




Lie Affgebras Vis-à-Vis Lie Algebras

Ryszard R. Andruszkiewicz, Tomasz Brzeziński , and
Krzysztof Radziszewski

Abstract. It is shown that any Lie affgebra, that is an algebraic system consisting of an affine space together with a bi-affine multiplication satisfying affine versions of the antisymmetry and Jacobi identity, is isomorphic to a Lie algebra together with an element and a specific generalized derivation (in the sense of Leger and Luks in *J Algebra* 228:165–203, 2000). These Lie algebraic data can be taken for the construction of a Lie affgebra or, conversely, they can be uniquely derived for any Lie algebra fibre of the Lie affgebra. The close relationship between Lie affgebras and (enriched by the additional data) Lie algebras can be employed to attempt a classification of the former by the latter. In particular, up to isomorphism, a complex Lie affgebra with a simple Lie algebra fibre \mathfrak{g} is fully determined by a scalar and an element of \mathfrak{g} fixed up to an automorphism of \mathfrak{g} , and it can be universally embedded in a trivial extension of \mathfrak{g} by a derivation. The study is illustrated by a number of examples that include all Lie affgebras with one-dimensional, non-abelian two-dimensional, $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{so}(3)$ Lie algebra fibres. Extensions of Lie affgebras by cocycles and their relation to cocycle extensions of tangent Lie algebras is briefly discussed and illustrated by Lie affgebras with the Witt and Virasoro algebra fibres.

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1. Introduction

The study of vector space valued Lie brackets on affine spaces or *Lie affgebras* was initiated in [10] and developed and applied to the investigation of differential geometry of bundles with affine fibres or *AV-geometry* [11, 13] with an

eye to the frame independent formulation of Lagrangian mechanics [12, 21]. A new point of view and extension of Lie affgebras in which no reference to a vector space is made or, indeed, no existence of the underlying vector space is presumed, was proposed in [6]. In this approach a vector space is an artefact rather than a fundamental ingredient of an affine space in the sense that any point of an affine space determines a vector space (with the chosen point playing the role of the zero vector), the *tangent space* or the *vector space fibre* at this point. The linearisation of the bi-affine Lie bracket on an affine space, or the bi-affine multiplication satisfying versions of antisymmetry and the Jacobi identity, on any vector space fibre determines the Lie algebra structure on this fibre. Thus a Lie affgebra can be seen as a *Lie algebra fibred affine space*. The aim of the present text is to reveal and explore the relationship between Lie affgebras and the corresponding Lie algebra structures on the vector space fibres.

The bi-affine map property of the Lie multiplication on an affine space yields the distributivity of the multiplication over the ternary heap operation arising from the translation of a point by a vector between two other points. The results of [1] show that in the associative case there is a very clear connection between abelian heaps with bi-heap associative operation or *trusses* and extensions of rings by integers: trusses are in one-to-one correspondence with a specific class of extensions of rings by *double homotheties* of Redei [20] or *self-permutable bimultiplications* of MacLane [17]. Due to different natures of extensions in the associative and Lie cases, one cannot expect such a straightforward correspondence between Lie algebra extensions and Lie affgebras save for exceptional cases (such as derivation Lie affgebras discussed in Sect. 4). On the other hand one can ask the following three questions. First, does the fact that a Lie algebra is a Lie algebra fibre of a Lie affgebra yield additional data on it? Second, are there specific data which make a Lie algebra a fibre of a Lie affgebra or, in other words, can one construct a unique Lie affgebra from a Lie algebra with additional structure? Third, do all Lie affgebras arise in that way from Lie algebras (with additional data)? We give affirmative and constructive answers to all these questions.

The paper is organized as follows. In Sect. 2 we recall and slightly modify the definition of a Lie affgebra from [6]. The main difference from [6] is that the Jacobi identity is now homogenized. Section 3 contains main results. First, Theorem 3.1 states that any Lie affgebra \mathfrak{g} determines and is determined by a Lie algebra (which is isomorphic to the Lie algebra fibre of the Lie affgebra) together with two linear maps λ and κ , and a constant. There is no restriction on the constant, while λ is a *generalized derivation* in the sense of [16] of the type $(\delta, \lambda, \lambda)$, with $\delta = \lambda - \kappa$. This necessarily means that κ is an element of the *quasicentroid* of \mathfrak{g} , $\text{QC}(\mathfrak{g})$, the Lie sub-algebra of linear endomorphisms of \mathfrak{g} extensively studied in [16]. Second, Theorem 3.4 asserts that a homomorphism between Lie affgebras is given by a homomorphism between Lie algebra fibres and a constant. The Lie algebra homomorphism intertwines the respective

elements of the quasacentroids, while intertwining general derivations up to the adjoint action by the constant. This allows for the formulation of clear criteria for isomorphisms between Lie affgebras.

In Sect. 4 we analyze Lie affgebras in which defining linear transformations λ and κ of \mathfrak{g} combine into a (no longer generalized) derivation $\delta = \lambda - \kappa$. This happens if and only if κ is an element of the *centroid* of \mathfrak{g} , $C(\mathfrak{g})$, a Lie sub-algebra of $QC(\mathfrak{g})$ studied in depth in [14, Chapter X]. Consequently, the combination of results of [14, Chapter X], [16] and [8] yields that over an algebraically closed field and up to isomorphism a Lie affgebra with a simple Lie algebra fibre \mathfrak{g} is fully determined by a scalar and an element of \mathfrak{g} , the latter given up to an automorphism of \mathfrak{g} . In bigger generality, when κ is a scalar multiple of the identity, we extend the existence of *Lie hulls* of [10], i.e. Lie algebras in which Lie affgebras are universally embedded as cosets of ideals, beyond the Lie affgebras with idempotent multiplications (which bijectively correspond to Lie affgebras of Grabowska, Grabowski and Urbański [10]). The construction of Lie affgebras from Lie algebras described in Sect. 3 is illustrated by a number of examples in Sect. 5. We classify all Lie affgebras with one-dimensional vector space fibres, non-abelian two-dimensional Lie algebra fibres and $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{so}(3)$ fibres. In Sect. 5.6 we demonstrate how the correspondence between Lie affgebras and Lie algebras can be used to conclude non-isomorphism of a class of matrix Lie affgebras over finite fields, thus justifying the necessity of the restriction of the characteristic of the field or equivalently the size of the matrices demanded in [7, Theorem 3.2]. The paper is concluded with Sect. 6 in which we discuss *cocycle extensions* of Lie affgebras and relate them to central extensions of their Lie algebra fibres. We illustrate this discussion by Lie affgebras with Witt and Virasoro algebra fibres.

Notation and Conventions

We work over a field \mathbb{F} of characteristic different from 2. The space of linear endomorphisms of a vector space V is denoted by $\text{Lin}(V)$.

The adjoint action of $a \in \mathfrak{g}$ on a Lie algebra \mathfrak{g} is denoted by $\text{ad}_a : \mathfrak{g} \rightarrow \mathfrak{g}$, $b \mapsto [a, b]$, where $[-, -]$ is the multiplication.

2. Lie Multiplications or Brackets on Affine Spaces

By an *abelian heap* [2, 19] we mean an algebraic system consisting of a set X together with a ternary operation $\langle -, -, - \rangle : X^3 \rightarrow X$ such that, for all $x_i \in X$, $i = 1, \dots, 5$,

$$\begin{aligned} \langle x_1, x_2, x_3 \rangle &= \langle x_3, x_2, x_1 \rangle, \langle x_1, x_1, x_2 \rangle = x_2, \langle \langle x_1, x_2, x_3 \rangle, x_4, x_5 \rangle \\ &= \langle x_1, x_2, \langle x_3, x_4, x_5 \rangle \rangle. \end{aligned}$$

As a consequence of these conditions, the distribution of (angled) brackets does not matter, and hence we write $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ for $\langle x_1, x_2, \langle x_3, x_4, x_5 \rangle$,

$x_5\rangle$). A homomorphism of heaps is a function $f : X \rightarrow Y$ preserving the operations in the sense that, for all $x_i \in X$, $f(\langle x_1, x_2, x_3 \rangle) = \langle f(x_1), f(x_2), f(x_3) \rangle$. By fixing an element $o \in X$ and reducing ternary operation of a non-empty heap to a binary operation $x + y := \langle x, o, y \rangle$, one obtains an abelian group (the retract of X at o). Conversely, any abelian group determines a unique abelian heap structure with the operation $\langle a, b, c \rangle = a - b + c$. A heap homomorphism $f : X \rightarrow Y$ uniquely defines the homomorphism of abelian groups between the respective retracts by the assignment $x \mapsto f(x) - f(o)$.

Definition 2.1. Following [18] (see also [4, Definition 4.8 & Proposition 4.9]) by an *affine space* over a field \mathbb{F} or an \mathbb{F} -*affine space* we mean an algebraic system consisting of a non-empty set \mathfrak{a} and two ternary operations

$$\begin{aligned} \langle -, -, - \rangle : \mathfrak{a}^3 &\rightarrow \mathfrak{a}, & (a, b, c) &\mapsto \langle a, b, c \rangle, \\ -\triangleright_a - : \mathbb{F} \times \mathfrak{a}^2 &\rightarrow \mathfrak{a}, & (\alpha, a, b) &\mapsto \alpha \triangleright_a b, \end{aligned}$$

such that,

- (a) $(\mathfrak{a}, \langle -, -, - \rangle)$ is an abelian heap;
- (b) for all $a, b \in \mathfrak{a}$ and $\alpha \in \mathbb{F}$, $\alpha \triangleright_a - : \mathfrak{a} \rightarrow \mathfrak{a}$ and $-\triangleright_a b : \mathbb{F} \rightarrow \mathfrak{a}$ are homomorphisms of heaps, where \mathbb{F} is the heap with the operation $\alpha - \beta + \gamma$;
- (c) for all $a \in \mathfrak{a}$, the map $-\triangleright_a - : \mathbb{F} \times \mathfrak{a} \rightarrow \mathfrak{a}$ is an action of the multiplicative monoid of \mathbb{F} , that is, for all $\alpha, \beta \in \mathbb{F}$ and $b \in \mathfrak{a}$, $(\alpha\beta) \triangleright_a b = \alpha \triangleright_a (\beta \triangleright_a b)$ and $1 \triangleright_a b = b$;
- (d) for all $a, b \in \mathfrak{a}$, $0 \triangleright_a b = a$;
- (e) the *base change property* is satisfied, that is, for all $\alpha \in \mathbb{F}$ and $a, b, c \in \mathfrak{a}$,

$$\alpha \triangleright_a b = \langle \alpha \triangleright_c b, \alpha \triangleright_c a, a \rangle.$$

For a given $a \in \mathfrak{a}$, the map $-\triangleright_a - : \mathbb{F} \times \mathfrak{a} \rightarrow \mathfrak{a}$ is called an *affine action* (of \mathbb{F} on \mathfrak{a}) *with the base a*.

