Theoretical Computer Science 1034 (2025) 115124

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

For rational numbers with Suppes-Ono division, equational validity is one-one equivalent with Diophantine unsolvability ☆

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ARTICLE INFO

Keywords:
Division by zero
Fracterm
Fracterm flattening
Diophantine equation
Decidability
1-1 degrees

ABSTRACT

Adding division to rings and fields leads to the question of how to deal with division by 0. From a plurality of options, we discuss in detail what we call *Suppes-Ono division* in which division by 0 produces 0. We explain the backstory of this semantic option and its associated notion of equality, and prove a result regarding the logical complexity of deciding equations over the rational numbers equipped with Suppes-Ono division. We prove that deciding the validity of the equations is computationally equivalent to the Diophantine Problem for the rational numbers, which is a longstanding open problem.

1. Introduction

Calculations on numerical data types with partial functions – division, tangents, logs, etc. – are commonplace. However, computing and reasoning with algebraic expressions containing such partial operators is complicated because they may not be defined. Indeed, logics for reasoning with partial operations are technically more involved than standard logics in which operators are total. For example, what it means for two algebraic expressions or terms t and r to be equal depends on semantic conventions governing their evaluation in a numerical algebra A, for which there are several options. Thus, if partial operators are made total then standard logics can be applied with their standard equality. In practical computation, applying an operator needs to produce a response.

The problem is clear in an arithmetical data type with a division operator $\frac{x}{y}$. The basic algebra of arithmetic is based on the axioms of commutative rings and fields. In fields there is just one place where a division operator is undefined, i.e., when $y = 0$. There are several semantic options for making division total. One option is to assume that $\frac{1}{0} = 0$, which is an option to be found in theorem provers and arguments in mathematical logic. What are the consequences of this hack for the data type?

With Q_0 we denote an algebra which consists of a field Q of rational numbers with operations $x + y, -x, x \cdot y$. To this, we also add a division operator $\frac{x}{y}$ to make an algebra called a *meadow* [9]. In the meadow Q_0 division is made total by defining

$$\frac{q}{0} = 0 \text{ for all } q \in Q.$$

We will refer to $q \in Q$ as a rational number, while acknowledging that there are many different representations and sets that may play the role of the rationals. The algebra Q_0 is a computable data type, i.e., a computable minimal algebra. Minimality, as originally

☆ This article belongs to Section B: Logic, semantics and theory of programming, Edited by Don Sannella.

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https://doi.org/10.1016/j.tcs.2025.115124
Received 12 June 2024; Received in revised form 18 December 2024; Accepted 12 February 2025

Available online 17 February 2025

introduced by the ADJ group (see, e.g., [13]), requires of a data type that it is an algebra generated by its constant elements and so is minimal in the sense that it has no proper substructures.

The signature Σ_m (with m for meadow) of Q_0 consists of the signature of rings with 1 augmented with a division operator. Thus, terms over Σ_m are more complicated than terms over fields – e.g., polynomials – as they involve nested divisions. Now, Q_0 induces a new equality relation $\sim$ on terms over the signature Σ_m:

$$t \sim r \iff Q_0 \models t = r.$$

We will refer to $\sim$ as *Suppes-Ono equality* and we will simply write $=$ instead of $\sim$ below as Suppes-Ono equality is meant throughout the paper.

Let $Eqn(Q_0)$ be the set of all equations over the signature Σ_m of meadows that are true in the meadow Q_0, i.e., the *equational theory* of Q_0. Since Q_0 is a computable data type, $Eqn(Q_0)$ is a computably enumerable set. Two related questions arise:

Is $Eqn(Q_0)$ decidable?
Is there a finite or computable axiomatisation E of Q_0 that is complete for the equational theory $Eqn(Q_0)$?

If there is a complete axiomatisation then it can be shown that $Eqn(Q_0)$ is decidable. In this note, we prove a theorem about the algorithmic complexity of this decision problem and the logical complexity of $Eqn(Q_0)$, namely:

Theorem. *The problem of solving diophantine equations in Q has the same one-one computable degree as deciding the invalidity of equations in Q_0.*

The Diophantine Problem for Q is long-standing open problem of number theory and arithmetic geometry, with many ramifications.[1] Its origins lie in Hilbert's Tenth Problem, to be found in his famed list of 23 problems posed in 1900; it asks for an algorithm that inputs a multi-variable polynomial $p(x_1, x_2, \dots, x_n)$ with integer coefficients and decides whether or not there exists a solution in integers $a_1, a_2, \dots, a_n$ such that $p(a_1, a_2, \dots, a_n) = 0$. In 1970, Yu. Matiyasevich showed that no such algorithm exists [16].

