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Generalised Weyl algebras and Segal–Bargmann transform for the Meixner class of orthogonal polynomials

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Submitted to Swansea University in fulfillment of the requirements for the Degree of Ph.D.

November 2024

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Date............................12 November 2024..

Acknowledgements

This dissertation could not be completed without the support of my supervisor, Prof. Eugene Lytvynov. I would especially like to sincerely thank my supervisor for his kindness, guidance and help throughout this dissertation period. He always encourages me. His suggestion is not only about research but also about my study life. Thanks for his goodwill.

I also would like to thank Dr. Dmitri Frankeinstein for his assistance when I was working as a teaching assistant. Additionally, I thank him for being the secondary supervisor.

Essentially, this research could not be done if I did not get financial support from the Doctoral Training Program (DTP), EPSRC, UKRI which co-operated with Faculty of Science and Engineering, Swansea University, the project reference 2602423, related to EP/T517987/1.

Finally, I would like to deeply thank my family (mommy, KhunTa, uncle Pom) for their love and giving motivation throughout my life. They always activate and encourage me in whatever way I choose. Also, I would like to thank myself dedicating for this research work. Last but not least, I would like to thank Swansea University for all experiences that I have gained here.

Chadaphorn Kodsueb (Punkumkerd)

Tuesday 12$^{\text{th}}$ November, 2024

Abstract

Meixner (1934) proved that there exist exactly five classes of orthogonal Sheffer sequences: Hermite polynomials that are orthogonal with respect to the Gaussian distribution, Charlier polynomials orthogonal with respect to Poisson distribution, Laguerre polynomials orthogonal with respect to gamma distribution, Meixner polynomials of the first kind orthogonal with respect to negative binomial (Pascal) distribution, and Meixner polynomials of the second kind (or Meixner–Pollaczak polynomials) orthogonal with respect to Meixner distribution. Bargmann (1961) constructed a Hilbert space of entire functions on the complex plane, called nowadays Fock or Segal–Bargmann space. In this space, the creation and annihilation operators act as multiplication by the variable and differentiation, respectively. These operators generate a Weyl algebra. The Segal–Bargmann transform provides a unitary isomorphism between the L^2-space of the Gaussian distribution and the Fock space. This construction was later extended to the case of the Poisson distribution. The present dissertation deals with the latter three sets of orthogonal Sheffer sequences: Laguerre and Meixner of both the first and the second kind. We discuss generalised Weyl algebras that are naturally associated with these polynomials. By using a set of nonlinear coherent states, we construct a generalised Segal–Bargmann transform which is a unitary isomorphism between the L^2-space of the orthogonality measure and a certain Fock space of entire functions on the complex plane. In a special case, such a Fock space was already studied by Alpay–Jørgensen–Seager–Volok (2013) and Alpay–Porat (2018).

Contents

Chapter 1

Introduction

Fock spaces play a fundamental role in quantum mechanics as well as in infinite-dimensional analysis and probability, both classical and noncommutative (quantum), see e.g. [47, Chapter II] and [45, Chapter IV]. Roughly speaking, a Fock space is an infinite orthogonal sum of symmetric or antisymmetric n-particle Hilbert spaces. There exists an alternative description of a symmetric Fock space as a space of holomorphic functions. Such a space is usually called the Segal–Bargmann space.

Let us briefly discuss the Segal–Bargmann construction in the one-dimensional case. In 1961, Bargmann [10] defined a Hilbert space $\mathbb{F}(\mathbb{C})$ as the closure of polynomials on $\mathbb{C}$ in the L^2-space $L^2(\mathbb{C},\nu)$, where ν is the Gaussian measure on $\mathbb{C}$ given by $\nu(dz) = \pi^{-1}\exp(-|z|^2)\,dA(z)$, $dA(z)$ being the Lebesgue measure on $\mathbb{C}$. The monomials $(z^n)_{n=0}^\infty$ form an orthogonal basis for $\mathbb{F}(\mathbb{C})$ with $(z^n, z^m)_{\mathbb{F}(\mathbb{C})} = n!\,\delta_{n,m}$. The $\mathbb{F}(\mathbb{C})$ consists of entire functions $\varphi(z) = \sum_{n=0}^\infty \varphi_n\, z^n$ that satisfy $\sum_{n=0}^\infty |\varphi_n|^2\, n! < \infty$. The $\mathbb{F}(\mathbb{C})$ is a reproducing kernel Hilbert space with reproducing kernel $\mathbb{K}(z,w) = \sum_{n=0}^\infty (n!)^{-1}\,(\bar{z}w)^n$.

Next, let μ be the standard Gaussian distribution on $\mathbb{R}$ and let $(h_n)_{n=0}^\infty$ be the sequence of monic Hermite polynomials that form an orthogonal basis for $L^2(\mathbb{R},\mu)$. The Segal–Bargmann transform is the unitary operator $\mathbb{S} : L^2(\mathbb{R},\mu) \to \mathbb{F}(\mathbb{C})$ that satisfies $(\mathbb{S}\, h_n)(z) = z^n$. This operator has a representation through the coherent states:

$$\mathbb{E}(\xi, z) = \sum_{n=0}^\infty \frac{1}{n!}\, z^n\, h_n(\xi) = \exp\left(-\frac{1}{2}(z^2 - 2\xi z)\right), \quad \xi \in \mathbb{R},\ z \in \mathbb{C}.$$

More precisely, for $f \in L^2(\mathbb{R}, \mu)$, one has

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(\xi)\, \mathbb{E}(\xi, z)\ \mu(d\xi), \quad z \in \mathbb{C}.$$

The coherent states $\mathbb{E}(\xi, z)$ are eigenfunctions in the ξ variable of the lowering operator in $L^2(\mathbb{R}, \mu)$ with eigenvalue z. More exactly, if we define an (unbounded) operator ∂^- in $L^2(\mathbb{R}, \mu)$ by $\partial^- h_n = n h_{n-1}$, then $\partial^-\, \mathbb{E}(\cdot, z) = z\, \mathbb{E}(\cdot, z)$. For real z, the $\mathbb{S}$-transform of a function $f \in L^2(\mathbb{R}, \mu)$ can also be written in the form

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(z + \xi)\, \mu(d\xi).$$

Let also ∂^+ denote the raising operator for the Hermite polynomials: $\partial^+ h_n = h_{n+1}$. Then, the operator of multiplication by the variable in the L^2-space $L^2(\mathbb{R}, \mu)$ has the form $\partial^+ + \partial^-$. Hence, under the Segal–Bargmann transforms $\mathbb{S}$, this operator goes over into the operator $Z + D$, where Z is the multiplication by the variable z in $\mathbb{F}(\mathbb{C})$, and D is the differentiation in $\mathbb{F}(\mathbb{C})$. In this setting, the operators Z and D are adjoint of each other. Note that these operators satisfy the commutation relation $[D, Z] = \mathbf{1}$, hence they are generators of a Weyl algebra, see e.g. [43, Chapter 5].

The above results admit an immediate extension to the case where μ_σ is the Gaussian measure on $\mathbb{R}$ with mean 0 and variance $\sigma > 0$, cf. [9]. The corresponding Segal–Bargmann space $\mathbb{F}_\sigma(\mathbb{C})$ is defined as the closure of polynomials in $L^2(\mathbb{C}, \nu_\sigma)$, where ν_σ is the Gaussian measure on $\mathbb{C}$ given by

$$\nu_\sigma(dz) = \frac{1}{\pi\sigma}\, \exp\left(-\frac{|z|^2}{\sigma} \right)\, dA(z). \tag{1.1}$$

The monomials $(z^n)_{n=0}^\infty$ form an orthogonal basis for $\mathbb{F}_\sigma(\mathbb{C})$ with

$$(z^n, z^m)_{\mathbb{F}_\sigma(\mathbb{C})} = \delta_{n,m}\, n!\, \sigma^n. \tag{1.2}$$

The $\mathbb{F}_\sigma(\mathbb{C})$ consists of entire functions $\varphi(z) = \sum_{n=0}^\infty \varphi_n\, z^n$ that satisfy $\sum_{n=0}^\infty |\varphi_n|^2\, n!\, \sigma^n < \infty$. Then $\mathbb{F}_\sigma(\mathbb{C})$ is a reproducing kernel Hilbert space with reproducing kernel $\mathbb{K}_\sigma(z, w) = \sum_{n=0}^\infty (n!\, \sigma^n)^{-1}\, (\bar{z}w)^n$. The corresponding Segal–Bargmann transform $\mathbb{S} : L^2(\mathbb{R}, \mu_\sigma) \to \mathbb{F}_\sigma(\mathbb{C})$ has the representation

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(\xi)\, \mathbb{E}_\sigma(\xi, z)\ \mu_\sigma(dz), \quad z \in \mathbb{C},$$

where

$$\mathbb{E}_\sigma(\xi,z)=\sum_{n=0}^\infty \frac{1}{n!\,\sigma^n}\,z^n\,h_n(\xi)=\exp\left(-\frac{1}{2\sigma}(z^2-2\xi z)\right)$$

are the corresponding coherent states. Here $(h_n)_{n=0}^\infty$ is the sequence of monic Hermite polynomials that forms an orthogonal basis for $L^2(\mathbb{R},\mu_\sigma)$. One has $(\mathbb{S}h_n)(z)=z^n$. Under the Segal–Bargmann transform $\mathbb{S}$, the operator of multiplication by the variable goes over to the operator $Z+\sigma D$ in $\mathbb{F}_\sigma(\mathbb{C})$.

The Segal–Bargmann transform for the Gaussian measure admits both multivariate and infinite-dimensional extensions, see [10] and e.g. [46, Section 3.3].

Asai et al. [9] constructed a counterpart of the Segal–Bargmann transform in the case of the Poisson distribution with parameter $\sigma>0$: $\mu_\sigma(d\xi)=e^{-\sigma}\sum_{n=0}^\infty (n!)^{-1}\,\sigma^n\,\delta_n(d\xi)$. Let now $(c_n)_{n=0}^\infty$ be the sequence of monic Charlier polynomials on $\mathbb{R}$ that form an orthogonal basis for $L^2(\mathbb{R},\mu_\sigma)$. Then, the Segal–Bargmann transform is a unitary operator $\mathbb{S}:L^2(\mathbb{R},\mu_\sigma)\to\mathbb{F}_\sigma(\mathbb{C})$ with $(\mathbb{S}\,c_n)(z)=z^n$. The corresponding coherent states are $\mathbb{E}_\sigma(\xi,z)=e^{-z}\left(1+\frac{z}{\sigma}\right)^\xi$. One has the property $\sigma\partial^-\,\mathbb{E}_\sigma(\cdot,z)=z\,\mathbb{E}_\sigma(\cdot,z)$, where ∂^- is the lowering operator for the Charlier polynomials $(c_n)_{n=0}^\infty$. Note that $\sigma\partial^-$ is the adjoint of the raising operator ∂^+ for the polynomials $(c_n)_{n=0}^\infty$.

A key difference with the Gaussian case is that, under the transformation $\mathbb{S}$, the operator of multiplication by the variable goes over to the operator $\mathcal{U}\mathcal{V}$, where

$$\mathcal{U}=Z+\sigma,\quad \mathcal{V}=D+\mathbf{1}. \tag{1.3}$$

Note that the operators $\mathcal{U}$ and $\mathcal{V}$ satisfy the commutation relation $[\mathcal{V},\mathcal{U}]=\mathbf{1}$, hence $\mathcal{U}$ and $\mathcal{V}$ generate a Weyl algebra.

We refer the reader to Chapters 5 and 6 of Mansour and Schork's book [43] for an extensive review of various topics related to normal (Wick) ordering in the Weyl algebra. In particular, of crucial importance for us is the theorem of Katriel [38], which provides a normal ordering for the operator $(\mathcal{U}\mathcal{V})^n$ through the powers $\mathcal{U}^k$ and $\mathcal{V}^k$ $(k=1,\dots,n)$ and Stirling numbers of the second kind, $S(n,k)$.

Meixner [44] proved that there exist exactly five classes of orthogonal Sheffer sequences: Hermite polynomials that are orthogonal with respect to the Gaussian distribution, Charlier polynomials orthogonal with respect to Poisson distribution, Laguerre polynomials orthogonal with respect to gamma distribution, Meixner polynomials of the first kind orthogonal with respect to negative binomial (Pascal) distribution, and Meixner polynomials of the second kind (or Meixner–Pollaczak polynomials) orthogonal with respect to Meixner distribution. In fact, a monic polynomial sequence $(s_n)_{n=0}^{\infty}$ is an orthogonal Sheffer sequence if and only if it satisfies the recurrence relation

$$zs_n(z) = s_{n+1}(z) + (\lambda n + l)s_n(z) + \big(\sigma n + \eta n(n-1)\big)s_{n-1}(z), \tag{1.4}$$

where $\lambda \in \mathbb{R}$, $l \in \mathbb{R}$, $\sigma > 0$ and $\eta \geq 0$. The constant l is not essential as it corresponds to the shift of the orthogonality measure by l. It is also convenient to introduce parameters $\alpha, \beta \in \mathbb{C}$ that satisfy $\alpha + \beta = \lambda, \alpha\beta = \eta$.

In this dissertation, we will be dealing with the case $\eta > 0$ (equivalently both $\alpha \neq 0$ and $\beta \neq 0$), which corresponds to the latter three cases of orthogonal Sheffer sequences: Laguerre polynomials ($\alpha = \beta > 0$), Meixner polynomials of the first kind ($0 < \beta < \alpha$) and Meixner polynomials of the second kind ($\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$, $\overline{\alpha} = \beta$). In the first two cases ($0 < \beta \leq \alpha$), we choose $l = \frac{\sigma}{\alpha}$, and in the third case, we choose $l = 0$. We denote by $\mu_{\alpha,\beta,\sigma}$ the orthogonality measure for the polynomial sequence $(s_n(z))_{n=0}^{\infty}$.

A starting point for our research was the following observation. Consider operators $\mathcal{U}$ and $\mathcal{V}$ acting on polynomials of complex variable and given by

$$\mathcal{U} = Z + \frac{\sigma}{\alpha}, \quad \mathcal{V} = \alpha D_\beta + \mathbf{1}, \tag{1.5}$$

compare with (1.3). In formula (1.5), we use the so-called h-derivative (see e.g. [35, Chapter 1]): for $h \in \mathbb{C}$,

$$(D_h f)(z) = \frac{f(z+h) - f(z)}{h}.$$

The operators $\mathcal{U}$ and $\mathcal{V}$ satisfy the commutation relation

$$[\mathcal{U}, \mathcal{V}] = -\beta\mathcal{V} - (\alpha - \beta). \tag{1.6}$$

Hence, they generate a generalised Weyl algebra, see e.g. [43, Chapter 8] and the references therein.

We advise the reader to compare the generalised Weyl algebra generated by the operators $\mathcal{U}$ and $\mathcal{V}$ with Feinsilver's finite difference algebra [19].

Let $((z \mid \beta)_n)_{n=0}^{\infty}$ denote the generalised factorials of variable z with increment β [31], i.e., $(z \mid \beta)_0 = 1$ and

$$(z \mid \beta)_n = z(z-\beta)\cdots(z-(n-1)\beta), \quad n \geq 1.$$

The D_β is the lowering operator for these polynomials: $D_\beta(z \mid \beta)_n = n(z \mid \beta)_{n-1}$. Then, the operator $\rho := \mathcal{U}\mathcal{V}$ satisfies

$$\rho(z \mid \beta)_n = (z \mid \beta)_{n+1} + \left(\lambda n + \frac{\sigma}{\alpha}\right)(z \mid \beta)_n + (\sigma n + \eta n(n-1))(z \mid \beta)_{n-1}, \tag{1.7}$$

compare with formula (1.4).

Let us remark that orthogonal Sheffer polynomial sequences with $\eta > 0$ also appear in studies related to the square of white noise algebra, see e.g. [1] and the references therein. It was shown in [2] that the square of white noise algebra contains a subalgebra generated by elements fulfilling the relations of Feinsilver's finite difference algebra [19], see also [14] and [15].

In Chapter 3, in view of formula (1.7), similarly to Katriel's theorem, we discuss the normal ordering for the operator ρ^n in terms of $\mathcal{U}^k$ and $\mathcal{V}^k$, compare with [43, Chapter 8]. This result allows us to derive explicit combinatorial formulas for $s_n(z)$ and for a representation of monomials z^n through the polynomials $s_k(z)$. In these formulas, we use Stirling numbers and Lah's numbers. As a corollary, we find useful combinatorial formulas for the moments of the orthogonality measure $\mu_{\alpha,\beta,\sigma}$, see Corollaries 4.19, 4.43 and 4.61.

The main results of the dissertation are in Chapter 4. We first explicitly construct an open unbounded domain $\mathcal{D}_{\alpha,\beta,\sigma}$ in $\mathbb{C}$ that contains 0. We define a reproducing kernel Hilbert space $\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ of analytic function on $\mathcal{D}_{\alpha,\beta,\sigma}$ that have representation $\varphi(z) = \sum_{n=0}^{\infty} \varphi_n\,(z \mid \beta)_n$ with coefficients $\varphi_n \in \mathbb{C}$ that satisfy $\sum_{n=0}^{\infty} |\varphi_n|^2\, n!\,(\sigma \mid -\eta)_n < \infty$. We construct a unitary operator $\mathcal{S} : L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ that satisfies $(\mathcal{S}s_n)(z) = (z \mid \beta)_n$. The operator $\mathcal{S}$ admits a representation

$$(\mathcal{S}f)(z) = \int_{\mathbb{R}} f(\xi)\, \mathcal{E}_{\alpha,\beta,\sigma}(\xi, z)\ \mu_{\alpha,\beta,\sigma}(d\xi),$$

where

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{(z \mid \beta)_n}{n!\,(\sigma \mid -\eta)_n}\, s_n(\xi),$$

and we derive an explicit formula for $\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z)$. For example, in the third (Meixner) case, the domain $\mathcal{D}_{\alpha,\beta,\sigma}$ is given by formula (4.214) below and the function $\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z)$ is given by (4.227).

Furthermore, we properly extend the definition of the probability distribution $\mu_{\alpha,\beta,\sigma}$ to the definition of a complex-valued measure $\mu_{\alpha,\beta,\zeta}$ on $\mathbb{R}$ where ζ is a parameter from an open domain in the complex plane that contains $(0, +\infty)$. In particular, for each $n \in \mathbb{N}$,

$$\zeta \mapsto \int_{\mathbb{R}} \xi^n \, \mu_{\alpha,\beta,\zeta}(d\xi) \quad \in \mathbb{C}$$

is an analytic extension of the function

$$(0, +\infty) \ni \sigma \mapsto \int_{\mathbb{R}} \xi^n \, \mu_{\alpha,\beta,\sigma}(d\xi) \in \mathbb{R}.$$

In the cases of the gamma distribution and the negative binomial distribution, we derive the following representation of the transformation $\mathcal{S}$: for $f \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$,

$$(\mathcal{S}f)(z) = \int_{\mathbb{R}} f(\xi) \; \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi). \tag{1.8}$$

A counterpart of this formula in the case of the Meixner distribution takes the form

$$(\mathcal{S}f)(z) = \int_{\mathbb{R}} f(\xi + z) \; \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi), \tag{1.9}$$

however the function f is taken from a certain dense subset of $L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$ such that f admits a unique extension to an analytic function, and so $f(\xi+z)$ is well-defined for z from a complex domain in $\mathbb{C}$. To prove formulas (1.8), (1.9), we use the obtained combinatorial formulas for the moments of $\mu_{\alpha,\beta,\sigma}$.

Next, for $\eta > 0$ and $\sigma > 0$, we define a Hilbert space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as the closure of polynomials on $\mathbb{C}$ in L^2-space $L^2(\mathbb{C}, \lambda_{\eta,\sigma})$. Here, $\lambda_{\eta,\sigma}$ is the random Gaussian measure $\nu_\xi(dz)$ (see (1.1)) where ξ is a random variable with values in $(0, +\infty)$ that has gamma distribution $\mu_{\eta,\eta,\eta\sigma}$. The monomials $(z^n)_{n=0}^\infty$ form an orthogonal basis for $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ with

$$(z^n, z^m)_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})} = \delta_{n,m} \, n! \, (\sigma \mid -\eta)_n, \tag{1.10}$$

compare with (1.2). The space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ consists of entire functions $\varphi(z) = \sum_{n=0}^\infty \varphi_n z^n$ whose coefficients satisfy $\sum_{n=0}^\infty |\varphi_n|^2 \, n! \, (\sigma \mid -\eta)_n < \infty$. The $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is a reproducing kernel Hilbert space with reproducing kernel

$$\mathbb{K}_{\eta,\sigma}(z,w)=\sum_{n=0}^{\infty}\frac{(\bar{z}w)^n}{n!\,(\sigma\mid-\eta)_n}.$$

In the limiting case $\eta=0$, we have $(\sigma\mid-\eta)_n=(\sigma\mid 0)_n=\sigma^n$, thus $\mathbb{F}_{0,\sigma}(\mathbb{C})$ is the classical Segal–Bargmann space $\mathbb{F}_\sigma(\mathbb{C})$. For $\eta=1$ and $\sigma=1$,

$$\mathbb{K}_{1,1}(z,w)=\sum_{n=0}^{\infty}\frac{(\bar{z}w)^n}{(n!)^2}.$$

The Hilbert space $\mathbb{F}_{1,1}(\mathbb{C})$ was studied by Alpay et al. [7, Section 9] and Alpay–Porat [8], see also [36, 37].

Next, we construct a unitary operator

$$\mathbb{T}:\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})\to\mathbb{F}_{\eta,\sigma}(\mathbb{C})$$

that satisfies $(\mathbb{T}(\cdot\mid\beta)_n)(z)=z^n$. We prove that this operator has a representation

$$(\mathbb{T}f)(z)=\int_{\mathbb{N}_0}f(\beta\xi)\,\pi_{\frac{z}{\beta}}(d\xi). \tag{1.11}$$

Here, for $\zeta\in\mathbb{C}$, $\pi_\zeta=e^{-\zeta}\sum_{n=0}^{\infty}(n!)^{-1}\,\zeta^n\,\delta_n$, the complex-valued Poisson measure with parameter ζ.

We define a generalised Segal–Bargmann transform as the unitary operator

$$\mathbb{S}:L^2(\mathbb{R},\mu_{\alpha,\beta,\sigma})\to\mathbb{F}_{\eta,\sigma}(\mathbb{C})$$

that satisfies $(\mathbb{S}s_n)(z)=z^n$. Thus, $\mathbb{S}=\mathbb{T}\mathcal{S}$. Hence, $\mathbb{S}$ admits a representation

$$(\mathbb{S}f)(z)=\int_{\mathbb{R}}f(\xi)\,\mathbb{E}_{\alpha,\beta,\sigma}(\xi,z)\;\mu_{\alpha,\beta,\sigma}(d\xi),$$

where

$$\begin{aligned}\mathbb{E}_{\alpha,\beta,\sigma}(\xi,z)&=\sum_{n=0}^{\infty}\frac{z^n}{n!\,(\sigma\mid-\eta)_n}\,s_n(\xi)\\&=\int_{\mathbb{N}_0}\mathcal{E}_{\alpha,\beta,\sigma}(\xi,\beta\zeta)\;\pi_{\frac{z}{\beta}}(d\zeta).\end{aligned} \tag{1.12}$$

It follows from (1.12) that $(\mathbb{E}_{\alpha,\beta,\sigma}(\cdot,z))_{z\in\mathbb{C}}$ are nonlinear coherent states corresponding to the sequence $(n!\,(\sigma\mid-\eta)_n)_{n=0}^{\infty}$, see e.g. [6, 24, 55]. For applications of (generalised) coherent states in physics, see e.g. [23, 48].

Let ∂^+ and ∂^- be the raising and lowering operators for the orthogonal Sheffer sequence $(s_n)_{n=0}^\infty$. Then, the adjoint of ∂^+ in $L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$ is the operator $A^- = \sigma\partial^- + \eta\,\partial^+\partial^-\partial^-$. The nonlinear coherent state $\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z)$ is an eigenfunction of A^- with eigenvalue z.

In the cases of the gamma distribution and the negative binomial distribution, in view of (1.8) and (1.11), we get

$$(\mathbb{S}f)(z) = \int f\ d\rho_{\alpha,\beta,\sigma,z}, \tag{1.13}$$

where

$$\rho_{\alpha,\beta,\sigma,z} = \int_{\mathbb{N}_0} \pi_{\frac{z}{\beta}}(dm)\ \mu_{\alpha,\beta,\,\eta m+\sigma}. \tag{1.14}$$

In particular, for $z > 0$, $\rho_{\alpha,\beta,\sigma,z}$ is the random gamma, respectively negative binomial distributions $\mu_{\alpha,\beta,\,\eta\zeta+\sigma}$ where ζ is a random variable having Poisson distribution with parameter $\frac{z}{\beta}$. Similarly, in view of (1.9), a counterpart of formula (1.13) in the case ofthe Meixner distribution takes the form

$$(\mathbb{S}f)(z) = \int f(\xi + z)\ \rho_{\alpha,\beta,\sigma,z}(d\xi),$$

where $\rho_{\alpha,\beta,\sigma,z}$ is still defined by (1.14). In particular, for $z = \beta r$ with $r > 0$, $\rho_{\alpha,\beta,\sigma,z}$ is the random Meixner distribution $\mu_{\alpha,\beta,\eta\zeta+\sigma}$ with ζ being a random variable that has Poisson distribution with parameter r.

Under the generalised Segal–Bargmann transform $\mathbb{S}$, the operator of multiplication by variable in $L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$ goes over to the operator $\mathbb{U}\mathbb{V}$ for $0 < \beta \le \alpha$ and $\mathbb{U}\mathbb{V} - \frac{\sigma}{\alpha}$ for $\alpha = \overline{\beta} \in \mathbb{C} \setminus \mathbb{R}$, respectively. Here,

$$\mathbb{U} = \mathbb{T}\mathcal{U}\mathbb{T}^{-1} = Z(\mathbf{1} + \beta D) + \frac{\sigma}{\alpha}, \quad \mathbb{V} = \mathbb{T}\mathcal{V}\mathbb{T}^{-1} = \mathbf{1} + \alpha D.$$

We also find explicit analytic formulas for the action of the operators ∂^+, ∂^-, $U = \mathbb{S}^{-1}\mathbb{U}\mathbb{S}$ and $V = \mathbb{S}^{-1}\mathbb{V}\mathbb{S}$ in $L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$.

The dissertation is organised as follows. In Chapter 2, we discuss preliminaries. In Section 2.1, we recall some main results of umbral calculus. Section 2.2 deals with umbral composition of Sheffer sequences. In Section 2.3, we recall some analytic facts related to Sheffer homeomorphisms. In Section 2.4, we recall Meixner's classification of orthogonal

Sheffer sequences. In Section 2.5, we briefly review reproducing kernel Hilbert spaces. In Sections 2.6 and 2.7, we discuss classical and generalised coherent states and the Segal–Bargmann transforms for the Gaussian and Poisson distributions. In Sections 2.8 and 2.9, we discuss Stirling numbers and their generalisations. In Sections 2.10 and 2.11, we review Wick ordering in the (classical) Weyl algebra and in generalised Weyl algebras.

In Chapter 3, we derive some results on Wick ordering in the special class of generalised Weyl algebras with generators satisfying (1.6).

As stated above, Chapter 4 contains the main results of the dissertation that we briefly described previously. Since each class of orthogonal Sheffer sequences requires its own analytic techniques, we discuss each of them separately: Poisson case in Section 4.1, gamma case ($\alpha = \beta > 0$) in Section 4.2-4.4, the negative binomial case ($0 < \beta < \alpha$) in Section 4.5, and finally the Meixner case ($\alpha = \overline{\beta} \in \mathbb{C} \setminus \mathbb{R}$) in Section 4.6.

Chapter 2

Preliminaries

2.1 Umbral calculus

This section is based on [51] and Chapter IV, Sections 3 and 4 of [40].

Umbral calculus provides a formalism for the systematic derivation and classification of many classical combinatorial identities for polynomial sequences, along with associated generating functions, expansions, duplication formulas, recurrence relations, inversions, Rodrigues representation. The term *umbral calculus* was coined by Sylvester from the word *umbra* (meaning *shadow* in Latin), and reflects the fact that, for many types of identities involving sequences of polynomials $(p_n)_{n=0}^{\infty}$, 'shadow' identities are obtained by replacing the usual differentiation $(z^n)' = nz^{n-1}$ with a 'shadow' differentiation (operation)

$$Dp_n = np_{n-1}.$$

Let $\mathfrak{F}$ denote either the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$. Let $\mathcal{P}(\mathfrak{F})$ denote field of polynomials on $\mathfrak{F}$ with coefficients from $\mathfrak{F}$.

Remark 2.1. Note that, in the literature, $\mathcal{P}(\mathfrak{F})$ is often denoted by $\mathfrak{F}[z]$. However, we prefer the notation $\mathcal{P}(\mathfrak{F})$ to keep an analogy with the infinite dimensional case considered below.

Definition 2.2. A *polynomial sequence* $(p_n(z))_{n=0}^{\infty}$ in $\mathcal{P}(\mathfrak{F})$ is a sequence of polynomials from $\mathcal{P}(\mathfrak{F})$ such that $p_n(z)$ has degree (exactly) n:

$$p_n(z) = \sum_{k=0}^{n} a_{kn}\, z^k, \quad a_{nn} \neq 0.$$

Definition 2.3. A polynomial sequence is called *monic* if $a_{nn} = 1$ for all $n \geq 0$.

Definition 2.4. We say that a polynomial sequence $(p_n(z))_{n=0}^{\infty}$ is of *binomial type*, or just *binomial sequence*, if it satisfies the (generalised) binomial identity: for all $n \in \mathbb{N}$,

$$p_n(z+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(z)\, p_{n-k}(y).$$

Definition 2.5. A *(linear) operator* on polynomials is a linear transformation acting from $\mathcal{P}(\mathfrak{F})$ to $\mathcal{P}(\mathfrak{F})$. We denote by $\mathcal{L}(\mathcal{P}(\mathfrak{F}))$ the vector space of all linear operators in $\mathcal{P}(\mathfrak{F})$.

Since a polynomial sequence $(p_n(z))_{n=0}^{\infty}$ gives a basis of $\mathcal{P}(\mathfrak{F})$, an operator $Q \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ is determined by the $Qp_n(z)$ for $n \geq 0$.

Definition 2.6. For $a \in \mathfrak{F}$, we denote by $E^a \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ *the operator of shift by* a:

$$E^a p(z) := p(z+a).$$

Proposition 2.7 (Boole's formula). *For $a \in \mathfrak{F}$, we have*

$$E^a = e^{aD} = \sum_{k=0}^{\infty} \frac{a^k}{k!} D^k,$$

where D is the differentiation, i.e., $Dz^n = nz^{n-1}$.

Definition 2.8. An operator $Q \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ is called *shift-invariant* if $QE^a = E^a Q$ for any element $a \in \mathfrak{F}$.

Definition 2.9. An operator $Q \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ is called a *delta operator* if Q is shift-invariant and $Qz = c$, where c is a nonzero constant.

Definition 2.10. For a polynomial sequence $(p_n(z))_{n=0}^{\infty}$, its *lowering operator* $Q \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ is defined by $Qp_n(z) = np_{n-1}(z)$.

Theorem 2.11. *Let $(p_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence and $Q \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ be the lowering operator for this sequence. Then the following statements are equivalent:*

(BT1) The sequence $(p_n(z))_{n=0}^{\infty}$ *is of binomial type.*

(BT2) The operator Q *is a delta operator.*

(BT3) The operator Q *is of the form:*

$$Q = B(D) = \sum_{n=1}^{\infty} b_n D^n,$$

where $B(t) = \sum_{n=1}^{\infty} b_n t^n$ *is a formal power series with* $b_1 = 1$.

(BT4) The polynomial sequence $(p_n(z))_{n=0}^{\infty}$ *has the (exponential) generating function of the form*

$$\sum_{n=0}^{\infty} p_n(z) \frac{t^n}{n!} = \exp(zA(t)), \tag{2.1}$$

where $A(t) = \sum_{n=1}^{\infty} a_n t^n$ *is a formal power series with* $a_1 = 1$. *Formula* (2.1) *is understood as an equality of formal power series in* t.

Remark 2.12. In fact, in Theorem 1.11 (see in [40]) the formal power series $A(t)$ and $B(t)$ are the compositional inverse of each other, i.e.,

$$A(B(t)) = B(A(t)) = t.$$

Remark 2.13. For each delta operator Q, there exits a unique binomial sequence $(p_n(z))_{n=0}^{\infty}$ for which Q is its lowering operator. Then $(p_n(z))_{n=0}^{\infty}$ is called a *basic sequence for* Q.

Theorem 2.14 (The first expansion theorem)**.** *Let* T *be a shift-invariant operator and* Q *be a delta operator with basic sequence* $(p_n(z))_{n=0}^{\infty}$. *Then*

$$T = G(Q) = \sum_{n=0}^{\infty} g_n Q^n, \tag{2.2}$$

where $G(t) = \sum_{n=1}^{\infty} g_n t^n$ *is a formal power series. Here*

$$g_n = \frac{1}{n!} (Tp_n)(0).$$

Let us consider two examples of binomial sequences, falling factorials and rising factorials, which are crucial for our work.

The *falling factorials* are defined by $(z)_0 := 1$ and for $n \geq 1$

$$(z)_n = z(z-1)\cdots(z-n+1).$$

This is a polynomial sequence of binomial type with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,(z)_n = \exp[z\,\log(1+t)],$$

where $\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{t^n}{n}$.

The *rising factorials* are defined by $(z)^{(0)} := 1$ and for $n \geq 1$

$$(z)^{(n)} = z(z+1)\cdots(z+n-1).$$

This is a polynomial sequence of binomial type with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,(z)^{(n)} = \exp[-z\,\log(1-t)].$$

Note that $(z)^{(n)} = (-1)^n\,(-z)_n$.

Definition 2.15. Let $(p_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence and let Q be its lowering operator (hence a delta operator). A monic polynomial sequence $(s_n(z))_{n=0}^{\infty}$ is called a *Sheffer sequence for* Q if Q is a lowering operator for $(s_n(z))_{n=0}^{\infty}$, i.e., $Qs_n(z) = ns_{n-1}(z)$.

Theorem 2.16. *Let $Q = B(D)$ be a delta operator with basic sequence $(p_n(z))_{n=0}^{\infty}$ that has generating function* (2.1)*. Let $(s_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence. Then the following statements are equivalent:*

(SS1) The sequence $(s_n(z))_{n=0}^{\infty}$ is a Sheffer sequence for Q.

(SS2) There exists a (unique, invertible) shift-invariant operator T such that

$$p_n(z) = Ts_n(z) \quad \textit{or} \quad s_n(z) = T^{-1}p_n(z).$$

(SS3) The polynomial sequence $(s_n(z))_{n=0}^{\infty}$ has the (exponential) generating function of the form

$$\sum_{n=0}^{\infty} s_n(z)\,\frac{t^n}{n!} = \exp(z\,A(t))\,C(t), \tag{2.3}$$

where $A(t) = \sum_{n=1}^{\infty} a_n t^n$ *is as in* (2.1) *and* $C(t) = \sum_{n=0}^{\infty} c_n t^n$ *is a formal power series with* $c_0 = 1$.

(SS4) The following modified binomial identity holds: for all $n \in \mathbb{N}$,

$$s_n(z+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(z)\, p_{n-k}(y).$$

(SS5) There exists a sequence $(\rho_n)_{n=0}^{\infty}$ *with* $\rho_n \in \mathfrak{F}$ *with* $\rho_0 = 1$ *such that*

$$s_n(z) = \sum_{k=0}^{n} \binom{n}{k} \rho_k\, p_{n-k}(z).$$

Remark 2.17. In the literature one mostly defines a Sheffer sequence $(s_n(z))_{n=0}^{\infty}$ as a (monic) polynomial sequence with generating function (2.3).

Corollary 2.18. *Let* $(s_n(z))_{n=0}^{\infty}$ *be a Sheffer sequence. Then* $(s_n(z))_{n=0}^{\infty}$ *is of binomial type if and only if* $s_n(0) = 0$ *for all* $n \in \mathbb{N}$.

2.2 Umbral composition

This section develops further ideas of the previous section on umbral calculus. It is based on Section 7 in [51], Chapter IV, Section 5 in [40], and [27].

Let $\mathbb{M}$ denote the space of all linear operators $\mathcal{A} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ such that

$$\begin{aligned} \mathcal{A}\mathbf{1} &= a_{0,0}, \\ \mathcal{A}z^n &= z^n + a_{n,n-1}\, z^{n-1} + a_{n,n-2}\, z^{n-2} + \cdots + a_{n,0}, \quad n \in \mathbb{N}. \end{aligned}$$

Thus, for each $\mathcal{A} \in \mathbb{M}$, $(\mathcal{A}z^n)_{n=0}^{\infty}$ is a monic polynomial sequence. Then, $\mathcal{A}$ maps any monic polynomial sequence $(p_n(z))_{n=0}^{\infty}$ to the monic polynomial sequence $(\mathcal{A}p_n(z))_{n=0}^{\infty}$. There is a one-to-one correspondence between linear operators $\mathcal{A}$ from $\mathbb{M}$ and monic polynomial sequences. Indeed, for any monic polynomial sequence $(p_n(z))_{n=0}^{\infty}$, we define $\mathcal{A} \in \mathbb{M}$ by

$$\mathcal{A}z^n = p_n(z),$$

and by the same formula $\mathcal{A} \in \mathbb{M}$ determines $(p_n(z))_{n=0}^{\infty}$. As easily seen, $\mathbb{M}$ is a group under the composition of linear maps.

Next, we define $\mathbb{U} \subset \mathbb{M}$ as the set of all linear operators $\mathcal{U} \in \mathbb{M}$ such that

$$\mathcal{U} z^n = p_n(z)$$

where $(p_n(z))_{n=0}^{\infty}$ is a polynomial sequence of binomial type. Elements of $\mathbb{U}$ are called *umbral operators.*

Proposition 2.19. *(i) Let $\mathcal{U} \in \mathbb{U}$. Then, for each binomial sequence $(q_n(z))_{n=0}^{\infty}$, $(\mathcal{U} q_n(z))_{n=0}^{\infty}$ is also a binomial sequence.*

(ii) $\mathbb{U}$ *is a subgroup of* $\mathbb{M}$*, and it is called the* umbral group.

(iii) Let $\mathcal{U}_1, \mathcal{U}_2 \in \mathbb{U}$, and let the generating function of $(\mathcal{U}_i\, z^n)_{n=0}^{\infty}$ be of the form

$$G_i(z,t) = \exp(z\, A_i(t)) \quad (i = 1, 2).$$

Then the generating function of $(\mathcal{U}_1 \mathcal{U}_2\, z^n)_{n=0}^{\infty}$ is of the form

$$G(z,t) = \exp(z\, A_1(A_2(t))).$$

We define $\mathbb{S} \subset \mathbb{M}$ as the set of all linear operators $\mathcal{S} \in \mathbb{M}$ such that $(\mathcal{S} z^n)_{n=0}^{\infty}$ is a Sheffer sequence.

Proposition 2.20. *(i) Let $\mathcal{S} \in \mathbb{S}$. Then, for each Sheffer sequence $(s_n(z))_{n=0}^{\infty}$, $(\mathcal{S} s_n(z))_{n=0}^{\infty}$ is also a Sheffer sequence.*

(ii) $\mathbb{S}$ *is a subgroup of* $\mathbb{M}$*, and it is called the* Sheffer group.

(iii) Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{S}$. Let the generating function of $(\mathcal{S}_i\, z^n)_{n=0}^{\infty}$ be of the form

$$G_i(z,t) = \exp(z\, A_i(t))\, C_i(t).$$

Then the generating function of $(\mathcal{S}_1 \mathcal{S}_2\, z^n)_{n=0}^{\infty}$ is of the form

$$G(z,t) = \exp(z\, A_1(A_2(t)))\, C_1(A_2(t))\, C_2(t).$$

(iv) $\mathbb{U}$ *is a normal subgroup of* $\mathbb{S}$.

The above statements allow one to define a group operation on monic polynomial sequences/ Sheffer sequences/ binomial sequences.

Definition 2.21. Let $\mathbf{p} = (p_n(z))_{n=0}^{\infty}$ and $\mathbf{q} = (q_n(z))_{n=0}^{\infty}$ be monic polynomial sequences. Define the monic polynomial sequence $\mathbf{r} = (r_n(z))_{n=0}^{\infty}$ by

$$r_n(z) = M_2 M_1 \, z^n,$$

where $M_1 z^n = p_n(z)$ and $M_2 z^n = q_n(z)$. One writes $\mathbf{r} = \mathbf{p}(\mathbf{q})$ and call $\mathbf{r}$ the *umbral composition* of $\mathbf{p}$ and $\mathbf{q}$.

Thus, the above groups $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{U}$ can also be interpreted as groups of monic polynomial sequences, respectively Sheffer sequences, respectively binomial sequences with group operation defined as the umbral composition.

2.3 Sheffer homeomorphisms in the one-dimensional case

In this section, we will discuss several result of Grabiner [26], see also [20].

Definition 2.22. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $\tau > 0$. One says that f is *of order at most τ and minimal type (when the order is equal to τ)* if f satisfies the estimate

$$\sup_{z \in \mathbb{C}} |f(z)| \exp(-t\,|z|^{\tau}) < \infty \quad \text{for all } \ t > 0.$$

One denotes by $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$, the vector space of all such functions.