An *affine map* $f : \mathfrak{a} \rightarrow \mathfrak{b}$ is a heap homomorphism preserving the actions in the sense that, for all $a, b \in \mathfrak{a}$ and $\alpha \in \mathbb{F}$,

$$f(\alpha \triangleright_a b) = \alpha \triangleright_{f(a)} f(b).$$

The set of affine maps from \mathfrak{a} to \mathfrak{b} is denoted by $\text{Aff}(\mathfrak{a}, \mathfrak{b})$.

Given affine spaces $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, a function $f : \mathfrak{a} \times \mathfrak{b} \rightarrow \mathfrak{c}$ is said to be *bi-affine*, if for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ the functions

$$\mathfrak{b} \rightarrow \mathfrak{c}, \quad b' \mapsto f(a, b'), \quad \text{and} \quad \mathfrak{a} \rightarrow \mathfrak{c}, \quad a' \mapsto f(a', b),$$

are affine maps.

Remark 2.2. Given an element o of an \mathbb{F} -affine space \mathfrak{a} , the retract of the heap \mathfrak{a} at o , i.e., the abelian group structure on \mathfrak{a} with addition $a + b = \langle a, o, b \rangle$ and the neutral element o , together with the operation

$$\mathbb{F} \times \mathfrak{a} \rightarrow \mathfrak{a}, \quad (\alpha, a) \mapsto \alpha a := \alpha \triangleright_o a, \tag{2.1}$$

is a vector space. We call it the *tangent space to \mathfrak{a} at o* or the *vector space fibre of \mathfrak{a} at o* and denote it by $T_o\mathfrak{a}$. Different choices of o lead to isomorphic tangent spaces. The vector space $T_o\mathfrak{a}$ acts freely and transitively on \mathfrak{a} by

$$\mathfrak{a} \times T_o\mathfrak{a} \rightarrow \mathfrak{a}, \quad (a, b) \mapsto \langle a, o, b \rangle.$$

Thanks to the base change property (e) in Definition 2.1, the affine action of \mathfrak{a} in terms of the linear structure of its vector space fibre at o (2.1) can be written as

$$\alpha \triangleright_a b = (1 - \alpha)a + \alpha b, \quad (2.2)$$

for all $a, b \in \mathfrak{a}$ and $\alpha \in \mathbb{F}$. (This formula is independent of the choice of o .) On the other hand, any vector space can be understood as an affine space with the affine action (2.2).

Every $f \in \text{Aff}(\mathfrak{a}, \mathfrak{b})$ induces the unique linear transformation $\hat{f} : T_o\mathfrak{a} \rightarrow T_o\mathfrak{b}$, by $\hat{f} : a \mapsto f(a) - f(o)$. Conversely, given any linear transformation $\hat{f} : T_o\mathfrak{a} \rightarrow T_o\mathfrak{b}$ and an element $b \in \mathfrak{b}$, the map $\mathfrak{a} \rightarrow \mathfrak{b}$, $a \mapsto \hat{f}(a) + b$ is an affine map. All elements of $\text{Aff}(\mathfrak{a}, \mathfrak{b})$ arise in that way. Iterating this correspondence one obtains a correspondence between bi-affine and bilinear maps.

The preceding discussion clarifies the connection between the traditional definition of an affine space as a set accompanied with a vector space (of free vectors) acting on it and Definition 2.1.

Remark 2.3. The conditions of Definition 2.1 imply that, for all $\alpha \in \mathbb{F}$ and $b \in \mathfrak{a}$, the map $\alpha \triangleright_{-} b : \mathfrak{a} \rightarrow \mathfrak{a}$, $a \mapsto \alpha \triangleright_a b$, is a homomorphism of heaps (so there is no need to state it explicitly as a part of the definition). Furthermore,

$$\alpha \triangleright_a a = a, \quad (2.3)$$

for all $\alpha \in \mathbb{F}$ and $a \in \mathfrak{a}$. For the proofs see e.g. [4, Lemma 3.5].

The main object of studies of this paper is given in the following

Definition 2.4. By an *affgebra* we understand an affine space together with a bi-affine binary operation, called a *multiplication*. An affgebra \mathfrak{a} is called a *Lie affgebra* if the multiplication $\{-, -\} : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{a}$ satisfies the following conditions:

(a) affine antisymmetry, that is, for all $a, b \in \mathfrak{a}$,

$$\langle \{a, b\}, \{a, a\}, \{b, a\} \rangle = \{b, b\}; \quad (2.4)$$

(b) the affine Jacobi identity, that is, for all $a, b, c \in \mathfrak{a}$,

$$\langle \{a, \{b, c\}\}, \{a, \{a, a\}\}, \{b, \{c, a\}\}, \{b, \{b, b\}\}, \{c, \{a, b\}\} \rangle = \{c, \{c, c\}\}. \quad (2.5)$$

The multiplication in a Lie affgebra is often referred to as an *affine Lie bracket*.

A homomorphism of (Lie) affgebras is an affine map that preserves multiplications. It is straightforward to check that the homomorphic image of an affgebra (resp. Lie affgebra) is an affgebra (resp. Lie affgebra).

Remark 2.5. A Lie affgebra of Definition 2.4 is a slight modification of the notion introduced in [6, Definition 3.1] in so far as the inhomogeneous Jacobi identity

$$\langle \{a, \{b, c\}\}, \{a, a\}, \{b, \{c, a\}\}, \{b, b\}, \{c, \{a, b\}\} \rangle = \{c, c\}. \tag{2.6}$$

there is now replaced by its homogeneous version (2.5). Note that (2.6) implies that, for all $a \in \mathfrak{a}$, necessarily $3\{a, a\} = 3\{a, \{a, a\}\}$ in any abelian group retract of the heap $(\mathfrak{a}, \langle -, -, - \rangle)$. Thus if the characteristic of the field is different from three, (2.6) implies (2.5).

Needless to say, as in [6, Definition 3.1], there is a version of (2.5) in which the nested brackets $\{-, \{-, -\}\}$ are replaced by $\{\{-, -\}, -\}$, but clearly results obtained in one convention are easily translated to the other one.

The key fact motivating this paper that can also be considered as a strong support for Definition 2.4 is an observation made in [6, Theorem 3.15] that any tangent space to a Lie affgebra inherits a natural Lie algebra structure. This observation is not affected by the homogenisation of the Jacobi identity.

Theorem 2.6. *Let \mathfrak{a} be a Lie affgebra with multiplication $\{-, -\}$. Then, for all o in \mathfrak{a} , $T_o\mathfrak{a}$ is a Lie algebra with the multiplication*

$$[a, b] = \{a, b\} - \{a, o\} + \{o, o\} - \{o, b\}, \tag{2.7}$$

for all $a, b \in T_o\mathfrak{a}$. We call $T_o\mathfrak{a}$ with this multiplication a Lie algebra tangent to \mathfrak{a} at o or simply a tangent Lie algebra or a Lie algebra fibre of \mathfrak{a} .

Proof. Note that the multiplication (2.7) is the unique bilinear map associated to the bi-affine map $\{-, -\}$ by the step by step linearisation that connects affine maps with the linear ones on the retracts as described in Remark 2.2. In view of (2.4), for all $a \in \mathfrak{a}$,

$$[a, a] = \{a, a\} - \{a, o\} + \{o, o\} - \{o, a\} = o,$$

and hence the multiplication (2.7) is antisymmetric by bilinearity. The Jacobi identity is proven by a straightforward direct calculation that uses the affine Jacobi identity (2.5) written in $T_o\mathfrak{a}$,

$$\{a, \{b, c\}\} - \{a, \{a, a\}\} + \{b, \{c, a\}\} - \{b, \{b, b\}\} + \{c, \{a, b\}\} - \{c, \{c, c\}\} = o, \tag{2.8}$$

for all $a, b, c \in \mathfrak{a}$. □

All tangent Lie algebras to a given Lie affgebra are mutually isomorphic (as Lie algebras).

Example 2.7. Similarly to vector spaces that can also be seen as affine spaces, any Lie algebra \mathfrak{g} is a Lie affgebra with the multiplications, for all $a, b, s \in \mathfrak{g}$,

$$\{a, b\} = [a, b] + b + s. \tag{2.9}$$

Any tangent Lie algebra to this Lie affgebra is isomorphic to \mathfrak{g} . These are not all Lie affgebra structures with tangent \mathfrak{g} . For example, given any $\zeta \in \mathbb{F}$, an affine space \mathfrak{a} and $\chi \in \text{Aff}(\mathfrak{a}, \mathfrak{a})$, one can define the Lie affgebra $\mathfrak{a}_{\zeta, \chi}$ with the affine action multiplication

$$\{a, b\} = \langle \zeta \triangleright_a b, a, \chi(a) \rangle. \quad (2.10)$$

If the characteristic of a field is different from 3 and $\zeta \neq 0$, then one easily checks that the multiplication (2.10) satisfies the non-homogeneous affine Jacobi identity (2.6) if and only if $\chi = \text{id}$ in which case the multiplication takes the form

$$\{a, b\} = \zeta \triangleright_a b. \quad (2.11)$$

As explained in [6], if \mathfrak{a} has at least two elements, $\mathfrak{a}_{\zeta, \text{id}} \cong \mathfrak{a}_{\xi, \text{id}}$ if and only if $\zeta = \xi$. On the other hand, for all $\zeta \in \mathbb{F}$ and $o \in \mathfrak{a}$, $T_o \mathfrak{a}_{\zeta, \chi}$ is an abelian Lie algebra, i.e. $[a, b] = o$, for all $a, b \in T_o \mathfrak{a}_{\zeta, \chi}$.

It is one of the aims of this paper to study in detail the relationship between Lie affgebras and their tangent Lie algebras.

3. Lie Affgebras with a Prescribed Tangent Lie Algebra

The first main result of this paper reveals close connection between Lie affgebras and generalized derivations of Lie algebras in the sense of Leger and Luks [16].

Theorem 3.1. *Let \mathfrak{g} be a Lie algebra and $\kappa, \lambda \in \text{Lin}(\mathfrak{g})$ be such that, for all $a, b \in \mathfrak{g}$,*

$$\lambda([a, b]) = [\lambda(a), b] - [a, \kappa(b)] + [a, \lambda(b)]. \quad (3.1)$$

Then, for all $s \in \mathfrak{g}$, \mathfrak{g} is a Lie affgebra with the affine space structure

$$\langle a, b, c \rangle = a - b + c, \quad \alpha \triangleright_a b = \alpha b + (1 - \alpha)a, \quad \text{for all } a, b, c \in \mathfrak{g}, \alpha \in \mathbb{F},$$

and the multiplication, for all $a, b \in \mathfrak{g}$,

$$\{a, b\} = [a, b] + \kappa(a) + \lambda(b - a) + s. \quad (3.2)$$

We denote this Lie affgebra by $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$. Furthermore, for all $o \in \mathfrak{g}$

$$T_o \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s) \cong \mathfrak{g}.$$

Conversely, for any Lie affgebra \mathfrak{a} and any $o \in \mathfrak{a}$, there exist κ, λ, s necessarily satisfying (3.1) and such that $\mathfrak{a} = \mathfrak{a}(T_o \mathfrak{a}; \kappa, \lambda, s)$.

