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RESEARCH ARTICLE

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Ground states of a non-local variational problem and Thomas–Fermi limit for the Choquard equation

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Abstract

We study non-negative optimisers of a Gagliardo–Nirenberg-type inequality

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \leqslant C\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{p\theta}\left(\int_{\mathbb{R}^N}|u|^q dx\right)^{2p(1-\theta)/q},$$

which involves the non-local Riesz energy with $0<\alpha<N$, $p>\frac{N+\alpha}{N}$, $q>\frac{2Np}{N+\alpha}$ and $\theta=\frac{(N+\alpha)q-2Np}{Np(q-2)}$. For $p=2$, the equivalent problem has been studied in connection with the Keller–Segel diffusion–aggregation models in the past few decades. The general case $p\neq 2$ considered here appears in the study of Thomas–Fermi limit regime for the Choquard equations with local repulsion. We establish, for the first time, optimal ranges of parameters for the validity of the above interpolation inequality, discuss the existence and qualitative properties of the non-negative maximisers and in some special cases estimate the optimal constant. For $p=2$, it is known that the maximisers are Hölder continuous and compactly supported on a ball. We show that for $p < 2$, the maximisers are smooth functions supported on $\mathbb{R}^N$, while for $p > 2$, the maximisers consist of a characteristic function of a ball and a non-constant non-increasing Hölder continuous function supported on the same ball. We use these qualitative properties of the maximisers to establish the validity of the Thomas–Fermi approximations for the Choquard equations with local repulsion. The results are verified numerically with extensive examples.

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1 | INTRODUCTION

1.1 | Background

Our starting point is the Choquard-type equation

$$-\Delta w + \varepsilon w + |w|^{q-2}w = (I_\alpha * |w|^p)|w|^{p-2}w \quad \text{in } \mathbb{R}^N, \tag{P_ε}$$

where $w : \mathbb{R}^N \to \mathbb{R}$ is an unknown function, $N \geqslant 3$, $p > 1$, $q > 2$ and $\varepsilon \geqslant 0$. By $I_\alpha(x) := A_\alpha |x|^{\alpha-N}$, we denote the Riesz potential with $\alpha \in (0, N)$, and $*$ stands for the standard convolution in $\mathbb{R}^N$. The choice of the *Riesz constant* $A_\alpha := \frac{\Gamma((N-\alpha)/2)}{\pi^{N/2}2^\alpha\Gamma(\alpha/2)}$ ensures that $I_\alpha(x)$ could be interpreted as the Green's function of the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$, and that the semigroup property $I_{\alpha+\beta} = I_\alpha * I_\beta$ holds for all $\alpha, \beta \in (0, N)$ such that $\alpha + \beta < N$, see, for example, [17, pp. 73–74].

When $N = 3$, $\alpha = 2$, $p = 2$ and $q = 4$, under the name of Gross–Pitaevskii–Poisson equation, (P_ε) was proposed in cosmology as a model to describe the Cold Dark Matter made of axions or bosons in the form of self-gravitating Bose–Einstein Condensate at absolute zero temperatures [5, 6, 14, 19, 33]. The non-local convolution term on the right-hand side of (P_ε) represents the gravitational attraction between bosonic particles. The local term $|w|^{q-2}w$ accounts for the repulsive short-range quantum force self-interaction between bosons. Similar models appear in the literature under the names Ultralight Axion Dark Matter and Fuzzy Dark Matter, see [6] for a history survey. More generally, (P_ε) can be seen as a Hartree-type non-linear Schrödinger equation with an attractive long-range interaction and repulsive short-range interactions. While for most of the relevant physical applications the parameter p is chosen to be 2, the cases with $p \neq 2$ appear in several relativistic models of the density functional theory [2–4].

By a *ground state* of (P_ε), we understand a non-negative weak solution $w \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ which has a minimal energy of the functional

$$\mathcal{I}_\varepsilon(w) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |w|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |w|^q dx - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p dx \tag{1.1}$$

amongst all non-trivial finite energy solutions of (P_ε). The following was proved in [28, Theorem 1.1], under optimal or near optimal assumptions on the parameters.

Theorem 1.1. *Let* $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > 2$*, or* $p \geqslant \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha}$*. Then, for each* $\varepsilon > 0$*, (*P_ε*) admits a positive, radial, monotone decreasing ground state solution* $w_\varepsilon \in H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$*. Moreover, there exists a positive constant* C *depending on* N*,* α*,* p*,* q *and* ε*, such that*

- *if* $p > 2$*,*

$$\lim_{|x|\to\infty} w_\varepsilon(x)|x|^{\frac{N-1}{2}} e^{\sqrt{\varepsilon}|x|} = C;$$

- *if* $p = 2$*,*

$$\lim_{|x|\to\infty} w_\varepsilon(x)|x|^{\frac{N-1}{2}} \exp\left(\int_{\rho_\varepsilon}^{|x|} \sqrt{\varepsilon - \frac{A_\alpha \|w_\varepsilon\|_2^2}{s^{N-\alpha}}}\, ds\right) = C, \qquad \rho_\varepsilon = \left(\varepsilon^{-1} A_\alpha \|w_\varepsilon\|_2^2\right)^{\frac{1}{N-\alpha}};$$

- *if* $p < 2$*,*

$$\lim_{x\to\infty} w_\varepsilon(x)|x|^{\frac{N-\alpha}{2-p}} = \left(\varepsilon^{-1} A_\alpha \|w_\varepsilon\|_p^p\right)^{\frac{1}{2-p}}.$$

In addition to the existence of ground states for every fixed $\varepsilon > 0$, in [28], the authors have identified and studied several limit regimes for ground states of (P_ε), as $\varepsilon \to 0$ or $\varepsilon \to \infty$. One of the relevant limit regimes is associated with the rescaling

$$u_\varepsilon(x) := \varepsilon^{-\frac{1}{q-2}} w\left(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\right), \tag{1.2}$$

which converts (P_ε) to the equation

$$-\varepsilon^\nu \Delta u_\varepsilon + u_\varepsilon + |u_\varepsilon|^{q-2} u_\varepsilon = (I_\alpha * |u_\varepsilon|^p)|u_\varepsilon|^{p-2} u_\varepsilon \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\nu = \frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}$. The *Thomas–Fermi limit regime* for the Choquard equation (P_ε) is the scenario when $\varepsilon \to 0$ and $\nu > 0$, or $\varepsilon \to \infty$ and $\nu < 0$. In this regime, ε^ν approaches zero and the *formal* limit equation for Equation (1.3) is the *Thomas–Fermi*-type integral equation

$$u + |u|^{q-2} u = (I_\alpha * |u|^p)|u|^{p-2} u \quad \text{in } \mathbb{R}^N. \tag{TF}$$

When $p = 2$ and $\alpha = 2$, equations equivalent to (*TF*) are well known in astrophysical literature, cf. [13, p. 92], and their mathematical analysis goes back to [1, 26]. More recently, (*TF*) with $p = 2$ and general $\alpha \in (0, N)$ was studied in [7, 10] (existence of solutions) and [8, 11, 12] (uniqueness), all in the context of Keller–Segel models. See also [28, Theorem 2.6] which proves the existence of a ground state for (*TF*) with $p = 2$ for the optimal range $q > \frac{4N}{N+\alpha}$, extending some of the existence results in [7, 10].

In [28, Theorems 2.7 and 3.2], for the special case $p = 2$ and $\alpha = 2$, the authors established the convergence of the rescaled ground states u_ε of (1.3) to the ground states u of (*TF*) in the

Thomas–Fermi regime, thus justifying the formal analysis of the rescaling (1.2). Recall that for $p = 2$, the limit ground state of (*TF*) is compactly supported on a ball, so that the rescaled ground states u_ε develop a sharp "corner" near the boundary of the support of the limit ground state. This phenomenon is well known in astrophysics, where the radius of support of the limit ground state provides approximate radius of the astrophysical object. In the context of self-gravitating Bose–Einstein Condensate models, the Thomas–Fermi limit regime (under the name of *Thomas–Fermi approximations*) was used as the key tool in the astrophysical studies of the Gross–Pitaevskii–Poisson equation ($\alpha = 2$, $p = 2$, $q = 4$) in [5, 14, 33].

The main goal of this work is to study the existence and qualitative properties of ground states for (*TF*) for $p \neq 2$ and $\alpha \in (0, N)$, under optimal assumptions on the parameters, and to use these properties to establish the validity of the Thomas–Fermi approximations for the Choquard equations with local repulsion. In particular, we are going to prove that:

- if $p < 2$, ground states for (*TF*) are positive smooth functions supported on $\mathbb{R}^N$;
- if $p > 2$, ground states for (*TF*) are discontinuous and represented as a linear combination of a characteristic function of a ball, and a non-constant non-increasing Hölder continuous function supported on the same ball.

As a comparison, for the special case $p = 2$, it is well known in [7, 10] that ground states for (*TF*) are Hölder continuous and compactly supported on a ball. We also establish qualitative properties of the ground states, including decay at infinity for $p < 2$, and regularity near the boundary of the support for $p > 2$. This information becomes crucial in the proofs of convergence of the rescaled ground states of (P_ε) to the limit profiles governed by (*TF*).

1.2 | Variational setup for (*TF*) and main results

Solutions of the Thomas–Fermi equation (*TF*) correspond, at least formally, to the critical points of the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q\, dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p),$$

where, and in what follows, $\mathcal{D}_\alpha$ denotes the Coulomb interaction

$$\mathcal{D}_\alpha(f, g) := A_\alpha \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x)\, g(y)}{|x-y|^{N-\alpha}} dx\, dy,$$

here A_α is the Riesz constant. Throughout this work, we assume the following restrictions on the parameters:

$$\frac{1}{q} < \frac{N+\alpha}{2Np} < \frac{1}{2}. \tag{1.4}$$

Then, using the Hardy–Littlewood–Sobolev (HLS) and Hölder's inequalities, we can control the Coulomb term, that is,

$$\mathcal{D}_\alpha(|u|^p, |u|^p) \leqslant \mathscr{C}_{N,\alpha}\|u\|^{2p}_{\frac{2Np}{N+\alpha}} \leqslant \mathscr{C}_{N,\alpha}\|u\|_2^{2p\theta}\|u\|_q^{2p(1-\theta)}. \tag{1.5}$$

Here, $\mathscr{C}_{N,\alpha} = \frac{\Gamma((N-\alpha)/2)}{2^\alpha \pi^{\alpha/2}\Gamma((N+\alpha)/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{\alpha/N}$ is the sharp constant in the Hardy–Littlewood–Sobolev inequality [24, 25], and $\theta \in (0,1)$ satisfies the condition

$$\frac{N+\alpha}{2Np} = \frac{\theta}{2} + \frac{1-\theta}{q}, \quad \text{or } \theta = \frac{(N+\alpha)q - 2Np}{Np(q-2)}. \tag{1.6}$$

Therefore, the conditions in (1.4) ensure that the energy E is continuous and Fréchet differentiable on $L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$, and its critical points are solutions of (*TF*). Moreover, critical points of E (and solutions of (*TF*)) satisfy the Nehari identity

$$\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |u|^q dx = \mathcal{D}_\alpha(|u|^p, |u|^p). \tag{1.7}$$

By a *ground state* of (*TF*), we understand a non-negative solution $u \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ of (*TF*) which has a minimal energy E amongst all functions in the Pohožaev manifold $\mathscr{P}$, defined as

$$\mathscr{P} = \left\{u \neq 0 : u \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N),\ \mathcal{P}(u) = 0\right\}, \tag{1.8}$$

where the functional $\mathcal{P}$ is given by

$$\mathcal{P}(u) := \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p).$$

Since the energy E is not bounded from below (by replacing u with $u(\cdot/\lambda)$ with $\lambda \to +\infty$), constrained minimisation techniques are better suited for the construction of ground states. Moreover, the Pohožaev manifold $\mathscr{P}$ is preferred over the Nehari manifold characterised by Equation (1.7), primarily because of simplifications due to the common expressions $\frac{1}{2}\|u\|_2^2 + \frac{1}{q}\|u\|_q^q$ appearing in both $E(u)$ and $\mathcal{P}(u)$, as demonstrated in Section 4.

Another way to construct ground states of (*TF*) is to look for maximisers of the Gagliardo–Nirenberg quotient associated to the interpolation inequality (1.5), that is,

$$\mathscr{C}_{N,\alpha,p,q} := \sup\left\{\frac{\mathcal{D}_\alpha(|v|^p, |v|^p)}{\|v\|_2^{2p\theta}\|v\|_q^{2p(1-\theta)}} : v \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N), v \neq 0\right\}. \tag{1.9}$$

From (1.5), it is clear that $\mathscr{C}_{N,\alpha,p,q} \leqslant \mathscr{C}_{N,\alpha}$. Note that the quotient in Equation (1.9) is invariant w.r.t. translation, dilation and scaling; every maximiser for $\mathscr{C}_{N,\alpha,p,q}$ (if it exists) can be rescaled to a ground states solutions of (*TF*) (see Lemma 2.3 below).

Using symmetric rearrangements, Strauss' radial bounds and Helly's selection principle for radial functions, we prove the following result.

Theorem 1.2. *Let $N \geqslant 1$, $\alpha \in (0,N)$, $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Then, there exists a non-negative radial non-increasing maximiser $u_* \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ for $\mathscr{C}_{N,\alpha,p,q}$, which is also a ground state of the Thomas–Fermi equation (TF).*

Remark 1.1. While the precise value of $\mathscr{C}_{N,\alpha,p,q}$ is not known in general, we prove below (see Proposition 2.1) that for fixed admissible values of N, p and q,

$$\lim_{\alpha\to 0}\mathscr{C}_{N,\alpha,p,q}=1,\qquad \lim_{\alpha\to N}A_\alpha^{-1}\mathscr{C}_{N,\alpha,p,q}=1, \tag{1.10}$$

here A_α is the Riesz constant. For specific combination of parameters, we can estimate $\mathscr{C}_{N,\alpha,p,q}$ by looking at the ansatz $v(x)=\lambda(1+|x|^2/\mu^2)^{-\gamma}$. The invariance of the quotient (1.9) with respect to λ and μ suggests the choice $\lambda=\mu=1$ for simplicity, leading to the governing equation

$$c_1 v + c_2|v|^{q-2}v=(I_\alpha * |v|^p)|v|^{p-2}v, \tag{1.11}$$

with some $c_1, c_2>0$, which is equivalent to (*TF*) after a scaling. Using the well-known Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k,$$

with the Pochhammer symbol $(a)_k=a(a+1)\cdots(a+k-1)$ and the convention $(a)_0=1$, $|v|^p$ can be written as

$$(1+|x|^2)^{-\gamma p}=\sum_{k=0}^{\infty}\frac{(\gamma p)_k}{k!}(-|x|^2)^k=\sum_{k=0}^{\infty}\frac{(\gamma p)_k(N/2)_k}{k!(N/2)_k}(-|x|^2)^k={}_2F_1\left(\gamma p,\frac{N}{2};\frac{N}{2};-|x|^2\right).$$

This enables the explicit representation

$$I_\alpha*(1+|x|^2)^{-\gamma p}=\frac{\Gamma(\gamma p-\alpha/2)\Gamma((N-\alpha)/2)}{2^\alpha\Gamma(\gamma p)\Gamma(N/2)}\,{}_2F_1\left(\gamma p-\frac{\alpha}{2},\frac{N-\alpha}{2};\frac{N}{2};-|x|^2\right)$$

similar to that in [23], and we can show that a non-trivial solution of (1.11) corresponds to the function $v(x)=(1+|x|^2)^{-(N+1)/2}$ with†

$$p=\frac{N+\alpha+2}{N+1},\qquad q=\frac{2(N+2)}{N+1}. \tag{1.12}$$

In the absence of uniqueness results for (1.11), we cannot conclude that this solution is an optimiser for (1.9); however, the optimal constant can be estimated (more details in Appendix A) as

$$\mathscr{C}_{N,\alpha,p,q}\geqslant\frac{N(N+\alpha+2)}{\pi^{\alpha/2}2^{\alpha+1}(N+2)}\frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2+1)}\left(\frac{N+2}{2(N+1)}\frac{\Gamma(N+1)}{\Gamma(N/2+1)}\right)^{\alpha/N}. \tag{1.13}$$

We conjecture that $v(x)=(1+|x|^2)^{-(N+1)/2}$ is an optimiser for $\mathscr{C}_{N,\alpha,p,q}$ for the specific values of p and q in (1.12), and (1.13) is actually an equality.

† A symbolic solver in Maple or Mathematica is recommended. The same v, after appropriate rescaling, satisfies (*TF*) with another set of parameters $p=\frac{N+\alpha+2}{N+2}$, $q=\frac{2N+2}{N+2}$, but the corresponding quotient in (1.9) is not bounded from above because the condition $p>(N+\alpha)/N$ in Theorem (1.2) is violated.