For each $t > 0$,

$$\|f\|_{\tau,t} := \sup_{z \in \mathbb{C}} |f(z)| \exp(-t\,|z|^{\tau}) \tag{2.4}$$

is a norm on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$. Clearly, for any $0 < t_1 < t_2$,

$$\|f\|_{\tau,t_2} \le \|f\|_{\tau,t_1}. \tag{2.5}$$

For each $t > 0$, let $B_{\tau,t}$ denote the Banach space obtained as the completion of $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ in the norm $\|\cdot\|_{\tau,t}$. Due to (2.5), for any $0 < t_1 < t_2$,

$$B_{\tau,t_1} \subset B_{\tau,t_2}$$

and the embedding of B_{τ,t_1} into B_{τ,t_2} is continuous.

Note that, as a set,

$$\mathcal{E}^{\tau}_{\min}(\mathbb{C}) = \bigcap_{t>0} B_{\tau,t}.$$

It follows from Proposition 2.23 below that $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ is not an empty set. In particular, every polynomial on $\mathbb{C}$ belongs to $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$.

One defines the projective limit topology on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ given by the norms (2.4). This means that one chooses the coarsest locally convex topology on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ for which the embedding of $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ into $B_{\tau,t}$ is continuous for each $t > 0$. Thus,

$$\mathcal{E}^{\tau}_{\min}(\mathbb{C}) = \operatorname{proj}\lim_{t\to 0} B_{\tau,t}.$$

In particular, $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ is a Fréchet space.

Note that a sequence $(f_k)_{k=1}^{\infty}$ converges to f in $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ if and only if $f_k \to f$ in each $B_{\tau,t}$. Also note that, for $0 < \tau_1 < \tau_2$, we have

$$\mathcal{E}^{\tau_1}_{\min}(\mathbb{C}) \subset \mathcal{E}^{\tau_2}_{\min}(\mathbb{C}),$$

and the embedding of $\mathcal{E}^{\tau_1}_{\min}(\mathbb{C})$ into $\mathcal{E}^{\tau_2}_{\min}(\mathbb{C})$ is continuous.

There exists an alternative description of the topology on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$.

Proposition 2.23. *Let $\tau > 0$.*

(i) An entire function of f belongs to $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ if and only if

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

where the numbers $f_n \in \mathbb{C}$ satisfy

$$\sum_{n=0}^{\infty} |f_n|^2 (n!)^{\frac{2}{\tau}} 2^{nl} < \infty \quad \textit{for all } l \in \mathbb{N}.$$

(ii) For each $l \in \mathbb{N}$, denote by $E_{\tau,l}$ the Hilbert space of all entire functions

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

such that

$$\mathcal{N}_{\tau,l}(f) := \left(\sum_{n=0}^{\infty} |f_n|^2 (n!)^{\frac{2}{\tau}} 2^{nl} \right)^{\frac{1}{2}} < \infty. \tag{2.6}$$

(The inner product in $E_{\tau,l}$ is such that $\mathcal{N}_{\tau,l}$ is the corresponding norm.) Then, as a set,

$$\mathcal{E}^{\tau}_{\min}(\mathbb{C}) = \bigcap_{l\in\mathbb{N}} E_{\tau,l}$$

and the topology on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ coincides with the projective limit of the space $E_{\tau,l}$ for $l \in \mathbb{N}$. Thus

$$\mathcal{E}^{\tau}_{\min}(\mathbb{C}) = \operatorname{proj}\lim_{l\to\infty} E_{\tau,l}.$$

Theorem 2.24. *Let $(s_n(z))_{n=1}^{\infty}$ be a Sheffer sequence with generating function (2.3) such that $A(t)$ and $C(t)$ are holomorphic in a neighborhood of zero. Let $\tau \in (0,1]$. Then the Sheffer operator $\mathcal{S} : \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ given by*

$$\mathcal{S}z^n = s_n(z)$$

extends by continuity to a linear self-homeomorphism of the space $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$. In particular, each function $f \in \mathcal{E}^{\tau}_{\min}(\mathbb{C})$ admits a unique representation

$$f(z) = \sum_{n=0}^{\infty} f_n\, s_n(z), \tag{2.7}$$

where the series on the right-hand side of formula (2.7) converges in $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$.

Corollary 2.25. *Let $(s^{(i)}_n(z))_{n=1}^{\infty}$, $i = 1, 2$, be two Sheffer sequences which satisfy the condition of Theorem 2.24. Let $\tau \in (0,1]$. Then, the Sheffer operator $\mathcal{S} : \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ defined by*

$$\mathcal{S}s^{(1)}_n(z) = s^{(2)}_n(z)$$

extends by continuity to a linear self-homeomorphism of the space $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$.

Corollary 2.26. *Let $\tau \in (0,1]$. Let $(s_n(z))_{n=0}^{\infty}$ be a Sheffer sequence satisfying the condition of Theorem 2.24.*

(i) An entire function of f belongs to $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ if and only if it can be represented on the form (2.7), where

$$\sum_{n=0}^{\infty} |f_n|^2\, (n!)^{\frac{2}{\tau}}\, 2^{nl} < \infty \quad \textit{for all } l \in \mathbb{N}. \tag{2.8}$$

(ii) For each $l \in \mathbb{N}$, denote by $H_{\tau,l}$ the completion of $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ in the Hilbertian norm

$$|||f|||_{\tau,l} := \left(\sum_{n=0}^{\infty} |f_n|^2 \, (n!)^{\frac{2}{\tau}} \, 2^{nl} \right)^{\frac{1}{2}}, \tag{2.9}$$

where f_n ($n \in \mathbb{N}_0$) are the coefficient from (2.7)*. Then,*

$$\mathcal{E}^{\tau}_{\min}(\mathbb{C}) = \operatorname*{proj\,lim}_{l \to \infty} H_{\tau,l}. \tag{2.10}$$

2.4 The Meixner class of orthogonal polynomials

This section is based on [44] and Chapter I, Section 4 and Chapter V, Section 4 of [16].

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$.

Definition 2.27*.* Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\int_{\mathbb{R}} |z|^n \, \mu(dz) < \infty \quad \text{for all } n \geq 0.$$

A polynomial sequence $(p_n(z))_{n=0}^{\infty}$ is called *orthogonal with respect to μ* if

$$\int_{\mathbb{R}} p_m(z) \, p_n(z) \; \mu(dz) = 0 \quad \text{if } m \neq n.$$

We will say that a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is concentrated on a finite set if there exists a finite set $\{z_1, \dots, z_n\} \subset \mathbb{R}$ with $\mu(\{z_1, \dots, z_n\}) = 1$.

Theorem 2.28 (Favard)**.** *Let $(p_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence. Then $(p_n(z))_{n=0}^{\infty}$ is orthogonal with respect to a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that is not concentrated on a finite set if and only if there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ ($n \geq 0$) such that the sequence $(p_n(z))_{n=0}^{\infty}$ satisfies the recurrence relation*

$$zp_n(z) = p_{n+1}(z) + a_n \, p_n(z) + b_n \, p_{n-1}(z), \quad n \geq 0.$$

Here $p_{-1}(z) := 0$.

Meixner [44] found all monic orthogonal Sheffer sequences.

Theorem 2.29 (Meixner). *Let $(s_n(z))_{n=0}^{\infty}$ be a monic Sheffer sequence, equivalently $(s_n(z))_{n=0}^{\infty}$ has generating function* (2.3)*. Then $(s_n(z))_{n=0}^{\infty}$ is orthogonal with respect to a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that is not concentrated on a finite set if and only if there exist constants $\lambda \in \mathbb{R}$, $l \in \mathbb{R}$, $\sigma > 0$ and $\eta \geq 0$ such that the sequence $(s_n(z))_{n=0}^{\infty}$ satisfies the recurrence relation*

$$zs_n(z) = s_{n+1}(z) + (\lambda n + l)s_n(z) + \big(\sigma n + \eta n(n-1)\big)s_{n-1}(z). \tag{2.11}$$

The set of all orthogonal monic Sheffer sequences is divided into five classes. Note that the constant l corresponds to a shift of the orthogonality measure μ by l. Hence, below we will choose the constant l depending on which choice is more convenient for us.

Let λ and η be as in Theorem 2.29. Define $\alpha, \beta \in \mathbb{C}$ that satisfy, for all $t \in \mathbb{R}$,

$$1 + \lambda t + \eta t^2 = (1 + \alpha t)(1 + \beta t), \tag{2.12}$$

equivalently

$$\alpha + \beta = \lambda, \quad \alpha\beta = \eta. \tag{2.13}$$

We denote by

$$G(z,t) := \exp(zA(t))\, C(t)$$

the generating function of the orthogonal Sheffer sequence $(s_n(z))_{n=0}^{\infty}$. Let also

$$F(t) := \int_{\mathbb{R}} e^{izt}\mu(dz) = \frac{1}{C(B(it))} \tag{2.14}$$

denote the Fourier transform of the orthogonality measure μ.

Remark 2.30. There are two ways to understand equality (2.14). First, we state that $F(t) = \frac{1}{C(B(it))}$ for all $t \in \mathbb{R}$ such that the function $C(B(it))$ is well-defined. Second, we formally expand

$$\int_{\mathbb{R}} e^{izt}\mu(dz) = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \int_{\mathbb{R}} z^n \, \mu(dz)$$

and then we state that

$$1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \int_{\mathbb{R}} z^n \, \mu(dz) = \frac{1}{C(B(it))}$$

as an equality of formal power series in t.

Below $\sigma > 0$ is arbitrary.

Case 1 (Gaussian). Let $\lambda = 0$, $\eta = 0$, which implies that $\alpha = \beta = 0$. Choose $l = 0$. Then

$$\begin{aligned} A(t) &= B(t) = t, \\ C(t) &= \exp\Big(-\frac{1}{2}\sigma t^2\Big), \\ G(z,t) &= \exp(zt - \frac{1}{2}\sigma t^2), \\ F(t) &= \exp\Big(-\frac{1}{2}\sigma t^2\Big). \end{aligned} \tag{2.15}$$

Hence, $(s_n(z))_{n=0}^{\infty}$ is a sequence of *Hermite polynomials* and μ is $\mathcal{N}(0,\sigma)$, the *Gaussian (normal) distribution with mean* 0 *and variance* σ:

$$d\mu(z) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{z^2}{2\sigma}}\, dz. \tag{2.16}$$

Case 2 (Poisson). Let $\lambda \neq 0$, $\eta = 0$, which implies $\alpha = \lambda \neq 0$ and $\beta = 0$. Choose $l = \frac{\sigma}{\alpha}$. Then

$$\begin{aligned} A(t) &= \frac{1}{\alpha}\log(1 + t\alpha), \\ B(t) &= \frac{1}{\alpha}(e^{t\alpha} - 1), \\ C(t) &= \exp\Big(-\frac{\sigma t}{\alpha}\Big), \\ G(z,t) &= \exp\Big(\frac{z}{\alpha}\log(1+t\alpha) - \frac{\sigma t}{\alpha}\Big), \\ F(t) &= \exp\Big(\frac{\sigma}{\alpha^2}(e^{i\alpha t} - 1)\Big). \end{aligned}$$

Thus, $(s_n(z))_{n=0}^{\infty}$ is a sequence of *Charlier polynomials.* The explicit form of the measure μ is

$$\mu(dz) = \exp\Big(-\frac{\sigma}{\alpha^2}\Big)\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{\sigma}{\alpha^2}\Big)^n \delta_{\alpha n}(dz), \tag{2.17}$$

where δ_y denotes the Dirac measure with mass at y.

Assume that a random variable X has Poisson distribution with parameter $\frac{\sigma}{\alpha^2}$. Then αX has distribution μ. In particular, in the case $\alpha = 1$, μ is the usual *Poisson distribution with parameter* σ.

Case 3 (Gamma). Let $\eta > 0$ and $\lambda^2 - 4\eta = 0$, which implies $\alpha = \beta \neq 0$ and $\alpha, \beta \in \mathbb{R}$, furthermore $\lambda = 2\alpha \neq 0$, $\eta = \alpha^2 > 0$. Choose $l = \frac{\sigma}{\alpha}$. Then

$$\begin{aligned} A(t) &= \frac{t}{1+\alpha t}, \\ B(t) &= \frac{t}{1-\alpha t}, \\ C(t) &= (1+\alpha t)^{-\frac{\sigma}{\alpha^2}}, \\ G(z,t) &= \exp\left(\frac{zt}{1+\alpha t}\right)(1+\alpha t)^{-\frac{\sigma}{\alpha^2}}, \\ F(t) &= (1-i\alpha t)^{-\frac{\sigma}{\alpha^2}}. \end{aligned}$$

Thus, $(s_n(z))_{n=0}^{\infty}$ is a sequence of *Laguerre polynomials.* The explicit form of the measure μ, for $\alpha > 0$, is

$$\mu(dz) = \mathbf{1}_{(0,+\infty)}(z) \frac{1}{\Gamma(\frac{\sigma}{\alpha^2})} \left(\frac{1}{\alpha}\right)^{\frac{\sigma}{\alpha^2}} z^{-1+\frac{\sigma}{\alpha^2}} e^{-\frac{z}{\alpha}} \, dz. \tag{2.18}$$

Recall that the *gamma distribution* with parameters a, b is defined by

$$\mathrm{Gamma}(a,b) = \mathbf{1}_{(0,+\infty)}(z) \frac{z^{a-1} e^{-bz} b^a}{\Gamma(a)} \, dz.$$

Therefore $\mu = \mathrm{Gamma}(\frac{\sigma}{\alpha^2}, \frac{1}{\alpha})$.

Case 4 (Negative binomial). Let $\eta > 0$ and $\lambda^2 - 4\eta > 0$, which corresponds to $\alpha \neq \beta$ and either $\alpha > 0$, $\beta > 0$ or $\alpha < 0$, $\beta < 0$. Suppose that $|\alpha| > |\beta|$. Choose $l = \frac{\sigma}{\alpha}$. Then

$$\begin{aligned} A(t) &= \frac{1}{\beta-\alpha} \log\left(\frac{1+\beta t}{1+\alpha t}\right), \\ B(t) &= \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha e^{(\beta-\alpha)t}}, \\ C(t) &= (1+\beta t)^{-\frac{\sigma}{\alpha\beta}}, \\ G(z,t) &= \left(\frac{1+\beta t}{1+\alpha t}\right)^{\frac{z}{\beta-\alpha}} (1+\beta t)^{-\frac{\sigma}{\alpha\beta}}, \\ F(t) &= \left(\frac{(\beta-\alpha)e^{(\beta-\alpha)it}}{\beta - \alpha e^{(\beta-\alpha)it}}\right)^{\frac{\sigma}{\alpha\beta}}. \end{aligned}$$

So $(s_n(z))_{n=0}^{\infty}$ is a sequence of the *Meixner polynomials of the first kind.* The explicit form of the measure μ is

$$\mu(dz) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{\sigma}{\alpha\beta}} \sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{\sigma}{\alpha\beta}\right)^{(n)} \delta_{(\alpha-\beta)n}(dz). \tag{2.19}$$

where the rising factorial $(a)^{(n)} := a(a+1)\cdots(a+n-1)$, $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $(a)^{(0)} := 1$. Thus, μ is equal to the distribution of a random variable $(\alpha-\beta)X$, where X has a *negative binomial distribution*, also called the *Pascal distribution*.

Case 5 (Meixner). Let $\eta > 0$ and $\lambda^2 - 4\eta < 0$, which implies $\alpha = \overline{\beta}$, $\Im(\alpha) \neq 0$. We suppose that $\Im(\alpha) > \Im(\beta)$ and choose $l = 0$. Then the formulas for $A(t)$ and $B(t)$ have the same form was in Case 4, and

$$C(t) = \left(\frac{(1+\alpha t)^{\frac{1}{\alpha}}}{(1+\beta t)^{\frac{1}{\beta}}} \right)^{\frac{\sigma}{\alpha-\beta}},$$
$$G(z,t) = \left(\frac{1+\beta t}{1+\alpha t} \right)^{\frac{z}{\beta-\alpha}} \left(\frac{(1+\alpha t)^{\frac{1}{\alpha}}}{(1+\beta t)^{\frac{1}{\beta}}} \right)^{\frac{\sigma}{\beta-\alpha}}$$
$$F(t) = \left(\frac{\beta-\alpha}{\beta e^{i\alpha t} - \alpha e^{i\beta t}} \right)^{\frac{\sigma}{\alpha\beta}}.$$

The $(s_n(z))_{n=0}^{\infty}$ now is a sequence of the *Meixner–Pollaczak polynomials*, or in other terms, the *Meixner polynomials of the second kind*.

The explicit form of the measure μ can be calculated as follows. According to e.g. [53, 54], a general *Meixner distribution* on $\mathbb{R}$ is of the form

$$\frac{(2\cos(\frac{b}{2}))^{2d}}{2a\pi\,\Gamma(2d)} \exp\left(\frac{b(z-m)}{a} \right) \left| \Gamma\big(d + \frac{i(z-m)}{a}\big) \right|^2 dz, \tag{2.20}$$

where $a > 0$, $b \in (-\pi, \pi)$, $d > 0$, $m \in \mathbb{R}$.

It follows from [44, p.13] that the measure μ is the Meixner distribution with parameters $a = 2\Im(\alpha)$, $b = \pi - 2\operatorname{Arg}(\alpha)$ (where $\operatorname{Arg}(\alpha) \in (0,\pi)$), $d = \frac{\sigma}{2|\alpha|^2}$ and $m = -\frac{\sigma\Re(\alpha)}{|\alpha|^2}$. Thus, μ has the form

$$\mu(dz) = C \exp\left(\frac{(\pi - 2\operatorname{Arg}(\alpha))z}{2\Im(\alpha)} \right) \left| \Gamma\left(\frac{iz}{2\Im(\alpha)} + \frac{i\sigma}{2\alpha\Im(\alpha)} \right) \right|^2 dz, \tag{2.21}$$

where C is the normalising constant

$$C = \frac{(2\cos(\frac{b}{2}))^{2d}}{2a\pi\,\Gamma(2d)} \exp\left(-\frac{bm}{a} \right), \tag{2.22}$$

where the constant a, b, d and m are chosen as above.

2.5 Reproducing kernel Hilbert spaces

This section is based on [30, Appendix A.3], [13, Sections 1.2, 1.3].

Let $\mathcal{D}$ be a topological space. Let $C(\mathcal{D})$ denote the space of complex-valued continuous functions on $\mathcal{D}$. Let $\mathbb{F}(\mathcal{D})$ be a complex Hilbert space such that, as a set, $\mathbb{F}(\mathcal{D}) \subset C(\mathcal{D})$. We assume that if $\varphi \neq 0$ as an element of $\mathbb{F}(\mathcal{D})$, then $\varphi \neq 0$ as an element of $C(\mathcal{D})$.

One says that $\mathbb{F}(\mathcal{D})$ is a *reproducing kernel Hilbert space* if, for each $z \in \mathcal{D}$, the functional

$$\mathbb{F}(\mathcal{D}) \ni \varphi \mapsto \ L_z\varphi := \varphi(z) \ \in \mathbb{C} \tag{2.23}$$

is continuous. Hence, for each $z \in \mathcal{D}$, there exists $\mathbb{K}_z \in \mathbb{F}(\mathcal{D})$ such that

$$L_z\varphi = (\varphi, \mathbb{K}_z)_{\mathbb{F}(\mathcal{D})}, \quad \varphi \in \mathbb{F}(\mathcal{D}). \tag{2.24}$$

By (2.23) and (2.24), for each $\varphi \in \mathbb{F}(\mathcal{D})$ and $z \in \mathcal{D}$,

$$\varphi(z) = (\varphi, \mathbb{K}_z)_{\mathbb{F}(\mathcal{D})}. \tag{2.25}$$

Since $\mathbb{K}_z$ is an element of $\mathbb{F}(\mathcal{D})$, we have, for all $w \in \mathcal{D}$,

$$\mathbb{K}_z(w) = (\mathbb{K}_z, \mathbb{K}_w)_{\mathbb{F}(\mathcal{D})}.$$

Then, the function

$$\mathbb{K}(z, w) := \mathbb{K}_z(w) = (\mathbb{K}_z, \mathbb{K}_w)_{\mathbb{F}(\mathcal{D})} \tag{2.26}$$

is called the *reproducing kernel* for $\mathbb{F}(\mathcal{D})$.

Note that, by (2.25) and (2.26), for each $\varphi \in \mathbb{F}(\mathcal{D})$ and $z \in \mathcal{D}$,

$$\varphi(z) = (\varphi, \mathbb{K}(z, \cdot))_{\mathbb{F}(\mathcal{D})}. \tag{2.27}$$

Example 2.31. Assume that ν is a probability measure on the Borel σ-algebra on $\mathcal{D}$ and assume that $\mathbb{F}(\mathcal{D})$ is a subspace of $L^2(\mathcal{D}, \nu)$ and is a reproducing kernel Hilbert space. Let $\mathbb{K}(z, w)$ be its reproducing kernel. Then, by (2.27), for each $\varphi \in \mathbb{F}(\mathcal{D})$ and $z \in \mathcal{D}$,

$$\varphi(z) = \int_{\mathcal{D}} \varphi(w) \, \overline{\mathbb{K}(z, w)} \ \nu(dw).$$

2.6 (Generalised) coherent states and (generalised) Segal–Bargmann transforms

The material of this section is mostly based on [5, Section I], see also [4, Section 5.3]. For applications of coherent states in physics, see e.g. [23, 48]. Nonlinear coherent states are discussed e.g. in [6, 24, 55].

Let $\mathbb{H}$ be a separable complex Hilbert space and let $\{e_n\}_{n=0}^{\infty}$ be a fixed orthonormal basis for $\mathbb{H}$. Let $\mathcal{D}$ be an open domain in $\mathbb{C}$, let ν be a measure on $\mathcal{D}$, and let $\{\Phi_n\}_{n=0}^{\infty}$ be an orthonormal system in $L^2(\mathcal{D}, \nu)$. We assume that each Φ_n is a continuous function on $\mathcal{D}$ and

$$\sum_{n=0}^{\infty} |\Phi_n(z)|^2 < \infty \qquad \forall z \in \mathcal{D}. \tag{2.28}$$

Denote by $\mathbb{F}(\mathcal{D})$ the subspace of $L^2(\mathcal{D}, \nu)$ constructed as the closed linear span of $\{\Phi_n\}_{n=0}^{\infty}$. Thus, $\{\Phi_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathbb{F}(\mathcal{D})$.

A *generalised coherent state* is defined, for each $z \in \mathcal{D}$, by

$$\eta_z = \sum_{n=0}^{\infty} \Phi_n(z)\, e_n.$$

By (2.28), we have $\eta_z \in \mathbb{H}$ for each $z \in \mathcal{D}$.

If we additionally assume that

$$\sum_{n=0}^{\infty} |\Phi_n(z)|^2 = 1 \quad \forall z \in \mathcal{D},$$

then $\|\eta_z\|_{\mathbb{H}} = 1$ and the generalised coherent state is called *canonical*, or *normalized.*

In many cases, generalised coherent states are constructed as eigenvectors for a certain annihilation operator A^- in the system. More precisely, for each $z \in \mathcal{D}$, η_z is an element of the domain of A^- and

$$A^- \eta_z = z\eta_z.$$

A *generalised Segal–Bargmann transform* is then defined as a unitary operator

$$\mathbb{S} : \mathbb{H} \to \mathbb{F}(\mathcal{D})$$

satisfying

$$\mathbb{S}\, e_n = \Phi_n, \qquad n \geq 0.$$

For each $z \in \mathcal{D}$, denote

$$\widehat{\eta}_z := \sum_{n=0}^{\infty} \overline{\Phi_n(z)}\, e_n. \tag{2.29}$$

Then $\widehat{\eta}_z \in \mathbb{H}$ and, for each $n \in \mathbb{N}_0$,

$$(e_n, \widehat{\eta}_z)_{\mathbb{H}} = \Phi_n(z).$$

Thus, for each $f \in \mathbb{H}$,

$$(\mathbb{S}f)(z) = (f, \widehat{\eta}_z)_{\mathbb{H}}. \tag{2.30}$$

For each $\varphi \in \mathbb{F}(\mathcal{D})$, let $f := \mathbb{S}^{-1}\varphi$. Then, by (2.30),

$$\begin{aligned}
\varphi(z) &= (\mathbb{S}^{-1}\varphi, \widehat{\eta}_z)_{\mathbb{H}} \\
&= (\mathbb{S}^{*}\varphi, \widehat{\eta}_z)_{\mathbb{H}} \\
&= (\varphi, \mathbb{S}\,\widehat{\eta}_z)_{\mathbb{F}(\mathcal{D})} \\
&= (\varphi, \mathbb{K}_z)_{\mathbb{F}(\mathcal{D})},
\end{aligned}$$

where $\mathbb{K}_z := \mathbb{S}\,\widehat{\eta}_z$. Thus, $\mathbb{F}(\mathcal{D})$ is a reproducing kernel Hilbert space, with the reproducing kernel

$$\begin{aligned}
\mathbb{K}(z, w) &= (\mathbb{K}_z, \mathbb{K}_w)_{\mathbb{F}(\mathcal{D})} \\
&= (\mathbb{S}\,\widehat{\eta}_z, \mathbb{S}\,\widehat{\eta}_w)_{\mathbb{F}(\mathcal{D})} \\
&= (\widehat{\eta}_z, \widehat{\eta}_w)_{\mathbb{H}}.
\end{aligned}$$

Let us now consider a subclass of generalised coherent states which are called *nonlinear coherent states*.

Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that the power series

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{\rho_n} \tag{2.31}$$

has a non-zero radius of convergence $L \in (0, \infty]$. Let $\mathcal{D} := \{z \in \mathbb{C} \mid |z| < L\}$. We assume that there exists a probability measure ν on $\mathcal{D}$ such that $\{\frac{z^n}{\sqrt{\rho_n}}\}_{n=0}^{\infty}$ is an orthonormal system with respect to ν, i.e., for $m, n \geq 0$,

$$\int_{\mathcal{D}} z^m\, \bar{z}^n\; \nu(dz) = \delta_{m,n}\, \rho_n.$$

Assume that the probability measure ν has density with respect to the Lebesgue measure $dA(z) = dx\,dy$, where $z = x + iy$ $(x, y \in \mathbb{R})$. Then, the above assumption means that there exists a probability measure λ on $[0, L)$ satisfying

$$\int_0^L r^{2n}\, d\lambda(r) = \rho_n.$$

In the latter case, the measure ν on $\mathcal{D}$ is of the form

$$\nu(dz) = \frac{1}{2\pi}\, d\theta\, d\lambda(r),$$

where $z = re^{i\theta}$.

Let $\{e_n\}_{n=0}^{\infty}$ be a fixed orthonormal basis in a Hilbert space $\mathbb{H}$. The *(normalised) nonlinear coherent states corresponding to the sequence* $\{\rho_n\}_{n=0}^{\infty}$ are defined by

$$\eta_z := \mathcal{N}^{-\frac{1}{2}}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}}\, e_n, \qquad z \in \mathcal{D},$$

where

$$\mathcal{N}(|z|^2) := \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_n}.$$

Denote

$$\Phi_n(z) := \mathcal{N}^{-\frac{1}{2}}(|z|^2)\, \frac{z^n}{\sqrt{\rho_n}}. \tag{2.32}$$

Then $\{\Phi_n\}_{n=0}^{\infty}$ is an orthonormal system in $L^2(\mathcal{D}, \mathcal{N}(|z|^2)\, d\nu(z))$. Hence, η_z are generalised coherent states. Note that, by (2.29) and (2.32),

$$\widehat{\eta}_z = \eta_{\bar{z}}.$$

Hence, the corresponding Segal–Bargmann transform $\mathbb{S} : \mathbb{H} \to \mathbb{F}(\mathcal{D})$ is of the form

$$(\mathbb{S}f)(z) = (f, \eta_{\bar{z}})_{\mathbb{H}}.$$

The reproducing kernel in the Hilbert space $\mathbb{F}(\mathcal{D})$ is of the form

$$\begin{aligned}
\mathbb{K}(z, w) &= (\widehat{\eta}_z, \widehat{\eta}_w)_{\mathbb{H}} \\
&= (\eta_{\bar{z}}, \eta_{\bar{w}})_{\mathbb{H}} \\
&= \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{\rho_n}
\end{aligned}$$

$$= \mathcal{N}^{-1}(|z|^2)\, E(\bar{z}w),$$

where

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{\rho_n}.$$

For our purposes, we will need non-normalised nonlinear coherent states. Under the above assumption, these are defined by

$$\eta_z = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}}\, e_n, \tag{2.33}$$

so that

$$\Phi_n(z) = \frac{z^n}{\sqrt{\rho_n}}. \tag{2.34}$$

Since $\{\Phi_n\}_{n=0}^{\infty}$ is an orthonormal system in $L^2(\mathcal{D}, \nu)$, $\mathbb{F}(\mathcal{D})$ is defined as the closed linear span of $\{\frac{z^n}{\sqrt{\rho_n}}\}_{n=0}^{\infty}$ in $L^2(\mathcal{D}, \nu)$, i.e., $\mathbb{F}(\mathcal{D})$ is the closure of polynomials $\mathcal{P}(\mathbb{C})$ in $L^2(\mathcal{D}, \nu)$. The corresponding Segal–Bargmann transform $\mathbb{S} : \mathbb{H} \to \mathbb{F}(\mathcal{D})$ is of the form

$$(\mathbb{S}f)(z) = (f, \eta_{\bar{z}})_{\mathbb{H}}$$

and the reproducing kernel of the Hilbert space $\mathbb{F}(\mathcal{D})$ is

$$\mathbb{K}(z, w) = E(\bar{z}w).$$

Note that

$$(\mathbb{S}e_n)(z) = \frac{z^n}{\sqrt{\rho_n}},$$

and so

$$(\mathbb{S}\sqrt{\rho_n}\, e_n)(z) = z^n.$$

Let us also make the following observation. For each $n \in \mathbb{N}_0$, define $c_0 = 0$ and $c_n = \frac{\rho_n}{\rho_{n-1}}$ for $n \geq 1$. In particular, $\rho_n = c_1 c_2 \dots c_n$. Define a (possibly unbounded) linear operation A^- in $\mathbb{H}$ satisfying

$$A^- e_n = \sqrt{c_n}\, e_{n-1}. \tag{2.35}$$

Then, at least heuristically,

$$A^- \eta_z = A^- \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}}\, e_n = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}}\, A^- e_n$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}} \sqrt{c_n}\, e_{n-1} = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\rho_n}} \sqrt{\frac{\rho_n}{\rho_{n-1}}}\, e_{n-1}$$
$$= \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\rho_{n-1}}}\, e_{n-1} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{\sqrt{\rho_n}}\, e_n = z\eta_z.$$

Thus, each η_z is an eigenvector for A^- with eigenvalue z.

Let A^+ denote the adjoint operator of A^- in $\mathbb{H}$. Since $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathbb{H}$, (2.35) implies that

$$A^+ e_n = \sqrt{c_{n+1}}\, e_{n+1}, \quad n \in \mathbb{N}_0.$$

Let

$$\widetilde{A^+} := \mathbb{S}\, A^+\, \mathbb{S}^{-1}.$$

Then, for all $\varphi \in \mathbb{F}(\mathcal{D})$ and $z \in \mathcal{D}$,

$$\begin{aligned} (\widetilde{A^+}\varphi)(z) &= (\mathbb{S}\, A^+\, \mathbb{S}^{-1}\, \varphi)(z) = (A^+\, \mathbb{S}^{-1}\, \varphi, \eta_{\bar{z}})_{\mathbb{H}} \\ &= (\mathbb{S}^{-1}\, \varphi, A^- \eta_{\bar{z}})_{\mathbb{H}} = (\mathbb{S}^{-1}\, \varphi, \bar{z}\eta_{\bar{z}})_{\mathbb{H}} \\ &= z\, (\mathbb{S}^{-1}\, \varphi, \eta_{\bar{z}})_{\mathbb{H}} = z\, (\mathbb{S}\,\mathbb{S}^{-1}\, \varphi)(z) = z\, \varphi(z). \end{aligned} \tag{2.36}$$

Therefore, $\widetilde{A^+}$ acts as the operator of multiplication by z in $\mathbb{F}(\mathcal{D})$.

2.7 Coherent states and the Segal–Bargmann transform for the Gaussian and Poisson distributions

We will now discuss the classical Segal–Bargmann transform (in the one-dimensional case) for the Gaussian distribution and its extension to the Poisson distribution. We mostly use materials from Section 3.3 in [46] and [9].

2.7.1 Gaussian case

Let $\sigma > 0$. Let ν_σ be the Gaussian measure on $\mathbb{C}$ given by

$$\nu_\sigma(dz) = \frac{1}{\pi\sigma} \exp\left(- \frac{|z|^2}{\sigma} \right) dA(z). \tag{2.37}$$

It is known that, for any $m, n \in \mathbb{N}_0$,

$$\int_{\mathbb{C}} z^m \, \overline{z^n} \, \nu_\sigma(dz) = \delta_{m,n} \, n! \, \sigma^n, \tag{2.38}$$

i.e.

$$(z^m, z^n)_{L^2(\mathbb{C},\nu_\sigma)} = \delta_{m,n} \, n! \, \sigma^n.$$

Let $\mathbb{F}_\sigma(\mathbb{C})$ denote the Hilbert space of all entire functions $u : \mathbb{C} \to \mathbb{C}$,

$$u(z) = \sum_{n=0}^{\infty} c_n z^n \tag{2.39}$$

with $c_n \in \mathbb{C}$ satisfying

$$\sum_{n=0}^{\infty} |c_n|^2 \, n! \, \sigma^n < \infty, \tag{2.40}$$

and inner product of $u(z) = \sum_{n=0}^{\infty} c_n z^n, \ v(z) = \sum_{n=0}^{\infty} d_n z^n$ given by

$$(u, v) = \int_{\mathbb{C}} u(z)\overline{v(z)} \ \nu_\sigma(dz) = \sum_{n=0}^{\infty} c_n \, \overline{d_n} \, n! \, \sigma^n.$$

Remark 2.32. Note that, under the condition (2.40),

$$\sum_{n=0}^{\infty} |c_n||z|^n = \sum_{n=0}^{\infty} |c_n|\sqrt{n! \, \sigma^n} \, \frac{1}{\sqrt{n! \, \sigma^n}} |z|^n \le \left(\sum_{n=0}^{\infty} |c_n|^2 \, n! \, \sigma^n\right)^{1/2} \left(\sum_{m=0}^{\infty} \frac{|z|^m}{m! \, \sigma^m}\right)^{1/2} < \infty.$$

Hence, the function $u(z)$ given by (2.39) is indeed entire.

The space $\mathbb{F}_\sigma(\mathbb{C})$ is a proper subspace of the Hilbert space $L^2(\mathbb{C}, \nu_\sigma)$.

Let μ_σ be the Gaussian measure on $\mathbb{R}$ given by

$$\mu_\sigma(d\xi) = \frac{1}{\sqrt{2\pi\sigma}} \, e^{-\frac{\xi^2}{2\sigma}} \ d\xi,$$

i.e., μ_σ is the measure from (2.16).

Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Hermite polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, s_n(\xi) = \exp\left(\xi t - \frac{1}{2}\sigma t^2\right). \tag{2.41}$$

Then

$$(s_n, s_m)_{L^2(\mathbb{R},\mu_\sigma)} = \delta_{m,n} \, n! \, \sigma^n, \tag{2.42}$$

hence the polynomials $(\frac{1}{\sqrt{n!}\,\sigma^{n/2}}\, s_n(z))_{n=0}^{\infty}$ form an orthonormal basis in $L^2(\mathbb{R},\mu_\sigma)$. Here $L^2(\mathbb{R},\mu_\sigma)$ is the L^2-space of all μ_σ-square-integrable functions $f:\mathbb{R}\to\mathbb{C}$.

Thus, in the notations of Section 2.6, we choose $\mathcal{D}=\mathbb{C}$, $\nu=\nu_\sigma$, $\rho_n=n!\,\sigma^n$, $\mathbb{F}(\mathbb{C})=\mathbb{F}_\sigma(\mathbb{C})$, and $\mathbb{H}=L^2(\mathbb{R},\mu_\sigma)$. Then, for each $z\in\mathbb{C}$, the corresponding coherent state is given by

$$\mathbb{E}_\sigma(\xi,z):=\eta_z(\xi)=\sum_{n=0}^{\infty}\frac{1}{n!\,\sigma^n}\,z^n s_n(\xi)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{\sigma}\right)^n s_n(\xi),\quad \xi\in\mathbb{R}. \tag{2.43}$$

The corresponding Segal–Bargmann transform (also called the $\mathbb{S}$-transform) is the unitary operator

$$\mathbb{S}:L^2(\mathbb{R},\mu_\sigma)\to\mathbb{F}_\sigma(\mathbb{C}) \tag{2.44}$$

satisfying

$$(\mathbb{S}s_n)(z)=z^n. \tag{2.45}$$

and

$$(\mathbb{S}f)(z)=\int_{\mathbb{R}}f(\xi)\,\mathbb{E}_\sigma(\xi,z)\;\mu_\sigma(d\xi). \tag{2.46}$$

Another way to interpret the $\mathbb{S}$-transform is by looking at the explicit form of $\mathbb{E}_\sigma(\xi,z)$. By (2.41) and (2.43),

$$\begin{aligned}\mathbb{E}_\sigma(\xi,z)&=\exp\left(\frac{\xi z}{\sigma}-\frac{1}{2}\sigma\frac{z^2}{\sigma^2}\right)\\&=\exp\left(-\frac{z^2-2\xi z}{2\sigma}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}}f(\xi)\,\mathbb{E}_\sigma(\xi,z)\;\mu_\sigma(d\xi)&=\int_{\mathbb{R}}f(\xi)\,\exp\left(-\frac{z^2-2\xi z}{2\sigma}\right)\frac{1}{\sqrt{2\pi\sigma}}\,\exp\left(-\frac{\xi^2}{2\sigma}\right)\,d\xi\\&=\int_{\mathbb{R}}f(\xi)\,\frac{1}{\sqrt{2\pi\sigma}}\,\exp\left(-\frac{z^2-2\xi z+\xi^2}{2\sigma}\right)\,d\xi\\&=\int_{\mathbb{R}}f(\xi)\,\frac{1}{\sqrt{2\pi\sigma}}\,\exp\left(-\frac{(\xi-z)^2}{2\sigma}\right)\,d\xi\\&=\int_{\mathbb{R}}f(\xi+z)\,\frac{1}{\sqrt{2\pi\sigma}}\,\exp\left(-\frac{\xi^2}{2\sigma}\right)\,d\xi.\end{aligned} \tag{2.47}$$

Hence, by (2.46), for each $z\in\mathbb{R}$,

$$(\mathbb{S}f)(z)=\int_{\mathbb{R}}f(\xi+z)\,\mu_\sigma(d\xi)$$

$$= \int_{\mathbb{R}} f(\xi)\, \mu_\sigma(-z + d\xi). \tag{2.48}$$

where $\mu(-z + d\xi)$ is the push-forward of the measure μ under the transformation

$$\mathbb{R} \ni \xi \mapsto -z + \xi \in \mathbb{R}.$$

Recall also that an entire function is completely determined by its values on $\mathbb{R}$.

2.7.2 Poisson case

Let us now discuss an extension of this construction to the case of Poisson distribution [9].

Let now a measure μ_σ be of the form

$$\mu_\sigma(d\xi) = \exp(-\sigma) \sum_{n=0}^{\infty} \frac{1}{n!}\, \sigma^n\, \delta_n(d\xi),$$

i.e., μ_σ is the Poisson distribution with parameter σ. Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Charlier polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, s_n(\xi) = \exp(\xi\, \log(1+t) - \sigma t). \tag{2.49}$$

Then formula (2.42) also holds in this case. Hence, we can again define a Segal–Bargmann transform by (2.44) and (2.45) (with different μ_σ and s_n). Again, formula (2.46) holds with the function $\mathbb{E}_\sigma(\xi, z)$ defined by (2.43). Using (2.43) and (2.49), one can explicitly calculate the function $\mathbb{E}_\sigma(\xi, z)$:

$$\begin{aligned} \mathbb{E}_\sigma(\xi, z) &= \exp\left(\xi \log\left(1 + \frac{z}{\sigma}\right) - \sigma \frac{z}{\sigma}\right) \\ &= \exp\left(\xi \log\left(1 + \frac{z}{\sigma}\right) - z\right) \\ &= e^{-z}\left(1 + \frac{z}{\sigma}\right)^{\xi}. \end{aligned}$$

2.8 Stirling numbers

In this section, we will present definitions and results on Stirling numbers based on Chapter IX, Section 2, Chapter XII, Section 1 and Chapter XIV, Section 1 of [49].

We define the *falling factorials* $((z)_n)_{n=0}^{\infty}$ as the polynomial sequence of binomial type with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (z)_n = \exp[z \log(1+t)] = (1+t)^z. \tag{2.50}$$

Explicitly, $(z)_0 = 1$ and $(z)_n = z(z-1)\cdots(z-n+1)$ for $n \in \mathbb{N}$.

By (2.50), the lowering operator Q for $(z)_n$, i.e., the operator Q satisfying $Q(z)_n = n\,(z)_{n-1}$, is of the form

$$Q = e^D - \mathbf{1} = E^1 - \mathbf{1}.$$

We will denote this operator by D_1. Thus,

$$D_1\, p(z) = p(z+1) - p(z).$$

We define the *rising factorials* $((z)^{(n)})_{n=0}^{\infty}$ as the polynomial sequence of binomial type with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (z)^{(n)} = \exp[-z \log(1-t)] = (1-t)^{-z}. \tag{2.51}$$

Explicitly, $(z)^{(0)} = 1$ and $(z)^{(n)} = z(z+1)(z+2)\cdots(z+n-1)$ for $n \in \mathbb{N}$.

