4.1, Proposition 4.2]) satisfies the Pohožaev identity

$$\frac{(d-2)^2}{2}\|\nabla u_0\|_2^2 + \varepsilon\frac{d^2}{2}\|u_0\|_2^2 - \frac{(d+\alpha)^2}{2p}\int_{\mathbb{R}^d}(I_\alpha * |u_0|^p)|u_0|^p\,dx + \frac{d^2}{q}\|u_0\|_q^q = 0.$$

The above equation, together with $\mathscr{P}_\varepsilon(u_0) = 0$, implies that

$$\frac{d-2}{2}\|\nabla u_0\|_2^2 + \frac{d+\alpha)\alpha}{2p}\int_{\mathbb{R}^d}(I_\alpha * |u_0|^p)|u_0|^p\,dx = 0,$$

which contradict $u_0 \neq 0$.
STEP 5: $\mathcal{I}'_\varepsilon(u_0) = 0$, i.e., u_0 is a weak solution of (P_ε).
Proof of Step 5: By the Lagrange multiplier rule, there exists $\mu \in \mathbb{R}$ such that $\mathcal{I}'_\varepsilon(u_0) = \mu\mathcal{P}'_\varepsilon(u_0)$. We claim that $\mu = 0$. As a matter of fact, since $\mathcal{I}'_\varepsilon(u_0) = \mu\mathcal{P}'_\varepsilon(u_0)$, then u_0 satisfies in the weak sense the following equation,

$$-(\mu(d-2)-1)\Delta u + (\mu d-1)\varepsilon u - (\mu(d+\alpha)-1)(I_\alpha * |u|^p)|u|^{p-2}u + (\mu d-1)|u|^{q-2}u = 0 \text{ in } \mathbb{R}^d. \tag{4.1.8}$$

Then, again by [82, Propositions 4.1-4.2], u_0 satisfies the Pohožaev identity

$$\begin{aligned} &\frac{(\mu(d-2)-1)(d-2)}{2}\|\nabla u_0\|_2^2+\frac{\varepsilon(\mu d-1)d}{2}\|u_0\|_2^2 \\ &-\frac{(\mu(d+\alpha)-1)(d+\alpha)}{2p}\int_{\mathbb{R}^d}(I_\alpha*|u_0|^p)|u_0|^p\,dx+\frac{(\mu d-1)d}{q}\|u_0\|_q^q=0. \end{aligned} \tag{4.1.9}$$

Finally, by $\mathcal{P}_\varepsilon(u_0)=0$ we conclude that

$$\mu(d-2)\|\nabla u_0\|_2^2+\frac{\mu\alpha(d+\alpha)}{2p}\int_{\mathbb{R}^d}(I_\alpha*|u_0|^p)|u_0|^p\,dx=0,$$

which means that $\mu=0$. Therefore $\mathcal{I}_\varepsilon'(u_0)=0$.

4.1.2 Variational setup for (*TF*) and main results

Now, having proved the existence of ground state solutions for (P_ε), we focus on finding ground states for the limit equation (*TF*). Solutions of the Thomas–Fermi equation (*TF*) correspond, at least formally, to the critical points of the energy

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^d}|u|^2dx+\frac{1}{q}\int_{\mathbb{R}^d}|u|^q\,dx-\frac{1}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p), \tag{4.1.10}$$

where we recall that $\mathcal{D}_\alpha$ denotes the Coulomb interaction

$$\mathcal{D}_\alpha(f,g):=A_\alpha\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{f(x)\,g(y)}{|x-y|^{d-\alpha}}dx\,dy.$$

Throughout this work, we assume that the following restrictions on the parameters

$$\frac{1}{q}<\frac{d+\alpha}{2dp}<\frac{1}{2}. \tag{4.1.11}$$

Then using the Hardy-Littlewood-Sobolev (HLS) and Hölder's inequalities we can control the Coulomb term, i.e.,

$$\mathcal{D}_\alpha(|u|^p,|u|^p)\le \mathscr{C}_{d,\alpha}\|u\|^{2p}_{L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)}\le \mathscr{C}_{d,\alpha}\|u\|^{2p\theta}_{L^2(\mathbb{R}^d)}\|u\|^{2p(1-\theta)}_{L^q(\mathbb{R}^d)}. \tag{4.1.12}$$

Here $\mathscr{C}_{d,\alpha} = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{\alpha/2}\Gamma((d+\alpha)/2)}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{\alpha/d}$ is the sharp constant in the Hardy–Littlewood–Sobolev inequality [77, 78], and $\theta \in (0,1)$ satisfies the condition

$$\frac{d+\alpha}{2dp} = \frac{\theta}{2} + \frac{1-\theta}{q}, \quad \text{or } \theta = \frac{(d+\alpha)q - 2dp}{dp(q-2)}. \tag{4.1.13}$$

Therefore, the conditions in (4.1.11) ensure that the energy E is continuous and Fréchet differentiable on $L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, and its critical points are solutions of (*TF*). Moreover, critical points of E (and solutions of (*TF*)) satisfy the Nehari identity

$$\int_{\mathbb{R}^d} |u|^2 dx + \int_{\mathbb{R}^d} |u|^q dx = \mathcal{D}_\alpha(|u|^p, |u|^p). \tag{4.1.14}$$

By a *ground state* of (*TF*) we understand a function $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ solving (*TF*) which has a minimal energy E amongst all functions in the Pohožaev manifold $\mathscr{P}$, defined as

$$\mathscr{P} := \left\{u \neq 0 : u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), \mathcal{P}(u) = 0\right\}, \tag{4.1.15}$$

where the functional $\mathcal{P}$ is given by

$$\mathcal{P}(u) := \frac{d}{2}\int_{\mathbb{R}^d} |u|^2 dx + \frac{d}{q}\int_{\mathbb{R}^d} |u|^q dx - \frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p).$$

Since the energy E is not bounded from below (by replacing u with $u(\cdot/\lambda)$ with $\lambda \to +\infty$), constrained minimization techniques are better suited for the construction of ground states. Moreover, the Pohožaev manifold $\mathscr{P}$ is preferred over the Nehari manifold characterised by Eq. (4.1.14), primarily because of simplifications due to the common expressions $\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^d} |u|^q dx$ appearing in both $E(u)$ and $\mathcal{P}(u)$, as demonstrated in Section 4.5.

Another way to construct ground states of (*TF*) is to look for maximizers of the Gagliardo–Nirenberg quotient associated to the interpolation inequality (4.1.12), i.e.,

$$\mathscr{C}_{d,\alpha,p,q} := \sup\left\{\frac{\mathcal{D}_\alpha(|v|^p, |v|^p)}{\|v\|^{2p\theta}_{L^2(\mathbb{R}^d)}\|v\|^{2p(1-\theta)}_{L^q(\mathbb{R}^d)}} : v \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), v \neq 0\right\}. \tag{4.1.16}$$

From (4.1.12), it is clear that $\mathscr{C}_{d,\alpha,p,q} \leq \mathscr{C}_{d,\alpha}$. Note that the quotient in Eq. (4.1.16) is invariant w.r.t. translation, dilation and scaling; every maximizer for $\mathscr{C}_{d,\alpha,p,q}$ (if it exists) can be rescaled to a ground states solutions of (*TF*) (see Lemma 4.3.6 below).

Using symmetric rearrangements, Strauss' radial bounds and Helly's selection principle for radial functions, we prove the following result.

Theorem 4.1.2. *Let $d \geq 1$, $\alpha \in (0, d)$, $p > \frac{d+\alpha}{d}$ and $q > \frac{2dp}{d+\alpha}$. Then there exists a non-negative radial non-increasing maximizer $u_* \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $\mathscr{C}_{d,\alpha,p,q}$, which is also a ground state solution of the Thomas–Fermi equation (TF).*

Remark 4.1.1. While the precise value of $\mathscr{C}_{d,\alpha,p,q}$ is not known in general, we prove below that for fixed admissible values of d, p and q,

$$\lim_{\alpha\to 0} \mathscr{C}_{d,\alpha,p,q} = 1, \qquad \lim_{\alpha\to d} A_\alpha^{-1}\mathscr{C}_{d,\alpha,p,q} = 1,$$

here A_α is the Riesz constant (see Proposition 4.2.1). We further refer to [66] for a specific combination of parameters where we can identify $\mathscr{C}_{d,\alpha,p,q}$ by looking for optimizers taking the simple ansatz $v(x) = \lambda(1 + |x|^2/\mu^2)^{-\gamma}$ for some positive constants λ, μ and γ.

Remark 4.1.2. The substitution $\rho = |u|^p$, $m = q/p$ and $n = 2/p$ leads to an equivalent formulation of the quotient in Eq. (4.1.16), i.e.,

$$\mathscr{C}_{d,\alpha,m,n} := \sup\left\{\frac{\mathcal{D}_\alpha(\rho,\rho)}{\left(\int_{\mathbb{R}^d}\rho^n dx\right)^{\frac{2\theta}{n}}\left(\int_{\mathbb{R}^d}\rho^m dx\right)^{\frac{2(1-\theta)}{m}}} : 0 \leq \rho \in L^n(\mathbb{R}^d)\cap L^m(\mathbb{R}^d), \rho \neq 0\right\}.$$

The corresponding interpolation inequality then takes the form

$$A_\alpha \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\rho(x)|\,|\rho(y)|}{|x-y|^{d-\alpha}} dx\,dy \leq \mathscr{C}_{d,\alpha,m,n}\left(\int_{\mathbb{R}^d}|\rho|^n dx\right)^{\frac{2\theta}{n}}\left(\int_{\mathbb{R}^d}|\rho|^m dx\right)^{\frac{2(1-\theta)}{m}},$$

for all $\rho \in L^n(\mathbb{R}^d)\cap L^m(\mathbb{R}^d)$, where $0 < n < \frac{2d}{d+\alpha} < m$. This can be seen as a standard interpolation associated to the Hardy–Littlewood–Sobolev inequality, which however includes *sublinear* exponents $n < 1$. Relevant variational problems with $n = 1$ can be found in the early works by P.-L. Lions [80; 81, Section II] and in the context of diffusion-aggregation models in the recent papers [29, 33]. We are not aware of any works where the case $n \neq 1$ was considered.

Mathematically, the most striking phenomenon related to (*TF*) is how the behaviour of the support of ground states to (*TF*) depends on the value of p. Our main result is the following.

Theorem 4.1.3. *Let $d \geq 1$, $\alpha \in (0, d)$, $p > \frac{d+\alpha}{d}$, $q > \frac{2dp}{d+\alpha}$. Then every non negative radial ground state solution $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ of (TF) is C^∞ in the set $\{x \in \mathbb{R}^d : u(x) > 0\}$ and:*

(*a*) *if $p < 2$, then $\{u > 0\} = \mathbb{R}^d$ and $\lim_{x\to\infty} u(x)|x|^{\frac{d-\alpha}{2-p}} = \left(A_\alpha \int_{\mathbb{R}^d} u^p\, dx\right)^{\frac{1}{2-p}}$, where A_α is the Riesz constant;*

(*b*) *if $p = 2$, then u is Hölder continuous and $\{u > 0\} = B_R$ for some $R > 0$;*

(*c*) *if $p > 2$, then $\{u > 0\} = B_R$ for some $R > 0$ and*

$$u = \lambda \chi_{B_R} + \phi,$$

where $\lambda \geq \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}}$ and $\phi : B_R \to \mathbb{R}$ is a Hölder continuous radial non-increasing function such that $\phi(0) > 0$ and $\lim_{|x|\to R^-} \phi(|x|) = 0$.

In the case $p = 2$, some results are known at least since the early paper by P.-L. Lions [80]. However other results derived here are new even in this classical case. In the case $d = 3$, $\alpha = 2$, $p = 2$ and $q = 4$ the (unique) non negative radial ground state of (*TF*) is known explicitly and is given by the function

$$u(x) = \chi_{B_\pi}(x)\sqrt{\tfrac{\sin(|x|)}{|x|}}. \tag{4.1.17}$$

This is (up to the physical constants) the Thomas–Fermi approximation solution for self–gravitating BEC observed in [20, 38, 106] and the support radius $R = \pi$ is the approximate radius of the BEC star. Note that $u \notin H^1(\mathbb{R}^3)$. For $p \geq 2$ and general values of d, α and q the radius of the support of a ground state of (*TF*) can be easily estimated. For the sake of completeness, we refer also to Remark 4.2.1 for a short a proof of the fact that u defined by (4.1.17) solves (*TF*). In particular, in Corollaries 4.4.3 and 4.4.4 we show that for fixed admissible d, p and q, the radius of the support of the ground states diverges to $+\infty$ as $\alpha \to 0$, and shrinks to zero as $\alpha \to d$. In Lemma 4.4.2 and 4.4.3 we obtain quantitative estimates on the Hölder continuity of the ground state near the boundary of the support when $p \geq 2$.

We refer to [66, Figures 2 and 4] for numerical simulations carried out in Matlab. In particular, such numerical experiments suggests that for $p > 2$ the jump near the boundary λ is always strictly larger than $\lambda_* = (\frac{p-2}{q-p})^{1/(q-2)}$. If established analytically, this would also rule out the second option in the regularity estimate (4.4.14) implying that Hölder regularity of the ground state near the boundary of the support is always of order $\tau \in (0, \min\{\alpha, 1\})$.

Note again that, the regularity results concerning the case $p \neq 2$ stated in Theorem 4.1.3 are new even in the classical case $\alpha = 2$. Similarly, as far as concerns the convergence of rescaled ground states of (4.1.3) to a ground state of (*TF*), [82, Theorem 2.7] is the only outcome the author is aware of. In this last result, several additional restrictions appear in order to establish that ground states for (*TF*) belong to $H^1(\mathbb{R}^d)$, making the proof easier. If $p = 2$ such a regularity can be deduced (under some restrictions on the parameters) from Theorem 4.1.3–(b). However, if $p > 2$, Theorem 4.1.3–(c) implies that such regularity is never achieved because of the discontinuity at the boundary of the support.

4.1.3 Thomas–Fermi limit profiles for Choquard equation (P_ε)

Next we prove that in the relevant asymptotic regimes, ground states w of (P_ε) described in Theorem 4.1.1 converge, after the rescaling

$$u_\varepsilon(x) := \varepsilon^{-\frac{1}{q-2}} w_\varepsilon\big(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\big)$$

towards a ground state of the Thomas–Fermi equation (*TF*). To identify the asymptotic regimes observe that the rescaling (4.1.2) transforms the Choquard energy $\mathcal{I}_\varepsilon(u)$ in such a way that

$$\mathcal{J}_\varepsilon(w_\varepsilon) = \varepsilon^{\frac{q(d+\alpha)-2dp}{\alpha(q-2)}} \mathcal{I}_\varepsilon(u_\varepsilon),$$

where we denoted

$$\mathcal{J}_\varepsilon(w) := \frac{1}{2}\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} |w|^2 dx + \frac{1}{q}\int_{\mathbb{R}^d} |w|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|w|^p, |w|^p). \tag{4.1.18}$$

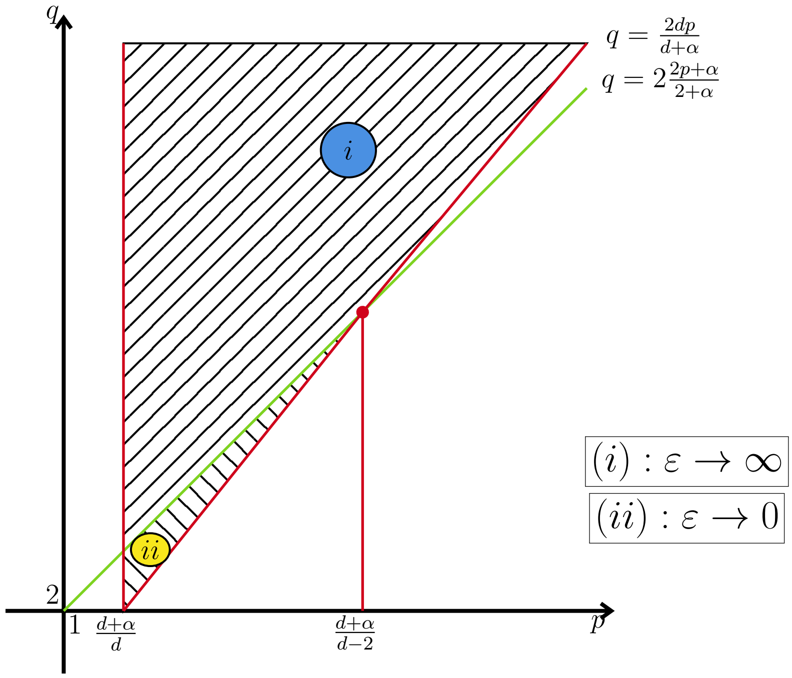

Figure 4.1

Then formally, $\mathcal{J}_0(u) = E(u)$ and we note that if $\varepsilon \to 0$ and $q < 2\frac{2p+\alpha}{2+\alpha}$, or if $\varepsilon \to \infty$ and $q > 2\frac{2p+\alpha}{2+\alpha}$ then $\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \to 0$. Combined with the existence range of the ground state of (P_ε) in Theorem 4.1.1, this formally identifies the Thomas–Fermi limit regimes. In Section 4.5 we prove the following result, that confirms our reasoning based on formal asymptotics and covers the ranges of $\alpha \neq 2$ and $p \neq 2$ that were left missing in [82, Theorem 2.7 and 3.2].