Remark 1.2. The substitution $\rho = |u|^p$, $m = q/p$ and $n = 2/p$ leads to an equivalent formulation of the quotient in Equation (1.9), that is,

$$\mathscr{C}_{N,\alpha,m,n} := \sup \left\{ \frac{\mathcal{D}_\alpha(\rho,\rho)}{\left(\int_{\mathbb{R}^N} \rho^n dx\right)^{\frac{2\theta}{n}} \left(\int_{\mathbb{R}^N} \rho^m dx\right)^{\frac{2(1-\theta)}{m}}} : 0 \leqslant \rho \in L^n(\mathbb{R}^N) \cap L^m(\mathbb{R}^N), \rho \neq 0 \right\}.$$

The corresponding interpolation inequality then takes the form

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\rho(x)|\,|\rho(y)|}{|x-y|^{d-\alpha}} dx\, dy \leqslant C_{N,\alpha,m,n} \left(\int_{\mathbb{R}^N} |\rho|^n dx \right)^{\frac{2\theta}{n}} \left(\int_{\mathbb{R}^N} |\rho|^m dx \right)^{\frac{2(1-\theta)}{m}}, \tag{1.14}$$

for all $\rho \in L^n(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$, where $0 < n < \frac{2N}{N+\alpha} < m$, and where by $L^n(\mathbb{R}^N)$, we denote the class of n-integrable functions with any $n > 0$. Equation (1.14) can be seen as a standard interpolation associated to the Hardy–Littlewood–Sobolev inequality, which, however, includes *sublinear* exponents $n < 1$. Relevant variational problems with $n = 1$ can be found in the early works by Auchmuty and Beals [1] and P.-L. Lions [26]; [27, Section II] with $\alpha = 2$. The case $\alpha \in (0, N)$ in the context of diffusion-aggregation models was studied in the form (1.14) in the recent papers [7, 8, 10–12]. We are not aware of any works where the case $n \neq 1$ was considered.

Mathematically, the most striking phenomenon related to (*TF*) is how the behaviour of the support of ground states to (*TF*) depends on the value of p. Our main result in this work is the following theorem, which is formulated for radial ground states only, since (*TF*) is translation invariant.

Theorem 1.3. *Let $N \geqslant 1$, $\alpha \in (0, N)$, $p > \frac{N+\alpha}{N}$, $q > \frac{2Np}{N+\alpha}$. Then, every non-negative radial non-increasing ground state $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ of* (TF) *is C^∞ in the set $\{x \in \mathbb{R}^N : u(|x|) > 0\}$ and:*

(a) *if $p < 2$, then $\{u > 0\} = \mathbb{R}^N$ and $\lim_{|x|\to\infty} u(|x|)|x|^{\frac{N-\alpha}{2-p}} = \left(A_\alpha \int_{\mathbb{R}^N} u^p\, dx\right)^{\frac{1}{2-p}}$, where A_α is the Riesz constant;*
(b) *if $p = 2$, then $u : \mathbb{R}^N \to \mathbb{R}$ is a Hölder continuous and $\{u > 0\} = B_R$ for some $R > 0$;*
(c) *if $p > 2$, then $\{u > 0\} = B_R$ for some $R > 0$ and*

$$u = \lambda \chi_{B_R} + \phi,$$

where $\lambda \geqslant \left(\frac{p-2}{q-p}\right)^{1/(q-2)}$ is a constant, and where $\phi : B_R \to \mathbb{R}$ is a Hölder continuous radial non-increasing function such that $\phi(0) > 0$ and $\lim_{|x|\to R} \phi(|x|) = 0$.

The case $p = 2$ of Theorem 1.3 is well studied. The existence and qualitative properties of the ground states of (*TF*) in the case $N = 3$, $\alpha = 2$, $p = 2$ and $q > 8/3$ are classical and goes back to [1, 26]. The case $N \geqslant 2$, $\alpha \in (0, N)$ and $q \geqslant q_c := 2(2 - \alpha/N)$ is a recent study by J. Carrillo et al. [7, 10] in the context of Keller–Siegel systems. The case $p = 2$ and $q > \frac{4N}{N+\alpha}$ appeared in [28, Theorem 2.6]. In some special cases, it is known that a bounded radially non-increasing ground state of (*TF*) is unique. For $p = 2$ and $\alpha = 2$, this follows from [20, Lemma 5]. For $p = 2$ and $\alpha < 2$, this is the recent result in [12, Theorem 1.1 and Proposition 5.4] (see also [8, 11, 15] and further references therein). For $p \neq 2$, or for $p = 2$ and $\alpha > 2$, the uniqueness seems to be open at present.

In the case $N = 3$, $\alpha = 2$, $p = 2$ and $q = 4$, the (unique) non-negative radial ground state of (TF) is known explicitly and is given by the function

$$u(x) = \chi_{B_\pi}(x)\sqrt{\frac{\sin(|x|)}{|x|}}. \tag{1.15}$$

This is (up to the physical constants) the Thomas–Fermi approximation solution for self–gravitating BEC observed in [5, 14, 33] and the support radius $R = \pi$ is the approximate radius of the BEC star. Note that $u \notin H^1(\mathbb{R}^3)$. For $p \geqslant 2$ and general values of N, α and q the radius of the support of a ground state of (TF) can be easily estimated. In particular, in Corollaries 3.3 and 3.4 we show that for fixed admissible N, p and q, the radius of the support of the ground states diverges to $+\infty$ as $\alpha \to 0$, and shrinks to zero as $\alpha \to N$. In Lemmas 3.2 and 3.3, we obtain quantitative estimates on the Hölder continuity of the ground state near the boundary of the support when $p \geqslant 2$.

1.3 | Thomas–Fermi limit for the Choquard equation (P_ε)

Next, we prove that in the relevant asymptotic regimes, ground states w of (P_ε) described in Theorem 1.1 converge, after the rescaling (1.2), towards a ground state of the Thomas–Fermi equation (TF). To identify the asymptotic regimes, observe that the rescaling (1.2) transforms the Choquard energy $\mathcal{I}_\varepsilon(u)$ in such a way that

$$\mathcal{J}_\varepsilon(v) = \varepsilon^{\frac{q(N+\alpha)-2Np}{\alpha(q-2)}}\mathcal{I}_\varepsilon(u),$$

where we denote

$$\mathcal{J}_\varepsilon(v) := \frac{1}{2}\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}|v|^2dx + \frac{1}{q}\int_{\mathbb{R}^N}|v|^qdx - \frac{1}{2p}\mathcal{D}_\alpha(|v|^p,|v|^p). \tag{1.16}$$

We note that if $\varepsilon \to 0$ and $q < \frac{2(2p+\alpha)}{2+\alpha}$, or if $\varepsilon \to \infty$ and $q > \frac{2(2p+\alpha)}{2+\alpha}$, then $\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \to 0$, and *formally*, the limit energy for $\mathcal{J}_\varepsilon$ coincides with the Thomas–Fermi energy E. Combined with the existence range of the ground state of (P_ε) in Theorem 1.1, this formally identifies the Thomas–Fermi limit regimes (see Figure 1). In Section 4, we prove the following result, which confirms our reasoning based on formal asymptotics and which covers the ranges of $\alpha \neq 2$ and $p \neq 2$ left missing in [28, Theorem 2.7 and 3.2].

Theorem 1.4. *Let $N \geqslant 3$ and $\alpha \in (0, N)$. Assume that either of the following holds:*

(i) $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > \frac{2(2p+\alpha)}{2+\alpha}$, *or* $p > \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha}$;
(ii) $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $\frac{2Np}{N+\alpha} < q < \frac{2(2p+\alpha)}{2+\alpha}$.

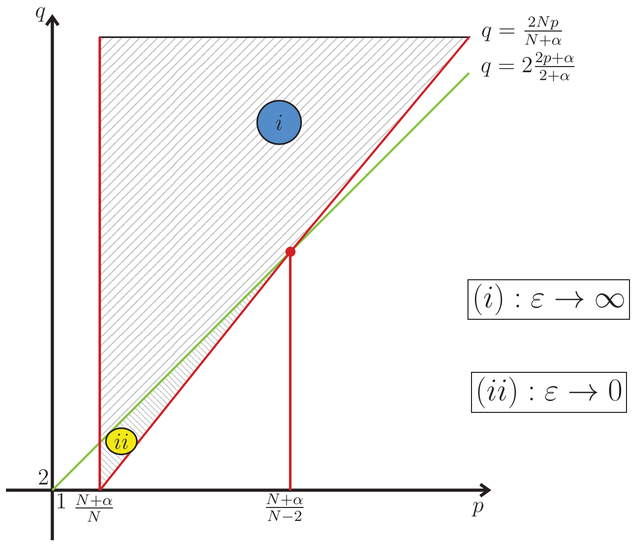

FIGURE 1 Thomas–Fermi limit regimes (i) and (ii) in Theorem 1.4.

Then, there exists a sequence of parameters $(\varepsilon_k)_{k\in\mathbb{N}}$ *with the corresponding ground states* (u_{ε_k}) *of* (P_{ε_k}) *such that*

$$\varepsilon_k \to \infty \; if \; (i) \; holds, or \; \varepsilon_k \to 0 \; if \; (ii) \; holds,$$

and the rescaled sequence of ground states of (P_ε)

$$u_{\varepsilon_k}(x) := \varepsilon_k^{-\frac{1}{q-2}} w_{\varepsilon_k}\left(\varepsilon_k^{-\frac{2p-q}{\alpha(q-2)}} x\right)$$

converges in $L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ *to a non-negative ground state of the Thomas–Fermi equation* (TF). *Moreover,* $\varepsilon_k^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}}\|\nabla u_{\varepsilon_k}\|_2^2 \to 0$ *as* $k\to\infty$.

Location of limit regimes (i) and (ii) on the (p,q) plane is outlined in Figure 1. Note that typically, the limit ground state of (*TF*) is not in $H^1(\mathbb{R}^N)$ and the quantity $\|\nabla u_{\varepsilon_k}\|_2^2$ in Theorem 1.4 blows up as $k\to\infty$. The qualitative properties and Hölder regularity of the ground state of (*TF*) established in Theorem 1.3 become crucial in the analysis of Thomas–Fermi convergence in Theorem 1.4.

Notations

For real-valued non-negative functions $f(t)$, $g(t)$ defined on a subset of $\mathbb{R}_+$, we write:

$f(t)\lesssim g(t)$ if there exists $C>0$ independent of t such that $f(t)\leqslant Cg(t)$;
$f(t)\gtrsim g(t)$ if $g(t)\lesssim f(t)$;
$f(t)\sim g(t)$ if $f(t)\lesssim g(t)$ and $f(t)\gtrsim g(t)$.

Bearing in mind that $f(t), g(t) \geqslant 0$, we write $f(t) = \mathcal{O}(g(t))$ if $f(t) \sim g(t)$, and $f(t) = o(g(t))$ if $\lim \frac{f(t)}{g(t)} = 0$. As usual, $B_R = \{x \in \mathbb{R}^N \,:\, |x| < R\}$ and $|B_R|$ is the volume of B_R. By C, c, c_1, and so on, we denote generic positive constants whose value may change from line to line.

2 | PROOF OF THEOREM 1.2

In what follows, unless specified otherwise, we always assume that $N \geqslant 1$ and $\alpha \in (0, N)$.

2.1 | Existence of an optimiser

For $u \in L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, denote the quotient

$$\mathcal{R}_\alpha(u) := \frac{\mathcal{D}_\alpha(|u|^p, |u|^p)}{\|u\|_2^{2p\theta} \|u\|_q^{2p(1-\theta)}}.$$

We are going to show that the best constant

$$\mathscr{C}_{N,\alpha,p,q} = \sup\left\{\mathcal{R}_\alpha(u) \,:\, u \in L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), u \neq 0\right\} \tag{2.1}$$

is achieved, following with some modifications the arguments in [9, Proposition 8].

Lemma 2.1. *Let $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Then, there exists a non-negative radial non-increasing maximiser $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for $\mathscr{C}_{N,\alpha,p,q}$.*

Proof. Let $(u_n) \subset L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a maximising sequence, such that $\mathcal{R}_\alpha(u_n) \to \mathscr{C}_{N,\alpha,p,q}$ as $n \to \infty$. Let u_n^* denote the Schwartz spherical rearrangements of $|u_n|$, centred at the origin. Then, $u_n^* = u_n^*(|x|)$ is non-negative radially symmetric non-increasing, and

$$\|u_n\|_2^2 = \|u_n^*\|_2^2, \quad \|u_n\|_q^q = \|u_n^*\|_q^q, \quad \mathcal{D}_\alpha(|u_n|^p, |u_n|^p) \leqslant \mathcal{D}_\alpha((u_n^*)^p, (u_n^*)^p). \tag{2.2}$$

Therefore, $\mathcal{R}_\alpha(u_n) \leqslant \mathcal{R}_\alpha(u_n^*)$ and (u_n^*) is also a maximising sequence of $\mathscr{C}_{N,\alpha,p,q}$. Without loss of generality, we can denote u_n^* by u_n in the rest of the proof.

By using the scaling invariance and homogeneity of $\mathcal{R}_\alpha$, we can assume that $\|u_n\|_2 = \|u_n\|_q = 1$. Indeed, since $\mathcal{R}_\alpha(\lambda u_n(\mu\,\cdot)) = \mathcal{R}_\alpha(u_n)$ for all $\lambda, \mu > 0$, we can choose λ_n and μ_n such that $\|\lambda_n u_n(\mu_n\,\cdot)\|_2 = \|\lambda_n u_n(\mu_n\,\cdot)\|_q = 1$. To simplify the notation, we still denote such rescaled maximising sequence by $(u_n)_n$. In particular, because of the normalisation just employed, we have

$$\mathcal{R}_\alpha(u_n) = \mathcal{D}_\alpha(u_n^p, u_n^p) \to \mathscr{C}_{N,\alpha,p,q}$$

as $n \to \infty$. Using Strauss' L^s–bounds [32] with $s = 2$ and $s = q$, we conclude that

$$u_n(x) \leqslant U(x) := C \min\{|x|^{-N/2}, |x|^{-N/q}\}.$$

Since $U \in L^s(\mathbb{R}^N)$ for $s \in (2, q)$, by Helly's selection principle and boundedness of u_n in $L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, (modulo extracting a subsequence) $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, for some $0 \leqslant u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. By the Lebesgue dominated convergence theorem, we see that for $s \in (2, q)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s dx = \int_{\mathbb{R}^N} |u|^s dx.$$

From the restriction $q > \frac{2Np}{N+\alpha} > p$, by the non-local Brezis–Lieb Lemma [29, Proposition 4.7], we conclude that

$$\lim_{n\to\infty} \mathcal{D}_\alpha(u_n^p, u_n^p) = \mathcal{D}_\alpha(u^p, u^p) = \mathscr{C}_{N,\alpha,p,q}.$$

By Fatou's lemma, we get that

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx \geqslant \int_{\mathbb{R}^N} |u|^2 dx > 0, \quad 1 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx \geqslant \int_{\mathbb{R}^N} |u|^q dx > 0.$$

We claim that $\int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u|^q dx = 1$. Otherwise, if $\|u\|_2 \|u\|_q < 1$, then

$$\mathscr{C}_{N,\alpha,p,q} \geqslant \mathcal{R}_\alpha(u) > \mathcal{D}_\alpha(u^p, u^p) = \lim_{n\to\infty} \mathcal{D}_\alpha(u_n^p, u_n^p) = \lim_{n\to\infty} \mathcal{R}_\alpha(u_n) = \mathscr{C}_{N,\alpha,p,q},$$

a contradiction. Therefore, our claim holds and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx = \int_{\mathbb{R}^N} |u|^q dx,$$

that is, u_n converges to u strongly in $L^s(\mathbb{R}^N)$ for $s \in [2, q]$, and $\mathscr{C}_{N,\alpha,p,q} = \mathcal{R}_\alpha(u)$. □

Next, we briefly discuss the asymptotic behaviours of the optimal constant $\mathscr{C}_{N,\alpha,p,q}$ when α approaches 0 or N.