The Diophantine Problem for rationals asks a similar question for the polynomial $p \in Q[x_1, x_2, \dots, x_n]$ and a solution $a_1, a_2, \dots, a_n \in Q$. That the decision problem for integers has been solved for over 50 years, underlines the fact that the unsolved rational number decision problem is a significant mathematical challenge. Thus, the decidability and completeness of $Eqn(Q_0)$ are clearly difficult problems.

Various forms of Hilbert's Tenth Problem are beautifully surveyed in [22].

2. Meadows and Suppes-Ono division

2.1. Fields and meadows

We start from the theory of fields. A *field* is a commutative ring F with 1 in which $0 \neq 1$ and for all $x \in F$,

$$x \neq 0 \text{ implies } \exists y[x \cdot y = 1].$$

A field is an algebra of the form

$$(F \mid 0, 1, x + y, -x, x \cdot y).$$

Let Σ_r be a signature for rings and fields.

In a ring R, for each x, if there is a y such that $x \cdot y = 1$ then x is called an *invertible element* and y is called the *inverse* of x. In every ring, the additive inverse 0 is not invertible as we can derive $0.x = 0$ from the axioms of commutative rings. Thus, a field is a ring in which all elements are invertible, except 0. Note rings and fields have the same three operations. Fields having only ring operations and no inverse or division was well established from the early days of abstract algebra [28]. To fields we add a division operator to make a meadow.

Definition 2.1. *A* meadow *is a partial algebra $F(\_/\_)$ obtained as an expansion of a field with a division function $\_/\_$ that works as usual on non-zero elements of the domain of F. It has the form*

$$(F \mid 0, 1, x + y, -x, x \cdot y, x/y).$$

Let $\Sigma_m = \Sigma_r \cup \{\_/\_\}$ be the signature of a meadow.

Definition 2.2. *A meadow has* Suppes-Ono division *if* $\frac{x}{0} = 0$.

[1] For instance, the problem is equivalent to deciding whether a variety X over Q has a rational point.

Recall that to qualify as a data type, an algebra must be minimal, i.e., generated by its constants and operations. For the field Q of rationals is *not* minimal and is not a data type for that reason. Division is needed to turn the classical field of rational numbers into a minimal algebra and so a data type.

2.2. Some history of $\frac{1}{0} = 0$

The idea of making division total can be found in old algebra texts but it rarely leads to any form of theory. However, the need for total operations to simplify logical reasoning and practical computation demands rigorous semantic analysis. In the modern era, Patrick Suppes, in [27], was first to formulate division by zero as a problem in logic for which a plurality of solutions exists, none of which may be fully satisfactory, however. Suppes conclusion in 1957 is his proposal of $\frac{1}{0} = 0$ as a plausible solution; he observed, correctly in hindsight, that mathematicians would not easily embrace this idea. In [1] a review is given of the discussion of division by zero by Suppes.

Systematic logical research on the consequences of adopting $\frac{1}{0} = 0$ starts with the work of Ono [17] in 1983. In view of these facts, we propose to refer to adopting $\frac{1}{0} = 0$ as *Suppes-Ono division* and the equality relation on arithmetical expressions which results from it as *Suppes-Ono equality*.

For a survey on options concerning division by zero we refer to [3].

Adopting $\frac{1}{0} = 0$ has been done by many researchers independently in the context of informatics. Systems for proof checking such as *Coq*, *Agda*, *Lean*, *Isabelle* have used $\frac{1}{0} = 0$ to simplify type checking. In such systems, $\frac{1}{0} = 0$ does not feature a proof rule which one expects to be used, nor does it give expression to some perspective on division. It provides merely a short-cut and adopting $\frac{1}{0} = 1, 2, \ldots$ would work just as well. Similar conventions with similar motivations of simplification can be found in model theory and in mathematical logic.

In [9], we used $\frac{1}{0} = 0$ in order to provide an finite equational initial algebra specification of an abstract data type of rational numbers, a longstanding open problem. In that paper, we also proposed the convention to refer to an arithmetical data type with an operator for division, or for inverse, as a *meadow*. Thus, Q_0 is a meadow of rational numbers, and using the terminology of [6], Q_0 is an *involutive meadow* because the inverse function $x^{-1} = \frac{1}{x}$ is an involution, i.e., $(x^{-1})^{-1} = x$. A good deal is known about involutive meadows but one important open problem is:

Problem 2.1. Can an equational specification of Q_0 be found which constitutes a confluent and terminating term rewriting system *without* the use of auxiliary operators?