Proof. Let \mathfrak{g} be a Lie algebra and κ, λ and s be as in the assumptions of theorem. Due to the fact that the Lie multiplication $[-, -]$ is a bilinear operation and κ, λ are linear functions, the multiplication (3.2) is a bi-affine map as the sum of a linear and constant parts. To prove affine antisymmetry let us take

any $a, b \in \mathfrak{g}$, then from the antisymmetry of the Lie multiplication and from the linearity of λ we obtain

$$\begin{aligned} \langle \{a, b\}, \{a, a\}, \{b, a\} \rangle &= \{a, b\} - \{a, a\} + \{b, a\} \\ &= [a, b] + \kappa(a) + \lambda(b - a) + s \\ &\quad - \kappa(a) - s + [b, a] + \kappa(b) + \lambda(a - b) + s \\ &= \kappa(b) + s = \{b, b\} \end{aligned}$$

Thus the affine antisymmetry holds with no restrictions on the data λ, κ and s .

To show the affine Jacobi identity (2.5) of the multiplication (3.2), let us first note that

$$\begin{aligned} \{a, \{b, c\}\} - \{a, \{a, a\}\} &= [a, [b, c]] + [a, \kappa(b)] + [a, \lambda(c)] - [a, \lambda(b)] \\ &\quad + \lambda([b, c]) + \lambda\kappa(b - a) + \lambda^2(c - b) - [a, \kappa(a)]. \end{aligned}$$

Hence the multiplication defined by the formula (3.2) satisfies the affine Jacobi identity (2.5) if and only if

$$\begin{aligned} 0 &= \langle \{a, \{b, c\}\}, \{a, \{a, a\}\}, \{b, \{c, a\}\}, \{b, \{b, b\}\}, \{c, \{a, b\}\} \rangle - \{c, \{c, c\}\} \\ &= \{a, \{b, c\}\} - \{a, \{a, a\}\} + \{b, \{c, a\}\} - \{b, \{b, b\}\} + \{c, \{a, b\}\} - \{c, \{c, c\}\} \\ &= [a, [b, c]] + [a, \kappa(b)] + [a, \lambda(c)] - [a, \lambda(b)] + \lambda([b, c]) - [a, \kappa(a)] \\ &\quad + [b, [c, a]] + [b, \kappa(c)] + [b, \lambda(a)] - [b, \lambda(c)] + \lambda([c, a]) - [b, \kappa(b)] \\ &\quad + [c, [a, b]] + [c, \kappa(a)] + [c, \lambda(b)] - [c, \lambda(a)] + \lambda([a, b]) - [c, \kappa(c)] \\ &= [a, \kappa(b)] + [a, \lambda(c)] - [a, \lambda(b)] + \lambda([b, c]) - [a, \kappa(a)] + [b, \kappa(c)] \\ &\quad + [b, \lambda(a)] - [b, \lambda(c)] \\ &\quad + \lambda([c, a]) - [b, \kappa(b)] + [c, \kappa(a)] + [c, \lambda(b)] - [c, \lambda(a)] + \lambda([a, b]) - [c, \kappa(c)]. \end{aligned}$$

For $b = c$ the above equality holds if and only if $[b - a, \kappa(b - a)] = 0$ or, equivalently,

$$[a, \kappa(a)] = 0, \tag{3.3}$$

for all $a \in \mathfrak{g}$. Next, setting $c = 0$ and in view of (3.3), we deduce the necessity of (3.1). Since (3.3) follows from (3.1), by setting $b = a$ in (3.1), the multiplication (3.2) satisfies the affine Jacobi identity.

Writing $[-, -]'$ for the multiplication in $T_o\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$, in terms of the vector space structure of \mathfrak{g} , we have

$$[a, b]' = [a, b] - [a, o] - [o, b] + o,$$

and hence the vector space isomorphism

$$\varphi : T_o\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s) \rightarrow \mathfrak{g}, \quad a \mapsto a - o,$$

is the required isomorphism of Lie algebras.

In the opposite direction, given a Lie algebra \mathfrak{a} , we need to indicate for any element $o \in \mathfrak{a}$ linear functions κ, λ acting on the tangent Lie algebra $T_o\mathfrak{a}$

and an element $s \in \mathfrak{a}$ which meet the condition (3.1). Comparing (3.2) with (2.7), we expect good candidates for these to be the following data:

$$\begin{aligned} s &= \{o, o\}, \\ \lambda &: T_o\mathfrak{a} \rightarrow T_o\mathfrak{a}, \quad a \mapsto \{o, a\} - \{o, o\}, \\ \kappa &: T_o\mathfrak{a} \rightarrow T_o\mathfrak{a}, \quad a \mapsto \{a, a\} - \{o, o\}. \end{aligned}$$

Since $\{o, -\}$ is an affine map, it is clear that λ is a linear transformation. In view of the affine antisymmetry of $\{-, -\}$, the map κ is equivalently given by

$$\kappa(a) = \{o, a\} - \{o, o\} + \{a, o\} - \{o, o\} = \lambda(a) + \{a, o\} - \{o, o\},$$

and hence κ is linear as the sum of linear maps.

In view of the affine antisymmetry the relation between the affine Lie multiplication $\{-, -\}$ and its tangent Lie multiplication $[-, -]$ at o (2.7) is expressed in terms of s , λ and κ by the formula (3.2), and the first part of the proof ensures that the condition (3.1) is fulfilled. \square

Following [16] we denote by $\Delta(\mathfrak{g})$ the set of triples $(\lambda, \lambda', \lambda'')$ of linear endomorphisms of \mathfrak{g} that satisfy the following condition, for all $a, b \in \mathfrak{g}$,

$$[\lambda(a), b] + [a, \lambda'(b)] = \lambda''([a, b]). \quad (3.4)$$

Any map λ for which there exist λ' and λ'' such that $(\lambda, \lambda', \lambda'') \in \Delta(\mathfrak{g})$ is called a *generalized derivation*. By slightly bending this terminology, in what follows we will refer either to λ or the triple $(\lambda, \lambda', \lambda'')$ as to a generalized derivation. Among all generalized derivations one distinguishes several important classes. First and most standard, λ is a *derivation* of \mathfrak{g} if and only if $(\lambda, \lambda, \lambda) \in \Delta(\mathfrak{g})$. A *quasi-centroid* of \mathfrak{g} , denoted by $\text{QC}(\mathfrak{g})$, is a vector space of all $\kappa \in \text{Lin}(\mathfrak{g})$, such that $(\kappa, -\kappa, 0) \in \Delta(\mathfrak{g})$. That is, $\kappa \in \text{QC}(\mathfrak{g})$ if and only if, for all $a, b \in \mathfrak{g}$,

$$[\kappa(a), b] = [a, \kappa(b)]. \quad (3.5)$$

Since we assume that $\text{char}\mathbb{F} \neq 2$, the condition (3.5) is equivalent to (3.3), i.e.

$$\text{QC}(\mathfrak{g}) = \{\kappa \in \text{Lin}(\mathfrak{g}) \mid [\kappa(a), a] = 0, \text{ for all } a \in \mathfrak{g}\}. \quad (3.6)$$

It is easily seen that $\text{QC}(\mathfrak{g})$ is a Lie algebra with the multiplication given by the commutator (defined with respect to the composition of endomorphisms). The structure of $\text{QC}(\mathfrak{g})$ is studied in detail in [16]. For example, if $Z(\mathfrak{g}) = 0$, then $\text{QC}(\mathfrak{g})$ is a commutative associative algebra (with respect to composition); see [16, Theorem 5.12]. Next, the *centroid* of \mathfrak{g} , denoted by $\text{C}(\mathfrak{g})$ is the space of linear endomorphisms κ such that $(0, \kappa, \kappa) \in \Delta(\mathfrak{g})$ or, equivalently by the antisymmetry of the Lie multiplication, $(\kappa, 0, \kappa) \in \Delta(\mathfrak{g})$. That is $\kappa \in \text{C}(\mathfrak{g})$ if and only if, for all $a, b \in \mathfrak{g}$,

$$\kappa([a, b]) = [\kappa(a), b] = [a, \kappa(b)]. \quad (3.7)$$

Obviously $\text{C}(\mathfrak{g}) \subseteq \text{QC}(\mathfrak{g})$ and it is clear that $\text{C}(\mathfrak{g})$ is an associative unital algebra with respect to composition. Centroids of Lie algebras are studied in

[14, Chapter X], where it is shown, for instance, that if \mathfrak{g} is a perfect Lie algebra (that is, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) then $C(\mathfrak{g})$ is a commutative algebra [14, Chapter X, Lemma 1], while if \mathfrak{g} is simple, then $C(\mathfrak{g})$ is a field [14, Chapter X, Theorem 1].

The interplay between the quasicentroid and the centroid of a Lie algebra \mathfrak{g} is studied in [16]. In particular, it is proven there that if $Z(\mathfrak{g}) = 0$, then $QC(\mathfrak{g}) = C(\mathfrak{g}) \oplus A$, where $A^2 = 0$ [16, Theorem 5.12], Furthermore, if in addition $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $QC(\mathfrak{g}) = C(\mathfrak{g})$, [16, Theorem 5.28].

The following very likely well-known lemma (see e.g. references in [15]) is proven by straightforward arguments, hence its proof is omitted.

Lemma 3.2. *Let \mathfrak{g} be a Lie algebra, $\lambda, \delta \in \text{Lin}(\mathfrak{g})$, and set $\kappa := \lambda - \delta$.*

(1) *The following statements are equivalent:*

- (i) $(\delta, \lambda, \lambda) \in \Delta(\mathfrak{g})$;
- (ii) for all $\alpha \in C(\mathfrak{g})$, $(\delta, \lambda + \alpha, \lambda + \alpha) \in \Delta(\mathfrak{g})$;
- (iii) for all derivations α of \mathfrak{g} , $(\delta + \alpha, \lambda + \alpha, \lambda + \alpha) \in \Delta(\mathfrak{g})$;
- (iv) λ and κ satisfy (3.1).

(2) *If $(\delta, \lambda, \lambda) \in \Delta(\mathfrak{g})$, then:*

- (i) for all positive integers n , $\kappa^n \in QC(\mathfrak{g})$;
- (ii) for all $a \in \mathfrak{g}$, the vector space $\mathfrak{g}_\kappa(a)$ spanned by $\{\kappa^n(a) \mid n \in \mathbb{N}\}$ is an abelian Lie subalgebra of \mathfrak{g} ;
- (iii) δ is a derivation if and only if $\kappa \in C(\mathfrak{g})$.