Proposition 2.1. *Assume that $N \geqslant 1$, $\alpha \in (0, N)$. If $p > 1$ and $q \geqslant 2p$, then*

$$\lim_{\alpha\to 0} \mathscr{C}_{N,\alpha,p,q} = 1. \tag{2.3}$$

Furthermore, if $p \geqslant 2$ and $q > p$, then

$$\lim_{\alpha\to N} A_\alpha^{-1} \mathscr{C}_{N,\alpha,p,q} = 1. \tag{2.4}$$

Proof. First of all, we notice that $p > 1$ implies $p > \frac{N+\alpha}{N}$ for every α sufficiently close to zero, and $2p > \frac{2Np}{N+\alpha}$ for every $\alpha \in (0, N)$. Similarly, the assumption $q > p$ ensures that $q > \frac{2Np}{N+\alpha}$ for every α sufficiently close to N, and clearly $p \geqslant 2 > \frac{N+\alpha}{N}$. Thus, under our assumptions on p and q, if α is sufficiently close to zero or N, the optimal constant $\mathscr{C}_{N,\alpha,p,q}$ is well defined. Firstly, we prove that

$$\limsup_{\alpha\to 0} \mathscr{C}_{N,\alpha,p,q} \leqslant 1.$$

By the HLS inequality and standard properties of the Gamma function, we conclude that

$$\mathscr{C}_{N,\alpha,p,q} \leqslant \mathscr{C}_{N,\alpha} = (2\sqrt{\pi})^{-\alpha} \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)} \right)^{\frac{\alpha}{N}} \to 1 \quad \text{as } \alpha \to 0, \tag{2.5}$$

where $\mathscr{C}_{N,\alpha}$ is the sharp constant in the HLS inequality (1.5).

To derive a lower bound of $\mathscr{C}_{N,\alpha,p,q}$, we take $u = \chi_{B_1}$ as the trial function. Then, by the explicit expression for the Riesz potential of a characteristic function given in Equation (3.20) below,

$$\mathscr{C}_{N,\alpha,p,q} \geqslant |B_1|^{-p\theta - \frac{2p(1-\theta)}{q}} \frac{\Gamma((N-\alpha)/2)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(N/2)} \int_{B_1} {}_2F_1(-\alpha/2, (N-\alpha)/2; N/2; |x|^2) dx, \tag{2.6}$$

where $\theta = \theta(\alpha)$ is defined in Equation (1.6). Next, we note the following limits (with fixed p, q):

$$\lim_{\alpha \to 0} \left(p\theta(\alpha) + \frac{2p(1-\theta(\alpha))}{q} \right) = 1;$$

$$\lim_{\alpha \to 0} {}_2F_1(-\alpha/2, (N-\alpha)/2; N/2; |x|^2) = 1, \quad \text{for every } |x| < 1.$$

Thus Fatou's lemma yields

$$\liminf_{\alpha \to 0} \mathscr{C}_{N,\alpha,p,q} \geqslant |B_1|^{-1} \int_{B_1} \liminf_{\alpha \to 0} {}_2F_1(-\alpha/2, (N-\alpha)/2; N/2; |x|^2) dx = 1,$$

which concludes the proof of (2.3).

To derive the limit (2.4), we notice that

$$\lim_{\alpha \to N} \left(p\theta(\alpha) + \frac{2p(1-\theta(\alpha))}{q} \right) = 2;$$

$$\lim_{\alpha \to N} {}_2F_1(-\alpha/2, (N-\alpha)/2; N/2; |x|^2) = 1, \quad \text{for every } |x| < 1,$$

with the same $\theta = \theta(\alpha)$ as in Equation (1.6). Hence, by Equation (2.6) and Fatou's Lemma, we deduce

$$\liminf_{\alpha \to N} A_\alpha^{-1} \mathscr{C}_{N,\alpha,p,q} \geqslant |B_1|^{-1} \pi^{\frac{N}{2}} \liminf_{\alpha \to N} \frac{1}{\Gamma(1+\frac{\alpha}{2})} = |B_1|^{-1} \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} = 1.$$

Finally, similarly (2.5) and using the explicit form of A_α and $\mathscr{C}_{N,\alpha}$, we obtain

$$\limsup_{\alpha \to N} A_\alpha^{-1} \mathscr{C}_{N,\alpha,p,q} \leqslant \limsup_{\alpha \to N} A_\alpha^{-1} \mathscr{C}_{N,\alpha} \leqslant 1,$$

which concludes the proof of (2.4). □

In Section 3, we will show that if $p \geqslant 2$, then maximisers for $\mathscr{C}_{N,\alpha,p,q}$ have compact support. Estimates in Proposition 2.1 will be then used to estimate the radius of support of the maximisers as $\alpha \to 0$ and $\alpha \to N$.

2.2 | Maximiser has full support if $p < 2$

Our next observation is that in the case $p < 2$, maximisers for $\mathscr{C}_{N,\alpha,p,q}$ have full support $\mathbb{R}^N$.

Lemma 2.2. *Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing maximiser for $\mathscr{C}_{N,\alpha,p,q}$. If $p < 2$, then* $\mathrm{Supp}(u) = \mathbb{R}^N$.

Proof. Without loss of generality, we can assume that $\|u\|_2 = \|u\|_q = 1$. Arguing by contradiction, assume that there exists an open set $A \subset \mathbb{R}^N$ with $A \cap \mathrm{Supp}(u) = \emptyset$ and $0 < |A| < +\infty$. For $\epsilon > 0$, consider the family of trial functions $v_\epsilon := u + \epsilon\chi_A \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. We obtain

$$
\begin{aligned}
\mathcal{R}_\alpha(v_\epsilon) &= \frac{\mathcal{D}_\alpha(|u+\epsilon\chi_A|^p, |u+\epsilon\chi_A|^p)}{\left(\int_{\mathbb{R}^N}(u^2+\epsilon^2\chi_A)dx\right)^{p\theta}\left(\int_{\mathbb{R}^N}(u^q+\epsilon^q\chi_A)dx\right)^{\frac{2p(1-\theta)}{q}}} \\
&\geqslant \frac{\mathcal{D}_\alpha(|u|^p,|u|^p) + 2\epsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A) + \epsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A)}{(1+p\theta\epsilon^2|A|)\left(1+\epsilon^q\frac{2p(1-\theta)}{q}|A|\right)} \\
&\geqslant \frac{\mathcal{D}_\alpha(|u|^p,|u|^p) + 2\epsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A) + \epsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A)}{1+C\epsilon^2} \\
&\geqslant \mathscr{C}_{N,\alpha,p,q} + \frac{2\epsilon^p\mathcal{D}_\alpha(|u|^p,\chi_A) + \epsilon^{2p}\mathcal{D}_\alpha(\chi_A,\chi_A) - C\epsilon^2}{1+C\epsilon^2}.
\end{aligned}
$$

Because $p < 2$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, we have $\mathcal{R}_\alpha(v_\epsilon) > \mathscr{C}_{N,\alpha,p,q}$, which contradicts to the fact that u is a maximiser. □

2.3 | Connection with ground states of (*TF*)

A direct computation shows that the Euler–Lagrange equation of $\mathcal{R}_\alpha(u)$ (or equivalently $\log \mathcal{R}_\alpha(u)$) for $u \in L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ has the form

$$
Au + B|u|^{q-2}u = C(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{2.7}
$$

where

$$
A = \frac{2p\theta}{\|u\|_2^2}, \quad B = \frac{2p(1-\theta)}{\|u\|_q^q}, \quad C = \frac{2p}{\mathcal{D}_\alpha(|u|^p,|u|^p)}.
$$

In particular, maximisers of $\mathscr{C}_{N,\alpha,p,q}$ constructed in the proof of Lemma 2.1 are weak solutions of Equation (2.7) and, after a rescaling, of (*TF*). Indeed, given a maximiser u for $\mathscr{C}_{N,\alpha,p,q}$, for $\lambda, \mu > 0$, consider the two-parameter family of functions $u_{\lambda,\mu}(x) = \lambda u(\mu x)$. In view of the scaling invariance and homogeneity of $\mathcal{R}_\alpha$, we know that $\mathcal{R}_\alpha(u) = \mathcal{R}_\alpha(u_{\lambda,\mu})$. Therefore, if we set λ_* and μ_* such that

$$\lambda_*^{q-2} = \left(\frac{1-\theta}{\theta}\right)\frac{\|u\|_2^2}{\|u\|_q^q}, \qquad \mu_*^\alpha = \left(\frac{1-\theta}{\lambda_*^{q-2p}}\right)\frac{\mathcal{D}_\alpha(|u|^p,|u|^p)}{\|u\|_q^q}, \tag{2.8}$$

we obtain $A = B = C$, and hence, u_{λ_*,μ_*} is a solution of (TF). In the next lemma, we prove that u_{λ_*,μ_*} is a ground state of (TF), that is, u_{λ_*,μ_*} belongs to the Pohožaev manifold $\mathscr{P}$, defined in (1.8), and $E(u_{\lambda_*,\mu_*}) = \sigma_*$, where

$$\sigma_* := \inf_{u\in\mathscr{P}} E(u) \tag{2.9}$$

is the *ground state energy* of (TF).

Lemma 2.3. *Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing maximiser for $\mathscr{C}_{N,\alpha,p,q}$. Then, the function u_{λ_*,μ_*}, where λ_* and μ_* are defined by (2.8), is a ground state of (TF).*

Proof. For the brevity of notation, let us set $u_* = u_{\lambda_*,\mu_*}$ for any maximiser u of the quotient $\mathcal{R}_\alpha$. From (2.8), we deduce directly that

$$\frac{N}{2}\|u_*\|_2^2 + \frac{N}{q}\|u_*\|_q^q - \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u_*|^p,|u_*|^p) = \frac{N}{\theta}\left(\frac{\theta}{2} + \frac{1-\theta}{q} - \frac{N+\alpha}{2Np}\right)\|u_*\|_2^2 = 0,$$

that is, $u_* \in \mathscr{P}$. We only need to prove that $E(u_*) = \sigma_*$, where σ_* is the ground state energy defined in (2.9). To this aim, we explore the relation between the optimal constant $\mathscr{C}_{N,\alpha,p,q}$ and the ground state energy σ_*.

As an intermediate step, consider the functional

$$\mathcal{E}(w) = \frac{1}{2}\|w\|_2^2 + \frac{1}{q}\|w\|_q^q$$

on the set

$$\mathcal{A} := \left\{w \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N) \,:\, \mathcal{D}_\alpha(|w|^p,|w|^p) = 1\right\},$$

and note that $\mathcal{A}$ is invariant with respect to the rescaling $w_t(\cdot) = t^{-\frac{N+\alpha}{2p}} w(\cdot/t)$.

For a given $w \in \mathcal{A}$, by optimising the quantity $\mathcal{E}(w_t)$ with respect to t (see [21] for the details), we deduce that

$$\mathcal{E}(w) \geqslant \mathcal{E}(w_{t^*}) = \left(\|w\|_2^{2p\theta}\|w\|_q^{2p(1-\theta)}\right)^{\frac{N}{N+\alpha}}\theta_*,$$

with

$$t^* = \left(\frac{(N+\alpha)q - 2Np}{q(Np-N-\alpha)}\frac{\|w\|_q^q}{\|w\|_2^2}\right)^{\frac{2p}{(q-2)(N+\alpha)}}, \quad \theta_* = \left(\frac{1-\theta}{\theta}\right)^{\frac{q\theta}{2(1-\theta)+q\theta}}\left(\frac{N+\alpha}{2Np(1-\theta)}\right). \tag{2.10}$$

As a consequence,

$$\inf_{w\in\mathcal{A}} \mathcal{E}(w) = \inf_{w\in\mathcal{A}} \theta_*\left(\|w\|_2^{2p\theta}\|w\|_q^{2p(1-\theta)}\right)^{\frac{N}{N+\alpha}} = \theta_*\mathscr{C}_{N,\alpha,p,q}^{-\frac{N}{N+\alpha}}. \tag{2.11}$$

On the other hand, functions in $\mathscr{P}$ and $\mathcal{A}$ can be one-to-one mapped to each other with a spatial scaling. That is, for any $u\in\mathscr{P}$, we have $u(\cdot/\tau_u)\in\mathcal{A}$ for $\tau_u := \left(\mathcal{D}_\alpha(|u|^p,|u|^p)\right)^{-1/(N+\alpha)} = \left(\frac{N+\alpha}{2Np\mathcal{E}(u)}\right)^{1/\alpha}$; and for any $w\in\mathcal{A}$, we have $w(\cdot/t_w)\in\mathscr{P}$ with $t_w := \left(\frac{2Np\mathcal{E}(w)}{N+\alpha}\right)^{1/\alpha}$. Therefore, given $u\in\mathscr{P}$ and $w=u(\cdot/\tau_u)\in\mathcal{A}$, we have

$$E(u) = \mathcal{E}(u) - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p,|u|^p) = \frac{\alpha}{N+\alpha}\left(\frac{2Np}{N+\alpha}\right)^{\frac{N}{\alpha}}\mathcal{E}(w)^{\frac{N+\alpha}{\alpha}}. \tag{2.12}$$

Using the one-to-one correspondence between functions in $\mathscr{P}$ and $\mathcal{A}$ and taking the infimum in (2.12), we obtain (see [21] for further details),

$$\sigma_* = \frac{\alpha}{N+\alpha}\left(\frac{2Np}{N+\alpha}\right)^{\frac{N}{\alpha}}\left(\inf_{w\in\mathcal{A}}\mathcal{E}(w)\right)^{\frac{N+\alpha}{\alpha}} = \alpha(2Np)^{\frac{N}{\alpha}}\left(\frac{\theta_*}{N+\alpha}\right)^{\frac{N+\alpha}{\alpha}}\mathscr{C}_{N,\alpha,p,q}^{-\frac{N}{\alpha}}, \tag{2.13}$$

establishing the relation between these optimal values.

In order to conclude, we only need to show that the relation (2.13) is satisfied for $u_*\in\mathscr{P}$. In fact, the choice of λ_* and μ_* in (2.8) implies

$$\|u_*\|_q^q = \frac{1-\theta}{\theta}\|u_*\|_2^2, \quad \mathcal{D}_\alpha(|u_*|^p,|u_*|^p) = \frac{1}{\theta}\|u_*\|_2^2,$$

which enables us to write both $E(u_*)$ and $\mathcal{R}_\alpha(u_*)$ in terms of $\|u_*\|_2^2$. That is,

$$E(u_*) = \frac{\alpha(q-2)}{2(N+\alpha)q-4Np}\|u_*\|_2^2, \quad \text{and} \quad \mathcal{R}_\alpha(u_*) = \frac{1}{\theta}\left(\frac{\theta}{1-\theta}\right)^{2p(1-\theta)/q}\|u_*\|_2^{-2\alpha/N}.$$

By eliminating $\|u_*\|_2$ from the above two relations, we infer that

$$E(u_*) = \alpha(2Np)^{\frac{N}{\alpha}}\left(\frac{\theta_*}{N+\alpha}\right)^{\frac{N+\alpha}{\alpha}}\mathcal{R}_\alpha(u_*)^{-\frac{N}{\alpha}},$$

which in view of (2.13) and $\mathcal{R}_\alpha(u_*) = \mathscr{C}_{N,\alpha,p,q}$ implies $E(u_*)=\sigma_*$. □

Proof of Theorem 1.2. Follows from Lemmas 2.1 and 2.3. □

3 | REGULARITY, DECAY AND SUPPORT

In this section, we establish qualitative properties of the ground states of (TF), described in Theorem 1.3 and, additionally, obtain a quantitative characterisation of the Hölder continuity of the ground states.

3.1 | Regularity, decay and support properties

Recall that if $s \in (1, \frac{N}{\alpha})$ and $\frac{1}{t} = \frac{1}{s} - \frac{\alpha}{N}$, then the HLS inequality implies that the operator

$$I_\alpha * (\cdot) : L^s(\mathbb{R}^N) \to L^t(\mathbb{R}^N)$$

is bounded [24]. We first establish the following fact about the far field behaviour of $I_\alpha * u^p$: if the non-negative function u decays fast enough, then $I_\alpha * u^p$ decays algebraically like the Riesz potential I_α itself.

Lemma 3.1. *Assume that $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing solution of (TF). Then, there exists $\epsilon > 0$ such that $u \in L^{p-\epsilon}(\mathbb{R}^N)$ and*

$$\lim_{|x|\to\infty} \frac{I_\alpha * u^p}{I_\alpha(x)\int_{\mathbb{R}^N} u^p dx} = 1. \tag{3.1}$$

Proof. We first prove that $u \in L^{p-\epsilon}(\mathbb{R}^N)$ some $\epsilon > 0$, which is trivial if $p > 2$. Otherwise if $p \in (\frac{N+\alpha}{N}, 2]$, we can show that $u \in L^{s_n}(\mathbb{R}^N)$ for a sequence (s_n) of positive decreasing exponents eventually smaller than p.

