AN ALGEBRAIC FRAMEWORK FOR GEOMETRICALLY CONTINUOUS SPLINES

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Abstract. Geometrically continuous splines are piecewise polynomial functions defined on a collection of patches which are stitched together through transition maps. They are called G^r-splines if, after composition with the transition maps, they are continuously differentiable functions to order r on each pair of patches with stitched boundaries. This type of spline has been used to represent smooth shapes with complex topology for which (parametric) spline functions on fixed partitions are not sufficient. In this article, we develop new algebraic tools to analyze G^r-spline spaces. We define G^r-domains and transition maps using an algebraic approach, and establish an algebraic criterion to determine whether a piecewise function is G^r-continuous on the given domain. In the proposed framework, we construct a chain complex whose top homology is isomorphic to the G^r-spline space. This complex generalizes Billera-Schenck-Stillman homological complex used to study parametric splines. Additionally, we show how previous constructions of G^r-splines fit into this new algebraic framework, and present an algorithm to construct a bases for G^r-spline spaces. We illustrate how our algebraic approach works with concrete examples and prove a dimension formula for the G^r-spline space in terms of invariants to the chain complex. In some special cases, explicit dimension formulas in terms of the degree of splines are also given.

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2020 *Mathematics Subject Classification.* 13D02, 68W30, 41A15.

Key words and phrases. Multivariate spline functions, algebraic transition maps, dimension and bases of spline spaces, geometrically continuous splines, geometric continuity.

A. Mantzaflaris and N. Villamizar wish to thank the support by The Alliance Hubert Curien Programme, project number: 515492678.

N. Villamizar and B. Yuan were supported by the UK Engineering and Physical Sciences Research Council (EPSRC) New Investigator Award EP/V012835/1.

1. Introduction

Spline functions are mathematical representations built upon simpler pieces (usually defined by low-degree polynomials) which are glued together forming a smooth curve, surface or volume. Splines constitute an appealing tool for shape representation not only for the simplicity of their construction [24, 51], but because they are a fundamental component of the approximation of solutions of partial differential equations by the finite element method [63]. They play a central role in geometric modeling [25], in approximation theory [39], and in novel fields such as isogeometric analysis [17, 35].

Splines have been traditionally studied within the realm of numerical analysis and computational mathematics [19, 39]. They are defined as piecewise polynomial functions defined on a partition of a real domain which are continuously differentiable up to some order $r \geqslant 0$ on the whole domain. These splines are called C^r-splines, or *parametric* C^r-continuous splines. If the polynomial degree is bounded then the set of C^r-splines on a given domain is a finite dimensional real vector space. In the areas where splines are applied, it is important to be able to construct a basis, often with prescribed properties, for the space of C^r-splines on a given partitioned domain and fixed maximal polynomial degree. A more basic task which aids in the construction of a basis is simply to compute the dimension of the spline space. A formula for the dimension of C^1-spline spaces defined on triangulations of planar domains was first proposed by Strang [62] and proved for polynomial degree $d \geqslant 2$ for *generic* vertex positions by Billera in [8]. The seminal work in [8] unveiled fascinating connections between the study of splines and homological algebra techniques, putting spline theory at the interface between commutative algebra, geometric modeling, and numerical analysis. The problem of computing the dimension of spline spaces on planar triangulations has received considerable attention using a wide variety of techniques. Early works using Bernstein-Bézier methods from approximation theory include [1, 2, 34, 60], see also [39] and the references therein. Some extensions of the algebraic approach introduced in [8] to compute the dimension of bivariate spline spaces include [10, 46, 56, 57]. One important feature of C^r-splines on triangulations is that the formula of the dimension of the space of splines of degree less than or equal to d for $d \geqslant 3r+2$ is a lower bound on the dimension of the space for any degree $d \geqslant 0$ [59]. If $r+1 \leqslant d \leqslant 3r+1$, to find an explicit formula for the dimension of bivariate spline spaces remains an open problem. In this direction, some remarkable results have been proved using tools from rigidity theory [66], and homological and commutative algebra [55, 58, 64, 68]. The literature on computing the dimension of trivariate splines on tetrahedral partitions is much less conclusive. The dimension has been computed if $r = 0$ (see [4] or [9]), and also if $r = 1$, $d \geqslant 8$, and the partition is generic [5]. For $r > 1$, bounds on the dimension of trivariate spline spaces have been proved in [3, 22, 23, 40, 47].