In [8] this question is analysed for an arbitrary computable data type, and answered positively though permitting the use of auxiliary functions.

In the same style, equational specifications for quite different arithmetical data types – wheels of rationals and transrationals – addressing the partiality of division by adding new elements including $\perp, \infty, +\infty, -\infty$ to the rationals have been developed in [10] and [11].

2.3. Fracterms

Working with division introduces fractions. Following [4] we use *fracterm* rather than fraction for an expression $\frac{t}{r}$ over Σ_m having division as its leading function symbol. By doing so, the ambiguity of fraction is avoided, in which the word sometimes refers to a number without any syntactic bias, and on other occasions refers to an expression. A fracterm is a structured entity comprising a numerator, a denominator and a function symbol as parts, and thereby a fracterm is definitely not a number!

Now, a fracterm $\frac{t}{r}$ which contains a single occurrence of division only is called *flat*. Of course, in practical work with rationals we expect that terms can be transformed into a flat fracterms, e.g.,

$$\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{8}{9}.$$

But this is *not* the case for involutive meadows. However, we do have:

Theorem 2.1. *Each arithmetical expression t over Σ_m can be computably transformed into an equivalent expression of the form*

$$\frac{t_1}{r_1} + \ldots + \frac{t_n}{r_n}$$

where the t_i and r_i do not involve division. Furthermore, no bound on the number n of flat summands can be assumed.

Proof. The computable transformation to sums of flat fracterms is proved in [5]; the lack of a finite bound is shown in [7]. □

A polynomial $p(x_1, \ldots, p_n)$ with integer coefficients can be understood as a division-free expression or term over Σ_m.

2.4. Computable reductions and 1-1 degrees

In the theory of computability, general decision problems are classified by various concepts of degrees of unsolvability, following Emil Post's [23]. These take the form of comparing sets A and B by a reduction wherein deciding A can be done via deciding B in some way. The most well known are the Turing degrees and the many-one degrees. Through the theory of numbered sets and algebras these notions, which are formulated for computability on the natural numbers, can be applied to any sets of elements [15,25,26].

Let U be any numbered set. Following [25], this means there is a surjection $\alpha : \Omega \subseteq \mathbb{N} \to U$ which is called a *numbering* of U.

Definition 2.3. *Let $A, B \subset U$. A is* many-one reducible *to B if there is a computable function $f : U \to U$ such that for all $x \in U$,*

$$x \in A \iff f(x) \in B.$$

We write $A \leqslant_m B$.

A is many-one equivalent *to B if $A \leqslant_m B$ and $B \leqslant_m A$. We write $A \equiv_m B$.*

The concept is finer than the Turing degree and can be further refined as follows:

Definition 2.4. *Let $A, B \subset U$. A is* one-one reducible *to B if there is an injective or one-to-one computable function $f : U \to U$ such that for all $x \in U$,*

$$x \in A \iff f(x) \in B.$$

We write $A \leqslant_1 B$.

A is one-one equivalent *to B if $A \leqslant_1 B$ and $B \leqslant_1 A$. We write $A \equiv_1 B$.*

One-one equivalence is the finest classification of relative computability in use.

Let $\overline{A}$ be the complement of A in U. We will use this fact

Lemma 2.1. *Suppose $A \leqslant_1 \overline{B}$ and $B \leqslant_1 \overline{A}$. Then $A \equiv_1 \overline{B}$.*

Proof. We have $A \leqslant_1 \overline{B}$ and $B \leqslant_1 \overline{A}$. Choose 1-1 function g such that $b \in B$ iff $g(b) \in \overline{A}$, then g provides a 1-1 reduction from $\overline{B}$ to A as well, combined we have $A \equiv_1 \overline{B}$. □

3. On Diophantine equations over Q

A *Diophantine equation* is an equation of the form

$$p(x_1, \ldots, x_n) = 0$$

with p an integer polynomial. The question whether or not $p(x_1, \ldots, x_n) = 0$ has a solution in the rationals Q is clearly computably enumerable: all Diophantine equations which have a rational solution can be effectively enumerated. The decidability of solvability of Diophantine equations over Q is a long standing open problem. For an easy introduction to the problem see [22], or for more information we mention [14] and [21].