Theorem 4.1.4. *Let $d \geq 3$ and $\alpha \in (0, d)$. Assume that either (i) or (ii) holds:*

i) $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ *and* $q > 2\frac{2p+\alpha}{2+\alpha}$, *or* $p > \frac{d+\alpha}{d-2}$ *and* $q > \frac{2dp}{d+\alpha}$;

ii) $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ *and* $\frac{2dp}{d+\alpha} < q < 2\frac{2p+\alpha}{2+\alpha}$.

Then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a sequence of ground states (u_{ε_k}) of (P_{ε_k}) such that

$$\varepsilon_k \to \infty \textit{ if } (i) \textit{ holds, or } \varepsilon_k \to 0 \textit{ if } (ii) \textit{ holds,}$$

and the rescaled sequence of ground states of (P_ε)

$$u_{\varepsilon_k}(x) := \varepsilon_k^{-\frac{1}{q-2}} w_{\varepsilon_k}\big(\varepsilon_k^{-\frac{2p-q}{\alpha(q-2)}} x\big)$$

converges in $L^2(\mathbb{R}^d)$ *and* $L^q(\mathbb{R}^d)$ *to a non negative ground state solution of the Thomas-Fermi equation* (TF). *Moreover,* $\varepsilon_k^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \|\nabla u_{\varepsilon_k}\|^2_{L^2(\mathbb{R}^d)} \to 0$ *as* $k \to +\infty$.

4.2 Proof of Theorem 4.1.2.

Assume that $d \geq 1$, $\alpha \in (0, d)$ and $p \geq \frac{d+\alpha}{d}$ and $q > \frac{2dp}{d+\alpha}$. For $u \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, denote

$$\mathcal{R}(u) := \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|^{2p\theta}_{L^2(\mathbb{R}^d)} \|u\|^{2p(1-\theta)}_{L^q(\mathbb{R}^d)}}.$$

We are going to show that the best constant

$$\mathscr{C}_{d,\alpha,p,q} = \sup\{\mathcal{R}(u) : u \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), u \neq 0\} \tag{4.2.1}$$

is achieved. We follow with some modifications the arguments in [57, proof of Proposition 8].

Let $\{u_n\}_n$ be a sequence such that $\mathcal{R}(u_n) \to \mathscr{C}_{d,\alpha,p,q}$ as $n \to \infty$. Let u_n^* denote the Schwartz spherical rearrangement of $|u_n|$. Then u_n^* is non negative radially symmetric nonincreasing, and

$$\|u_n\|^2_{L^2(\mathbb{R}^d)} = \|u_n^*\|^2_{L^2(\mathbb{R}^d)}, \ \|u_n\|^q_{L^q(\mathbb{R}^d)} = \|u_n^*\|^q_{L^q(\mathbb{R}^d)}, \ \mathcal{D}_\alpha(|u_n|^p, |u_n|^p) \leq \mathcal{D}_\alpha((u_n^*)^p, (u_n^*)^p). \tag{4.2.2}$$

Therefore $\mathcal{R}(u_n) \leq \mathcal{R}(u_n^*)$ and $\{u_n^*\}$ is also a maximizing sequence of $\mathscr{C}_{d,\alpha,p,q}$. Without loss of generality, we can denote u_n^* by u_n in the rest of the proof.

By using the scaling invariance and homogeneity of $\mathcal{R}$ we can assume that $\|u_n\|_{L^2(\mathbb{R}^d)} = \|u_n\|_{L^q(\mathbb{R}^d)} = 1$, so that

$$\mathcal{R}(u_n) = \mathcal{D}_\alpha(u_n^p, u_n^p) \to \mathscr{C}_{d,\alpha,p,q}$$

as $n \to \infty$. Using Strauss' L^s–bounds [104] with $s = 2$ and $s = q$, we conclude that

$$u_n(|x|) \leq U(x) := C \min\{|x|^{-d/2}, |x|^{-d/q}\}.$$

Since $U \in L^s(\mathbb{R}^d)$ for $s \in (2, q)$, by Helly's selection principle there exists $0 \leq u \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^d$ as $n \to \infty$. By the Lebesgue dominated convergence, we see that for $s \in (2, q)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^s dx = \int_{\mathbb{R}^d} |u|^s dx.$$

Note that $q > \frac{2dp}{d+\alpha} > p$. Then by the non local Brezis-Lieb Lemma [85, Proposition 4.7] we conclude that

$$\lim_{n\to\infty} \mathcal{D}_\alpha(u_n^p, u_n^p) = \mathcal{D}_\alpha(u^p, u^p) = \mathscr{C}_{d,\alpha,p,q},$$

which in particular, implies that $u \neq 0$. By Fatou's Lemma, we get that

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^2 dx \geq \int_{\mathbb{R}^d} |u|^2 dx > 0, \quad 1 = \lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^q dx \geq \int_{\mathbb{R}^d} |u|^q dx > 0.$$

We claim that $\int_{\mathbb{R}^d} |u|^2 dx = \int_{\mathbb{R}^d} |u|^q dx = 1$. By assuming $\|u\|_{L^2(\mathbb{R}^d)} \|u\|_{L^q(\mathbb{R}^d)} < 1$, then

$$\mathscr{C}_{d,\alpha,p,q} \geq \mathcal{R}(u) > \mathcal{D}_\alpha(u^p, u^p) = \lim_{n\to\infty} \mathcal{D}_\alpha(u_n^p, u_n^p) = \lim_{n\to\infty} \mathcal{R}(u_n) = \mathscr{C}_{d,\alpha,p,q},$$

a contradiction. Therefore, our claim holds and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^2 dx = \int_{\mathbb{R}^d} |u|^2 dx, \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^q dx = \int_{\mathbb{R}^d} |u|^q dx,$$

that is, $u_n \to u$ strongly in $L^s(\mathbb{R}^d)$ for $s \in [2, q]$. Furthermore, we have $\mathscr{C}_{d,\alpha,p,q} = \mathcal{R}(u)$. Let u^* be the Schwartz spherical rearrangement of u, then by (4.2.2) we obtain

$$\mathscr{C}_{d,\alpha,p,q} \geq \mathcal{R}(u^*) \geq \mathcal{R}(u) = \mathscr{C}_{d,\alpha,p,q},$$

which means that u^* is also a maximizer of $\mathscr{C}_{d,\alpha,p,q}$. □

Next we briefly discuss the asymptotic behaviours of the optimal constant $\mathscr{C}_{d,\alpha,p,q}$ when α approaches 0 or d.

Proposition 4.2.1. *Assume that $d \geq 1$, $\alpha \in (0, d)$. If $p > 1$ and $q \geq 2p$ then*

$$\lim_{\alpha\to 0} \mathscr{C}_{d,\alpha,p,q} = 1. \tag{4.2.3}$$

Furthermore, if $p \geq 2$ and $q > p$ then

$$\lim_{\alpha \to d} A_\alpha^{-1} \mathscr{C}_{d,\alpha,p,q} = 1. \tag{4.2.4}$$

Proof. First of all we notice that $p > 1$ implies $p > \frac{d+\alpha}{d}$ for every α sufficiently close to zero, and $2p > \frac{2dp}{d+\alpha}$ for every $\alpha \in (0, d)$. Similarly, the assumption $q > p$ ensures that $q > \frac{2dp}{d+\alpha}$ for every α sufficiently close to d, and clearly $p \geq 2 > \frac{d+\alpha}{d}$. Thus, under our assumptions on p and q, if α is sufficiently close to zero or N, the optimal constant $\mathscr{C}_{d,\alpha,p,q}$ is well defined. Next, we begin by proving that

$$\limsup_{\alpha \to 0} \mathscr{C}_{d,\alpha,p,q} \leq 1.$$

By the HLS inequality and standard properties of the Gamma function, we conclude that

$$\mathscr{C}_{d,\alpha,p,q} \leq \mathscr{C}_{d,\alpha} = (2\sqrt{\pi})^{-\alpha} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{\alpha}{d}} \to 1 \quad \text{as } \alpha \to 0, \tag{4.2.5}$$

where $\mathscr{C}_{d,\alpha}$ is the sharp constant in the HLS inequality (4.1.12).

On the other hand, we set $u = \chi_{B_1}$ to estimate the lower bound of $\mathscr{C}_{d,\alpha,p,q}$. Then, by the explicit expression for the Riesz potential of a characteristic function give in Eq. (4.4.19) below,

$$\mathscr{C}_{d,\alpha,p,q} \geq |B_1|^{-p\theta - \frac{2p(1-\theta)}{q}} \frac{\Gamma((d-\alpha)/2)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(d/2)} \int_{B_1} {}_2F_1(-\alpha/2, (d-\alpha)/2; d/2; |x|^2) dx, \tag{4.2.6}$$

where $\theta = \theta(\alpha)$ is defined in Eq. (4.1.13). Hence, by noting the following limits (with fixed p, q),

$$\begin{aligned} &\lim_{\alpha \to 0} \left(p\theta(\alpha) + \frac{2p(1-\theta(\alpha))}{q} \right) = 1; \\ &\lim_{\alpha \to 0} {}_2F_1(-\alpha/2, (d-\alpha)/2; d/2; |x|^2) = 1, \quad \text{for every } |x| < 1, \end{aligned} \tag{4.2.7}$$

Fatou's lemma yields

$$\liminf_{\alpha \to 0} \mathscr{C}_{d,\alpha,p,q} \geq |B_1|^{-1} \int_{B_1} \liminf_{\alpha \to 0} {}_2F_1(-\alpha/2, (d-\alpha)/2; d/2; |x|^2) dx = 1,$$

concluding the proof of (4.2.3). Note that (4.2.7) holds since for $|x| < 1$ the hypergeometric function is continuous with respect to the other parameters and so

$$\lim_{\alpha\to 0} {}_2F_1(-\alpha/2, (d-\alpha)/2; d/2; |x|^2) = {}_2F_1(0, d/2; d/2; |x|^2) = 1$$

where we used the representation formula,

$${}_2F_1(-m, b; c; |x|^2) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n} |x|^{2n}, \quad m \in \mathbb{N} \cup \{0\},$$

with $(\cdot)_n$ denoting the (rising) Pochhammer symbol. Next, for the other limit, we notice that

$$\lim_{\alpha\to d} \left(p\theta(\alpha) + \frac{2p(1-\theta(\alpha))}{q} \right) = 2;$$
$$\lim_{\alpha\to d} {}_2F_1(-\alpha/2, (d-\alpha)/2; d/2; |x|^2) = 1, \quad \text{for every } |x| < 1,$$

with the same $\theta = \theta(\alpha)$ defined by Eq. (4.1.13). Hence, by Eq. (4.2.6) and Fatou's Lemma we deduce that

$$\liminf_{\alpha\to d} A_\alpha^{-1} \mathscr{C}_{d,\alpha,p,q} \geq |B_1|^{-1} \pi^{\frac{d}{2}} \liminf_{\alpha\to d} \frac{1}{\Gamma(1+\frac{\alpha}{2})} = |B_1|^{-1} \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})} = 1.$$

Finally, similarly to the relation in (4.2.5), the constant from HLS inequality leads to

$$\limsup_{\alpha\to d} A_\alpha^{-1} \mathscr{C}_{d,\alpha,p,q} \leq \limsup_{\alpha\to d} A_\alpha^{-1} \mathscr{C}_{d,\alpha} \leq 1,$$

concluding the proof of (4.2.4). □

Lemma 4.2.1. *Let u be a non negative, radially, symmetric and nonincreasing maximizer of $\mathscr{C}_{d,\alpha,p,q}$ with $\frac{d+\alpha}{d} < p < 2$ and $q > \frac{2dp}{d+\alpha}$, then* $\mathrm{Supp}(u) = \mathbb{R}^d$.

Proof. Without loss of generality we can assume that $\|u\|_2 = \|u\|_q = 1$. Arguing by contradiction, we assume that there exists a set $A \subset \mathbb{R}^N$ with $A \cap \mathrm{Supp}(u) = \emptyset$ and $0 < |A| < +\infty$. Then we test the function $v := u + \varepsilon\chi_A$ with $\varepsilon > 0$, where χ_A is the characteristic function,

$$\begin{aligned}\mathcal{R}(v)=\frac{\mathcal{D}_\alpha(|v|^p,|v|^p)}{\|v\|^{2p\theta}_{L^2(\mathbb{R}^d)}\|v\|^{2p(1-\theta)}_{L^q(\mathbb{R}^d)}}&=\frac{\mathcal{D}_\alpha(|u+\varepsilon\chi_A|^p,|u+\varepsilon\chi_A|^p)}{(\int_{\mathbb{R}^d}(u^2+\varepsilon^2\chi_A)dx)^{p\theta}(\int_{\mathbb{R}^d}(u^q+\varepsilon^q\chi_A)dx)^{\frac{2p(1-\theta)}{q}}}\\&=\frac{\mathcal{D}_\alpha(|u|^p,|u|^p)+2\varepsilon^p D_\alpha(|u|^p,\chi_A)+\varepsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A)}{(\int_{\mathbb{R}^d}(u^2+\varepsilon^2\chi_A)dx)^{p\theta}(\int_{\mathbb{R}^d}(u^q+\varepsilon^q\chi_A)dx)^{\frac{2p(1-\theta)}{q}}}\\&\geq\frac{\mathcal{D}_\alpha(|u|^p,|u|^p)+2\varepsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A)+\varepsilon^{2p}\mathcal{D}_\alpha\chi_A,\chi_A)}{(1+p\theta\varepsilon^2|A|)(1+\varepsilon^q\frac{2p(1-\theta)|A|}{q})}\\&\geq\frac{\mathcal{D}_\alpha(|u|^p,|u|^p)+2\varepsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A)+\varepsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A)}{1+C\varepsilon^2}\\&\geq\mathscr{C}_{d,\alpha,p,q}+\frac{2\varepsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A)+\varepsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A)-C\varepsilon^2}{1+C\varepsilon^2},\end{aligned}$$

therefore there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, we have $\mathcal{R}(v)>\mathscr{C}_{d,\alpha,p,q}=\mathcal{R}(u)$, which contradicts that u is a maximizer. □

Remark 4.2.1. In what follows, before investigating several equivalent variational settings for ground state solutions of (*TF*), we prove by simple computations that for $d=3$, $\alpha=2$, $p=2$ and $q=4$ the function u defined by (4.1.17) solves (*TF*). One way is to directly employ the formula in [66, Section 5, eq. (5.2)]. On the other hand, one can simply argue as follows.

Since u defined by

$$u(x):=\chi_{B_\pi}(x)\sqrt{\tfrac{\sin(|x|)}{|x|}}$$

is bounded, the Riesz potential I_2*u^2 is continuous and smooth in $\mathbb{R}^d\setminus\partial B_\pi$, see e.g. Proposition 3.3.1. Furthermore, by combining Lemma 3.5.1 with Lemma 4.6.2 we infer that I_2*u^2 is a radially non increasing function solving

$$\begin{cases}-\Delta(I_2*u^2)=0 \text{ in } B_\pi^c,\\ \lim_{|x|\to+\infty}|x|(I_2*u^2)(x)=A_2\|u^2\|_{L^1(\mathbb{R}^d)}.\end{cases}\tag{4.2.8}$$

From (4.2.8) we deduce that

$$(I_2*u^2)(x)=\frac{D}{|x|}\quad\text{if } |x|>\pi$$

for some $D>0$. Then, we can find D in order to satisfy the condition at infinity. This leads to $D=\pi$. Similarly, since $f:=I_2*u^2$ is a bounded radial solution of

$-\Delta f = u^2$ in B_π, we further derive that

$$f''(r) + \frac{2}{r}f'(r) + \frac{\sin(r)}{r} = 0 \quad \text{in } (0,\pi),$$

i.e.,

$$(I_2 * u^2)(x) = C + \frac{\sin|x|}{|x|} \quad \text{in } B_\pi,$$

for some positive constant C. Finally, by continuity we conclude that

$$(I_2 * u^2)(x) = \begin{cases} 1 + \frac{\sin|x|}{|x|} & \text{if } |x| < \pi, \\ \frac{\pi}{|x|} & \text{if } |x| \geq \pi. \end{cases} \tag{4.2.9}$$

Hence, from (4.2.9) it's now straightforward to check that $I_2 * u^2 = 1 + u^2$ in B_π which is equivalent to (*TF*).

4.3 Equivalent variational settings.

Finding optimizers of the Gagliardo–Nirenberg quotient is only one possible approach to construct solutions of (*TF*). Historically, the most traditional approach consists in minimizing the nonconvex quantity

$$E_0(u) := \frac{1}{q}\int_{\mathbb{R}^d} |u|^q \, dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

subject to the L^2-norm constraint, which leads to the family of problems

$$M_c := \inf\left\{E_0(u) \;:\; \|u\|^2_{L^2(\mathbb{R}^d)} = c,\, u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\right\} \qquad (c > 0), \tag{4.3.1}$$

where c has the meaning of the *mass* of the density $\rho := |u|^2$. This is exactly the problem studied in the case $p = 2$ (in terms of ρ) by Auchmuty and Beals [11] and P.-L. Lions [80; 81, Section II] for $\alpha = 2$, and in [29, 33] for $\alpha \in (0, d)$. By the mass preserving scaling, $M_c = -\infty$ for all $q < q_c := 2\big(p - \frac{\alpha}{N}\big)$, that is q_c is the L^2-critical exponent for E_0. Note that under our assumptions $q_c > \frac{2dp}{d+\alpha}$, so the minimization problem for M_c is not well-posed in the range $q \in \big(\frac{2dp}{d+\alpha}, q_c\big)$, which is however covered by the existence result of Theorem 4.1.2.