The main concept that we investigate in this paper is the geometric continuity of *functions*. Classically, spline functions are defined in terms of parametric continuity. Given a collection of functions on a cell partition of a planar domain, it defines a C^k continuous spline, if for each edge shared by two cells of the partition, there exists an affine change of coordinates in one of the two cells, such that after this change of coordinates the Taylor expansions of the function on both sides of the common edge coincide up to order k. We generalize this notion by considering more general changes of coordinates or transition maps (not necessarily affine). In this case, the derivatives up to order k on each side of the edge are linearly dependent, which explains why we call these functions G^k geometrically continuous splines. If such functions are used to parametrize surfaces, the geometric continuity of the functions implies the continuity of the tangent planes along the edges, which is also known as geometric continuity of the surface.

Geometric continuity for surfaces is essential for Computer-Aided Design (CAD), where typically a sophisticated design is obtained as a smooth surface described by many patches joined together. Indeed, the concept originated in the automobile and aviation industry, see e.g. the early report [54]. Parametric continuity (of first order) is instead the condition that the parametrization of the patches matches with continuity of the first order derivatives in parametric coordinates (corresponding to identity transition maps in our setting). Since parametric continuity is not a necessary nor a sufficient condition for the smoothness of the represented surface, the study of geometrically continuous representations emerged as a tool for flexible representations of CAD models. Geometrically, G^1-splines are defined by the property that the tangent planes of each patch match at that interface. These conditions involve the derivatives of the parametric coordinates composed with the transition map assigned to each interface.

The concept of geometric continuity builds on the theory of differential manifolds. The first general definition of splines based on transition maps or reparametrizations was presented in [21], where it was used for building smooth surfaces from a collection of 2-dimensional patches. Subsequently, the construction of geometrically continuous splines received considerable attention using a wide variety of techniques, here we list only a few representative references. Some early works include [27, 31], see also the survey [49] and the references therein. The construction of G^1- and G^2-spline surfaces has received considerable attention; see for instance [15, 43, 45, 48, 50, 52, 53]. In particular, G^1-splines have been used for surface fitting and surface reconstruction in geometric modeling; see [42, 61]. Manifold-based constructions of G^r-splines have been studied in [20, 28, 29, 30, 33, 65, 67]. See [6], where rational geometrically continuous spline are considered.

Research on geometrically continuous splines has been revived during the last decade due to its importance for solving partial differential equations by means of isogeometric analysis, see for instance [16, 35], and the references therein. In particular, smooth splines are used for the discretization of

partial differential equations and yield outstanding benefits compared to piecewise C^0 polynomial approximations: superior approximation properties and numerical stability, as well as the possibility to directly discretize high-order differential equations, such as the ones arising in thin-shell theory, in fracture models, in phase-field based multiphase flows, and in geometric flows on surfaces [35].

The computation of the dimension and construction of basis functions has been considered mainly for G^1-continuous splines. The Bernstein–Bézier techniques for the construction of smooth surfaces on planar unstructured quad-meshes with linear transition maps are developed in [7]. For linear transition maps computed from the parametrizations of quad-meshes of planar domains, the space of G^1-smooth isogeometric spline functions is studied (dimension, basis) in [36, 37, 38]. An analogous definition of transition maps is used in [11] to construct G^1-splines on volumetric two-patch domains. In [44], a construction of G^1-surfaces approximating Catmull–Clark subdivision surfaces using quadratic transition maps is proposed. Algebraic properties of piecewise polynomials and transition maps are used in [45] to construct G^1-splines for sufficiently large polynomial degree. In [45], the authors define a *topological surface* in the context of geometric continuity as a collection of polygons along with an identification between pairs of edges of these polygons. The splines on a topological surface are defined as piecewise polynomial functions which are differentiable after composition with transition maps associated to pairs of identified edges. The analysis of the dimension of G^1-splines in [45] leads to the construction of basis functions when the topological surface is composed of quadrangular and triangular patches. The construction of topological surfaces was extended to G^1-splines on quad meshes with 4-split macro patches in [13]. A second basis construction for G^1-splines on topological surfaces is introduced in [14] where it is used for fitting of points clouds in isogeometric analysis for the solution of diffusion equations.