The first assertion of Lemma (3.2) allows one to state Theorem 3.1 in the following (equivalent) form

Corollary 3.3. *There is a one-to-one correspondence between Lie affgebras with Lie algebra fibre \mathfrak{g} and generalized derivations $(\delta, \lambda, \lambda)$ of \mathfrak{g} supplemented with an element of \mathfrak{g} .*

We note in passing that in terms of $(\delta, \lambda, \lambda) \in \Delta(\mathfrak{g})$ and $s \in \mathfrak{g}$ the multiplication (3.2) comes out as

$$\{a, b\} = [a, b] - \delta(a) + \lambda(b) + s. \quad (3.8)$$

Furthermore, the multiplication

$$\{a, b\}' := [a, b] - \delta(b) + \lambda(a) + s.$$

satisfies the alternative version of the affine Jacobi identity mentioned in the last paragraph of Remark 2.5 (i.e. with the nesting of the brackets $\{\{-, -\}'$, $- \}'$). So both versions of the Lie affgebra are obtained with exactly the same sets of data on a Lie algebra, thus confirming the claim made at the end of Remark 2.5 about the conceptual equivalence of both approaches.

Theorem 3.1 allows one to make a foray into the classification of Lie affgebras with a prescribed tangent Lie algebra (we will illustrate this in Sect. 5). To make this strategy feasible however, first one needs to determine homomorphisms between Lie affgebras $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$.

Theorem 3.4. *A function $\varphi : \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s) \rightarrow \mathfrak{a}(\mathfrak{g}'; \kappa', \lambda', s')$ is a homomorphism of Lie affgebras if and only if there exist a Lie algebra homomorphism $\psi : \mathfrak{g} \rightarrow \mathfrak{g}'$ and $q' \in \mathfrak{g}'$, such that*

$$\psi\kappa = \kappa'\psi, \quad (3.9a)$$

$$\psi\lambda = (\text{ad}_{q'} + \lambda')\psi, \quad (3.9b)$$

$$\psi(s) = s' - q' + \kappa'(q'). \quad (3.9c)$$

Proof. A function $\varphi : \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s) \rightarrow \mathfrak{a}(\mathfrak{g}'; \kappa', \lambda', s')$ is a homomorphism of affine spaces if and only if the function

$$\psi : \mathfrak{g} \rightarrow \mathfrak{g}', \quad a \mapsto \varphi(a) - \varphi(0),$$

is a linear transformation. Set $q' = \varphi(0)$.

Written explicitly the Lie affgebra homomorphism condition, for all $a, b \in \mathfrak{g}$,

$$\varphi(\{a, b\}) = \{\varphi(a), \varphi(b)\},$$

boils down to

$$\begin{aligned} \psi([a, b]) + \psi\kappa(a) + \psi\lambda(b - a) + \psi(s) + q' &= [\psi(a), \psi(b)] + [q', \psi(b - a)] \\ &+ \kappa'\psi(a) + \kappa(q') + \lambda'(\psi(b) - \psi(a)) + s'. \end{aligned} \quad (3.10)$$

Setting $a = b = 0$ we obtain (3.9c), and thus (3.10) reduces to

$$\begin{aligned} \psi([a, b]) + \psi\kappa(a) + \psi\lambda(b - a) &= [\psi(a), \psi(b)] \\ &+ [q', \psi(b - a)] + \kappa'\psi(a) + \lambda'(\psi(b) - \psi(a)). \end{aligned} \quad (3.11)$$

Setting $a = b$ in (3.11) we obtain (3.9a) and thus further simplification of (3.10),

$$\begin{aligned} \psi([a, b]) + \psi\lambda(b - a) &= [\psi(a), \psi(b)] + [q', \psi(b - a)] \\ &+ \lambda'(\psi(b) - \psi(a)). \end{aligned} \quad (3.12)$$

Finally, setting $a = 0$ yields (3.9b) as well as the fact that ψ is a homomorphism of Lie algebras.

The converse is straightforward. \square

Corollary 3.5. *Let $\mathfrak{a} = \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ and $\mathfrak{a}' = \mathfrak{a}(\mathfrak{g}'; \kappa', \lambda', s')$. Then \mathfrak{a} is isomorphic to \mathfrak{a}' if and only if there exist a Lie algebra isomorphism $\Psi : \mathfrak{g} \rightarrow \mathfrak{g}'$ and an element $q \in \mathfrak{g}$ such that*

$$\kappa' = \Psi\kappa\Psi^{-1}, \quad (3.13a)$$

$$\lambda' = \Psi(\lambda - \text{ad}_q)\Psi^{-1}, \quad (3.13b)$$

$$s' = \Psi(s + q - \kappa(q)). \quad (3.13c)$$

Proof. This follows immediately from Theorem 3.4 as the affine map $\Phi : \mathfrak{a} \rightarrow \mathfrak{a}'$ is an isomorphism if and only if its linear part $\Psi = \Phi - \Phi(0) : \mathfrak{g} \rightarrow \mathfrak{g}'$ is an isomorphism of vector spaces. Rearranging equations (3.9), setting $q = \Psi^{-1}(q')$,

where $q' = \Phi(0)$ and noting that $\text{ad}_{q'} = \Psi \text{ad}_q \Psi^{-1}$ we obtain conditions (3.13). \square

We observe in passing that in terms of generalized derivations $(\delta, \lambda, \lambda)$ and $(\delta', \lambda', \lambda')$ (where $\delta = \lambda - \kappa$ and $\delta' = \lambda' - \kappa'$) Eqs. (3.9a), (3.13a) take the forms

$$\psi\delta = (\text{ad}_{q'} + \delta')\psi, \quad \delta' = \Psi(\delta - \text{ad}_q)\Psi^{-1},$$

respectively.

As an immediate consequence of Corollary 3.5 we obtain the following criterion of non-isomorphism of Lie affgebras.

Corollary 3.6. *If Lie affgebras \mathfrak{a} and \mathfrak{b} have non-isomorphic Lie algebra fibres, then they are not isomorphic.*

Theorem 3.4 allows one also to identify simple Lie affgebras. Following the universal algebra (or category theory) convention, we say that a (Lie) affgebra \mathfrak{a} is *simple* provided every homomorphism with \mathfrak{a} as the domain is either constant or an injective function.

Corollary 3.7. *If \mathfrak{g} is a simple Lie algebra, then $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ is a simple Lie affgebra.*

Proof. Consider a Lie affgebra homomorphism $\varphi : \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s) \rightarrow \mathfrak{a}'$. By Theorem 3.1, for any $o \in \mathfrak{a}'$, $\mathfrak{a}' = \mathfrak{a}(T_o \mathfrak{a}'; \kappa', \lambda', s')$. In view of Theorem 3.4, $\varphi = \psi + q'$, where $\psi : \mathfrak{g} \rightarrow T_o \mathfrak{a}'$ is a Lie algebra homomorphism satisfying conditions (3.9). Since \mathfrak{g} is a simple Lie algebra ψ is either the zero map, in which case φ is constant, or injective, in which case also φ is injective. \square

Given a vector space V and its subspace U , any coset $v + U \subseteq V$ is an affine subspace of V in a natural way:

$$\begin{aligned} \langle v + u, v + u', v + u'' \rangle &= v + (u - u' + u''), \\ \alpha \triangleright_{v+u}(v + u') &= v + (1 - \alpha)u + \alpha u', \end{aligned}$$

for all $u, u', u'' \in U$. The criterion when a coset of a subspace of a Lie algebra \mathfrak{g} yields a Lie subaffgebra of $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ is given in the following

Proposition 3.8. *Let $\mathfrak{a} = \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ be a Lie affgebra and let \mathfrak{b} be an affine subspace of \mathfrak{a} . Then \mathfrak{b} is a Lie subaffgebra of \mathfrak{a} if and only if there exist $a \in \mathfrak{b}$ and a Lie subalgebra \mathfrak{h} of \mathfrak{a} , such that $\mathfrak{b} = a + \mathfrak{h}$ and*

- (a) $\kappa(a) - a + s \in \mathfrak{h}$;
- (b) $\kappa(\mathfrak{h}) \subseteq \mathfrak{h}$;
- (c) $(\lambda + \text{ad}_a)(\mathfrak{h}) \subseteq \mathfrak{h}$.

Furthermore,

$$\mathfrak{b} \cong \mathfrak{a}(\mathfrak{h}; \kappa, \lambda + \text{ad}_a, \kappa(a) - a + s),$$

as Lie affgebras.

Proof. An affine subspace \mathfrak{b} is necessarily of the form $a + \mathfrak{h}$, where \mathfrak{h} is a vector subspace of \mathfrak{g} . Then $\mathfrak{b} = a + \mathfrak{h}$ is a Lie subaffgebra of \mathfrak{a} if and only if, for all $v, w \in V$,

$$\begin{aligned} \{a + v, a + w\} &= [v, w] + [a, w] + [v, a] + \kappa(a) \\ &\quad + \kappa(v) + \lambda(w - v) + s \in a + \mathfrak{h}. \end{aligned} \quad (3.14)$$

Setting $v = w = 0$ in (3.14) we obtain property (a). In view of (a), setting $v = w$ yields (b), while setting $w = 0$ yields (c). All this then necessarily implies that $[v, w] \in \mathfrak{h}$, hence \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . Conversely, if \mathfrak{h} is a Lie subalgebra of \mathfrak{g} and (a)–(c) hold, $\{a + v, a + w\} \in a + \mathfrak{h}$, so $a + \mathfrak{h}$ is a Lie subaffgebra of \mathfrak{a} .

Properties (a)–(c) imply that, when restricted to \mathfrak{h} , $\kappa, \lambda + \text{ad}_a \in \text{Lin}(\mathfrak{h})$, while $\kappa(a) - a + s \in \mathfrak{h}$. Since λ satisfies (3.1), its restriction to \mathfrak{h} shifted by the inner derivation ad_a also satisfies property (3.1) by Lemma 3.2. Hence $\mathfrak{a}(\mathfrak{h}; \kappa, \lambda + \text{ad}_a, \kappa(a) - a + s)$ is a Lie affgebra. Its isomorphism with $a + \mathfrak{h}$ is given by $v \mapsto a + v$. \square

4. Derivation-Type Lie Affgebras and Lie Algebra Hulls

As explained in [6, Proposition 3.5] there is a bijective correspondence between idempotent Lie affgebra multiplications on an affine space \mathfrak{a} and Lie multiplications valued in the tangent vector space at any given element of \mathfrak{a} that satisfy Lie affgebra axioms as stated in [10]. On the other hand [10, Theorem 10] establishes that the latter can be lifted up to a Lie multiplication on a vector space hull of \mathfrak{a} , that is, a universally defined vector space in which \mathfrak{a} is a hyperplane of codimension one. The resulting Lie algebra is called the *Lie algebra hull* of \mathfrak{a} . In this section we extend Lie algebra hulls to a wider class of Lie affgebras and interpret them as extensions of a tangent Lie algebra by derivation.