By (2.51), the lowering operator Q for $(z)^{(n)}$ is of the form

$$Q = \mathbf{1} - e^{-D} = \mathbf{1} - E^{-1}.$$

Note that $(z)^{(n)} = (-1)^n\,(-z)_n$. We will denote this operator by D_{-1}. Thus,

$$D_{-1}\, p(z) = p(z) - p(z-1).$$

Stirling numbers express the coefficients in expansions of falling and rising factorials as polynomials.

Definition 2.33. The *Stirling numbers of the first kind*, $s(n,k)$, are the coefficient of the expansion

$$(z)_n = \sum_{k=1}^{n} s(n,k)\, z^k.$$

Remark 2.34. One also defines $s(n,k)$ for all $n, k \in \mathbb{N}_0$ by setting $s(n,k){:=}0$ if $1 \le k \le n$ does not hold.

The (exponential) generating function of $s(n,k)$ is

$$\sum_{n=k}^{\infty} s(n,k)\,\frac{t^n}{n!} = \frac{1}{k!}\,(\log(1+t))^k$$

for $k \geq 1$.

Definition 2.35. The *unsigned Stirling numbers of the first kind* are defined by

$$c(n,k) = |s(n,k)| = (-1)^{n-k} s(n,k).$$

It holds that

$$(z)^{(n)} = \sum_{k=1}^{n} c(n,k)\,z^k.$$

The $c(n,k)$ is the number of all permutations $\pi \in \mathfrak{S}_n$ with exactly k cycles.

The numbers $c(n,k)$ satisfy the recurrence relation $c(1,1) = 1$ and

$$c(n+1,k) = c(n,k-1) + n\,c(n,k).$$

Definition 2.36. The *Stirling numbers of the second kind*, $S(n,k)$, are defined as the coefficient in the expansion

$$z^n = \sum_{k=1}^{n} S(n,k)\,(z)_k.$$

Remark 2.37. Similarly, one defines $S(n,k) := 0$ if $1 \leq k \leq n$ does not hold.

The Stirling numbers of the second kind have the (exponential) generating function as follows:

$$\sum_{n=k}^{\infty} S(n,k)\,\frac{t^n}{n!} = \frac{1}{k!}\,(e^t - 1)^k$$

for $k \geq 1$.

The $S(n,k)$ is equal to the number of all partitions of a set of n elements into k unordered non-empty subsets.

The numbers $S(n,k)$ satisfy the recurrence relation $S(1,1) = 1$ and

$$S(n+1,k) = k\,S(n,k) + S(n,k-1).$$

It holds that

$$S(n,k) = \frac{1}{k!}\,D_1^k x^n\big|_{x=0}.$$

The following formula, called the *orthogonality property*, holds: for all $1 \le i \le n$,

$$\sum_{k=i}^{n} S(n,k)\, s(k,i) = \sum_{k=i}^{n} s(n,k)\, S(k,i) = \delta_{n,i}. \tag{2.52}$$

Here $\delta_{n,i}$ is the Kronecker delta.

One defines the monic polynomial sequence of *Touchard* (or *exponential*) *polynomials* by

$$T_n(z) := \sum_{k=1}^{n} S(n,k)\, z^k. \tag{2.53}$$

This is a polynomial sequence of binomial type with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, T_n(z) = \exp(z(e^t - 1)). \tag{2.54}$$

Definition 2.38. One defines the *Lah numbers* as the coefficients expressing the falling factorials in terms of the rising factorials and vice versa:

$$(z)^{(n)} = \sum_{k=1}^{n} L(n,k)\, (z)_k \quad \text{and} \quad (z)_n = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)\, (z)^{(k)}. \tag{2.55}$$

Explicitly,

$$L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!}.$$

The Lah numbers are sometimes called *Stirling numbers of the third kind.*

The $L(n,k)$ is the number of all partitions of a set of n elements into k non-empty ordered sets.

The Lah numbers satisfy the recurrence relation $L(1,1) = 1$ and

$$L(n+1,k) = (n+k)\, L(n,k) + L(n,k-1).$$

Also,

$$L(n,k) = \sum_{i=k}^{n} c(n,i)\, S(i,k).$$

2.9 Generalised Stirling numbers of Hsu and Shiue

The main reference for this section is the paper of Hsu and Shiue [31]. We also refer to [28, 29, 35] and Chapter IV, Section 2 of [43].

Let $\alpha \in \mathfrak{F}$, where, as above, $\mathfrak{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.39. Denote $(z \mid \alpha)_0 := 1$ and

$$(z \mid \alpha)_n := z(z-\alpha)(z-2\alpha)\cdots(z-(n-1)\alpha), \quad n \in \mathbb{N}.$$

Then the sequence $((z \mid \alpha)_n)_{n=0}^{\infty}$ is called the *generalised factorial of variable z with increment α.*

Thus, for $\alpha = 1$, we get $(z \mid 1)_n = (z)_n$, the falling factorial, and for $\alpha = -1$, $(z \mid -1)_n = (z)^{(n)}$, the rising factorial.

Promptly, we have the following relations:

$$(z \mid \alpha)_n = \alpha^n \left(\frac{z}{\alpha}\right)_n \quad \text{and} \quad (z \mid -\alpha)_n = \alpha^n \left(\frac{z}{\alpha}\right)^{(n)} \quad \text{for } \alpha \neq 0. \tag{2.56}$$

By (2.50) and (2.56),

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (z \mid \alpha)_n = \exp\left[\frac{z}{\alpha}\log(1+\alpha t)\right]. \tag{2.57}$$

Therefore, we conclude that $((z \mid \alpha)_n)_{n=0}^{\infty}$ is a sequence of polynomials of binomial type.

Recall the difference operators D_1 and D_{-1}. Then, we unify them into the *divided difference operator*, or *h-derivative* D_h defined by

$$D_h := \frac{1}{h}(E^h - \mathbf{1}), \tag{2.58}$$

equivalently

$$D_h\, p(z) = \frac{1}{h}\big(p(z+h) - p(z)\big).$$

Then, for $h = \alpha$,

$$D_\alpha (z \mid \alpha)_n = n\,(z \mid \alpha)_{n-1}, \tag{2.59}$$

i.e., D_α is the lowering operator for $(z \mid \alpha)_n$.

Note also that

$$D_\alpha^k (z \mid \alpha)_j \Big|_{z=0} = \alpha^k\, k!\, \delta_{kj}.$$

Definition 2.40. Let $n \in \mathbb{N}_0$ and $\alpha, \beta, r \in \mathfrak{F}$ (where $\mathfrak{F} = \mathbb{R}$ or $\mathbb{C}$). *Generalised Stirling numbers, denoted by $S(n, k; \alpha, \beta, r)$, are defined by*

$$(z \mid \alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha, \beta, r)\,(z - r \mid \beta)_k. \tag{2.60}$$

We also define the Stirling-type pair by

$$\{S^{(1)}, S^{(2)}\} \equiv \{S^{(1)}(n,k), S^{(2)}(n,k)\} := \{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}.$$

Thus,

$$(z \mid \alpha)_n = \sum_{k=0}^{n} S^{(1)}(n,k)\,(z-r \mid \beta)_k \tag{2.61}$$

and

$$(z \mid \beta)_n = \sum_{k=0}^{n} S^{(2)}(n,k)\,(z+r \mid \alpha)_k. \tag{2.62}$$

Obviously, the $(1,0,0)$-pair is the classical Stirling number pair $\{s(n,k), S(n,k)\}$. Note that the binomial coefficients are of the form

$$S(n,k;0,0,1) = \binom{n}{k}.$$

In addition, the Lah numbers $\{L(n,k), (-1)^{n-k}L(n,k)\}$ form the $(-1,1,0)$-pair while $\{c(n,k), (-1)^{n-k}S(n,k)\}$ form the $(-1,0,0)$-pair.

The orthogonality relations are of the form

$$\sum_{k=n}^{m} S^{(1)}(m,k)\,S^{(2)}(k,n) = \sum_{k=n}^{m} S^{(2)}(m,k)\,S^{(1)}(k,n) = \delta_{mn}\,.$$

For convenience, when α, β and r are fixed, we will just write $S(n,k)$ for $S(n,k;\alpha,\beta,r)$.

The definition (2.61) implies the following:

$$S(0,0)=1, \quad S(n,n)=1 \text{ for all } n, \quad S(1,0)=r, \quad S(n,0)=(r \mid \alpha)_n \tag{2.63}$$

and the following recurrence formula holds, for $1 \le k \le n$,

$$S(n+1,k) = S(n,k-1) + (k\beta - n\alpha + r)\,S(n,k).$$

The (vertical) generating function of $S(n,k)$ has the following form. If $\alpha\beta \neq 0$, then

$$\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}(1+\alpha t)^{\frac{r}{\alpha}}\left(\frac{(1+\alpha t)^{\frac{\beta}{\alpha}}-1}{\beta}\right)^{k}. \tag{2.64}$$

If $\beta \neq 0$ and $\alpha = 0$, one finds the generating function of $S(n, k)$ by taking the limit in (2.64) as $\alpha \to 0$ and using the fact that

$$\lim_{\alpha \to 0} (1 + \alpha t)^{\frac{1}{\alpha}} = e^t.$$

If $\alpha \neq 0$ and $\beta = 0$, one finds the generating function of $S(n, k)$ by taking the limit in (2.64) as $\beta \to 0$ and using the fact that

$$\lim_{\beta \to 0} \frac{c^\beta - 1}{\beta} = \log c.$$

Remark 2.41. Setting $S_n(z) = \sum_{k=0}^{n} S(n, k)\, z^k$, the generating function of the monic polynomial sequence $(S_n(z))_{n=0}^{\infty}$ is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, S_n(z) = (1 + \alpha t)^{\frac{r}{\alpha}} \exp\left[\left((1 + \alpha t)^{\frac{\beta}{\alpha}} - 1\right) \frac{z}{\beta}\right].$$

Furthermore, for the polynomial sequence $(S_n(z))_{n=0}^{\infty}$, the *Dobinski-type formula* holds:

$$S_n(z) = e^{-\frac{z}{\beta}} \sum_{k=0}^{\infty} \frac{(z/\beta)^k}{k!}\, (k\beta + r \mid \alpha)_n.$$

2.10 Wick ordering in the Weyl algebra

In this section, we will discuss elements of Wick ordering in a Weyl algebra. Our presentation is mostly based on Chapter 1, Section 1.2 in [43].

Definition 2.42. Let $h \in \mathbb{C} \setminus \{0\}$. The *Weyl algebra* $\mathbb{A}_h$ is defined as the free complex algebra in two generators U and V satisfying the commutation relation

$$UV - VU = h. \tag{2.65}$$

Here and below we identify the constant h with $h \cdot \mathbf{1}$, where $\mathbf{1}$ is the identity for multiplication in $\mathbb{A}_h$.

Remark 2.43. In physics literature, the Weyl algebra is also called the *Heisenberg algebra* or the *Heisenberg–Weyl algebra*.

Note that the *commutator of operators U and V* is defined by

$$[U,V]=UV-VU.$$

Thus, formula (2.65) can be written as

$$[U,V]=h.$$

Remark 2.44. Note that if U and V satisfy (2.65), then, for any constants $\alpha,\beta\in\mathbb{C}$,

$$[U+\alpha,V+\beta]=h.$$

In quantum mechanics, one uses the canonical commutation relations (CCR) to describe boson. In its simplest form, the CCR algebra is generated by two operators, a *creation operator* a^+ and an *annihilation operator* a^- such that a^- is the adjoint of a^+ and

$$a^-a^+-a^+a^-=\mathbf{1}.$$

Thus, the CCR algebra is the Weyl algebra $\mathbb{A}_1$, with $U=a^-$, $V=a^+$, and we additionally assume that $(a^+)^*=a^-$.

In the Fock representation of this simplest CCR algebra, the annihilation operator a^- lowers the number of particles in a given state by one, whereas the creation operator a^+ increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

Definition 2.45. In the Weyl algebra $\mathbb{A}_h$ a product of operators U and V is said to be *normally ordered*, or *Wick ordered* if all operators V are on the left and all operators U are on the right. The process of putting a product into a normal order is called *normal ordering*, or *Wick ordering.*

Remark 2.46. An *anti-normal ordering* is analogously defined, where all operators U are on the left and all operators V are on the right.

A simple representation of (2.65) with $h=1$ is given by operators U and V acting in $\mathcal{P}(\mathbb{C})$ as follows: $U=D$ is the operator of differentiation, $V=Z$ is the operator of multiplication by the variable z, i.e.,

$$Dz^n=nz^{n-1},\quad Zz^n=z^{n+1}.$$

The operator $\mathfrak{D} := ZD$ is called the *Euler operator* [3].

Theorem 2.47 (Grunert's formula)**.** *We have*

$$\mathfrak{D}^n = \sum_{k=1}^{n} S(n,k)\, Z^k D^k$$

where $S(n,k)$ is the Stirling numbers of the second kind.

The following theorem is a generalization of Grunert's formula, see [38].

Theorem 2.48 (Katriel's formula)**.** *Let U and V satisfy* (2.65) *with $h = 1$. Then*

$$(VU)^n = \sum_{k=1}^{n} S(n,k)\, V^k U^k.$$

More generally, for U and V satisfying (2.65) *with $h \in \mathbb{C}$, we have*

$$(VU)^n = \sum_{k=1}^{n} S(n,k)\, h^{n-k}\, V^k U^k. \tag{2.66}$$

A converse formula to (2.66) is given by

$$V^n U^n = \sum_{k=1}^{n} s(n,k)\, h^{n-k}\, (VU)^k, \tag{2.67}$$

where $s(n,k)$ are Stirling numbers of the first kind. For $h = 1$, formula (2.67) gives

$$V^n U^n = \sum_{k=1}^{n} s(n,k)\, (VU)^k = (VU)_n.$$

2.11 Generalised Weyl algebra

This section is mostly based on Chapter I, Section 3; Chapter VI, Section 1 and Chapter VIII, Sections 1-3, 5 in [43].

Definition 2.49. Let $f \in \mathcal{P}(\mathbb{C})$. The *generalised Weyl algebra* $\mathbb{A}^{\langle f \rangle}$ is the complex free algebra in two generators U and V satisfying the commutation relation

$$UV - VU = f(V). \tag{2.68}$$

In the special case where $f(z) = hz^s$, with $s \in \mathbb{N}_0$ and $h \in \mathbb{C}$, $h \neq 0$, the algebra $\mathbb{A}^{\langle f \rangle}$ will be denoted by $\mathbb{A}_{s;h}$.

Note that, if $s = 0$, then $\mathbb{A}_{0;h}$ becomes the Weyl algebra $\mathbb{A}_h$.

For $s = 1$, the commutation relation (2.68) in the algebra $\mathbb{A}_{1;h}$ becomes

$$UV - VU = hV. \tag{2.69}$$

Some aspects of normal ordering in $\mathbb{A}_{1;h}$ were discussed, from physical point of view in [17, 18, 39, 59, 60, 61, 62]. From mathematical point of view, the algebra $\mathbb{A}_{1;h}$ was already studied by Littlewood [42] in 1933. Later on Sack [52] considered the algebra $\mathbb{A}_{1;h}$ in the context of 'shift operators', and an operational interpretation was given by $[\mathfrak{D}, D] = -D$, equivalently $[\mathfrak{D}, Z] = Z$, where $\mathfrak{D} = ZD$ is the Euler operator. Important ordering results in $\mathbb{A}_{1;h}$ were obtained by Viskov in [57].

Note that a simple representation of the algebra $\mathbb{A}^{\langle f\rangle}$ can be constructed by considering operators U and V acting in $\mathcal{P}(\mathbb{C})$ by

$$U = f(Z)\,D, \quad V = Z.$$

The following proposition was proved by Irving [32], see also Viskov [58].

Proposition 2.50. *Let $p \in \mathcal{P}(\mathbb{C})$. In $\mathbb{A}^{\langle f\rangle}$, one has the Wick ordering result:*

$$U\,p(V) = p(V)\,U + p'(V)\,f(V).$$

Let δ be an operator acting on polynomials of the variable V as follows:

$$\delta\,p(V) = f(V)\,p'(V). \tag{2.70}$$

In particular,

$$\begin{aligned}
\delta V &= f(V),\\
\delta^2 V &= f(V)\,f'(V),\\
\delta^3 V &= f(V)\,f'(V)^2 + f(V)^2\,f''(V).
\end{aligned}$$

The following proposition is shown in [25].

Proposition 2.51. *In $\mathbb{A}^{\langle f\rangle}$, one has the Wick ordering result:*

$$U^n\,V = \sum_{k=0}^{n} \binom{n}{k} \left(\delta^{n-k} V\right) U^k,$$

where δ is the operator acting on polynomials of the variable V defined by (2.70)*. Here, we use the convention $\delta^0 V = V$.*

Remark 2.52. In the case where V appears in higher degree, one has:

$$U^n V^m = \sum_{k=0}^{n} \binom{n}{k} \left(\delta^{n-k} V^m\right) U^k.$$

In the shift algebra $\mathbb{A}_{1;h}$, $\delta V = hV$, hence Proposition 2.51 implies:

$$U^n V = V \sum_{k=0}^{n} \binom{n}{k} h^{n-k}\, U^k = V\,(U+h)^n.$$

More generally, if $f(V) = hV + g$, then $\delta V = hV + g$. Hence,

$$\delta^n V = h^{n-1}\,(hV+g),$$

and so Proposition 2.51 implies:

$$U^n\, V = \sum_{k=0}^{n} \binom{n}{k} h^{n-k-1}\,(hV+g)\, U^k.$$

The following proposition was proved in [32], see also [12].

Proposition 2.53. *In $\mathbb{A}^{\langle f \rangle}$, one has*

$$V\, U^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} U^{n-k} \left(\delta^k V\right).$$

More generally,

$$V^m\, U^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} U^{n-k} \left(\delta^k V^m\right). \tag{2.71}$$

Definition 2.54. Let $h \in \mathbb{C} \setminus \{0\}$ and let $s \in \mathbb{N}_0$. Let U and V satisfy $UV - VU = hV^s$, i.e., they generate $\mathbb{A}_{s;h}$. The *generalised Stirling numbers* $\mathfrak{S}_{s;h}(n,k)$ are defined for $n, k \in \mathbb{N}$ with $1 \le k \le n$ as the Wick ordering coefficients of $(VU)^n$, that is, by

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k)\, V^{s(n-k)+k}\, U^k.$$

By Katriel's theorem, $\mathfrak{S}_{0;1}(n,k) = S(n,k)$. More generally, $\mathfrak{S}_{0;h}(n,k) = h^{n-k}\, S(n,k)$.

The following proposition is shown by Viskov in [57].

Proposition 2.55. *In $\mathbb{A}_{1;1}$, one has*

$$\begin{aligned}(VU)^n &= \sum_{k=1}^{n} c(n,k)\, V^n\, U^k \\ &= V^n\, U\,(U+1)\ldots(U+(n-1)) \\ &= V^n\,(U)^{(n)}.\end{aligned}$$

More generally, in $\mathbb{A}_{1;h}$, one has

$$\begin{aligned}(VU)^n &= \sum_{k=1}^{n} h^{n-k}\, c(n,k)\, V^n\, U^k \\ &= V^n\, U\,(U+h)(U+2h)\cdots(U+h(n-1)) \\ &= V^n\,(U \mid -h)_n.\end{aligned}$$

By Proposition 2.55,

$$\mathfrak{S}_{1;1}(n,k) = c(n,k) = (-1)^{n-k}\, s(n,k),$$

more generally

$$\mathfrak{S}_{1;h}(n,k) = h^{n-k}\, c(n,k) = (-h)^{n-k}\, s(n,k). \tag{2.72}$$

Proposition 2.56. *For $h \in \mathbb{C} \setminus \{0\}$ and $s \in \mathbb{N}_0$, the generalised Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ satisfy the recurrence relation*

$$\mathfrak{S}_{s;h}(n+1,k) = \mathfrak{S}_{s;h}(n,k-1) + h[k+s(n-k)]\,\mathfrak{S}_{s;h}(n,k) \tag{2.73}$$

with the initial condition $\mathfrak{S}_{s;h}(1,1) = 1$, and $\mathfrak{S}_{s;h}(n,0) = 0$ for all $n \in \mathbb{N}_0$.

Remark 2.57. For $s = 1$, the equation (2.73) reduces to

$$\mathfrak{S}_{1;h}(n+1,k) = \mathfrak{S}_{1;h}(n,k-1) + hn\,\mathfrak{S}_{1;h}(n,k),$$

which is, for $h = 1$, precisely the recurrence relation of the signless Stirling numbers of the first kind, $c(n,k)$. For an arbitrary h, the generalised Stirling numbers are rescaled Stirling numbers of the first kind, see (2.72).

Theorem 2.58. *The generalised Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ are equal to the generalised Stirling numbers of Hsu and Shiue, $S(n,k;\alpha,\beta,r)$, with $\alpha = -hs$, $\beta = h(1-s)$ and $r = 0$, that is,*

$$\mathfrak{S}_{s;h}(n,k) = S(n,k;-hs,h(1-s),0).$$

Conversely, if $r = 0$ and $\alpha \neq \beta$, then the generalised Stirling numbers $S(n,k;\alpha,\beta,0)$ of Hsu and Shiue correspond to the case $s = \frac{\alpha}{\alpha-\beta}$ and $h = \beta - \alpha$ of the generalised Stirling numbers $\mathfrak{S}_{s;h}(n,k)$, that is,

$$S(n,k;\alpha,\beta,0) = \mathfrak{S}_{\frac{\alpha}{\alpha-\beta};\beta-\alpha}(n,k), \quad \alpha \neq \beta.$$

Finally we mention that Al-Salam and Ismail [52] proved the following result about anti-normal ordering.

Proposition 2.59. *In $\mathbb{A}_{1;-1}$, one has:*

$$(VU)^n = \left[\sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (U)_k\right] V^n.$$

Chapter 3

Normal ordering in a special class of generalised Weyl algebras

3.1 A special class of generalised Stirling numbers of Hsu and Shiue

Recall the generalised Stirling numbers of Hsu and Shiue, see Definition 2.40. For our considerations below, we will need these numbers in the special case when $\beta = -\alpha$ in (2.60). Below we prove a proposition that provides explicit formulas for the Hsu–Shiue numbers in this special case.

Proposition 3.1. *Let $a \in \mathbb{C}$ and $r \in \mathbb{C}$. Let the numbers $S(n, k; a, -a, r)$ satisfy*

$$(z + r \mid a)_n = \sum_{k=0}^{n} S(n, k; a, -a, r)(z \mid -a)_k.$$

Then,

$$S(n, 0; a, -a, r) = (r \mid a)_n \tag{3.1}$$

and for $k = 1, \dots, n$,

$$\begin{aligned} &S(n, k; a, -a, r) \\ &\quad = \sum_{j=0}^{n-k} \binom{n}{j} (-a)^{n-j-k} \, L(n-j, k) \, (r \mid a)_j \end{aligned} \tag{3.2}$$

$$= (-a)^{n-k}\, L(n,k) + \sum_{i=1}^{k} \left(\sum_{j=i}^{k} \binom{n}{j} (-a)^{n-k-i}\, L(n-j,k)\, s(j,i) \right) (-r)^i. \tag{3.3}$$

Proof. Recall that $((z \mid a)_n)_{n=0}^{\infty}$ is a polynomial sequence of binomial type. Hence, using (2.55) and (2.56), we get:

$$\begin{aligned}
(z + r \mid a)_n &= \sum_{k=0}^{n} \binom{n}{k} (z \mid a)_k\, (r \mid a)_{n-k} \\
&= (r \mid a)_n + \sum_{k=1}^{n} \binom{n}{k} (z \mid a)_k\, (r \mid a)_{n-k} \\
&= (r \mid a)_n + \sum_{k=1}^{n} \binom{n}{k} a^k \left(\frac{z}{a}\right)_k (r \mid a)_{n-k} \\
&= (r \mid a)_n + \sum_{k=1}^{n} \binom{n}{k} a^k\, (r \mid a)_{n-k} \times \sum_{i=1}^{k} (-1)^{k-i}\, L(k,i) \left(\frac{z}{a}\right)^{(i)} \\
&= (r \mid a)_n + \sum_{k=1}^{n} \binom{n}{k} a^k\, (r \mid a)_{n-k} \times \sum_{i=1}^{k} (-1)^{k-i} L(k,i)\, a^{-i} (z \mid -a)_i \\
&= (r \mid a)_n + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \binom{n}{k} a^{k-i}\, (r \mid a)_{n-k}\, (-1)^{k-i}\, L(k,i) \right) (z \mid -a)_i \\
&= (r \mid a)_n + \sum_{k=1}^{n} \left(\sum_{i=k}^{n} \binom{n}{i} (-a)^{i-k}\, (r \mid a)_{n-i}\, L(i,k) \right) (z \mid -a)_k.
\end{aligned}$$

Hence, (3.1) holds and for $k = 1, \ldots, n$,

$$\begin{aligned}
S(n,k; a, -a, r) &= \sum_{i=k}^{n} \binom{n}{i} (-a)^{i-k}\, (r \mid a)_{n-i}\, L(i,k) \\
&= \sum_{j=0}^{n-k} \binom{n}{n-j} (-a)^{n-j-k}\, L(n-j,k)\, (r \mid a)_j \\
&= \sum_{j=0}^{n-k} \binom{n}{j} (-a)^{n-j-k}\, L(n-j,k)\, (r \mid a)_j,
\end{aligned}$$

which proves (3.2). Furthermore, we continue the above calculations as follows:

$$\begin{aligned}
&= \sum_{j=0}^{n-k} \binom{n}{j} (-a)^{n-j-k}\, L(n-j,k)\, a^j \left(\frac{r}{a}\right)_j \\
&= (-a)^{n-k}\, L(n,k) + \sum_{j=1}^{k} \binom{n}{j} (-a)^{n-k}\, L(n-j,k) \times \sum_{i=1}^{j} s(j,i) \left(\frac{r}{a}\right)^i
\end{aligned}$$

$$= (-a)^{n-k} L(n,k) + \sum_{j=1}^{k} \sum_{i=1}^{j} \binom{n}{j} (-1)^i (-a)^{n-k-i} L(n-j,k)\, s(j,i) r^i$$

$$= (-a)^{n-k} L(n,k) + \sum_{i=1}^{k} \left(\sum_{j=i}^{k} \binom{n}{j} (-a)^{n-k-i} L(n-j,k)\, s(j,i) \right) (-r)^i,$$

which proves (3.3). □

3.2 Results on normal ordering in a class of generalised Weyl algebras

In this section, we will present a result on normal ordering in a class of generalised Weyl algebras.

Recall Definition 2.49. Let $a, b \in \mathbb{C}$ and consider the generalised Weyl algebras $\mathbb{A}^{\langle f \rangle}$ with $f(V) = -aV - b$. Thus, we are interested in the complex free algebras in two generators U and V satisfying the commutation relation

$$UV - VU = -aV - b,$$

equivalently

$$VU - UV = aV + b, \tag{3.4}$$

compare with (2.69).

Proposition 3.2. *For $n \in \mathbb{N}$, we have*

$$(UV)^n = \sum_{k=1}^{n} b^{n-k} S(n,k)\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^k$$
$$= \sum_{k=1}^{n} b^{n-k} S(n,k)\, (U \mid -a)_k\, V^k. \tag{3.5}$$

Remark 3.3. Note that, in the existent literature, one would normally consider the normal ordering of $(VU)^n$ in which all operators V are to the right of the operators U, while we have the opposite situation.

Proof of Proposition 3.2. We start with the following

Lemma 3.4. *We have*

$$V^n U = (U + na)\, V^n + nb\, V^{n-1} \tag{3.6}$$

Proof. Formula (3.6) is a special case of formula (2.71), which states, in particular, that

$$V^n U = UV^n - \delta(V^n) = UV^n + (aV + b)\, n\, V^{n-1}.$$

Nevertheless, for the reader's convenience, let us present a complete proof of (3.6) by induction.

For $n = 1$, formula (3.6) states

$$VU = (U + a)\, V + b,$$

which is just (3.4). Assume that (3.6) holds for n and let us prove it for $n + 1$.

We have, by (3.4),

$$\begin{aligned}
V^{n+1}U &= V(V^n U)\\
&= V[(U + na)V^n + nbV^{n-1}]\\
&= VUV^n + na\, V^{n+1} + nb\, V^n\\
&= (UV + aV + b)V^n + na\, V^{n+1} + nb\, V^n\\
&= UV^{n+1} + (n + 1)\, a\, V^{n+1} + (n + 1)\, b\, V^n.
\end{aligned}$$

□

Now we prove (3.5) by induction. For $n = 1$, (3.5) becomes the tautology $UV = UV$. Assume that (3.5) holds for n and let us prove it for $n + 1$. We have, by Lemma 3.4,

$$\begin{aligned}
(UV)^{n+1} &= (UV)^n\, (UV)\\
&= \sum_{k=1}^{n} U(U + a)(U + 2a) \cdots (U + (k - 1)a)\, V^k\, UV\, b^{n-k}\, S(n, k)\\
&= \sum_{k=1}^{n} b^{n-k}\, S(n, k)\, U(U + a)(U + 2a) \cdots (U + (k - 1)a)\, [(U + ka)\, V^k + kb\, V^{k-1}]\, V\\
&= \sum_{k=1}^{n} b^{n-k}\, S(n, k)\, U(U + a)(U + 2a) \cdots (U + ka)\, V^{k+1}\\
&\quad + \sum_{k=1}^{n} k\, b^{n-k+1}\, S(n, k)\, U(U + a)(U + 2a) \cdots (U + (k - 1)a)\, V^k
\end{aligned} \tag{3.7}$$

Recall that $S(n,0)=0$ and $S(n,n+1)=0$. Hence, we continue (3.7) as follows:

$$
\begin{aligned}
&= \sum_{k=0}^{n} S(n,k)\, b^{n-k}\, U(U+a)(U+2a)\cdots(U+ka)\, V^{k+1} \\
&\quad + \sum_{k=1}^{n+1} S(n,k)\, k\, b^{n-k+1}\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^{k} \\
&= \sum_{k=1}^{n+1} S(n,k-1)\, b^{n-k+1}\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^{k} \\
&\quad + \sum_{k=1}^{n+1} S(n,k)\, k\, b^{n-k+1}\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^{k} \\
&= \sum_{k=1}^{n+1} \big(S(n,k-1)+k\, S(n,k)\big)\, b^{n+1-k}\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^{k} \\
&= \sum_{k=1}^{n+1} b^{n+1-k}\, S(n+1,k)\, U(U+a)(U+2a)\cdots(U+(k-1)a)\, V^{k}. \qquad \square
\end{aligned}
$$

Chapter 4

Wick ordering and generalised Segal–Bargman transforms for the Meixner class of orthogonal polynomials

4.1 Poisson case

In this section, we will re-visit the (generalised) Segal–Bargmann transform for the Poisson distribution, compare with [9].

For $\varkappa > 0$, we denote by $\pi_\varkappa$ the Poisson distribution with parameter $\varkappa$:

$$\pi_\varkappa(dz) = e^{-\varkappa} \sum_{n=0}^{\infty} \frac{\varkappa^n}{n!}\, \delta_n(dz).$$

For $a \in \mathbb{C} \setminus \{0\}$, we denote by $\pi_{a,\varkappa}$ the push-forward of $\pi_\varkappa$ under the map $z \mapsto az$. Thus,

$$\pi_{a,\varkappa}(dz) = e^{-\varkappa} \sum_{n=0}^{\infty} \frac{\varkappa^n}{n!}\, \delta_{an}(dz),$$

and $\pi_{a,\varkappa}$ is concentrated on the set $a\mathbb{N}_0 = \{an \mid n \in \mathbb{N}_0\}$. In particular, $\pi_{1,\varkappa} = \pi_\varkappa$.

Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $\sigma > 0$. Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Charlier polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, s_n(z) = \exp\left(\frac{z}{\alpha}\, \log(1+t\alpha) - \frac{\sigma t}{\alpha}\right), \tag{4.1}$$

and let $\mu_{\alpha,0,\sigma}$ be its orthogonality measure given by (2.17), i.e.,

$$\mu_{\alpha,0,\sigma}(dz) = \exp\left(-\frac{\sigma}{\alpha^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\sigma}{\alpha^2}\right)^n \delta_{\alpha n}(dz). \tag{4.2}$$

Thus,

$$\mu_{\alpha,0,\sigma} = \pi_{\alpha,\frac{\sigma}{\alpha^2}}.$$

Recall that $(s_n(z))_{n=0}^{\infty}$ satisfy the recurrence relation

$$zs_n(z) = s_{n+1}(z) + \left(\alpha n + \frac{\sigma}{\alpha}\right) s_n(z) + \sigma n\, s_{n-1}(z). \tag{4.3}$$

We define operators $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ by

$$\begin{aligned}
\mathcal{U} &= Z + \frac{\sigma}{\alpha},\\
\mathcal{V} &= \alpha D + \mathbf{1} = \frac{\alpha}{\sigma}\left(\sigma D + \frac{\sigma}{\alpha}\right),
\end{aligned}$$

where

$$(Zp)(z) = z\, p(z), \quad (Dp)(z) = p'(z).$$

We have

$$[\mathcal{V}, \mathcal{U}] = \left[\alpha D + \mathbf{1}, Z + \frac{\sigma}{\alpha}\right] = [\alpha D, Z] = \alpha.$$

Hence, $\mathcal{V}$ and $\mathcal{U}$ are generators of Weyl algebra $\mathbb{A}_\alpha$.

Let

$$\rho = \mathcal{U}\mathcal{V} = Z + \alpha Z D + \frac{\sigma}{\alpha} + \sigma D.$$

Hence,

$$\rho z^n = z^{n+1} + \left(\alpha n + \frac{\sigma}{\alpha}\right) z^n + \sigma n z^{n-1}. \tag{4.4}$$

Consider the Sheffer operator $\mathcal{I} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ defined by

$$\mathcal{I} z^n = s_n(z). \tag{4.5}$$

Then, by (4.3), (4.4) and (4.5),

$$\mathcal{I}\rho = Z\mathcal{I}, \tag{4.6}$$

equivalently

$$Z = \mathcal{I}\rho\mathcal{I}^{-1}.$$

By Katriel's theorem (formula (2.66)),

$$\rho^n = \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\,\mathcal{U}^k\,\mathcal{V}^k.$$

Hence,

$$\rho^n 1 = \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\,\mathcal{U}^k 1 = \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\left(z+\frac{\sigma}{\alpha}\right)^k. \tag{4.7}$$

Denote

$$T_{\alpha;n}(z) := \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\,z^k = \alpha^n \sum_{k=1}^{n} S(n,k)\left(\frac{z}{\alpha}\right)^k = \alpha^n\,T_n\left(\frac{z}{\alpha}\right), \tag{4.8}$$

where $(T_n(z))_{n=0}^{\infty}$ is the sequence of Touchard polynomials, see (2.53).

Lemma 4.1. $(T_{\alpha;n}(z))_{n=0}^{\infty}$ *is a polynomial sequence of binomial type with generating function*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,T_{\alpha;n}(z) = \exp\left(z\,\frac{e^{t\alpha}-1}{\alpha}\right).$$

Proof. By (2.54) and (4.8),

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,T_{\alpha;n}(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(t\alpha)^n\,T_n\left(\frac{z}{\alpha}\right) = \exp\left(z\,\frac{e^{t\alpha}-1}{\alpha}\right). \qquad \square$$

Thus, by (4.7) and (4.8),

$$\rho^n 1 = T_{\alpha;n}\left(z+\frac{\sigma}{\alpha}\right). \tag{4.9}$$

Also, by (4.7), we have

$$\begin{aligned}
\rho^n 1 &= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k} \sum_{i=0}^{k} \binom{k}{i} z^i \left(\frac{\sigma}{\alpha}\right)^{k-i} \\
&= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-2k}\,\sigma^k + \sum_{k=1}^{n}\sum_{i=1}^{k} S(n,k)\binom{k}{i}\alpha^{n-2k+i}\,z^i\,\sigma^{k-i}
\end{aligned}$$

$$= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-2k}\,\sigma^k + \sum_{i=1}^{n}\left(\sum_{k=i}^{n} S(n,k)\binom{k}{i}\alpha^{n-2k+i}\,\sigma^{k-i}\right)z^i$$

$$= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-2k}\,\sigma^k + \sum_{i=1}^{n}\left(\sum_{k=0}^{n-i} S(n,k+i)\binom{k+i}{i}\alpha^{n-2k-i}\,\sigma^{k}\right)z^i. \tag{4.10}$$

We have, by (4.6),

$$\mathcal{I}\rho^n 1 = Z^n\mathcal{I}1 = z^n. \tag{4.11}$$

Hence, (4.5) and (4.10) imply

$$z^n = \sum_{k=1}^{n} S(n,k)\,\alpha^{n-2k}\,\sigma^k + \sum_{i=1}^{n}\left(\sum_{k=0}^{n-i} S(n,k+i)\binom{k+i}{i}\alpha^{n-2k-i}\,\sigma^{k}\right)s_i(z). \tag{4.12}$$

Remark 4.2. Note that, in formula (4.12), the coefficient by $s_i(z)$ is a polynomial of σ of degree $n-i$.

Next, let $\mathcal{T}_\alpha \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ be the umbral operator defined by

$$\mathcal{T}_\alpha z^n := T_{\alpha;n}(z).$$

Define the Sheffer operator $\mathbb{S} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ by

$$\mathbb{S}z^n = T_{\alpha;n}\left(z+\frac{\sigma}{\alpha}\right),$$

i.e.

$$\mathbb{S} = E_{\frac{\sigma}{\alpha}}\mathcal{T}_\alpha. \tag{4.13}$$

By (4.9) and (4.11)

$$\mathcal{I}T_{\alpha;n}\left(z+\frac{\sigma}{\alpha}\right) = z^n,$$

and so

$$\mathcal{I}\mathbb{S}z^n = z^n. \tag{4.14}$$

Since both $\mathcal{I}, \mathbb{S} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ are bijective, formula (4.14) implies that

$$\mathcal{I} = \mathbb{S}^{-1}. \tag{4.15}$$

Since $\mathcal{T}_\alpha$ is an umbral operator, it is invertible. Hence, by (4.13),

$$\mathcal{I} = \mathcal{T}_\alpha^{-1}\,E_{-\frac{\sigma}{\alpha}}. \tag{4.16}$$

Lemma 4.3. *We have*

$$\mathcal{T}_\alpha^{-1} z^n = \sum_{k=1}^{n} s(n,k)\,\alpha^{n-k}\,z^k \tag{4.17}$$

$$= (z \mid \alpha)_k. \tag{4.18}$$

Proof. Let $\mathcal{R}_\alpha$ be the operator defined by

$$\mathcal{R}_\alpha z^n = \sum_{k=1}^{n} s(n,k)\,\alpha^{n-k}\,z^k.$$

Let us show that

$$\mathcal{R}_\alpha \mathcal{T}_\alpha z^n = z^n,$$

which will imply that $\mathcal{R}_\alpha = \mathcal{T}_\alpha^{-1}$, and so formula (4.17) will hold.