The main contribution in this paper is to develop an algebraic framework to study spline functions which are defined on a collection of patches instead of fixed partitions of real domains. The collection of patches are *stitched* together through transition maps, and the continuity conditions are imposed after composition of the piecewise functions with the transition maps. The splines defined on this collection of patches are called *geometric continuous splines* or *G-splines*. If $r \geqslant 0$ is an integer, the set of G-splines which (after the composition with the transition maps) are continuously differentiable functions to order r on each pair of patches with stitched boundaries, are called *G^r-continuous*, or simply *G^r-splines*.

We provide an algebraic framework for geometrically continuous G^r-splines for any order of continuity $r \geqslant 0$. This method extends the results on G^1-splines in [45], and generalizes the algebraic approach for (parametric) splines pioneered in [8].

We begin with a characterization of the domains over which the piecewise polynomials are defined. We define them as a couple (Δ, Φ) and call them *G^r-domains*. They consist of a cell complex Δ with a coordinate system associated to each maximal face, and a collection Φ of *algebraic transition maps* between adjacent patches. Unlike transition maps in differential geometry, which are diffeomorphisms between open sets, an algebraic transition map is a homomorphism between quotients of polynomial rings. It is a "stitch" rather than "gluing". We define G^r-splines as geometrically continuous piecewise polynomial functions on G^r-domains.

This algebraic setting for G^r-splines has several benefits. On one hand, it allows the use of algebraic tools to study the dimension and to construct a basis for G^r-spline spaces. Namely, we construct a chain complex of vector spaces whose top homology is isomorphic to the G^r-spline space. This chain complex allows us to analyze the dimension of G^r-spline spaces, as we illustrate for some special cases in Section 6. More importantly, the algebraic framework enables us to present an algorithm (Algorithm 1) to construct a basis for G^r-spline spaces over any given G^r-domain. Algorithm 1 is insensitive to the shape of patches, as it admits arbitrary shapes as well as triangles and rectangular patches, making the choice of domain very flexible. On the other hand, we do not lose the perspective gained from the framework in [45]. More precisely, we prove in Theorem 3.8 that if a G^1-domain is obtained from a topological surface, then the G^1-spline space in this paper and the space of differentiable functions on the topological space in [45] are the same. Hence, the dimension formulas obtained in [45] still apply in the new context. As a result, G^r-splines can be defined and studied in a pure algebraic setting. In particular, when the transition maps are algebraic, the criterion for a function to be G^r-continuous is algebraic. We formulate concrete examples of algebraic transition maps. As a byproduct of exploring the relation between the concept of geometrically continuous splines in different frameworks, we establish a way of obtaining transition maps from certain kinds of differential manifolds.

This article is organized as follows. In Section 2, we introduce the algebraic language to define G^r-splines. We define G^r-domains and G^r-continuity in Section 2.1. We construct a chain complex whose top homology is isomorphic to the G^r-spline space in Section 2.2, and consider the bounded degree case of this chain complex in Section 2.3. We show that our construction yields the celebrated Billera-Schenck-Stillman complex as a special case in Appendix A. The main result in Section 2.4 is Lemma 2.21, which not only leads us to the construction of a basis for G^r-spline spaces over any given G^r-domain in Algorithm 1, but also serves as a general technique in computing generators for certain subspaces of a quotient ring. In Section 3, we explain how to obtain a G^1-spline space from a geometrically continuous spline space as defined in [45]. The main result in this section is Theorem 3.8, where we prove that these two notions of geometrically continuous splines can be identified. Hence, results in [45] and [13] apply in the new

algebraic framework. We describe some interesting families of transition maps in Section 4, including concrete examples of transition maps obtained from manifolds. Section 5 and 6 are devoted to the two dimensional case. In Section 5, we obtain a formula for the Euler characteristic of the chain complex constructed in Section 2.3 for G^1-splines of bounded polynomial degree. These computations are also an application of Lemma 2.21. We analyze the homology terms of the chain complex in Section 6. The formula for the Euler characteristic together with explicit computations of the dimension of the homology terms yield a dimension formula for the dimension of G^1-spline spaces defined on two-adjacent patches, or on a star of vertex and continuity conditions given by symmetric gluing data. We conclude this section with Example 6.10, where using Algorithm 1 and `Macaulay2` [26], we compute the exact dimension of a G^1-spline space defined on the surface of a cube for certain polynomial degrees. Finally, we give some concluding remarks in Section 7.