Definition 4.1. A Lie affgebra \mathfrak{a} is said to be *derivation-type* if $\mathfrak{a} \cong \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ such that $\delta := \lambda - \kappa$ is a derivation of \mathfrak{g} .

Lemma 3.2 immediately implies that a Lie affgebra $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ is a derivation-type Lie affgebra if and only if $\kappa \in C(\mathfrak{g})$. Thus the study of derivation-type Lie affgebras in large part boils down to the study of centroids of Lie algebras.

Proposition 4.2.

- (1) Any finite dimensional Lie affgebra with a centreless perfect Lie algebra fibre is a derivation-type Lie affgebra.
- (2) If \mathbb{F} is an algebraically closed field and \mathfrak{g} is a simple Lie algebra, then:
 - (i) Any Lie affgebra with \mathfrak{g} as the tangent Lie algebra is necessarily isomorphic to $\mathfrak{a}(\mathfrak{g}; \kappa \text{ id}, \kappa \text{ id}, s)$, for some $s \in \mathfrak{g}$ and $\kappa \in \mathbb{F}$. In particular, the multiplication is

$$\{a, b\} = [a, b] + \kappa b + s, \quad (4.1)$$

for all $a, b \in \mathfrak{g}$.
 (ii) Furthermore,

$$\mathfrak{a}(\mathfrak{g}; \kappa \text{ id}, \kappa \text{ id}, s) \cong \mathfrak{a}(\mathfrak{g}; \kappa' \text{ id}, \kappa' \text{ id}, s'), \tag{4.2}$$

as Lie affgebras if and only if $\kappa' = \kappa$ and $s' = \Psi(s)$, where Ψ is an automorphism of \mathfrak{g} .

Proof. (1) The statement is a straightforward consequence of [16, Theorem 5.28], which states that if $Z(\mathfrak{g}) = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $\text{QC}(\mathfrak{g}) = \text{C}(\mathfrak{g})$.

(2)(i) Since a simple Lie algebra satisfies the hypothesis of the first statement, $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ is a derivation-type Lie affgebra. All derivations of a simple Lie algebra are inner, hence

$$\lambda - \kappa = \text{ad}_q,$$

for some $q \in \mathfrak{g}$. Choosing this q in Corollary 3.5 and using (3.13b), we obtain $\lambda' = \kappa'$, as required. The formula for the multiplication follows immediately from (3.2). Furthermore, since $\kappa \in \text{C}(\mathfrak{g})$, necessarily $\kappa \circ \text{ad}_a = \text{ad}_a \circ \kappa$, for all $a \in \mathfrak{g}$ and by the simplicity of \mathfrak{g} and Schur’s lemma, κ is a scalar multiple of the identity (compare [14, Chapter X, Theorem 1] or [8, Corollary 5.5]).

(2)(ii) Since \mathfrak{g} has the trivial centre, the constant q in Corollary 3.5 is necessary zero, as any other choice would separate λ from κ . Thus isomorphisms (4.2) correspond to Lie algebra automorphisms Ψ of \mathfrak{g} , which in view of Corollary 3.5 do not modify κ while changing s to $s' = \Psi(s)$. \square

The automorphisms of simple complex Lie algebras are well-known; see e.g. [14, Chapter IX]. In particular, in the case of series A_n , i.e., Lie algebras of traceless matrices $\mathfrak{sl}(n+1, \mathbb{C})$, the isomorphisms (4.2) take place if and only if there is a special linear matrix $A \in \text{SL}(n, \mathbb{C})$ such that $s' = A^{-1}sA$ in all cases or $s' = -A^{-1}s^T A$ if $n \geq 2$; see [14, Chapter IX, Theorem 5]. In the case of the Lie algebra of odd-dimensional antisymmetric matrices (the B_n series) of dimension at least 5 (i.e., $n \geq 2$) and even-dimensional antisymmetric matrices (the D_n series) of dimension at least 10 (i.e., $n \geq 5$), the isomorphisms (4.2) hold if and only if there is an orthogonal matrix O such that $s' = O^{-1}sO$; see [14, Chapter IX, Theorem 6]. Finally, in the case of the Lie algebras from the series C_n with $n \geq 3$, the correspondence (4.2) is given by symplectic matrices M , $s' = M^{-1}sM$; see again [14, Chapter IX, Theorem 6]. In Sect. 5.4 we give an explicit description of isomorphism classes of Lie affgebras with the Lie algebra fibres $\mathfrak{sl}(2, \mathbb{C})$.

Recall that if δ is a derivation of a Lie algebra \mathfrak{g} , then one can define the semi-direct product Lie algebra $\mathfrak{g}(\delta) := \{x + \alpha\delta \mid x \in \mathfrak{g}, \alpha \in \mathbb{F}\} \cong \mathfrak{g} \oplus \mathbb{F}$ with the multiplication

$$[x + \alpha\delta, y + \beta\delta] = [x, y] + \alpha\delta(y) - \beta\delta(x), \tag{4.3}$$

for all $x, y \in \mathfrak{g}$ and scalars $\alpha, \beta \in \mathbb{F}$. It is clear that \mathfrak{g} is an ideal in the Lie algebra $\mathfrak{g}(\delta)$.

Proposition 4.3. *Let \mathfrak{a} be a derivation-type Lie affgebra isomorphic to $\mathfrak{a}(\mathfrak{g}; \kappa \text{id}, \lambda, s)$, for some scalar $\kappa \in \mathbb{F}$. Set $\delta = \lambda - \kappa \text{id}$. Then $\mathfrak{a} \cong \mathfrak{g} + \delta$, where $\mathfrak{g} + \delta$ is the Lie subaffgebra of $\mathfrak{a}(\mathfrak{g}(\delta); \kappa \text{id}, \kappa \text{id}, s + (1 - \kappa)\delta)$.*

Proof. The multiplication in $\mathfrak{a}(\mathfrak{g}(\delta); \kappa \text{id}, \kappa \text{id}, s + (1 - \kappa)\delta)$ comes out as

$$\{a + \alpha\delta, b + \beta\delta\} = [a, b] + \alpha\delta(b) - \beta\delta(a) + \kappa b + s + (\kappa(\beta - 1) + 1)\delta.$$

Since the multiplication in $\mathfrak{a}(\mathfrak{g}; \kappa \text{id}, \lambda, s)$ is

$$\{a, b\} = [a, b] + \kappa b + \delta(b - a) + s,$$

we find that

$$\{a + \alpha\delta, b + \beta\delta\} = \{a, b\} + (\alpha - 1)\delta(b) + (1 - \beta)\delta(a) + (\kappa(\beta - 1) + 1)\delta.$$

In particular $\{a + \delta, b + \delta\} = \{a, b\} + \delta$ and the required isomorphism of Lie affgebras is $a \mapsto a + \delta$. □

Remark 4.4. In view of Proposition 4.2, Proposition 4.3 applies to all Lie affgebras with simple Lie algebra fibres. Proposition 4.2 implies further that, over an algebraically closed field, up to isomorphism, $\lambda = \kappa$ and hence $\delta = 0$. Thus, over an algebraically closed field, if \mathfrak{a} is fibred by a simple Lie algebra \mathfrak{g} , then the Lie hull of \mathfrak{a} is isomorphic to the trivial extension of \mathfrak{g} by δ so that

$$[a, \delta] = 0,$$

for all $a \in \mathfrak{g}$.

Proposition 4.5. *Let \mathfrak{a} be a Lie affgebra fibred by a Lie algebra \mathfrak{g} which does not contain a non-trivial abelian subalgebra. Then \mathfrak{a} is a derivation-type Lie affgebra satisfying the hypothesis of Proposition 4.3.*

Proof. Let $\mathfrak{a} \cong \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$. Take any basis $\{a_i\}$ of \mathfrak{a} and consider vector spaces $\mathfrak{g}_\kappa(a_i)$ spanned by $\{\kappa^n(a_i) \mid n \in \mathbb{N}\}$. In view of Lemma 3.2, each $\mathfrak{g}_\kappa(a_i)$ is an abelian subalgebra of \mathfrak{g} . Therefore, by hypothesis, $\mathfrak{g}_\kappa(a_i)$ are one-dimensional spaces, and there exist $\kappa_i \in \mathbb{F}$, such that $\kappa(a_i) = \kappa_i a_i$. Next, since $\kappa \in \text{QC}(\mathfrak{g})$, $\kappa_i[a_i, a_j] = \kappa_j[a_i, a_j]$. If $i \neq j$, then again by the hypothesis $[a_i, a_j] \neq 0$, and so $\kappa_i = \kappa_j$, for all i, j . This proves that κ is necessarily a multiple of the identity, and the statement follows. □

Corollary 4.6. *Let \mathfrak{a} be a Lie affgebra. The following statements are equivalent:*

- (1) *The multiplication $\{-, -\}$ is an idempotent operation, i.e., $\{a, a\} = a$, for all $a \in \mathfrak{a}$.*
- (2) *There exist a Lie algebra \mathfrak{g} and a derivation δ of \mathfrak{g} such that $\mathfrak{a} \cong \mathfrak{a}(\mathfrak{g}; \text{id}, \delta + \text{id}, 0)$.*
- (3) *For any $o \in \mathfrak{a}$, there is a derivation δ of the tangent Lie algebra $T_o\mathfrak{a}$ such that $\mathfrak{a} \cong T_o\mathfrak{a} + \delta$, where $T_o\mathfrak{a} + \delta$ is a Lie subaffgebra of a Lie affgebra $\mathfrak{a}(\delta) \cong \mathfrak{a} \times \mathbb{F}$ with the multiplication*

$$\{x, y\} = [x, y] + y,$$

for all $x, y \in (T_o\mathfrak{a})(\delta)$.

Proof. By Theorem 3.1, $\mathfrak{a} \cong \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$. In view of (3.2), $\{a, a\} = a$, for all $a \in \mathfrak{a}$ if and only if

$$a = \kappa(a) + s.$$

Since κ is a linear endomorphism, $s = 0$ and $\kappa(a) = a$, and hence \mathfrak{a} is a derivation-type Lie affgebra with derivation $\delta = \lambda - \text{id}$. This establishes the equivalence of (1) and (2).