Another approach to construct ground states of (*TF*) is to minimize the convex functional

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^d} |u|^2\,dx + \frac{1}{q}\int_{\mathbb{R}^d} |u|^q\,dx$$

subject to the non local constraint,

$$m_c =: \inf\Big\{\mathcal{E}(u)\,:\,\mathcal{D}_\alpha(|u|^p,|u|^p) = c,\, u \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)\Big\} \qquad (c>0). \quad (4.3.2)$$

In [82, Proposition 6.1] it has been proved that a minimizers for m_c exists in the full range $q \in \left(\frac{2dp}{d+\alpha},\infty\right)$, for every $c > 0$.

One more approach is to minimize the total energy E defined by (4.1.10) to the Pohožaev constraint,

$$\sigma := \inf\left\{E(u)\,:\,u \in \mathscr{P}\right\}, \qquad (4.3.3)$$

where

$$\mathscr{P} := \big\{0 \neq u \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d) : \mathcal{P}(u) = 0\big\}$$

and

$$\mathcal{P}(u) := \frac{d}{2}\int_{\mathbb{R}^d} |u|^2 dx + \frac{d}{q}\int_{\mathbb{R}^d} |u|^q dx - \frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p). \qquad (4.3.4)$$

To show that $\sigma > 0$ is well defined, we first show that $\mathscr{P} \neq \emptyset$.

Lemma 4.3.1. *For each $u \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)\setminus\{0\}$, there exists a unique $t_u > 0$ such that $u(x/t_u) \in \mathscr{P}$.*

Proof. For each $u \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)\setminus\{0\}$, let $u_t(x) = u(x/t)$, then we have

$$E(u_t) = t^d\Big(\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{q}\|u\|^q_{L^q(\mathbb{R}^d)}\Big) - \frac{t^{d+\alpha}}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p).$$

It is easy to see that there exists a unique $t_u > 0$ such that $E(u_{t_u}) = \max_{t>0} E(u_t)$ and $E(u_t)'|_{t_u} = 0$, which means that $u_{t_u} \in \mathscr{P}$. □

Let us define $M : L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d) \to \mathbb{R}_+$ as

$$M(u) = \|u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^q_{L^q(\mathbb{R}^d)}.$$

Just taking into account the definition of M and the norm $\|\cdot\|_{L^2\cap L^q}$[1], we can check

[1] Recall that $L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ in naturally endowed with the norm

$$\|\cdot\|_{L^2\cap L^q} = \|\cdot\|_{L^2(\mathbb{R}^d)} + \|\cdot\|_{L^q(\mathbb{R}^d)}$$

that

$$2^{-\frac{q}{2}}\|u\|^q_{L^2\cap L^q} \leq M(u) \leq \|u\|_{L^2\cap L^q}, \text{if either } M(u)\leq 1 \text{ or } \|u\|_{L^2\cap L^q}\leq 1. \tag{4.3.5}$$

Thus we can give the following estimate between M and V.

Lemma 4.3.2. *There exists $C>0$ such that, for all $u\in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$,*

$$\mathcal{D}_\alpha(|u|^p,|u|^p)\leq CM(u)^{\frac{d+\alpha}{d}}.$$

Proof. For each $u\in L^2\cap L^q(\mathbb{R}^d)\setminus\{0\}$, let u_t be defined in Lemma 4.3.1 with $t=M(u)^{-\frac{1}{d}}$, then $M(u_t)=t^dM(u)=1$. It follows from the HLS inequality, the embedding $L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)\hookrightarrow L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)$, and (4.3.5) that

$$\begin{aligned}\mathcal{D}_\alpha(|u|^p,|u|^p)) = t^{-(d+\alpha)}\mathcal{D}_\alpha(|u_t|^p,|u_t|^p) &\leq Ct^{-(d+\alpha)}\|u_t\|^{2p}_{L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)}\\ &\leq Ct^{-(d+\alpha)}\|u_t\|^{2p}_{L^2\cap L^q}\leq CM(u)^{\frac{d+\alpha}{d}},\end{aligned}$$

which is the desired conclusion. □

Lemma 4.3.3. *There exists $\eta>0$ such that $\mathcal{D}_\alpha(|u|^p,|u|^p)\geq\eta$ for all $u\in\mathscr{P}$. Moreover, $\sigma>0$.*

Proof. For each $u\in\mathscr{P}$, by Lemma 4.3.2, we see that there exists $C>0$ such that

$$0=\mathcal{P}(u)\geq\frac{d}{q}M(u)-\frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p)\geq C\mathcal{D}_\alpha(|u|^p,|u|^p)^{\frac{d}{d+\alpha}}-\frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p),$$

which implies that there exists $\eta>0$ such that $\mathcal{D}_\alpha(|u|^p,|u|^p)\geq\eta$. Since $u\in\mathscr{P}$, we have $E(u)=\frac{\alpha}{2dp}\mathcal{D}_\alpha(|u|^p,|u|^p)\geq\frac{\alpha\eta}{2dp}$. Therefore $\sigma\geq\frac{\alpha\eta}{2dp}>0$. □

Lemma 4.3.3 shows that σ make sense. The next result describes the relationships of m_c, $\mathscr{C}_{d,\alpha,p,q}$ and σ. As a matter of simplicity, we focus on the case $c=1$ only.

Lemma 4.3.4. *The above three constrained problems (4.2.1), (4.3.2) and (4.3.3) are equivalent, that is:*

$$\sigma=\alpha(2dp)^{\frac{d}{\alpha}}\left(\frac{m_1}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}},\quad m_1=\theta_*\mathscr{C}^{-\frac{d}{d+\alpha}}_{d,\alpha,p,q}, \tag{4.3.6}$$

where θ_* *is given by*

$$\theta_* := \Big(\frac{1-\theta}{\theta}\Big)^{\frac{q\theta}{2(1-\theta)+q\theta}}\Big(\frac{\theta}{2(1-\theta)}+\frac{1}{q}\Big).$$

In particular

$$\sigma = \alpha(2dp)^{\frac{d}{\alpha}}\left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}}\mathscr{C}_{d,\alpha,p,q}^{-\frac{d}{\alpha}}. \tag{4.3.7}$$

Proof. First of all, we show that (4.2.1) and (4.3.3) are equivalent up to rescaling. Indeed, for each $u \in \mathscr{P}$, by Lemma 4.3.3 we see that $\mathcal{D}_\alpha(|u|^p, |u|^p) > 0$. Then, there exists a unique $t_u = \left(\frac{1}{\mathcal{D}_\alpha(|u|^p,|u|^p)}\right)^{\frac{1}{d+\alpha}}$ such that

$$\mathcal{D}_\alpha(|u(\cdot/t_u)|^p, |u(\cdot/t_u)|^p) = t_u^{d+\alpha}\mathcal{D}_\alpha(|u|^p, |u|^p) = 1,$$

which means that $u(x/t_u) \in \mathcal{A}_1$ where by $\mathcal{A}_1$ we denote

$$\mathcal{A}_1 = \left\{u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) : \mathcal{D}_\alpha(|u|^p, |u|^p) = 1\right\}.$$

On the other hand, for each $v \in \mathcal{A}_1$ there exists a unique $t_v := \left(\frac{2dp\mathcal{E}(v)}{d+\alpha}\right)^{\frac{1}{\alpha}}$ such that $v(x/t_v) \in \mathscr{P}$. Moreover, let's notice that the following holds

$$\begin{aligned}&\mathscr{P} \longrightarrow \mathcal{A}_1 \longrightarrow \mathscr{P}\\ &u \longmapsto u\left(\tfrac{x}{t_u}\right) := v \longmapsto v\left(\tfrac{x}{t_v}\right) = u.\end{aligned} \tag{4.3.8}$$

As a matter of fact,

$$t_v = \left(\frac{2dp\,\mathcal{E}(v)}{d+\alpha}\right)^{\frac{1}{\alpha}} = \left(\frac{2dp\,\mathcal{E}(u)}{d+\alpha}\right)^{\frac{1}{\alpha}} t_u^{\frac{d}{\alpha}},$$

from which

$$t_v \cdot t_u = \left(\frac{2dp\,\mathcal{E}(u)}{d+\alpha}\right)^{\frac{1}{\alpha}} t_u^{\frac{d+\alpha}{\alpha}} = (\mathcal{D}_\alpha(|u|^p, |u|^p))^{\frac{1}{\alpha}} \cdot (\mathcal{D}_\alpha(|u|^p, |u|^p))^{-\frac{1}{\alpha}} = 1.$$

Similarly,

$$\begin{aligned}&\mathcal{A}_1 \longrightarrow \mathscr{P} \longrightarrow \mathcal{A}_1\\ &v \longmapsto v\left(\tfrac{x}{t_v}\right) := u \longmapsto u\left(\tfrac{x}{t_u}\right) = v\end{aligned} \tag{4.3.9}$$

since

$$t_u = \left(\frac{1}{\mathcal{D}_\alpha(|u|^p, |u|^p)}\right)^{\frac{1}{d+\alpha}} = \left(\frac{1}{\mathcal{D}_\alpha(|v(\cdot/t_v)|^p, |v(\cdot/t_v)|^p)}\right)^{\frac{1}{d+\alpha}} = t_v^{-1}.$$

Thus, for every $u \in \mathscr{P}$, we can find a unique $v \in \mathcal{A}_1$ such that $v(x/t_v) = u(x)$ and

$$\begin{aligned}
\sigma = \inf_{u\in\mathscr{P}} E(u) &= \inf_{v\in\mathcal{A}_1} E(v(x/t_v)) = \inf_{v\in\mathcal{A}_1} \left\{ t_v^d\,\mathcal{E}(v) - \frac{1}{2p} t_v^{d+\alpha} \right\} \\
&= \inf_{v\in\mathcal{A}_1} \frac{\alpha}{d+\alpha} \left(\frac{2dp\,\mathcal{E}(v)}{d+\alpha}\right)^{\frac{d}{\alpha}} \mathcal{E}(v) \\
&= \frac{\alpha}{d+\alpha} \left(\frac{2dp}{d+\alpha}\right)^{\frac{d}{\alpha}} \inf_{v\in\mathcal{A}_1} \mathcal{E}(v)^{\frac{d+\alpha}{\alpha}} \\
&= \frac{\alpha}{d+\alpha} \Big(\frac{2dp}{d+\alpha}\Big)^{\frac{d}{\alpha}} m_1^{\frac{d+\alpha}{\alpha}} \\
&= \alpha(2dp)^{\frac{d}{\alpha}} \Big(\frac{m_1}{d+\alpha}\Big)^{\frac{d+\alpha}{\alpha}}.
\end{aligned}$$

This implies that σ is achieved if and only if m_1 is achieved.
Now we show that the minimization problem (4.3.2) and (4.2.1) are equivalent. Indeed, given $u \in \mathcal{A}_1$, after a rescaling, $u_t(x) = t^{-\frac{d+\alpha}{2p}} u(\frac{x}{t}) \in \mathcal{A}_1$ for all $t > 0$. Minimizing $\mathcal{E}(u_t)$ with respect $t > 0$, after a direct computation, we find that

$$\mathcal{E}(u_{t_1}) = \min_{t>0} \mathcal{E}(u_t) = \theta_* \mathcal{R}(u)^{-\frac{q}{p(2(1-\theta)+q\theta)}} = \theta_* \mathcal{R}(u)^{-\frac{d}{d+\alpha}}, \tag{4.3.10}$$

where θ is given in (4.1.13) and

$$t_1 := \left(\frac{((d+\alpha)q - 2dp)\|u\|^q_{L^q(\mathbb{R}^d)}}{q(dp-d-\alpha)\|u\|^2_{L^2(\mathbb{R}^d)}}\right)^{\frac{2p}{(q-2)(d+\alpha)}}, \quad \theta_* := \Big(\frac{1-\theta}{\theta}\Big)^{\frac{q\theta}{2(1-\theta)+q\theta}} \Big(\frac{\theta}{2(1-\theta)} + \frac{1}{q}\Big).$$

Moreover, from (4.3.10) we get

$$\begin{aligned}
m_1 &= \inf_{u\in\mathcal{A}_1} \min_{t>0} \mathcal{E}(u_t) = \theta_* \inf_{u\in\mathcal{A}_1} \mathcal{R}(u)^{-\frac{d}{d+\alpha}} = \theta_* \Big(\sup_{\mathcal{A}_1} \mathcal{R}(u)\Big)^{-\frac{d}{d+\alpha}} \\
&= \theta_* \Big(\sup_{u\in L^2\cap L^q(\mathbb{R}^d)} \mathcal{R}(u)\Big)^{-\frac{d}{d+\alpha}} = \theta_* \mathscr{C}_{d,\alpha,p,q}^{-\frac{d}{d+\alpha}},
\end{aligned} \tag{4.3.11}$$

where we used that $\mathcal{R}$ is scaling invariance and so $\sup_{\mathcal{A}_1} \mathcal{R}(u) = \sup_{u\in L^2\cap L^q(\mathbb{R}^d)} \mathcal{R}(u)$. In particular, $\mathscr{C}_{d,\alpha,p,q}$ in the maximization problem (4.2.1) is achieved if and only if

m_1 is achieved in the minimization problem (4.3.2).

Finally, (4.3.7) follows simply by combining the relations in (4.3.6). □

4.3.1 Connection with ground states of (*TF*)

We recall that by a *ground state* of (*TF*) we understand a function $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ solving (*TF*) which has a minimal energy E defined by (4.1.10) amongst all functions belonging to the Pohožaev manifold $\mathscr{P}$.

In this subsection (see Lemma 4.3.6), taking into account Lemma 4.3.4, we explicitly compute the rescaling which transform any maximizer for $\mathscr{C}_{d,\alpha,p,q}$ to a ground state solution of (*TF*).

First, we recall that, the Euler–Lagrange equation of the quantity $\log \mathcal{R}(u)$ for $u \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ has the form

$$Au + B|u|^{q-2}u = C(I_\alpha * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^d, \tag{4.3.12}$$

where

$$A := \frac{2p\theta}{\|u\|^2_{L^2(\mathbb{R}^d)}}, \quad B := \frac{2p(1-\theta)}{\|u\|^q_{L^q(\mathbb{R}^d)}}, \quad C := \frac{2p}{\mathcal{D}_\alpha(|u|^p, |u|^p)}.$$

In particularly, maximizers of $\mathscr{C}_{d,\alpha,p,q}$ in Eq. (4.2.1) constructed in the proof of Theorem 4.1.3 are solutions of Eq. (4.3.12) and, after a rescaling, of (*TF*). Indeed, given an optimizer u for $\mathscr{C}_{d,\alpha,p,q}$, let's consider the function $u_{\lambda,\mu} = \lambda u(\mu x)$, for some $\lambda > 0, \mu > 0$. By the definition of θ (equation (4.1.13)) we have that $\mathcal{R}(u) = \mathcal{R}(u_{\lambda,\mu})$. Then, we can choose λ, μ such that $A = B = C$, the corresponding $u_{\lambda,\mu}$ (which is still a maximizer of $\mathscr{C}_{d,\alpha,p,q}$) satisfies (*TF*).

Lemma 4.3.5. *Let $\alpha \in (0, d)$, $d \geq 1$, $p > \frac{d+\alpha}{d}$ and $q > \frac{2dp}{d+\alpha}$. If u is a maximizer for $\mathscr{C}_{d,\alpha,p,q}$ then the function $u_{\lambda_*,\mu_*} = \lambda_* u(\mu_* x)$ with*

$$\lambda_* := \left(\left(\frac{1-\theta}{\theta} \right) \frac{\|u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^q_{L^q(\mathbb{R}^d)}} \right)^{\frac{1}{q-2}}, \tag{4.3.13}$$

$$\mu_* := \left(\left(\frac{1-\theta}{\lambda_*^{q-2p}} \right) \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|^q_{L^q(\mathbb{R}^d)}} \right)^{\frac{1}{\alpha}}, \tag{4.3.14}$$

is a solution of (TF).

Proof. Let u be a maximizer for $\mathscr{C}_{d,\alpha,p,q}$. Then, $u_{\lambda,\mu}$ remains a maximizer for $\mathscr{C}_{d,\alpha,p,q}$. Moreover, if we set $\lambda := \lambda_*$ where

$$\lambda_*^{q-2} := \left(\frac{1-\theta}{\theta}\right) \frac{\|u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^q_{L^q(\mathbb{R}^d)}},$$

we obtain that

$$\frac{2p\theta}{\|u_{\lambda_*,\mu}\|^2_{L^2(\mathbb{R}^d)}} = \frac{2p(1-\theta)}{\|u_{\lambda_*,\mu}\|^q_{L^q(\mathbb{R}^d)}}.$$

Furthermore, if we set $\mu := \mu_*$ where

$$\mu_*^{\alpha} := \left(\frac{1-\theta}{\lambda_*^{q-2p}}\right) \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|^q_{L^q(\mathbb{R}^d)}},$$

we get that

$$A = \frac{2p\theta}{\|u_{\lambda_*,\mu}\|^2_{L^2(\mathbb{R}^d)}} = B = \frac{2p(1-\theta)}{\|u_{\lambda_*,\mu_*}\|^q_{L^q(\mathbb{R}^d)}} = C = \frac{2p}{\mathcal{D}_\alpha(|u_{\lambda_*,\mu_*}|^p, |u_{\lambda_*,\mu_*}|^p)}, \tag{4.3.15}$$

concluding the proof. □

Lemma 4.3.6. *Let u be a maximizer of $\mathscr{C}_{d,\alpha,p,q}$. Then, the function u_{λ_*,μ_*} where λ_*, μ_* are defined respectively by (4.3.13) and (4.3.14) is a ground state solution of (TF).*

Proof. For the convenience of the reader let's set $u_* = u_{\lambda_*,\mu_*}$. From Lemma 4.3.5 we know that u_* is a solution of (*TF*). Then, by definition of ground state, it remains to prove that $\mathcal{P}(u_*) = 0$ and that u_* minimizes the energy E on $\mathscr{P}$.