Firstly, by Hölder's inequality, we see that

$$\int_{\mathbb{R}^N} |(I_\alpha * u^p)u^{p-1}|^\sigma dx \leqslant \left(\int_{\mathbb{R}^N} |(I_\alpha * u^p)|^{\sigma t} dx\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^N} u^{(p-1)\sigma r} dx\right)^{\frac{1}{r}}, \tag{3.2}$$

provided that $1/t + 1/r = 1$ for positive t and r. We want to find a sequence (s_n) of positive numbers, so that if $u \in L^{s_n}(\mathbb{R}^N)$, then $u \in L^{s_{n+1}}(\mathbb{R}^N)$. By choosing the parameters σ, t and r in Equation (3.2) so that

$$\sigma = s_{n+1}, \qquad (p-1)\sigma r = s_n, \qquad \frac{1}{\sigma t} = \frac{p}{s_n} - \frac{\alpha}{N}.$$

The last equation, arising from the HLS inequality, has to be supplied with the condition $\alpha/N < p/s_n < 1$, or $s_n \in (p, Np/\alpha)$. Therefore, the sequence (s_n) satisfies the recursion relation

$$\frac{1}{s_{n+1}} = \frac{1}{\sigma t} + \frac{1}{\sigma r} = \frac{p}{s_n} - \frac{\alpha}{N} + \frac{p-1}{s_n} = \frac{2p-1}{s_n} - \frac{\alpha}{N}. \tag{3.3}$$

With the (unstable) fixed point $s_* = 2N(p-1)/\alpha > 2$ (as $p > (N+\alpha)/N$), the general term can be written as

$$\frac{1}{s_n} = (2p-1)^n\left(\frac{1}{s_0} - \frac{1}{s_*}\right) + \frac{1}{s_*}. \tag{3.4}$$

If s_0 is chosen to be any number inside the interval $(2, s_*)$, then s_n is monotonically decreasing to zero. Therefore, we can look for the largest integer n_0 such that $s_{n_0} > p$. If $s_{n_0+1} < p$, then $u \in L^{p-\epsilon}$ for any positive $\epsilon < p - s_{n_0+1}$. Otherwise, if $s_{n_0+1} = p$, we can always choose s_0 slightly smaller to get $s_{n_0+1} < p$, since by the recursion relation Equation (3.4), s_n depends continuously and monotonically on the initial value $s_0 \in (2, s_*)$.

Consequently, since u is radially symmetric and non-increasing, by the Strauss' $L^{p-\epsilon}$–bound we have a faster decay estimate

$$u(|x|) \leqslant C|x|^{-\frac{N}{p-\epsilon}} \quad (|x| > 0).$$

Then, by [22, Lemma 6.1], we obtain the desired limit (3.1). □

Corollary 3.1. *Assume that* $\frac{N+\alpha}{N} < p < 2$ *and* $q > \frac{2Np}{N+\alpha}$. *Let* $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *be a non-negative radial non-increasing solution of (TF). Then, u satisfies the following algebraic decay rate:*

$$\lim_{x\to\infty} u(x)|x|^{\frac{N-\alpha}{2-p}} = \left(A_\alpha \int_{\mathbb{R}^N} u^p\, dx\right)^{\frac{1}{2-p}}, \tag{3.5}$$

where $A_\alpha > 0$ *is the Riesz constant. Furthermore, we have that* $u \in L^1(\mathbb{R}^N)$.

Proof. By Lemma 3.1, $I_\alpha * u^p = I_\alpha(x) \int_{\mathbb{R}^N} u^p dx\big(1 + o(1)\big)$ as $|x| \to \infty$. Hence, the governing equation (TF) implies that

$$\lim_{|x|\to\infty} u(x)^{2-p}|x|^{N-\alpha}(1 + u^{q-2}(x)) = \lim_{|x|\to\infty} |x|^{N-\alpha} I_\alpha * u^p = A_\alpha \int_{\mathbb{R}^N} u^p\, dx.$$

From the monotonicity of u and the fact that $q > 2$, we conclude that $u^{q-2}(|x|)$ vanishes at infinity, and therefore,

$$\lim_{|x|\to\infty} u^{2-p}(x)|x|^{N-\alpha} = A_\alpha \int_{\mathbb{R}^N} u^p\, dx,$$

which is equivalent to (3.5). From the condition $\frac{N+\alpha}{N} < p < 2$, the power $\frac{N-\alpha}{2-p}$ is strictly larger than N. That is, u decays faster than $|x|^{-N}$ and hence $u \in L^1(\mathbb{R}^N)$. □

Lemma 3.2. *Assume that* $p > \frac{N+\alpha}{N}$ *and* $q > \frac{2Np}{N+\alpha}$. *Let* $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *be a non-negative solution of (TF). Then,* $u \in L^\infty(\mathbb{R}^N)$ *and*

$$I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^N) \quad \textit{for every} \quad \tau \in (0, \min\{\alpha, 1\}). \tag{3.6}$$

In particular, if $p \leqslant 2$, *the following hold:*

(i) *If* $p < 2$, *then u is Hölder continuous in* $\{u > 0\}$ *of order* τ, *for every* $\tau \in (0, \min\{\alpha, 1\})$.
(ii) *If* $p = 2$, *then u is Hölder continuous in* $\{u > 0\}$ *of order* $\kappa(q)$, *where* $\kappa(q)$ *is defined by*

$$\kappa(q) = \begin{cases} \tau, & \text{if } q \leqslant 3, \\ \frac{\tau}{q-2}, & \text{if } q > 3, \end{cases} \tag{3.7}$$

for every $\tau \in (0, \min\{\alpha, 1\})$.

Proof. Assume $u \in L^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, where $s \in \left(p, \frac{Np}{\alpha}\right)$. Note that $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ implies that $I_\alpha * u^p$ is almost everywhere finite on $\mathbb{R}^N$. Moreover, by the HLS inequality, if $s \in (p, Np/\alpha)$, then $I_\alpha * u^p \in L^\tau(\mathbb{R}^N)$ where

$$\frac{1}{\tau} = \frac{p}{s} - \frac{\alpha}{N} > 0. \tag{3.8}$$

Then, (3.2) implies that $(I_\alpha * u^p)u^{p-1} \in L^\sigma(\mathbb{R}^N)$ for $\sigma \geqslant 1$ and $(I_\alpha * u^p)u^{p-1} \in \mathcal{L}^{1/\sigma}(\mathbb{R}^N)$ for $\sigma \in (0,1)$.[†]

Next, we split the argument in three cases:

CASE 1: $q > \frac{Np}{\alpha}$. In this case, $u \in L^{\bar{q}}(\mathbb{R}^N)$ for some $\bar{q} \in (\frac{Np}{\alpha}, q]$ such that $\alpha - \frac{N}{\bar{q}} < 1$. Then, $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$ and is Hölder continuous of order $\alpha - \frac{N}{\bar{q}}$ (cf [16, Theorem 2]).

CASE 2: $q = \frac{Np}{\alpha}$. Since in this case, there exists $\epsilon > 0$ small such that $u \in L^{p(\frac{N}{\alpha}-\epsilon)}(\mathbb{R}^N)$, from (3.8), we get $(I_\alpha * u^p)u^{p-1} \in L^\sigma(\mathbb{R}^N)$ where

$$\frac{1}{\sigma} = \frac{2p-1}{p(N/\alpha - \epsilon)} - \frac{\alpha}{N}.$$

Thus, recalling that

$$u^{q-1} \leqslant u + u^{q-1} = (I_\alpha * u^p)u^{p-1} \quad \text{a.e. in } \mathbb{R}^N,$$

$u \in L^{(q-1)\sigma}(\mathbb{R}^N)$ and $(q-1)\sigma > \frac{Np}{\alpha}$ provided $0 < \epsilon < \frac{N}{\alpha}(1 - \frac{2p-1}{p+q-1})$. Thus, $u^p \in L^{\frac{N}{\alpha}-\epsilon}(\mathbb{R}^N) \cap L^{\frac{(q-1)\sigma}{p}}(\mathbb{R}^N)$, therefore, $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$ and is Hölder continuous of order γ for some $\gamma \in (0,1]$.

CASE 3: $q \in (\frac{2Np}{N+\alpha}, \frac{Np}{\alpha})$. Let us set $s_0 := q > p$ and

$$\frac{1}{s_{n+1}} := \frac{1}{(q-1)\sigma_n} = \frac{2p-1}{(q-1)s_n} - \frac{\alpha}{N(q-1)}. \tag{3.9}$$

Then $u^p \in L^{\frac{s_0}{p}}$ and $(I_\alpha * u^p)u^{p-1} \in L^{\sigma_0}(\mathbb{R}^N)$. Hence, we conclude that $u \in L^{(q-1)\sigma_0}(\mathbb{R}^N) = L^{s_1}(\mathbb{R}^N)$. Note that $q > \frac{2Np}{N+\alpha}$ implies $s_1 > s_0 = q$. In particular, if $s_n < \frac{Np}{\alpha}$, an induction argument yields $s_n < s_{n+1}$. This proves that (s_n) is monotone increasing, as long as s_n is between q and Np/α.

We claim that, after finite steps, there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} \geqslant \frac{Np}{\alpha}$ and $s_n < \frac{Np}{\alpha}$ for all $n < n_0$. If not, we can obtain a sequence (s_n) satisfying (3.9) and $q < s_n < \frac{Np}{\alpha}$ for all $n \in \mathbb{N}$. By the monotonicity of this sequence, we also conclude that s_n converges to the unique fixed point $s_* = N(2p-q)/\alpha > q$, which contradicts the condition $q > 2Np/(N+\alpha)$.

Then, if $s_{n_0} > \frac{Np}{\alpha}$, we conclude that $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$ and is Hölder continuous of order $\alpha - \frac{N}{s_{n_0}}$. If $s_{n_0} = \frac{Np}{\alpha}$, we can argue as in the previous case and we still obtain boundedness and Hölder regularity of $I_\alpha * u^p$.

Next, from $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$ and the relation

$$u^{2-p} + u^{q-p} = I_\alpha * u^p \quad \text{a.e. in } \{u > 0\}, \tag{3.10}$$

[†] We denote $\mathcal{L}^t(\mathbb{R}^N) = \{f : \mathbb{R}^n \to \mathbb{R} : \int |f|^t dx < \infty\}$, where $t \in (0,1)$. Note that $\mathcal{L}^t(\mathbb{R}^N)$ is no longer a normed space for $t \in (0,1)$ because the triangle inequality does not hold.

we conclude that $u \in L^\infty(\mathbb{R}^N)$. Therefore, $u \in L^s(\mathbb{R}^N)$ for all $s \geqslant 2$ from which $I_\alpha * u^p$ is Hölder continuous of order τ for any $\tau \in (0, \min\{1, \alpha\})$.

Furthermore, if $p < 2$, since the function $f(t) = t^{2-p} + t^{q-p}$ has a differentiable inverse on $(0, \infty)$ and $u \in L^\infty(\mathbb{R}^N)$, it follows from (3.10) that u has the same Hölder regularity as $I_\alpha * u^p$ in $\{u > 0\}$.

Similarly, if $p = 2$, the function $f(t) = 1 + t^{q-2}$ has a differentiable inverse on $(0, +\infty)$ if $q \leqslant 3$, and a locally Hölder inverse of order $\frac{1}{q-2}$ if $q > 3$. Then, again, from the boundedness of u and (3.10), we obtain (3.7). □

Corollary 3.2. *Assume that $p \geqslant 2$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing solution of (TF). Then, u is compactly supported.*

Proof. Since u is radially non-increasing, by an abuse of notation, we still denote $u(r) = u(|x|)$ where $r = |x|$. Now, since $u(r)$ is a non-increasing function, it can have at most a countable number points of discontinuity. Then, without loss of generality, if r' is a discontinuity point, we define

$$u(r') := \lim_{r \to r'^+} u(r). \tag{3.11}$$

Note that the above limit exists by monotonicity of $u(r)$ and, by doing this, we are only modifying u on a set of measure zero. In fact,

$$u(r') = \liminf_{r \to r'} u(r), \tag{3.12}$$

which makes u a lower semi-continuous function, and the set $\{u > 0\}$ is open.

Arguing by contradiction, we assume that $\{u > 0\} = \mathbb{R}^N$. Since u is non-negative and satisfies (*TF*), we have

$$1 \leqslant 1 + u^{q-2} = (I_\alpha * u^p)u^{p-2} \quad \forall\, x \in \mathbb{R}^N. \tag{3.13}$$

On the other hand, $I_\alpha * u^p$ vanishes at infinity by (3.1), and $u \in L^\infty(\mathbb{R}^N)$ by Lemma 3.2. Hence, there exist $R > 0$ such that $(I_\alpha * u^p)u^{p-2} < 1$ in B_R^c, a contradiction to (3.13). □

Next we show that when $p > 2$, non-negative solutions of (*TF*) are discontinuous at the boundary of the support and Hölder continuous inside the support.

Lemma 3.3. *Assume that $p > 2$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing solution of (TF). Then, there exists*

$$\lambda \geqslant \lambda_* := \left(\frac{p-2}{q-p}\right)^{\frac{1}{q-2}} \tag{3.14}$$

such that $\{u > 0\} = \{u > \lambda\}$ and u is Hölder continuous of order $\kappa(p, q, \lambda)$ in $\{u > 0\}$, where

$$\kappa(p, q, \lambda) = \begin{cases} \tau, & \text{if } \lambda > \lambda_*, \\ \frac{\tau}{2}, & \text{if } \lambda = \lambda_*, \end{cases} \tag{3.15}$$

for every $\tau \in (0, \min\{\alpha, 1\})$.

Proof. Set $B_{R_*} := \{u > 0\}$, where $R_* < \infty$ in view of Corollary 3.2. Assume by contradiction that there exists a sequence $(r_n) \subset (0, R_*)$ such that $r_n \to R_*$ and $u(r_n) \to 0$. Then, by (3.10),

$$(I_\alpha * u^p)(r_n) = u(r_n)^{2-p} + u(r_n)^{q-p} \to \infty. \tag{3.16}$$

However, from Lemma 3.2, the left-hand side of (3.16) is bounded which leads to a contradiction. We have therefore proved that u is far away from zero inside its support, or equivalently that there exists $\lambda > 0$ such that $\{u > 0\} = \{u > \lambda\}$.

In what follows, we prove continuity of u in B_{R_*}. First, we recall that $I_\alpha * u^p$ is Hölder continuous by Lemma 3.2, and is radially non-increasing, since u is radially non-increasing. Next, we define the quantities

$$\lambda = \lim_{r \to R_*^-} u(r), \quad \gamma = u(0). \tag{3.17}$$

Note that λ_*, defined in (3.14), is the unique minimum of the function f defined by

$$f(t) = t^{2-p} + t^{q-p} \qquad (t \in (0, +\infty)). \tag{3.18}$$

To prove the continuity of u, as we will see shortly, it is enough to prove that $\lambda \geqslant \lambda_*$. To this aim, we split the proof into two steps.

Step 1: $\gamma > \lambda_*$. Assume by contradiction that $\gamma \leqslant \lambda_*$. Since u is radially non-increasing, $u(r) \leqslant \lambda_*$ for every $r \in (0, R_*)$. Furthermore, since the function $f(t) = t^{2-p} + t^{q-p}$ is decreasing in the interval $(0, \lambda_*]$, we deduce that $f(u(r))$ is non-decreasing in $(0, R_*)$. Thus, from the equality $f(u(r)) = (I_\alpha * u^p)(r)$ and monotonicity of $I_\alpha * u^p$, and injectivity of f (or strict monotonicity of f) the function u must be constant inside the support. Namely, $u(x) = \gamma \chi_{B_{R_*}}(x)$ and, from (3.10),

$$\gamma^{2-p} + \gamma^{q-p} = \gamma^p I_\alpha * \chi_{B_{R_*}} \quad \text{in } B_{R_*}. \tag{3.19}$$

In fact, $I_\alpha * \chi_{B_{R_*}}$ can be written in terms of the Gauss Hypergeometric function as

$$(I_\alpha * \chi_{B_{R_*}})(x) = \frac{\Gamma((N-\alpha)/2) R_*^\alpha}{2^\alpha \Gamma(1+\alpha/2)\Gamma(N/2)} {}_2F_1\left(-\frac{\alpha}{2}, \frac{N-\alpha}{2}; \frac{N}{2}; \frac{|x|^2}{R_*^2}\right), \tag{3.20}$$

which is never a constant for $\alpha \in (0, N)$. We have therefore proved that $\gamma > \lambda_*$.

Step 2: $\lambda \geqslant \lambda_*$. Assume that $\lambda < \lambda_*$. First of all, we notice that u cannot be continuous in $(0, R_*)$. As a matter of fact, if u is continuous, since by Step 1, we have that $\lambda_* \in (\lambda, \gamma)$, the value λ_* is achieved by u. Namely, there exists $\bar{r} \in (0, R_*)$ such that $u(\bar{r}) = \lambda_*$. Arguing as before, since $u(r)$ is non-increasing, f is decreasing in $[\lambda, \lambda_*]$, and $(I_\alpha * u^p)(r)$ is non-increasing. We infer that $u(r)$ is constant for every $r \in [\bar{r}, R_*)$. However, this implies that

$$\lambda = \lim_{r \to R_*^-} u(r) = u(\bar{r}) = \lambda_*,$$

which is a contradiction.