We have, by using the orthogonality property of Stirling numbers,

$$\begin{aligned}
\mathcal{R}_\alpha \mathcal{T}_\alpha z^n &= \mathcal{R}_\alpha \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\,z^k \\
&= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k}\,\mathcal{R}_\alpha z^k \\
&= \sum_{k=1}^{n} S(n,k)\,\alpha^{n-k} \sum_{i=1}^{k} s(k,i)\,\alpha^{k-i}\,z^i \\
&= \sum_{k=1}^{n} \sum_{i=1}^{k} S(n,k)\,s(k,i)\,\alpha^{n-i}\,z^i \\
&= \sum_{i=1}^{n} \left(\sum_{k=i}^{n} S(n,k)\,s(k,i) \right) \alpha^{n-i}\,z^i \\
&= \sum_{i=1}^{n} \delta_{n,i}\,\alpha^{n-i}\,z^i = z^n.
\end{aligned}$$

Formula (4.18) follows from Definition 2.33 and (2.56). □

Lemma 4.4. *We have*

$$s_n(z) = \left(-\frac{\sigma}{\alpha} \right)^n + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-i} \binom{n}{k} s(n-k,i)\,\alpha^{n-2k-i}\,(-1)^k \sigma^k \right) z^i.$$

Proof. By (4.5), (4.16) and (4.17), we have

$$s_n(z) = \mathcal{I} z^n = \mathcal{T}_\alpha^{-1}\,E_{-\frac{\sigma}{\alpha}}\,z^n$$

$$
\begin{aligned}
&= \mathcal{T}_\alpha^{-1}\Big(z - \frac{\sigma}{\alpha}\Big)^n \\
&= \mathcal{T}_\alpha^{-1}\sum_{k=0}^{n}\binom{n}{k} z^k\Big(-\frac{\sigma}{\alpha}\Big)^{n-k} \\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{k=1}^{n}\binom{n}{k}\Big(-\frac{\sigma}{\alpha}\Big)^{n-k}\mathcal{T}_\alpha^{-1} z^k \qquad (4.19)\\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{k=1}^{n}\binom{n}{k}\Big(-\frac{\sigma}{\alpha}\Big)^{n-k}\sum_{i=1}^{k} s(k,i)\,\alpha^{k-i}\,z^i \\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{k=1}^{n}\sum_{i=1}^{k} s(k,i)\binom{n}{k}(-\sigma)^{n-k}\,\alpha^{2k-n-i}\,z^i \\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{i=1}^{k}\bigg(\sum_{k=i}^{n}\binom{n}{k}\,s(k,i)\,(-\sigma)^{n-k}\,\alpha^{2k-n-i}\bigg)z^i \\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{i=1}^{n}\bigg(\sum_{k=0}^{n-i}\binom{n}{k}\,s(n-k,i)\,\alpha^{n-2k-i}\,(-1)^k\sigma^k\bigg)z^i. \qquad \square
\end{aligned}
$$

Theorem 4.5. *Let $\alpha \neq 0$ and $\sigma > 0$. Let $(s_n(z))_{n=0}^\infty$ be the sequence of Charlier polynomials with generating function*

$$
\sum_{n=0}^{\infty}\frac{t^n}{n!}\,s_n(z) = \exp\left(\frac{z}{\alpha}\,\log(1+t\alpha) - \frac{\sigma t}{\alpha}\right), \qquad (4.20)
$$

equivalently satisfying the recurrence relation

$$
zs_n(z) = s_{n+1}(z) + \Big(\alpha n + \frac{\sigma}{\alpha}\Big)s_n(z) + \sigma n\, s_{n-1}(z).
$$

Then we have

$$
z^n = \sum_{k=1}^{n} S(n,k)\,\alpha^{n-2k}\,\sigma^k + \sum_{i=1}^{n}\bigg(\sum_{k=0}^{n-i} S(n,k+i)\binom{k+i}{i}\alpha^{n-2k-i}\,\sigma^k\bigg)s_i(z), \qquad (4.21)
$$

and

$$
\begin{aligned}
s_n(z) &= \sum_{k=0}^{n}\binom{n}{k}\Big(-\frac{\sigma}{\alpha}\Big)^{n-k}(z\mid\alpha)_k \qquad (4.22)\\
&= \Big(-\frac{\sigma}{\alpha}\Big)^n + \sum_{i=1}^{n}\bigg(\sum_{k=0}^{n-i}\binom{n}{k}\,s(n-k,i)\,\alpha^{n-2k-i}\,(-1)^k\sigma^k\bigg)z^i. \qquad (4.23)
\end{aligned}
$$

In particular, in the Poisson case, i.e., when $\alpha = 1$, we have

$$
z^n = \sum_{k=1}^{n} S(n,k)\,\sigma^k + \sum_{i=1}^{n}\bigg(\sum_{k=0}^{n-i} S(n,k+i)\binom{k+i}{i}\,\sigma^k\bigg)s_i(z), \qquad (4.24)
$$

and

$$\begin{aligned} s_n(z) &= \sum_{k=0}^{n} \binom{n}{k} (-\sigma)^{n-k} (z)_k \\ &= (-\sigma)^n + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-i} \binom{n}{k} s(n-k,i) (-\sigma)^k \right) z^i. \end{aligned}$$

Proof. We already proved formulas (4.21) and (4.23). Formula (4.22) follows from (4.18) and (4.19). □

Remark 4.6. Formulas (4.22) and (4.23) are known, see e.g. Section 3.3 in [50]. The reader is advised to compare formula (4.24) with an expansion of $\binom{z}{n}$ in the Charlier polynomials discussed in Example 1 on page 479 of [34].

Recall that, by (4.5) and (4.15), the operator $\mathbb{S} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ satisfies

$$(\mathbb{S} s_n)(z) = z^n. \tag{4.25}$$

Hence, similarly to Subsection 2.7, we can extend $\mathbb{S}$ to the unitary operator

$$\mathbb{S} : L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma}) \to \mathbb{F}_\sigma(\mathbb{C}),$$

which is a (generalised) Segal–Bargmann transform.

Analogously to (2.43), we consider, for $\xi \in \alpha\mathbb{N}_0$ and $z \in \mathbb{C}$, the corresponding coherent state

$$\begin{aligned} \mathbb{E}_{\alpha,0,\sigma}(\xi, z) = \eta_z(\xi) &= \sum_{n=0}^{\infty} \frac{1}{n!} s_n(\xi) \left(\frac{z}{\sigma}\right)^n \\ &= \left(1 + \frac{\alpha z}{\sigma}\right)^{\frac{\xi}{\alpha}} \exp\left(-\frac{z}{\sigma}\right). \end{aligned} \tag{4.26}$$

Hence, by (2.17) and (2.46), for $f \in L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$,

$$\begin{aligned} (\mathbb{S}f)(z) &= \int_{\alpha\mathbb{N}_0} f(\xi)\, \mathbb{E}_{\alpha,0,\sigma}(\xi, z)\, \mu_{\alpha,0,\sigma}(d\xi) \\ &= \sum_{n=0}^{\infty} f(\alpha n) \left(1 + \frac{\alpha z}{\sigma}\right)^n \exp\left(-\frac{z}{\alpha}\right) \exp\left(-\frac{\sigma}{\alpha^2}\right) \frac{1}{n!} \left(\frac{\sigma}{\alpha^2}\right)^n \\ &= \exp\left(-\frac{z}{\alpha} - \frac{\sigma}{\alpha^2}\right) \sum_{n=0}^{\infty} f(\alpha n) \frac{1}{n!} \left(\frac{\sigma}{\alpha^2} + \frac{z}{\alpha}\right)^n \\ &= \exp\left(-\frac{z}{\alpha} - \frac{\sigma}{\alpha^2}\right) \sum_{n=0}^{\infty} f(\alpha n) \frac{1}{n!} \left(\frac{\sigma + \alpha z}{\alpha^2}\right)^n. \end{aligned}$$

Lemma 4.7. *For each $f \in L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$, the series*

$$\sum_{n=0}^{\infty} f(\alpha n)\,\frac{1}{n!}\left(\frac{\sigma+\alpha z}{\alpha^2}\right)^n \tag{4.27}$$

converges absolutely for all $z \in \mathbb{C}$.

Proof. We have, using the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{n=0}^{\infty} |f(\alpha n)|\,\frac{1}{n!}\left|\frac{\sigma+\alpha z}{\alpha^2}\right|^n &\le \sum_{n=0}^{\infty} |f(\alpha n)|\,\frac{1}{n!}\left(\frac{\sigma+|\alpha z|}{\alpha^2}\right)^n\\ &= \sum_{n=0}^{\infty} |f(\alpha n)|\,\left(\frac{1}{n!}\right)^{\frac12}\left(\frac{\sigma}{\alpha^2}\right)^{\frac{n}{2}} \times \left(\frac{1}{n!}\right)^{\frac12}\left(\frac{\sigma+|\alpha z|}{\alpha^2}\right)^n\left(\frac{\sigma}{\alpha^2}\right)^{-\frac{n}{2}}\\ &\le \left(\sum_{n=0}^{\infty} |f(\alpha n)|^2\,\frac{1}{n!}\left(\frac{\sigma}{\alpha^2}\right)^n\right)^{\frac12}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{(\sigma+|\alpha z|)^2}{\alpha^2\sigma}\right)^n\right)^{\frac12} < \infty.\end{aligned}$$

□

Let $z \in \mathbb{R}$, $z > -\frac{\sigma}{\alpha}$. By Lemma 4.7, each $f \in L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$ is integrable with respect to the measure $\mu_{\alpha,0,\sigma+\alpha z}$ and we have

$$(\mathbb{S}f)(z) = \int_{\alpha\mathbb{N}_0} f(\xi)\,\mu_{\alpha,0,\sigma+\alpha z}(d\xi). \tag{4.28}$$

Definition 4.8. For each $\varkappa \in \mathbb{C}$, we define a complex-valued (or signed for $z \in (-\infty, 0)$) measure on $\mathbb{N}_0$ by

$$\pi_\varkappa(d\xi) = e^{-\varkappa}\sum_{n=0}^{\infty}\frac{\varkappa^n}{n!}\,\delta_n(d\xi).$$

(Note that $\pi_\varkappa(\mathbb{N}_0) = 1$.) Similarly, for each $a \in \mathbb{C}\setminus\{0\}$, we define a complex-valued measure on $a\mathbb{N}_0$ by

$$\pi_{a,\varkappa}(d\xi) = e^{-\varkappa}\sum_{n=0}^{\infty}\frac{\varkappa^n}{n!}\,\delta_{an}(d\xi).$$

(Again with $\pi_{a,\varkappa}(a\mathbb{N}_0) = 1$.)

Now, for each $z \in \mathbb{C}$, we define

$$\mu_{\alpha,0,z}(d\xi) = \pi_{\alpha,\frac{z}{\alpha^2}}(d\xi) = \exp\left(-\frac{z}{\alpha^2}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{\alpha^2}\right)^n\delta_{\alpha n}(d\xi) \tag{4.29}$$

is a complex-valued measure on $\alpha\mathbb{N}_0$ with $\mu_{\alpha,0,z}(\alpha\mathbb{N}_0) = 1$. Hence, by Lemma 4.7, formula (4.28) holds for all $z \in \mathbb{C}$. Thus, we have proved the following

Theorem 4.9. *Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Charlier polynomials as in Theorem 4.5 and let $\mu_{\alpha,0,\sigma}$ be its orthogonality measure, given by* (4.2)*. Let, for each $z \in \mathbb{C}$, $\mu_{\alpha,0,z}$ be defined by* (4.29)*. Then the Segal–Bargmann transform*

$$\mathbb{S} : L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma}) \to \mathbb{F}_\sigma(\mathbb{C})$$

is a unitary operator given by

$$(\mathbb{S}f)(z) = \int_{\alpha\mathbb{N}_0} f(\xi)\, \mu_{\alpha,0,\sigma+\alpha z}(d\xi) \tag{4.30}$$

for all $z \in \mathbb{C}$. In particular, for $z \in \mathbb{R}$, $z > -\frac{\sigma}{\alpha}$, the integration on the right-hand side of formula (4.30) *is with respect to a probability measure.*

Particularly, for $\alpha = 1$ (the Poisson case), we have

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} f(\xi)\, \mu_{1,0,\sigma+z}\,(d\xi) = \int_{\mathbb{N}_0} f(\xi)\, \pi_{\sigma+z}\,(d\xi).$$

Remark 4.10. The reader is advised to compare Theorem 4.9 with formula (2.48) that holds in the Gaussian case. Note that, in the Gaussian case, for $z \in \mathbb{R}$, $\mu_\sigma(-z + d\xi)$ is the Gaussian distribution with mean z and variance σ. On the other hand, in the Poisson case, for $z > -\frac{\sigma}{\alpha}$, $\mu_{\alpha,0,\sigma+\alpha z}$ is a distribution that has mean $\frac{\sigma}{\alpha} + z$ and variance $\sigma + \alpha z$.

Remark 4.11. In the Gaussian case, we may also interpret the $\mathbb{S}$-transform for all $z \in \mathbb{C}\setminus\mathbb{R}$ as integration with respect to a complex-valued measure. Indeed, for $z \in \mathbb{C} \setminus \mathbb{R}$, define the complex-valued measure

$$\mu_{\sigma,z}(d\xi) := \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(\xi - z)^2}{2\sigma}\right) d\xi.$$

Note that, by (2.47)

$$\mu_{\sigma,z}(\mathbb{R}) = \mathbb{S}1 = 1.$$

Next, we note that, for each $f \in L^2(\mathbb{R}, \mu_\sigma)$,

$$\int_{\mathbb{R}} |f(\xi)| \left|\exp\left(-\frac{(\xi - z)^2}{2\sigma}\right)\right| d\xi < \infty.$$

Indeed, let $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$. Then

$$\left|\exp\left(-\frac{(\xi - z)^2}{2\sigma}\right)\right| = \exp\left(-\frac{\Re[(\xi - z)^2]}{2\sigma}\right)$$

$$
\begin{aligned}
&= \exp\left(-\frac{\Re((\xi - z_1 - iz_2)^2)}{2\sigma}\right)\\
&= \exp\left(-\frac{(\xi - z_1)^2 - {z_2}^2}{2\sigma}\right)\\
&= \exp\left(-\frac{\xi^2}{2\sigma} + \frac{2\xi z_1 - {z_1}^2 + {z_2}^2}{2\sigma}\right).
\end{aligned}
$$

Hence, by Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}} |f(\xi)| \left|\exp\left(-\frac{(\xi - z)^2}{2\sigma}\right)\right| \, d\xi\\
&\quad \le \left(\int_{\mathbb{R}} |f(\xi)|^2 \exp\left(-\frac{\xi^2}{2\sigma}\right) \, d\xi\right)^{1/2} \times \left(\int_{\mathbb{R}} \exp\left(\frac{\xi^2}{2\sigma}\right) \left|\exp\left(-\frac{(\xi - z)^2}{2\sigma}\right)\right|^2 \, d\xi\right)^{1/2}\\
&\quad = \|f\|_{L^2(\mathbb{R},\mu_\sigma)} \left(\int_{\mathbb{R}} \exp\left(\frac{\xi^2}{2\sigma} - \frac{\xi^2}{\sigma} + \frac{2\xi z_1 - {z_1}^2 + {z_2}^2}{\sigma}\right) \, d\xi\right)^{1/2}\\
&\quad = \|f\|_{L^2(\mathbb{R},\mu_\sigma)} \left(\int_{\mathbb{R}} \exp\left(-\frac{\xi^2}{2\sigma} + \frac{2\xi z_1 - {z_1}^2 + {z_2}^2}{\sigma}\right) \, d\xi\right)^{1/2} < \infty.
\end{aligned}
$$

Therefore, by (2.47), for all $f \in L^2(\mathbb{R}, \mu_\sigma)$ and all $z \in \mathbb{C}$,

$$
(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(\xi) \, \mu_{\sigma,z}(d\xi).
$$

Let ∂^- denote the lowering operator for the polynomial sequence $(s_n(z))_{n=0}^\infty$, i.e.

$$
\partial^- s_n = n s_{n-1}.
$$

Let ∂^+ denote the raising operator for the polynomial sequence $(s_n(z))_{n=0}^\infty$, i.e.

$$
\partial^+ s_n = s_{n+1}.
$$

The restriction to $\mathcal{P}(\mathbb{R})$ of the adjoint operator of ∂^+ in $L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$ is $\sigma\partial^-$.

Hence,

$$
\|s_n\|^2_{L^2(\alpha\mathbb{N}_0,\mu_{\alpha,0,\sigma})} = n! \, \sigma^n.
$$

Thus, a function $f : \alpha\mathbb{N}_0 \to \mathbb{C}$ belongs to $L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$ if and only if it can be uniquely represented in the form

$$
f(z) = \sum_{n=0}^\infty f_n \, s_n(z), \tag{4.31}
$$

where $f_n \in \mathbb{C}$ ($n \in \mathbb{N}_0$) satisfy

$$\|f\|^2_{L^2(\alpha\mathbb{N}_0,\mu_{\alpha,0,\sigma})} = \sum_{n=0}^{\infty} |f_n|^2\, n!\, \sigma^n < \infty.$$

By (4.20), the Charlier polynomials $(s_n(z))_{n=0}^{\infty}$ satisfy the condition of Theorem 2.24. Hence, by Corollary 2.26, for $\tau \in (0,1]$, each function $f \in \mathcal{E}^{\tau}_{\min}(\mathbb{C})$ admits a unique representation as in (4.31) with

$$\sum_{n=0}^{\infty} |f_n|^2\, (n!)^{\frac{2}{\tau}}\, 2^{nl} < \infty \quad \text{for all } l \in \mathbb{N},$$

and furthermore, the topology on $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ is given by (2.9), (2.10). For each $\tau \in (0,1]$ and $l \in \mathbb{N}$, there exists $C > 0$ such that

$$\left\| \sum_{n=0}^{\infty} f_n s_n(z) \right\|_{L^2(\alpha\mathbb{N}_0,\mu_{\alpha,0,\sigma})} \leq \quad C \, \Big|\!\Big|\!\Big| \sum_{n=0}^{\infty} f_n s_n(z) \Big|\!\Big|\!\Big|_{\tau,l}.$$

Hence, for each $f \in \mathcal{E}^{\tau}_{\min}(\mathbb{C})$, the restriction of f to $\alpha\mathbb{N}_0$ determines a function from $L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$, and furthermore, if a restriction of f to $\alpha\mathbb{N}_0$ is identically equal to zero, then the function f is identically equal to zero on $\mathbb{C}$. Thus, we obtain an embedding of $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ into $L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$ and this embedding is continuous.

Thus, Theorem 2.24 and formula (4.25) imply

Proposition 4.12. *Let the conditions of Theorem 4.9 be satisfied. Then, for each* $\tau \in (0,1]$*, the Fréchet space* $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ *is continuously embedded into* $L^2(\alpha\mathbb{N}_0, \mu_{\alpha,0,\sigma})$ *(in the above explained sense). Furthermore, the operator* $\mathbb{S}$ *from Theorem 4.9 restricted to* $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$ *is a self-homeomorphism of* $\mathcal{E}^{\tau}_{\min}(\mathbb{C})$*.*

Next, denote

$$U := \mathcal{I}\mathcal{U}\mathbb{S}, \quad V := \mathcal{I}\mathcal{V}\mathbb{S}. \tag{4.32}$$

Since $\rho = \mathcal{U}\mathcal{V}$ and $Z = \mathcal{I}\rho\mathbb{S}$,

$$Z = UV.$$

Since

$$\partial^+ = \mathcal{I}Z\mathbb{S}, \quad \partial^- = \mathcal{I}D\mathbb{S},$$

we get

$$U = \partial^+ + \frac{\sigma}{\alpha}, \quad V = \alpha\partial^- + \mathbf{1} = \frac{\alpha}{\sigma}\left(\sigma\partial^- + \frac{\sigma}{\alpha}\right).$$

Note that the adjoint of $\partial^+ + \frac{\sigma}{\alpha}$ is $\sigma\partial^- + \frac{\sigma}{\alpha}$.

Proposition 4.13. *The operators ∂^- and V are closable and let us keep the notation ∂^- and V for their respective closures. Then, for each $z \in \mathbb{C}$, the coherent state $\eta_z(\xi)$ given by* (4.26) *belongs to the domain of ∂^- (which coincides with the domain of V) and*

$$\begin{aligned}(\sigma\partial^- \eta_z)(\xi) &= z\,\eta_z(\xi),\\ (V\,\eta_z)(\xi) &= \frac{\alpha}{\sigma}\left(z + \frac{\sigma}{\alpha}\right)\eta_z(\xi).\end{aligned} \tag{4.33}$$

Proof. Since the adjoint of ∂^- is densely defined (it coincides with $\frac{1}{\sigma}\,\partial^+$ on the polynomials), ∂^- is closable, hence also V is closable. As easily seen, for each $z \in \mathbb{C}$, $\eta_z(\xi)$ belongs to the domain of ∂^-.

We then have

$$\begin{aligned}(\sigma\partial^- \eta_z)(\xi) &= \sigma\partial^- \sum_{n=0}^{\infty} \frac{z^n}{n!\,\sigma^n}\, s_n(\xi) = \sum_{n=1}^{\infty} \frac{\sigma z^n n}{n!\,\sigma^n}\, s_{n-1}(\xi)\\ &= z\sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!\,\sigma^{n-1}}\, s_{n-1}(\xi) = z\sum_{n=0}^{\infty} \frac{z^n}{n!\,\sigma^n}\, s_n(\xi) = z\,\eta_z(\xi),\end{aligned}$$

which in turn also implies (4.33). □

Let us find the explicit form of U and V. By Theorem 2.11, Remark 2.12 and Theorem 2.16,

$$\partial^- = B(D), \tag{4.34}$$

where $B(t)$ is the compositional inverse of $A(t) = \frac{1}{\alpha}\log(1 + t\alpha)$. Hence,

$$B(t) = \frac{1}{\alpha}(e^{\alpha t} - 1).$$

Therefore, by (4.34) and Boole's formula,

$$\partial^- = \frac{1}{\alpha}(e^{\alpha D} - 1) = \frac{1}{\alpha}(E^{\alpha} - 1).$$

Hence,

$$V = \alpha\partial^- + 1 = E^{\alpha}.$$

Next, since

$$Z = UV = UE^{\alpha},$$

by multiplying this equality by $E_{-\alpha}$ on the right, we get

$$U = ZE^{-\alpha}.$$

Thus, we have proved

Proposition 4.14. *Under the condition of Theorem 4.5, we get, for U and V defined by* (4.32),

$$U = ZE^{-\alpha}, \quad V = E^{\alpha}.$$

Below, we will also need the following results.

Proposition 4.15. *For each $a \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$, we have*

$$\int_{a\mathbb{N}_0} (\xi \mid a)_n \ \pi_{a,\frac{z}{a}}(d\xi) = \int_{\mathbb{N}_0} (a\xi \mid a)_n \ \pi_{\frac{z}{a}}(d\xi) = z^n. \tag{4.35}$$

Proof. By (4.21) and (2.53),

$$\begin{aligned}\int_{\alpha\mathbb{N}_0} \xi^n \, \mu_{\alpha,0,\sigma}(d\xi) &= \sum_{k=1}^{n} S(n,k)\, \alpha^{n-2k}\, \sigma^k \\ &= \alpha^n \sum_{k=1}^{n} S(n,k) \left(\frac{\sigma}{\alpha^2}\right)^k \\ &= \alpha^n\, T_n\left(\frac{\sigma}{\alpha^2}\right),\end{aligned}$$

and so

$$\int_{\alpha\mathbb{N}_0} \left(\frac{\xi}{\alpha}\right)^n \ \pi_{\alpha,\sigma}(d\xi) = T_n\left(\frac{\sigma}{\alpha^2}\right).$$

Hence, by the orthogonality property of Stirling numbers, see (2.52), we get

$$\begin{aligned}\int_{\alpha\mathbb{N}_0} \left(\frac{\xi}{\alpha}\right)_n \ \mu_{\alpha,0,\sigma}(d\xi) &= \int_{\alpha\mathbb{N}_0} \sum_{k=1}^{n} s(n,k) \left(\frac{\xi}{\alpha}\right)^k \ \mu_{\alpha,0,\sigma}(d\xi) \\ &= \sum_{k=1}^{n} s(n,k)\ T_k\left(\frac{\sigma}{\alpha^2}\right) \\ &= \sum_{k=1}^{n} s(n,k) \sum_{i=1}^{k} S(k,i) \left(\frac{\sigma}{\alpha^2}\right)^i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left(\sum_{k=i}^{n} s(n,k)\, S(k,i) \right) \left(\frac{\sigma}{\alpha^2} \right)^i \\
&= \left(\frac{\sigma}{\alpha^2} \right)^n .
\end{aligned}
$$

From here

$$
\begin{aligned}
\int_{\alpha\mathbb{N}_0} (\xi \mid \alpha)_n \ \mu_{\alpha,0,\sigma}\,(d\xi) &= \alpha^n \int_{\alpha\mathbb{N}_0} \left(\frac{\xi}{\alpha} \right)_n \ \mu_{\alpha,0,\sigma}\,(d\xi) \\
&= \alpha^n \left(\frac{\sigma}{\alpha^2} \right)^n = \left(\frac{\sigma}{\alpha} \right)^n ,
\end{aligned}
$$

and so, for each $z > 0$

$$
\int_{\alpha\mathbb{N}_0} (\xi \mid \alpha)_n \ \mu_{\alpha,0,\alpha z}\,(d\xi) = z^n. \tag{4.36}
$$

As easily seen,

$$
\mathbb{C} \ni z \mapsto \int_{\alpha\mathbb{N}_0} (\xi \mid \alpha)_n \ \mu_{\alpha,0,\alpha z}\,(d\xi)
$$

is entire function. Hence, formula (4.36) holds for all $z \in \mathbb{C}$ by the uniqueness theorem for holomorphic functions, see e.g. [33, Theorem 22.12].

Thus, we have proved formula (4.35) for $a > 0$ and $z \in \mathbb{C}$. But, as easily seen, for a fixed $z \in \mathbb{C}$, the left-hand side of formula (4.35) determines a holomorphic functions of $a \in \mathbb{C} \setminus \{0\}$. Applying again the uniqueness theorem, we conclude that formula (4.35) holds for all $a \in \mathbb{C} \setminus \{0\}$. $\square$

4.2 Gamma case

Let $\alpha = \beta > 0$ and $\sigma > 0$. Thus, $\lambda = 2\alpha$ and $\eta = \alpha^2$. Let also $l = \frac{\sigma}{\alpha}$.

Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Laguerre polynomials with generating function

$$
\sum_{n=0}^{\infty} \frac{t^n}{n!}\, s_n(z) = G(z,t) = \exp\left(\frac{zt}{1+\alpha t} - \frac{\sigma}{\alpha^2} \log(1+\alpha t) \right). \tag{4.37}
$$

Hence, $(s_n(z))_{n=0}^{\infty}$ satisfies the recurrence relation

$$
z s_n(z) = s_{n+1}(z) + (\lambda n + l)\, s_n(z) + (\sigma n + \eta n(n-1))\, s_{n-1}(z). \tag{4.38}
$$

We define $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ by

$$
\mathcal{U} := Z + \frac{\sigma}{\alpha},
$$

$$\mathcal{V} := \alpha D_\alpha + \mathbf{1} = E^\alpha, \tag{4.39}$$

where

$$D_\alpha\, p(z) = \frac{p(z+\alpha) - p(z)}{\alpha}.$$

Lemma 4.16. *The operators $\mathcal{U}$ and $\mathcal{V}$ generate a generalised Weyl algebra with the commutation relation*

$$[\mathcal{U}, \mathcal{V}] = -\alpha\mathcal{V}.$$

Proof. We have

$$[\mathcal{U}, \mathcal{V}] = [Z + \frac{\sigma}{\alpha}, E^\alpha] = [Z, E^\alpha] = -\alpha E^\alpha = -\alpha\mathcal{V}.$$

□

Let $\rho = \mathcal{U}\mathcal{V}$. Then

$$\begin{aligned}\rho &= \left(Z + \frac{\sigma}{\alpha}\right)\left(\alpha D_\alpha + \mathbf{1}\right) \\ &= \alpha Z D_\alpha + Z + \sigma D_\alpha + \frac{\sigma}{\alpha}.\end{aligned}$$

Lemma 4.17. *Define $\mathcal{I} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ by*

$$\mathcal{I}(z \mid \alpha)_n = s_n(z). \tag{4.40}$$

Then

$$\mathcal{I}\rho = Z\mathcal{I},$$

equivalently

$$Z = \mathcal{I}\rho\mathcal{I}^{-1}.$$

Proof. By (2.59),

$$\begin{aligned} z(z \mid \alpha)_n &= z\, z(z-\alpha)(z-2\alpha)\cdots(z-(n-1)\alpha) \\ &= (z - n\alpha + n\alpha)\, z(z-\alpha)(z-2\alpha)\cdots(z-(n-1)\alpha) \\ &= z(z-\alpha)(z-2\alpha)\cdots(z-n\alpha) + n\alpha\, z(z-\alpha)(z-2\alpha)\cdots(z-(n-1)\alpha) \\ &= (z \mid \alpha)_{n+1} + n\alpha(z \mid \alpha)_n. \end{aligned} \tag{4.41}$$

Therefore,

$$
\begin{aligned}
\rho(z \mid \alpha)_n &= \alpha n\, z(z \mid \alpha)_{n-1} + z(z \mid \alpha)_n + \sigma n(z \mid \alpha)_{n-1} + \frac{\sigma}{\alpha}\,(z \mid \alpha)_n \\
&= \alpha n(z \mid \alpha)_n + \alpha n\alpha(n-1)(z \mid \alpha)_{n-1} + (z \mid \alpha)_{n+1} + n\alpha(z \mid \alpha)_n + \sigma n(z \mid \alpha)_{n-1} \\
&\quad + \frac{\sigma}{\alpha}\,(z \mid \alpha)_n \\
&= (z \mid \alpha)_{n+1} + \left(2\alpha n + \frac{\sigma}{\alpha}\right)(z \mid \alpha)_n + (\sigma n + \sigma^2 n(n-1))(z \mid \alpha)_{n-1} \\
&= (z \mid \alpha)_{n+1} + (\lambda n + l)(z \mid \alpha)_n + (\sigma n + \eta\, n(n-1))(z \mid \alpha)_{n-1}. \qquad (4.42)
\end{aligned}
$$

□

Consider $\rho^n = (\mathcal{U}\mathcal{V})^n$. By Proposition 3.2 with $a = \alpha$ and $b = 0$,

$$\rho^n = (\mathcal{U} \mid -\alpha)_n\, \mathcal{V}^n.$$

Hence,

$$
\begin{aligned}
\rho^n 1 = (\mathcal{U} \mid -\alpha)_n\, \mathcal{V}^n\, 1 &= (\mathcal{U} \mid -\alpha)_n\, 1 \\
&= \left(Z + \frac{\sigma}{\alpha} \mid -\alpha\right)_n 1 = \left(z + \frac{\sigma}{\alpha} \mid -\alpha\right)_n. \qquad (4.43)
\end{aligned}
$$

Note that $\left(\left(z + \frac{\sigma}{\alpha} \mid -\alpha\right)_n\right)_{n=0}^{\infty}$ is a Sheffer sequence.

By the definition of generalised Stirling numbers of Hsu and Shiue, see (2.60), we have

$$\left(z + \frac{\sigma}{\alpha} \mid -\alpha\right)_n = \sum_{k=0}^{n} S(n, k; -\alpha, \alpha, \frac{\sigma}{\alpha})\,(z \mid \alpha)_k. \qquad (4.44)$$

Note that

$$\mathcal{I}\rho^n 1 = Z^n \mathcal{I}1 = Z^n 1 = z^n. \qquad (4.45)$$

Hence,

$$\mathcal{I}\left(z + \frac{\sigma}{\alpha} \mid -\alpha\right)_n = z^n. \qquad (4.46)$$

Next, we have

$$
\begin{aligned}
\left(\frac{\sigma}{\alpha} \mid -\alpha\right)_n &= \frac{\sigma}{\alpha}\left(\frac{\sigma}{\alpha} + \alpha\right)\left(\frac{\sigma}{\alpha} + 2\alpha\right)\cdots\left(\frac{\sigma}{\alpha} + (n-1)\alpha\right) \\
&= \frac{1}{\alpha^n}\,\sigma(\sigma + \alpha^2)(\sigma + 2\alpha^2)\cdots(\sigma + (n-1)\alpha^2)
\end{aligned}
$$

$$= \frac{1}{\alpha^n} (\sigma \mid -\alpha^2)_n = \frac{1}{\alpha^n} (\sigma \mid -\eta)_n. \tag{4.47}$$

Applying $\mathcal{I}$ to (4.44) and using (4.43), (4.47) and Proposition 3.1, we get

$$\begin{aligned}
z^n &= \sum_{k=0}^{n} S(n, k; -\alpha, \alpha, \frac{\sigma}{\alpha})\, s_k(z) \\
&= \left(\frac{\sigma}{\alpha} \mid -\alpha\right)_n + \sum_{k=1}^{n} \left[\sum_{j=0}^{n-k} \binom{n}{j} \alpha^{n-j-k} L(n-j, k) \left(\frac{\sigma}{\alpha} \mid -\alpha\right)_j \right] s_k(z) \\
&= \frac{1}{\alpha^n} (\sigma \mid -\eta)_n + \sum_{k=1}^{n} \left[\sum_{j=0}^{n-k} \binom{n}{j} \alpha^{n-j-k} L(n-j, k) \frac{1}{\alpha^j} (\sigma \mid -\eta)_j \right] s_k(z) \\
&= \frac{1}{\alpha^n} (\sigma \mid -\eta)_n + \sum_{k=1}^{n} \left[\sum_{j=0}^{n-k} \binom{n}{j} \alpha^{n-2j-k} L(n-j, k) (\sigma \mid -\eta)_j \right] s_k(z).
\end{aligned}$$

By the definition of the unsigned Stirling numbers of the first kind, we have

$$(z \mid -\eta)_j = \eta^j \left(\frac{z}{\eta}\right)^{(j)} = \eta^j \sum_{i=1}^{j} c(j, i) \left(\frac{z}{\eta}\right)^i = \sum_{i=1}^{j} c(j, i)\, \eta^{j-i} z^i. \tag{4.48}$$

Therefore, we also have

$$\begin{aligned}
z^n &= \sum_{i=1}^{n} c(n, i)\, \alpha^{n-2i} \sigma^i + \sum_{k=1}^{n} \Bigg[\alpha^{n-k} L(n, k) \\
&\quad + \sum_{j=1}^{n-k} \binom{n}{j} \alpha^{n-2j-k} L(n-j, k) \sum_{i=1}^{j} c(j, i)\, \eta^{j-i} \sigma^i \Bigg] s_k(z) \\
&= \sum_{i=1}^{n} c(n, i)\, \alpha^{n-2i} \sigma^i + \sum_{k=1}^{n} \Bigg[\alpha^{n-k} L(n, k) \\
&\quad + \sum_{i=1}^{n-k} \left(\sum_{j=i}^{n-k} \binom{n}{j} \alpha^{n-k-2i} L(n-j, k)\, c(j, i) \right) \sigma^i \Bigg] s_k(z).
\end{aligned} \tag{4.49}$$

Now, let find the explicit form of $s_n(z)$.

From (4.44) and the definition of Stirling-type pair, see (2.62), we also have

$$(z \mid \alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha, -\alpha, -\frac{\sigma}{\alpha})(z + \frac{\sigma}{\alpha} \mid -\alpha)_k. \tag{4.50}$$

Hence, by (4.40) and (4.46),

$$s_n(z) = \sum_{k=0}^{n} S(n, k; \alpha, -\alpha, -\frac{\sigma}{\alpha})\, z^k. \tag{4.51}$$

Similarly to (4.47), we have

$$\left(-\frac{\sigma}{\alpha} \mid \alpha\right)_n = (-\alpha)^{-n} (\sigma \mid -\alpha^2)_n = (-\alpha)^{-n} (\sigma \mid -\eta)_n. \tag{4.52}$$

By (3.1), (3.2) and (4.52),

$$S(n, 0; \alpha, -\alpha, -\frac{\sigma}{\alpha}) = (-\frac{\sigma}{\alpha} \mid \alpha)_n \tag{4.53}$$

and for $k = 1, \ldots, n$,

$$\begin{aligned} S(n, k; \alpha, -\alpha, -\frac{\sigma}{\alpha}) &= \sum_{j=0}^{n-k} \binom{n}{j} (-\alpha)^{n-j-k} L(n-j, k) (-\frac{\sigma}{\alpha} \mid \alpha)_j \\ &= \sum_{j=0}^{n-k} \binom{n}{j} (-\alpha)^{n-j-k} L(n-j, k) \left(-\frac{1}{\alpha}\right)^n (\sigma \mid -\eta)_n. \end{aligned} \tag{4.54}$$

By (4.51), (4.53) and (4.54),

$$s_n(z) = (-\alpha)^{-n} (\sigma \mid -\eta)_n + \sum_{k=1}^{n} \left(\sum_{j=0}^{n-k} \binom{n}{j} (-\alpha)^{n-j-k} L(n-j, k) \left(-\frac{1}{\alpha}\right)^n (\sigma \mid -\eta)_n \right) z^k. \tag{4.55}$$

Also, by (3.3),

$$\begin{aligned} &S(n, k; \alpha, -\alpha, -\frac{\sigma}{\alpha}) \\ &\quad = (-\alpha)^{n-k} L(n, k) + \sum_{i=1}^{k} \left(\sum_{j=i}^{k} \binom{n}{j} (-1)^{n-k-i} \alpha^{n-k-2i} L(n-j, k)\, s(j, i) \right) \sigma^i, \end{aligned}$$

and so, by (4.48),

$$\begin{aligned} s_n(z) &= (-1)^n \sum_{i=1}^{n} c(n, i)\, \alpha^{n-2i}\, \sigma^i \\ &\quad + \sum_{k=1}^{n} \left[(-\alpha)^{n-k} L(n, k) + \sum_{i=1}^{k} \left(\sum_{j=i}^{k} \binom{n}{j} (-1)^{n-k-i}\, \alpha^{n-k-2i} L(n-j, k)\, s(j, i) \right) \sigma^i \right] z^k. \end{aligned} \tag{4.56}$$

Let us now sum up the obtained formulas.

Theorem 4.18. *Let $\alpha > 0$ and $\sigma > 0$. Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of Laguerre polynomials with generating function*

$$G(z, t) = \exp\left(\frac{zt}{1 + \alpha t} - \frac{\sigma}{\alpha^2} \log(1 + \alpha t) \right), \tag{4.57}$$

equivalently satisfying the recurrence relation

$$zs_n(z) = (\lambda n + l)\, s_n(z) + (\sigma n + \eta n(n-1))\, s_{n-1}(z),$$

where $\lambda = 2\alpha$, $\eta = \alpha^2$ *and* $l = \frac{\sigma}{\alpha}$*. Then we have*

$$\begin{aligned} z^n &= \sum_{k=0}^{n} S(n,k;-\alpha,\alpha,\frac{\sigma}{\alpha})\, s_n(z) &(4.58)\\ &= \alpha^{-n}(\sigma \mid -\eta)_n + \sum_{k=1}^{n}\left[\sum_{j=0}^{n-k}\binom{n}{j}\alpha^{n-2j-k}L(n-j,k)(\sigma \mid -\eta)_j\right]s_k(z) &(4.59)\\ &= \sum_{i=1}^{n} c(n,i)\,\alpha^{n-2i}\,\sigma^i + \sum_{k=1}^{n}\Bigg[\alpha^{n-k}L(n,k)\\ &\quad + \sum_{i=1}^{n-k}\left(\sum_{j=i}^{n-k}\binom{n}{j}\alpha^{n-k-2i}L(n-j,k)\,c(j,i)\right)\sigma^i\Bigg]s_k(z), &(4.60)\end{aligned}$$

and

$$\begin{aligned} s_n(z) &= \sum_{k=0}^{n} S(n,k;\alpha,-\alpha,-\frac{\sigma}{\alpha})z^n &(4.61)\\ &= \alpha^{-n}(\sigma \mid -\eta)_n + \sum_{k=1}^{n}\left[\sum_{j=0}^{n-k}\binom{n}{j}(-\alpha)^{n-2j-k}L(n-j,k)\,(\sigma \mid -\eta)_j\right]z^k &(4.62)\\ &= (-1)^n\sum_{i=1}^{n} c(n,i)\,\alpha^{n-2i}\,\sigma^i\\ &\quad + \sum_{k=1}^{n}\left[(-\alpha)^{n-k}\,L(n,k) + \sum_{i=1}^{k}\left(\sum_{j=i}^{k}\binom{n}{j}\,(-1)^{n-k-i}\,\alpha^{n-k-2i}\,L(n-j,k)\,s(j,i)\right)\sigma^i\right]z^k. &(4.63)\end{aligned}$$

Let $\mu_{\alpha,\alpha,\sigma}$ denote the orthogonality measure for $(s_n(z))_{n=0}^{\infty}$, thus

$$\mu_{\alpha,\alpha,\sigma}(dz) = \mathbf{1}_{(0,+\infty)}(z)\,\frac{1}{\Gamma(\frac{\sigma}{\alpha^2})}\left(\frac{1}{\alpha}\right)^{\frac{\sigma}{\alpha^2}} z^{-1+\frac{\sigma}{\alpha^2}}\,e^{-\frac{z}{\alpha}}\,dz. \tag{4.64}$$

Corollary 4.19. *We have*

$$\int_0^\infty z^n\ \mu_{\alpha,\alpha,\sigma}(dz) = \left(\frac{\sigma}{\alpha} \mid -\alpha\right)_n. \tag{4.65}$$

Proof. The result follows from (4.52) and (4.59). □

Lemma 4.20. *We have, for each $n \in \mathbb{N}$*

$$\|s_n\|^2{}_{L^2(\mu_{\alpha,\alpha,\sigma})} = n!\,(\sigma \mid -\eta)_n.$$

Proof. It follows from the theory of orthogonal polynomials, see e.g. [16, Chapter I, Section 4] and (4.38) that

$$\begin{aligned}\|s_n\|^2{}_{L^2(\mu_{\alpha,\alpha,\sigma})} &= \prod_{k=1}^{n} \big(\sigma k + \eta k(k-1)\big) = \prod_{k=1}^{n} \big[k\big(\sigma + \eta(k-1)\big)\big] \\ &= n! \prod_{k=1}^{n} \big(\sigma + \eta(k-1)\big) = n!\,(\sigma \mid -\eta)_n. \qquad \square\end{aligned}$$

By (4.43),

$$\rho^n 1 = \left(\frac{\alpha z + \sigma}{\alpha} \mid -\alpha\right)_n.$$

Assume $z > -\frac{\sigma}{\alpha}$, equivalently $\sigma + \alpha z > 0$. Then

$$(\rho^n 1)(z) = \int_0^\infty \xi^n \; \mu_{\alpha,\alpha,\alpha z+\sigma}(d\xi). \tag{4.66}$$

Define $\mathcal{S} = \mathcal{I}^{-1}$, i.e., by (4.40),

$$(\mathcal{S}s_n)(z) = (z \mid \alpha)_n. \tag{4.67}$$

By (4.45) and (4.66), for $z > -\frac{\sigma}{\alpha}$,

$$(\mathcal{S}\xi^n)(z) = \int_0^\infty \xi^n \; \mu_{\alpha,\alpha,\alpha z+\sigma}(d\xi).$$

Hence, for each polynomial $p \in \mathcal{P}(\mathfrak{F})$ and $z > -\frac{\sigma}{\alpha}$,

$$(\mathcal{S}p)(z) = \int_0^\infty p(\xi) \; \mu_{\alpha,\alpha,\alpha z+\sigma}(d\xi). \tag{4.68}$$

Our next aim is to extend formula (4.68) to a set of complex z.