First of all, from the equality (4.3.15), we deduce the following relations

$$\|u_*\|^q_{L^q(\mathbb{R}^d)} = \left(\frac{1-\theta}{\theta}\right) \|u_*\|^2_{L^2(\mathbb{R}^d)}, \quad \mathcal{D}_\alpha(|u_*|^p, |u_*|^p) = \frac{\|u_*\|^2_{L^2(\mathbb{R}^d)}}{\theta}. \tag{4.3.16}$$

Next, taking into account (4.3.16) and (4.1.13), we deduce that

$$\mathcal{P}(u_*) = \|u_*\|^2_{L^2(\mathbb{R}^d)} \left(\frac{d}{2} + \frac{d(1-\theta)}{q\theta} - \frac{d+\alpha}{2p\theta}\right) = 0.$$

Moreover, by Lemma 4.3.4 we obtain the relation

$$\inf_{u\in\mathscr{P}} E(u) = \sigma = \alpha(2dp)^{\frac{d}{\alpha}} \left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}} \mathscr{C}_{d,\alpha,p,q}^{-\frac{d}{\alpha}}, \tag{4.3.17}$$

where θ_* is defined by

$$\theta_* := \left(\frac{1-\theta}{\theta}\right)^{\frac{q\theta}{2(1-\theta)+q\theta}} \left(\frac{\theta}{2(1-\theta)} + \frac{1}{q}\right). \tag{4.3.18}$$

We claim that u_* satisfies

$$E(u_*) = \alpha(2dp)^{\frac{d}{\alpha}} \left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}} \mathcal{R}(u_*)^{-\frac{d}{\alpha}} \tag{4.3.19}$$

which, in view of (4.3.17), proves that u_* has minimal energy on $\mathscr{P}$.
By (4.3.16) we obtain that

$$E(u_*) = \frac{\alpha}{2dp\theta} \|u_*\|^2_{L^2(\mathbb{R}^d)}, \tag{4.3.20}$$

and

$$\mathcal{R}(u_*)^{-\frac{d}{\alpha}} = \theta^{\frac{d}{\alpha}} \left(\frac{1-\theta}{\theta}\right)^{\frac{2dp(1-\theta)}{q\alpha}} \|u_*\|_{L^2(\mathbb{R}^d)}^{\left(2p\theta - \frac{4p(1-\theta)}{q}\right)\frac{d}{\alpha}}. \tag{4.3.21}$$

Then, by (4.1.13), we notice that

$$\left(2p\theta - \frac{4p(1-\theta)}{q}\right)\frac{d}{\alpha} = 2$$

from which

$$\mathcal{R}(u_*)^{-\frac{d}{\alpha}} = \theta^{\frac{d}{\alpha}} \left(\frac{1-\theta}{\theta}\right)^{\frac{2dp(1-\theta)}{q\alpha}} \|u_*\|^2_{L^2(\mathbb{R}^d)}. \tag{4.3.22}$$

In order to conclude it's enough to prove the following equality

$$\theta^{\frac{d}{\alpha}}\left(\frac{1-\theta}{\theta}\right)^{\frac{2dp(1-\theta)}{q\alpha}} \alpha(2dp)^{\frac{d}{\alpha}}(d+\alpha)^{-\frac{d+\alpha}{\alpha}}\left(\frac{1-\theta}{\theta}\right)^{\frac{q\theta}{2(1-\theta)+q\theta}}\left[\left(\frac{\theta}{2(1-\theta)}+\frac{1}{q}\right)\right]^{\frac{d+\alpha}{\alpha}} = \frac{\alpha}{2dp\theta}. \tag{4.3.23}$$

First we notice that (again by (4.1.13))

$$\left(\frac{\theta}{2(1-\theta)} + \frac{1}{q}\right) = \frac{d+\alpha}{2dp(1-\theta)},$$

from which (4.3.23) is equivalent to

$$\left(\frac{\theta}{1-\theta}\right)^{\frac{d+\alpha}{\alpha}-\frac{2dp(1-\theta)}{q\alpha}-\frac{q\theta(d+\alpha)}{2\alpha(1-\theta)+q\theta\alpha}} = 1, \tag{4.3.24}$$

which is true since (again by (4.1.13))

$$\begin{aligned}\frac{d+\alpha}{\alpha}-\frac{2dp(1-\theta)}{q\alpha}-\frac{q\theta(d+\alpha)}{2\alpha(1-\theta)+q\theta\alpha} &= \frac{d+\alpha}{\alpha}-\frac{2dp(1-\theta)}{q\alpha}-\frac{dp\theta}{\alpha} \\ &= 0.\end{aligned} \tag{4.3.25}$$

□

Remark 4.3.1. Note that the proof of Lemma 4.3.6 seems more complicated than needed because of a technical reason we will explain in this remark. As we have already proved, from (4.3.16) we immediately conclude that $u_* \in \mathscr{P}$. In particular, if we could say that $\mathscr{P}$ is a natural constraint i.e., if u is a critical point for E in $\mathscr{P}$ then $E'(u) = 0$, we would get the thesis. Indeed, if $\mathscr{P}$ is a natural constraint and u_* is not a minimizer, by arguing as in Section 4.1.1 we can find a minimizer u_0 such that

$$\sigma = E(u_0) < E(u_*).$$

On the other hand, by the assumption on $\mathscr{P}$ being a natural constraint, u_0 solves (*TF*) and in particular, it satisfies the Nehari identity (4.1.14). However, by arguing as in the first part of Lemma 4.3.6 we infer that u_0 satisfies

$$E(u_0) = \alpha(2dp)^{\frac{d}{\alpha}}\left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}}\mathcal{R}(u_0)^{-\frac{d}{\alpha}} < E(u_*) = \alpha(2dp)^{\frac{d}{\alpha}}\left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}}\mathcal{R}(u_*)^{-\frac{d}{\alpha}},$$

but this contradicts the fact that u_* maximizes the quotient $\mathcal{R}$. Taking into account the above argument it remains to prove that $\mathscr{P}$ is a natural constraint. Nevertheless, by looking at the proof of the existence in Section 4.1.1 (see again [82, Corollary 4.1, Proposition 4.2]) some regularity is needed. While in the previous framework such regularity is obtained because of the presence of the diffusion term $-\Delta$, we already know that we can not expect this regularity to be achieved in our Thomas–Fermi setting (see for instance Corollary 4.4.2 and Lemma 4.4.3 where we in fact prove that a solution of (*TF*) is not even continuous if $p > 2$).

Lemma 4.3.7. *Let u be a minimizer for E in $\mathscr{P}$. Then, the function $u(x/t_u)$ is a minimizer for $\mathcal{E}$ in $\mathcal{A}_1$, where t_u is defined by*

$$t_u := \left(\frac{1}{\mathcal{D}_\alpha(|u|^p, |u|^p)}\right)^{\frac{1}{d+\alpha}}. \tag{4.3.26}$$

Proof. Let u be a minimizer for E in $\mathscr{P}$. It's easy to see easy that the function defined by $v(x) = u(x/t_u)$ belongs to $\mathcal{A}_1$. Moreover, in view of Lemma 4.3.4 we only need to prove that

$$E(u) = \alpha(2dp)^{\frac{d}{\alpha}}\left(\frac{\mathcal{E}(v)}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}}. \tag{4.3.27}$$

To this aim we first note that

$$\begin{aligned}\mathcal{E}(v) &= \left(\frac{1}{\mathcal{D}_\alpha(|u|^p, |u|^p)}\right)^{\frac{d}{d+\alpha}}\left[\frac{1}{2}\int_{\mathbb{R}^d}|u|^2 + \frac{1}{q}\int_{\mathbb{R}^d}|u|^q\right]\\ &= \frac{d+\alpha}{2dp}\mathcal{D}_\alpha(|u|^p, |u|^p).\end{aligned} \tag{4.3.28}$$

Moreover, by combining the equality

$$E(u) = \sigma = \frac{\alpha}{2dp}\mathcal{D}_\alpha(|u|^p, |u|^p)$$

with (4.3.28), we infer

$$\mathcal{E}(v) = \left(\frac{1}{2dp}\right)^{\frac{d}{d+\alpha}}(d+\alpha)\left(\frac{\sigma}{\alpha}\right)^{\frac{\alpha}{d+\alpha}}$$

which is equivalent to (4.3.27). □

Remark 4.3.2. Note that, by combining Lemma 4.3.6 with (4.3.7), if u is a maximizer of $\mathscr{C}_{N,\alpha,p,q}$, then function $u_{\lambda,\mu}$ defined by

$$u_{\lambda,\mu}(x) := \lambda u(\mu x),$$

with

$$\lambda := \left(\left(\frac{1-\theta}{\theta}\right)\frac{\|u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^q_{L^q(\mathbb{R}^d)}}\right)^{\frac{1}{q-2}}, \quad \mu := \left(\left(\frac{1-\theta}{\theta}\right)\frac{\|u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^q_{L^q(\mathbb{R}^d)}}\right)^{\frac{2p}{(d+\alpha)(q-2)}}(\mathcal{D}_\alpha(|u|^p, |u|^p))^{\frac{1}{d+\alpha}}$$

is a minimizer for $\mathcal{E}$ in $\mathcal{A}_1$.

Remark 4.3.3. We point out that, although the trivial inequality $\mathscr{C}_{d,\alpha,p,q} \leq \mathscr{C}_{d,\alpha}$, cf (4.1.12), the equality never holds. This can be easily seen from the Euler–Lagrange equation (4.3.12). As a matter of fact, assume by contradiction that $\mathscr{C}_{d,\alpha,p,q} = \mathscr{C}_{d,\alpha}$. Then, the optimizer u_* constructed in the proof of Theorem 4.1.2 satisfies

$$\mathscr{C}_{d,\alpha} = \mathscr{C}_{d,\alpha,p,q} = \frac{\mathcal{D}_\alpha(u_*^p, u_*^p)}{\|u_*\|_{L^2(\mathbb{R}^d)}^{2p\theta} \|u_*\|_{L^q(\mathbb{R}^d)}^{2p(1-\theta)}} \leq \frac{\mathcal{D}_\alpha(u_*^p, u_*^p)}{\|u_*^p\|^2_{L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)}} \leq \mathscr{C}_{d,\alpha}, \tag{4.3.29}$$

i.e., u_*^p is an optimizer for HLS inequality. In particular, since u_* is non negative and radially non increasing we infer (see [78, Theorem 4.3])

$$u_*(x) = \frac{\lambda}{(\gamma^2 + |x|^2)^{\frac{d+\alpha}{2p}}}, \quad \gamma > 0,\ \lambda > 0. \tag{4.3.30}$$

Next, we claim that if u_* is of the form (4.3.30) then it can not satisfy the Euler–Lagrange equation (4.3.12). Indeed, if it was true we would obtain the relation

$$2p\theta u_* + 2p(1-\theta)u_*^{q-1} = \frac{2p}{\mathscr{C}_{d,\alpha,p,q}}(I_\alpha * u_*^p)u_*^{p-1}. \tag{4.3.31}$$

Therefore, by Lemma 3.5.1 and the fact that $q > 2$ we obtain that

$$\frac{2p\theta\lambda}{|x|^{\frac{d+\alpha}{2p}}} = \frac{2pA_\alpha}{\mathscr{C}_{d,\alpha,p,q}} \frac{\|u_*^p\|_{L^1(\mathbb{R}^d)}}{|x|^{d-\alpha+\frac{(d+\alpha)(p-1)}{p}}} + o\left(\frac{1}{|x|^{\frac{d+\alpha}{2p}}}\right) \quad \text{as } |x| \to +\infty. \tag{4.3.32}$$

Thus, to have the equality in (4.3.32) we must at least have

$$\frac{d+\alpha}{p} = d - \alpha + \frac{(d+\alpha)(p-1)}{p}, \tag{4.3.33}$$

which is always false because of the assumption $p > \frac{d+\alpha}{d}$. We have therefore proved that $\mathscr{C}_{d,\alpha,p,q} < \mathscr{C}_{d,\alpha}$.

4.4 Regularity, decay and support

In this section we prove some results mainly addressing regularity and support properties of radially non increasing ground state solutions for (*TF*). Furthermore, we prove that such ground state solutions decay polynomially if $p < 2$ and are compactly supported if $p \geq 2$.

4.4.1 Decay properties

Recall that if $s \in (1, \frac{d}{\alpha})$ and $\frac{1}{t} = \frac{1}{s} - \frac{\alpha}{d}$, then

$$I_\alpha * (\cdot) : L^s(\mathbb{R}^d) \to L^t(\mathbb{R}^d)$$

is bounded. We first establish the following fact about the far field behaviour of $I_\alpha * u^p$: if the non negative function u decays fast enough, then $I_\alpha * u^p$ decays algebraically like the Riesz potential I_α itself.

Lemma 4.4.1. *Assume that $p > \frac{d+\alpha}{d}$ and $q > \frac{2dp}{d+\alpha}$. Let $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ be a non negative radial non increasing solution of (TF). Then there exists $\epsilon > 0$ such that $u \in L^{p-\epsilon}(\mathbb{R}^d)$ and*

$$\lim_{|x|\to\infty} \frac{I_\alpha * u^p}{I_\alpha(x) \int_{\mathbb{R}^d} u^p dx} = 1. \tag{4.4.1}$$

Proof. We first prove that $u \in L^{p-\epsilon}(\mathbb{R}^d)$ some $\epsilon > 0$, which is trivial if $p > 2$. Otherwise if $p \in (\frac{d+\alpha}{d}, 2]$, we can show that $u \in L^{s_n}(\mathbb{R}^d)$ for a sequence (s_n) of positive decreasing exponents eventually smaller than p.

First, by Hölder inequality, we see that

$$\int_{\mathbb{R}^d} |(I_\alpha * u^p)u^{p-1}|^\sigma dx \leq \Big(\int_{\mathbb{R}^d} |(I_\alpha * u^p)|^{\sigma t} dx\Big)^{\frac{1}{t}} \Big(\int_{\mathbb{R}^d} u^{(p-1)\sigma r} dx\Big)^{\frac{1}{r}}, \tag{4.4.2}$$

provided that $1/t + 1/r = 1$ for positive t and r. We want to find a sequence (s_n) of positive numbers, so that if $u \in L^{s_n}(\mathbb{R}^d)$, then $u \in L^{s_{n+1}}(\mathbb{R}^d)$. By choosing the parameters σ, t and r in Eq. (4.4.2) so that

$$\sigma := s_{n+1}, \qquad (p-1)\sigma r := s_n, \qquad \frac{1}{\sigma t} = \frac{p}{s_n} - \frac{\alpha}{d}.$$

The last equation, arising from the HLS inequality, has to be supplied with the condition $\alpha/d < p/s_n < 1$, or $s_n \in (p, dp/\alpha)$. Therefore, the sequence (s_n) satisfies the recursion relation

$$\frac{1}{s_{n+1}} = \frac{1}{\sigma t} + \frac{1}{\sigma r} = \frac{p}{s_n} - \frac{\alpha}{d} + \frac{p-1}{s_n} = \frac{2p-1}{s_n} - \frac{\alpha}{d}. \tag{4.4.3}$$

With the (unstable) fixed point $s_* = 2d(p-1)/\alpha > 2$ (as $p > (d+\alpha)/d$), the general term can be written as

$$\frac{1}{s_n} = (2p-1)^n \left(\frac{1}{s_0} - \frac{1}{s_*}\right) + \frac{1}{s_*}. \tag{4.4.4}$$

If s_0 is chosen to be any number inside the interval $(2, s_*)$, then s_n is monotonically decreasing to zero. Therefore, we can look for the largest integer n_0 such that $s_{n_0} > p$. If $s_{n_0+1} < p$, then $u \in L^{p-\epsilon}$ for any positive $\epsilon < p - s_{n_0+1}$. Otherwise, if $s_{n_0+1} = p$, we can always choose s_0 slightly smaller to get $s_{n_0+1} < p$, since by the recursion relation Eq. (4.4.4), s_n depends continuously and monotonically

Consequently, by the Strauss' $L^{p-\epsilon}$–bound we have the following faster decay estimate for the radially symmetric and non increasing function u:

$$u(|x|) \leq C|x|^{-\frac{d}{p-\epsilon}} \quad (|x| > 0).$$

Then, by Lemma 3.5.1, we get the desired limit (4.4.1). □

Corollary 4.4.1. *Assume that* $\frac{d+\alpha}{d} < p < 2$ *and* $q > \frac{2dp}{d+\alpha}$*. Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a non negative radial non increasing solution of (TF). Then* u *satisfies the following algebraic decay rate*

$$\lim_{x\to\infty} u(x)|x|^{\frac{d-\alpha}{2-p}} = \left(A_\alpha \int_{\mathbb{R}^d} u^p\, dx\right)^{\frac{1}{2-p}}, \tag{4.4.5}$$

where $A_\alpha > 0$ *is the Riesz constant. Furthermore, we have that* $u \in L^1(\mathbb{R}^d)$*.*

Proof. By Lemma 4.4.1, $I_\alpha * u^p = I_\alpha(x) \int_{\mathbb{R}^d} u^p dx\big(1 + o(1)\big)$ as $|x| \to \infty$. Hence, the governing equation (TF) implies that

$$\lim_{|x|\to\infty} u(x)^{2-p}|x|^{d-\alpha}(1 + u^{q-2}(x)) = \lim_{|x|\to\infty} |x|^{d-\alpha} I_\alpha * u^p = A_\alpha \int_{\mathbb{R}^d} u^p\, dx.$$

From the monotonicity of u and the fact that $q > 2$, we conclude that $u^{q-2}(|x|)$ vanishes at infinity, and therefore

$$\lim_{|x|\to\infty} u^{2-p}(x)|x|^{d-\alpha} = A_\alpha \int_{\mathbb{R}^d} u^p\, dx,$$

which is equivalent to (4.4.5). From the condition $\frac{d+\alpha}{d} < p < 2$, the power $\frac{d-\alpha}{2-p}$ is strictly larger than d. That is, u decays faster than $|x|^{-d}$ and hence $u \in L^1(\mathbb{R}^d)$. □

4.4.2 Regularity and support

In the subsequent results we prove up to the boundary regularity of radially non increasing ground state solutions for (*TF*). In particular, we derive smoothness in the interior of their supports, cf. Lemma 4.4.4.