Next, we show that if r' is a discontinuity point of $u(r)$, we must have that $u(r') \in [\lambda, \lambda_*]$. Indeed, by (3.12), if $u(r') = L \in (\lambda_*, \gamma]$, then for every $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that $u(r) \geqslant L - \epsilon$ for every $r \in (r' - \delta, r' + \delta)$. In particular, if we choose ϵ such that $L - \epsilon > \lambda_*$, we deduce that $u((r' - \delta, r' + \delta)) \subset (\lambda_*, \gamma]$. But in this interval, the function f is invertible with continuous inverse. Then,

$$u(r) = f^{-1}((I_\alpha * u^p)(r)) \quad \forall r \in (r' - \delta, r' + \delta),$$

which, in particular, implies continuity of u at r' and this is a contradiction.

Next, in view of monotonicity of $u(r)$, we conclude that $u([r', R_*)) \subset [\lambda, \lambda_*]$. Then, since in $[\lambda, \lambda_*]$ the function f is decreasing, again monotonicity of u implies that $u(r) = \lambda$ for every $r \in [r', R_*]$. Finally, it remains to prove that this is not possible and this will imply that $\lambda \geqslant \lambda_*$.

Since we have assumed that $u(r)$ is non-increasing and constant in $[r', R_*]$, there exists $\lambda_1 > \lambda$ such that

$$u^p = \lambda^p \chi_{B_{R_*}} + \phi + (\lambda_1^p - \lambda^p)\chi_{B_{r'}}, \tag{3.21}$$

where ϕ is a radially non-increasing function such that $\phi(r) = 0$ if $r \geqslant r'$. Thus, by combining (3.10) with (3.21), we obtain the equality

$$\lambda^{2-p} + \lambda^{q-p} = \lambda^p (I_\alpha * \chi_{B_{R_*}})(r) + (\lambda_1^p - \lambda^p)(I_\alpha * \chi_{B_{r'}})(r) + (I_\alpha * \phi)(r) \quad \forall r \in (r', R_*). \tag{3.22}$$

However, again by (3.20), the right-hand side of (3.22) is decreasing in $(R_* - \epsilon, R_*)$ for some $\epsilon > 0$ small enough and this contradicts (3.22).

We have therefore proved that $\lambda \geqslant \lambda_*$. Then, on the set $[\lambda_*, \infty)$, the function f has an inverse $f^{-1} : [\lambda_*, \infty) \to [\Lambda_*, \infty)$, where we denote $\Lambda_* := f(\lambda_*)$. We conclude that

$$u = f^{-1}(I_\alpha * u^p) \quad \text{in } B_{R_*}. \tag{3.23}$$

To prove that the desired Hölder exponent given by (3.15), we consider two different cases.

Case a): $\lambda > \lambda_*$. In this case, f is a Lipschitz function with Lipschitz inverse in the set

$$u(B_{R_*}) := \left\{ u(x) : x \in B_{R_*} \right\},$$

and by Lemma 3.2, we have $u = f^{-1}(I_\alpha * u^p) \in C^{0,\tau}(B_{R_*})$ for every $\tau \in (0, \min\{1, \alpha\})$.

Case b): $\lambda = \lambda_*$. In this case, let us notice that $f''(\lambda_*) = p(q-p)^2 \lambda_*^{q-p-2} > 0$, which means that if $\epsilon > 0$ is small enough, the following expansion holds:

$$f(t) = f(\lambda_*) + \frac{1}{2} f''(\lambda_*)(t - \lambda_*)^2 + o\big((t - \lambda_*)^2\big) \quad \forall t \in (\lambda_* - \epsilon, \lambda_* + \epsilon). \tag{3.24}$$

Let f^{-1} be the inverse of f on $[\lambda_*, \infty)$. Then, if for $s \geqslant \Lambda_*$, we set $t := f^{-1}(s)$, by (3.24), we obtain

$$\lim_{s \to \Lambda_*^+} \frac{|f^{-1}(\Lambda_*) - f^{-1}(s)|}{|\Lambda_* - s|^{\frac{1}{2}}} = \lim_{t \to \lambda_*^+} \frac{|\lambda_* - t|}{\left|\frac{1}{2} f''(\lambda_*)(t - \lambda_*)^2 + o(t - \lambda_*)^2\right|^{\frac{1}{2}}} = \sqrt{\frac{2}{f''(\lambda_*)}},$$

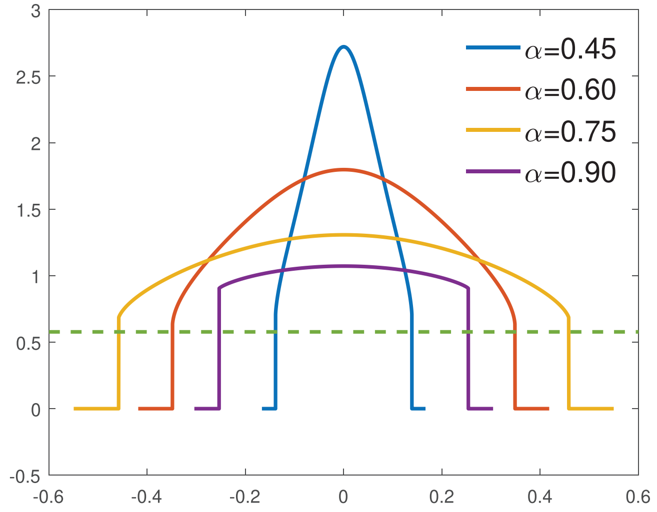

FIGURE 2 The jump near the boundary compared with λ^*(the dashed line) in one dimension with $p = 2.5, q = 4$ and different α.

which proves that f^{-1} is Hölder continuous of order $1/2$. Then, using Lemma 3.2, we obtain $u = f^{-1}(I_\alpha * u^p) \in C^{0,\frac{\tau}{2}}(B_{R_*})$ for every $\tau \in (0, \min\{1, \alpha\})$. □

Remark 3.1. Although the above discussion about the regularity depends on the jump λ of the solution near the boundary of the support, compared with $\lambda_* = \left(\frac{p-2}{q-p}\right)^{1/(q-2)}$. Numerical experiments suggests that λ is always strictly larger than λ_*, as shown in Figure 2, although the difference becomes smaller for smaller α.

Finally, similarly to [10, Theorem 10], we show that non-negative solutions of (*TF*) are smooth inside the support.

Lemma 3.4. *Assume that $p \geqslant 2$ and $q > \frac{2Np}{N+\alpha}$. Let $u \in L^2 \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing solution of (TF). Then, $u \in C^\infty$ inside its support.*

Proof. Assume first that $0 < \alpha < 2$. As in Lemma 3.3, denote by $B_{R_*} = \{u > 0\}$. Let $x \in B_{R_*}$ and $B_R(x)$ be a ball centred at x and radius R, such that $\overline{B_R(x)} \subset B_{R_*}$. By Lemma 3.2, we know that $I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^N)$ for every $\tau \in (0, \min\{\alpha, 1\})$. Then, as in the proof of Lemma 3.3,

$$u = f^{-1}(I_\alpha * u^p), \tag{3.25}$$

where f^{-1} is the inverse of f on $[\lambda_*, \infty)$. In particular, since in $\overline{B_R(x)}$, the function u is away from λ if $p > 2$ (respectively, from 0 if $p = 2$), then u (and so u^p) has the regularity of $I_\alpha * u^p$. Namely, $u^p \in C^{0,\tau}(\overline{B_R(x)})$ for every $\tau \in (0, \min\{\alpha, 1\})$. Then, [30, Theorem 1.1, Corollary 3.5] yields $I_\alpha * u^p \in C^{0,\tau+\alpha}(\overline{B_{R/2}(x)})$ for every $\tau \in (0, \min\{\alpha, 1\})$, provided that $\tau + \alpha$ is not an integer. Here, $C^{0,\tau+\alpha} := C^{\gamma',\gamma''}$ with $\tau + \alpha = \gamma' + \gamma''$ and γ' is the largest integer less or equal than $\tau + \alpha$. Hence, again, by (3.25), we conclude that u has the regularity of $I_\alpha * u^p$. By iterating the above argument, for every $k \in \mathbb{N}$, we can find $j \in \mathbb{N}$ such that $\tau + j\alpha$ is non-integer and bigger than k. This proves that $u \in C^k(\overline{B_{R/2^j}(x)})$. Since x was arbitrary, this implies that u is smooth inside its support.

If $2 \leqslant \alpha < N$, we can argue again similarly to the proof of [10, Theorem 10]. □

3.2 | Support estimates for $p > 2$

The following two statements follow from estimates in Proposition 2.1.

Corollary 3.3. *Let $p > 2$ and $q > 2p$. Let $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing ground state of (TF) and R_* be the radius of the support of u. Then $R_* \to +\infty$ as $\alpha \to 0$.*

Proof. By combining the Nehari identity (1.7) and Pohožaev identity $\mathcal{P}(u) = 0$, we get

$$\|u\|_2^2 = \frac{(N+\alpha)q - 2pN}{q(Np - N - \alpha)} \|u\|_q^q. \tag{3.26}$$

As a result, the minimal energy can also be represented using $\|u\|_q$, that is,

$$\sigma_* = \inf_{v \in \mathscr{P}} E(v) = E(u) = \frac{\alpha(q-2)}{2(Np - N - \alpha)q} \|u\|_q^q. \tag{3.27}$$

In what follows we will write $\sigma_* = \sigma_*(\alpha)$ stressing the dependence on α. Moreover, by the monotonicity of $u(|x|)$ in $|x|$,

$$\|u\|_q^q < u^q(0)|B_{R_*}|. \tag{3.28}$$

Evaluating the governing equation (*TF*) at 0, we estimate

$$u^{q-p}(0) \leqslant u^{2-p}(0) + u^{q-p}(0) = (I_\alpha * u^p)(0), \tag{3.29}$$

from which

$$u(0) \leqslant ((I_\alpha * u^p)(0))^{\frac{1}{q-p}} = \left(A_\alpha \int_{B_{R_*}} u^p(y)|y|^{\alpha - N} dy \right)^{\frac{1}{q-p}} \leqslant u^{\frac{p}{q-p}}(0) \left(A_\alpha \omega_N \frac{R_*^\alpha}{\alpha} \right)^{\frac{1}{q-p}},$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. The previous inequality leads to the bound

$$u(0) \leqslant \left(A_\alpha \omega_N \frac{R_*^\alpha}{\alpha} \right)^{\frac{1}{q-2p}}.$$

This estimate, when combined with (3.28), turns the relation (3.27) into

$$|B_{R_*}| R_*^{\frac{\alpha q}{q-2p}} \geqslant \frac{2q(Np - N - \alpha)}{\alpha(q-2)} \left(\frac{A_\alpha \omega_N}{\alpha} \right)^{-\frac{q}{q-2p}} \sigma_*(\alpha). \tag{3.30}$$

Next, if we consider $\theta_* = \theta_*(\alpha)$ defined as in (2.10), we have that

$$\lim_{\alpha\to 0}\theta_*(\alpha) = \left(\frac{q(p-1)}{q-2p}\right)^{\frac{q-2p}{2(p-1)+q-2p}}\left(\frac{q-2p}{2q(p-1)}+\frac{1}{q}\right) = \overline{\theta_*}. \tag{3.31}$$

Furthermore, under our assumptions on p and q, we have $\lim_{\alpha\to 0} 2p\theta_*(\alpha) = 2p\overline{\theta_*} > 1$. Moreover, from (3.30), we deduce that

$$R_* \geqslant \omega_N^{-\frac{2(q-p)}{q(N+\alpha)-2Np}}\left(\frac{2q(Np-N-\alpha)}{(q-2)}\right)^{\frac{q-2p}{q(N+\alpha)-2Np}}\left(\frac{A_\alpha}{\alpha}\right)^{-\frac{q}{q(N+\alpha)-2Np}}\left(\frac{\sigma_*(\alpha)}{\alpha}\right)^{\frac{q-2p}{q(N+\alpha)-2Np}}. \tag{3.32}$$

Since $\lim_{\alpha\to 0}\alpha^{-1}A_\alpha = \omega_N^{-1}$, it remains to prove that $\lim_{\alpha\to 0}\alpha^{-1}\sigma_*(\alpha) = \infty$. To do so, from (2.13), (3.30) and Proposition 2.1, we infer

$$\lim_{\alpha\to 0}\frac{\sigma_*(\alpha)}{\alpha} = \frac{\overline{\theta_*}}{N}\lim_{\alpha\to 0}\left(\frac{N}{N+\alpha}\frac{2p\theta_*(\alpha)}{\mathscr{C}_{N,\alpha,p,q}}\right)^{\frac{N}{\alpha}} = \infty, \tag{3.33}$$

where in the last equality, we used that

$$\lim_{\alpha\to 0}\left(\frac{N}{N+\alpha}\frac{2p\theta_*(\alpha)}{\mathscr{C}_{N,\alpha,p,q}}\right) = 2p\overline{\theta_*} > 1.$$

This concludes the proof. □

Corollary 3.4. *Let $p > 2$. Let $u \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ be a non-negative radial non-increasing ground state of (TF) and R_* be the radius of the support of u. Then, $R_* \to 0$ as $\alpha \to N$.*

Proof. By combining (3.27), Lemma 3.3 with the monotonicity of u, we obtain that

$$|B_{R_*}| \leqslant \left(\frac{q-p}{p-2}\right)^{\frac{q}{q-2}}\frac{2(Np-N-\alpha)q}{\alpha(q-2)}\sigma_*(\alpha). \tag{3.34}$$

Furthermore, by combining Proposition 2.1 with (2.13), we conclude that $\lim_{\alpha\to N}\sigma_*(\alpha) = 0$. Thus, the conclusion follows in view of (3.34). □

3.3 | Gradient estimates for $p < 2$

In the rest of the section, we consider the case $\frac{N+\alpha}{N} < p < 2$. Recall that in this case, non-negative radial non-increasing solutions $u \in L^2(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ of (*TF*) are supported on $\mathbb{R}^N$. Our aim is to show that $\nabla u \in L^2(\mathbb{R}^N;\mathbb{R}^N)$.

Note that for $\alpha > 1$, the gradient $\nabla I_\alpha * u^p$ is well defined, while for $0 < \alpha \leqslant 1$, it becomes a singular integral and is defined via the Cauchy principal value, namely

$$\nabla(I_\alpha * u^p) = \begin{cases} (\nabla I_\alpha) * u^p, & \alpha > 1,\\ \displaystyle\int_{\mathbb{R}^N}\nabla|x-y|^{\alpha-N}(|u(y)|^p - |u(x)|^p)dy, & 0 < \alpha \leqslant 1, \end{cases} \tag{3.35}$$

cf. [10, eq. (1.2)]. Recall the following result from [10, Lemma 1].

Lemma 3.5. *Assume that* $u \geqslant 0$ *and* $u^p \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then,*

(i) *If* $0 < \alpha < N$, *then* $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$.
(ii) *If* $0 < \alpha \leqslant 1$ *and* $u^p \in C^{0,\gamma}(\mathbb{R}^N)$ *with* $\gamma \in (1-\alpha, 1)$, *or if* $1 < \alpha < N$, *then* $\nabla(I_\alpha * u^p) \in L^\infty(\mathbb{R}^N)$, *that is,* $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^N)$.

Using the estimates of Lemma 3.5, we first show that positive solutions of (*TF*) are globally Lipschitz.

Lemma 3.6. *Assume that* $\frac{N+\alpha}{N} < p < 2$ *and* $q > \frac{2Np}{N+\alpha}$. *Let* $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *be a positive radial non-increasing solution of* (TF). *Then,* $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^N)$ *and* $u \in W^{1,\infty}(\mathbb{R}^N)$.

Proof. By Lemma 2.2 and Lemma 3.2, we know that $\mathrm{Supp}(u) = \mathbb{R}^N$ and $u^p \in L^\infty(\mathbb{R}^N)$.

If $1 < \alpha < N$, we can apply Lemma 3.5 to conclude that $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^N)$. Next, since $\mathrm{Supp}(u) = \mathbb{R}^N$, u satisfies the equivalent governing equation

$$u^{2-p} + u^{q-p} = I_\alpha * u^p \quad \text{in } \mathbb{R}^N. \tag{3.36}$$

Finally, as we have already noticed in the proof of Lemma 3.2, the function $f(t) = t^{2-p} + t^{q-p}$ has a differentiable inverse on $(0, +\infty)$ under our assumptions on p and q. From (3.36), we conclude that $u \in W^{1,\infty}(\mathbb{R}^N)$.