By (4.64) and (4.68), we have, for a polynomial p and $z > -\frac{\sigma}{\alpha}$,

$$\begin{aligned}(\mathcal{S}p)(z) &= \int_0^\infty \frac{1}{\Gamma(\frac{\sigma+\alpha z}{\alpha^2})} \left(\frac{1}{\alpha}\right)^{\frac{\sigma+\alpha z}{\alpha^2}} \xi^{-1+\frac{\sigma+\alpha z}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\, p(\xi)\; d\xi \\ &= \frac{1}{\Gamma(\frac{\sigma+\alpha z}{\alpha^2})}\, \alpha^{-\frac{\sigma+\alpha z}{\alpha^2}} \int_0^\infty p(\xi)\, \xi^{-1+\frac{\sigma+\alpha z}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\; d\xi.\end{aligned}$$

We will now define a complex-valued gamma measure.

Recall that, for $z \in \mathbb{C}$ with $\Re(z) > 0$, the gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\; dt,$$

and furthermore $\Gamma(z)$ is holomorphic on $\{z \in \mathbb{C} \mid \Re(z) > 0\}$. Note that, for all $z \in \mathbb{C}$ with $\Re(z) > 0$, $\Gamma(z) \neq 0$.

In view of (4.64), we define, for each $z \in \mathbb{C}$ with $\Re(z) > 0$, a complex-valued measure $\mu_{\alpha,\alpha,z}$ on $\mathbb{R}$ by

$$\mu_{\alpha,\alpha,z}\,(d\xi) = \mathbf{1}_{(0,+\infty)}(\xi)\, \frac{1}{\Gamma(\frac{z}{\alpha^2})}\, \alpha^{-\frac{z}{\alpha^2}}\, \xi^{-1+\frac{z}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\; d\xi. \tag{4.69}$$

Theorem 4.21. *For each polynomial $p \in \mathcal{P}(\mathfrak{F})$, the function*

$$z \mapsto \int_0^\infty p(\xi)\; \mu_{\alpha,\alpha,z}\,(d\xi) = \int_0^\infty p(\xi)\; \frac{1}{\Gamma(\frac{z}{\alpha^2})}\, \alpha^{-\frac{z}{\alpha^2}}\, \xi^{-1+\frac{z}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\; d\xi. \tag{4.70}$$

is well-defined and holomorphic on $\{z \in \mathbb{C} \mid \Re(z) > 0\}$.

Proof. First, we note that

$$\begin{aligned}
&\int_0^\infty |p(\xi)|\, \left|\xi^{-1+\frac{z}{\alpha^2}}\right|\, e^{-\frac{\xi}{\alpha}}\, d\xi \\
&\quad = \int_0^\infty |p(\xi)|\, \left|\xi^{-1+\frac{\Re(z)}{\alpha^2}} \cdot \xi^{\frac{i\Im(z)}{\alpha^2}}\right|\, e^{-\frac{\xi}{\alpha}}\, d\xi \\
&\quad = \int_0^\infty |p(\xi)|\, \xi^{-1+\frac{\Re(z)}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\, d\xi < \infty.
\end{aligned}$$

Thus, the function (4.70) is well-defined.

It is sufficient to prove that the function (4.70) is holomorphic on $\{z \in \mathbb{C} \mid \delta < \Re(z) < \Delta\}$ for any $0 < \delta < \Delta$.

We have

$$\frac{\partial}{\partial z}\left(p(\xi)\, \xi^{-1+\frac{z}{\alpha^2}}\, e^{-\frac{\xi}{\alpha}}\right) = p(\xi)\, \xi^{-1+\frac{z}{\alpha^2}}\, \log(\xi)\, e^{-\frac{\xi}{\alpha}} =: f(\xi, z).$$

By the theorem on differentiability of an integral depending on parameter, it is sufficient to prove that there exists a function $g(\xi)$ such that

$$|f(\xi, z)| \leq g(\xi) \quad \text{for all } \xi > 0,\; z \in \mathbb{C},\; \delta < z < \Delta,$$

and

$$\int_0^\infty g(\xi)\; d\xi\; < \infty. \tag{4.71}$$

To this end, we need an estimate on $\log(\xi)$.

Lemma 4.22. *(i) For $\xi \geq 1$,*

$$\log(\xi) \leq \xi.$$

(ii) For each $\varepsilon > 0$, there exists $C > 0$ such that, for all $\xi \in (0,1]$,

$$|\log(\xi)| \leq C\,\xi^{-\epsilon}.$$

Proof. These are well-known estimates, but we prefer to present their proof for the reader's convenience.

(i) Consider the function $f(\xi) = \xi - \log(\xi)$ for $\xi \geq 1$.
Then $f(1) = 1$ and $f'(\xi) = 1 - \frac{1}{\xi} \geq 0$ for $\xi \geq 1$. Hence, $f(\xi) > 0$ for all $\xi \geq 1$.

(ii) Fix $\varepsilon > 0$. We have to find $C > 0$ such that

$$-\log(\xi) \leq C\,\xi^{-\epsilon}.$$

It is sufficient to prove that the function $-\xi^{\varepsilon}\log(\xi)$ has a finite limit at zero.

By L'Hôpital's rule,

$$\lim_{\xi\to 0}(-1)\,\xi^{\varepsilon}\log(\xi) = \lim_{\xi\to 0}\frac{-\log(\xi)}{\xi^{-\varepsilon}} = \lim_{\xi\to 0}\frac{-\frac{1}{\xi}}{-\varepsilon\,\xi^{-\varepsilon-1}} = \varepsilon^{-1}\lim_{\xi\to 0}\xi^{\varepsilon} = 0,$$

which proves the statement. □

Hence, by Lemma 4.22,

$$\begin{aligned}
|f(\xi,z)| &\leq C\,\mathbf{1}_{(0,1]}(\xi)\,|p(\xi)|\,\xi^{-1+\frac{\Re(z)}{\alpha^2}}\,\xi^{-\varepsilon} \\
&\qquad + \mathbf{1}_{[1,\infty)}(\xi)\,|p(\xi)|\,\xi^{-1+\frac{\Re(z)}{\alpha^2}}\,\xi\,e^{-\frac{\xi}{\alpha}} \\
&\leq C\,\mathbf{1}_{(0,1]}(\xi)\,|p(\xi)|\,\xi^{-1-\varepsilon+\frac{\delta}{\alpha^2}} \\
&\qquad + \mathbf{1}_{[1,\infty)}(\xi)\,|p(\xi)|\,\xi^{\frac{D}{\alpha^2}}\,e^{-\frac{\xi}{\alpha}} =: g(\xi).
\end{aligned}$$

Then (4.71) holds if we choose $0 < \varepsilon < \frac{\delta}{\alpha^2}$. □

Corollary 4.23. *For each $z \in \mathbb{C}$, $\Re(z) > 0$,*

$$\int_0^{\infty} \mu_{\alpha,\alpha,z}(d\xi) = 1. \tag{4.72}$$

Proof. Since formula (4.72) holds for all $z>0$ and the function $z\mapsto\int_0^\infty \mu_{\alpha,\alpha,z}(d\xi)$ is holomorphic on $\{z\in\mathbb{C}\mid\Re(z)>0\}$, the statement follows from the identity theorem for holomorphic functions (see e.g. [22, Theorem III.3.2]). □

Corollary 4.24. *For each polynomial $p\in\mathcal{P}(\mathfrak{F})$ and $z\in\mathbb{C}$, $\Re(z)>-\frac{\sigma}{\alpha}$,*

$$(\mathcal{S}p)(z)=\int_0^\infty p(\xi)\;\mu_{\alpha,\alpha,\alpha z+\sigma}\,(d\xi).$$

Proof. By (4.67), or each polynomial $p\in\mathcal{P}(\mathfrak{F})$, $(\mathcal{S}p)(z)$ is an entire function. Hence, the statement follows from (4.68), Theorem 4.21 and the identity theorem for holomorphic functions. □

Our next aim it to define $(\mathcal{S}f)(z)$ for each $f\in L^2(\mu_{\alpha,\alpha,\sigma})$. So let

$$f(z)=\sum_{n=0}^\infty f_n\,s_n(z)\in L^2(\mu_{\alpha,\alpha,\sigma}),$$

equivalently

$$\sum_{n=0}^\infty |f_n|^2\,n!\,(\sigma\mid-\eta)_n<\infty, \tag{4.73}$$

see Lemma 4.20. In view of (4.67), we define

$$(\mathcal{S}f)(z)=\sum_{n=0}^\infty f_n(z\mid\alpha)_n \tag{4.74}$$

for all $z\in\mathbb{C}$ such that the series on the right-hand side of (4.74) converges.

Theorem 4.25. *(i) Let $\mathcal{D}_{\alpha,\alpha,\sigma}:=\{z\in\mathbb{C}\mid\Re(z)>-\frac{\sigma}{2\alpha}\}$. Let $(f_n)_{n=0}^\infty$ be a sequence of complex numbers such that* (4.73) *holds. Then the series $\sum_{n=0}^\infty f_n(z\mid\alpha)_n$ converges uniformly on compact sets in $\mathcal{D}_{\alpha,\alpha,\sigma}$, hence it is a holomorphic function on $\mathcal{D}_{\alpha,\alpha,\sigma}$.*

(ii) Denote by $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ the vector space of all holomorphic functions on $\mathcal{D}_{\alpha,\alpha,\sigma}$ that have the representation

$$\varphi(z)=\sum_{n=0}^\infty f_n\,(z\mid\alpha)_n, \tag{4.75}$$

with $(f_n)_{n=0}^\infty$ satisfying (4.73). *Consider $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ as a Hilbert space, equipped with inner product*

$$(\varphi,\psi)_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})}:=\sum_{n=0}^\infty f_n\,\overline{g_n}\,n!\,(\sigma\mid-\alpha^2)_n$$

for $\varphi(z) = \sum_{n=0}^{\infty} f_n(z \mid \alpha)_n$, $\psi(z) = \sum_{n=0}^{\infty} g_n(z \mid \alpha)_n \in \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$. *Let*

$$\mathcal{S} : L^2(\mu_{\alpha,\alpha,\sigma}) \to \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$$

be the unitary operator defined by

$$(\mathcal{S}s_n)(z) = (z \mid \alpha)_n.$$

Then, for each $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$,

$$(\mathcal{S}f)(z) = \int_0^\infty f \, d\mu_{\alpha,\alpha,\alpha z+\sigma} \, . \tag{4.76}$$

(iii) In formula (4.75),

$$f_n = \frac{1}{n!}(D_\alpha^n \varphi)(0) \tag{4.77}$$

$$= \frac{(-1)^n}{n! \, \alpha^n} \sum_{l=0}^{n} (-1)^k \binom{n}{k} \varphi(\alpha k). \tag{4.78}$$

In particular, the function φ *is completely determined by its values on the set* $\{\alpha k \mid k \in \mathbb{N}_0\}$.

(iv) The $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ *is a reproducing kernel Hilbert space with reproducing kernel*

$$\mathcal{K}_{\alpha,\alpha,\sigma}(z, w) = \sum_{n=0}^{\infty} \frac{(\bar{z} \mid \alpha)_n \, (w \mid \alpha)_n}{n! \, (\sigma \mid -\alpha^2)_n}, \quad z, w \in \mathcal{D}_{\alpha,\alpha,\sigma}.$$

(v) Define

$$\mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{(z \mid \alpha)_n}{n! \, (\sigma \mid -\alpha^2)_n} \, s_n(\xi), \quad \xi \in (0, \infty), \ z \in \mathcal{D}_{\alpha,\alpha,\sigma}. \tag{4.79}$$

Then, for each $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$, *we have* $\mathcal{E}_{\alpha,\alpha,\sigma}(\cdot, z) \in L^2(\mu_{\alpha,\alpha,\sigma})$ *and*

$$(\mathcal{S}f)(z) = (f, \mathcal{E}_{\alpha,\alpha,\sigma}(\cdot, \bar{z}))_{L^2(\mu_{\alpha,\alpha,\sigma})}$$

$$= \int_0^\infty f(\xi) \, \mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z) \ \mu_{\alpha,\alpha,\sigma}(d\xi). \tag{4.80}$$

(vi) The function $\mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z)$ *defined by* (4.79) *admits the following explicit formula, for* $\xi \in (0, \infty)$ *and* $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$,

$$\mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z) = \frac{\Gamma\left(\frac{\sigma}{\alpha^2}\right)}{\Gamma\left(\frac{\alpha z+\sigma}{\alpha^2}\right)} \, \alpha^{-\frac{z}{\alpha}} \, \xi^{\frac{z}{\alpha}}. \tag{4.81}$$

Proof. (i) and (ii). First we prove that, for each $f \in L^2(\mu_{\alpha,\alpha,\sigma})$, the integral on the right-hand side of (4.76) is well-defined. We have

$$\mu_{\alpha,\alpha,\alpha z+\sigma}\,(d\xi) = \mathbf{1}_{(0,+\infty)}\,\frac{1}{\Gamma(\frac{\alpha z+\sigma}{\alpha^2})}\,\alpha^{-\frac{\alpha z+\sigma}{\alpha^2}}\,\xi^{-1+\frac{\alpha z+\sigma}{\alpha^2}}\,e^{-\frac{\xi}{\alpha}}\,d\xi.$$

Let $-\frac{\sigma}{2\alpha} < \delta < \Delta < +\infty$. Then, for each $z \in \mathbb{C}$, $\delta \le \Re(z) \le \Delta$, we have

$$\begin{aligned}
&\int_0^\infty |f(\xi)|\,|\xi^{-1+\frac{\alpha z+\sigma}{\alpha^2}}|\,e^{-\frac{\xi}{\alpha}}\,d\xi\\
&\quad= \int_0^\infty |f(\xi)|\,\xi^{-1+\frac{\alpha\Re(z)+\sigma}{\alpha^2}}\,e^{-\frac{\xi}{\alpha}}\,d\xi\\
&\quad= \int_0^\infty |f(\xi)|\,\xi^{\frac{\Re(z)}{\alpha}}\,\xi^{-1+\frac{\sigma}{\alpha^2}}\,e^{-\frac{\xi}{\alpha}}\,d\xi\\
&\quad\le \left(\int_0^\infty |f(\xi)|^2\,\xi^{-1+\frac{\sigma}{\alpha^2}}\,e^{-\frac{\xi}{\alpha}}\,d\xi\right)^{\frac12}\left(\int_0^\infty \xi^{-1+\frac{\sigma}{\alpha^2}+\frac{2\Re(z)}{\alpha}}\,e^{-\frac{\xi}{\alpha}}\,d\xi\right)^{\frac12}\\
&\quad\le C\,\|f\|_{L^2(\mu_{\alpha,\alpha,\sigma})},
\end{aligned}\tag{4.82}$$

where the constant $C > 0$ depends on δ and Δ and is independent of f.

Next, we have

$$f(z) = \lim_{n\to\infty} p_n(z),$$

where $p_n(z) = \sum_{i=0}^n f_i\,s_i(z)$ and the convergence is in $L^2(\mu_{\alpha,\alpha,\sigma})$. By Corollary 4.24,

$$(\mathcal{S}p_n)(z) = \sum_{i=0}^n f_i\,(z \mid \alpha)_i = \int_0^\infty p_n\,d\mu_{\alpha,\alpha,\alpha z+\sigma},$$

and obviously $(\mathcal{S}p_n)(z)$ is holomorphic on $\mathcal{D}_{\alpha,\alpha,\sigma}$. Formula (4.82) implies that $(\mathcal{S}p_n)(z)$ converges to $\int_0^\infty f\,d\mu_{\alpha,\alpha,\alpha z+\sigma}$ uniformly on compact sets in $\mathcal{D}_{\alpha,\alpha,\sigma}$. Therefore, formula (4.76) indeed holds and $(\mathcal{S}f)(z)$ is holomorphic on $\mathcal{D}_{\alpha,\alpha,\sigma}$. This proves statements (i) and (ii) of the theorem.

(iii) Formula (4.77) is obvious since D_α is the lowering operator for $(z \mid \alpha)_n$. Formula (4.78) is a generalization of Euler's formula for Stirling numbers of the second kind, compare with Remark 4.4 in [21]. We will now prove it by induction: we need to prove that, for a function φ,

$$(D_\alpha^n\varphi)(0) = \frac{(-1)^n}{\alpha^n}\sum_{k=0}^n (-1)^k\binom{n}{k}\varphi(\alpha k),$$

which is equivalent to the formula

$$(D_\alpha^n\varphi)(z) = \frac{(-1)^n}{\alpha^n}\sum_{k=0}^n(-1)^k\binom{n}{k}\varphi(z+\alpha k). \tag{4.83}$$

For $n=1$, formula (4.83) obviously holds.

Assume (4.83) holds for n, and let us prove it for $n+1$:

$$\begin{aligned}
(D_\alpha^{n+1}\varphi)(z) &= (D_\alpha\, D_\alpha^n\varphi)(z)\\
&= \frac{1}{\alpha}\left[(D_\alpha^n\varphi)(z+\alpha) - (D_\alpha^n\varphi)(z)\right]\\
&= \frac{1}{\alpha}\left[\frac{(-1)^n}{\alpha^n}\sum_{k=0}^n(-1)^k\binom{n}{k}\varphi(z+\alpha(k+1)) - \frac{(-1)^n}{\alpha^n}\sum_{k=0}^n(-1)^k\binom{n}{k}\varphi(z+\alpha k)\right]\\
&= \frac{(-1)^{n+1}}{\alpha^{n+1}}\left[\sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}\varphi(z+\alpha k) + \sum_{k=0}^n(-1)^k\binom{n}{k}\varphi(z+\alpha k)\right]\\
&= \frac{(-1)^{n+1}}{\alpha^{n+1}}\Bigg[\varphi(z) + \sum_{k=1}^n(-1)^k\left[\binom{n}{k-1}+\binom{n}{k}\right]\varphi(z+\alpha k)\\
&\qquad\qquad + (-1)^{n+1}\varphi(z+\alpha(n+1))\Bigg]\\
&= \frac{(-1)^{n+1}}{\alpha^{n+1}}\left[\varphi(z) + \sum_{k=1}^n(-1)^k\binom{n+1}{k}\varphi(z+\alpha k) + (-1)^{n+1}\varphi(z+\alpha(n+1))\right]\\
&= \frac{(-1)^{n+1}}{\alpha^{n+1}}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\varphi(z+\alpha k).
\end{aligned}$$

(iv) Since each function from $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ is holomorphic on $\mathcal{D}_{\alpha,\alpha,\sigma}$, we have the inclusion $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma}) \subset C(\mathcal{D}_{\alpha,\alpha,\sigma})$. Next, we have to prove that if $\varphi \neq 0$ in $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$, then $\varphi \neq 0$ as an element of $C(\mathcal{D}_{\alpha,\alpha,\sigma})$. Assume the contrary: $\varphi \neq 0$ in $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ but $\varphi = 0$ in $C(\mathcal{D}_{\alpha,\alpha,\sigma})$. In particular, $\varphi(\alpha k) = 0$ for all $k \in \mathbb{N}_0$. But then part (iii) (formula (4.78)) implies that $\varphi = 0$ in $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$, which is a contradiction.

Next, formulas (4.76) and (4.82) imply, for each $f \in L^2(\mu_{\alpha,\alpha,\sigma})$ and $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$,

$$|(\mathcal{S}f)(z)| \le C\,\|f\|_{L^2(\mu_{\alpha,\alpha,\sigma})}. \tag{4.84}$$

Since $\mathcal{S} : L^2(\mu_{\alpha,\alpha,\sigma}) \to \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ is a unitary operator, formula (4.84) implies, for each $\varphi \in \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ and $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$,

$$|\varphi(z)| \le C\,\|\varphi\|_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})}.$$

Hence, for each $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$, the functional

$$\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma}) \ni \varphi \mapsto \varphi(z) \in \mathbb{C}$$

is continuous. Hence, $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ is a reproducing kernel Hilbert space.

Denote by $\mathcal{K}_z$ the element of $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ that satisfies

$$\varphi(z) = (\varphi, \mathcal{K}_z)_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})} \qquad \forall \varphi \in \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma}).$$

Obviously, $\mathcal{K}_z(w) = \sum_{n=0}^{\infty} g_n \,(w \mid \alpha)_n$ for some $g_n \in \mathbb{C}$. Since

$$((\cdot \mid \alpha)_n, \mathcal{K}_z(\cdot))_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})} = (z \mid \alpha)_n$$

and

$$((\cdot \mid \alpha)_n, \mathcal{K}_z(\cdot))_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})} = \overline{g_n}\, n!\,(\sigma \mid -\alpha^2)_n,$$

we get

$$\overline{g_n} = \frac{(z \mid \alpha)_n}{n!\,(\sigma \mid -\alpha^2)_n}$$

and so

$$g_n = \frac{(\overline{z} \mid \alpha)_n}{n!\,(\sigma \mid -\alpha^2)_n}.$$

Hence, the reproducing kernel is given by

$$\begin{aligned}\mathcal{K}_{\alpha,\alpha,\sigma}(z, w) &= \mathcal{K}_z(w)\\ &= \sum_{n=0}^{\infty} \frac{(\overline{z} \mid \alpha)_n\,(w \mid \alpha)_n}{n!\,(\sigma \mid -\alpha^2)_n}.\end{aligned}$$

(v) By part (iv), for each $f \in L^2(\mu_{\alpha,\alpha,\sigma})$ and $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$,

$$\begin{aligned}(\mathcal{S}f)(z) &= (\mathcal{S}f, \mathcal{K}_{\alpha,\alpha,\sigma}(z,\cdot))_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})}\\ &= (f, \mathcal{S}^{-1}\,\mathcal{K}_{\alpha,\alpha,\sigma}(z,\cdot))_{L^2(\mu_{\alpha,\alpha,\sigma})}.\end{aligned} \tag{4.85}$$

By (4.79),

$$\begin{aligned}\mathcal{S}^{-1}\,\mathcal{K}_{\alpha,\alpha,\sigma}(z,\cdot) &= \mathcal{S}^{-1} \sum_{n=0}^{\infty} \frac{(\overline{z} \mid \alpha)_n\,(\cdot \mid \alpha)_n}{n!\,(\sigma \mid -\alpha^2)_n}\\ &= \sum_{n=0}^{\infty} \frac{(\overline{z} \mid \alpha)_n\, s_n(\cdot)}{n!\,(\sigma \mid -\alpha^2)_n}\end{aligned}$$

$$= \mathcal{E}_{\alpha,\alpha,\sigma}(\cdot, \overline{z}). \tag{4.86}$$

Formulas (4.85) and (4.86) imply statement (v).

(vi) Choose an arbitrary Borel-measurable subset A of $(0, +\infty)$ and let $f = \chi_A \in L^2(\mu_{\alpha,\alpha,\sigma})$ be the indicator function of A. By (4.76) and (4.80),

$$\mu_{\alpha,\alpha,\alpha z+\sigma}(A) = \int_A \mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z)\ \mu_{\alpha,\alpha,\sigma}(d\xi).$$

Hence, for almost all $\xi \in (0, +\infty)$,

$$\mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z) = \frac{d\mu_{\alpha,\alpha,\alpha z+\sigma}}{d\mu_{\alpha,\alpha,\sigma}}\,(\xi). \tag{4.87}$$

Hence, by (4.69), for almost all $\xi \in (0, +\infty)$ and all $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$, formula (4.81) holds. But since the function $\mathcal{E}_{\alpha,\alpha,\sigma}(\xi, z)$ is continuous in ξ, the formula (4.81) holds for all $\xi \in (0, +\infty)$ and for all $z \in \mathcal{D}_{\alpha,\alpha,\sigma}$. □

4.3 The space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$

For $\eta > 0$ and $\sigma > 0$, we denote by $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ the vector space of all entire functions

$$\varphi : \mathbb{C} \to \mathbb{C}, \quad \varphi(z) = \sum_{n=0}^{\infty} f_n z^n$$

with the coefficients $(f_n)_{n=0}^{\infty}$ satisfying (4.73). Consider $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as a Hilbert space equipped with the inner product

$$(\varphi, \psi)_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\, \overline{g_n}\, n!\, (\sigma \mid -\eta)_n$$

for $\varphi(z) = \sum_{n=0}^{\infty} f_n z^n$, $\psi(z) = \sum_{n=0}^{\infty} g_n z^n \in \mathbb{F}_{\eta,\sigma}(\mathbb{C})$. This is a reproducing kernel Hilbert space with reproducing kernel

$$\mathbb{K}_{\eta,\sigma}(z, w) = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{n!\, (\sigma \mid -\eta)_n}.$$

Remark 4.26. For $\eta = 1$ and $\sigma = 1$, the inner product in $\mathbb{F}_{\eta,\sigma}(\mathbb{C}) = \mathbb{F}_{1,1}(\mathbb{C})$ becomes

$$(\varphi, \psi)_{\mathbb{F}_{1,1}(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\, \overline{g_n}\, (n!)^2.$$

This Hilbert space was studied in [7, Section 9] and [8], see also [37, Lemma 4] and [36].

Proposition 4.27. *Let $\eta > 0$ and $\sigma > 0$. Consider the gamma distribution*

$$\mu_{\eta,\eta,\,\eta\sigma}(dx) = \mathbf{1}_{(0,+\infty)}(x)\,\frac{1}{\Gamma(\frac{\sigma}{\eta})}\left(\frac{1}{\eta}\right)^{\frac{\sigma}{\eta}} x^{-1+\frac{\sigma}{\eta}}\,e^{-\frac{x}{\eta}}\;dx.$$

Let ξ be a random variable that has gamma distribution $\mu_{\eta,\eta,\eta\sigma}$. Let $\lambda_{\eta,\sigma}$ denote the Gaussian measure on $\mathbb{C}$ with random variable ξ:

$$\lambda_{\eta,\sigma}(dz) := \int_0^\infty \nu_\xi(dz)\;\mu_{\eta,\eta,\,\eta\sigma}(d\xi) = \Lambda_{\eta,\sigma}(z)\;dA(z).$$

Here, for $\varsigma > 0$, ν_ς denotes the Gaussian measure as in formula (2.37) *and*

$$\begin{aligned}\Lambda_{\eta,\sigma}(z) &= \frac{1}{\pi}\int_0^\infty \frac{1}{\xi}\,\exp\left(-\frac{|z|^2}{\xi}\right)\mu_{\eta,\eta,\,\eta\sigma}(d\xi)\\ &= \frac{1}{\pi\,\Gamma(\frac{\sigma}{\eta})}\left(\frac{1}{\eta}\right)^{\frac{\sigma}{\eta}}\int_0^\infty \exp\left(-\frac{|z|^2}{\xi}-\frac{\xi}{\eta}\right)\xi^{-2+\frac{\sigma}{\eta}}\;d\xi.\end{aligned} \tag{4.88}$$

Then, for any $m, n \in \mathbb{N}_0$,

$$\int_{\mathbb{C}} z^m\,\overline{z^n}\;\lambda_{\eta,\sigma}(dz) = \delta_{m,n}\,n!\,(\sigma \mid -\eta)_n = (z^m, z^n)_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})}.$$

Thus, $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is the closed subspace of $L^2(\mathbb{C},\lambda_{\eta,\sigma})$ constructed as the closure of polynomials on $\mathbb{C}$.

Proof. By (2.38) and (4.65),

$$\begin{aligned}\int_{\mathbb{C}} z^m\,\overline{z^n}\;\lambda_{\eta,\sigma}(dz) &= \int_0^\infty\int_{\mathbb{C}} z^m\,\overline{z^n}\;\nu_\xi(dz)\;\mu_{\eta,\eta,\,\eta\sigma}(d\xi)\\ &= \delta_{m,n}\,n!\int_0^\infty \xi^n\,\mu_{\eta,\eta,\,\eta\sigma}(d\xi)\\ &= \delta_{m,n}\,n!\left(\frac{\eta\sigma}{\eta}\mid -\eta\right)_n = \delta_{m,n}\,n!\;(\sigma \mid -\eta)_n\,.\end{aligned}$$

□

Following [8], let us recall some basic facts about the Mellin transform and the Mellin convolution.

Let $f : (0,+\infty) \to \mathbb{R}$ be such that, for some interval $(a,b) \subset \mathbb{R}$, the function $f(x)\,x^{c-1}$ is integrable on $(0,+\infty)$ for all $c \in (a,b)$. Then the Mellin transform of f is defined by

$$\mathcal{M}(f)(c) = \int_0^\infty x^{c-1}\,f(x)\;dx, \quad c \in (a,b).$$

Obviously, for $f(x) = e^{-x}$,

$$\mathcal{M}(f)(c) = \int_0^\infty x^{c-1}\, e^{-x}\, dx = \Gamma(c), \quad c > 0.$$

More generally, for $\eta > 0$ and $f(x) = e^{-\frac{x}{\eta}}$, we have

$$\begin{aligned}\mathcal{M}(f)(c) &= \int_0^\infty x^{c-1}\, e^{-\frac{x}{\eta}}\, dx \\ &= \eta^{c-1} \int_0^\infty x^{c-1}\, e^{-x}\, \eta\, dx = \eta^c\, \Gamma(c), \quad c > 0.\end{aligned}$$

The Mellin convolution of functions f and g is the function $f * g$ that satisfies

$$\mathcal{M}(f * g)(c) = (\mathcal{M}(f)(c))(\mathcal{M}(g)(c)).$$

Explicitly, the function $f * g$ is given by

$$\begin{aligned}(f * g)(x) &= \int_0^\infty f\left(\frac{x}{t}\right) g(t)\, \frac{1}{t}\, dt \\ &= \int_0^\infty f(t)\, g\left(\frac{x}{t}\right) \frac{1}{t}\, dt, \quad x > 0. \end{aligned} \tag{4.89}$$

Proposition 4.28. *Assume that $\sigma = \eta$. Then the function $\Lambda_{\eta,\sigma}$ given by* (4.88) *has the form*

$$\Lambda_{\eta,\sigma}(z) = \frac{1}{\pi\eta}\, \psi(|z|^2),$$

where

$$\psi(z) = (f_1 * f_2)(z)$$

with $f_1(z) = e^{-z}$ and $f_2(z) = e^{-\frac{z}{\eta}}$. In particular, for $\sigma = \eta = 1$,

$$\Lambda_{1,1}(z) = \frac{1}{\pi}\, (f_1 * f_1)(|z|^2).$$

Proof. Immediate by formulas (4.88) and (4.89). ☐

Remark 4.29. In the special case $\sigma = \eta = 1$, Proposition 4.28 was proved in [7, 8]. Furthermore, by [7, Proposition 9.2],

$$\Lambda_{1,1}(z) = \frac{1}{\pi} \int_{\mathbb{R}} e^{-|z|\, 2\cosh(t)}\, dt, \quad z > 0.$$

Note that

$$\frac{1}{\pi} \int_{\mathbb{R}} \exp(-r\, \cosh(t))\, dt, \quad r > 0$$

is the modified Bessel function of the second kind of order 0, see e.g. [56, p.419].

4.4 The gamma case (continued)

In this section, we continue the studies of the gamma case.

Recall Definition 4.8.

Proposition 4.30. *Define a unitary operator* $\mathbb{T} : \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{T}(\cdot \mid \alpha)_n)(z) = z^n.$$

Then, for each $f \in \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ *and* $z \in \mathbb{C}$,

$$(\mathbb{T}f)(z) = \int_{\mathbb{N}_0} f(\alpha\xi)\ \pi_{\frac{z}{\alpha}}(d\xi). \tag{4.90}$$

Proof. By Proposition 4.15, formula (4.90) holds for each polynomials $f(z)$.

Let $f(z) = \sum_{n=0}^{\infty} f_n(z \mid \alpha)_n \in \mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$. Consider

$$\int_{\mathbb{N}_0} \sum_{n=0}^{\infty} |f_n(\xi \mid \alpha)_n| \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha|z|}{\alpha^2}\right)^k \delta_{\alpha k}\ (d\xi) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |f_n|\ |(\alpha k \mid \alpha)_n|\ \frac{1}{k!}\,\frac{|z|^k}{\alpha^k}. \tag{4.91}$$

Noting that, for all $k, n \in \mathbb{N}_0$,

$$(\alpha k \mid \alpha)_n = \alpha^n\,(k)_n\ \geq 0,$$

and using Proposition 4.15, we continue (4.91) as follows:

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |f_n|\,\alpha^n\,(k)_n\,\frac{1}{k!}\,\frac{|z|^k}{\alpha^k} \\
&= \sum_{n=0}^{\infty} |f_n|\,\alpha^n\ \sum_{k=0}^{\infty} (k)_n\,\frac{1}{k!}\,\frac{|z|^k}{\alpha^k} \\
&= \sum_{n=0}^{\infty} |f_n|\,\alpha^n\ \exp\left(\frac{|z|}{\alpha}\right) \int_{\mathbb{N}_0} (\xi)_n\,\pi_{\frac{|z|}{\alpha}}(d\xi) \\
&= \exp\left(\frac{|z|}{\alpha}\right) \sum_{n=0}^{\infty} |f_n|\,\alpha^n\,\frac{|z|^n}{\alpha^n} \\
&= \exp\left(\frac{|z|}{\alpha}\right) \sum_{n=0}^{\infty} |f_n|\,|z|^n \\
&\leq \exp\left(\frac{|z|}{\alpha}\right) \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!\,(\sigma \mid -\eta)_n}\right)^{1/2} \|f\|_{\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})}.
\end{aligned}$$

It follows that the integral on the right-hand side of formula (4.90) is well-defined and (4.90) holds. □

Theorem 4.31. *Define a unitary operator* $\mathbb{S} : L^2(\mu_{\alpha,\alpha,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{S}s_n)(z) = z^n.$$

Then we have:

(i) For each $f \in L^2(\mu_{\alpha,\alpha,\sigma})$,

$$(\mathbb{S}f)(z) = \int_0^\infty f(\xi)\ \rho_{\alpha,\alpha,\sigma,z}(d\xi), \tag{4.92}$$

where

$$\rho_{\alpha,\alpha,\sigma,z}(d\xi) := \int_{\mathbb{N}_0} \mu_{\alpha,\alpha,\eta m+\sigma}(d\xi)\ \pi_{\frac{z}{\alpha}}(dm). \tag{4.93}$$

In particular, for each $z > 0$, $\rho_{\alpha,\alpha,\sigma,z}$ *is the random gamma distribution* $\mu_{\alpha,\alpha,\eta\zeta+\sigma}$, *where* ζ *is a random variable that has Poisson distribution* $\pi_{\frac{z}{\alpha}}$.

(ii) The operator $\mathbb{S}$ *is a generalised Segal–Bargmann transform constructed through the nonlinear coherent states*

$$\mathbb{E}_{\alpha,\alpha,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,(\sigma \mid -\eta)_n}\, s_n(\xi) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,(\sigma \mid -\alpha^2)_n}\, s_n(\xi), \tag{4.94}$$

i.e.,

$$\begin{aligned}(\mathbb{S}f)(z) &= (f, \mathbb{E}_{\alpha,\alpha,\sigma}(\cdot, \bar{z}))_{L^2(\mu_{\alpha,\alpha,\sigma})} \\ &= \int_0^\infty f(\xi)\, \mathbb{E}_{\alpha,\alpha,\sigma}(\xi, z)\ \mu_{\alpha,\alpha,\sigma}(d\xi).\end{aligned} \tag{4.95}$$

(iii) It holds that

$$\mathbb{E}_{\alpha,\alpha,\sigma}(\xi, z) = \int_{\mathbb{N}_0} \mathcal{E}_{\alpha,\alpha,\sigma}(\xi, \alpha\zeta)\ \pi_{\frac{z}{\alpha}}(d\zeta),$$

where the function $\mathcal{E}_{\alpha,\alpha,\sigma}$ *is given by formula* (4.81).

Proof. (i) We need to note that $\mathbb{S} = \mathbb{T}\mathcal{S}$ and apply Theorem 4.25 (iii) and Proposition 4.30.

(ii) The statement follows immediately from Section 2.6, the fact that

$$\|s_n\|^2_{L^2(\mu_{\alpha,\alpha,\sigma})} = n!\,(\sigma \mid -\eta)_n,$$

and Proposition 4.27.

(iii) Similarly to (4.87), we conclude from (4.92) and (4.95) that

$$\mathbb{E}_{\alpha,\alpha,\sigma}(\xi, z) = \frac{d\rho_{\alpha,\alpha,\sigma,z}}{d\mu_{\alpha,\alpha,\sigma}}(\xi).$$

Hence, by (4.87) and (4.93),

$$\begin{aligned} \mathbb{E}_{\alpha,\alpha,\sigma}(\xi, z) &= \int_{\alpha\mathbb{N}_0} \frac{d\mu_{\alpha,\alpha,\eta\zeta+\sigma}}{d\mu_{\alpha,\alpha,\sigma}}(\xi)\ \pi_{\frac{z}{\alpha}}(d\zeta) \\ &= \int_{\alpha\mathbb{N}_0} \mathcal{E}_{\alpha,\alpha,\sigma}(\xi, \alpha\zeta)\ \pi_{\frac{z}{\alpha}}(d\zeta). \qquad \square \end{aligned}$$

Remark 4.32. Note that our constructions in this section can be summarised in the following commutative diagram:

$$\begin{tikzcd} L^2(\mu_{\alpha,\beta,\sigma}) \arrow[r, "\mathbb{S}"] \arrow[d, "\mathcal{S}"] & \mathbb{F}_{\eta,\sigma}(\mathbb{C}) \\ \mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma}) \text{ or } \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C}) \arrow[ur, "\mathbb{T}"] & \end{tikzcd}$$

Remark 4.33. In accordance with the notations of Section 2.6, we use

$$\rho_n = n!\,(\sigma \mid -\eta)_n$$

for the construction of nonlinear coherent states in Theorem 4.31. In the special case where $\eta = 1$ and $\sigma = 2j$ with $j \in \{1, \frac{1}{2}, 2, \frac{2}{3}, \dots\}$,

$$\rho_n = n!\,(2j)^{(n)}.$$

Nonlinear coherent states with such a choice of ρ_n are called the *Barut–Girardello states* [11], see also [6, Section 1.1.3]. Such states appeared in [11] in a study of coherent states associated with the Lie algebra of the group $SU(1,1)$.

Similarly to Proposition 4.12, we have

Proposition 4.34. *The Fréchet space $\mathcal{E}^1_{\min}(\mathbb{C})$ is continuously embedded into the following spaces: $L^2(\mu_{\alpha,\alpha,\sigma})$, $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ and $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$. Furthermore, the operators $\mathcal{S}$, $\mathbb{S}$ and $\mathbb{T}$ restricted to $\mathcal{E}^\tau_{\min}(\mathbb{C})$ $(\tau \in (0,1])$ are self-homeomorphisms of $\mathcal{E}^\tau_{\min}(\mathbb{C})$.*

Proof. By (4.37) and (2.56), the polynomial sequences $(s_n(z))_{n=0}^\infty$, $((z \mid \alpha)_n)_{n=0}^\infty$ and $(z^n)_{n=0}^\infty$ satisfy the condition of Theorem 2.24.

Next, for each $f(z) = \sum_{n=0}^\infty f_n\, s_n(z)$, we have, by (2.6),

$$\begin{aligned}\sum_{n=0}^\infty |f_n|^2\, n!\, (\sigma \mid -\eta)_n &\le \sum_{n=0}^\infty |f_n|^2\, (n!)^2 \left(\max\{\eta,\sigma\}\right)^n\\ &\le \sum_{n=0}^\infty |f_n|^2\, (n!)^2\, 2^{nl} = \mathcal{N}_{1,l}^2\,(f),\end{aligned}$$

where $l \in \mathbb{N}$ is chosen so that $2^l \ge \max\{\eta,\sigma\}$. Hence, $\mathcal{E}^1_{\min}(\mathbb{C})$ is continuously embedded into $L^2(\mu_{\alpha,\alpha,\sigma})$ and hence also in $\mathcal{F}_{\alpha,\alpha,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ and $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$. Now, the theorem easily follows. □

Let ∂^+ and ∂^- denote the raising and lowering operators for the polynomials $(s_n(z))_{n=0}^\infty$:

$$\partial^+ s_n = s_{n+1}, \quad \partial^- s_n = n s_{n-1}. \tag{4.96}$$

Denote

$$U := \mathcal{I}\mathcal{U}\mathcal{S}, \quad V := \mathcal{I}\mathcal{V}\mathcal{S}.$$

Proposition 4.35. *We have*

$$Z = UV$$

and

$$U = \partial^+ + \alpha\, \partial^+\partial^- + \frac{\sigma}{\alpha} = \partial^+(1+\alpha\,\partial^-) + \frac{\sigma}{\alpha}, \tag{4.97}$$

$$V = 1 + \alpha\,\partial^-. \tag{4.98}$$

Proof. Immediate by (2.59), (4.39) and (4.41). □

By Theorem 2.16 and (4.37),

$$\partial^- = \frac{D}{1-\alpha\, D}.$$

Hence,

$$V = 1 + \alpha\,\partial^- = \frac{1}{1-\alpha\, D}. \tag{4.99}$$

Theorem 4.36. *The operators Z, D, ∂^-, ∂^+, U and V may be extended by continuity to $\mathcal{E}^1_{\min}(\mathbb{C})$. Furthermore, for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$,*

$$(Vf)(z) = \int_0^\infty f(z+s)\ \mu_{\alpha,\alpha,\alpha^2}(ds) = \int_0^\infty (E^s f)(z)\ \mu_{\alpha,\alpha,\alpha^2}(ds), \tag{4.100}$$

$$(\partial^- f)(z) = \int_0^\infty \big(f(z+s) - f(s)\big)\,\frac{1}{\alpha}\,\mu_{\alpha,\alpha,\alpha^2}(ds) \tag{4.101}$$

and

$$U = Z(1-\alpha D)\,f(z), \tag{4.102}$$

$$\partial^+ = Z(1-\alpha D)^2 - \frac{\sigma}{\alpha}(1-\alpha D). \tag{4.103}$$

Proof. As easily seen, the operator D acts continuously in each Hilbert space $E_{1,l}$. Hence it is continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$, see Proposition 2.23. Similarly, the operator Z acts continuously from each Hilbert space $E_{1,l}$ into $E_{1,l-1}$, hence it is continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$. Next, ∂^- acts continuously in each Hilbert space $H_{1,l}$, hence it is continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$, see Corollary 2.26 (ii). By (4.98), V is then continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$. Similarly, the operator ∂^+ acts continuously from each Hilbert space $H_{1,l}$ into $H_{1,l-1}$, hence it is continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$. By (4.97), U is then continuous in $\mathcal{E}^1_{\min}(\mathbb{C})$.