Lemma 4.4.2. *Assume that $p > \frac{d+\alpha}{d}$ and $q > \frac{2dp}{d+\alpha}$. Let $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ be a non negative solution of (TF). Then $u \in L^\infty(\mathbb{R}^d)$ and*

$$I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^d) \quad \textit{for every} \quad \tau \in (0, \min\{\alpha, 1\}). \tag{4.4.6}$$

In particular, if $p \leq 2$ the following hold:

(*i*) *If $p < 2$ then u is Hölder continuous in $\{u > 0\}$ of order τ, for every $\tau \in (0, \min\{\alpha, 1\})$.*

(*ii*) *If $p = 2$ then u is Hölder continuous in $\{u > 0\}$ of order $\kappa(q)$, where $\kappa(q)$ is defined by*

$$\kappa(q) := \begin{cases} \tau, & \text{if } q \leq 3, \\ \frac{\tau}{q-2}, & \text{if } q > 3, \end{cases} \tag{4.4.7}$$

for every $\tau \in (0, \min\{\alpha, 1\})$.

Proof. Assume $u \in L^s(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, where $s \in \left(p, \frac{dp}{\alpha}\right)$. Note that $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ implies that $I_\alpha * u^p$ is almost everywhere finite on $\mathbb{R}^d$. Moreover, by the HLS inequality, if $s \in (p, dp/\alpha)$ then $I_\alpha * u^p \in L^\tau(\mathbb{R}^d)$ where

$$\frac{1}{\tau} := \frac{p}{s} - \frac{\alpha}{d} > 0. \tag{4.4.8}$$

Then (4.4.2) implies that $(I_\alpha * u^p)u^{p-1} \in L^\sigma(\mathbb{R}^d)$ for $\sigma \geq 1$ and $(I_\alpha * u^p)u^{p-1} \in \mathcal{L}^{1/\sigma}(\mathbb{R}^d)$ for $\sigma \in (0,1)$.[2]

Next, we split the argument in three cases:

Case 1: $q > \frac{dp}{\alpha}$. In this case, $u \in L^{\bar{q}}(\mathbb{R}^d)$ for some $\bar{q} \in \left(\frac{dp}{\alpha}, q\right]$ such that $\alpha - \frac{d}{\bar{q}} < 1$. Then $I_\alpha * u^p \in L^\infty(\mathbb{R}^d)$ and is Hölder continuous of order $\alpha - \frac{d}{\bar{q}}$ (cf Lemma 1.4.3).

[2] We denote $\mathcal{L}^t(\mathbb{R}^N) = \{f : \mathbb{R}^n \to \mathbb{R} : \int |f|^t dx < \infty\}$, where $t \in (0,1)$. Note that $\mathcal{L}^t(\mathbb{R}^N)$ is no longer a normed space for $t \in (0,1)$ because the triangle inequality does not hold.

CASE 2: $q = \frac{dp}{\alpha}$. Since in this case there exists $\epsilon > 0$ small such that $u \in L^{p\left(\frac{d}{\alpha}-\epsilon\right)}(\mathbb{R}^d)$, from (4.4.8) we get $(I_\alpha * u^p)u^{p-1} \in L^\sigma(\mathbb{R}^d)$ where

$$\frac{1}{\sigma} := \frac{2p-1}{p\,(d/\alpha - \epsilon)} - \frac{\alpha}{d}.$$

Thus, recalling that

$$u^{q-1} \leq u + u^{q-1} = (I_\alpha * u^p)u^{p-1} \quad \text{a.e. in } \mathbb{R}^d,$$

$u \in L^{(q-1)\sigma}(\mathbb{R}^N)$ and $(q-1)\sigma > \frac{dp}{\alpha}$ provided $0 < \epsilon < \frac{d}{\alpha}\left(1 - \frac{2p-1}{p+q-1}\right)$. Thus, $u^p \in L^{\frac{d}{\alpha}-\epsilon}(\mathbb{R}^d) \cap L^{\frac{(q-1)\sigma}{p}}(\mathbb{R}^d)$, therefore $I_\alpha * u^p \in L^\infty(\mathbb{R}^d)$ and is Hölder continuous of order γ for some $\gamma \in (0, 1]$.

CASE 3: $q \in (\frac{2dp}{d+\alpha}, \frac{dp}{\alpha})$. Let's set $s_0 := q > p$ and

$$\frac{1}{s_{n+1}} := \frac{1}{(q-1)\sigma_n} = \frac{2p-1}{(q-1)s_n} - \frac{\alpha}{N(q-1)}. \tag{4.4.9}$$

Then $u^p \in L^{\frac{s_0}{p}}$ and $(I_\alpha * u^p)u^{p-1} \in L^{\sigma_0}(\mathbb{R}^d)$. Thus, $u \in L^{(q-1)\sigma_0}(\mathbb{R}^d) = L^{s_1}(\mathbb{R}^d)$. Note that $q > \frac{2dp}{d+\alpha}$ implies $s_1 > s_0 = q$. In particular, if $s_n < \frac{dp}{\alpha}$, an induction argument yields $s_n < s_{n+1}$. This proves that $(s_n)_n$ is monotone increasing, as long as s_n is between q and dp/α.

We claim that, after finite steps, there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} \geq \frac{dp}{\alpha}$ and $s_n < \frac{dp}{\alpha}$ for all $n < n_0$. If not, we can obtain a sequence $\{s_n\}$ satisfying (4.4.9) and $q < s_n < \frac{dp}{\alpha}$ for all $n \in \mathbb{N}$. By the monotonicity of this sequence, we also conclude that s_n converges to the unique fixed point $s_* := d(2p-q)/\alpha > q$, which contradicts the condition $q > 2dp/(d+\alpha)$.

Then, if $s_{n_0} > \frac{dp}{\alpha}$ we conclude that $I_\alpha * u^p \in L^\infty(\mathbb{R}^d)$ and is Hölder continuous of order $\alpha - \frac{d}{s_{n_0}}$. If $s_{n_0} = \frac{dp}{\alpha}$ we can argue as in the previous case and we still obtain boundedness and Hölder regularity of $I_\alpha * u^p$.

Next, from $I_\alpha * u^p \in L^\infty(\mathbb{R}^d)$ and the relation

$$u^{2-p} + u^{q-p} = I_\alpha * u^p \quad \text{a.e. in } \{u > 0\}, \tag{4.4.10}$$

we conclude that $u \in L^\infty(\mathbb{R}^d)$. Therefore, $u \in L^s(\mathbb{R}^N)$ for all $s \geq 2$ from which $I_\alpha * u^p$ is Hölder continuous of order τ for any $\tau \in (0, \min\{1, \alpha\})$.

Furthermore, if $p < 2$, since the function $f(t) = t^{2-p} + t^{q-p}$ has a differentiable inverse on $(0, \infty)$ and $u \in L^\infty(\mathbb{R}^d)$, it follows from (4.4.10) that u has the same Hölder regularity as $I_\alpha * u^p$ in $\{u > 0\}$.

Similarly, if $p = 2$, the function $f(t) = 1 + t^{q-2}$ has a differentiable inverse on $(0, +\infty)$ if $q \leq 3$, and a locally Hölder inverse of order $\frac{1}{q-2}$ if $q > 3$. Then again from the boundedness of u and (4.4.10) we obtain (4.4.7). □

Corollary 4.4.2. *Assume that* $p \geq 2$ *and* $q > \frac{2dp}{d+\alpha}$. *Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a non negative radial non increasing solution of (TF). Then* u *is compactly supported.*

Proof. Since u is radially non increasing, by an abuse of notation we still denote $u(r) = u(|x|)$ where $r = |x|$. Now, since $u(r)$ is a non increasing function, it can have at most a countable number points of discontinuity. Then, without loss of generality, if r' is a discontinuity point, we define

$$u(r') := \lim_{r \to r'^+} u(r). \tag{4.4.11}$$

Note that the above limit exists by monotonicity of $u(r)$ and, by doing this, we are only modifying u on a set of measure zero. In fact,

$$u(r') = \liminf_{r \to r'} u(r), \tag{4.4.12}$$

which makes u a lower semi-continuous function, and the set $\{u > 0\}$ is open.

Arguing by contradiction, we assume that $\{u > 0\} = \mathbb{R}^d$. Since u is non negative and satisfies (TF), we have

$$1 \leq 1 + u^{q-2} = (I_\alpha * u^p)u^{p-2} \quad \forall\, x \in \mathbb{R}^d. \tag{4.4.13}$$

On the other hand, $I_\alpha * u^p$ vanishes at infinity by (4.4.1), and $u \in L^\infty(\mathbb{R}^d)$ by Lemma 4.4.2. Hence there exist $R > 0$ such that $(I_\alpha * u^p)u^{p-2} < 1$ in B_R^c, a contradiction to (4.4.13). □

Next we show that when $p > 2$, non negative solutions of (TF) are discontinuous at the boundary of the support and Hölder continuous inside the support.

Lemma 4.4.3. *Assume that* $p > 2$ *and* $q > \frac{2dp}{d+\alpha}$. *Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a non negative radial non increasing solution of (TF). Then there exists* $\lambda > 0$ *such that*

$\{u > 0\} = \{u > \lambda\}$ *and* u *is Hölder continuous of order* $\kappa(p, q, \lambda)$ *in* $\{u > 0\}$*, where*

$$\kappa(p, q, \lambda) := \begin{cases} \tau, & \text{if } \lambda > \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}}, \\ \frac{\tau}{2}, & \text{if } \lambda = \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}}, \end{cases} \tag{4.4.14}$$

for every $\tau \in (0, \min\{\alpha, 1\})$*.*

Proof. Set $B_{R_*} := \{u > 0\}$, where $R_* < \infty$ in view of Corollary 4.4.2. Assume by contradiction that there exists a sequence $\{r_n\}_n \subset (0, R_*)$ such that $r_n \to R_*$ and $u(r_n) \to 0$. Then, by (4.4.10),

$$(I_\alpha * u^p)(r_n) = u(r_n)^{2-p} + u(r_n)^{q-p} \to \infty. \tag{4.4.15}$$

However, from Lemma 4.4.2, the left hand side of (4.4.15) is bounded which leads to a contradiction. We have therefore proved that u is far away from zero inside its support, or equivalently that there exists $\lambda > 0$ such that $\{u > 0\} = \{u > \lambda\}$.

In what follows, we prove continuity of u in B_{R_*}. First, we recall that $I_\alpha * u^p$ is Hölder continuous by Lemma 4.4.2, and is radially non increasing, since u is radially non increasing. Next, we define the following quantities:

$$\lambda_* := \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}}, \quad \lambda := \lim_{r \to R_*^-} u(r), \quad \gamma := u(0). \tag{4.4.16}$$

Note that λ_* is the unique minimum of the function f defined on $(0, +\infty)$ by

$$f(t) := t^{2-p} + t^{q-p}. \tag{4.4.17}$$

To prove the continuity of u, as we will see shortly, it is enough to prove that $\lambda \geq \lambda_*$. To this aim, we split the proof into two steps.

Step 1: $\gamma > \lambda_*$. Assume by contradiction that $\gamma \leq \lambda_*$. Since u is radially non increasing, $u(r) \leq \lambda_*$ for every $r \in (0, R_*)$. Furthermore, since the function $f(t) = t^{2-p} + t^{q-p}$ is decreasing in the interval $(0, \lambda_*]$, we deduce that $f(u(r))$ is non decreasing in $(0, R_*)$. Thus, from the equality $f(u(r)) = (I_\alpha * u^p)(r)$ and monotonicity of $I_\alpha * u^p$, and injectivity of f (or strict monotonicity of f) the function

u must be constant inside the support. Namely, $u(x) = \gamma\chi_{B_{R_*}}(x)$ and, from (4.4.10),

$$\gamma^{2-p} + \gamma^{q-p} = \gamma^p I_\alpha * \chi_{B_{R_*}} \quad \text{in} \;\; B_{R_*}. \tag{4.4.18}$$

In fact, $I_\alpha * \chi_{B_{R_*}}$ can be written in terms of the Gauss Hypergeometric function as

$$(I_\alpha * \chi_{B_{R_*}})(x) = \frac{\Gamma((d-\alpha)/2)R_*^\alpha}{2^\alpha\Gamma(1+\alpha/2)\Gamma(d/2)} {}_2F_1\left(-\frac{\alpha}{2}, \frac{d-\alpha}{2}; \frac{d}{2}; \frac{|x|^2}{R_*^2}\right), \tag{4.4.19}$$

which is never a constant for $\alpha \in (0, d)$. We have therefore proved that $\gamma > \lambda_*$.

STEP 2: $\lambda \geq \lambda_*$. Assume that $\lambda < \lambda_*$. First of all, we notice that u can not be continuous in $(0, R_*)$. As a matter of fact, if u is continuous, since by Step 1 we have that $\lambda_* \in (\lambda, \gamma)$, the value λ_* is achieved by u. Namely, there exists $\bar{r} \in (0, R_*)$ such that $u(\bar{r}) = \lambda_*$. Arguing as before, since $u(r)$ is non increasing, f is decreasing in $[\lambda, \lambda_*]$, and $(I_\alpha * u^p)(r)$ is non increasing. We infer that $u(r)$ is constant for every $r \in [\bar{r}, R_*)$. However, this implies that

$$\lambda = \lim_{r \to R_*^-} u(r) = u(\bar{r}) = \lambda_*,$$

which is a contradiction.

Next, we show that if r' is a discontinuity point of $u(r)$, we must have that $u(r') \in [\lambda, \lambda_*]$. Indeed, by (4.4.12), if $u(r') = L \in (\lambda_*, \gamma]$ then for every $\epsilon > 0$ sufficiently small there exists $\delta > 0$ such that $u(r) \geq L - \epsilon$ for every $r \in (r' - \delta, r' + \delta)$. In particular, if we choose ϵ such that $L - \epsilon > \lambda_*$, we deduce that $u((r' - \delta, r' + \delta)) \subset (\lambda_*, \gamma]$. But in this interval the function f is invertible with continuous inverse. Then,

$$u(r) = f^{-1}\big((I_\alpha * u^p)(r)\big) \quad \forall r \in (r' - \delta, r' + \delta),$$

which in particular implies continuity of u at r' and this is a contradiction.

Next, in view of monotonicity of $u(r)$, we conclude that $u([r', R_*)) \subset [\lambda, \lambda_*]$. Then, since in $[\lambda, \lambda_*]$ the function f is decreasing, again monotonicity of u implies that $u(r) = \lambda$ for every $r \in [r', R_*]$. Finally, it remains to prove that this is not possible and this will imply that $\lambda \geq \lambda_*$.

Since we have assumed that $u(r)$ is non increasing and constant in $[r', R_*]$, there exists $\lambda_1 > \lambda$ such that

$$u^p = \lambda^p \chi_{B_{R_*}} + \phi + (\lambda_1^p - \lambda^p)\chi_{B_{r'}}, \tag{4.4.20}$$

where ϕ is a radially non increasing function such that $\phi(r) = 0$ if $r \geq r'$. Thus, by combining (4.4.10) with (4.4.20) we obtain the equality

$$\lambda^{2-p} + \lambda^{q-p} = \lambda^p (I_\alpha * \chi_{B_{R_*}})(r) + (\lambda_1^p - \lambda^p)(I_\alpha * \chi_{B_{r'}})(r) + (I_\alpha * \phi)(r) \quad \forall r \in (r', R_*). \tag{4.4.21}$$

However, again by (4.4.19), the right hand side of (4.4.21) is decreasing in $(R_* - \epsilon, R_*)$ for some $\epsilon > 0$ small enough and this contradicts (4.4.21).

We have therefore proved that $\lambda \geq \lambda_*$. Then on the set $[\lambda_*, \infty)$ the function f has an inverse $f^{-1} : [\lambda_*, \infty) \to [\Lambda_*, \infty)$, where we denote $\Lambda_* := f(\lambda_*)$. We conclude that

$$u = f^{-1}(I_\alpha * u^p) \quad \text{in } B_{R_*}. \tag{4.4.22}$$

To prove that the desired Hölder exponent given by (4.4.14) we consider two different cases.

Case a): $\lambda > \lambda_*$. In this case f is a Lipschitz function with Lipschitz inverse in the set

$$u(B_{R_*}) := \{u(x) : x \in B_{R_*}\}$$

and by Lemma 4.4.2 we have $u = f^{-1}(I_\alpha * u^p) \in C^{0,\tau}(B_{R_*})$ for every $\tau \in (0, \min\{1, \alpha\})$.