2. The space of G^r–splines

In this section we introduce G^r-continuity and geometrically continuous G^r-spline functions. We present a chain complex for G^r-splines, which can be seen as a generalization of the Billera-Schenck-Stillman chain complex for (parametric) splines introduced in [8] and refined in [57]. First, we introduce the domain of definition of the G^r-spline functions which we call G^r-domains.

2.1. The G^r-domains and G^r-continuity conditions.

Recall that a cell complex Δ is *n-dimensional*, or an *n-complex*, if the largest dimension of any face in Δ is n (see [32] for basics on cell complexes). An n-dimensional cell complex is called *pure* if all its maximal faces (with respect to inclusion) are of dimension n. If $i \geqslant 0$, a face of a cell complex Δ of dimension i is called an i-face. We denote by Δ_i the collection of all i-faces of Δ, and say that two n-faces σ_1 and σ_2 are *adjacent* if they share an $(n-1)$-face.

Throughout this paper we assume that Δ is a pure n-dimensional cell complex. To each n-face σ of Δ we assign a system of coordinates (that is, a vector space isomorphic to $\mathbb{R}^n$) denoted $X(\sigma)$. More precisely, for each $\sigma \in \Delta_n$ we have a map $\psi_\sigma \colon \sigma \to X(\sigma)$ such that ψ_σ is a homeomorphism between σ and its image $\operatorname{Im} \psi_\sigma$. We refer to $X(\sigma)$ as the coordinates of σ, and write $\mathcal{R}(\sigma)$ for the corresponding coordinate ring, which is isomorphic to a polynomial ring in n-variables with real coefficients.

Definition 2.1. Let Δ be an n-dimensional cell complex with coordinates $X(\sigma)$ for each n-face $\sigma \in \Delta$. If $S \subseteq X(\sigma)$, we define $I_\sigma(S)$ as the set of all polynomials in $\mathcal{R}(\sigma)$ which vanish at p for every $p \in S$. Namely,

$$I_\sigma(S) = \big\{ f \in \mathcal{R}(\sigma) \colon f(p) = 0 \text{ for each point } p \in S \big\}.$$

Notice that $I_\sigma(S)$ is an ideal of $\mathcal{R}(\sigma)$ for every $\sigma \in \Delta_n$ and $S \subseteq X(\sigma)$. If $\alpha \subseteq \sigma$ is a k-face of an n-face $\sigma \in \Delta_n$, then $I_\sigma(\alpha) \subseteq \mathcal{R}(\sigma)$ is the ideal of all polynomials in $\mathcal{R}(\sigma)$ vanishing at $\psi_\sigma(\alpha) \subseteq X(\sigma)$. In the case, $S = \{p\}$

consists only of one point $p \in X(\sigma)$, we simply write $I_\sigma(p)$ for the ideal $I_\sigma(S)$. In particular, if the point $p \in \alpha \subseteq \sigma$ with $\alpha \in \Delta_k$, then $I_\sigma(\alpha) \subseteq I_\sigma(p)$.

Note that we do not require any face $\alpha \subseteq \sigma$ to lie on an algebraic subset of $X(\sigma)$. If $\alpha \subseteq \sigma$ is a k-face that lies on an algebraic subset of $X(\sigma)$, then $I_\sigma(\alpha) \neq \{0\}$, otherwise $I_\sigma(\alpha) = \{0\}$.

Definition 2.2 (Locally algebraic faces)**.** If Δ is an n-dimensional cell complex with coordinates $X(\sigma)$ for each n-face $\sigma \in \Delta$, we say that an $(n-1)$-face $\tau \in \Delta_{n-1}$ is *locally algebraic* if for each n-face σ containing τ the ideal $I_\sigma(\tau)$ is generated by an irreducible polynomial $f \in \mathcal{R}(\sigma)$. Notice that this property depends on the homeomorphism ψ_σ and the choice of the coordinate system $X(\sigma)$.