If $0 < \alpha \leqslant 1$, then Lemma 3.2 yields that $I_\alpha * u^p \in C^{0,\tau}(\mathbb{R}^N)$ for every $\tau \in (0, \alpha)$. Thus, again, from (3.36), the differentiability of the inverse f^{-1} on $(0, +\infty)$ and the boundedness of u, we infer $u \in C^{0,\tau}(\mathbb{R}^N)$ for every $\tau \in (0, \alpha)$. In particular, if $1/2 < \alpha \leqslant 1$, we can ensure that $\tau > 1 - \alpha$, and hence, $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^N)$ by Lemma 3.5. Then, arguing as before, $u \in W^{1,\infty}(\mathbb{R}^N)$. For $0 < \alpha < 1/2$, on the other hand, we need to use bootstrapping argument. Let us fix $n \in \mathbb{N}$, $n \geqslant 2$ such that $\frac{1}{n+1} < \alpha < \frac{1}{n}$ and let us choose $\tau > 0$ small enough such that $\tau + n\alpha < 1$ (note that this is possible because of the definition of n). Then, we define $\tau_n := \tau + (n-1)\alpha$. Then, by Equation (3.36) together with the locally Lipschitz continuity of the inverse of f, we can apply [31, Proposition 2.8] n-times to conclude that $I_\alpha * u^p \in C^{0,\tau_{n+1}}(\mathbb{R}^N)$. By our choice of n, we have the two-sided inequality $1 - \alpha < \tau_{n+1} < 1$. Hence, by Equation (3.36) again, we deduce that u (and, in particular, u^p) belongs to $C^{0,\tau_{n+1}}(\mathbb{R}^N)$. To conclude, by Lemma 3.2, we conclude that $I_\alpha * u^p \in W^{1,\infty}(\mathbb{R}^N)$, and that u has the same regularity. Finally, if $\alpha = \frac{1}{2}$, it is sufficient to start the above iterations with $\tau - \epsilon$, for some $\epsilon > 0$ small enough such that $1 - \alpha < \tau_{n+1} - \epsilon < 1$. □

Next, we show that positive solutions are actually arbitrarily smooth.

Lemma 3.7. *Assume that* $\frac{N+\alpha}{N} < p < 2$ *and* $q > \frac{2Np}{N+\alpha}$. *Let* $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *be a positive radial non-increasing solution of* (TF). *Then,* $u \in C^\infty(\mathbb{R}^N)$.

Proof. This follows from Equation (3.36) as in the proof of Lemma 3.4. □

Next, we establish a gradient estimate on the non-negative solutions of (*TF*).

Lemma 3.8. *Assume that* $\frac{N+\alpha}{N} < p < 2$ *and* $q > \frac{2Np}{N+\alpha}$. *Let* $u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *be a non-negative radial non-increasing solution of* (TF). *Then,* $\nabla u \in L^1(\mathbb{R}^N)$. *In particular,* $u \in H^1(\mathbb{R}^N)$.

Proof. Assume first that $1 < \alpha < N$. From the expression in Equation (3.35), we deduce that

$$|\nabla(I_\alpha * u^p)(x)| \leqslant \int_{\mathbb{R}^N} |\nabla|x-y|^{\alpha-N}|u^p(y)dy$$
$$\leqslant (N-\alpha)A_\alpha \int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha+1}}dy = \frac{(N-\alpha)A_\alpha\|u\|_p^p}{|x|^{N-\alpha+1}} + o\left(\frac{1}{|x|^{N-\alpha+1}}\right), \quad (3.37)$$

for $|x|$ sufficiently large. Note also that the inverse f^{-1} is differentiable on $(0,+\infty)$ and

$$\left(f^{-1}\right)'(t) = t^{\frac{p-1}{2-p}} + o(t^{\frac{p-1}{2-p}}) \quad \text{as } t \to 0^+. \quad (3.38)$$

Hence, by using the chain rule in (3.36), Lemma 3.1, (3.37) and (3.38), we infer

$$|\nabla u(x)| = \left|\left(f^{-1}\right)'((I_\alpha * u^p)(x))\nabla(I_\alpha * u^p)(x)\right| \lesssim \frac{1}{|x|^{\frac{N-\alpha}{2-p}+1}} \quad \text{as } |x| \to +\infty. \quad (3.39)$$

Combining Lemma 3.6 with (3.39) yields $\nabla u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ which concludes the proof.

Assume now that $\alpha \in (0,1]$. From (3.35), arguing as in [7, Lemma 2.2], we have

$$\nabla(I_\alpha * u^p) \leqslant (N-\alpha)A_\alpha\left(\int_{|x-y|\leqslant 1} \frac{|u^p(y)-u^p(x)|}{|x-y|^{N-\alpha+1}} + \int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{N-\alpha+1}}\right). \quad (3.40)$$
$$= I_1 + I_2.$$

First we note that, since $\alpha \in (0,1]$, for every $\varepsilon \in (0,N)$, I_2 can be estimated as

$$\int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{N-\alpha+1}}dy \leqslant \int_{|x-y|>1} \frac{u^p(y)}{|x-y|^{N-\varepsilon}}dy$$
$$\leqslant \int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\varepsilon}}dy = \frac{\|u\|_p^p}{|x|^{N-\varepsilon}} + o\left(\frac{1}{|x|^{N-\varepsilon}}\right), \quad (3.41)$$

where for the last equality, we used the decay estimate (3.1) on u established in Lemma 3.1.

Let us fix $0 < \bar{\varepsilon} < \frac{(N-\alpha)(p-1)}{2-p}$. Since $\nabla(I_\alpha * u^p) \in L^\infty(\mathbb{R}^N)$, by applying the gradient operator to both sides of Equation (3.36), we deduce that

$$|\nabla u(x)| \lesssim \left|\left(f^{-1}\right)'((I_\alpha * u^p)(x))\right| \lesssim \frac{1}{|x|^{\frac{(N-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty, \quad (3.42)$$

where for the last inequality, we used Corollary 3.1 and Equation (3.38).

Next, we estimate I_1. By combining Equation (3.42) with Lemma 3.1, we conclude that

$$I_1 \lesssim \|\nabla u^p\|_{L^\infty(\overline{B_1(x)})} \int_{|x-y|\leqslant 1} \frac{dy}{|x-y|^{N-\alpha}}$$
$$\lesssim \|u^{p-1}\nabla u\|_{L^\infty(\overline{B_1(x)})} \lesssim \frac{1}{|x|^{\frac{2(N-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty. \quad (3.43)$$

Then, if $\frac{2(N-\alpha)(p-1)}{2-p} \leqslant N-\bar{\varepsilon}$, combining together Equation (3.41) with Equation (3.43) yields

$$\nabla(I_\alpha * u^p) \lesssim \frac{1}{|x|^{\frac{2(N-\alpha)(p-1)}{2-p}}} + \frac{1}{|x|^{N-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty. \tag{3.44}$$

On the other hand, if $\frac{2(N-\alpha)(p-1)}{2-p} > N-\bar{\varepsilon}$, by the same argument it, follows that

$$\nabla(I_\alpha * u^p) \lesssim \frac{1}{|x|^{N-\varepsilon}} \quad \text{as } |x| \to +\infty,$$

which, in turn, implies that

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{(N-\alpha)(p-1)}{2-p}+N-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty.$$

Then, by the choice of $\bar{\varepsilon}$, we conclude that $\nabla u \in L^1(\mathbb{R}^N)$. However, if Equation (3.44) holds, we can improve the inequality (3.42) to

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{3(N-\alpha)(p-1)}{2-p}}} \quad \text{as } |x| \to +\infty. \tag{3.45}$$

Then, we can iterate this argument, until we find the first positive integer k such that

$$\frac{2k(N-\alpha)(p-1)}{2-p} > N-\bar{\varepsilon}.$$

In this way, we obtain that

$$|\nabla u(x)| \lesssim \frac{1}{|x|^{\frac{(N-\alpha)(p-1)}{2-p}+N-\bar{\varepsilon}}} \quad \text{as } |x| \to +\infty,$$

which again implies $\nabla u \in L^1(\mathbb{R}^N)$ by the choice of $\bar{\varepsilon}$. □

4 | LIMIT PROFILES FOR THE CHOQUARD EQUATION

Throughout this section, we assume that $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q > \frac{2Np}{N+\alpha}$. As already highlighted in the Introduction, the rescaling $u(x) = \varepsilon^{-\frac{1}{q-2}} w\big(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\big)$ converts the Choquard problem (P_ε) into the equation

$$-\varepsilon^\nu \Delta u + u + |u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{4.1}$$

where we denoted $\nu := \frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}$. Notice that

(i) $\nu > 0$ if and only if $q < \frac{2(2p+\alpha)}{2+\alpha}$,
(ii) $\nu < 0$ if and only if $q > \frac{2(2p+\alpha)}{2+\alpha}$.

The energy that corresponds to the rescaled equation (4.1) is given by

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\varepsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p), \tag{4.2}$$

and its Pohožaev functional is defined by

$$\mathcal{P}_\varepsilon(u) = \frac{N-2}{2}\varepsilon^\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u|^p, |u|^p). \tag{4.3}$$

We note that

$$\mathcal{J}_\varepsilon(u) = \varepsilon^{\frac{q(N+\alpha)-2Np}{\alpha(q-2)}} \mathcal{I}_\varepsilon(w), \tag{4.4}$$

where $\mathcal{I}_\varepsilon(w)$ is the Choquard energy defined in (1.1). Following [28], we consider the rescaled minimisation problem

$$\sigma_\varepsilon := \inf_{u \in \mathscr{P}_\varepsilon} \mathcal{J}_\varepsilon(u), \tag{4.5}$$

where Pohožaev manifold $\mathscr{P}_\varepsilon$ is defined as $\mathscr{P}_\varepsilon = \left\{u \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), u \neq 0 \,:\, \mathcal{P}_\varepsilon(u) = 0\right\}$.

Given $\varepsilon > 0$, let $w_\varepsilon \in H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ be a positive, radial, monotonically decreasing ground state solution of (P_ε) (see Theorem 1.1). Define

$$u_\varepsilon(x) := \varepsilon^{-\frac{1}{q-2}} w_\varepsilon\left(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\right). \tag{4.6}$$

Then, $u_\varepsilon \in \mathscr{P}_\varepsilon$ and $\mathcal{J}_\varepsilon(u_\varepsilon) = \sigma_\varepsilon$, that is, u_ε is the minimiser of (4.5) and a positive ground state solution of the rescaled equation (4.1), see [28].

In this section, we shall prove Theorem 1.4, which states that u_ε converges as $\varepsilon \to +\infty$ and $\nu < 0$ (respectively as $\varepsilon \to 0$ and $\nu > 0$) to a non-negative radial non-increasing ground state solution $u_* \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ of the Thomas–Fermi equation (*TF*), constructed in Lemma 2.3 from a maximiser in Theorem 1.2. Recall that $E(u_*) = \sigma_*$, where

$$\sigma_* = \inf_{u \in \mathscr{P}} E(u) = \alpha(2Np)^{\frac{N}{\alpha}} \left(\frac{\theta_*}{N+\alpha}\right)^{\frac{N+\alpha}{\alpha}} \mathscr{C}_{N,\alpha,p,q}^{-\frac{N}{\alpha}}, \tag{4.7}$$

as described in (3.20). The essential step in our proof of convergence is to show that $\sigma_\varepsilon \to \sigma_*$.

In what follows we shall only consider the case $\varepsilon \to +\infty$ and $\nu < 0$, that is, $q > \frac{2(2p+\alpha)}{2+\alpha}$. The arguments in the case $\varepsilon \to 0$ and $\nu > 0$ are very similar.

Firstly, we study the easier case when the ground state solution $u_* \in D^1(\mathbb{R}^N)$, where for $N \geqslant 3$, we denote $D^1(\mathbb{R}^N) = \left\{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \,:\, \|\nabla u\|_2 < \infty\right\}$. In particular, $u_* \in D^1(\mathbb{R}^N)$ if $p < 2$, as proved in Lemma 3.8.

Lemma 4.1. *Assume* $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > \frac{2(2p+\alpha)}{2+\alpha}$. *If* $u_* \in D^1(\mathbb{R}^N)$, *then (1)* $\sigma_\varepsilon > \sigma_*$ *and (2)* $\sigma_\varepsilon \to \sigma_*$ *as* $\varepsilon \to +\infty$.

Proof. Let $u_\varepsilon \in \mathscr{P}_\varepsilon$ be the minimiser of the problem $\inf_{u\in\mathscr{P}_\varepsilon} \mathcal{J}_\varepsilon(u)$ such that $\mathcal{J}(u_\varepsilon) = \sigma_\varepsilon$. Then,

$$\mathcal{P}(u_\varepsilon) = \mathcal{P}_\varepsilon(u_\varepsilon) - \left(\frac{N-2}{2}\right)\varepsilon^\nu \|\nabla u_\varepsilon\|_2^2 < 0.$$

Let $w_{\varepsilon,t}(x) := u_\varepsilon\left(\frac{x}{t}\right)$, then we obtain

$$\mathcal{P}(w_{\varepsilon,t}) = \frac{Nt^N}{2}\|u_\varepsilon\|_2^2 + \frac{Nt^N}{q}\|u_\varepsilon\|_q^q - \frac{(N+\alpha)t^{N+\alpha}}{2p}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p), \tag{4.8}$$

and $\mathcal{P}(w_{\varepsilon,1}) = \mathcal{P}(u_\varepsilon) < 0$. On the other hand, the dependence of t of various terms in Equation (4.8) implies that $\mathcal{P}(w_{\varepsilon,t})$ is positive if $t > 0$ is small. Therefore, by the continuity of $t \mapsto \mathcal{P}(w_{\varepsilon,t})$, there exists $t_\varepsilon \in (0,1)$ such that $\mathcal{P}(w_{\varepsilon,t_\varepsilon}) = 0$ and hence $w_{\varepsilon,t_\varepsilon} \in \mathscr{P}$. Consequently,

$$\begin{aligned}\sigma_* \leqslant E(w_{\varepsilon,t_\varepsilon}) = \frac{\alpha t_\varepsilon^{N+\alpha}}{2Np}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p) &< \frac{\alpha}{2Np}\mathcal{D}_\alpha(|u_\varepsilon|^p, |u_\varepsilon|^p) + \frac{\varepsilon^\nu}{N}\|\nabla u_\varepsilon\|_2^2 \\ &= \mathcal{J}_\varepsilon(u_\varepsilon) = \sigma_\varepsilon,\end{aligned} \tag{4.9}$$

which proves the first part of the statement.

Now, let u_* be the ground state solution for (TF) obtained in Lemma 2.3. Then, by the assumption $u_* \in D^1(\mathbb{R}^N)$,

$$\mathcal{P}_\varepsilon(u_*) = \frac{(N-2)\varepsilon^\nu}{2}\|\nabla u_*\|_2^2 > 0. \tag{4.10}$$

Define the rescaled function $\omega_t(x) := u_*\left(\frac{x}{t}\right)$. Then, $\mathcal{P}_\varepsilon(\omega_t)$, expressed in term of u_* as

$$\mathcal{P}_\varepsilon(\omega_t) = \tfrac{N-2}{2}\varepsilon^\nu t^{N-2}\|\nabla u_*\|_2^2 + Nt^N\left(\frac{\|u_*\|_2^2}{2} + \frac{\|u_*\|_q^q}{q}\right) - \tfrac{N+\alpha}{2p}t^{N+\alpha}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p) \to -\infty,$$

as $t \to +\infty$. This implies the existence of $t_\varepsilon > 1$ such that $\mathcal{P}_\varepsilon(\omega_{t_\varepsilon}) = 0$. In particular, $t_\varepsilon \to 1$ because

$$1 < (t_\varepsilon)^\alpha \leqslant \frac{N\left(\frac{\|u_*\|_2^2}{2} + \frac{\|u_*\|_q^q}{q}\right) + \frac{N-2}{2}\varepsilon^\nu\|\nabla u_*\|_2^2}{\frac{N+\alpha}{2p}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p)} \to 1 \quad \text{as } \varepsilon \to +\infty.$$

Now, from $\varepsilon^\nu \to 0$ and $t_\varepsilon \to 1$ as $\varepsilon \to +\infty$, we conclude that

$$\begin{aligned}\sigma_\varepsilon &\leqslant \mathcal{J}_\varepsilon\left(u_*\left(\frac{x}{t_\varepsilon}\right)\right) \\ &= \frac{\varepsilon^\nu}{2}t_\varepsilon^{N-2}\|\nabla u_*\|_2^2 + \frac{t_\varepsilon^N}{2}\left(\|v_*\|_2^2 + \|u_*\|_q^q\right) - \frac{t_\varepsilon^{N+\alpha}}{2p}\mathcal{D}_\alpha(|u_*|^p, |u_*|^p) \to E(u_*) = \sigma_*.\end{aligned} \tag{4.11}$$

The assertion about the convergence $\sigma_\varepsilon \to \sigma_*$ now follows by combining (4.9) with (4.11). □

Next, we show that $\sigma_\varepsilon \to \sigma_*$ as $\varepsilon \to +\infty$ without assuming that $u_* \in D^1(\mathbb{R}^N)$. In fact, in the case $p \geqslant 2$, it is expected that $u_* \notin D^1(\mathbb{R}^N)$, as, for example, follows from Hölder estimates in Lemma 3.3.