Case b): $\lambda = \lambda_*$. In this case, let's notice that $f''(\lambda_*) = p(q-p)^2 \lambda_*^{q-p-2} > 0$, which means that if $\epsilon > 0$ is small enough, the following expansion holds

$$f(t) = f(\lambda_*) + \frac{1}{2} f''(\lambda_*)(t - \lambda_*)^2 + o\big((t - \lambda_*)^2\big) \quad \forall t \in (\lambda_* - \epsilon, \lambda_* + \epsilon). \tag{4.4.23}$$

Let f^{-1} be the inverse of f on $[\lambda_*, \infty)$. Then, if for $s \geq \Lambda_*$ we set $t := f^{-1}(s)$, by (4.4.23) we obtain

$$\lim_{s \to \Lambda_*^+} \frac{|f^{-1}(\Lambda_*) - f^{-1}(s)|}{|\Lambda_* - s|^{\frac{1}{2}}} = \lim_{t \to \lambda_*^+} \frac{|\lambda_* - t|}{\left|\frac{1}{2} f''(\lambda_*)(t - \lambda_*)^2 + o(t - \lambda_*)^2\right|^{\frac{1}{2}}} = \sqrt{\frac{2}{f''(\lambda_*)}},$$

which proves that f^{-1} is Hölder continuous of order $1/2$. Then using Lemma 4.4.2 we obtain $u = f^{-1}(I_\alpha * u^p) \in C^{0,\frac{\tau}{2}}(B_{R_*})$ for every $\tau \in (0, \min\{1, \alpha\})$. $\square$

Finally, similarly to [33, Theorem 10] we show that non negative solutions of (*TF*) are smooth inside their support.

Lemma 4.4.4. *Assume that $p \geq 2$ and $q > \frac{2dp}{d+\alpha}$. Let $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ be a non negative radial non increasing solution of (TF). Then $u \in C^\infty$ inside its support.*

Proof. Assume first that $0 < \alpha < 2$. As in Lemma 4.4.3, denote by $B_{R_*} = \{u > 0\}$. Let $x \in B_{R_*}$ and B be a ball centred at x, such that $\overline{B} \subset B_{R_*}$. By Lemma 4.4.2, we know that $I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^N)$ for every $\tau \in (0, \min\{\alpha, 1\})$. Then, as in the proof of Lemma 4.4.3,

$$u = f^{-1}(I_\alpha * u^p), \tag{4.4.24}$$

where f^{-1} is the inverse of f on $[\lambda_*, \infty)$. In particular, since in $\overline{B}$ the function u is away from λ if $p > 2$ (respectively from 0 if $p = 2$), then u (and so u^p) has the regularity of $I_\alpha * u^p$. Namely, $u^p \in C^{0,\tau}(\overline{B})$ for every $\tau \in (0, \min\{\alpha, 1\})$. Then, Proposition 3.3.1 yields $I_\alpha * u^p \in C^{\tau+\alpha}(\overline{B}_{1/2})$ for every $\tau \in (0, \min\{\alpha, 1\})$, provided that $\tau + \alpha$ is not an integer. Here by $B_{1/2}$ we denoted the ball centred at x with half of the radius of B. Hence, again by (4.4.24), we conclude that u has the regularity of $I_\alpha * u^p$. By iterating the above argument, for every $k \in \mathbb{N}$ we can find $j \in \mathbb{N}$ such that $\tau + j\alpha$ is non integer and bigger than k. This proves that $u \in C^k(\overline{B}_{1/2^j})$. Since x was arbitrary, this implies that u is smooth inside its support.

If $2 \leq \alpha < d$ it's enough to argue again in a similar way to the proof of [33, Theorem 10]. □

4.4.3 Support estimates for $p > 2$

The following two statements follow from Proposition 4.2.1.

Corollary 4.4.3. *Let $p > 2$, $q > 2p$ and R_* be the radius of the support of a radially non increasing ground state solution of (TF). Then $R_* \to +\infty$ as $\alpha \to 0$.*

Proof. By combining the Nehari identity (4.1.14) and Pohožaev identity $\mathcal{P}(u) = 0$, we get

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \frac{(d+\alpha)q - 2pd}{q(dp - d - \alpha)} \|u\|^q_{L^q(\mathbb{R}^d)}. \tag{4.4.25}$$

As a result, the minimal energy can also be represented using $\|u\|_{L^q(\mathbb{R}^d)}$, that is,

$$\sigma = \inf_{v \in \mathscr{P}} E(v) = E(u) = \frac{\alpha(q-2)}{2(dp - d - \alpha)q} \|u\|^q_{L^q(\mathbb{R}^d)}. \tag{4.4.26}$$

In what follows we will write $\sigma = \sigma(\alpha)$ stressing the dependence on α. Moreover, by the monotonicity of $u(x)$ in $|x|$,

$$\|u\|^q_{L^q(\mathbb{R}^d)} < u^q(0)|B_{R_*}|. \tag{4.4.27}$$

Evaluating the governing equation (*TF*) at 0, we get

$$u^{q-p}(0) \le u^{2-p}(0) + u^{q-p}(0) = (I_\alpha * u^p)(0), \tag{4.4.28}$$

from which

$$u(0) \le ((I_\alpha * u^p)(0))^{\frac{1}{q-p}} = \left(A_\alpha \int_{B_{R_*}} u^p(y)|y|^{\alpha-d}dy\right)^{\frac{1}{q-p}} \le u^{\frac{p}{q-p}}(0)\left(A_\alpha \omega_d \frac{R_*^\alpha}{\alpha}\right)^{\frac{1}{q-p}},$$

where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$. The previous inequality leads to the bound

$$u(0) \le \left(A_\alpha \omega_d \frac{R_*^\alpha}{\alpha}\right)^{\frac{1}{q-2p}}.$$

This estimate, when combined with (4.4.27), turns the relation (4.4.26) into

$$|B_{R_*}|R_*^{\frac{\alpha q}{q-2p}} \ge \frac{2q(dp-d-\alpha)}{\alpha(q-2)}\left(\frac{A_\alpha \omega_d}{\alpha}\right)^{-\frac{q}{q-2p}}\sigma(\alpha) \tag{4.4.29}$$

Next, if we consider $\theta_* = \theta_*(\alpha)$ defined as in (4.3.18), we have that

$$\lim_{\alpha\to 0}\theta_*(\alpha) = \left(\frac{q(p-1)}{q-2p}\right)^{\frac{q-2p}{2(p-1)+q-2p}}\left(\frac{q-2p}{2q(p-1)}+\frac{1}{q}\right) = \overline{\theta_*}. \tag{4.4.30}$$

Furthermore, under our assumptions on p and q, we have $\lim_{\alpha\to 0} 2p\theta_*(\alpha) = 2p\overline{\theta_*} > 1$. Moreover, from (4.4.29) we deduce that

$$R_* \ge \omega_d^{-\frac{2(q-p)}{q(d+\alpha)-2dp}}\left(\frac{2q(dp-d-\alpha)}{(q-2)}\right)^{\frac{q-2p}{q(d+\alpha)-2dp}}\left(\frac{A_\alpha}{\alpha}\right)^{-\frac{q}{q(d+\alpha)-2dp}}\left(\frac{\sigma(\alpha)}{\alpha}\right)^{\frac{q-2p}{q(d+\alpha)-2dp}}. \tag{4.4.31}$$

Since $\lim_{\alpha\to 0}\alpha^{-1}A_\alpha = \omega_d^{-1}$, it remains to prove that $\lim_{\alpha\to 0}\alpha^{-1}\sigma(\alpha) = \infty$. To do so, from (4.3.7), (4.4.29) and Proposition 4.2.1 we infer

$$\lim_{\alpha\to 0}\frac{\sigma(\alpha)}{\alpha} = \frac{\overline{\theta_*}}{d}\lim_{\alpha\to 0}\left(\frac{d}{d+\alpha}\frac{2p\theta_*(\alpha)}{\mathscr{C}_{d,\alpha,p,q}}\right)^{\frac{d}{\alpha}} = \infty, \tag{4.4.32}$$

where in the last equality we used that

$$\lim_{\alpha \to 0} \left(\frac{d}{d+\alpha} \frac{2p\theta_*(\alpha)}{\mathscr{C}_{d,\alpha,p,q}} \right) = 2p\overline{\theta_*} > 1.$$

This concludes the proof. □

Corollary 4.4.4. *Let $p > 2$ and R_* be the radius of the support of a radially non increasing ground state solution of (TF). Then, $R_* \to 0$ as $\alpha \to d$.*

Proof. By combining (4.4.26), Lemma 4.4.3 with the monotonicity of u we obtain that

$$|B_{R_*}| \leq \left(\frac{q-p}{p-2} \right)^{\frac{q}{q-2}} \frac{2(dp-d-\alpha)q}{\alpha(q-2)} \sigma(\alpha) \tag{4.4.33}$$

Furthermore, by combining Proposition 4.2.1 with (4.3.7) we conclude that

$$\lim_{\alpha \to d} \sigma(\alpha) = 0. \tag{4.4.34}$$

In view of (4.4.34) and (4.4.33) we obtain the thesis. □

4.4.4 Gradient estimates for $p < 2$

In the rest of the section we consider the case $\frac{d+\alpha}{d} < p < 2$. Recall that in this case non negative radial non increasing solutions $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ of (TF) are supported on $\mathbb{R}^d$. Our aim is to show that $\nabla u \in L^2(\mathbb{R}^d; \mathbb{R}^d)$.

Note that for $\alpha > 1$ the gradient $\nabla I_\alpha * u^p$ is well defined, while for $0 < \alpha \leq 1$ it becomes a singular integral and is defined via the Cauchy principal value, namely

$$\nabla(I_\alpha * u^p) = \begin{cases} (\nabla I_\alpha) * u^p, & \alpha > 1, \\ \displaystyle\int_{\mathbb{R}^N} \nabla |x-y|^{\alpha-d} (|u(y)|^p - |u(x)|^p) dy, & 0 < \alpha \leq 1, \end{cases} \tag{4.4.35}$$

cf. [33, eq. (1.2)]. Recall the following result from [33, Lemma 1].

Lemma 4.4.5. *Assume that $u \geq 0$ and $u^p \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then*

(i) If $0 < \alpha < d$, then $I_\alpha * u^p \in L^\infty(\mathbb{R}^d)$.

(ii) If $0 < \alpha \leq 1$ and $u^p \in C^{0,\gamma}(\mathbb{R}^d)$ with $\gamma \in (1-\alpha, 1)$, or if $1 < \alpha < d$ then $\nabla(I_\alpha * u^p) \in L^\infty(\mathbb{R}^d)$, i.e., $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^d)$.

Using the estimates of Lemma 4.4.5, we first show that positive solutions of (*TF*) are globally Lipschitz.

Lemma 4.4.6. *Assume that* $\frac{d+\alpha}{d} < p < 2$ *and* $q > \frac{2dp}{d+\alpha}$*. Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a positive radial non increasing solution of* (*TF*)*. Then* $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^d)$ *and* $u \in W^{1,\infty}(\mathbb{R}^d)$*.*

Proof. By Lemma 4.2.1 and Lemma 4.4.2, we know that $\mathrm{Supp}(u) = \mathbb{R}^d$ and $u^p \in L^\infty(\mathbb{R}^d)$.

If $1 < \alpha < d$ we can apply Lemma 4.4.5 to conclude that $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^d)$. Next, since $\mathrm{Supp}(u) = \mathbb{R}^d$, u satisfies the equivalent governing equation

$$u^{2-p} + u^{q-p} = I_\alpha * u^p \quad \text{in } \mathbb{R}^d. \tag{4.4.36}$$

Finally, as we have already noticed in the proof of Lemma 4.4.2, the function $f(t) = t^{2-p} + t^{q-p}$ has a differentiable inverse on $(0, +\infty)$ under our assumptions on p and q. From (4.4.36) we conclude that $u \in W^{1,\infty}(\mathbb{R}^d)$.

If $0 < \alpha \leq 1$ then Lemma 4.4.2 yields that $I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^d)$ for every $\tau \in (0, \alpha)$. Thus, again from (4.4.36), the differentiability of the inverse f^{-1} on $(0, +\infty)$ and the boundedness of u, we infer $u \in C^{0,\tau}(\mathbb{R}^d)$ for every $\tau \in (0, \alpha)$. In particular, if $1/2 < \alpha \leq 1$ we can ensure that $\tau > 1 - \alpha$, and hence $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^d)$ by Lemma 4.4.5. Then, arguing as before, $u \in W^{1,\infty}(\mathbb{R}^d)$. For $0 < \alpha < 1/2$, on the other hand, we need to use bootstrapping argument. Let us fix $n \in \mathbb{N}$, $n \geq 2$ such that $\frac{1}{n+1} < \alpha < \frac{1}{n}$ and let us choose $\tau > 0$ small enough such that $\tau + n\alpha < 1$ (note that this is possible because of the definition of n). Then, we define $\tau_n := \tau + (n-1)\alpha$. Then, by Eq. (4.4.36) together with the locally Lipschitz continuity of the inverse of f, we can apply Proposition 3.3.1 n–times to conclude that $I_\alpha * u^p \in C^{0,\tau_{n+1}}(\mathbb{R}^d)$. By our choice of n, we have the two sided inequality $1 - \alpha < \tau_{n+1} < 1$. Hence, by Eq. (4.4.36) again, we deduce that u (and in particular u^p) belongs to $C^{0,\tau_{n+1}}(\mathbb{R}^d)$. To conclude, by Lemma 4.4.2 we conclude that $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^d)$, and that u has the same regularity. Finally, if $\alpha = \frac{1}{2}$ it is sufficient to start the the above iterations with $\tau - \epsilon$, for some $\epsilon > 0$ small enough such that $1 - \alpha < \tau_{n+1} - \epsilon < 1$. □

Next we show that positive solutions are actually arbitrarily smooth.

Lemma 4.4.7. *Assume that* $\frac{d+\alpha}{d} < p < 2$ *and* $q > \frac{2dp}{d+\alpha}$*. Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a positive radial non increasing solution of* (*TF*)*. Then,* $u \in C^\infty(\mathbb{R}^d)$*.*

Proof. This follows from Eq. (4.4.36) as in the proof of Lemma 4.4.4. □

Next we establish a gradient estimate on the non negative solutions of (*TF*).

Lemma 4.4.8. *Assume that* $\frac{d+\alpha}{d} < p < 2$ *and* $q > \frac{2dp}{d+\alpha}$*. Let* $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *be a non negative radial non increasing solution of* (*TF*)*. Then,* $\nabla u \in L^1(\mathbb{R}^d)$*. In particular,* $u \in H^1(\mathbb{R}^d)$*.*

Proof. Assume first that $1 < \alpha < d$. From the expression in Eq. (4.4.35), we deduce that

$$\begin{aligned} |\nabla(I_\alpha * u^p)(x)| &\leq \int_{\mathbb{R}^d} \left|\nabla|x-y|^{\alpha-d}\right| u^p(y)dy \\ &\leq (d-\alpha)A_\alpha \int_{\mathbb{R}^d} \frac{u^p(y)}{|x-y|^{d-\alpha+1}}dy = \frac{(d-\alpha)A_\alpha\|u\|^p_{L^p(\mathbb{R}^d)}}{|x|^{d-\alpha+1}} + o\left(\frac{1}{|x|^{d-\alpha+1}}\right), \end{aligned} \tag{4.4.37}$$

for $|x|$ sufficiently large. Note also that the inverse f^{-1} is differentiable on $(0,+\infty)$ and

$$\left(f^{-1}\right)'(t) = t^{\frac{p-1}{2-p}} + o(t^{\frac{p-1}{2-p}}) \quad \text{as } t \to 0^+. \tag{4.4.38}$$

Hence, by using the chain rule in (4.4.36), Lemma 4.4.1, (4.4.37) and (4.4.38), we infer

$$|\nabla u(x)| = \left|\left(f^{-1}\right)'\big((I_\alpha * u^p)(x)\big)\nabla(I_\alpha * u^p)(x)\right| \lesssim \frac{1}{|x|^{\frac{d-\alpha}{2-p}+1}} \quad \text{as } |x| \to +\infty. \tag{4.4.39}$$

Combining Lemma 4.4.6 with (4.4.39) yields $\nabla u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ which concludes the proof.

Assume now that $\alpha \in (0,1]$. From (4.4.35), arguing as in [29, Lemma 2.2] we have

$$\begin{aligned} \nabla(I_\alpha * u^p) &\leq (d-\alpha)A_\alpha \left(\int_{|x-y|\leq 1} \frac{|u^p(y)-u^p(x)|}{|x-y|^{d-\alpha+1}} + \int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{d-\alpha+1}}\right) \\ &=: I_1 + I_2. \end{aligned} \tag{4.4.40}$$

First we note that, since $\alpha \in (0,1]$, for every $\varepsilon \in (0,d)$, I_2 can be estimated as

$$\int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{d-\alpha+1}} \leq \int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{d-\varepsilon}}$$
$$\leq \int_{\mathbb{R}^d} \frac{u^p(y)}{|x-y|^{d-\varepsilon}} dy = \frac{\|u\|^p_{L^p(\mathbb{R}^d)}}{|x|^{N-\varepsilon}} + o\left(\frac{1}{|x|^{d-\varepsilon}}\right), \quad (4.4.41)$$

where for the last equality we used the decay estimate (4.4.1) on u established in Lemma 4.4.1.