By (4.99), for each monomial z^n,

$$\begin{aligned} Vz^n &= \sum_{k=0}^n \frac{n!}{(n-k)!}\,\alpha^k\,z^{n-k} \\ &= \sum_{k=0}^n \frac{n!}{k!}\,\alpha^{n-k}\,z^k. \end{aligned} \tag{4.104}$$

On the another hand, by (4.65),

$$\begin{aligned} &\int_0^\infty (z+s)^n\ \mu_{\alpha,\alpha,\alpha^2}(ds) \\ &= \sum_{k=0}^n \binom{n}{k} z^k \int_0^\infty s^{n-k}\ \mu_{\alpha,\alpha,\alpha^2}(ds) \\ &= \sum_{k=0}^n \binom{n}{k} z^k\,\alpha^{n-k}\,(n-k)! \\ &= \sum_{k=0}^n \frac{n!}{k!}\,\alpha^{n-k}\,z^k. \end{aligned} \tag{4.105}$$

By (4.104) and (4.105), for each polynomials $p(z)$,

$$(Vp)(z) = \int_0^\infty p(z+s)\ \mu_{\alpha,\alpha,\alpha^2}(ds).$$

To finish the proof of (4.100), it is sufficient to show that, for each fixed $z \in \mathbb{C}$, the map

$$B_{1,t} \ni f \mapsto \int_0^\infty f(z+s)\ \mu_{\alpha,\alpha,\alpha^2}(ds)$$

is well-defined for some $t > 0$ and continuous. Since

$$|f(z)| \ \le\ \|f\|_{1,t} \exp(t|z|),$$

see (2.4), we have

$$\begin{aligned}\int_0^\infty |f(z+s)|\ \mu_{\alpha,\alpha,\alpha^2}(ds) &\ \le\ \|f\|_{1,t} \int_0^\infty \exp(t|z+s|)\ \mu_{\alpha,\alpha,\alpha^2}(ds)\\ &\ \le\ \|f\|_{1,t}\, e^{t|z|} \int_0^\infty e^{ts}\ \mu_{\alpha,\alpha,\alpha^2}(ds).\end{aligned} \tag{4.106}$$

For $t > \frac{1}{\alpha}$, we have

$$\int_0^\infty e^{ts}\ \mu_{\alpha,\alpha,\alpha^2}(ds) < \infty,$$

and now (4.100) easily follows.

Formula (4.101) follows from (4.98) and (4.100).

By (4.99),

$$Z = U(1-\alpha D)^{-1}.$$

Multiplying this equality by $(1-\alpha D)$ on the right, gives (4.102). Similarly, by (4.97) and (4.99),

$$\partial^+(1-\alpha D)^{-1} = U - \frac{\sigma}{\alpha} = Z(1-\alpha D) - \frac{\sigma}{\alpha},$$

which implies (4.103). $\square$

Since

$$\mathbb{S}\partial^-\mathbb{S}^{-1} = D, \quad \mathbb{S}\partial^+\mathbb{S}^{-1} = Z,$$

we also have, by (4.96)–(4.98), the following

Proposition 4.37. *Define*

$$\mathbb{U} := \mathbb{S}U\mathbb{S}^{-1} = \mathbb{T}\mathcal{U}\mathbb{T}^{-1}, \quad \mathbb{V} := \mathbb{S}V\mathbb{S}^{-1} = \mathbb{T}\mathcal{V}\mathbb{T}^{-1}.$$

Then

$$\mathbb{U} = Z(1+\alpha D) + \frac{\sigma}{\alpha}, \quad \mathbb{V} = 1 + \alpha D.$$

Similarly to Proposition 4.13 we have:

Proposition 4.38. *The operator*

$$A^- := \sigma\partial^- + \eta\,\partial^+\partial^-\partial^-$$

is closable and let us keep the notation A^- for its closure. Then, for each $z \in \mathbb{C}$, the coherent state $\mathbb{E}_{\alpha,\alpha,\sigma}(\cdot, z)$ given by (4.94) *belongs to the domain of A^- and*

$$A^-\,\mathbb{E}_{\alpha,\alpha,\sigma}(\cdot, z) = z\,\mathbb{E}_{\alpha,\alpha,\sigma}(\cdot, z).$$

Proof. Similar to the proof of Proposition 4.13. We only need to note that

$$A^- s_n = \big(\sigma n + \eta n(n-1)\big)s_{n-1} = n\big(\sigma + \eta(n-1)\big)s_{n-1}. \qquad \square$$

Remark 4.39. Note that ∂^+ is the restriction to $\mathcal{P}(\mathfrak{F})$ of the adjoint operator of A^-, hence formula $\mathbb{S}\,\partial^+\,\mathbb{S}^{-1} = Z$ agrees with formula (2.36) in Section 2.6.

4.5 Negative-binomial case

Let $\sigma > 0$ and $\alpha, \beta \in \mathbb{C}$. Define $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ by

$$\begin{aligned} \mathcal{U} &:= Z + \frac{\sigma}{\alpha}, \\ \mathcal{V} &:= \alpha D_\beta + 1. \end{aligned} \tag{4.107}$$

Lemma 4.40. *$\mathcal{U}$ and $\mathcal{V}$ generate a generalised Weyl algebra with the commutator relation*

$$[\mathcal{U}, \mathcal{V}] = -\beta\mathcal{V} - (\alpha - \beta). \tag{4.108}$$

Proof. We have

$$[\mathcal{U},\mathcal{V}] = [Z + \frac{\sigma}{\alpha}, \alpha D_\beta + 1] = [Z, \alpha D_\beta] = \alpha\,[Z, D_\beta].$$

We have, for each $p \in \mathcal{P}(\mathfrak{F})$,

$$\begin{aligned}(ZD_\beta\, p)(z) &= \frac{z}{\beta}\,[p(z+\beta) - p(z)],\\ (D_\beta Z\, p)(z) &= \frac{1}{\beta}\,[(z+\beta)\,p(z+\beta) - z\,p(z)]\\ &= \frac{z}{\beta}\,[p(z+\beta) - p(z)] + p(z+\beta).\end{aligned}$$

Hence,

$$\begin{aligned}[\mathcal{U},\mathcal{V}]\,p(z) = -\alpha(E^\beta p)(z) &= -\alpha\,p(z+\beta)\\ &= -\alpha[p(z+\beta) - p(z)] - \alpha\,p(z)\\ &= -\alpha\beta\,D_\beta\,p(z) - \alpha\,p(z)\\ &= -\beta(\alpha D_\beta + 1)\,p(z) + \beta\,p(z) - \alpha\,p(z)\\ &= -\beta\mathcal{V}\,p(z) - (\alpha - \beta)\,p(z).\end{aligned}$$

□

Let $\rho := \mathcal{U}\mathcal{V}$. By Lemma 4.40 and Proposition 3.2,

$$\rho^n = \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\,S(n,k)\,(\mathcal{U}\mid -\beta)_k\,\mathcal{V}^k. \tag{4.109}$$

Let now $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$ and $|\alpha| > |\beta|$, and let $l = \frac{\sigma}{\alpha}$. Let $(s_n(z))_{n=0}^\infty$ be the sequence of the Meixner polynomials of the first kind with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,s_n(z) = \left(\frac{1+\beta t}{1+\alpha t}\right)^{\frac{z}{\beta-\alpha}} (1+\beta t)^{-\frac{\sigma}{\alpha\beta}}. \tag{4.110}$$

Hence, these polynomials are orthogonal with respect to the negative binomial distribution (2.19). The polynomials $(s_n(z))_{n=0}^\infty$ satisfy the recurrence relation (4.38) with λ and η given by (2.13).

Lemma 4.41. *Define $\mathcal{I} \in \mathcal{L}(\mathcal{P}(\mathfrak{F}))$ by*

$$\mathcal{I}(z \mid \beta)_n = s_n(z). \tag{4.111}$$

Then,

$$Z = \mathcal{I}\rho\mathcal{I}^{-1}. \tag{4.112}$$

Proof. We have

$$\rho = \alpha Z D_\beta + Z + \sigma D_\beta + \frac{\sigma}{\alpha}.$$

By (4.41), similarly to (4.42), we get

$$\begin{aligned}
\rho(z \mid \beta)_n &= \alpha z n(z \mid \beta)_{n-1} + z(z \mid \beta)_n + \sigma n(z \mid \beta)_{n-1} + \frac{\sigma}{\alpha}(z \mid \beta)_n \\
&= \alpha n(z \mid \beta)_n + \alpha n\beta(n-1)(z \mid \beta)_{n-1} + (z \mid \beta)_{n+1} + n\beta(z \mid \beta)_n \\
&\quad + \sigma n(z \mid \beta)_{n-1} + \frac{\sigma}{\alpha}(z \mid \beta)_n \\
&= (z \mid \beta)_{n+1} + \left(n(\alpha+\beta) + \frac{\sigma}{\alpha}\right)(z \mid \beta)_n + (n\sigma + \alpha\beta n(n-1))(z \mid \beta)_{n-1} \\
&= (z \mid \beta)_{n+1} + \left(n\lambda + \frac{\sigma}{\alpha}\right)(z \mid \beta)_n + (n\sigma + \eta n(n-1))(z \mid \beta)_{n-1}. \qquad (4.113)
\end{aligned}$$

□

Theorem 4.42. *Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$ and $|\alpha| > |\beta|$. Let $(s_n(z))_{n=0}^{\infty}$ be the sequence of the Meixner polynomials of the first kind with generating function* (4.110)*, equivalently satisfying the recurrence relation* (4.38) *with $\lambda = \alpha + \beta$, $\eta = \alpha\beta$ and $l = \frac{\sigma}{\alpha}$. Then*

$$\begin{aligned}
z^n &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\, S(n,k)\, (\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\qquad + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} (\alpha-\beta)^{n-k}\, S(n,k)\, S(k,i;-\beta,\beta,\frac{\sigma}{\alpha}) \right) s_i(z) \qquad (4.114) \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, (\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\quad + \sum_{i=1}^{n} \left(\sum_{j=0}^{n-i} \left(\sum_{k=j}^{n-i} (\alpha-\beta)^{n-k-i}\, S(n,k+i) \binom{k+i}{j} \beta^{k-j} L(k+i-j,i) \right) (\frac{\sigma}{\alpha} \mid -\beta)_j \right) s_i(z),
\end{aligned}$$

and

$$\begin{aligned}
s_n(z) &= (-\frac{\sigma}{\alpha} \mid \beta)_n + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} S(n,k;\beta,-\beta,-\frac{\sigma}{\alpha})\, (\alpha-\beta)^{k-i}\, s(k,i) \right) z^i \qquad (4.115) \\
&= (-\frac{\sigma}{\alpha} \mid \beta)_n \\
&\quad + \sum_{i=1}^{n} \left(\sum_{j=0}^{n-i} \left(\sum_{k=j}^{n-i} \binom{n}{j} (-\beta)^{k-j}\, L(n-j,n-k)\, (\alpha-\beta)^{n-k-i}\, s(n-k,i) \right) (-\frac{\sigma}{\alpha} \mid \beta)_j \right) z^i.
\end{aligned}$$

Proof. By (4.109), we get

$$
\begin{aligned}
(\rho^n 1)(z) &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, (Z+\frac{\sigma}{\alpha} \mid -\beta)_k\, (\alpha D_\beta + 1)^k 1 \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, (z+\frac{\sigma}{\alpha} \mid -\beta)_k \qquad (4.116)\\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k) \sum_{i=0}^{k} S(k,i;-\beta,\beta,\frac{\sigma}{\alpha})(z \mid \beta)_i \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, S(k,0;-\beta,\beta,\frac{\sigma}{\alpha}) \\
&\quad + \sum_{k=1}^{n} \sum_{i=1}^{k} (\alpha-\beta)^{n-k} S(n,k)\, S(k,i;-\beta,\beta,\frac{\sigma}{\alpha})(z \mid \beta)_i \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\quad + \sum_{i=1}^{n} \sum_{k=i}^{n} (\alpha-\beta)^{n-k} S(n,k)\, S(k,i;-\beta,\beta,\frac{\sigma}{\alpha})(z \mid \beta)_i. \qquad (4.117)
\end{aligned}
$$

Applying $\mathcal{I}$ to (4.117) gives (4.114). Using (3.2) and (4.114), we have

$$
\begin{aligned}
z^n &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\quad + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} (\alpha-\beta)^{n-k} S(n,k) \sum_{j=0}^{k-i} \binom{k}{j} \beta^{k-j-i} L(k-j,i)(\frac{\sigma}{\alpha} \mid -\beta)_j \right) s_i(z) \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\quad + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-i} (\alpha-\beta)^{n-k-i} S(n,k+i) \sum_{j=0}^{k} \binom{k+i}{j} \beta^{k-j} L(k+i-j,i)(\frac{\sigma}{\alpha} \mid -\beta)_j \right) s_i(z) \\
&= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\frac{\sigma}{\alpha} \mid -\beta)_k \\
&\quad + \sum_{i=1}^{n} \left(\sum_{j=0}^{n-i} \left(\sum_{k=j}^{n-i} (\alpha-\beta)^{n-k-i} S(n,k+i) \binom{k+i}{j} \beta^{k-j} L(k+i-j,i) \right) (\frac{\sigma}{\alpha} \mid -\beta)_j \right) s_i(z).
\end{aligned}
$$

To prove formula (4.115), we proceed as follows. Denote

$$
p_n(z) = (z \mid \beta)_n, \quad q_n(z) = (z+\frac{\sigma}{\alpha} \mid -\beta)_n, \quad r_n(z) = (\rho^n 1)(z).
$$

By (4.116),

$$r_n(z) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, q_k(z).$$

By using the orthogonality of Stirling numbers, we conclude, similarly to the proof of Lemma 4.3, that

$$q_n(z) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} s(n,k)\, r_k(z).$$

Hence, by the definition of the generalised Stirling numbers of Hsu and Shiue and (2.63),

$$\begin{aligned}
p_n(z) &= \sum_{k=0}^{n} S(n,k;\beta,-\beta,-\frac{\sigma}{\alpha})\, q_k(z) \\
&= S(n,0;\beta,-\beta,-\frac{\sigma}{\alpha}) + \sum_{k=1}^{n} S(n,k;\beta,-\beta,-\frac{\sigma}{\alpha}) \left(\sum_{i=1}^{k} (\alpha-\beta)^{k-i} s(k,i)\, r_i(z) \right) \\
&= (-\frac{\sigma}{\alpha} \mid \beta)_n + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} S(n,k;\beta,-\beta,-\frac{\sigma}{\alpha})\, (\alpha-\beta)^{k-i}\, s(k,i) \right) r_i(z). \qquad (4.118)
\end{aligned}$$

Applying $\mathcal{I}$ to (4.118) gives (4.115). By (4.118) and (3.2), we get

$$\begin{aligned}
s_n(z) &= (-\frac{\sigma}{\alpha} \mid \beta)_n \\
&\quad + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \left(\sum_{j=0}^{n-k} \binom{n}{j} (-\beta)^{n-j-k} L(n-j,k)(-\frac{\sigma}{\alpha} \mid \beta)_j \right) (\alpha-\beta)^{k-i}\, s(k,i) \right) z^i \\
&= (-\frac{\sigma}{\alpha} \mid \beta)_n \\
&\quad + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-i} \sum_{j=0}^{k} \binom{n}{j} (-\beta)^{k-j} L(n-j,n-k)(-\frac{\sigma}{\alpha} \mid \beta)_j (\alpha-\beta)^{n-k-i}\, s(n-k,i) \right) z^i \\
&= (-\frac{\sigma}{\alpha} \mid \beta)_n \\
&\quad + \sum_{i=1}^{n} \left(\sum_{j=0}^{n-i} \left(\sum_{k=j}^{n-i} \binom{n}{j} (-\beta)^{k-j} L(n-j,n-k)(\alpha-\beta)^{n-k-i}\, s(n-k,i) \right) (-\frac{\sigma}{\alpha} \mid \beta)_j \right) z^i.
\end{aligned}$$

□

From now on assume $\alpha > \beta > 0$ (the case of $\alpha < \beta < 0$ is similar).

Corollary 4.43. *Let $\mu_{\alpha,\beta,\sigma}$ denote the orthogonality measure for $(s_n(z))_{n=0}^{\infty}$:*

$$\mu_{\alpha,\beta,\sigma}(dz) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{\sigma}{\alpha\beta}} \sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^{n} \frac{1}{n!} \left(\frac{\sigma}{\eta}\right)^{(n)} \delta_{(\alpha-\beta)n}(dz). \qquad (4.119)$$

Then we have

$$\int_{(\alpha-\beta)\mathbb{N}_0} z^n\ \mu_{\alpha,\beta,\sigma}(dz) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\, S(n,k) \left(\frac{\sigma}{\alpha} \mid -\beta\right)_k. \tag{4.120}$$

Proof. Immediate by (4.114). □

By (4.116),

$$\begin{aligned}(\rho^n 1)(z) &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\, S(n,k) \left(z + \frac{\sigma}{\alpha} \mid -\beta\right)_k \\ &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\, S(n,k) \left(\frac{\alpha z+\sigma}{\alpha} \mid -\beta\right)_k. \end{aligned} \tag{4.121}$$

Assume $\alpha z + \sigma > 0$, equivalently $z > -\frac{\sigma}{\alpha}$. Then, by (4.120) and (4.121),

$$(\rho^n 1)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} \xi^n\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi). \tag{4.122}$$

Define $\mathcal{S} := \mathcal{I}^{-1}$, i.e. by (4.111),

$$\mathcal{S} s_n(z) = (z \mid \beta)_n. \tag{4.123}$$

By (4.112),

$$(\mathcal{S}\xi^n)(z) = (\rho^n 1)(z).$$

Therefore, by (4.122),

$$(\mathcal{S}\xi^n)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} \xi^n\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi).$$

Hence, by linearity, for each $p \in \mathcal{P}(\mathfrak{F})$,

$$(\mathcal{S}p)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi), \quad z > -\frac{\sigma}{\alpha}. \tag{4.124}$$

Our next aim is to extend (4.124) to the case $z \in \mathbb{C}$. In view of (4.119), for each $z \in \mathbb{C}$, we define a complex-valued measure on $(\alpha-\beta)\mathbb{N}_0$ by

$$\mu_{\alpha,\beta,z}(d\xi) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{z}{\alpha\beta}} \sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{z}{\eta}\right)^{(n)} \delta_{(\alpha-\beta)n}(d\xi). \tag{4.125}$$

Proposition 4.44. *For each polynomial $p \in \mathcal{P}(\mathfrak{F})$, the function*

$$\mathbb{C} \ni z \mapsto \int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,z}(d\xi) \tag{4.126}$$

is well-defined and entire.

Proof. The following lemma implies that the function in (4.126) is well-defined.

Lemma 4.45. *Let $z \in \mathbb{C}$. For each $k \in \mathbb{N}$, the integral*

$$\int_{(\alpha-\beta)\mathbb{N}_0} \xi^k\ \mu_{\alpha,\beta,z}(d\xi)$$

converges absolutely, i.e.

$$\sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left| \left(\frac{z}{\alpha\beta}\right)^{(n)} \right| ((\alpha-\beta)n)^k < \infty. \tag{4.127}$$

Proof. We have

$$\begin{aligned}
\left| \left(\frac{z}{\eta}\right)^{(n)} \right| &= \left| \frac{z}{\eta}\left(\frac{z}{\eta}+1\right)\left(\frac{z}{\eta}+2\right)\cdots\left(\frac{z}{\eta}+(n-1)\right) \right| \\
&= \left|\frac{z}{\eta}\right| \left|\frac{z}{\eta}+1\right| \left|\frac{z}{\eta}+2\right| \cdots \left|\frac{z}{\eta}+(n-1)\right| \\
&\le \frac{|z|}{\eta}\left(\frac{|z|}{\eta}+1\right)\left(\frac{|z|}{\eta}+2\right)\cdots\left(\frac{|z|}{\eta}+(n-1)\right) \\
&= \left(\frac{|z|}{\eta}\right)^{(n)}.
\end{aligned}$$

Hence, the sum in (4.127) is bounded by

$$\sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{|z|}{\eta}\right)^{(n)} ((\alpha-\beta)n)^k \ < \ \infty,$$

because each monomial ξ^k is integreble with respect to the negative binomial distribution $\mu_{\alpha,\beta,|z|}$. □

Let $p \in \mathcal{P}(\mathfrak{F})$. For each $N \in \mathbb{N}$, the function

$$F_N(z) = \left(1-\frac{\beta}{\alpha}\right)^{\frac{z}{\eta}} \sum_{n=0}^{N} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{z}{\eta}\right)^{(n)} p((\alpha-\beta)n)$$

is entire. By Lemma 4.45 and its proof, the function F_N converges uniformly to $\int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,z}(d\xi)$ on compact sets in $\mathbb{C}$. Hence, the function (4.126) is entire. □

Corollary 4.46. *For each $z \in \mathbb{C}$,*

$$\mu_{\alpha,\beta,z}((\alpha-\beta)\mathbb{N}_0) = 1. \tag{4.128}$$

Proof. Since (4.128) holds for all $z > 0$, and $z \mapsto \mu_{\alpha,\beta,z}((\alpha-\beta)\mathbb{N}_0)$ is an entire function, (4.128) holds for all $z \in \mathbb{C}$ by the identity theorem for holomorphic functions. $\square$

Proposition 4.47. *For all $p \in \mathcal{P}(\mathfrak{F})$ and $z \in \mathbb{C}$,*

$$(\mathcal{S}p)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi). \tag{4.129}$$

Proof. By (4.124), formula (4.129) holds for all $z > -\frac{\sigma}{\alpha}$. By (4.123), $(\mathcal{S}p)(z)$ is an entire function, and by Proposition 4.44,

$$\mathbb{C} \ni z \mapsto \int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi)$$

is entire. Hence, (4.129) holds for all $z \in \mathbb{C}$ by the identity theorem. $\square$

Our next aim is to define $(\mathcal{S}f)(z)$ for each $f \in L^2(\mu_{\alpha,\beta,\sigma})$. So, let

$$f(z) = \sum_{n=0}^{\infty} f_n\, s_n(z) \quad \in L^2(\mu_{\alpha,\beta,\sigma}), \tag{4.130}$$

hence

$$\sum_{n=0}^{\infty} |f_n|^2\, n!\, (\sigma \mid -\eta)_n < \infty, \tag{4.131}$$

compare with Lemma 4.20. In view of (4.123), we define

$$(\mathcal{S}f)(z) = \sum_{n=0}^{\infty} f_n(z \mid \beta)_n \tag{4.132}$$

for all $z \in \mathbb{C}$ such that the series on the right-hand side of (4.132) converges.

Theorem 4.48. *(i) Let $(f_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that* (4.131) *holds. Then the series $\sum_{n=0}^{\infty} f_n\,(z \mid \beta)_n$ converges uniformly on compact sets in $\mathbb{C}$, hence it is an entire function.*

(ii) Denote by $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ the space of all entire functions that have the representation

$$\varphi(z) = \sum_{n=0}^{\infty} f_n\,(z \mid \beta)_n, \tag{4.133}$$

with $(f_n)_{n=0}^{\infty}$ *satisfying* (4.131). *Consider* $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ *as a Hilbert space equipped with inner product*

$$(\varphi,\psi)_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\, \overline{g_n}\, n!\, (\sigma \mid -\eta)_n \tag{4.134}$$

for $\varphi(z) = \sum_{n=0}^{\infty} f_n\,(z \mid \beta)_n$, $\psi(z) = \sum_{n=0}^{\infty} g_n\,(z \mid \beta)_n \in \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$. *Let*

$$\mathcal{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$$

be the unitary operator defined by $(\mathcal{S}s_n)(z) = (z \mid \beta)_n$. *Then, for all* $z \in \mathbb{C}$,

$$(\mathcal{S}f)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} f \; d\mu_{\alpha,\beta,\alpha z+\sigma}. \tag{4.135}$$

(iii) In formula (4.133),

$$\begin{aligned} f_n &= \frac{1}{n!}(D_\beta^n \varphi)(0) && (4.136)\\ &= \frac{(-1)^n}{n!\,\beta^n}\sum_{l=0}^{n}(-1)^k \binom{n}{k}\varphi(\beta k). && (4.137)\end{aligned}$$

In particular, the function φ *is completely determined by its values on the set* $\{\beta k \mid k \in \mathbb{N}_0\}$.

(iv) The $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ *is a reproducing kernel Hilbert space with reproducing kernel*

$$\mathcal{K}_{\alpha,\beta,\sigma}(z,w) = \sum_{n=0}^{\infty} \frac{(\bar{z} \mid \beta)_n\,(w \mid \beta)_n}{(\sigma \mid -\eta)_n}, \qquad z, w \in \mathbb{C}. \tag{4.138}$$

(v) Define

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{(z \mid \beta)_n}{n!\,(\sigma \mid -\eta)_n}\, s_n(\xi), \qquad \xi \in (\alpha-\beta)\mathbb{N}_0,\ z \in \mathbb{C}. \tag{4.139}$$

Then, for each $z \in \mathbb{C}$, *we have* $\mathcal{E}_{\alpha,\beta,\sigma}(\cdot, z) \in L^2(\mu_{\alpha,\beta,\sigma})$ *and*

$$\begin{aligned}(\mathcal{S}f)(z) &= (f, \mathcal{E}_{\alpha,\beta,\sigma}(\cdot,\bar{z}))_{L^2(\mu_{\alpha,\beta,\sigma})}\\ &= \int_{(\alpha-\beta)\mathbb{N}_0} f(\xi)\,\mathcal{E}_{\alpha,\beta,\sigma}(\xi,z)\;\mu_{\alpha,\beta,\sigma}(d\xi). && (4.140)\end{aligned}$$

(vi) The function $\mathcal{E}_{\alpha,\beta,\sigma}(\xi,z)$ *defined by* (4.139) *admits the following explicit formula, for* $\xi = (\alpha-\beta)n$ *with* $n \in \mathbb{N}_0$ *and* $z \in \mathbb{C}$:

$$\mathcal{E}_{\alpha,\beta,\sigma}((\alpha-\beta)n, z) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{z}{\beta}} \frac{(\alpha z + \sigma \mid -\eta)_n}{(\sigma \mid -\eta)_n}. \tag{4.141}$$

Proof. (i) and (iii) Let us first prove that the integral on the right-hand side of (4.135) is well-defined and, for a fixed $z \in \mathbb{C}$, it depends continuously on $f \in L^2(\mu_{\alpha,\beta,\sigma})$.

We have, for $f \in L^2(\mu_{\alpha,\beta,\sigma})$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} |f((\alpha-\beta)n)| \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left| \left(\frac{\sigma+\alpha z}{\eta}\right)^{(n)} \right| \\
&\quad \le \sum_{n=0}^{\infty} |f((\alpha-\beta)n)| \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{\sigma+\alpha|z|}{\eta}\right)^{(n)} \\
&\quad = \sum_{n=0}^{\infty} |f((\alpha-\beta)n)| \left[\left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{\sigma}{\eta}\right)^{(n)} \right]^{\frac{1}{2}} \\
&\qquad\qquad \times \left[\left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \frac{1}{\left(\frac{\sigma}{\eta}\right)^{(n)}} \right]^{\frac{1}{2}} \left(\frac{\sigma+\alpha|z|}{\eta}\right)^{(n)} \\
&\quad \le \left(1-\frac{\beta}{\alpha}\right)^{-\frac{\sigma}{\eta}} \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})} \left(\sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{\left[\left(\frac{\sigma+\alpha|z|}{\eta}\right)^{(n)}\right]^2}{n!\left(\frac{\sigma}{\eta}\right)^{(n)}} \right)^{\frac{1}{2}}. \qquad (4.142)
\end{aligned}$$

Lemma 4.49. *For any $a_1 > a_2 > 0$ and $0 < q < 1$,*

$$\sum_{n=0}^{\infty} q^n \frac{\left[(a_1)^{(n)}\right]^2}{n!\,(a_2)^{(n)}} < \infty.$$

Proof. Let $(r_n)_{n=0}^{\infty}$ be a sequence of positive real numbers such that, for each $q \in (0,1)$,

$$\sum_{n=1}^{\infty} q^n r_n < \infty.$$

We state that, for each $C > 1$, there exists a constant $\rho > 0$ such that

$$r_n \le \rho\, C^n \quad \text{for all } n \in \mathbb{N},$$

equivalently, the sequence $(\frac{r_n}{C^n})_{n=1}^{\infty}$ is bounded. Indeed, assume the contrary, i.e., there exists a sequence $(n_k)_{k=1}^{\infty}$ of increasing natural numbers such that

$$\frac{r_{n_k}}{C^{n_k}} \to +\infty \quad \text{as} \quad k \to \infty.$$

Hence, without loss of generality, we have

$$\frac{r_{n_k}}{C^{n_k}} \ge 1 \quad \text{for all } k \in \mathbb{N},$$

equivalently

$$r_{n_k} \;\geq\; C^{n_k}.$$

But then, for $q = C^{-1} < 1$,

$$\sum_{n=1}^{\infty} q^n \, r_n \;\geq\; \sum_{k=1}^{\infty} q^{n_k} \, C^{n_k} \;=\; \sum_{k=1}^{\infty} (qC)^{n_k} \;=\; +\infty.$$

But this contradicts the assumption that $\sum_{n=1}^{\infty} \; q^n \, r_n \;<\; \infty$.

It follows from the definition of a negative binomial distribution that, for each $0 < q < 1$ and $a_1 > 0$,

$$\sum_{n=0}^{\infty} q^n \, \frac{1}{n!} \, (a_1)^{(n)} < \infty.$$

Therefore, for each $\varepsilon > 0$,

$$\frac{(a_1)^{(n)}}{n!} \leq \rho_1 \, (1+\varepsilon)^n, \tag{4.143}$$

where the constant $\rho_1 > 0$ depends only on a_1 and ε. Next, for any $a_2 > 0$,

$$\begin{aligned}
(a_2)^{(n)} &= a_2(a_2+1)(a_2+2)\cdots(a_2+(n-1)) \\
&\geq a_2 \, 1 \cdot 2 \cdots (n-1) = a_2 \, (n-1)! \\
&= \frac{a_2}{n} \, n! \, .
\end{aligned} \tag{4.144}$$

Formulas (4.143) and (4.144) imply, for any $a_1 > a_2 > 0$ and $\varepsilon > 0$,

$$\begin{aligned}
\frac{(a_1)^{(n)}}{(a_2)^{(n)}} &\leq \frac{n}{a_2} \, \frac{(a_1)^{(n)}}{n!} \\
&\leq \rho_2 \, (1+2\varepsilon)^n,
\end{aligned} \tag{4.145}$$

where $\rho_2 > 0$ depends on a_1, a_2 and ε.

Hence, for each $\varepsilon > 0$, we have, by (4.143) and (4.145),

$$\sum_{n=0}^{\infty} q^n \, \frac{[(a_1)^{(n)}]^2}{n! \, (a_2)^{(n)}} \leq \rho_1 \rho_2 \sum_{n=0}^{\infty} \big[q(1+\varepsilon)(1+2\varepsilon) \big]^n < \infty,$$

if we choose $\varepsilon > 0$ so that $(1+\varepsilon)(1+2\varepsilon) < 1/q$. $\square$

By (4.142) and Lemma 4.49, for each $K > 0$ and $z \in \mathbb{C}$, $|z| \leq K$,

$$\sum_{n=0}^{\infty} |f((\alpha-\beta)n)| \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left| \left(\frac{\sigma + \alpha z}{\eta}\right)^{(n)} \right| \leq C \, \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})}, \tag{4.146}$$

where the constant C depends on α, β, σ and K.

Next, for $f(z) = \sum_{i=0}^{\infty} f_i\, s_i(z) \in L^2(\mu_{\alpha,\beta,\sigma})$, we have

$$f(z) = \lim_{n\to\infty} p_n(z),$$

where $p_n(z) = \sum_{i=0}^{n} f_i\, s_i(z)$ and the convergence is in $L^2(\mu_{\alpha,\beta,\sigma})$. By Proposition 4.47,

$$(\mathcal{S}p_n)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} p(\xi)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi),$$

and obviously $(\mathcal{S}p_n)(z) = \sum_{i=0}^{n} f_i\,(z \mid \beta)_i$ is an entire function. Formula (4.146) implies that, as $n \to \infty$, $(\mathcal{S}p_n)(z)$ converges to $\int_{(\alpha-\beta)\mathbb{N}_0} f(\xi)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi)$ uniformly on compact sets in $\mathbb{C}$. Therefore, formula (4.135) indeed holds for all $z \in \mathbb{C}$, and $(\mathcal{S}f)(z)$ is an entire function. This proves statements (i) and (ii) of the theorem.

(iii) This statement follows immediately from Theorem 4.25 (iii).

(iv) Similarly to the proof of Theorem 4.25 (iv), we show that $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C}) \subset C(\mathbb{C})$ and if $\varphi \neq 0$ in $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ then $\varphi \neq 0$ in $C(\mathbb{C})$.

By formulas (4.135) and (4.146), for each $z \in \mathbb{C}$, there exists $C > 0$ such that, for all $f \in L^2(\mu_{\alpha,\beta,\sigma})$,

$$|(\mathcal{S}f)(z)| \le C\, \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})}. \tag{4.147}$$

Since $\mathcal{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ is a unitary operator, formula (4.147) implies, for all $\varphi \in \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$,

$$|\varphi(z)| \le C\, \|\varphi\|_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})}.$$

Hence, $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ is a reproducing kernel Hilbert space.

Denote by $\mathcal{K}_z$ the element of $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ that satisfies

$$\varphi(z) = (\varphi, \mathcal{K}_z)_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})} \qquad \forall \varphi \in \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C}).$$

We have $\mathcal{K}_z(w) = \sum_{n=0}^{\infty} g_n\,(w \mid \beta)_n$ for some $g_n \in \mathbb{C}$. Since

$$\big((\cdot \mid \beta)_n, \mathcal{K}_z(\cdot)\big)_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})} = (z \mid \beta)_n$$

and

$$\big((\cdot \mid \beta)_n, \mathcal{K}_z(\cdot)\big)_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})} = \overline{g_n}\, n!\,(\sigma \mid -\eta)_n,$$

we have

$$\overline{g_n} = \frac{(z \mid \beta)_n}{n!\,(\sigma \mid -\eta)_n}, \tag{4.148}$$

equivalently

$$g_n = \frac{(\bar{z} \mid \beta)_n}{n!\,(\sigma \mid -\eta)_n}.$$

Hence, the reproducing kernel is given by (4.138).

(v) By part (iv), for each $f \in L^2(\mu_{\alpha,\beta,\sigma})$ and $z \in \mathbb{C}$,

$$\begin{aligned}(\mathcal{S}f)(z) &= (\mathcal{S}f, \mathcal{K}_{\alpha,\beta,\sigma}(z,\cdot))_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})}\\ &= (f, \mathcal{S}^{-1}\,\mathcal{K}_{\alpha,\beta,\sigma}(z,\cdot))_{L^2(\mu_{\alpha,\beta,\sigma})},\end{aligned} \tag{4.149}$$

and by (4.139),

$$\begin{aligned}\mathcal{S}^{-1}\,\mathcal{K}_{\alpha,\beta,\sigma}(z,\cdot) &= \mathcal{S}^{-1}\sum_{n=0}^{\infty}\frac{(\bar{z} \mid \beta)_n\,(\cdot \mid \beta)_n}{n!\,(\sigma \mid -\eta)_n}\\ &= \sum_{n=0}^{\infty}\frac{(\bar{z} \mid \beta)_n\, s_n(\cdot)}{n!\,(\sigma \mid -\eta)_n}\\ &= \mathcal{E}_{\alpha,\beta,\sigma}(\cdot, \bar{z}).\end{aligned} \tag{4.150}$$

Formulas (4.149) and (4.150) imply the statement.

(vi) Similarly to the proof of Theorem 4.25 (vi), we obtain from formulas (4.135) and (4.140), for $\xi \in (\alpha-\beta)\mathbb{N}_0$ and $z \in \mathbb{C}$,

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) = \frac{d\mu_{\alpha,\beta,\alpha z+\sigma}}{d\mu_{\alpha,\beta,\sigma}}(\xi). \tag{4.151}$$

Formula (4.141) follows immediately from (4.125) and (4.151). $\square$

Consider the Hilbert space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ defined in Section 4.3. By Proposition 4.27, $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is the closed subspace of $L^2(\mathbb{C}, \lambda_{\eta,\sigma})$ constructed as the closure of polynomials on $\mathbb{C}$.

Proposition 4.50. *Define a unitary operator* $\mathbb{T} : \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{T}(\cdot \mid \beta)_n)(z) = z^n. \tag{4.152}$$

Then, for each $f \in \mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ *and* $z \in \mathbb{C}$,

$$(\mathbb{T}f)(z) = \int_{\mathbb{N}_0} f(\beta\xi)\;\pi_{\frac{z}{\beta}}(d\xi). \tag{4.153}$$

Proof. The operator $\mathbb{T}$ defined by (4.152) is obviously unitary, see (4.134). So, we only need to check that the operator $\mathbb{T}$ given by (4.152) satisfies (4.153). Since $\alpha > \beta$, we have

$$(\sigma \mid -\eta)_n = (\sigma \mid -\alpha\beta)_n > (\sigma \mid -\beta^2)_n.$$

Hence, by the definition of $\mathcal{F}_{\beta,\beta,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$ (see Theorem 4.25) and the definition of $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ (see Theorem 4.48), we conclude that $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C}) \subset \mathcal{F}_{\beta,\beta,\sigma}(\mathcal{D}_{\alpha,\alpha,\sigma})$. But now formula (4.153) follows from Proposition 4.30. $\square$

Theorem 4.51. *Define a unitary operator* $\mathbb{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{S}s_n)(z) = z^n.$$

(i) For each $f \in L^2(\mu_{\alpha,\beta,\sigma})$,

$$(\mathbb{S}f)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} f(\xi)\ \rho_{\alpha,\beta,\sigma,z}(d\xi), \tag{4.154}$$

where

$$\rho_{\alpha,\beta,\sigma,z}(d\xi) = \int_{\mathbb{N}_0} \mu_{\alpha,\beta,\,\eta m+\sigma}(d\xi)\ \pi_{\frac{z}{\beta}}(dm). \tag{4.155}$$

In particular, for each $z > 0$, $\rho_{\alpha,\beta,\sigma,z}$ *is the random negative binomial distribution* $\mu_{\alpha,\beta,\,\eta\zeta+\sigma}$, *where* ζ *is a random variable having Poisson distribution* $\pi_{\frac{z}{\beta}}$.

(ii) The operator $\mathbb{S}$ *is a generalised Segal–Bargmann transform constructed through the nonlinear coherent state*

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,(\sigma \mid -\eta)_n}\, s_n(\xi), \tag{4.156}$$

i.e.,

$$\begin{aligned}(\mathbb{S}f)(z) &= (f, \mathbb{E}_{\alpha,\beta,\sigma}(\cdot, \bar{z}))_{L^2(\mu_{\alpha,\beta,\sigma})}\\ &= \int_{(\alpha-\beta)\mathbb{N}_0} f(\xi)\, \mathbb{E}_{\alpha,\beta,\sigma}(\xi, z)\ \mu_{\alpha,\beta,\sigma}(d\xi).\end{aligned} \tag{4.157}$$

(iii) We have

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi, z) = \int_{\mathbb{N}_0} \mathcal{E}_{\alpha,\beta,\sigma}(\xi, \beta\zeta)\ \pi_{\frac{z}{\beta}}(d\zeta), \tag{4.158}$$

where the function $\mathcal{E}_{\alpha,\beta,\sigma}$ *is given by formula* (4.141).

Proof. (i) We only need to note that $\mathbb{S} = \mathbb{T}\mathcal{S}$ and apply Theorem 4.48 (iii) and Proposition 4.50.

(ii) Similar to the proof of statement (ii) of Theorem 4.31.

(iii) By (4.154) and (4.157), we get

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi, z) = \frac{d\rho_{\alpha,\beta,\sigma,z}}{d\mu_{\alpha,\beta,\sigma}}(\xi).$$

Hence, by (4.155),

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi, z) = \int_{\mathbb{N}_0} \frac{d\mu_{\alpha,\beta,\eta\zeta+\sigma}}{d\mu_{\alpha,\beta,\sigma}}(\xi)\ \pi_{\frac{z}{\beta}}(d\zeta),$$

which, by (4.151), yields (4.158). □

The proof of the following proposition is similar to the proof of Proposition 4.34.