If all the $(n-1)$-faces of Δ are locally algebraic, and $\alpha \subseteq \sigma \in \Delta_n$ is a k-face for $0 \leqslant k \leqslant n-2$, then

$$I_\sigma(\alpha) = \sum_{\alpha \subseteq \tau \subseteq \sigma} I_\sigma(\tau)\,.$$

Remark 2.3. If $\tau = \sigma_1 \cap \sigma_2$ is the common $(n-1)$-face of two n-faces σ_1 and σ_2 of a cell complex Δ, by Definition 2.1, there are two ideals $I_{\sigma_1}(\tau) \subseteq \mathcal{R}(\sigma_1)$ and $I_{\sigma_2}(\tau) \subseteq \mathcal{R}(\sigma_2)$ associated to the face τ, one for each n-face containing τ. The subindices of the ideals indicate the local coordinates at the corresponding n-face. (These ideals should not be confused with the ideal $\mathcal{I}^r(\tau)$ which we introduce later in Section 2.2. The latter is a unique ideal associated to τ, which depends on both coordinate rings $\mathcal{R}(\sigma_1)$ and $\mathcal{R}(\sigma_1)$ and on the ideals $I_{\sigma_1}(\tau)$ and $I_{\sigma_2}(\tau)$, see Definition 2.13.)

Definition 2.4 (G^r-algebraic transition maps)**.** Let $r \geqslant 0$ be an integer, and take two n-faces σ_1 and σ_2 of a cell complex Δ such that $\sigma_1 \cap \sigma_2 = \alpha \neq \emptyset$. An $\mathbb{R}$-algebra homomorphism

$$\phi_{12}\colon \mathcal{R}(\sigma_1)/I_{\sigma_1}(\alpha)^{r+1} \to \mathcal{R}(\sigma_2)/I_{\sigma_2}(\alpha)^{r+1} \tag{2.1}$$

is called a *G^r-algebraic transition map* from σ_1 to σ_2, if

$$\phi_{12}^{-1}\big(I_{\sigma_2}(p)\big) = I_{\sigma_1}(p), \text{ for each point } p \in \alpha.$$

A ring homomorphism $\widetilde{\phi}_{12}\colon \mathcal{R}(\sigma_1) \to \mathcal{R}(\sigma_2)$ is called a *lift* of ϕ_{12} if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}(\sigma_1) & \xrightarrow{\widetilde{\phi}_{12}} & \mathcal{R}(\sigma_2) \\ \downarrow & & \downarrow \\ \mathcal{R}(\sigma_1)/I_{\sigma_1}(\alpha)^{r+1} & \xrightarrow{\phi_{12}} & \mathcal{R}(\sigma_2)/I_{\sigma_2}(\alpha)^{r+1}. \end{array}$$

Here, the vertical arrows are the natural quotient maps.

Definition 2.5 (Compatibility conditions)**.** Let Δ be a pure n-dimensional cell complex. A collection Φ of G^r-algebraic transition maps is *compatible* on Δ if the following conditions hold:

(C1) For each n-face $\sigma_i \in \Delta_n$, the only algebraic transition map ϕ_{ii} in Φ from σ_i to itself is the identity map on $\mathcal{R}(\sigma_i)$.

(C2) If the intersection of two n-faces $\sigma_1, \sigma_2 \in \Delta_n$ is a non-empty face $\alpha = \sigma_1 \cap \sigma_2$, then either ϕ_{12} or ϕ_{21} is in Φ. If both of them are in Φ, then $\phi_{12} = \phi_{21}^{-1}$. If $\phi_{12}, \phi'_{12} \in \Phi$ are transition maps from σ_1 to σ_2, then $\phi_{12} = \phi'_{12}$.