Let us fix $0 < \bar{\varepsilon} < \frac{(d-\alpha)(p-1)}{2-p}$. Since $\nabla(I_\alpha * u^p) \in L^\infty(\mathbb{R}^d)$, by applying the gradient operator to both sides of Eq. (4.4.36) we deduce that

$$|\nabla u(x)| \lesssim \left|(f^{-1})'((I_\alpha * u^p)(x))\right| \lesssim \frac{1}{|x|^{\frac{(d-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty, \quad (4.4.42)$$

where for the last inequality we used Corollary 4.4.1 and Eq. (4.4.38).

Next, we estimate I_1. By combining Eq. (4.4.42) with Lemma 4.4.1 we conclude that

$$I_1 \lesssim \|\nabla u^p\|_{L^\infty(\overline{B_1(x)})} \int_{|x-y|\leq 1} \frac{dy}{|x-y|^{d-\alpha}}$$
$$\lesssim \|u^{p-1}\nabla u\|_{L^\infty(\overline{B_1(x)})} \lesssim \frac{1}{|x|^{\frac{2(d-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty. \quad (4.4.43)$$

Then, if $\frac{2(d-\alpha)(p-1)}{2-p} \leq d-\bar{\varepsilon}$, combining together Eq. (4.4.41) with Eq. (4.4.43) yields

$$\nabla(I_\alpha * u^p) \lesssim \frac{1}{|x|^{\frac{2(d-\alpha)(p-1)}{2-p}}} + \frac{1}{|x|^{d-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty. \quad (4.4.44)$$

On the other hand, if $\frac{2(d-\alpha)(p-1)}{2-p} > d-\bar{\varepsilon}$, by the same argument it follows that

$$\nabla(I_\alpha * u^p) \lesssim \frac{1}{|x|^{d-\varepsilon}} \quad \text{as } |x| \to +\infty$$

which in turn implies that

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{(d-\alpha)(p-1)}{2-p}+d-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty.$$

Then, by the choice of $\bar{\varepsilon}$ we conclude that $\nabla u \in L^1(\mathbb{R}^d)$. However, if Eq. (4.4.44) holds, we can improve the inequality (4.4.42) to

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{3(d-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty. \tag{4.4.45}$$

Then, we can iterate this argument, until we find the first positive integer k such that

$$\frac{2k(d-\alpha)(p-1)}{2-p} > d - \bar{\varepsilon}.$$

In this way we obtain that

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{(d-\alpha)(p-1)}{2-p}+d-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty,$$

which again implies $\nabla u \in L^1(\mathbb{R}^d)$ by the choice of $\bar{\varepsilon}$. □

4.5 Limit profiles for the Choquard equation

Throughout this section we assume that $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ and $q > \frac{2dp}{d+\alpha}$. As already highlighted in the Introduction, the rescaling $u(x) = \varepsilon^{-\frac{1}{q-2}} w\big(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\big)$ converts the Choquard problem (P_ε) into the equation

$$-\varepsilon^\nu \Delta u + u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^d, \tag{4.5.1}$$

where we denoted

$$\nu := \frac{2(2p+\alpha) - q(2+\alpha)}{\alpha(q-2)}.$$

Notice that:

(*i*) $\nu > 0$ iff $q < 2\frac{2p+\alpha}{2+\alpha}$,

(*ii*) $\nu < 0$ iff $q > 2\frac{2p+\alpha}{2+\alpha}$.

The energy that corresponds to the rescaled equation (4.5.1) is given by

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\varepsilon^\nu \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^d} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p), \tag{4.5.2}$$

and the Pohožaev functional is defined by

$$\mathcal{P}_\varepsilon(u) := \frac{d-2}{2}\varepsilon^\nu \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{d}{2}\int_{\mathbb{R}^d} |u|^2 dx + \frac{d}{q}\int_{\mathbb{R}^d} |u|^q dx - \frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p). \tag{4.5.3}$$

We note that

$$\mathcal{J}_\varepsilon(u) = \varepsilon^{\frac{q(d+\alpha)-2dp}{\alpha(q-2)}} \mathcal{I}_\varepsilon(w), \tag{4.5.4}$$

where $\mathcal{I}_\varepsilon(w)$ is the Choquard energy defined in (4.1.1). Following [82], we consider the rescaled minimization problem

$$\sigma_\varepsilon := \inf_{u\in\mathscr{P}_\varepsilon} \mathcal{J}_\varepsilon(u), \tag{4.5.5}$$

where Pohožaev manifold $\mathscr{P}_\varepsilon$ is defined as

$$\mathscr{P}_\varepsilon := \left\{u \in H^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), u \neq 0 : \mathcal{P}_\varepsilon(u) = 0\right\}.$$

Given $\varepsilon > 0$, let $w_\varepsilon \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ be a positive, radial, monotonically decreasing ground state solution of (P_ε)(see Theorem 4.1.1). Define

$$u_\varepsilon(x) := \varepsilon^{-\frac{1}{q-2}} w_\varepsilon\big(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\big). \tag{4.5.6}$$

Then $u_\varepsilon \in \mathscr{P}_\varepsilon$ and $\mathcal{J}_\varepsilon(u_\varepsilon) = \sigma_\varepsilon$, that is u_ε is the minimizer of (4.5.5) and a positive ground state solution of the rescaled equation (4.5.1), see [82].

In this section we shall prove Theorem 4.1.4, which states that u_ε converges as $\varepsilon \to +\infty$ and $\nu < 0$ (respectively as $\varepsilon \to 0$ and $\nu > 0$) to a non negative radial non increasing ground state solution $u_* \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ of the Thomas–Fermi equation (*TF*), constructed in Lemma 4.3.6 from a maximizer in Theorem 4.1.2. Recall that $E(u_*) = \sigma$, where

$$\sigma = \inf_{u\in\mathscr{P}} E(u) = \alpha(2dp)^{\frac{d}{\alpha}} \left(\frac{\theta_*}{d+\alpha}\right)^{\frac{d+\alpha}{\alpha}} \mathscr{C}_{d,\alpha,p,q}^{-\frac{d}{\alpha}}, \tag{4.5.7}$$

as described in (4.4.19). The essential step in our proof of convergence is to show that $\sigma_\varepsilon \to \sigma$.

In what follows we shall only consider the case $\varepsilon \to +\infty$ and $\nu < 0$, i.e., $q > 2\frac{2p+\alpha}{2+\alpha}$. The arguments in the case $\varepsilon \to 0$ and $\nu > 0$ are very similar.

First we study the easier case when the ground state solution $u_* \in \mathring{H}^1(\mathbb{R}^d)$, where we recall that by $\mathring{H}^1(\mathbb{R}^d)$ we denote the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^d)}$, assuming $d \geq 3$. In particular, this is true if $p < 2$, as proved in Lemma 4.4.8.

Lemma 4.5.1. *Assume* $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ *and* $q > 2\frac{2p+\alpha}{2+\alpha}$. *If* $u_* \in \mathring{H}^1(\mathbb{R}^d)$ *then (1)* $\sigma_\varepsilon > \sigma$ *and (2)* σ_ε *converges to* σ *as* $\varepsilon \to +\infty$.

Proof. Let $u_\varepsilon \in \mathscr{P}_\varepsilon$ be the minimizer of the problem $\inf\limits_{u \in \mathscr{P}_\varepsilon} \mathcal{J}_\varepsilon(u)$ such that $\mathcal{J}(u_\varepsilon) = \sigma_\varepsilon$. Then,

$$\mathcal{P}(u_\varepsilon) = \mathcal{P}_\varepsilon(u_\varepsilon) - \left(\frac{d-2}{2}\right)\varepsilon^\nu \|\nabla u_\varepsilon\|^2_{L^2(\mathbb{R}^d)} < 0.$$

Let $w_{\varepsilon,t}(x) := u_\varepsilon\left(\frac{x}{t}\right)$, then we obtain

$$\mathcal{P}(w_{\varepsilon,t}) = \frac{dt^d}{2}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^d)} + \frac{dt^d}{q}\|u_\varepsilon\|^q_{L^q(\mathbb{R}^d)} - \frac{(d+\alpha)t^{d+\alpha}}{2p}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p), \tag{4.5.8}$$

and $\mathcal{P}(w_{\varepsilon,1}) = \mathcal{P}(u_\varepsilon) < 0$. On the other hand, the dependence of t of various terms in Eq. (4.5.8) implies that $\mathcal{P}(w_{\varepsilon,t})$ is positive if $t > 0$ is small. Therefore, by the continuity of $t \mapsto \mathcal{P}(w_{\varepsilon,t})$, there exists $t_\varepsilon \in (0,1)$ such that $\mathcal{P}(w_{\varepsilon,t_\varepsilon}) = 0$ and hence $w_{\varepsilon,t_\varepsilon} \in \mathscr{P}$. Consequently,

$$\begin{aligned}\sigma \leq E(w_{\varepsilon,t_\varepsilon}) &= \frac{\alpha t_\varepsilon^{d+\alpha}}{2dp}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p) < \frac{\alpha}{2dp}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p) + \frac{\varepsilon^\nu}{N}\|\nabla u_\varepsilon\|^2_{L^2(\mathbb{R}^d)} \\ &= \mathcal{J}_\varepsilon(u_\varepsilon) = \sigma_\varepsilon,\end{aligned} \tag{4.5.9}$$

which proves the first part of the statement.

Now, let u_* be the ground state solution for (TF) obtained in Lemma 4.3.6. Then, by the assumption $u_* \in \mathring{H}^1(\mathbb{R}^d)$,

$$\mathcal{P}_\varepsilon(u_*) = \frac{(d-2)\varepsilon^\nu}{2}\|\nabla u_*\|^2_{L^2(\mathbb{R}^d)} > 0. \tag{4.5.10}$$

Define the rescaled function $\omega_t(x) := u_*\left(\frac{x}{t}\right)$. Then $\mathcal{P}_\varepsilon(\omega_t)$, expressed in term of u_* as

$$\frac{(d-2)\varepsilon^\nu t^{d-2}}{2}\|\nabla u_*\|^2_{L^2(\mathbb{R}^d)} + dt^d\left(\frac{\|u_*\|^2_{L^2(\mathbb{R}^d)}}{2} + \frac{\|u_*\|^q_{L^q(\mathbb{R}^d)}}{q}\right) - \frac{(d+\alpha)t^{d+\alpha}}{2p}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p),$$

goes to $-\infty$ when t goes to $+\infty$. This implies the existence of $t_\varepsilon > 1$, such that $\mathcal{P}_\varepsilon(\omega_{t_\varepsilon}) = 0$. In particular, we have $t_\varepsilon \to 1$ because

$$1 < (t_\varepsilon)^\alpha \leq \frac{d\left(\frac{\|u_*\|^2_{L^2(\mathbb{R}^d)}}{2} + \frac{\|u_*\|^q_{L^q(\mathbb{R}^d)}}{q}\right) + \frac{d-2}{2}\varepsilon^\nu \|\nabla u_*\|^2_{L^2(\mathbb{R}^d)}}{\frac{d+\alpha}{2p}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p)} \to 1 \quad \text{as } \varepsilon \to +\infty.$$

Now, from $\varepsilon^\nu \to 0$ and $t_\varepsilon \to 1$ as $\varepsilon \to +\infty$, we conclude that

$$\begin{aligned}\sigma_\varepsilon &\leq \mathcal{J}_\varepsilon\left(u_*\left(\tfrac{x}{t_\varepsilon}\right)\right)\\ &= \frac{\varepsilon^\nu}{2} t_\varepsilon^{d-2}\|\nabla u_*\|^2_{L^2(\mathbb{R}^d)} + \frac{t_\varepsilon^d}{2}\left(\|v_*\|^2_{L^2(\mathbb{R}^d)} + \|u_*\|^q_{L^q(\mathbb{R}^d)}\right) - \frac{t_\varepsilon^{d+\alpha}}{2p}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p)\\ &\to E(u_*) = \sigma.\end{aligned} \tag{4.5.11}$$

The assertion about the convergence $\sigma_\varepsilon \to \sigma$ now follows by combining (4.5.9) with (4.5.11). □

Next we show $\sigma_\varepsilon \to \sigma$ as $\varepsilon \to +\infty$ without assuming that $u_* \in \mathring{H}^1(\mathbb{R}^d)$. In fact, this is expected for the case $p > 2$ as proved in Theorem 4.1.3.

Lemma 4.5.2. *Assume* $\frac{d+\alpha}{d} < p < \frac{N+\alpha}{d-2}$ *and* $q > 2\frac{2p+\alpha}{2+\alpha}$. *Then there exists a sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *such that* $\varepsilon_k \to \infty$ *and* $0 < \sigma_{\varepsilon_k} - \sigma \to 0$, *as* $k \to \infty$.

Proof. If $p < 2$ then the assertion follows by combining Lemma 4.4.8 with Lemma 4.5.1.

Assume that $p \geq 2$. Again, if the ground state solution u_* of (*TF*) constructed in Lemma 4.3.6 belongs to $\mathring{H}^1(\mathbb{R}^d)$ we conclude by Lemma 4.5.1. If not, (for example for $p > 2$), we argue as follows.

First note that arguing as in (4.5.9) in the first part of the proof of Lemma 4.5.1, we conclude that $\sigma_\varepsilon \geq \sigma$. It remains then to prove that $\sigma_\varepsilon \to \sigma$ by constructing an sequence of approximate minimizers of $\mathcal{J}_\varepsilon$ from u_*, which is achieved by truncating u_* (to avoid the singularity near the boundary) on a length scale s depending on ε.

Given $s \geq 0$ small, we introduce the cut-off function $\eta_s \in C_c^\infty(\mathbb{R}^d)$ such that $\eta_s(x) = 1$ for $|x| \leq R_* - s$, $0 < \eta_s(x) < 1$ for $R_* - s < |x| \leq R_* - \frac{s}{2}$, $\eta_s(x) = 0$ for $|x| \geq R_* - \frac{s}{2}$. Furthermore, $|\eta_s'(x)| \leq \frac{4}{s}$ and $|\eta_s'(x)| \geq \frac{1}{2s}$ for $R_* - \frac{4s}{5} < |x| < R_* - \frac{3s}{5}$. Set

$$\psi_s(x) := \eta_s(x)u_*(x).$$

By the definition of η_s, since $u_* \in L^\infty(\mathbb{R}^d)$ and it is supported in B_{R_*}, for every $1 \leq r < \infty$ we have

$$\int_{\mathbb{R}^d} |\psi_s^p(x) - u_*^p(x)|^r = \int_{R_*-s\leq|x|\leq R_*} |u_*(x)|^{pr}(1-\eta_s^p(x))^r \\ \leq \|u_*\|^{pr}_{L^\infty(\mathbb{R}^d)} |A_{R_*-s,R_*}| = \mathcal{O}(s)\,, \quad (4.5.12)$$

where $|A_{R_*-s,R_*}|$ is the volume of $B_{R_*} \setminus \overline{B}_{R_*-s}$. Further, by combining the Hardy-Littlewood-Sobolev inequality with (4.5.12), we obtain

$$0 \leq \mathcal{D}_\alpha(|u_*|^p, |u_*|^p) - \mathcal{D}_\alpha(|\psi_s|^p, |\psi_s|^p) = \mathcal{D}_\alpha(|u_*|^p + |\psi_s|^p, |u_*|^p - |\psi_s|^p) \\ \leq C\|u_*^p + \psi_s^p\|_{L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)} \|u_*^p - \psi_s^p\|_{L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d)} = \mathcal{O}\left(s^{\frac{d+\alpha}{2d}}\right).$$

To summarise, the following holds:

$$\mathcal{D}_\alpha(|\psi_s|^p, |\psi_s|^p) = \mathcal{D}_\alpha(|u_*|^p, |u_*|^p) - \mathcal{O}(s^{\frac{d+\alpha}{2d}}), \quad (4.5.13)$$

$$\|\psi_s\|^q_{L^q(\mathbb{R}^d)} = \|u_*\|^q_{L^q(\mathbb{R}^d)} - \mathcal{O}(s), \quad (4.5.14)$$

$$\|\psi_s\|^2_{L^2(\mathbb{R}^d)} = \|u_*\|^2_{L^2(\mathbb{R}^d)} - \mathcal{O}(s), \quad (4.5.15)$$

where here by $\mathcal{O}(s)$ denotes a *non negative* function such that $\mathcal{O}(s) \leq Cs$ for every $s > 0$ small enough and for a constant $C > 0$ independent of s.

Note that by Lemma 4.4.4, the function ψ_s is smooth and, since $u_* \notin \mathring{H}^1(\mathbb{R}^d)$, the quantity $\|\nabla\psi_s\|^2_{L^2(\mathbb{R}^d)}$ blow up as $s \to 0^+$. In particular, there exists a decreasing sequence $(s_k)_{k\in\mathbb{N}}$ converging to zero such that $\|\nabla\psi_{s_k}\|^2_{L^2(\mathbb{R}^d)}$ diverges monotonically to infinity. Hence, we can define a piecewise linear, monotonically increasing, continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $\lim_{s\to 0^+} f\left(\frac{1}{s}\right) = +\infty$ and

$$f\left(\frac{1}{s_k}\right) = \|\nabla\psi_{s_k}\|^2_{L^2(\mathbb{R}^d)}.$$

In what follows we are going to describe a way we can control the rate of blow up of $f\left(\frac{1}{s_k}\right)$ in terms of quantities in (4.5.13)–(4.5.15).