Proposition 4.52. *The Fréchet space $\mathcal{E}^1_{\min}(\mathbb{C})$ is continuously embedded into the following spaces: $L^2(\mu_{\alpha,\beta,\sigma})$, $\mathcal{F}_{\alpha,\beta,\sigma}(\mathbb{C})$ and $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$. Furthermore, the operators $\mathcal{S}$, $\mathbb{S}$ and $\mathbb{T}$ restricted to $\mathcal{E}^\tau_{\min}(\mathbb{C})$ ($\tau \in (0, 1]$) are self-homeomorphisms of $\mathcal{E}^\tau_{\min}(\mathbb{C})$.*

Let ∂^+ and ∂^- denote the raising and lowering operators for the polynomials $(s_n(z))_{n=0}^\infty$, compare with formula (4.96). Denote

$$U := \mathcal{I}\mathcal{U}\mathcal{S}, \quad V := \mathcal{I}\mathcal{V}\mathcal{S}. \tag{4.159}$$

Similarly to Proposition 4.35, we have

Proposition 4.53. *We have*

$$Z = UV$$

and

$$U = \partial^+ + \beta\,\partial^+\,\partial^- + \frac{\sigma}{\alpha} = \partial^+(1 + \beta\,\partial^-) + \frac{\sigma}{\alpha}, \tag{4.160}$$

$$V = \alpha\,\partial^- + 1. \tag{4.161}$$

Proposition 4.54. *We have*

$$\partial^- = \frac{D_{\beta-\alpha}}{1 - \alpha D_{\beta-\alpha}} \tag{4.162}$$

$$= \frac{D_{\alpha-\beta}}{1 - \beta D_{\alpha-\beta}}. \tag{4.163}$$

Proof. By Theorem 2.11, Remark 2.12 and Theorem 2.16,

$$\partial^- = B(D), \tag{4.164}$$

where $B(t) = A^{\langle -1\rangle}(t)$, i.e. $B(t)$ is the compositional inverse of $A(t)$.

From Case 4 (Negative binomial) in Section 2.4, we have

$$\begin{aligned}
B(t) &= \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha e^{(\beta-\alpha)t}} \\
&= \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha(e^{(\beta-\alpha)t} - 1) - \alpha} \\
&= \frac{e^{(\beta-\alpha)t} - 1}{(\beta-\alpha) - \alpha(e^{(\beta-\alpha)t} - 1)} \\
&= \frac{\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}}{1 - \alpha\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}}.
\end{aligned} \tag{4.165}$$

By (4.164), (4.165) and Boole's formula, this implies (4.162).

Similarly

$$\begin{aligned}
B(t) &= \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha e^{(\beta-\alpha)t}} \\
&= \frac{1 - e^{(\alpha-\beta)t}}{\beta e^{(\alpha-\beta)t} - \alpha} \\
&= \frac{e^{(\alpha-\beta)t} - 1}{\alpha - \beta(e^{(\alpha-\beta)t} - 1) - \beta} \\
&= \frac{e^{(\alpha-\beta)t} - 1}{(\alpha-\beta) - \beta(e^{(\alpha-\beta)t} - 1)} \\
&= \frac{\frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta}}{1 - \beta\frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta}},
\end{aligned}$$

which implies (4.163). □

Recall that the operator Z acts continuously in $\mathcal{E}^1_{\min}(\mathbb{C})$, see Theorem 4.36.

Theorem 4.55. *The operators D_ζ for each $\zeta \in \mathbb{C}$, and ∂^-, ∂^+, U, V may be extended by continuity to $\mathcal{E}^1_{\min}(\mathbb{C})$. Furthermore, for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$, we have*

$$(\partial^- f)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} \big(f(z+s) - f(z)\big)\,\frac{1}{\beta}\,\mu_{\alpha,\beta,\eta}(ds), \tag{4.166}$$

$$(Vf)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} \left(f(z+s) - f(z) \cdot \frac{\alpha-\beta}{\alpha} \right) \frac{\alpha}{\beta}\, \mu_{\alpha,\beta,\eta}(ds), \tag{4.167}$$

and

$$U = Z(1 - \alpha D_{\beta-\alpha}) \tag{4.168}$$

$$\partial^+ = Z(1 - \alpha D_{\beta-\alpha})(1 - \beta D_{\alpha-\beta}) - \frac{\sigma}{\alpha}\,(1 - \beta D_{\alpha-\beta}). \tag{4.169}$$

Proof. The continuity of D_ζ for each $\zeta \in \mathbb{C}$ in $\mathcal{E}^1_{\min}(\mathbb{C})$ easily follows from Corollary 2.26 applied to the falling factorials $((z \mid \zeta)_n)_{n=0}^\infty$. The continuity of ∂^-, ∂^+, U, V in $\mathcal{E}^1_{\min}(\mathbb{C})$ can be shown analogously to the proof of Theorem 4.36.

We start with verifying (4.166). By (4.163), we have

$$\begin{aligned} \partial^- &= \frac{D_{\alpha-\beta}}{1-\beta D_{\alpha-\beta}} = D_{\alpha-\beta} \sum_{n=0}^\infty \beta^n\, D^n_{\alpha-\beta} \\ &= \sum_{n=0}^\infty \beta^n\, D^{n+1}_{\alpha-\beta} = \sum_{n=1}^\infty \beta^{n-1}\, D^n_{\alpha-\beta}. \end{aligned}$$

Hence,

$$\begin{aligned} \partial^-(z \mid \alpha-\beta)_n &= \sum_{k=1}^\infty \beta^{k-1}\, D^k_{\alpha-\beta}\, (z \mid \alpha-\beta)_n \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!}\, \beta^{k-1}\, (z \mid \alpha-\beta)_{n-k} \\ &= \sum_{k=0}^{n-1} \frac{n!}{k!}\, \beta^{n-k-1}\, (z \mid \alpha-\beta)_k. \end{aligned} \tag{4.170}$$

On the other hand,

$$\begin{aligned} &\int_{(\alpha-\beta)\mathbb{N}_0} \big[(z+s \mid \alpha-\beta)_n - (z \mid \alpha-\beta)_n \big]\, \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds) \\ &\quad = \int_{(\alpha-\beta)\mathbb{N}_0} \left[\sum_{k=0}^n \binom{n}{k} (z \mid \alpha-\beta)_k\, (s \mid \alpha-\beta)_{n-k} - (z \mid \alpha-\beta)_n \right] \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds) \\ &\quad = \int_{(\alpha-\beta)\mathbb{N}_0} \left[\sum_{k=0}^{n-1} \binom{n}{k} (z \mid \alpha-\beta)_k\, (s \mid \alpha-\beta)_{n-k} \right] \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds) \\ &\quad = \sum_{k=0}^{n-1} \frac{n!}{k!}\, (z \mid \alpha-\beta)_k \;\cdot\; \frac{1}{\beta\,(n-k)!} \int_{(\alpha-\beta)\mathbb{N}_0} (s \mid \alpha-\beta)_{n-k}\; \mu_{\alpha,\beta,\eta}(ds). \end{aligned} \tag{4.171}$$

Lemma 4.56. *We have*

$$\int_{(\alpha-\beta)\mathbb{N}_0} (s \mid \alpha-\beta)_n \ \mu_{\alpha,\beta,\sigma}(ds) = \beta^n \left(\frac{\sigma}{\eta}\right)^{(n)}. \tag{4.172}$$

Proof. By (2.52), Definition 2.33 and (4.120), we have

$$\begin{aligned}
&\int_{(\alpha-\beta)\mathbb{N}_0} (s \mid \alpha-\beta)_n \ \mu_{\alpha,\beta,\sigma}(ds) \\
&= \int_{(\alpha-\beta)\mathbb{N}_0} (\alpha-\beta)^n \left(\frac{s}{\alpha-\beta}\right)_n \ \mu_{\alpha,\beta,\sigma}(ds) \\
&= \sum_{k=1}^{n} s(n,k)\,(\alpha-\beta)^{n-k} \int_{(\alpha-\beta)\mathbb{N}_0} s^k \ \mu_{\alpha,\beta,\sigma}(ds) \\
&= \sum_{k=1}^{n} s(n,k)\,(\alpha-\beta)^{n-k} \sum_{i=1}^{k} (\alpha-\beta)^{k-i}\, S(k,i) \left(\frac{\sigma}{\alpha} \mid -\beta\right)_i \\
&= \sum_{i=1}^{n} \sum_{k=i}^{n} s(n,k)\,(\alpha-\beta)^{n-i}\, S(k,i) \left(\frac{\sigma}{\alpha} \mid -\beta\right)_i \\
&= \sum_{i=1}^{n} (\alpha-\beta)^{n-i} \left(\frac{\sigma}{\alpha} \mid -\beta\right)_i \sum_{k=i}^{n} s(n,k)\, S(k,i) \\
&= \sum_{i=1}^{n} (\alpha-\beta)^{n-i} \left(\frac{\sigma}{\alpha} \mid -\beta\right)_i \delta_{n,i} \\
&= \left(\frac{\sigma}{\alpha} \mid -\beta\right)_n \\
&= \beta^n \left(\frac{\sigma}{\alpha\beta} \mid -1\right)_n = \beta^n \left(\frac{\sigma}{\eta}\right)^{(n)}.
\end{aligned}$$

□

By (4.172),

$$\int_{(\alpha-\beta)\mathbb{N}_0} (s \mid \alpha-\beta)_n \ \mu_{\alpha,\beta,\eta}(ds) = \beta^n\, n!\,. \tag{4.173}$$

By (4.170), (4.171) and (4.173), we get

$$\partial^-(z \mid \alpha-\beta)_n = \int_{(\alpha-\beta)\mathbb{N}_0} [(z+s \mid \alpha-\beta)_n - (z \mid \alpha-\beta)_n]\, \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds).$$

Hence, (4.166) holds for each $f \in \mathcal{P}(\mathbb{C})$. The extension of this formula to $f \in \mathcal{E}^1_{\min}(\mathbb{C})$ is similar to the proof of Theorem 4.36

Next, by (4.161), for $f \in \mathcal{E}^1_{\min}(\mathbb{C})$, we have

$$(Vf)(z) = \alpha \int_{(\alpha-\beta)\mathbb{N}_0} \big(f(z+s) - f(z)\big)\, \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds) + \int_0^\infty f(z)\ \mu_{\alpha,\beta,\eta}(ds)$$

$$
\begin{aligned}
&= \int_{(\alpha-\beta)\mathbb{N}_0} \big(f(z+s)-f(z)\big)\,\frac{\alpha}{\beta}\,\mu_{\alpha,\beta,\eta}(ds) + \int_0^\infty f(z)\,\frac{\beta}{\alpha}\cdot\frac{\alpha}{\beta}\,\mu_{\alpha,\beta,\eta}(ds)\\
&= \int_{(\alpha-\beta)\mathbb{N}_0} \left(f(z+s)-f(z)\left(1-\frac{\beta}{\alpha}\right)\right)\frac{\alpha}{\beta}\,\mu_{\alpha,\beta,\eta}(ds)\\
&= \int_{(\alpha-\beta)\mathbb{N}_0} \left(f(z+s)-f(z)\cdot\frac{\alpha-\beta}{\alpha}\right)\frac{\alpha}{\beta}\,\mu_{\alpha,\beta,\eta}(ds), \qquad (4.174)
\end{aligned}
$$

which is (4.167).

From (4.161) and (4.162), it follows that

$$
\begin{aligned}
V &= \alpha\,\frac{D_{\beta-\alpha}}{1-\alpha D_{\beta-\alpha}} + 1 = \frac{\alpha D_{\beta-\alpha} + (1-\alpha D_{\beta-\alpha})}{1-\alpha D_{\beta-\alpha}}\\
&= \frac{1}{1-\alpha D_{\beta-\alpha}} = (1-\alpha D_{\beta-\alpha})^{-1}. \qquad (4.175)
\end{aligned}
$$

Since

$$
Z = UV = U\,(1-\alpha D_{\beta-\alpha})^{-1}. \qquad (4.176)
$$

Multiplying this equality by $(1-\alpha D_{\beta-\alpha})$ from the right gives formula (4.168).

Finally, by (4.163),

$$
\begin{aligned}
1+\beta\partial^- &= 1+\beta\,\frac{D_{\alpha-\beta}}{1-\beta D_{\alpha-\beta}}\\
&= \frac{(1-\beta D_{\alpha-\beta}) + \beta D_{\alpha-\beta}}{1-\beta D_{\alpha-\beta}}\\
&= \frac{1}{1-\beta D_{\alpha-\beta}} = (1-\beta D_{\alpha-\beta})^{-1}. \qquad (4.177)
\end{aligned}
$$

Hence, by (4.160),

$$
U = \partial^+ (1-\beta D_{\alpha-\beta})^{-1} + \frac{\sigma}{\alpha}. \qquad (4.178)
$$

By (4.168) and (4.178),

$$
Z(1-\alpha D_{\beta-\alpha}) = \partial^+ (1-\beta D_{\alpha-\beta})^{-1} + \frac{\sigma}{\alpha}. \qquad (4.179)
$$

Multiplying this equality by $(1-\beta D_{\alpha-\beta})$ from the right gives formula (4.169). □

Since

$$
\mathbb{S}\partial^-\mathbb{S}^{-1} = D, \quad \mathbb{S}\partial^+\mathbb{S}^{-1} = Z,
$$

we have, by (4.159)–(4.161), the following

Proposition 4.57. *Define*

$$\mathbb{U} := \mathbb{S}U\mathbb{S}^{-1} = \mathbb{T}\mathcal{U}\mathbb{T}^{-1}, \quad \mathbb{V} := \mathbb{S}V\mathbb{S}^{-1} = \mathbb{T}\mathcal{V}\mathbb{T}^{-1}.$$

Then

$$\mathbb{U} = Z(1+\beta D) + \frac{\sigma}{\alpha}, \quad \mathbb{V} = 1 + \alpha D.$$

Similarly to Proposition 4.38, we have

Proposition 4.58. *The operator*

$$A^- := \sigma\partial^- + \eta\,\partial^+\partial^-\partial^-$$

is closable and let us keep the notation A^- for its closure. Then, for each $z \in \mathbb{C}$, the coherent state $\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z)$ given by (4.156) *belongs to the domain of A^- and*

$$A^-\,\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z) = z\,\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z).$$

4.6 Meixner case

Let $\sigma > 0$ and $\alpha, \beta \in \mathbb{C}$, $\beta = \overline{\alpha}$, $\Im(\alpha) > 0$. Define $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ by (4.107). By Lemma 4.40, $\mathcal{U}$ and $\mathcal{V}$ generate a generalised Weyl algebra with the commutation relation (4.108). Let as before $\rho = \mathcal{U}\mathcal{V}$. Then ρ^n is given by formula (4.109).

Let $(\widetilde{s}_n(z))_{n=0}^\infty$ be a sequence of monic polynomials on $\mathbb{C}$ satisfying the recurrence relation

$$z\,\widetilde{s}_n(z) = \widetilde{s}_{n+1}(z) + (\lambda n + \frac{\sigma}{\alpha})\,\widetilde{s}_n(z) + (\sigma n + \eta n(n-1))\,\widetilde{s}_{n-1}(z), \quad n \in \mathbb{N}_0, \tag{4.180}$$

where $\widetilde{s}_{-1}(z) = 0$, $\lambda = \alpha + \beta = 2\,\Re(\alpha)$, $\eta = \alpha\beta = |\alpha|^2$.

Let also $(s_n(z))_{n=0}^\infty$ be the sequence of Meixner polynomials satisfying the recurrence relation

$$zs_n(z) = s_{n+1}(z) + \lambda n\, s_n(z) + (\sigma n + \eta n(n-1))\, s_{n-1}(z). \tag{4.181}$$

By Section 2.4, $(s_n(z))_{n=0}^\infty$ are orthogonal with respect to the Meixner distribution on $\mathbb{R}$:

$$\mu_{\alpha,\beta,\sigma}(dx) = C_{\alpha,\beta,\sigma}\,\exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right)\left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\sigma}{2\alpha\Im(\alpha)}\right)\right|^2 dx, \tag{4.182}$$

where $\mathrm{Arg}(\alpha) \in (0, \pi)$ and the constant $C_{\alpha,\beta,\sigma}$ is given by

$$C_{\alpha,\beta,\sigma} = \frac{\left(2\cos\left(\frac{\pi}{2} - \mathrm{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \mathrm{Arg}(\alpha))\sigma\Re(\alpha)}{\Im(\alpha)\eta}\right). \tag{4.183}$$

By (4.180),

$$\left(z - \frac{\sigma}{\alpha}\right)\widetilde{s}_n(z) = \widetilde{s}_{n+1}(z) + \lambda n\,\widetilde{s}_n(z) + (\sigma n + \eta n(n-1))\,\widetilde{s}_{n-1}(z),$$

and so

$$z\widetilde{s}_n\left(z + \frac{\sigma}{\alpha}\right) = \widetilde{s}_{n+1}\left(z + \frac{\sigma}{\alpha}\right) + \lambda n\widetilde{s}_n\left(z + \frac{\sigma}{\alpha}\right) + (\sigma n + \eta n(n-1))\widetilde{s}_{n-1}\left(z + \frac{\sigma}{\alpha}\right).$$

Hence, by (4.181), we see that, for all $z \in \mathbb{C}$,

$$\widetilde{s}_n\left(z + \frac{\sigma}{\alpha}\right) = s_n(z), \tag{4.184}$$

in particular, for all $x \in \mathbb{R}$,

$$\widetilde{s}_n\left(x + \frac{\sigma}{\alpha}\right) = s_n(x). \tag{4.185}$$

Define

$$\mathbb{X}_{\alpha,\beta,\sigma} = \left\{z \in \mathbb{C} \mid z = x + \frac{\sigma}{\alpha},\ x \in \mathbb{R}\right\} = \mathbb{R} - i\,\frac{\sigma\,\Im(\alpha)}{\eta}.$$

Let $\widetilde{\mu}_{\alpha,\beta,\sigma}(dz)$ denote the probability measure on $\mathbb{X}_{\alpha,\beta,\sigma}$ which is the push-forward of $\mu_{\alpha,\beta,\sigma}$ under the map

$$\mathbb{R} \ni x \mapsto x + \frac{\sigma}{\alpha} \in \mathbb{X}_{\alpha,\beta,\sigma}. \tag{4.186}$$

Note also, that, by (4.185), $\widetilde{s}_n(z) \in \mathcal{P}(\mathbb{R})$ for each $z \in \mathbb{X}_{\alpha,\beta,\sigma}$.

Proposition 4.59. *The measure $\widetilde{\mu}_{\alpha,\beta,\sigma}$ on $\mathbb{X}_{\alpha,\beta,\sigma}$ is given by*

$$\widetilde{\mu}_{\alpha,\beta,\sigma}(dz) = \widetilde{C}_{\alpha,\beta,\sigma}\,\exp\left(\frac{(\frac{\pi}{2} - \mathrm{Arg}(\alpha))z}{\Im(\alpha)}\right)\left|\Gamma\left(\frac{iz}{2\,\Im(\alpha)}\right)\right|^2 dz, \tag{4.187}$$

where

$$\widetilde{C}_{\alpha,\beta,\sigma} = \frac{\left(2\cos\left(\frac{\pi}{2} - \mathrm{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(\frac{i(\frac{\pi}{2} - \mathrm{Arg}(\alpha))\sigma}{\eta}\right). \tag{4.188}$$

Note that $dz = d\,\Re(z)$ on $\mathbb{X}_{\alpha,\beta,\sigma}$.

Furthermore, $(\widetilde{s}_n(z))_{n=0}^\infty$ is a sequence of monic orthogonal polynomials on $\mathbb{X}_{\alpha,\beta,\sigma}$ with respect to the measure $\widetilde{\mu}_{\alpha,\beta,\sigma}$. More exactly,

$$\int_{\mathbb{X}_{\alpha,\beta,\sigma}} \widetilde{s}_m(z)\,\widetilde{s}_n(z)\ \widetilde{\mu}_{\alpha,\beta,\sigma}(dz) = \int_{\mathbb{R}} s_m(z)\,s_n(z)\ \mu_{\alpha,\beta,\sigma}(dx) = \delta_{m,n}\,n!\,(\sigma \mid -\eta)_n. \tag{4.189}$$

Proof. By (4.185) and the definition of $\widetilde{\mu}_{\alpha,\beta,\sigma}$, we have, for all $m, n \in \mathbb{N}_0$,

$$\begin{aligned}
&\int_{\mathbb{X}_{\alpha,\beta,\sigma}} \widetilde{s}_m(z)\, \widetilde{s}_n(z)\; \widetilde{\mu}_{\alpha,\beta,\sigma}(dz) \\
&\qquad = \int_{\mathbb{R}} \widetilde{s}_m\left(x + \frac{\sigma}{\alpha}\right)\, \widetilde{s}_n\left(x + \frac{\sigma}{\alpha}\right)\; \mu_{\alpha,\beta,\sigma}(dx) \\
&\qquad = \int_{\mathbb{R}} s_m(x)\, s_n(x)\; \mu_{\alpha,\beta,\sigma}(dx) = \delta_{m,n}\, n!\, (\sigma \mid -\eta)_n,
\end{aligned}$$

which proves (4.189).

Now, we will derive the explicit form of the measure $\widetilde{\mu}_{\alpha,\beta,\sigma}$. The push-forward of the Lebesgue measure dx under the map (4.186) is the measure $dz = d\Re(z)$ on $\mathbb{X}_{\alpha,\beta,\sigma}$. By construction, the density of the measure $\widetilde{\mu}_{\alpha,\beta,\sigma}$ with respect to dz is the density of the measure $\mu_{\alpha,\beta,\sigma}$ at the point $z - \frac{\sigma}{\alpha}$. We calculate:

$$\begin{aligned}
\left|\Gamma\left(\frac{i(z-\frac{\sigma}{\alpha})}{2\Im(\alpha)} + \frac{i\sigma}{2\alpha\Im(\alpha)}\right)\right|^2 &= \left|\Gamma\left(\frac{iz}{2\Im(\alpha)} - \frac{i\sigma}{2\alpha\Im(\alpha)} + \frac{i\sigma}{2\alpha\Im(\alpha)}\right)\right|^2 \\
&= \left|\Gamma\left(\frac{iz}{2\Im(\alpha)}\right)\right|^2, \qquad (4.190)
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))(z - \frac{\sigma}{\alpha})}{\Im(\alpha)}\right) \\
&\quad = \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))z}{\Im(\alpha)}\right) \exp\left(-\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\sigma}{\Im(\alpha)\,\alpha}\right) \\
&\quad = \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))z}{\Im(\alpha)}\right) \exp\left(-\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\sigma\beta}{\Im(\alpha)\,\eta}\right). \qquad (4.191)
\end{aligned}$$

By (4.182), (4.183), (4.190), and (4.191), formula (4.187) holds with

$$\begin{aligned}
\widetilde{C}_{\alpha,\beta,\sigma} &= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\sigma\Re(\alpha)}{\Im(\alpha)\,\eta}\right) \exp\left(-\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\sigma\beta}{\Im(\alpha)\,\eta}\right) \\
&= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\sigma(\Re(\alpha) - \beta)}{\Im(\alpha)\,\eta}\right) \\
&= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\sigma\, i\Im(\alpha)}{\Im(\alpha)\,\eta}\right) \\
&= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})} \exp\left(i\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\sigma}{\eta}\right). \qquad (4.192)
\end{aligned}$$

$\square$

Define $\widetilde{\mathcal{I}} \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ by

$$\widetilde{\mathcal{I}}(z \mid \beta)_n = \widetilde{s}_n(z).$$

Then, a straightforward generalization of Lemma 4.41 gives

$$Z = \widetilde{\mathcal{I}}\rho\widetilde{\mathcal{I}}^{-1}. \tag{4.193}$$

Proposition 4.60. *We have*

$$\begin{aligned} z^n = &\sum_{k=1}^{n}(\alpha-\beta)^{n-k}S(n,k)(\frac{\sigma}{\alpha} \mid -\beta)_k \\ &+ \sum_{i=1}^{n}\left(\sum_{k=i}^{n}(\alpha-\beta)^{n-k}S(n,k)\,S(k,i;-\beta,\beta,\frac{\sigma}{\alpha})\right)\widetilde{s}_i(z), \end{aligned} \tag{4.194}$$

and

$$\begin{aligned} \widetilde{s}_n(z) = &(-\frac{\sigma}{\alpha} \mid \beta)_n \\ &+ \sum_{i=1}^{n}\left(\sum_{k=i}^{n}S(n,k;\beta,-\beta,-\frac{\sigma}{\alpha})(\alpha-\beta)^{k-i}\,s(k,i)\right)z^i. \end{aligned} \tag{4.195}$$

Proof. Essentially identical to the proof of Theorem 4.42. □

Corollary 4.61. *We have*

$$\begin{aligned} \int_{\mathbb{X}_{\alpha,\beta,\sigma}} z^n\ \widetilde{\mu}_{\alpha,\beta,\sigma}(dz) &= \int_{\mathbb{R}}\left(x+\frac{\sigma}{\alpha}\right)^n\ \mu_{\alpha,\beta,\sigma}(dx) \\ &= \sum_{k=1}^{n}(\alpha-\beta)^{n-k}\,S(n,k)\left(\frac{\sigma}{\alpha} \mid -\beta\right)_k. \end{aligned}$$

Proof. Immediate from formulas (4.189) and (4.194). □

Define $\widetilde{\mathcal{S}} \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ as $\widetilde{\mathcal{S}} := \widetilde{\mathcal{I}}^{-1}$. Then

$$(\widetilde{\mathcal{S}}\,\widetilde{s}_n)(z) = (z \mid \beta)_n \tag{4.196}$$

and

$$\widetilde{\mathcal{S}}Z\widetilde{\mathcal{S}}^{-1} = \rho, \tag{4.197}$$

and so

$$(\widetilde{\mathcal{S}}\xi^n)(z) = (\rho^n 1)(z). \tag{4.198}$$

Corollary 4.62. *Let $z \in \mathbb{C}$ be the form $z = \beta r$, where $r > -\frac{\sigma}{\eta}$. Then,*

$$(\widetilde{\mathcal{S}}p)(z) = \int_{\mathbb{X}_{\alpha,\beta,\alpha z+\sigma}} p(\xi)\ \widetilde{\mu}_{\alpha,\beta,\alpha z+\sigma}(d\xi)$$
$$= \int_{\mathbb{R}} p\left(\xi + z + \frac{\sigma}{\alpha}\right)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi)$$

for all $p \in \mathcal{P}(\mathbb{C})$.

Proof. First, we note that, for z as in the lemma, we have $\alpha z + \sigma = \eta r + \sigma \in (0, \infty)$.

It is sufficient to prove that, for each $n \in \mathbb{N}$,

$$(\widetilde{\mathcal{S}}\xi^n)(z) = \int_{\mathbb{X}_{\alpha,\beta,\alpha z+\sigma}} \xi^n\ \widetilde{\mu}_{\alpha,\beta,\alpha z+\sigma}(d\xi).$$

But this follows from formulas (4.198), (4.116) (which holds in our case) and Corollary 4.61. □

Below, we will assume that $\Re(\alpha) \geq 0$, the case $\Re(\alpha) < 0$ can be done similarly.

Lemma 4.63. *Let $g_{\alpha,\beta,\sigma}(x)$ denote the density of the measure $\mu_{\alpha,\beta,\sigma}$, i.e.,*

$$g_{\alpha,\beta,\sigma}(x) = \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\sigma}{\eta})}\ \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\sigma\,\Re(\alpha)}{\Im(\alpha)\,\eta}\right)$$
$$\times \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right)\left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\sigma}{2\alpha\,\Im(\alpha)}\right)\right|^2.$$

Denote

$$D_{\alpha,\beta} = \left\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0,\ |\Im(\zeta)| < \Re(\zeta)\,\frac{\Im(\alpha)}{\Re(\alpha)}\right\}$$

if $\Re(\alpha) > 0$, and

$$D_{\alpha,\beta} = \{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\}$$

if $\Re(\alpha) = 0$.

Then for each fixed $x \in \mathbb{R}$, the function

$$(0, +\infty) \ni \sigma \mapsto g_{\alpha,\beta,\sigma}(x) \in \mathbb{R}$$

admits a unique analytic extension to a function

$$D_{\alpha,\beta} \ni \zeta \mapsto g_{\alpha,\beta,\zeta}(x) \in \mathbb{C}.$$

Explicitly, for $\zeta \in D_{\alpha,\beta}$, we have

$$
\begin{aligned}
g_{\alpha,\beta,\zeta}(x) = {} & \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\zeta}{\eta}}}{4\Im(\alpha)\,\pi\,\Gamma(\frac{\zeta}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\zeta\,\Re(\alpha)}{\Im(\alpha)\,\eta}\right) \\
& \times \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right) \Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta\beta}{2\eta\,\Im(\alpha)}\right) \Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta\alpha}{2\eta\,\Im(\alpha)}\right). \qquad (4.199)
\end{aligned}
$$

Proof. Since $\operatorname{Arg}(\alpha) \in (0, \pi)$, we have $\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right) > 0$. Therefore, the function

$$\mathbb{C} \ni \zeta \mapsto \left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\zeta}{\eta}} \in \mathbb{C}$$

is entire. Next, the function

$$\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\} \ni \zeta \mapsto \Gamma\left(\frac{\zeta}{\eta}\right) \in \mathbb{C}$$

is analytic and has no zeroes. Therefore, the function

$$\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\} \ni \zeta \mapsto \frac{1}{\Gamma(\frac{\zeta}{\eta})} \in \mathbb{C}$$

is analytic. The function

$$\mathbb{C} \ni \zeta \mapsto \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\Re(\alpha)\,\zeta}{\Im(\alpha)\,\eta}\right) \in \mathbb{C}$$

is entire.

Using the property $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ of the complex gamma function, we write, for $\zeta > 0$,

$$\left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta}{2\alpha\Im(\alpha)}\right)\right|^2 = \Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta}{2\alpha\,\Im(\alpha)}\right) \Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta}{2\beta\,\Im(\alpha)}\right).$$

We note that, for $\zeta \in D_{\alpha,\beta}$, the following inequalities hold:

$$
\begin{aligned}
\Re(\zeta)\,\Im(\alpha) - \Im(\zeta)\,\Re(\alpha) > 0, \\
\Re(\zeta)\,\Im(\alpha) + \Im(\zeta)\,\Re(\alpha) > 0.
\end{aligned}
$$

Hence, the mappings

$$
\begin{aligned}
D_{\alpha,\beta} \ni \zeta \mapsto & \frac{ix}{2\Im(\alpha)} + \frac{i\zeta}{2\alpha\,\Im(\alpha)} = \frac{ix}{2\Im(\alpha)} + \frac{i\zeta\beta}{2\eta\,\Im(\alpha)}, \\
D_{\alpha,\beta} \ni \zeta \mapsto & -\frac{ix}{2\Im(\alpha)} - \frac{i\zeta}{2\beta\,\Im(\alpha)} = -\frac{ix}{2\Im(\alpha)} - \frac{i\zeta\alpha}{2\eta\,\Im(\alpha)}.
\end{aligned}
$$

map the domain $D_{\alpha,\beta}$ into the domain $\{z \in \mathbb{C} \mid \Re(z) > 0\}$, on which the gamma function $\Gamma(z)$ is analytic. Hence, the function

$$\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta}{2\alpha\,\Im(\alpha)}\right)\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta}{2\beta\,\Im(\alpha)}\right)$$

is analytic on $D_{\alpha,\beta}$. Now, the lemma follows from the theorem on uniqueness of analytic function. $\square$

Below, for each $\zeta \in D_{\alpha,\beta}$, we will denote by $\mu_{\alpha,\beta,\zeta}(dx)$ the complex-valued measure on $\mathbb{R}$ that has density $g_{\alpha,\beta,\zeta}(x)$ with respect to the Lebesgue measure dx.

The following proposition is taken from [41, p.15].

Proposition 4.64. *The following asymptotic formula holds, for $d, x \in \mathbb{R}$:*

$$|\Gamma(d+ix)| = \sqrt{2\pi}\,\exp\left(-\frac{\pi}{2}|x|\right)\,|x|^{d-\frac{1}{2}}\,(1 + E(d,x)),$$

whereas, for each fixed $K > 0$, $E(d,x) \to 0$ uniformly in the strip $\{|d| \le K\}$ as $|x| \to \infty$.

Lemma 4.65. *Let $K > 1$.*

(i) There exists a constant $C > 0$ such that, for all $d \in [\frac{1}{K}, K]$ and $x \in \mathbb{R}$,

$$|\Gamma(d+ix)| \le C\exp\left(-\frac{\pi}{2}|x|\right)(1+|x|)^{K-\frac{1}{2}}.$$

(ii) For each $r < \frac{\pi}{2}$, there exists a constant $C_r > 0$ such that, for all $d \in [\frac{1}{K}, K]$ and $x \in \mathbb{R}$,

$$|\Gamma(d+ix)| \le C_r \exp(-r|x|).$$

Proof. (i) The result follows from Proposition 4.64 if we take into account that the function

$$[\frac{1}{K}, K] \times \mathbb{R} \ni (d,x) \mapsto |\Gamma(d+ix)| \in \mathbb{R}$$

is continuous, hence bounded on any set $[\frac{1}{K}, K] \times [-L, L]$ $(L > 0)$, whereas

$$\exp\left(-\frac{\pi}{2}|x|\right)(1+|x|)^{K-\frac{1}{2}} \ge 1, \quad x \in \mathbb{R}.$$

(ii) Immediate from (i). $\square$

Lemma 4.66. *For each $\zeta \in D_{\alpha,\beta}$ and $n \in \mathbb{N}_0$,*

$$\int_{\mathbb{R}} |x|^n \, |g_{\alpha,\beta,\zeta}(x)| \; dx < \infty.$$

Proof. In view of (4.199), we need to prove that

$$\int_{\mathbb{R}} |x|^n \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right) \left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta\beta}{2\eta\Im(\alpha)}\right)\right| \times \left|\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta\alpha}{2\eta\Im(\alpha)}\right)\right| \; dx < \infty. \tag{4.200}$$

We have

$$\begin{aligned}
\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta\beta}{2\eta\,\Im(\alpha)}\right) &= \Gamma\left(d_1 + i\Big(\frac{x}{2\Im(\alpha)} + l_1\Big)\right), \\
\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta\alpha}{2\eta\,\Im(\alpha)}\right) &= \Gamma\left(d_2 + i\Big(-\frac{x}{2\Im(\alpha)} + l_2\Big)\right),
\end{aligned} \tag{4.201}$$

where

$$\begin{aligned}
d_1 &= \frac{\Re(\zeta)\Im(\alpha) - \Im(\zeta)\Re(\alpha)}{2\eta\Im(\alpha)}, \\
l_1 &= \frac{\Re(\zeta)\Re(\alpha) + \Im(\zeta)\Im(\alpha)}{2\eta\Im(\alpha)}, \\
d_2 &= \frac{\Re(\zeta)\Im(\alpha) + \Im(\zeta)\Re(\alpha)}{2\eta\Im(\alpha)}, \\
l_2 &= \frac{-\Re(\zeta)\Re(\alpha) + \Im(\zeta)\Im(\alpha)}{2\eta\Im(\alpha)}.
\end{aligned} \tag{4.202}$$

In Lemma 4.65 (ii), choose $K > 0$ such that $d_1, d_2 \in [\frac{1}{K}, K]$. Then we conclude that, for $\frac{\pi}{2} - \operatorname{Arg}(\alpha) < r < \frac{\pi}{2}$, the integral in (4.200) is bounded by

$$\begin{aligned}
&C_r^2 \int_{\mathbb{R}} |x|^n \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,|x|}{\Im(\alpha)}\right) \\
&\qquad \times \exp\left(-r\left|\frac{x}{2\Im(\alpha)} + l_1\right|\right) \exp\left(-r\left|\frac{x}{2\Im(\alpha)} - l_2\right|\right) \; dx \\
&\leq C_r^2 e^{r(|l_1|+|l_2|)} \int_{\mathbb{R}} |x|^n \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))}{\Im(\alpha)}\,|x|\right) \exp\left(-\frac{r}{\Im(\alpha)}\,|x|\right) \; dx \\
&= C_r^2 e^{r(|l_1|+|l_2|)} \int_{\mathbb{R}} |x|^n \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha) - r)}{\Im(\alpha)}\,|x|\right) \; dx < \infty.
\end{aligned} \tag{4.203}$$

□

Lemma 4.67. *For each $n \in \mathbb{N}_0$, the map*

$$D_{\alpha,\beta} \ni \zeta \mapsto \int_{\mathbb{R}} x^n \, \mu_{\alpha,\beta,\zeta}(dx) \in \mathbb{C}$$

is analytic.

Proof. Define

$$\begin{aligned}\widehat{g}_{\alpha,\beta,\zeta}(x) = \exp&\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right) \\ &\times \Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta\beta}{2\eta\,\Im(\alpha)}\right) \Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta\alpha}{2\eta\,\Im(\alpha)}\right).\end{aligned}$$

It is sufficient to prove that, for each $n \in \mathbb{N}_0$, the map

$$D_{\alpha,\beta} \ni \zeta \mapsto \int_{\mathbb{R}} x^n \, \widehat{g}_{\alpha,\beta,\zeta}(x) \, dx \tag{4.204}$$

is analytic.

Let $\zeta \in D_{\alpha,\beta}$ be fixed. Let $R > 0$ be chosen so that

$$B(\zeta, R) = \{z \in \mathbb{C} \mid |z - \zeta| \leq R\} \subset D_{\alpha,\beta}.$$

To prove the differentiability of the map, it is sufficient to prove that

$$\int_{\mathbb{R}} |x|^n \sup_{z \in B(\zeta, \frac{R}{2})} \left| \frac{\partial}{\partial z} \widehat{g}_{\alpha,\beta,z}(x) \right| dx < \infty. \tag{4.205}$$

By Cauchy's integral formula,

$$\sup_{z \in B(\zeta, \frac{R}{2})} \left| \frac{\partial}{\partial z} \widehat{g}_{\alpha,\beta,z}(x) \right| \leq \sup_{z \in B(\sigma, R)} |\widehat{g}_{\alpha,\beta,z}(x)|. \tag{4.206}$$

Indeed, for each $z \in B(\sigma, \frac{R}{2})$,

$$\frac{\partial}{\partial z} \widehat{g}_{\alpha,\beta,z}(x) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\widehat{g}_{\alpha,\beta,z'}(x)}{z' - z} \, dz',$$

where γ is the circle $|z' - z| = \frac{R}{2}$ oriented counterclockwise. Hence,

$$\left| \frac{\partial}{\partial z} \widehat{g}_{\alpha,\beta,z}(x) \right| \leq \sup_{|z' - z| = \frac{R}{2}} |\widehat{g}_{\alpha,\beta,z'}(x)|,$$

which implies (4.206).

By (4.205) and (4.206), it is sufficient to prove that, for each $n \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} |x|^n \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))x}{\Im(\alpha)}\right) \sup_{z\in B(\zeta,R)} \left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{iz\beta}{2\eta\,\Im(\alpha)}\right)\right| \times \sup_{z\in B(\zeta,R)} \left|\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{iz\alpha}{2\eta\,\Im(\alpha)}\right)\right| dx < \infty. \tag{4.207}$$

Similarly to (4.202), denote

$$\begin{aligned} d_1(z) &= \frac{\Re(z)\Im(\alpha) - \Im(z)\Re(\alpha)}{2\eta\Im(\alpha)}, \\ l_1(z) &= \frac{\Re(z)\Re(\alpha) + \Im(z)\Im(\alpha)}{2\eta\Im(\alpha)}, \\ d_2(z) &= \frac{\Re(z)\Im(\alpha) + \Im(z)\Re(\alpha)}{2\eta\Im(\alpha)}, \\ l_2(z) &= \frac{-\Re(z)\Re(\alpha) + \Im(z)\Im(\alpha)}{2\eta\Im(\alpha)}. \end{aligned}$$

By (4.200) and (4.201),

$$\begin{aligned} &\sup_{z\in B(\zeta,R)} \left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{iz\beta}{2\eta\,\Im(\alpha)}\right)\right| \\ &\quad = \sup_{z\in B(\zeta,R)} \left|\Gamma\left(d_1(z) + i\big(\frac{x}{2\Im(\alpha)} + l_1(z)\big)\right)\right|, \\ &\sup_{z\in B(\zeta,R)} \left|\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{iz\alpha}{2\eta\Im(\alpha)}\right)\right| \\ &\quad = \sup_{z\in B(\zeta,R)} \left|\Gamma\left(d_2(z) + i\big(-\frac{x}{2\Im(\alpha)} + l_2(z)\big)\right)\right|. \end{aligned} \tag{4.208}$$

Choose $K > 0$ so that, for all $z \in B(\zeta, R)$, both $d_1(z)$ and $d_2(z)$ belong to $[-\frac{1}{K}, K]$. Denote

$$\mathcal{L}_1 = \max_{z\in B(\zeta,R)} |l_1(z)|, \qquad \mathcal{L}_2 = \max_{z\in B(\zeta,R)} |l_2(z)|.$$

Then, by Lemma 4.65, (ii), for $\frac{\pi}{2} - \operatorname{Arg}(\alpha) < r < \frac{\pi}{2}$ and $x \in \mathbb{R}$,

$$\begin{aligned} \sup_{z\in B(\zeta,R)} \left|\Gamma\left(d_1(z) + i\big(\frac{x}{2\Im(\alpha)} + l_1(z)\big)\right)\right| &\le C_r e^{r\mathcal{L}_1} e^{-r\frac{|x|}{2\Im(\alpha)}}, \\ \sup_{z\in B(\zeta,R)} \left|\Gamma\left(d_2(z) + i\big(-\frac{x}{2\Im(\alpha)} + l_2(z)\big)\right)\right| &\le C_r e^{r\mathcal{L}_2} e^{-r\frac{|x|}{2\Im(\alpha)}}. \end{aligned} \tag{4.209}$$

Now (4.207) follows from (4.208) and (4.209). □

For a fixed $\sigma > 0$, we define $\widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$ to be the pre-image of $D_{\alpha,\beta}$ under the map $\mathbb{C} \ni z \mapsto \alpha z + \sigma$, i.e.,

$$\widetilde{\mathcal{D}}_{\alpha,\beta,\sigma} = \left\{ z \in \mathbb{C} \mid z = \frac{\beta}{\eta}(\zeta - \sigma) \text{ where } \zeta \in D_{\alpha,\beta} \right\}.$$

Lemma 4.68. *$\widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$ is an open connected subset of $\mathbb{C}$ such that*

$$\left\{ z \in \mathbb{C} \mid z = \beta r \text{ where } r > -\frac{\sigma}{\eta} \right\} \subset \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}. \tag{4.210}$$

Furthermore, if $\Re(\alpha) = 0$ then $\mathbb{R} \subset \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$, and if $\Re(\alpha) > 0$ then

$$\left(-\frac{\sigma}{2\Re(\alpha)}, \infty \right) \subset \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}. \tag{4.211}$$

Proof. We only need to show formulas (4.210) and (4.211).