Assuming (2), by Theorem 3.1, given any $o \in \mathfrak{a}$ we can choose $\mathfrak{g} = T_o\mathfrak{a}$. Then $\mathfrak{a}(\delta) = \mathfrak{a}(\mathfrak{g}; \text{id}, \text{id}, 0)$ and (3) follows by Proposition 4.3. The implication (3) \implies (1) is immediate. \square

Example 4.7. The multiplication (2.11) of the Lie affgebra $\mathfrak{a}_{\zeta, \text{id}}$ described in Example 2.7 is an idempotent operation by (2.3), and hence Corollary 4.6 can be applied. The map λ comes out as $\lambda(a) = \zeta a$, so that $\delta(a) = (\zeta - 1)a$ and the multiplication of the hull of $\mathfrak{a}_{\zeta, \text{id}}$ reads

$$[a + \alpha\delta, b + \beta\delta] = (\zeta - 1)(ab - \beta a),$$

for all $a, b \in \mathfrak{a}$ and $\alpha, \beta \in \mathbb{F}$.

5. Examples

5.1. One-Dimensional Lie Affgebras

All one-dimensional Lie affgebras must have the unique one-dimensional abelian Lie algebra as their Lie algebra fibres. Let e be a fixed basis of \mathfrak{g} . There are no restrictions on λ and κ , which are both scalar multiples of identity, say $\lambda(e) = \lambda e$ and $\kappa(e) = \kappa e$, $\kappa, \lambda \in \mathbb{F}$, and are not affected by an automorphism Ψ of \mathfrak{g} , which itself is a non-zero scalar multiple of identity, say $\Psi(e) = \psi e$. The adjoint action is trivial. Let $s = \sigma e$ and $q = \xi e$. Then s can be transferred to

$$s' = \psi(\sigma + (1 - \kappa)\xi)e.$$

If $\kappa \neq 1$, ξ can be chosen so that $s' = 0$. If $\kappa = 1$ then either $\sigma = 0$ or it can be normalized to 1 by choosing ψ . Therefore, there are two families of non-isomorphic Lie affgebra structures on a one dimensional (affine) space with corresponding multiplications

$$\{a, b\} = \begin{cases} (\kappa - \lambda)a + \lambda b & \text{or} \\ (1 - \lambda)a + \lambda b + e. \end{cases}$$

All these are derivation-type Lie affgebras which fulfil the hypothesis of Proposition 4.3. Since the derivations are $\delta(e) = (\lambda - \kappa)e$, the Lie hulls are spanned by e and δ with the multiplication

$$[e, \delta] = (\kappa - \lambda)e,$$

and thus exhaust all non-isomorphic classes of two-dimensional Lie algebras: either $[e, \delta] = 0$ or $[e, \delta] = e$.

5.2. Lie Affgebras with Tangent Abelian Lie Algebras

This is more of a non-example rather than an example, but we include this discussion here as an illustration of the richness of the variety of Lie affgebras fibred by the same Lie algebra. We take \mathfrak{g} to be an n -dimensional vector space with the trivial multiplication $[a, b] = 0$. The conditions (3.1) and (3.7) are empty, so any linear endomorphisms κ and λ and a vector $s \in \mathfrak{g}$ give a derivation-type Lie affgebra $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$, with the multiplication

$$\{a, b\} = \kappa(a) + \lambda(b - a) + s, \quad (5.1)$$

for all $a, b \in \mathfrak{g}$.

Any linear automorphism Ψ of \mathfrak{g} is a Lie algebra homomorphism, the adjoint action is trivial and hence the equivalence classes of Lie affgebras are controlled by the general similarity relation between κ and κ' or λ and λ' , and depend on the multiplicity of eigenvalue 1 of κ or, equivalently, the rank of $\text{id} - \kappa$. If this rank is equal to n , then q can be chosen so that $s' = 0$ and thus the constant term from (5.1) can be removed. If the rank is equal to r , then the constant term will depend on $n - r$ free parameters.

5.3. Two Dimensional Lie Affgebras with Non-abelian Lie Algebra Fibres

Let \mathfrak{b} be the Borel subalgebra of $\mathfrak{sl}(2, \mathbb{C})$, that is, \mathfrak{b} is the 2-dimensional Lie algebra with a basis $E : e_1, e_2$ and the multiplication $[e_1, e_2] = e_1$. By Proposition 4.5 κ is necessarily a scalar multiple of identity, so that $\kappa \in \mathbb{C}(\mathfrak{b})$. Therefore, $\delta = \lambda - \kappa$ is a derivation by Lemma 3.2. One easily checks that all derivations of \mathfrak{b} are inner, and hence $\lambda = \kappa + \text{ad}_q$, where $q \in \mathfrak{b}$. In view of this and Eq. (3.13b) in Corollary 3.5, any Lie affgebra over \mathfrak{b} is isomorphic to one with $\lambda = \kappa = \gamma \text{id}$, where $\gamma \in \mathbb{C}$.

Since κ and λ are scalar multiples of identity, they are preserved by any automorphism Ψ of \mathfrak{b} . One easily finds that, in the basis E ,

$$\Psi(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

where $a, b \in \mathbb{C}$, $a \neq 0$. With no loss of generality we can choose $q = 0$ in (3.13c), so that $s' = \Psi(s)$, or in the basis E ,

$$\begin{pmatrix} s'_1 \\ s'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} as_1 + bs_2 \\ s_2 \end{pmatrix}.$$

If $s_2 \neq 0$, then b can be chosen so that $s'_1 = 0$. Otherwise, either $s_1 = 0$ and hence $s' = 0$ or a can be chosen in such a way that $s'_1 = 1$. Put together we obtain the following two families of non-isomorphic Lie affgebras

$$\begin{aligned} \mathfrak{a}(\mathfrak{b}; \gamma \text{id}, \gamma \text{id}, \sigma e_2) &: \quad \{x, y\} = [x, y] + \gamma y + \sigma e_2, \\ \mathfrak{a}(\mathfrak{b}; \gamma \text{id}, \gamma \text{id}, e_1) &: \quad \{x, y\} = [x, y] + \gamma y + e_1, \end{aligned}$$

where $\gamma, \sigma \in \mathbb{C}$. As up to isomorphism \mathfrak{b} is the only non-abelian two-dimensional Lie algebra, this classifies (up to isomorphism) all two-dimensional Lie affgebras with a non-abelian tangent Lie algebra.

By Proposition 4.5, all these affgebras are derivation-type affgebras that fulfill hypothesis of Proposition 4.3, and since $\lambda = \kappa$ the derivation $\lambda - \kappa$ is trivial. Consequently, the Lie algebra hull is spanned by e_1, e_2, δ , with the multiplication

$$[e_1, e_2] = e_1, \quad [\delta, e_1] = 0, \quad [\delta, e_2] = 0.$$

5.4. Lie Affgebras Fibred by $\mathfrak{sl}(2, \mathbb{C})$

Let us write the $\mathfrak{sl}(2, \mathbb{C})$ algebra in the Chevalley basis $C : e, h, f$,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{5.2}$$

Since $\mathfrak{sl}(2, \mathbb{C})$ is a simple Lie algebra over an algebraically closed field, by Proposition 4.2 every Lie affgebra with $\mathfrak{sl}(2, \mathbb{C})$ as a Lie algebra fibre is isomorphic to one with the multiplication

$$\{x, y\} = [x, y] + \gamma y + s, \quad \gamma \in \mathbb{C}, s \in \mathfrak{sl}(2, \mathbb{C}).$$

Equivalence classes corresponding to different choices of s are controlled by the automorphisms of $\mathfrak{sl}(2, \mathbb{C})$ through (3.13c). Any such automorphism is given by the conjugation with an element of the Lie group $SL(2, \mathbb{C})$, and thus,

$$\begin{pmatrix} s'_2 & s'_1 \\ s'_3 & -s'_2 \end{pmatrix} \sim \begin{pmatrix} s_2 & s_1 \\ s_3 & -s_2 \end{pmatrix},$$

provided there exists $u \in SL(2, \mathbb{C})$ such that

$$\begin{pmatrix} s'_2 & s'_1 \\ s'_3 & -s'_2 \end{pmatrix} = u^{-1} \begin{pmatrix} s_2 & s_1 \\ s_3 & -s_2 \end{pmatrix} u.$$

First consider,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} s_2 & s_1 \\ s_3 & -s_2 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s_2 - as_3 & 2as_2 + s_1 - a^2s_3 \\ s_3 & -s_2 + as_3 \end{pmatrix}.$$

If $s_3 \neq 0$ diagonal terms can be eliminated by a suitable choice of a . If $s_3 = 0$ we can obtain a diagonal matrix (as long as $s_2 \neq 0$) or a strictly upper triangular one. Put together we obtain that

$$s' = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} = \sigma h \quad \text{or} \quad s' = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix}.$$

The second case can be reduced further by applying

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 0 & b^{-2}\sigma_1 \\ b^2\sigma_2 & 0 \end{pmatrix}.$$

Depending on the nullity of the σ_i in addition to the zero matrix we obtain three possibilities

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f, \quad \begin{pmatrix} 0 & 1 \\ \sigma^2 & 0 \end{pmatrix}, \quad \sigma \neq 0.$$

The last matrix belongs to the diagonal class, since

$$\begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2\sigma} \\ \sigma & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ \sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2\sigma} \\ \sigma & \frac{1}{2} \end{pmatrix}.$$

Therefore isomorphism classes of Lie affgebras with the tangent Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ split into three cases

$$\begin{aligned} \mathfrak{a}(\mathfrak{sl}(2, \mathbb{C}); \gamma \text{ id}, \gamma \text{ id}, e) &: \{x, y\} = [x, y] + \gamma y + e, \\ \mathfrak{a}(\mathfrak{sl}(2, \mathbb{C}); \gamma \text{ id}, \gamma \text{ id}, \sigma h) &: \{x, y\} = [x, y] + \gamma y + \sigma h, \\ \mathfrak{a}(\mathfrak{sl}(2, \mathbb{C}); \gamma \text{ id}, \gamma \text{ id}, f) &: \{x, y\} = [x, y] + \gamma y + f, \end{aligned}$$

where $\gamma, \sigma \in \mathbb{C}$.

5.5. Lie Affgebras Fibred by $\mathfrak{so}(3)$

Let x_1, x_2, x_3 be the standard basis of the real Lie algebra $\mathfrak{so}(3)$, so that,

$$[x_i, x_j] = \varepsilon_{ijk} x_k,$$

where ε_{ijk} is the antisymmetric Levi-Civita symbol. Since $\mathfrak{so}(3)$ has no non-trivial abelian subalgebras, κ is a multiple of identity by Proposition 4.5. Since $\mathfrak{so}(3)$ is a simple Lie algebra, all derivations are inner, which allows one to choose $\lambda = \kappa = \gamma \text{id}$, where $\gamma \in \mathbb{R}$.