Lemma 4.2. *Assume* $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > \frac{2(2p+\alpha)}{2+\alpha}$. *Then, there exists a sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *such that* $\varepsilon_k \to \infty$ *and* $0 < \sigma_{\varepsilon_k} - \sigma_* \to 0$, *as* $k \to \infty$.

Proof. If $p < 2$, then the assertion follows by combining Lemma 3.8 with Lemma 4.1.

Assume that $p \geqslant 2$. Again, if the ground state solution u_* of (*TF*) constructed in Lemma 2.3 belongs to $D^1(\mathbb{R}^N)$, we conclude by Lemma 4.1. If not (e.g. for $p > 2$), we argue as follows.

Firstly note that arguing as in (4.9) in the first part of the proof of Lemma 4.1, we conclude that $\sigma_\varepsilon \geqslant \sigma_*$. It remains then to prove that $\sigma_\varepsilon \to \sigma_*$ by constructing an sequence of approximate minimisers of $\mathcal{J}_\varepsilon$ from u_*, which is achieved by truncating u_* (to avoid the singularity near the boundary) on a length scale s depending on ε.

Given $s \geqslant 0$ small, we introduce the cut-off function $\eta_s \in C_c^\infty(\mathbb{R}^N)$ such that $\eta_s(x) = 1$ for $|x| \leqslant R_* - s$, $0 < \eta_s(x) < 1$ for $R_* - s < |x| \leqslant R_* - \frac{s}{2}$, $\eta_s(x) = 0$ for $|x| \geqslant R_* - \frac{s}{2}$. Furthermore, $|\eta_s'(x)| \leqslant \frac{4}{s}$ and $|\eta_s'(x)| \geqslant \frac{1}{2s}$ for $R_* - \frac{4s}{5} < |x| < R_* - \frac{3s}{5}$. Set

$$\psi_s(x) := \eta_s(x) u_*(x).$$

By the definition of η_s, since $u_* \in L^\infty(\mathbb{R}^N)$ and it is supported in B_{R_*}, for every $1 \leqslant r < \infty$, we have

$$\int_{\mathbb{R}^N} |\psi_s^p(x) - u_*^p(x)|^r = \int_{R_*-s\leqslant|x|\leqslant R_*} |u_*(x)|^{pr}(1-\eta_s^p(x))^r \leqslant \|u_*\|^{pr}_{L^\infty(\mathbb{R}^N)} |A_{R_*-s,R_*}| = \mathcal{O}(s), \quad (4.12)$$

where $|A_{R_*-s,R_*}|$ is the volume of $B_{R_*} \setminus \overline{B}_{R_*-s}$. Further, by combining the Hardy–Littlewood–Sobolev inequality with (4.12), we obtain

$$0 \leqslant \mathcal{D}_\alpha(|u_*|^p, |u_*|^p) - \mathcal{D}_\alpha(|\psi_s|^p, |\psi_s|^p) = \mathcal{D}_\alpha(|u_*|^p + |\psi_s|^p, |u_*|^p - |\psi_s|^p) \leqslant C\|u_*^p + \psi_s^p\|_{\frac{2N}{N+\alpha}} \|u_*^p - \psi_s^p\|_{\frac{2N}{N+\alpha}} = \mathcal{O}\left(s^{\frac{N+\alpha}{2N}}\right).$$

To summarise, the following holds:

$$\mathcal{D}_\alpha(|\psi_s|^p, |\psi_s|^p) = \mathcal{D}_\alpha(|u_*|^p, |u_*|^p) - \mathcal{O}(s^{\frac{N+\alpha}{2N}}), \quad (4.13)$$

$$\|\psi_s\|_q^q = \|u_*\|_q^q - \mathcal{O}(s), \quad (4.14)$$

$$\|\psi_s\|_2^2 = \|u_*\|_2^2 - \mathcal{O}(s), \quad (4.15)$$

and we recall (see Section 1.3) that here $\mathcal{O}(s)$ denotes a *non-negative* function such that $\mathcal{O}(s) \leqslant Cs$ for every $s > 0$ small enough and for a constant $C > 0$ independent of s.

Note that by Lemma 3.4, the function ψ_s is smooth and, since $u_* \notin D^1(\mathbb{R}^N)$, the quantity $\|\nabla\psi_s\|_2^2$ blow up as $s \to 0^+$. In particular, there exists a decreasing sequence $(s_k)_k$ converging to zero such that $\|\nabla\psi_{s_k}\|_2^2$ diverges monotonically to infinity. Hence, we can define a piecewise linear, monotonically increasing, continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $\lim_{s\to 0^+} f\left(\frac{1}{s}\right) = +\infty$ and

$$f\left(\frac{1}{s_k}\right) = \|\nabla\psi_{s_k}\|_2^2.$$

We are going to describe a way we can control the rate of blow up of $f\left(\frac{1}{s_k}\right)$ in terms of the quantities in (4.13)–(4.15).

In what follows the parameter s_k will be defined as a function of ε_k, so that $f(1/s_k) = \|\nabla\psi_{s_k}\|_2^2$ blows up at a slower rate than $\varepsilon_k^{-\nu}$, to ensure the convergence $\sigma_{\varepsilon_k} \to \sigma_*$. To do this, we set

$$s_k := \frac{1}{g(\varepsilon_k)}, \tag{4.16}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable function such that $\lim_{\varepsilon\to+\infty} g(\varepsilon) = +\infty$, to be chosen later. Then, for all sufficiently large k, we have

$$\|\nabla\psi_{\frac{1}{g(\varepsilon_k)}}\|_2^2 = f(g(\varepsilon_k)) \nearrow +\infty \quad \text{as } k \to +\infty.$$

For the sake of notation simplicity, we further denote

$$\psi_{\varepsilon_k^{-1}} := \psi_{\frac{1}{g(\varepsilon_k)}}.$$

Combining together (4.13), (4.14) with (4.15), we have that

$$\mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k^{-1}}) = \left(\frac{N-2}{2}\right)\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2 + \underbrace{\mathcal{P}(u_*)}_{=0} - \mathcal{O}\left(\frac{1}{g(\varepsilon_k)}\right) + \mathcal{O}\left(\left(\frac{1}{g(\varepsilon_k)}\right)^{\frac{N+\alpha}{2N}}\right), \tag{4.17}$$

We claim that $\mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k^{-1}}) > 0$ for a suitable choice of the function g when k is sufficiently large. Indeed, if g satisfies the condition

$$\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = +\infty, \tag{4.18}$$

from (4.17), we obtain that

$$\begin{aligned}\mathcal{P}_{\varepsilon_k}(\psi_{\varepsilon_k^{-1}}) &\geqslant \left(\frac{N-2}{2}\right)\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2 - \mathcal{O}\left(\frac{1}{g(\varepsilon_k)}\right)\\ &= \left(\frac{N-2}{2}\right)\varepsilon_k^{\nu} f(g(\varepsilon_k)) - \mathcal{O}\left(\frac{1}{g(\varepsilon_k)}\right) > 0,\end{aligned} \tag{4.19}$$

provided that k is large enough. Next, the equality

$$\mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\frac{x}{t}\right)\right) =$$

$$= \frac{N-2}{2}t^{N-2}\varepsilon_k^{\nu}\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2 + Nt^N\left(\frac{\|\psi_{\varepsilon_k^{-1}}\|_2^2}{2} + \frac{\|\psi_{\varepsilon_k^{-1}}\|_q^q}{q}\right) - \frac{N+\alpha}{2p}t^{N+\alpha}\mathcal{D}_\alpha(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p)$$

implies that

$$\lim_{t\to+\infty} \mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\frac{x}{t}\right)\right) = -\infty. \tag{4.20}$$

Thus, by combining (4.19) with (4.20), for every k sufficiently large, there exists $t_{\varepsilon_k} > 1$ such that

$$\mathcal{P}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\frac{x}{t_{\varepsilon_k}}\right)\right) = 0. \tag{4.21}$$

In particular, by using (4.13), (4.14) and (4.15), if g satisfies the second condition

$$\lim_{\varepsilon\to+\infty} \varepsilon^\nu f(g(\varepsilon)) = 0, \tag{4.22}$$

we then can conclude that $t_{\varepsilon_k} \to 1$, since

$$1 < (t_{\varepsilon_k})^\alpha \leqslant \frac{N\left(\frac{1}{2}\|\psi_{\varepsilon_k^{-1}}\|_2^2 + \frac{1}{q}\|\psi_{\varepsilon_k^{-1}}\|_q^q\right) + \frac{N-2}{2}\varepsilon_k^\nu\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2}{\frac{N+\alpha}{2p}\mathcal{D}_\alpha(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p)} \to 1 \quad \text{as } k\to+\infty. \tag{4.23}$$

To summarise, to deduce (4.19) and (4.23), we need to construct a function g that satisfies:

(i) $\lim_{\varepsilon\to+\infty} \varepsilon^\nu f(g(\varepsilon)) = 0$;
(ii) $\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^\nu f(g(\varepsilon)) = +\infty$.

The existence of such function g is guaranteed by Lemma B.1 in the Appendix. Moreover,

$$\begin{aligned}\sigma_{\varepsilon_k} &\leqslant \mathcal{J}_{\varepsilon_k}\left(\psi_{\varepsilon_k^{-1}}\left(\frac{x}{t_{\varepsilon_k}}\right)\right)\\ &= \frac{t_{\varepsilon_k}^{N-2}}{2}\varepsilon_k^\nu\|\nabla\psi_{\varepsilon_k^{-1}}\|_2^2 + t_{\varepsilon_k}^N\left(\frac{1}{2}\|\psi_{\varepsilon_k^{-1}}\|_2^2 + \frac{1}{q}\|\psi_{\varepsilon_k^{-1}}\|_q^q\right) - \frac{t_{\varepsilon_k}^{N+\alpha}}{2p}\mathcal{D}_\alpha(|\psi_{\varepsilon_k^{-1}}|^p, |\psi_{\varepsilon_k^{-1}}|^p).\end{aligned} \tag{4.24}$$

Finally, in view of (4.13), (4.14), (4.15), (4.22) and since $t_{\varepsilon_k} \to 1$, the right-hand side of (4.24) converges to σ_* as $k\to\infty$. This implies that $\sigma_{\varepsilon_k} \to \sigma_*$ as $k\to+\infty$. □

Once the convergence of σ_ε towards σ_* is proved, we can show that the term $\varepsilon^\nu\|\nabla u_\varepsilon\|_2^2$ also vanishes in the same limit.

Corollary 4.1. *Assume that $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q > \frac{2(2p+\alpha)}{2+\alpha}$. Then, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a sequence of ground states (u_{ε_k}) of (P_{ε_k}) such that $\varepsilon_k \to \infty$ and*

$$\varepsilon_k^\nu\|\nabla u_{\varepsilon_k}\|_2^2 \to 0 \quad \textit{as } k\to+\infty.$$

Proof. Arguing as in the first part of Lemma 4.1, there exists $t_\varepsilon \in (0,1)$ such that $u_\varepsilon\left(\frac{x}{t_\varepsilon}\right) \in \mathscr{P}$. Now, let us consider the sequence $(t_{\varepsilon_k})_k$, corresponding to the sequence $(\varepsilon_k)_k$ in Lemma 4.2. We first prove that, up to a subsequence, $t_{\varepsilon_k} \to 1$ as $k\to+\infty$. Since $(t_{\varepsilon_k})_k$ is bounded, up to a subsequence

$t_{\varepsilon_k} \to t_0 \in [0,1]$. Assume by contradiction that $t_0 < 1$. Then,

$$\begin{aligned}\sigma_* &\leqslant E\left(u_{\varepsilon_k}\left(\frac{x}{t_{\varepsilon_k}}\right)\right) = \frac{\alpha t_{\varepsilon_k}^{N+\alpha}}{2Np}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \\ &\leqslant t_{\varepsilon_k}^{N+\alpha}\mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) = t_{\varepsilon_k}^{N+\alpha}\sigma_{\varepsilon_k} \to t_0^{N+\alpha}\sigma_* < \sigma_*,\end{aligned} \tag{4.25}$$

a contradiction. Therefore, we have proved that $t_{\varepsilon_k} \to 1$, and furthermore,

$$\mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) = \frac{\varepsilon_k^\nu}{2}\|\nabla u_{\varepsilon_k}\|_2^2 + \frac{\alpha}{2Np}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \to \sigma_*.$$

In particular, $(\varepsilon_k^\nu\|\nabla u_{\varepsilon_k}\|_2^2)_k$, $\left(\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p)\right)_k$ are bounded sequences and the same holds for $\left(\|u_{\varepsilon_k}\|_2\right)_k$ and $\left(\|u_{\varepsilon_k}\|_q\right)_k$. Therefore, by combining the equation

$$\begin{aligned}0 &= \tfrac{N}{2}t_{\varepsilon_k}^N\|u_{\varepsilon_k}\|_2^2 + \tfrac{N}{q}t_{\varepsilon_k}^N\|u_{\varepsilon_k}\|_q^q - \tfrac{N+\alpha}{2p}t_{\varepsilon_k}^{N+\alpha}\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p) \\ &= -\tfrac{N-2}{2}\varepsilon^\nu\|\nabla u_{\varepsilon_k}\|_2^2 + N(t_{\varepsilon_k}^N - 1)\left(\frac{\|u_{\varepsilon_k}\|_2^2}{2} + \frac{\|u_{\varepsilon_k}\|_q^q}{q}\right) - \tfrac{(N+\alpha)}{2p}(t_{\varepsilon_k}^{N+\alpha} - 1)\mathcal{D}_\alpha(|u_{\varepsilon_k}|^p, |u_{\varepsilon_k}|^p),\end{aligned}$$

with boundedness of the above sequences and $t_{\varepsilon_k} \to 1$, we obtain $\lim_{k\to+\infty} \varepsilon_k^\nu\|\nabla u_{\varepsilon_k}\|_2^2 = 0$. ☐

Proof of Theorem 1.4. By Lemma 4.2 and Corollary 4.1, there exists a sequence $(\varepsilon_k)_k$ such that $(u_{\varepsilon_k}(x/t_{\varepsilon_k})) \subset \mathscr{P}$ is a bounded radially non-increasing minimising sequence for the functional E. Then, if we set $v_{\varepsilon_k}(x) := u_{\varepsilon_k}(x/t_{\varepsilon_k})$, by arguing as in the proof of Lemma 2.1, there exists $\bar{v} \in L^2 \cap L^q(\mathbb{R}^N)$ such that

$$\begin{aligned}v_{\varepsilon_k} &\to \bar{v} \quad \text{in } L^s(\mathbb{R}^N),\ \forall s \in (2,q), \\ v_{\varepsilon_k} &\to \bar{v} \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

We claim that $\bar{v} \in \mathscr{P}$. Indeed, assume by contradiction that $\mathcal{P}(\bar{v}) \neq 0$. Since $(v_{\varepsilon_k})_k$ is a minimising sequence for σ_*, by the non-local Brezis–Lieb Lemma [29, Proposition 4.7], we derive

$$\mathcal{D}_\alpha(v_{\varepsilon_k}^p, v_{\varepsilon_k}^p) \to \mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) = \frac{2Np}{\alpha}\sigma_*, \tag{4.26}$$

and by the weak lower semi-continuity of the norms, we have $\mathcal{P}(\bar{v}) < 0$. Furthermore, it is easy to see that there exists $t_0 \in (0,1)$ such that $\bar{v}(x/t_0) \in \mathscr{P}$. However, this implies that

$$\sigma_* \leqslant E(\bar{v}(x/t_0)) = \frac{\alpha}{2Np}\mathcal{D}_\alpha(\bar{v}(x/t_0)^p, \bar{v}(x/t_0)^p) = \frac{t_0^{N+\alpha}\alpha}{2Np}\mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) < \frac{\alpha}{2Np}\mathcal{D}_\alpha(\bar{v}^p, \bar{v}^p) = \sigma_*,$$

that is a contradiction. Hence, $\bar{v} \in \mathscr{P}$.