Let $r \in \mathbb{R}$ and $z = \beta r$. Then

$$\alpha z + \sigma = \alpha\beta r + \sigma = \eta r + \sigma \quad \in (0, +\infty)$$

if $r > -\frac{\sigma}{\eta}$. Thus, (4.210) holds.

Next, for $r \in \mathbb{R}$,

$$\alpha r + \sigma = \Re(\alpha)\, r + \sigma + i\, \Im(\alpha)\, r.$$

If $\Re(\alpha) = 0$, then $\Re(\alpha r + \sigma) = \sigma > 0$, hence $r \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$. If $\Re(\alpha) > 0$, then $\Re(\alpha r + \sigma) = \Re(\alpha)\, r + \sigma > 0$ if $r > -\frac{\sigma}{\Re(\alpha)}$. Furthermore, the inequality

$$|\Im(\alpha)\, r| < (\Re(\alpha)\, r + \sigma)\, \frac{\Im(\alpha)}{\Re(\alpha)}$$

is equivalent to

$$|r| < r + \frac{\sigma}{\Re(\alpha)},$$

which holds if and only if $r > -\frac{\sigma}{2\,\Re(\alpha)}$. $\square$

Proposition 4.69. *Let $p \in \mathcal{P}(\mathbb{C})$. Then, for each $z \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$,*

$$(\widetilde{\mathcal{S}}p)(z) = \int_{\mathbb{R}} p\left(\xi + z + \frac{\sigma}{\alpha} \right) \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi). \tag{4.212}$$

Proof. By Corollary 4.62, formula (4.212) holds for $z = \beta r$ with $r > -\frac{\sigma}{\eta}$. Note that $(\widetilde{\mathcal{S}}\xi^n)(z)$ is an entire function of z. On the other hand, by Lemma 4.67, the function

$$\widetilde{\mathcal{D}}_{\alpha,\beta,\sigma} \ni z \mapsto \int_{\mathbb{R}} p\left(\xi + z + \frac{\sigma}{\alpha}\right) \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi) \tag{4.213}$$

is analytic. Hence, by (4.210) and the theorem on uniqueness of analytic function, formula (4.212) holds for all $z \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$. □

Next, we define $\mathcal{S} \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ by

$$(\mathcal{S}s_n)(z) = (z \mid \beta)_n.$$

Proposition 4.70. *Let $p \in \mathcal{P}(\mathbb{C})$. Then, for each $z \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$,*

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(\xi + z)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi).$$

Proof. By (4.184) and Proposition 4.69,

$$\begin{aligned}\int_{\mathbb{R}} & s_n(\xi + z)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi)\\ &= \int_{\mathbb{R}} \widetilde{s}_n\left(\xi + z + \frac{\sigma}{\alpha}\right) \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi)\\ &= (z \mid \beta)_n,\end{aligned}$$

which implies the statement. □

Proposition 4.71. *Let $p \in \mathcal{P}(\mathbb{C})$. Then, for each $x \in \left(-\frac{\sigma}{2\Re(\alpha)}, +\infty\right)$,*

$$(\mathcal{S}p)(x) = \int_{\mathbb{R}} p(\xi)\, \mathcal{G}_{\alpha,\beta,\sigma}(x,\xi)\ d\xi,$$

where

$$\begin{aligned}\mathcal{G}_{\alpha,\beta,\sigma}(x,\xi) &:= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\alpha x+\sigma}{\eta}}}{4\Im(\alpha)\pi\,\Gamma(\frac{\alpha x+\sigma}{\eta})} \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))(\alpha x + \sigma)\,\Re(\alpha)}{\Im(\alpha)\,\eta}\right)\\ &\times \exp\left(\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))(\xi - x)}{\Im(\alpha)}\right)\Gamma\left(\frac{i\xi}{2\,\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\,\Im(\alpha)}\right)\Gamma\left(\frac{-i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{x\alpha}{\eta}\right).\end{aligned}$$

Proof. Using formula (4.211) and Proposition 4.70, we get, for each $x \in (-\frac{\sigma}{2\Re(\alpha)}, +\infty)$ (here and below we denote $-\frac{\sigma}{2\Re(\alpha)} = -\infty$ if $\Re(\alpha) = 0$),

$$(\mathcal{S}p)(x) = \int_{\mathbb{R}} p(\xi + x)\ g_{\alpha,\beta,\alpha x+\sigma}(\xi)\, d\xi$$

$$= \int_{\mathbb{R}} p(x)\; g_{\alpha,\beta,\alpha x+\sigma}(\xi - x)\, d\xi.$$

It remains to note that

$$\mathcal{G}_{\alpha,\beta,\sigma}(x,\xi) = g_{\alpha,\beta,\alpha x+\sigma}(\xi - x).$$

Indeed,

$$\begin{aligned}
&\Gamma\left(\frac{i(\xi - x)}{2\,\Im(\alpha)} + \frac{i(\alpha x+\sigma)\beta}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(\frac{i\xi}{2\,\Im(\alpha)} - \frac{ix}{2\,\Im(\alpha)} + \frac{i\eta x}{2\eta\,\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(\frac{i\xi}{2\,\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\,\Im(\alpha)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma\left(-\frac{i(\xi - x)}{2\,\Im(\alpha)} - \frac{i(\alpha x+\sigma)\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{ix}{2\Im(\alpha)} - \frac{i\alpha^2 x}{2\eta\,\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{ix}{2\Im(\alpha)}\left(1 - \frac{\alpha^2}{\eta}\right) - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{ix}{2\Im(\alpha)} \cdot \frac{\alpha\beta - \alpha^2}{\eta} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{ix}{2\Im(\alpha)} \cdot \frac{\alpha(\beta - \alpha)}{\eta} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{ix\alpha(-2i)\Im(\alpha)}{2\Im(\alpha)\,\eta} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right)\\
&\quad = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} + \frac{x\alpha}{\eta} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right).
\end{aligned}$$

□

Lemma 4.72. *Define*

$$\mathcal{D}_{\alpha,\beta,\sigma} = \left\{ z \in \mathbb{C} \mid \Im(z) < \frac{1}{\Im(\alpha)}\left(\Re(z)\Re(\alpha) + \frac{\sigma}{2}\right)\right\}. \tag{4.214}$$

Then, for each fixed $\xi \in \mathbb{R}$, the function

$$\left(-\frac{\sigma}{2\Re(\alpha)}, +\infty\right) \ni x \mapsto \mathcal{G}_{\alpha,\beta,\sigma}(x,\xi) \in \mathbb{C}$$

admits a unique analytic extension to a function

$$\mathcal{D}_{\alpha,\beta,\sigma} \ni z \mapsto \mathcal{G}_{\alpha,\beta,\sigma}(z,\xi),$$

where

$$\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi) = \frac{\big(2\cos\big(\frac{\pi}{2}-\operatorname{Arg}(\alpha)\big)\big)^{\frac{\alpha z+\sigma}{\eta}}}{4\Im(\alpha)\pi\,\Gamma(\frac{\alpha z+\sigma}{\eta})}\ \exp\left(\frac{(\frac{\pi}{2}-\operatorname{Arg}(\alpha))(\alpha z+\sigma)\,\Re(\alpha)}{\Im(\alpha)\,\eta}\right)$$
$$\times\exp\left(\frac{(\frac{\pi}{2}-\operatorname{Arg}(\alpha))(\xi-z)}{\Im(\alpha)}\right)\Gamma\left(\frac{i\xi}{2\,\Im(\alpha)}+\frac{i\sigma\beta}{2\eta\,\Im(\alpha)}\right)\Gamma\left(-\frac{i\xi}{2\Im(\alpha)}-\frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}+\frac{z\alpha}{\eta}\right). \tag{4.215}$$

Proof. First, we note that

$$\begin{aligned}\alpha z+\sigma &= \big(\Re(\alpha)+i\Im(\alpha)\big)(\Re(z)+i\Im(z))+\sigma\\ &= (\Re(\alpha)\Re(z)-\Im(\alpha)\Im(z)+\sigma\big)+i\big(\Im(\alpha)\Re(z)+\Re(\alpha)\Im(z)\big)\end{aligned}$$

and

$$\Re(\alpha)\Re(z)-\Im(\alpha)\Im(z)+\sigma>0$$

for $z\in\mathcal{D}_{\alpha,\beta,\sigma}$.

Next, let us show that

$$\Re\bigg(-\frac{i\xi}{2\Im(\alpha)}-\frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}+\frac{z\alpha}{\eta}\bigg)>0 \quad \text{for } z\in\mathcal{D}_{\alpha,\beta,\sigma}. \tag{4.216}$$

For this, it is sufficient to check that

$$\Re\left(-\frac{i\sigma\alpha}{2\Im(\alpha)}+z\alpha\right)>0.$$

In fact, we have

$$\begin{aligned}\Re\left(-\frac{i\sigma\alpha}{2\Im(\alpha)}+z\alpha\right) &= \Re\left(-\frac{i\sigma\alpha}{2\Im(\alpha)}\right)+\Re\left(z\alpha\right)\\ &= \frac{\sigma}{2}+\Re(\alpha)\Re(z)-\Im(\alpha)\Im(z)>0\end{aligned}$$

for $z\in\mathcal{D}_{\alpha,\beta,\sigma}$.

Now, the lemma follows from the theorem on uniqueness of analytic function. □

Lemma 4.73. *For each $n\in\mathbb{N}_0$, the map*

$$\mathcal{D}_{\alpha,\beta,\sigma}\ni z\mapsto\int_{\mathbb{R}}\xi^n\ \mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)\,d\xi$$

is analytic.

Proof. The proof of this lemma is similar to the proof of Lemma 4.67, thus we omit it. □

Proposition 4.74. *Let $p \in \mathcal{P}(\mathbb{C})$. Then, for each $z \in \mathcal{D}_{\alpha,\beta,\sigma}$,*

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(\xi)\ \mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)\, d\xi,$$

where $\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)$ is given by (4.215).

Proof. We note that

$$\left(-\frac{\sigma}{2\Re(\alpha)}, +\infty\right) \subset \mathcal{D}_{\alpha,\beta,\sigma}.$$

Hence, by Proposition 4.71, Lemma 4.73 and the theorem on uniqueness of analytic function, the result follows. □

Proposition 4.75. *Let $z_0 \in \mathcal{D}_{\alpha,\beta,\sigma}$. Let $R > 0$ be such that the closed ball*

$$B(z_0, R) = \{z \in \mathbb{C} \mid |z - z_0| \le R\}$$

is a subset of $\mathcal{D}_{\alpha,\beta,\sigma}$. Then, there exists a constant $C > 0$ such that, for all $f \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$,

$$\sup_{z \in B(z_0,R)} \int_{\mathbb{R}} |f(\xi)|\, |\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)|\ d\xi \le C\, \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})}.$$

Proof. We have, by the Cauchy–Schwarz inequality, for each $z \in B(z_0, R)$,

$$\begin{aligned}
\int_{\mathbb{R}} |f(\xi)|\, |\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)|\, d\xi &= \int_{\mathbb{R}} |f(\xi)| \left| \frac{\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)}{g_{\alpha,\beta,\sigma}(\xi)} \right| \mu_{\alpha,\beta,\sigma}(d\xi) \\
&\le \left(\int_{\mathbb{R}} |f(\xi)|^2\ \mu_{\alpha,\beta,\sigma}(d\xi) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left| \frac{\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)}{g_{\alpha,\beta,\sigma}(\xi)} \right|^2 \mu_{\alpha,\beta,\sigma}(d\xi) \right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(\mathbb{R},\, \mu_{\alpha,\beta,\sigma})} \left(\int_{\mathbb{R}} \frac{|\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)|^2}{g_{\alpha,\beta,\sigma}(\xi)}\ d\xi \right)^{\frac{1}{2}}. \qquad (4.217)
\end{aligned}$$

We have, by (4.199) and (4.215),

$$\begin{aligned}
\frac{\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)^2}{g_{\alpha,\beta,\sigma}(\xi)} &= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{2\alpha z - \sigma}{\eta}}}{4\Im(\alpha)\pi} \cdot \frac{\Gamma(\frac{\sigma}{\eta})}{\Gamma^2\left(\frac{\alpha z + \sigma}{\eta}\right)} \\
&\times \exp\left[\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\, \Re(\alpha)\, (2\alpha z + \sigma)}{\Im(\alpha)\eta} + \frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))(\xi - 2z)}{\Im(\alpha)} \right] \\
&\times \Gamma\left(\frac{i\xi}{2\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\, \Im(\alpha)} \right) \cdot \frac{\Gamma^2\left(-\frac{i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\, \Im(\alpha)} + \frac{z\alpha}{\eta} \right)}{\Gamma\left(-\frac{i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\, \Im(\alpha)} \right)}. \qquad (4.218)
\end{aligned}$$

Since $\Gamma(\xi) = \Gamma(\bar{\xi})$, we have

$$\Gamma\left(\frac{i\xi}{2\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\,\Im(\alpha)}\right) = \Gamma\left(-\frac{i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)}\right).$$

Hence, by (4.218),

$$\begin{aligned}\frac{\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)^2}{g_{\alpha,\beta,\sigma}(\xi)} &= \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{2\alpha z-\sigma}{\eta}}}{4\Im(\alpha)\pi} \cdot \frac{\Gamma(\frac{\sigma}{\eta})}{\Gamma^2\left(\frac{\alpha z+\sigma}{\eta}\right)} \\ &\times \exp\left[\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\,\Re(\alpha)\,(2\alpha z+\sigma)}{\Im(\alpha)\eta} + \frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))(\xi - 2z)}{\Im(\alpha)}\right] \\ &\times \Gamma^2\left(-\frac{i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta}\right). \qquad (4.219)\end{aligned}$$

Thus, by (4.218) and (4.219), to show (4.217), it is sufficient to prove that

$$\sup_{z\in B(z_0,R)} \int_{\mathbb{R}} \exp\left[\frac{(\frac{\pi}{2} - \operatorname{Arg}(\alpha))\xi}{\Im(\alpha)}\right] \cdot \left|\Gamma^2\left(-\frac{i\xi}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta}\right)\right| d\xi < \infty. \qquad (4.220)$$

Using the change of variable $\xi' = -\frac{\xi}{2\Im(\alpha)}$, we see that (4.220) is equivalent to

$$\sup_{z\in B(z_0,R)} \int_{\mathbb{R}} \exp\left[-2\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\xi\right] \cdot \left|\Gamma^2\left(i\xi - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta}\right)\right| d\xi < \infty. \qquad (4.221)$$

For $\xi \in \mathbb{R}$ and $z \in B(z_0, R)$,

$$\begin{aligned}&i\xi - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta} \\ &\quad = i\xi - i\frac{\sigma\Re(\alpha)}{2\eta\,\Im(\alpha)} + \frac{\sigma}{2\eta} + \Re\left(\frac{z\alpha}{\eta}\right) + i\,\Im\left(\frac{z\alpha}{\eta}\right),\end{aligned}$$

where

$$\begin{aligned}\Re\left(\frac{z\alpha}{\eta}\right) &= \frac{\Re(z)\Re(\alpha) - \Im(z)\Im(\alpha)}{\eta}, \\ \Im\left(\frac{z\alpha}{\eta}\right) &= \frac{\Im(z)\Re(\alpha) + \Re(z)\Im(\alpha)}{\eta}.\end{aligned}$$

Define

$$\begin{aligned}d_1 = \frac{\sigma}{2\eta}, \quad l_1 = -\frac{\sigma\Re(\alpha)}{2\eta\,\Im(\alpha)}, \\ d_2(z) = \Re\left(\frac{z\alpha}{\eta}\right), \quad l_2(z) = \Im\left(\frac{z\alpha}{\eta}\right).\end{aligned}$$

Then,

$$\left|\Gamma\left(i\xi - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta}\right)\right| = \big|\Gamma\big(i(\xi + l_1 + l_2(z)) + d_1 + d_2(z)\big)\big|.$$

Recall (4.216). Choose $K > 1$ such that

$$d_1 + d_2(z) \in \left[\frac{1}{K}, K\right] \quad \text{for for all } z \in B(z_0, R).$$

Hence, by Lemma 4.65, (i), there exists a constant $C > 0$ such that, for all $\xi \in \mathbb{R}$ and $z \in B(z_0, R)$,

$$\begin{aligned} &\big|\Gamma\big(i(\xi + l_1 + l_2(z)) + d_1 + d_2(z)\big)\big| \\ &\quad \le C \exp\left(-\frac{\pi}{2}|\xi + l_1 + l_2(z)|\right)\big(1 + |\xi + l_1 + l_2(z)|\big)^{K-\frac{1}{2}}. \end{aligned} \tag{4.222}$$

This in turn implies that, for a constant $C' > 0$ and for all $\xi \in \mathbb{R}$ and $z \in B(z_0, R)$,

$$\begin{aligned} &\big|\Gamma\big(i(\xi + l_1 + l_2(z)) + d_1 + d_2(z)\big)\big| \\ &\quad \le C' \exp\left(-\frac{\pi}{2}|\xi|\right)\big(1 + |\xi|\big)^{K}. \end{aligned}$$

Therefore, for all $\xi \in \mathbb{R}$ and $z \in B(z_0, R)$,

$$\begin{aligned} &\exp\left[-2\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\xi\right] \cdot \left|\Gamma^2\left(i\xi - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta}\right)\right| \\ &\quad \le \exp\left[2\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)|\xi|\right] \cdot (C')^2 \exp\left(-\pi|\xi|\right)\big(1 + |\xi|\big)^{2K} \\ &\quad = (C')^2 \exp\big(-2\operatorname{Arg}(\alpha)|\xi|\big)\big(1 + |\xi|\big)^{2K}, \end{aligned}$$

which implies (4.221). □

Theorem 4.76. *(i) Let $(f_n)_{n=0}^\infty$ be a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} |f_n|^2\, n!\, (\sigma \mid -\eta)_n < \infty. \tag{4.223}$$

Then the series $\sum_{n=0}^\infty f_n\,(z \mid \beta)_n$ converges uniformly on compact sets in $\mathcal{D}_{\alpha,\beta,\sigma}$, and it is an analytic function on $\mathcal{D}_{\alpha,\beta,\sigma}$.

(ii) Denote by $\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ the space of all analytic functions on $\mathcal{D}_{\alpha,\beta,\sigma}$ that have representation

$$\varphi(z) = \sum_{n=0}^{\infty} f_n\,(z \mid \beta)_n, \tag{4.224}$$

with $(f_n)_{n=0}^{\infty}$ *satisfying* (4.223). *Consider* $\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ *as a Hilbert space equipped with inner product*

$$(\varphi, \psi)_{\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})} = \sum_{n=0}^{\infty} f_n \, \overline{g_n} \, n! \, (\sigma \mid -\eta)_n \tag{4.225}$$

with $\varphi(z) = \sum_{n=0}^{\infty} f_n \, (z \mid \beta)_n$, $\psi(z) = \sum_{n=0}^{\infty} g_n \, (z \mid \beta)_n \in \mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$. *Let*

$$\mathcal{S} : L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$$

be the unitary operator defined by $(\mathcal{S} s_n)(z) = (z \mid \beta)_n$. *Then, for all* $z \in \mathcal{D}_{\alpha,\beta,\sigma}$,

$$(\mathcal{S} f)(z) = \int_{\mathbb{R}} f(\xi) \; \mathcal{G}_{\alpha,\beta,\sigma}(z, \xi) \, d\xi,$$

where $\mathcal{G}_{\alpha,\beta,\sigma}(z,\xi)$ *is given by* (4.215).

(iii) In formula (4.224),

$$\begin{aligned} f_n &= \frac{1}{n!} (D_\beta^n \varphi)(0) \\ &= \frac{(-1)^n}{n! \, \beta^n} \sum_{l=0}^{n} (-1)^k \binom{n}{k} \varphi(\beta k). \end{aligned}$$

In particular, the function φ *is completely determined by its values on the set* $\{\beta k \mid k \in \mathbb{N}_0\}$.

(iv) The $\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ *is a reproducing kernel Hilbert space with reproducing kernel*

$$\mathcal{K}_{\alpha,\beta,\sigma}(z, w) = \sum_{n=0}^{\infty} \frac{(\bar{z} \mid \alpha)_n \, (w \mid \beta)_n}{(\sigma \mid -\eta)_n}, \qquad z, w \in \mathcal{D}_{\alpha,\beta,\sigma}.$$

(v) Define

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) = \sum_{n=0}^{\infty} \frac{(z \mid \beta)_n}{n! \, (\sigma \mid -\eta)_n} \, s_n(\xi), \qquad \xi \in \mathbb{R}, \;\; z \in \mathcal{D}_{\alpha,\beta,\sigma}. \tag{4.226}$$

Then, for each $z \in \mathcal{D}_{\alpha,\beta,\sigma}$, *we have* $\mathcal{E}_{\alpha,\beta,\sigma}(\cdot, z) \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$ *and*

$$\begin{aligned} (\mathcal{S} f)(z) &= (f, \overline{\mathcal{E}_{\alpha,\beta,\sigma}(\cdot, z)})_{L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})} \\ &= \int_{\mathbb{R}} f(\xi) \, \mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) \; \mu_{\alpha,\beta,\sigma}(d\xi). \end{aligned}$$

(vi) The function $\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z)$ *defined by* (4.226) *admits the following explicit formula, for* $\xi \in \mathbb{R}$ *and* $z \in \mathcal{D}_{\alpha,\beta,\sigma}$,

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi, z) = \Big(2 \cos\big(\frac{\pi}{2} - \mathrm{Arg}(\alpha)\big)\Big)^{\frac{\alpha z}{\eta}} \, \frac{\Gamma\left(\frac{\sigma}{\eta}\right)}{\Gamma\left(\frac{\sigma + \alpha z}{\eta}\right)}$$

$$\times \exp\left(\frac{i(\frac{\pi}{2}-\mathrm{Arg}(\alpha))\alpha z}{\eta}\right)\frac{\Gamma\left(-\frac{i\xi}{2\Im(\alpha)}-\frac{i\sigma\alpha}{2\eta\Im(\alpha)}+\frac{\alpha z}{\eta}\right)}{\Gamma\left(-\frac{i\xi}{2\Im(\alpha)}-\frac{i\sigma\alpha}{2\eta\Im(\alpha)}\right)}. \tag{4.227}$$

Proof. The proof is similar to that of Theorem 4.25 and Theorem 4.48, so we only outline the main new points.

Parts (i) and (ii) follow from Propositions 4.74 and 4.75.

To show part (iii), we note that the proof of part (iii) of Theorem 4.25 admits an immediate extension to the case where $\beta\in\mathbb{C}$. In fact, it easily follows from formula (4.214) that $r\beta\in\mathcal{D}_{\alpha,\beta,\sigma}$ for $r>-\frac{\sigma}{2|\alpha|^2}$, hence $\beta k\in\mathcal{D}_{\alpha,\beta,\sigma}$ for all $k\in\mathbb{N}_0$.

The proof of part (iv) is similar to the proof of Theorem 4.48, (iv). We only need to note that formula (4.148) implies

$$g_n=\frac{(\bar z\mid\alpha)_n}{n!\,(\sigma\mid-\eta)_n}.$$

(v) Immediate.

(vi) We note that

$$\mathcal{E}_{\alpha,\beta,\sigma}(\xi,z)=\frac{\mathcal{G}_{\alpha,\beta,\sigma}(\xi)}{g_{\alpha,\beta,\sigma}(\xi)},$$

hence the statement follows from (4.199) and (4.215). □

Similarly to Proposition 4.50, we have

Proposition 4.77. *Define a unitary operator* $\mathbb{T}:\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})\to\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{T}(\cdot\mid\beta)_n)(z)=z^n. \tag{4.228}$$

Then, for each $f\in\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ *and* $z\in\mathbb{C}$,

$$(\mathbb{T}f)(z)=\int_{\mathbb{N}_0}f(\beta\xi)\;\pi_{\frac{z}{\beta}}(d\xi). \tag{4.229}$$

Theorem 4.78. *Define a unitary operator* $\mathbb{S}:L^2(\mathbb{R},\mu_{\alpha,\beta,\sigma})\to\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ *by*

$$(\mathbb{S}s_n)(z)=z^n.$$

(i) For each $f\in L^2(\mathbb{R},\mu_{\alpha,\beta,\sigma})$,

$$(\mathbb{S}f)(z)=\int_{\mathbb{R}}f(\xi)\;\mathbb{G}_{\alpha,\beta,\sigma}(z,\xi)\,d\xi,\quad z\in\mathbb{C}, \tag{4.230}$$

where

$$\mathbb{G}_{\alpha,\beta,\sigma}(z,\xi) = \int_{\mathbb{N}_0} \mathcal{G}_{\alpha,\beta,\sigma}(\beta\zeta,\xi)\ \pi_{\frac{z}{\beta}}(d\zeta) \tag{4.231}$$

where the function $\mathcal{G}_{\alpha,\beta,\sigma}$ *is given by* (4.215).

(ii) The operator $\mathbb{S}$ *is a generalised Segal–Bargmann transform constructed through the nonlinear coherent state*

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,(\sigma \mid -\eta)_n}\, s_n(\xi), \tag{4.232}$$

i.e.,

$$\begin{aligned}(\mathbb{S}f)(z) &= (f, \mathbb{E}_{\alpha,\beta,\sigma}(\cdot,\bar{z}))_{L^2(\mathbb{R},\mu_{\alpha,\beta,\sigma})}\\ &= \int_{\mathbb{R}} f(\xi)\,\mathbb{E}_{\alpha,\beta,\sigma}(\xi,z)\ \mu_{\alpha,\beta,\sigma}(d\xi).\end{aligned} \tag{4.233}$$

Here

$$\mathbb{E}_{\alpha,\beta,\sigma}(\xi,z) = \int_{\mathbb{N}_0} \mathcal{E}_{\alpha,\beta,\sigma}(\xi,\beta\zeta)\ \pi_{\frac{z}{\beta}}(d\zeta), \tag{4.234}$$

where the function $\mathcal{E}_{\alpha,\beta,\sigma}$ *is given by formula* (4.227).

Proof. Since $\mathbb{S} = \mathbb{T}\mathcal{S}$, the theorem immediately follows from Theorem 4.76 and Proposition 4.77. □

Similarly to Propositon 4.52, we have

Proposition 4.79. *The Fréchet space* $\mathcal{E}^1_{\min}(\mathbb{C})$ *is continuously embedded into the following spaces:* $L^2(\mu_{\alpha,\beta,\sigma})$*,* $\mathcal{F}_{\alpha,\beta,\sigma}(\mathcal{D}_{\alpha,\beta,\sigma})$ *and* $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$*. Furthermore, the operators* $\mathcal{S}$*,* $\mathbb{S}$ *and* $\mathbb{T}$ *restricted to* $\mathcal{E}^\tau_{\min}(\mathbb{C})\,(\tau \in (0,1])$ *are self-homeomorphisms of* $\mathcal{E}^\tau_{\min}(\mathbb{C})$*.*

Proposition 4.80. *(i) Let* $f \in \mathcal{E}^1_{\min}(\mathbb{C})$*. Then, for each* $z \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$*,*

$$(\mathcal{S}f)(z) = \int_{\mathbb{R}} f(\xi + z)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi). \tag{4.235}$$

(ii) For each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$*,*

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(\xi + z)\ \rho_{\alpha,\beta,\sigma,z}(d\xi),$$

where

$$\rho_{\alpha,\beta,\sigma,z}(d\xi) = \int_{\mathbb{N}_0} \mu_{\alpha,\beta,\eta m+\sigma}(d\xi)\ \pi_{\frac{z}{\beta}}(dm).$$

In particular, for each $z = \beta r$ with $r > 0$, $\rho_{\alpha,\beta,\sigma,z}$ is the random Meixner distribution $\mu_{\alpha,\beta,\eta\zeta+\sigma}$, where ζ is a random variable having having Poisson distribution π_r.

Proof. (i) By Proposition 4.70, formula (4.235) holds for each $f \in \mathcal{P}(\mathbb{C})$.

Let $f(z) = \sum_{n=0}^{\infty} f_n\, s_n(z) \in \mathcal{E}^1_{\mathrm{min}}(\mathbb{C})$. Define $p_k(z) = \sum_{n=0}^{k} f_n\, s_n(z) \in \mathcal{P}(\mathbb{C})$. Then, $p_k \to f$ in $\mathcal{E}^1_{\mathrm{min}}(\mathbb{C})$ as $k \to \infty$. Hence, for each $z \in \mathbb{C}$, $(\mathcal{S}p_k)(z) \to (\mathcal{S}f)(z)$ as $k \to \infty$. By (2.4), for each $t > 0$, there exists a constant $C_t > 0$ such that

$$\sup_{k\in\mathbb{N}} \sup_{z\in\mathbb{C}} |p_k(z)| \le C_t\, e^{t|z|}.$$

Furthermore, for each $z \in \mathbb{C}$ and $\xi \in \mathbb{R}$,

$$p_k(\xi + z) \to f(\xi + z) \quad \text{as} \quad k \to \infty.$$

Hence, by the dominated convergence theorem, to prove that, for each $z \in \widetilde{\mathcal{D}}_{\alpha,\beta,\sigma}$,

$$\begin{aligned}\int_{\mathbb{R}} p_k(\xi + z)\ \mu_{\alpha,\beta,\alpha z+\sigma}(d\xi) &= \int_{\mathbb{R}} p_k(\xi + z)\ g_{\alpha,\beta,\alpha z+\sigma}(\xi)\, d\xi\\ &\to \int_{\mathbb{R}} f(\xi + z)\ g_{\alpha,\beta,\alpha z+\sigma}(\xi)\, d\xi \quad \text{as} \quad k \to \infty,\end{aligned}$$

it is sufficient to show that, for some $t > 0$,

$$\int_{\mathbb{R}} e^{t|z|}\, |g_{\alpha,\beta,\alpha z+\sigma}(\xi)|\ d\xi < \infty.$$

But this fact follows from the proof of Lemma 4.67, see in particular formula (4.203).

(ii) Immediate by (4.210), Proposition 4.77 and part (i). □

Let ∂^+ and ∂^- denote the raising and lowering operators for the polynomials $(s_n(z))_{n=0}^{\infty}$, compare with formula (4.96). Denote

$$U := \mathcal{IUS}, \quad V := \mathcal{IVS}.$$

Similarly to Proposition 4.53, we get

Proposition 4.81. *We have*

$$U = \partial^+ + \beta\, \partial^+\, \partial^- + \frac{\sigma}{\alpha} = \partial^+(1 + \beta\, \partial^-) + \frac{\sigma}{\alpha}, \tag{4.236}$$

$$V = \alpha\, \partial^- + 1, \tag{4.237}$$

and

$$UV = Z + \frac{\sigma}{\alpha}.$$

Theorem 4.82. *The operators ∂^-, ∂^+, U, V may be extended by continuity to $\mathcal{E}^1_{\min}(\mathbb{C})$. Furthermore, for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$, we have*

$$(\partial^- f)(z) = \int_{\mathbb{R}} \left(f\left(z + s + \frac{\eta}{\alpha} \right) - f(z) \right) \frac{1}{\beta}\, \mu_{\alpha,\beta,\eta}(ds), \tag{4.238}$$

$$(Vf)(z) = \int_{\mathbb{R}} \left(f\left(z + s + \frac{\eta}{\alpha} \right) - f(z) \cdot \frac{2i\Im(\alpha)}{\alpha} \right) \frac{\alpha}{\beta}\, \mu_{\alpha,\beta,\eta}(ds), \tag{4.239}$$

and

$$U = \left(Z + \frac{\sigma}{\alpha} \right) (1 - \alpha D_{-2i\Im(\alpha)}) \tag{4.240}$$

$$\partial^+ = \left(Z + \frac{\sigma}{\alpha} \right) (1 - \alpha D_{-2i\Im(\alpha)})(1 - \beta D_{2i\Im(\alpha)}) - \frac{\sigma}{\alpha}\, (1 - \beta D_{2i\Im(\alpha)}). \tag{4.241}$$

Proof. Let us similarly introduce operators $\widetilde{\partial}^+$ and $\widetilde{\partial}^-$ for the polynomials $(\widetilde{s}_n(z))_{n=0}^\infty$ which satisfy

$$(\widetilde{\partial}^+ \widetilde{s}_n)(z) = \widetilde{s}_{n+1}(z), \quad (\widetilde{\partial}^- \widetilde{s}_n)(z) = n\, \widetilde{s}_{n-1}(z). \tag{4.242}$$

Lemma 4.83. *We have*

$$\partial^- = \widetilde{\partial}^-, \tag{4.243}$$

$$\partial^+ = E^{\frac{\sigma}{\alpha}}\, \widetilde{\partial}^+\, E^{-\frac{\sigma}{\alpha}}. \tag{4.244}$$

Proof. By (4.184),

$$s_n(z) = (E^{\frac{\sigma}{\alpha}}\, \widetilde{s}_n)(z). \tag{4.245}$$

From this and (4.242), formula (4.244) follows.

Similarly,

$$\partial^- = E^{\frac{\sigma}{\alpha}}\, \widetilde{\partial}^-\, E^{-\frac{\sigma}{\alpha}}.$$

But ∂^- is the lowering operator for a Sheffer sequence, hence it is shift-invariant. Thus,

$$\partial^- = E^{\frac{\sigma}{\alpha}}\, E^{-\frac{\sigma}{\alpha}}\, \widetilde{\partial}^- = \widetilde{\partial}^-. \qquad \square$$

Define

$$\widetilde{U} := \widetilde{\mathcal{I}}\mathcal{U}\widetilde{\mathcal{S}}, \quad \widetilde{V} := \widetilde{\mathcal{I}}\mathcal{V}\widetilde{\mathcal{S}}.$$

Similarly to Proposition 4.53, we see that

$$\widetilde{U} = \widetilde{\partial}^+(1 + \beta\partial^-) + \frac{\sigma}{\alpha},$$

$$\widetilde{V} = \alpha\,\partial^- + 1.$$

Similarly to Proposition 4.54, we get

Lemma 4.84. *We have*

$$\begin{aligned}\widetilde{\partial}^- &= \frac{D_{\beta-\alpha}}{1-\alpha D_{\beta-\alpha}} = \frac{D_{-2i\Im(\alpha)}}{1-\alpha\, D_{-2i\Im(\alpha)}}\\ &= \frac{D_{\alpha-\beta}}{1-\beta D_{\alpha-\beta}} = \frac{D_{2i\Im(\alpha)}}{1-\beta\, D_{2i\Im(\alpha)}}.\end{aligned}$$

Lemma 4.85. *The operators ∂^-, $\widetilde{\partial}^+$, $\widetilde{U}$, $\widetilde{V}$ may be extended by continuity to $\mathcal{E}^1_{\min}(\mathbb{C})$, and furthermore, for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$,*

$$(\widetilde{\partial}^- f)(z) = \int_{\mathbb{X}_{\alpha,\beta,\eta}} \big(f(z+\zeta) - f(z)\big)\,\frac{1}{\beta}\,\widetilde{\mu}_{\alpha,\beta,\eta}\,(d\zeta), \tag{4.246}$$

$$(\widetilde{V} f)(z) = \int_{\mathbb{X}_{\alpha,\beta,\eta}} \left(f(z+\zeta) - f(z)\cdot\frac{2i\,\Im(\alpha)}{\alpha}\right)\,\frac{\alpha}{\beta}\,\widetilde{\mu}_{\alpha,\beta,\eta}(d\zeta), \tag{4.247}$$

$$\widetilde{U} = Z(1-\alpha D_{-2i\,\Im(\alpha)}), \tag{4.248}$$

$$\widetilde{\partial}^+ = Z(1-\alpha D_{-2i\,\Im(\alpha)})(1-\beta D_{2i\,\Im(\alpha)}) - \frac{\sigma}{\alpha}\,(1-\beta D_{2i\,\Im(\alpha)}). \tag{4.249}$$

Proof. The proof is similar to that of Theorem 4.55. Indeed, similarly to (4.170), we have

$$\partial^-(z \mid \alpha-\beta)_n = \sum_{k=0}^{n-1} \frac{n!}{k!}\,\beta^{n-k-1}\,(z \mid \alpha-\beta)_k, \tag{4.250}$$

and similarly to (4.171),

$$\begin{aligned}&\int_{\mathbb{X}_{\alpha,\beta,\eta}} \big[(z+\zeta \mid \alpha-\beta)_n - (z \mid \alpha-\beta)_n\big]\,\frac{1}{\beta}\,\widetilde{\mu}_{\alpha,\beta,\eta}(d\zeta)\\ &\qquad = \sum_{k=0}^{n-1} \frac{n!}{k!}\,(z \mid \alpha-\beta)_k \;\cdot\; \frac{1}{\beta\,(n-k)!}\int_{\mathbb{X}_{\alpha,\beta,\eta}} (\zeta \mid \alpha-\beta)_{n-k}\,\widetilde{\mu}_{\alpha,\beta,\eta}(d\zeta).\end{aligned} \tag{4.251}$$

Next, similarly to Lemma 4.56, we have

$$\int_{\mathbb{X}_{\alpha,\beta,\eta}} (\zeta \mid \alpha-\beta)_n\,\widetilde{\mu}_{\alpha,\beta,\sigma}\,(d\zeta) = \beta^n\left(\frac{\sigma}{\eta}\right)^{(n)}. \tag{4.252}$$

Indeed, to show (4.252), one only needs to use Corollary 4.61 instead of (4.120). Now, by (4.250)–(4.252), formula (4.246) easily follows.

Next, similarly to (4.174), we prove (4.247).

Finally, formulas (4.175)–(4.179) obviously remain true in our case if we replace U, V and ∂^+ with $\widetilde{U}$, $\widetilde{V}$ and $\widetilde{\partial}^+$, respectively. This proves formulas (4.248) and (4.249). □

Formulas (4.243), (4.246), (4.247) and the definition of $\mu_{\alpha,\beta,\eta}$ imply (4.238) and (4.239).

By (4.245), we have

$$\mathcal{S} = \widetilde{\mathcal{S}}\, E^{-\frac{\sigma}{\alpha}}, \qquad \mathcal{I} = E^{\frac{\sigma}{\alpha}}\, \widetilde{\mathcal{I}}.$$

Therefore,

$$U = \mathcal{I}\mathcal{U}\mathcal{S} = E^{\frac{\sigma}{\alpha}} \widetilde{\mathcal{I}}\mathcal{U}\, \widetilde{\mathcal{S}} E^{-\frac{\sigma}{\alpha}} = E^{\frac{\sigma}{\alpha}}\, \widetilde{U}\, E^{-\frac{\sigma}{\alpha}}.$$

Hence, by (4.248),

$$U = E^{\frac{\sigma}{\alpha}}\, Z(1 - \alpha D_{-2i\,\Im(\alpha)})\, E^{-\frac{\sigma}{\alpha}}.$$

Since the operator $1 - \alpha D_{-2i\,\Im(\alpha)}$ is shift-invariant, we get

$$U = E^{\frac{\sigma}{\alpha}}\, Z\, E^{-\frac{\sigma}{\alpha}}(1 - \alpha D_{-2i\,\Im(\alpha)}). \tag{4.253}$$

Since

$$(E^{\frac{\sigma}{\alpha}}\, Z\, E^{-\frac{\sigma}{\alpha}})(z) = E^{\frac{\sigma}{\alpha}} \left(z\, f(z - \frac{\sigma}{\alpha}) \right) = \left(z + \frac{\sigma}{\alpha} \right) f(z),$$

we have

$$E^{\frac{\sigma}{\alpha}}\, Z\, E^{-\frac{\sigma}{\alpha}} = Z + \frac{\sigma}{\alpha},$$

and so formula (4.240) follows from (4.253). The proof of (4.241) is similar. $\square$

Since

$$\mathbb{S}\partial^- \mathbb{S}^{-1} = D, \qquad \mathbb{S}\partial^+ \mathbb{S}^{-1} = Z,$$

we have similarly to Proposition 4.57, the following

Proposition 4.86. *Define*

$$\mathbb{U} = \mathbb{S}U\mathbb{S}^{-1} = \mathbb{T}\mathcal{U}\mathbb{T}^{-1}, \quad \mathbb{V} = \mathbb{S}V\mathbb{S}^{-1} = \mathbb{T}\mathcal{V}\mathbb{T}^{-1}.$$

Then

$$\mathbb{U} = Z(1 + \beta D) + \frac{\sigma}{\alpha}, \quad \mathbb{V} = 1 + \alpha D.$$

Similarly to Propositions 4.38 and 4.58, we have

Proposition 4.87. *The operator*

$$A^- := \sigma\partial^- + \eta\,\partial^+\partial^-\partial^-$$

is closable and let us keep the notation A^- for its closure. Then, for each $z \in \mathbb{C}$, the coherent state $\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z)$ given by (4.232) belongs to the domain of A^- and

$$A^-\,\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z) = z\,\mathbb{E}_{\alpha,\beta,\sigma}(\cdot, z).$$

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