The automorphisms of $\mathfrak{so}(3)$ are given by the conjugation by elements of the rotation group $SO(3)$. By suitably rotating the axes, one can transform a general element of $\mathfrak{so}(3)$ to σx_1 , with the uniquely determined $\sigma \in \mathbb{R}$. Thus any real Lie affgebra fibred by $\mathfrak{so}(3)$ is isomorphic to one with the multiplication, for all $x, y \in \mathfrak{so}(3)$,

$$\{x, y\} = [x, y] + \gamma y + \sigma x_1,$$

where $\gamma, \sigma \in \mathbb{R}$.

5.6. Normalized Affine Matrices

The Lie affgebras of general and special normalized affine matrices were introduced in [5] as affine subspaces of $\mathfrak{gl}(n+1, \mathbb{F})$ and $\mathfrak{sl}(n+1, \mathbb{F})$, respectively given by

$$\begin{aligned} \mathfrak{gna}(n, \mathbb{F}) &:= \{X \in \mathfrak{gl}(n+1, \mathbb{F}) \mid XE = EX = E\}, \\ \mathfrak{sna}(n, \mathbb{F}) &:= \{X \in \mathfrak{sl}(n+1, \mathbb{F}) \mid XE = EX = E\}, \end{aligned}$$

where E is the matrix in which all entries are equal to 1, that is (the sum of all $(n+1) \times (n+1)$ matrix units E_{ij}). In other words $\mathfrak{gna}(n, \mathbb{F})$ and $\mathfrak{sna}(n, \mathbb{F})$ contain matrices normalized so that the sum of all entries in each row and column is equal to 1. It has been shown in [7] that, in the terminology of the present text,

$$\mathfrak{gna}(n, \mathbb{F}) \cong \mathfrak{a}(\mathfrak{gl}_0(n, \mathbb{F}); \text{id}, \text{id}, 0), \quad \mathfrak{sna}(n, \mathbb{F}) \cong \mathfrak{a}(\mathfrak{sl}_0(n, \mathbb{F}); \text{id}, \text{id}, 0),$$

where the Lie algebras are defined as subalgebras of $\mathfrak{gl}(n+1, \mathbb{F})$ and $\mathfrak{sl}(n+1, \mathbb{F})$, respectively, by

$$\begin{aligned} \mathfrak{gl}_0(n, \mathbb{F}) &:= \{X \in \mathfrak{gl}(n+1, \mathbb{F}) \mid XE = EX = 0\}, \\ \mathfrak{sl}_0(n, \mathbb{F}) &:= \{X \in \mathfrak{sl}(n+1, \mathbb{F}) \mid XE = EX = 0\}. \end{aligned}$$

Furthermore, if the characteristic of \mathbb{F} does not divide n or $n+1$, $\mathfrak{gna}(n, \mathbb{F})$ and $\mathfrak{sna}(n, \mathbb{F})$ fit into Proposition 3.8 with $\mathfrak{h} \cong \mathfrak{gl}_0(n, \mathbb{F}) \cong \mathfrak{gl}(n, \mathbb{F})$ and $\mathfrak{h} \cong \mathfrak{sl}_0(n, \mathbb{F}) \cong \mathfrak{sl}(n, \mathbb{F})$, respectively. Thus

$$\mathfrak{gna}(n, \mathbb{F}) \cong \mathfrak{a}(\mathfrak{gl}(n, \mathbb{F}); \text{id}, \text{id}, 0), \quad \mathfrak{sna}(n, \mathbb{F}) \cong \mathfrak{a}(\mathfrak{sl}(n, \mathbb{F}); \text{id}, \text{id}, 0).$$

The proof of these isomorphisms given in [7, Theorem 3.2] is based on constructing explicit similarity transformations whose existence relies on the fact that $n+1$ is not a multiple of the characteristic of \mathbb{F} . Here we show that indeed this assumption is necessary.

Assume that $\text{char}\mathbb{F} \mid n+1$. Obviously, $\mathfrak{gna}(n, \mathbb{F}) = I_{n+1} + \mathfrak{gl}_0(n, \mathbb{F})$, where I_{n+1} is the identity matrix, and the conditions of Proposition 3.8 are satisfied with $\mathfrak{h} \cong \mathfrak{gl}_0(n, \mathbb{F})$. Since $\text{Tr}I_{n+1} = 0$, $I_{n+1} \in \mathfrak{sna}(n, \mathbb{F})$, and so $\mathfrak{sna}(n, \mathbb{F}) = I_{n+1} + \mathfrak{sl}_0(n, \mathbb{F})$. Furthermore, $E \in \mathfrak{sl}_0(n, \mathbb{F})$ is a central element. On the other hand $\mathfrak{sl}(n, \mathbb{F})$ has a trivial centre, and so $\mathfrak{sl}_0(n, \mathbb{F}) \not\cong \mathfrak{sl}(n, \mathbb{F})$ as Lie algebras. By Corollary 3.6,

$$\mathfrak{sna}(n, \mathbb{F}) \not\cong \mathfrak{a}(\mathfrak{sl}(n, \mathbb{F}); \text{id}, \text{id}, 0). \tag{5.3}$$

Next, consider the matrices

$$A_{ij} = E_{ij} - E_{in+1} - E_{n+1j} + E_{n+1n+1},$$

$i, j = 1, \dots, n$, which form a basis for $\mathfrak{gl}_0(n, \mathbb{F})$. Since the characteristic of \mathbb{F} is a factor of $n+1$, $E = \sum_{i,j} A_{ij}$, and

$$\sum_{i \neq j} [A_{ii}, A_{ij}] = \sum_{i \neq j} (2A_{ij} - A_{ii}) = 2 \sum_{i,j} A_{ij} = 2E.$$

Thus, if $\text{char}\mathbb{F} \neq 2$, $E \in [\mathfrak{gl}_0(n, \mathbb{F}), \mathfrak{gl}_0(n, \mathbb{F})]$. In the case of the even characteristic one finds that

$$[\mathfrak{gl}_0(n, \mathbb{F}), \mathfrak{gl}_0(n, \mathbb{F})] \ni \sum_{i < j} [A_{ii}, A_{jj}] = \sum_{i < j} (A_{ij} + A_{ji}) = E - I_{n+1}.$$

Therefore, irrespective of the parity of $\text{char}\mathbb{F}$, $[\mathfrak{gl}_0(n, \mathbb{F}), \mathfrak{gl}_0(n, \mathbb{F})]$ has non-trivial intersection with the centre of $\mathfrak{gl}_0(n, \mathbb{F})$. Since there are no non-trivial central elements of $\mathfrak{gl}(n, \mathbb{F})$ that are also in $[\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F})]$, we conclude that

$$\mathfrak{gna}(n, \mathbb{F}) \not\cong \mathfrak{a}(\mathfrak{gl}(n, \mathbb{F}); \text{id}, \text{id}, 0). \tag{5.4}$$

We now look at $\mathfrak{sna}(n, \mathbb{F})$ in case $\text{char}\mathbb{F} \mid n$. Let

$$A = E_{12} + E_{23} + \dots + E_{nn+1} + E_{n+11} \in \mathfrak{sna}(n, \mathbb{F}).$$

Since $AE - EA = 0$, for all $X \in \mathfrak{sl}_0(n, \mathbb{F})$, $AX - XA \in \mathfrak{sl}_0(n, \mathbb{F})$, and hence

$$\mathfrak{sna}(n, \mathbb{F}) = A + \mathfrak{sl}_0(n, \mathbb{F}) \cong \mathfrak{a}(\mathfrak{sl}_0(n, \mathbb{F}); \text{id}, \text{id} + \text{ad}_A, 0),$$

by Proposition 3.8. Following [7] we can introduce the $(n + 1) \times (n + 1)$ -matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5.5}$$

Since $\text{char}(\mathbb{F}) \nmid n + 1$, P is invertible with the inverse

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

As shown in [7], for all $X \in \mathfrak{sl}_0(n, \mathbb{F})$,

$$P^{-1}XP \in \begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$B := P^{-1}AP = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & -1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Using the similarity transformation given by P we thus obtain,

$$\mathfrak{sn}\mathfrak{a}(n, \mathbb{F}) \cong B + \begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix} \cong \mathfrak{a} \left(\begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix}; \text{id}, \text{id} + \text{ad}_B, 0 \right).$$

We will now show that

$$\mathfrak{sn}\mathfrak{a}(n, \mathbb{F}) \not\cong \mathfrak{a}(\mathfrak{sl}(n, \mathbb{F}); \text{id}, \text{id}, 0) \cong \mathfrak{a} \left(\begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix}; \text{id}, \text{id}, 0 \right).$$

In view of Corollary 3.5 the affgebras are isomorphic if and only if there exists $Q \in \begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix}$ such that, for all $X \in \begin{pmatrix} \mathfrak{sl}(n, \mathbb{F}) & 0 \\ 0 & 0 \end{pmatrix}$, $[B + Q, X] = 0$. Since $Z(\mathfrak{sl}(n, \mathbb{F})) = \mathbb{F}I_n$, and B has 1 as the $(n + 1, n + 1)$ -entry, there is no Q such that

$$B + Q \in \begin{pmatrix} Z(\mathfrak{sl}(n, \mathbb{F})) & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently,

$$\mathfrak{sn}\mathfrak{a}(n, \mathbb{F}) \not\cong \mathfrak{a}(\mathfrak{sl}(n, \mathbb{F}); \text{id}, \text{id}, 0).$$

6. An Application: Central Extensions of Lie Affgebras

Let \mathfrak{g} be a Lie algebra. Recall that an antisymmetric bilinear form $\pi : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ is a 2-cocycle (in the Chevalley-Eilenberg complex [9]) provided

$$\pi(a, [b, c]) + \pi(b, [c, a]) + \pi(c, [a, b]) = 0, \tag{6.1}$$

for all $a, b, c \in \mathfrak{g}$. Given a 2-cocycle π on \mathfrak{g} one can define the central extension of \mathfrak{g} by π as a vector space $\mathfrak{g} \oplus \mathbb{F}\langle z \rangle$ with the Lie multiplication, for all $a, b \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$,

$$[a + \alpha z, b + \beta z] = [a, b] + \pi(a, b)z;$$

see [9]. The central extension by a 2-cocycle π is denoted by $\mathfrak{g}(\pi)$. In this section we develop central extensions of Lie affgebras, by exploring the structure of $\mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$.

Lemma 6.1. *Let \mathfrak{a} be a Lie affgebra, and let $\omega : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{F}$ be a bi-affine map such that, for all $a, b, c \in \mathfrak{a}$*

$$\omega(a, b) - \omega(a, a) + \omega(b, a) = \omega(b, b), \tag{6.2a}$$

$$\omega(a, \{b, c\}) - \omega(a, \{a, a\}) + \omega(b, \{c, a\}) - \omega(b, \{b, b\}) + \omega(c, \{a, b\}) = \omega(c, \{c, c\}). \tag{6.2b}$$

Then the product affine space $\mathfrak{a} \times \mathbb{F}$ is a Lie affgebra with the multiplication

$$\{(a, \alpha), (b, \beta)\} = (\{a, b\}, \omega(a, b)), \tag{6.3}$$

for all $a, b \in \mathfrak{a}$ and $\alpha, \beta \in \mathbb{F}$. We denote this Lie affgebra by $\mathfrak{a}(\omega)$ and call it a cocycle extension of \mathfrak{a} .