We will now show that solvability of Diophantine equations $p = 0$ over Q and determination of validity of equations $t = r$ over Q_0 are problems with the same degree of unsolvability. In fact both problems have the same one-one degree of unsolvability as the complement of the other problem, using the terminology of [24].

Theorem 3.1. *Solvability of Diophantine equations $p = 0$ over Q and determination of invalidity of equations $t = r$ over Q_0 are problems with the same one-one degree of unsolvability.*

In view of Lemma 2.1 the proof consists of two propositions:

Proposition 3.1. *Solvability of Diophantine equations $p = 0$ over Q is $1-1$ reducible to the complement of validity of equations $t = r$ over Q_0.*

Proof. $p(x_1, \ldots, x_n) = 0$ is solvable over Q if, and only if, it is *not* the case that $Q_0 \models \frac{p(x_1,\ldots,x_n)}{p(x_1,\ldots,x_n)} = 1$. □

Proposition 3.2. *The validity of equations $t = r$ over Q_0 is $1-1$ reducible to non-solvability of Diophantine equations $p = 0$ over Q.*

Proof. First, notice that we may assume $r \equiv 0$ because $Q_0 \models t = r \iff Q_0 \models t - r = 0$ so that equational validity is one-one equivalent with equational validity where the right hand side of the equation is the constant 0.

We will show that an equation $t = 0$ is equivalent in Q_0 to a finite and effectively computable conjunction of conditional formulae of the form

$$(u = 0 \wedge v \neq 0) \rightarrow w = 0$$

where u, v and w are division free terms.

Let $\psi_n(u, v, w, u_1, \dots, u_n, v_1, \dots, v_n)$ be formulae with the following form:

$$u = 0 \wedge v \neq 0 \rightarrow w + \frac{u_1}{v_1} + \dots + \frac{u_n}{v_n} = 0.$$

By Theorem 2.1, t can be written as a sum $\frac{t_1}{r_1} + \dots + \frac{t_n}{r_n}$ with the t_i and r_i division free expressions. Thus, the equation $t = 0$ can be written equivalently in the following form $\psi(0, 1, 0, t_1, \dots, t_n, r_1, \dots, r_n)$, i.e.,

$$0 = 0 \wedge 1 \neq 0 \rightarrow 0 + \frac{t_1}{r_1} + \dots + \frac{t_n}{r_n} = 0.$$

Next, we show that each substitution instance with division free terms of $\psi_{n+1}(u, v, w, u_1, \dots, u_n, v_1, \dots, v_n)$ is equivalent to a conjunction of two substitution instances with division free terms of $\psi_n(u, v, w, u_1, \dots, u_n, v_1, \dots, v_n)$. It suffices to show this for the most general case:

Indeed $Q_0 \models u = 0 \wedge v \neq 0 \rightarrow w + \frac{u_1}{v_1} + \dots + \frac{u_{n+1}}{v_{n+1}} = 0$ if, and only if,

(i) $Q_0 \models u = 0 \wedge v_1 = 0 \wedge v \neq 0 \rightarrow w + \frac{u_2}{v_2} + \dots + \frac{u_{n+1}}{v_{n+1}} = 0$, and

(ii) $Q_0 \models u = 0 \wedge v_1 \neq 0 \wedge v \neq 0 \rightarrow w + \frac{u_1}{v_1} + \dots + \frac{u_{n+1}}{v_{n+1}} = 0$.

Further, condition (i) is equivalent to:

$$Q_0 \models (u^2 + v_1^2 = 0) \wedge v \neq 0 \rightarrow w + \frac{u_2}{v_2} + \dots + \frac{u_{n+1}}{v_{n+1}} = 0.$$

Condition (ii) is equivalent to:

$$Q_0 \models u = 0 \wedge (v \cdot v_1) \neq 0 \rightarrow (w \cdot v_1 + u_1) + \frac{u_2 \cdot v_1}{v_2} + \dots + \frac{u_n \cdot v_1}{v_n} = 0.$$

In both cases a substitution instance of $\psi_n(u, v, w, u_1, \dots, u_n, v_1, \dots, v_n)$ has been obtained. With induction on n, it follows that validity of $t = r$ boils down to a conjunction of substitution instances with division free terms of assertions of the form

$$u = 0 \wedge v \neq 0 \rightarrow w = 0.$$

Noticing that $v \neq 0 \rightarrow w = 0 \iff v \cdot w = 0$ we find that

$$(u = 0 \wedge v \neq 0 \rightarrow w = 0) \iff (u = 0 \rightarrow v \cdot w = 0).$$

The validity (the conjunction of) of a finite set of assertions $\phi_1, \dots \phi_k$ of the form

$$\phi_i \equiv p_i = 0 \rightarrow q_i = 0$$

must be checked with p_i and q_i division free terms, and this must be done by inspecting a single element from the complement of Diophantine solvability over Q.