Consequently, combining the standard Brezis–Lieb lemma with (4.26) yields

$$\sigma_* = \lim_{k\to+\infty} E(v_{\varepsilon_k}) = E(\bar{v}) + \lim_{k\to+\infty}\left(\frac{\|v_{\varepsilon_k} - \bar{v}\|_2^2}{2} + \frac{\|v_{\varepsilon_k} - \bar{v}\|_q^q}{q}\right) \geqslant \sigma_*.$$

This proves that $E(\bar{v}) = \sigma_*$ and $(v_{\varepsilon_k})_k$ converges to $\bar{v}$ in $L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, that is, $\bar{v}$ is a ground state solution of (*TF*). Finally, from $t_{\varepsilon_k} \to 1$, we further conclude that $(u_{\varepsilon_k})_k$ converges to $\bar{v}$ as well, and $\bar{v}$ solves (*TF*). □

5 | NUMERICAL APPROXIMATIONS OF THE OPTIMISERS

In this section, numerical approximations of the optimisers will be briefly discussed, complementing the theoretical results presented above. In general, it is difficult to find solutions to non-linear equations directly, while ground states as the limits of associated evolution equations are much easier to deal with. For instance, one might look for ground states to (*TF*) by looking at solutions of the parabolic flow

$$u_t = (I_\alpha * |u|^p)|u|^{p-2}u - u - |u|^{q-2}u,$$

or gradient (ascent) flow associated with the Rayleigh quotient $\mathcal{R}_\alpha(u)$. However, the corresponding solutions usually become singular (concentrating at one point) or zero (and spreading on the whole space). Instead, we consider the alternative equation

$$u_t = \lambda(t)(I_\alpha * |u|^p)|u|^{p-2}u - u - |u|^{q-2}u, \tag{5.1}$$

where the weight $\lambda(t)$ is chosen to make sure that the total interaction energy $\mathcal{D}_\alpha(|u|^p, |u|^p)$ is conserved. In other words, the condition that $0 = \frac{d}{dt}\mathcal{D}_\alpha(|u|^p, |u|^p)$ implies the choice

$$\lambda(t) = \left[\int (I_\alpha * |u|^p)^2 |u|^{2p-2} dx\right]^{-1} \int I_\alpha * |u|^p (|u|^p + |u|^{p+q-2}) dx.$$

Therefore, with fixed (and conserved) $\mathcal{D}_\alpha(|u|^p, |u|^p)$, at t goes to infinity, the corresponding solution becomes stationary and $\lambda(t)$ is expected to converge to a constant $\lambda(\infty)$. This stationary solution can then be normalised to a solution of (*TF*).

As usual, the main computational bottleneck in solving Equation (5.1) lies in the evaluation of the Riesz potential $I_\alpha * \rho$ for some function ρ. When $\rho(x) = \rho(|x|)$ is assumed to be radially symmetric, so is $I_\alpha * \rho(x)$, and

$$\begin{aligned} I_\alpha * \rho(|x|) &= A_\alpha \int_{\mathbb{R}^N} |x-y|^{\alpha-N} \rho(y) dy \\ &= A_\alpha (N-1)\omega_{N-1} \int_0^\infty s^{N-1}\rho(s) \int_0^\pi (r^2 + s^2 - 2rs\cos\theta)^{(\alpha-N)/2} \sin^{N-2}\theta d\theta ds, \end{aligned}$$

where $r = |x|$ and ω_N is the surface area of the unit ball in $\mathbb{R}^N$. In terms of the Gauss hypergeometric function, the previous double integral can be simplified, so that $I_\alpha * \rho(x)$ becomes

$$\frac{2^{1-\alpha}\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N}{2})} \int_0^\infty s^{N-1}\rho(s)(r^2+s^2)^{(\alpha-N)/2} {}_2F_1\left(\frac{N-\alpha}{4}, \frac{N-\alpha}{4} + \frac{1}{2}; \frac{N}{2}; \frac{4r^2s^2}{(r^2+s^2)^2}\right) ds. \tag{5.2}$$

At any $r = |x|$, the evaluation of this Riesz potential is essentially reduced to a numerical quadrature, although the integral in (5.2) becomes singular around $r = s$ for $\alpha \in (0, 1]$ (but still integrable for smooth function ρ). In three dimension, the Riesz potential can be simplified using the fact that

$$\int_0^\pi (r^2 + s^2 - 2rs\cos\theta)^{(\alpha-3)/2}\sin\theta d\theta = \begin{cases} \frac{(r+s)^{\alpha-1} - |r-s|^{\alpha-1}}{rs(\alpha-1)}, & \alpha \neq 1, \\ \frac{\ln(r+s) - \ln|r-s|}{rs}, & \alpha = 1. \end{cases}$$

To summarise, $I_\alpha * \rho(|x|)$ at $r = |x|$ can be written as

$$\int_0^\infty s^{N-1}\rho(s)k_{\alpha,N}(r,s)ds \tag{5.3}$$

for some kernel $K_{\alpha,N}$. All the examples shown in this paper are in one or three dimensions only, while the computation in general dimension using (5.3) is less accurate, because of the double integral involved and the evaluation of ${}_2F_1$ in the integrand. The numerical quadrature is performed using linear interpolation of ρ. That is, given $\rho(r_i)$ at the discrete points $\{r_i\}_{i=0}^\infty$, the integral in (5.3) is approximated by

$$\sum_{i=0}^\infty \int_{s_i}^{s_{i+1}} s^{N-1}k_{\alpha,N}(r,s)\left[\frac{s_{i+1}-s}{s_{i+1}-s_i}\rho(s_i) + \frac{s-s_i}{s_{i+1}-s_i}\rho(s_{i+1})\right]ds,$$

while the resulting integrals can usually be evaluated exactly, regardless of the singularity in the kernel $k_{\alpha,N}$.

If the optimiser is supported on a finite domain (i.e. $p \geqslant 2$), the numerical quadrature can be performed using standard techniques. When the computational domain is large enough, the support of the solution will be selected automatically by the evolution equation (5.1). When the optimiser is supported on the whole space (i.e. $p < 2$), the solution is expressed on a non-uniform mesh, and the expected algebraic decay rate $O(|x|^{-(N-\alpha)/(2-p)})$ of the optimisers proved in Corollary 3.1 can be used as a boundary condition at infinity, to improve the accuracy of the integral in (5.3) by taking into the contribution for $s \in [L,\infty)$ with a computational domain $[0, L]$.

The optimiser for $p = 4 > 2$, $\alpha = 2.5$ and $q = 8$ in three dimension is shown in Figure 3, which is discontinuous on the boundary of the support.

For $p = 1.5$, $\alpha = 1$ and $q = 2.5$ in three dimensions, the optimiser is plotted in Figure 4. This solution coincides with the explicit family with parameters specified in Equation (1.12), with the expected algebraic decay rate $O\big(|x|^{-(N-\alpha)/(2-p)}\big)$.

6 | CONCLUSION AND OPEN QUESTIONS

In this paper, a special class of Thomas–Fermi-type problem was studied, which appears as a limit regime for the Choquard equations with local repulsion. The associated governing equation (TF) was shown to have ground state solutions that correspond to sharp optimisers of a quotient involving the interaction energy and classical Banach norms. Regularity properties and other qualitative information of solutions to this governing equations were investigated. The convergence of ground states of the original Choquard equation in the relevant regimes to a ground state of Thomas–Fermi equation was also proved.

Below we list several open questions related to the results in the present work.

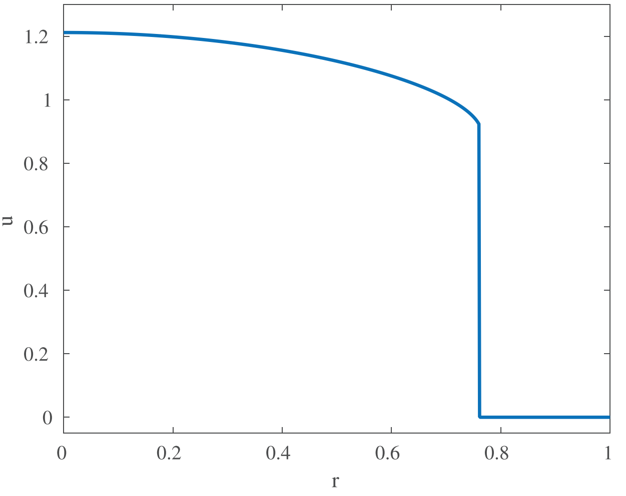

FIGURE 3 The optimiser for $p = 4, \alpha = 2.5$ and $q = 8$ in three dimensions, which is expected to be compactly supported.

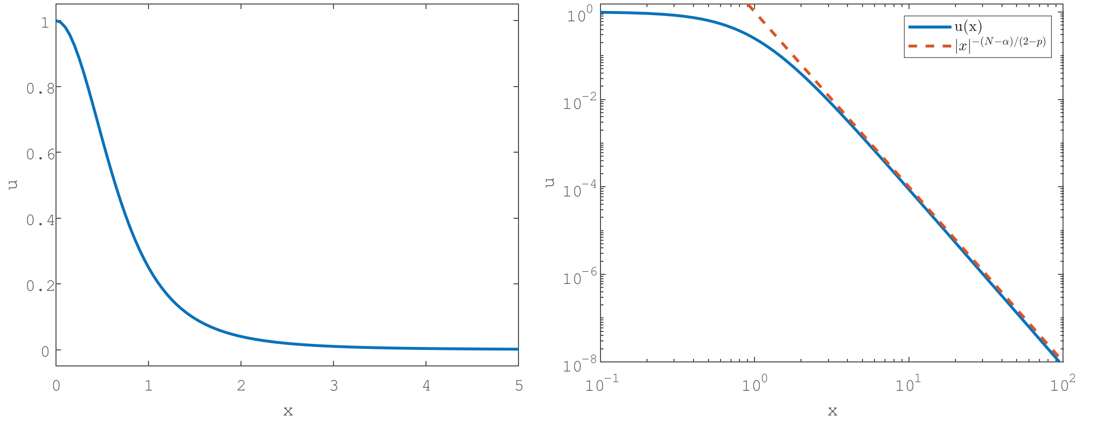

FIGURE 4 The ground state or optimiser for $p = 1.5 < 2, \alpha = 1$ and $q = 2.5$ in three dimensions, both near the origin or away from the origin.

6.1 | Uniqueness

The uniqueness of the ground state of (TF) beyond the case $p = 2$ and $\alpha \leqslant 2$ studied in [8, 11, 12, 20] seems to be open at present. If the uniqueness in the case $p < 2$ is known, this would imply that (1.14) provides a sharp constant rather than a lower bound, see Remark 1.2.

6.2 | Sharp regularity

Numerical experiments suggests that for $p > 2$, the jump near the boundary λ is always strictly larger than $\lambda_* = (\frac{p-2}{q-p})^{1/(q-2)}$, see Remark 3.1. If established analytically, this would also rule out the second option in the regularity estimate (3.15), implying that Hölder regularity of the ground state near the boundary of the support is always of order $\tau \in (0, \min\{\alpha, 1\})$.

6.3 | Limit profiles as $\alpha \to 0$ and $\alpha \to N$

The fact that the (normalised) sharp constant $\mathscr{C}_{N,\alpha,p,q}$ approaches 1 in the two limiting cases as α approaches 0 or N, see (1.10), suggests that after a rescaling ground states of (TF) converge to characteristic functions over a ball. However, numerical experiments indicate that the limits become singular because ground states becomes degenerate or singular. It would be interesting to investigate (singular) limits of the ground states of (TF) as $\alpha \to 0$ and $\alpha \to N$ analytically.

APPENDIX A: ESTIMATE ON $\mathscr{C}_{N,\alpha,p,q}$ FOR $p = \frac{N+\alpha+2}{N+1}$ AND $q = \frac{2(N+2)}{N+1}$

Here, we evaluate the estimate (1.13) on $\mathscr{C}_{N,\alpha,p,q}$ associated with the one-parameter solutions of (TF) discussed in Remark 1.1. First using the surface area $2\pi^{N/2}/\Gamma(N/2)$ of the unit sphere in $\mathbb{R}^N$, we have for $d > N/2$ the following special integral:

$$\int_{\mathbb{R}^N} (1+|x|^2)^{-d} dx = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty r^{N-1}(1+r^2)^{-d} dr = \frac{\pi^{N/2}\Gamma(d-N/2)}{\Gamma(d)}.$$

This implies that for the solution $v(x) = (1+|x|^2)^{-(N+2)/2}$ with $p = (N+\alpha+2)/(N+1)$ and $q = 2(N+2)/(N+1)$,

$$\|v\|_2^2 = \frac{\pi^{N/2}\Gamma(N/2+1)}{\Gamma(N+1)}, \qquad \|v\|_q^q = \frac{\pi^{N/2}\Gamma(N/2+2)}{\Gamma(N+2)}.$$

The Riesz potential $I_\alpha * v^p$ can also be evaluated [18, 23] as

$$\begin{aligned} I_\alpha * v^p(x) &= 2^{-\alpha}\frac{\Gamma(N/2+1)\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha+2)/2)\Gamma(N/2)}\, {}_2F_1\left(\frac{N}{2}+1, \frac{N-\alpha}{2}; \frac{N}{2}; -|x|^2\right) \\ &= 2^{-1-\alpha}\frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha+2)/2)}(1+|x|^2)^{-1-N/2+\alpha/2}(N+\alpha|x|^2), \end{aligned}$$

and consequently, the interaction energy is

$$\begin{aligned} \mathcal{D}_\alpha(v^p, v^p) &= 2^{-1-\alpha}\frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha+2)/2)} \int_{\mathbb{R}^N} (1+|x|^2)^{-N-2}(N+\alpha|x|^2) dx \\ &= \pi^{N/2}\frac{N(N+\alpha+2)}{2^{\alpha+2}} \frac{\Gamma((N-\alpha)/2)\Gamma(N/2+1)}{\Gamma((N+\alpha)/2+1)\Gamma(N+2)}. \end{aligned}$$

Putting all these together, with $\theta = \frac{2\alpha(N+1)}{N(N+\alpha+2)}$, we obtain

$$\begin{aligned} \mathscr{C}_{N,\alpha,p,q} \geqslant \mathcal{R}_\alpha(v) &= \frac{\mathcal{D}_\alpha(v^p, v^p)}{\|v\|_2^{2p\theta}\|v\|_q^{2p(1-\theta)}} \\ &= \frac{N(N+\alpha+2)}{\pi^{\alpha/2}2^{\alpha+1}(N+2)} \frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2+1)} \left(\frac{N+2}{2(N+1)}\frac{\Gamma(N+1)}{\Gamma(N/2+1)}\right)^{\alpha/N}, \end{aligned}$$

which establishes (1.13).

APPENDIX B: A CALCULUS LEMMA

Here, we prove a technical calculus lemma that was used in the proof of Lemma 4.2.

Lemma B.1. *For every $\nu < 0$, and for every continuous and strictly monotone increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $\lim_{\varepsilon\to+\infty} f(\varepsilon) = +\infty$, there exists a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\lim_{\varepsilon\to+\infty} g(\varepsilon) = +\infty, \tag{B1}$$

$$\lim_{\varepsilon\to+\infty} \varepsilon^{\nu} f(g(\varepsilon)) = 0, \tag{B2}$$

$$\lim_{\varepsilon\to+\infty} g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = +\infty. \tag{B3}$$

Proof. Let H be the function defined by

$$H(\varepsilon) = \min\left\{\log(\varepsilon), \sqrt{f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)}\right\}, \quad \text{for } \varepsilon > e^{-\frac{1}{\nu}}, \tag{B4}$$

where $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of f. Clearly,

$$\lim_{\varepsilon\to+\infty} H(\varepsilon) = +\infty. \tag{B5}$$

Note that such H is continuous and monotone increasing since both $\sqrt{f^{-1}(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)})}$ and $\log(\varepsilon)$ are continuous and monotone increasing functions for $\varepsilon > e^{-\frac{1}{\nu}}$. Hence, we define g as follows:

$$g(\varepsilon) := f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right) \quad \text{for } \varepsilon > e^{-\frac{1}{\nu}}, \tag{B6}$$

and we extend to a continuous non-negative function defined on $\mathbb{R}_+$. Note that from (B4), monotonicity and unboundedness of f^{-1}, we have

$$g(\varepsilon) \geqslant f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right) \longrightarrow +\infty \quad \text{as } \varepsilon \to +\infty.$$

Hence, (B1) holds. Furthermore, from (B5), we obtain

$$\lim_{\varepsilon\to+\infty} \varepsilon^{\nu} f(g(\varepsilon)) = \lim_{\varepsilon\to+\infty} \frac{1}{H(\varepsilon)} = 0,$$

which proves (B2). Finally, again from (B4) and (B6), it holds

$$g(\varepsilon)\varepsilon^{\nu} f(g(\varepsilon)) = \frac{1}{H(\varepsilon)} \cdot f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right) \geqslant \left(f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)\right)^{-\frac{1}{2}} f^{-1}\left(\frac{\varepsilon^{-\nu}}{H(\varepsilon)}\right)$$
$$\geqslant \sqrt{f^{-1}\left(\frac{\varepsilon^{-\nu}}{\log(\varepsilon)}\right)} \longrightarrow +\infty, \quad \text{as } \varepsilon \to +\infty,$$

which proves (B3). □

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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