Proof. The statement follows immediately from the observation that equations (6.2) express the affine antisymmetry and the Jacobi identity of the second factor in $\mathfrak{a} \times \mathbb{F}$. □

Proposition 6.2. *Let $\mathfrak{a} = \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$ and $\delta = \lambda - \kappa$. Then $\omega : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{F}$ satisfies conditions (6.2) if and only if there exist a 2-cocycle π on \mathfrak{g} , linear forms ρ, σ on \mathfrak{g} , and $\tau \in \mathbb{F}$ such that*

$$\omega(a, b) = \pi(a, b) + \rho(a) + \sigma(b) + \tau \tag{6.4}$$

and

$$\sigma([a, b]) = \pi(a, \delta(b)) + \pi(\lambda(a), b). \tag{6.5}$$

Furthermore,

$$\mathfrak{a}(\omega) \cong \mathfrak{a}(\mathfrak{g}(\pi); \hat{\kappa}, \hat{\lambda}, \hat{s}), \tag{6.6}$$

where

$$\begin{aligned} \hat{\kappa}(a + \alpha z) &= \kappa(a) + (\rho(a) + \sigma(a))z, & \hat{\lambda}(a + \alpha z) &= \lambda(a) + \sigma(a)z, \\ \hat{s} &= s + \tau z. \end{aligned} \tag{6.7}$$

Proof. The proof is analogous to the proof of Theorem 3.1. The maps π, ρ, σ and the element w are uniquely determined by setting

$$\rho(a) = \omega(a, 0) - \omega(0, 0), \quad \sigma(a) = \omega(0, a) - \omega(0, 0), \quad \tau = \omega(0, 0),$$

and

$$\pi(a, b) = \omega(a, b) - \omega(a, 0) - \omega(0, b) + \omega(0, 0).$$

The map π is skew-symmetric if and only if ω satisfies condition (6.2a).

Next,

$$\begin{aligned} \omega(a, \{b, c\}) - \omega(a, \{a, a\}) &= \pi(a, [b, c]) - \pi(a, \delta(b)) + \pi(a, \lambda(c)) + \sigma([b, c]) \\ &\quad - \pi(a, \lambda(a)) + \pi(a, \delta(a)) - \sigma\lambda(a) + \sigma\lambda(c) \\ &\quad + \sigma\delta(a) - \sigma\delta(b). \end{aligned}$$

Hence (6.2b) is satisfied if and only if

$$\begin{aligned} 0 &= \pi(a, [b, c]) - \pi(a, \delta(b)) + \pi(a, \lambda(c)) + \sigma([b, c]) - \pi(a, \lambda(a)) + \pi(a, \delta(a)) \\ &\quad + \pi(b, [c, a]) - \pi(b, \delta(c)) + \pi(b, \lambda(a)) + \sigma([c, a]) - \pi(b, \lambda(b)) + \pi(b, \delta(b)) \\ &\quad + \pi(c, [a, b]) - \pi(c, \delta(a)) + \pi(c, \lambda(b)) + \sigma([a, b]) - \pi(c, \lambda(c)) + \pi(c, \delta(c)). \end{aligned}$$

By step-by-step elimination, setting first $c = 0$ and then also $b = 0$, we arrive at the necessity of the 2-cocycle condition for π and (6.5). The sufficiency is clear.

The isomorphism (6.6) and the structure data (6.7) are obtained by comparing the expression for the multiplication (6.3) with the form of ω in (6.4) taken into account and the general formula for the multiplication in a Lie affgebra with the Lie algebra fibre $\mathfrak{g}(\pi)$ in Theorem 3.1. \square

Remark 6.3. When evaluated at $b = a$, the condition (6.5) implies that $\pi(a, \kappa(a)) = 0$, for all $a \in \mathfrak{g}$. This puts constraints on the possibility of extending \mathfrak{a} with a given cocycle π on \mathfrak{g} . As an illustration of this consider an extension of the two-dimensional Lie algebra \mathfrak{g} generated by q and p by the cocycle $\pi(q, p) = 1$. The resulting Lie algebra $\mathfrak{g}(\pi)$ is the Heisenberg algebra, generated by q, p, z with multiplications

$$[q, p] = z, \quad [q, z] = [p, z] = 0.$$

As discussed in Example 5.2, any κ, λ, s give rise to a Lie affgebra $\mathfrak{a} = \mathfrak{a}(\mathfrak{g}; \kappa, \lambda, s)$. On the other hand, only maps δ and λ , and in consequence κ that are diagonal in the basis $\{q, p\}$ satisfy (6.5).

Example 6.4. The isomorphism classes of cocycle extensions of a Lie algebra \mathfrak{g} are classified by the second cohomology group of \mathfrak{g} , [9, Theorem 26.2]. In case \mathfrak{g} is a finite-dimensional simple Lie algebra, this group is trivial by [9, Theorem 21.1], and hence – up to isomorphism – we can choose $\pi = 0$. Since σ in Proposition 6.2 must satisfy (6.5), $\sigma([a, b]) = 0$, for all $a, b \in \mathfrak{g}$. However, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and so $\sigma = 0$. Thus up to coboundaries in the cohomology of \mathfrak{g} ,

$$\omega(a, b) = \rho(a) + \tau.$$

Observing that, $\hat{\kappa}(z) = 0$, one can eliminate τ from the expression for \hat{s} in (6.7) by setting $q = -\tau z$ in (3.13c). This will not change $\hat{\lambda}$ by the centrality of z in $\mathfrak{g}(\pi)$. We conclude that all extensions of Lie affgebras \mathfrak{a} with simple finite dimensional Lie algebra fibres \mathfrak{g} are isomorphic to $\mathfrak{a}(\mathfrak{g}(0); \hat{\kappa}, \hat{\lambda}, \hat{s})$, where

$$\hat{\kappa}(a + \alpha z) = \kappa a + \rho(a)z, \quad \hat{\lambda}(a + \alpha z) = \kappa a, \quad \hat{s} = s, \tag{6.8}$$

for some $\kappa \in \mathbb{F}$, $s \in \mathfrak{g}$ and a linear functional ρ on \mathfrak{g} .

Example 6.5. The Witt Lie algebra \mathfrak{W}_1 of holomorphic vector fields on the Riemann sphere is spanned by the set $\{L_m \mid m \in \mathbb{Z}\}$ with multiplication

$$[L_m, L_n] = (m - n)L_{m+n}. \tag{6.9}$$

Setting $\kappa(L_m) = \sum_n \kappa_m^n L_n$ and exploring (3.3) and (6.9) one finds that the operator κ is diagonal. Next the property (3.5) yields that κ is necessarily the scalar multiple of identity. We denote this scalar by κ as well. Thus any Lie affgebra with \mathfrak{W}_1 Lie algebra fibres is of derivation type. As the first Chevalley-Eilenberg cohomology of \mathfrak{W}_1 with values in \mathfrak{W}_1 is trivial (see e.g. [22]), all derivations are inner and so, up to isomorphism we can choose $\lambda = \kappa$. The automorphism group of \mathfrak{W}_1 consists of two families of maps Ψ_+^t and Ψ_-^t , indexed by non-zero complex numbers $t \in \mathbb{C}^\times$ and given by

$$\Psi_\pm^t(L_m) = \pm t^{\pm m} L_{\pm m};$$

see e.g. [3]. In summary, up to isomorphism all Lie affgebras with \mathfrak{W}_1 Lie algebra fibres have the multiplications

$$\{L_m, L_n\} = (m - n)L_{m+n} + \kappa L_n + \sum_k s_k L_k, \tag{6.10}$$

where $\mathfrak{a}(\mathfrak{W}_1; \kappa \text{id}, \kappa \text{id}, \sum_k s_k L_k) \cong \mathfrak{a}(\mathfrak{W}_1; \kappa \text{id}, \kappa \text{id}, \sum_k s'_k L_k)$ provided that there exists $t \in \mathbb{C}^\times$ such that either $s'_k = t^k s_k$ or $s'_k = -t^{-k} s_{-k}$, for all $k \in \mathbb{Z}$.

The algebra \mathfrak{W}_1 has one-dimensional \mathbb{C} -valued second cohomology space, with the standard choice of the representing 2-cocycle

$$\pi(L_m, L_n) = \frac{m^3 - m}{12} \delta_{m, -n}. \tag{6.11}$$

The resulting central extension of \mathfrak{W}_1 is the Virasoro algebra \mathfrak{V} , generated by L_m , $m \in \mathbb{Z}$ and the central element c with the multiplications

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m, -n} c. \tag{6.12}$$

The cocycle extensions of $\mathfrak{a}(\mathfrak{W}_1; \kappa \text{id}, \kappa \text{id}, s)$ fall into two classes. Then condition (6.5) yields necessarily

$$2m\sigma(L_0) = \sigma([L_m, L_{-m}]) = \kappa\pi(L_m, L_{-m}) = \frac{\kappa(m^3 - m)}{12},$$

for all $m \in \mathbb{Z}$. Hence σ does not exist if $\kappa \neq 0$, while if $\kappa = 0$, $\sigma(L_0) = 0$. One then finds furthermore that $\sigma(L_m) = 0$, for all $m \in \mathbb{Z}$. Thus only $\mathfrak{a}(\mathfrak{W}_1; 0, 0, s)$ admits central extensions and these have the multiplications

$$\{L_m, L_n\} = (m - n)L_{m+n} + \left(\frac{m^3 - m}{12} \delta_{m, -n} + \varrho(L_m) + \tau \right) z + s,$$

$$\{L_m, z\} = (\varrho(L_m) + \tau) z + s, \quad \{z, L_n\} = \{z, z\} = \tau z + s,$$

for $\varrho \in \text{Lin}(\mathfrak{W}_1)$ and $\tau \in \mathbb{C}$.

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Declarations

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Ryszard R. Andruszkiewicz and Tomasz Brzeziński
Faculty of Mathematics
University of Białystok
K. Ciołkowskiego 1 M
15-245 Białystok
Poland
e-mail: randrusz@math.uwb.edu.pl

Tomasz Brzeziński
Department of Mathematics
Swansea University
Fabian Way
Swansea SA1 8EN
UK
e-mail: T.Brzezinski@swansea.ac.uk

Krzysztof Radziszewski
Doctoral School and Faculty of Mathematics
University of Białystok
K. Ciołkowskiego
15-245 Białystok
Poland
e-mail: K.Radziszewski@uwb.edu.pl

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