Without loss of generality, we may assume that the conditional equations ϕ_i contain pairwise disjoint sets of free variables. We first consider a single conditional equation and we notice that for division free expressions p and q, $Q_0 \not\models p = 0 \rightarrow q = 0$ if, and only if, for some valuation σ, $Q_0, \sigma \models p = 0 \wedge q \neq 0$.

This is equivalent to the following Diophantine equation having a solution, where the y_i and z_i are pairwise distinct new variables, i.e., variables that do not occur in p and in q:

$$p^2 + ((y_1^2 + y_2^2 + y_3^2 + y_4^2 + 1) \cdot q^2 - (z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1))^2 = 0 \quad (E)$$

Now, with the help of Dirichlet's theorem, it is easy to see that $\frac{y_1^2+y_2^2+y_3^2+y_4^2+1}{z_1^2+z_2^2+z_3^2+z_4^2+1}$ ranges over all and only positive rationals. As a consequence, if $p = 0$ and $q \neq 0$ there are positive integers r and s such that $q^2 = \frac{r}{s}$ and natural numbers $y_1, \dots, y_4$ and $z_1, \dots, z_4$ such that

$$r = y_1^2 + y_2^2 + y_3^2 + y_4^2 + 1 \text{ and } s = z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1$$

so that

$$(y_1^2 + y_2^2 + y_3^2 + y_4^2 + 1) \cdot q^2 - (z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1) = 0.$$

With $p = 0$ equation E above follows. Conversely, the validity of E implies that $p = 0$ and $q \neq 0$, the latter because both $y_1^2 + y_2^2 + y_3^2 + y_4^2 + 1$ and s do not take the value 0.

Next, we notice that if $r_1 = 0, \ldots, r_m = 0$ are Diophantine equations each involving pairwise disjoint sets of variables then the Diophantine system $r_1 = 0, \ldots, r_m = 0$ is solvable in Q if, and only if, the single equation $r_1^2 + \ldots + r_m^2 = 0$ has a solution in Q.

Taking these facts together, it follows that $Q_0 \models t = 0$ if, and only if, the Diophantine equation $r_1^2 + \ldots + r_m^2 = 0$ is not solvable over Q which establishes the required one-one reduction. □

4. Concluding remarks

A completeness theorem for an equational axiomatisation E of an algebra or class of algebras confirms that the properties in E (and in logical equivalent sets of axioms) fully characterise the algebra or class. To be precise, they do so in terms of what can be expressed in the language of equations and proved using the rules of equational logic (and, equivalently, those of first order logic). If the axiomatisation E is finite or computable then the equational theory of the algebra or class is decidable.

The rational numbers Q are *the* fundamental data type for measurement and computation. Units of measurements are based upon the data type of rationals, and all practical calculations are made with rationals. Our equivalence theorem measures the computability of the equational theory of Q_0 precisely using 1-1 degrees. This implies that to prove the existence or non-existence of a finite or computable axiom system for Q_0 capable of capturing the equational theory of Q_0 via a completeness theorem is a difficult open problem, because the diophantine problem for Q is a difficult open problem [22]. Further, given the known undecidability of the diophantine problem for the natural numbers N, it is plausible that no such axiomatisation exists.

For another option for a total semantics of division, where $\frac{1}{0} = \perp$, an error value $\perp$ added to Q, a finite complete equational axiomatisation does exist [12].

Our interest stems from the study of abstract data types for computer arithmetic. However, a group in Japan pursues the consequences of adopting $\frac{1}{0} = 0$ as an option for improving mathematical practice; we mention, e.g., [18–20]. In these papers the phrase "Division-by-Zero Calculus" is used for the various implications of adopting $\frac{1}{0} = 0$.

Working with Suppes-Ono division has advantages as well as disadvantages for computing but we do not expect that Suppes-Ono division will be adopted for daily mathematical practice at a significant scale. An example of theoretical work making use Suppes-Ono equality can be found in [2].

CRediT authorship contribution statement

Jan A. Bergstra: Writing – original draft, Investigation, Formal analysis, Conceptualization. **John V. Tucker:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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