Next the parameter s_k will be defined as a function of ε_k, so that the function $f(\frac{1}{s_k}) = \|\nabla\psi_{s_k}\|^2_{L^2(\mathbb{R}^d)}$ blows up at a slower rate than $\varepsilon_k^{-\nu}$ and hence the convergence of σ_{ε_k} towards σ. To do this, we set

$$s_k := \frac{1}{g(\varepsilon_k)}, \tag{4.5.16}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable function such that $\lim_{\varepsilon\to+\infty} g(\varepsilon) = +\infty$, to be chosen later. Then for all sufficiently large k we have

$$\|\nabla\psi_{\frac{1}{g(\varepsilon_k)}}\|^2_{L^2(\mathbb{R}^d)} = f(g(\varepsilon_k)) \nearrow +\infty \quad \text{as } k \to +\infty.$$

For the sake of notation simplicity we further denote

$$\psi_{\varepsilon_k^{-1}} := \psi_{\frac{1}{g(\varepsilon_k)}}.$$

Combining together (4.5.13), (4.5.14) with (4.5.15), we have that

$$\mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k^{-1}}) = \left(\frac{d-2}{2}\right)\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)} + \underbrace{\mathcal{P}(u_*)}_{=0} - \mathcal{O}\left(\tfrac{1}{g(\varepsilon_k)}\right) + \mathcal{O}\left(\left(\tfrac{1}{g(\varepsilon_k)}\right)^{\frac{d+\alpha}{2d}}\right), \tag{4.5.17}$$

We claim that $\mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k{}^{-1}}) > 0$ for a suitable choice of the function g when k is sufficiently large. Indeed, if g satisfies the condition

$$\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = +\infty, \tag{4.5.18}$$

from (4.5.17) we obtain that

$$\begin{aligned} \mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k^{-1}}) &\geq \left(\frac{d-2}{2}\right)\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)} - \mathcal{O}\left(\tfrac{1}{g(\varepsilon_k)}\right) \\ &= \left(\frac{d-2}{2}\right)\varepsilon_k^{\nu} f(g(\varepsilon_k)) - \mathcal{O}\left(\tfrac{1}{g(\varepsilon_k)}\right) > 0, \end{aligned} \tag{4.5.19}$$

provided that k is large enough. Next, the equality

$$\begin{aligned} &\mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\tfrac{x}{t}\right)\right) = \\ &= t^{d-2}\frac{(d-2)}{2}\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)} + dt^d\left(\frac{\|\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)}}{2} + \frac{\|\psi_{\varepsilon_k^{-1}}\|^q_{L^q(\mathbb{R}^d)}}{q}\right) \\ &- \frac{(d+\alpha)t^{d+\alpha}}{2p}\mathcal{D}_\alpha(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p) \end{aligned}$$

implies that

$$\lim_{t\to+\infty} \mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\tfrac{x}{t}\right)\right) = -\infty. \tag{4.5.20}$$

Thus, by combining (4.5.19) with (4.5.20), for every k sufficiently large there exists $t_{\varepsilon_k} > 1$ such that

$$\mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\tfrac{x}{t_{\varepsilon_k}}\right)\right) = 0. \tag{4.5.21}$$

In particular, by using (4.5.13), (4.5.14) and (4.5.15), if g satisfies the second condition

$$\lim_{\varepsilon\to+\infty} \varepsilon^{\nu} f(g(\varepsilon)) = 0, \tag{4.5.22}$$

we then can obtain $t_{\varepsilon_k} \to 1$, since as $k \to +\infty$ we have

$$1 < (t_{\varepsilon_k})^{\alpha} \le \frac{d\left(\frac{1}{2}\|\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{q}\|\psi_{\varepsilon_k^{-1}}\|^q_{L^q(\mathbb{R}^d)}\right) + \frac{d-2}{2}\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)}}{\frac{d+\alpha}{2p}\mathcal{D}_{\alpha}(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p)} \to 1. \tag{4.5.23}$$

To summarise, to deduce (4.5.19) and (4.5.23) we need a function g that satisfies:

(i) $\lim_{\varepsilon\to+\infty} \varepsilon^{\nu} f(g(\varepsilon)) = 0$;

(ii) $\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = +\infty$.

The existence of such function g is guaranteed by Lemma 4.6.3 in the Appendix. Moreover,

$$\begin{aligned}\sigma_{\varepsilon_k} &\le \mathcal{J}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\tfrac{x}{t_{\varepsilon_k}}\right)\right)\\ &= \frac{t_{\varepsilon_k}^{d-2}}{2}\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2 + t_{\varepsilon_k}^{d}\left(\frac{1}{2}\|\psi_{\varepsilon_k^{-1}}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{q}\|\psi_{\varepsilon_k^{-1}}\|^q_{L^q(\mathbb{R}^d)}\right) - \frac{t_{\varepsilon_k}^{d+\alpha}}{2p}\mathcal{D}_{\alpha}(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p).\end{aligned} \tag{4.5.24}$$

Finally, in view of (4.5.13), (4.5.14), (4.5.15), (4.5.22) and the limit $t_{\varepsilon_k} \to 1$, the right hand side of (4.5.24) converges to σ as $k \to \infty$. This implies that $\sigma_{\varepsilon_k} \to \sigma$ as $k \to +\infty$. □

Once the convergence of σ_{ε} towards σ is proved, we can show that the term $\varepsilon^{\nu}\|\nabla u_{\varepsilon}\|^2_{L^2(\mathbb{R}^d)}$ can also be safely ignored in the same limit.

Corollary 4.5.1. *Assume that* $\frac{d+\alpha}{d} < p < \frac{d+\alpha}{d-2}$ *and* $q > 2\frac{2p+\alpha}{2+\alpha}$. *Then there exists a sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *and a sequence of ground states* (u_{ε_k}) *of* (P_{ε_k}) *such that* $\varepsilon_k \to \infty$

and

$$\varepsilon_k^\nu \|\nabla u_{\varepsilon_k}\|_2^2 \to 0 \quad \text{as } k \to +\infty.$$

Proof. Arguing as in the first part of Lemma 4.5.1, there exists $t_\varepsilon \in (0,1)$ such that $u_\varepsilon\left(\frac{x}{t_\varepsilon}\right) \in \mathscr{P}$. Now, let's consider the sequence $(t_{\varepsilon_k})_k$, corresponding ot the same sequence $(\varepsilon_k)_k$ in Lemma 4.5.2. We first prove that, up to a subsequence, $t_{\varepsilon_k} \to 1$ as $k \to +\infty$. Since $(t_{\varepsilon_k})_k$ is bounded, up to a subsequence $t_{\varepsilon_k} \to t_0 \in [0,1]$. Assume by contradiction that $t_0 < 1$. Then

$$\begin{aligned} \sigma \le E\left(u_{\varepsilon_k}\left(\tfrac{x}{t_{\varepsilon_k}}\right)\right) &= \frac{\alpha t_{\varepsilon_k}^{d+\alpha}}{2dp}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \\ &\le t_{\varepsilon_k}^{d+\alpha}\mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) = t_{\varepsilon_k}^{d+\alpha}\sigma_{\varepsilon_k} \to t_0^{d+\alpha}\sigma < \sigma, \end{aligned} \tag{4.5.25}$$

a contradiction. Therefore, we have proved that $t_{\varepsilon_k} \to 1$ and furthermore,

$$\mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) = \frac{\varepsilon_k^\nu}{2}\|\nabla u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2dp}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \to \sigma.$$

In particular, $(\varepsilon_k^\nu\|\nabla u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2)_k$, $(\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p))_k$ are bounded sequences and the same holds for $\left(\|u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}\right)_k$ and $\left(\|u_{\varepsilon_k}\|_{L^q(\mathbb{R}^d)}\right)_k$. Therefore, by combining the equation

$$\begin{aligned} 0 &= \frac{dt_{\varepsilon_k}^d}{2}\|u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2 + \frac{dt_{\varepsilon_k}^d}{q}\|u_{\varepsilon_k}\|_{L^q(\mathbb{R}^d)}^q - \frac{(d+\alpha)t_\varepsilon^{d+\alpha}}{2p}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \\ &= -\frac{(d-2)\varepsilon^\nu}{2}\|\nabla u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2 + d(t_{\varepsilon_k}^d - 1)\left(\frac{\|u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2}{2} + \frac{\|u_{\varepsilon_k}\|_{L^q(\mathbb{R}^d)}^q}{q}\right) \\ &\qquad - \frac{(d+\alpha)}{2p}(t_{\varepsilon_k}^{d+\alpha} - 1)\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p), \end{aligned} \tag{4.5.26}$$

with boundedness of the above sequences and $t_{\varepsilon_k} \to 1$, we obtain

$$\lim_{k\to+\infty} \varepsilon_k^\nu\|\nabla u_{\varepsilon_k}\|_{L^2(\mathbb{R}^d)}^2 = 0.$$

□

4.5.1 Proof of Theorem 4.1.4

By Lemma 4.5.2 and Corollary 4.5.1, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $(u_{\varepsilon_k}(x/t_{\varepsilon_k})) \subset \mathscr{P}$ is a bounded radially non increasing minimizing sequence for the functional E. Then, if we set $v_{\varepsilon_k}(x) := u_{\varepsilon_k}(x/t_{\varepsilon_k})$, by arguing as in the proof of Theorem 4.1.2 there exists $\bar{v} \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ such that

$$\begin{aligned} v_{\varepsilon_k} &\to \bar{v} \quad \text{in } L^s(\mathbb{R}^d), \ \forall s \in (2,q), \\ v_{\varepsilon_k} &\to \bar{v} \quad \text{a.e. in } \mathbb{R}^d. \end{aligned}$$

Then, we claim that $\bar{v} \in \mathscr{P}$. As a matter of fact, assume by contradiction that $\mathcal{P}(\bar{v}) \neq 0$. Since v_{ε_k} is a minimizing sequence for σ, by the non local Brezis–Lieb Lemma we derive that

$$\mathcal{D}_\alpha(v_{\varepsilon_k}^p, v_{\varepsilon_k}^p) \to \mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) = \frac{2dp\sigma}{\alpha} \tag{4.5.27}$$

and, by the weak lower semi-continuity of the norms, we clearly have that $\mathcal{P}(\bar{v}) < 0$. Furthermore, it's easy to see that there exists $t_0 \in (0,1)$ such that $\bar{v}(x/t_0) \in \mathscr{P}$. However this implies that

$$\sigma \le E(\bar{v}(x/t_0)) = \frac{\alpha}{2dp}\mathcal{D}_\alpha(\bar{v}(x/t_0)^p, \bar{v}(x/t_0)^p) = \frac{t_0^{d+\alpha}\alpha}{2dp}\mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) < \frac{\alpha}{2dp}\mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) = \sigma,$$

that is a contradiction. Hence $\bar{v} \in \mathscr{P}$. Consequently, combining the standard Brezis–Lieb lemma with (4.5.27) yields

$$\sigma = \lim_{k\to+\infty} E(v_{\varepsilon_k}) = E(\bar{v}) + \lim_{k\to+\infty}\left(\frac{\|v_{\varepsilon_k} - \bar{v}\|_{L^2(\mathbb{R}^d)}^2}{2} + \frac{\|v_{\varepsilon_k} - \bar{v}\|_{L^q(\mathbb{R}^d)}^q}{q}\right) \ge \sigma,$$

proving that $E(\bar{v}) = \sigma$ and v_{ε_k} converges to $\bar{v}$ in $L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ i.e., $\bar{v}$ is a ground state solution of (*TF*). Finally, from $t_{\varepsilon_k} \to 1$ we further conclude that u_{ε_k} converges to $\bar{v}$ as well. □

4.6 Appendix

Here we prove some technical results that have been employed in this chapter. We begin by the differentiability of $\mathcal{D}_\alpha(|u|^p, |u|^p)$. Later on, we prove a calculus lemma that was used in the proof of Lemma 4.5.2.

Lemma 4.6.1. *Let $u \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $p > \frac{d+\alpha}{d}$, $q > \frac{2dp}{d+\alpha}$ and $\alpha \in (0,d)$. The function $\mathcal{D}_\alpha(|u|^p, |u|^p)$ defined is Frechet diffentiable and moreover, for every*

$\varphi \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$

$$\lim_{t\to 0} \frac{\mathcal{D}_\alpha(|u+t\varphi|^p, |u+t\varphi|^p) - \mathcal{D}_\alpha(|u|^p, |u|^p)}{t} = 2p \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}u\varphi. \quad (4.6.1)$$

Proof. To prove (4.6.1) by adding and subtracting the term $\int_{\mathbb{R}^d}(I_\alpha * |u+t\varphi|^p)|u|^p$ we derive that

$$\frac{\mathcal{D}_\alpha(|u+t\varphi|^p, |u+t\varphi|^p) - \mathcal{D}_\alpha(|u|^p, |u|^p)}{t} = I_1 + I_2$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^d} I_\alpha * |u+t\varphi|^p \left(\frac{|u+t\varphi|^p - |u|^p}{t}\right), \\ I_2 &:= \int_{\mathbb{R}^d} |u|^p \left(\frac{I_\alpha * |u+t\varphi|^p - I_\alpha * |u|^p}{t}\right). \end{aligned} \quad (4.6.2)$$

Now, if $|t| < 1$, by HLS inequality we have

$$I_1 \leq 2^{2(p-1)} \left(I_\alpha * |u|^p + I_\alpha * |\varphi|^p\right) \left(|u|^p + |\varphi|^p\right) \in L^1(\mathbb{R}^d).$$

Then, by dominated convergence theorem and differentiability of the L^p–norm

$$\lim_{t\to 0} I_1(t) = p \int_{\mathbb{R}^d} I_\alpha * |u|^p |u|^{p-2} u\varphi.$$

By a similar argument we deduce that

$$\lim_{t\to 0} I_2(t) = p \int_{\mathbb{R}^d} I_\alpha * |u|^p |u|^{p-2} u\varphi.$$

Next, to conclude the desired Frechèt differentiability it's enough to notice that the linear operator generated by $(I_\alpha * |u|^p)|u|^{p-2}u$ is continuous on $L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. $\square$

Lemma 4.6.2. *Let $f, g : \mathbb{R}^d \to \mathbb{R}_+$ be two radially non increasing functions. If $f * g \in L^s(\mathbb{R}^d)$ for some $s \in (1, \infty)$ then $f * g$ is radially non increasing. In particular, if $g \in L^t(\mathbb{R}^d)$ for some $1 < t < \frac{d}{\alpha}$ then $I_\alpha * g$ is radially non increasing.*

Proof. Since $f * g$ is non negative we clearly have

$$\|f * g\|_{L^s(\mathbb{R}^d)} = \left| \sup_{\|\varphi\|_{L^{s'}(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} (f * g)(x)\varphi(x)dx \right| = \sup_{\|\varphi\|_{L^{s'}(\mathbb{R}^d)}=1, \varphi\geq 0} \int_{\mathbb{R}^d} (f * g)(x)\varphi(x)dx$$
$$\leq \sup_{\|\varphi\|_{L^{s'}(\mathbb{R}^d)}=1, \varphi\geq 0} \int_{\mathbb{R}^d} (f * g)(x)\varphi^*(x)dx \leq \|f * g\|_{L^s(\mathbb{R}^d)},$$

where the first inequality follows from [78, Theorem 3.7]. This shows that the supremum is achieved by a non negative radially non increasing function φ^* and, such function satisfies

$$a(\varphi^*(x))^{s'} = b((f * g)(x))^s, \quad a, b > 0.$$

In particular, $f * g$ is radially non increasing. The second part of the statement follows simply by recalling again that $I_\alpha * g \in L^{\frac{dt}{d-t\alpha}}(\mathbb{R}^d)$. □

Lemma 4.6.3. *For every $\nu < 0$, and for every continuous and strictly monotone increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $\lim_{\varepsilon\to+\infty} f(\varepsilon) = +\infty$, there exists a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\lim_{\varepsilon\to+\infty} g(\varepsilon) = +\infty, \tag{4.6.3}$$

$$\lim_{\varepsilon\to+\infty} \varepsilon^\nu f(g(\varepsilon)) = 0, \tag{4.6.4}$$

$$\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^\nu f(g(\varepsilon)) = +\infty. \tag{4.6.5}$$

Proof. Let H be the function defined by

$$H(\varepsilon) := \min\left\{ \log(\varepsilon), \sqrt{f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)} \right\}, \quad \text{for } \varepsilon > e^{-\frac{1}{\nu}}, \tag{4.6.6}$$

where $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of f. Clearly,

$$\lim_{\varepsilon\to+\infty} H(\varepsilon) = +\infty. \tag{4.6.7}$$

Note that such H is continuous and monotone increasing since both of the functions $\sqrt{f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)}$ and $\log(\varepsilon)$ are continuous and monotone increasing for $\varepsilon > e^{-\frac{1}{\nu}}$.

Hence, we define g as follows:

$$g(\varepsilon) := f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right) \quad \text{for } \varepsilon > e^{-\frac{1}{\nu}} \tag{4.6.8}$$

and we extend to a continuous non negative function defined on $\mathbb{R}_+$. Note that from (4.6.6), monotonicity and unboundedness of f^{-1}, we have

$$g(\varepsilon) \geq f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right) \longrightarrow +\infty \quad \text{as } \varepsilon \to +\infty.$$

Hence, (4.6.3) holds. Furthermore, from (4.6.7) we obtain

$$\lim_{\varepsilon\to+\infty} \varepsilon^{\nu} f(g(\varepsilon)) = \lim_{\varepsilon\to+\infty} \frac{1}{H(\varepsilon)} = 0,$$

which proves (4.6.4). Finally, again from (4.6.6) and (4.6.8) it holds

$$\begin{aligned} g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = \frac{1}{H(\varepsilon)} \cdot f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right) &\geq \left(f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)\right)^{-\frac{1}{2}} f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right) \\ &\geq \sqrt{f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)} \longrightarrow +\infty, \qquad \text{as } \varepsilon \to +\infty, \end{aligned}$$

which proves (4.6.5). □

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