(C3) For any triple of n-faces $\sigma_1, \sigma_2, \sigma_3 \in \Delta_n$, if their intersection is a non-empty face $\beta = \sigma_1 \cap \sigma_2 \cap \sigma_3$, and $\phi_{12}, \phi_{23} \in \Phi$, then $\phi_{13} \in \Phi$ and each of the three maps has a natural restriction

$$\overline{\phi_{ij}} \colon \mathcal{R}(\sigma_i)/I_{\sigma_i}(\beta)^{r+1} \to \mathcal{R}(\sigma_j)/I_{\sigma_j}(\beta)^{r+1},$$

where $(i,j) = (1,2), (2,3), (1,3)$ such that $\overline{\phi_{23}} \circ \overline{\phi_{12}} = \overline{\phi_{13}}$.

Remark 2.6. Notice that in Definition 2.5, we do not assume that the transition maps ϕ_{ij} in the collection Φ are invertible, so we do not always have $\phi_{ij} \in \Phi \implies \phi_{ji} \in \Phi$. However, if both ϕ_{ij} and ϕ_{ji} are in Φ, then they are invertible, and they satisfy $\phi_{ij} = \phi_{ji}^{-1}$, which is in (C2). The reason, as we show in the next example, is that a G^r-algebraic transition map may not be invertible.

Example 2.7. Let Δ be a 2-dimensional polyhedral complex composed of two square faces σ_1 and σ_2 sharing a common edge τ. Let (u_i, v_i) be the coordinates in $X(\sigma_i)$ for the 2-dimensional faces σ_i, $i = 1, 2$. Up to a change of coordinates, we may assume that the ideals $I_{\sigma_i}(\tau)$ vanishing at $\psi_{\sigma_i}(\tau)$ are given by $I_{\sigma_i}(\tau) = \langle u_i \rangle \subseteq \mathcal{R}(\sigma_i) = \mathbb{R}[u_i, v_i]$, as shown in Figure 1 of Section 4. Then, the map ϕ_{12} defined by

$$\begin{aligned} \phi_{12} \colon \mathbb{R}[u_1, v_1]/\langle u_1^2 \rangle &\to \mathbb{R}[u_2, v_2]/\langle v_2^2 \rangle, \\ u_1 &\mapsto v_2, \\ v_1 &\mapsto u_2^3 \end{aligned}$$

is a non-invertible G^1-algebraic transition map from σ_1 to σ_2. The map ϕ_{12}, together with the identity maps on σ_1 and σ_2, form a collection Φ of compatible G^1-algebraic transition maps on Δ. $\diamond$

We now introduce the domains where the geometrically continuous splines are defined. Recall that an n-dimensional *manifold* is a second-countable Hausdorff space in which every point has neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of the closed n-dimensional upper half-space for points on the boundary [41].

Definition 2.8 (G^r-domains)**.** A *G^r-domain* (Δ, Φ) is an n-dimensional cell complex Δ which is homeomorphic to an n-dimensional manifold, together with a compatible collection of G^r-algebraic transition maps Φ.

In Section 4 we collect several examples of G^r-domains.

Remark 2.9. Notice that in a G^r-domain (Δ, Φ) in Definition 2.8, the collection of transition maps Φ satisfies the compatibility conditions (C1)–(C3),

hence Φ is determined by the transition maps for each pair of adjacent n-faces of Δ. In the examples below we describe Φ by giving this subset of transition maps.

Next, we construct the space of G^r-spline functions, which will be piecewise polynomial functions over G^r-domains that satisfy certain continuity conditions.

We restrict the definition of G^r-splines to G^r-domains. Hereafter, they are the only domains we consider in this paper.

Definition 2.10 (G^r-continuity along a face)**.** Let Δ be an n-dimensional cell complex, and $\sigma_1, \sigma_2 \in \Delta_n$ two n-faces such that $\alpha = \sigma_1 \cap \sigma_2$ is a non-empty face of Δ. Suppose ϕ_{12} is a G^r-algebraic transition map from σ_1 to σ_2 (as in Definition 2.4). We say that $f \in \mathcal{R}(\sigma_1)$ and $g \in \mathcal{R}(\sigma_2)$ *join with G^r-continuity along α with respect to ϕ_{12}* if

$$\phi_{12}\big(\bar{f}\big) = \bar{g},$$

where $\bar{f} = f + I_{\sigma_1}(\alpha)^{r+1}$ and $\bar{g} = g + I_{\sigma_2}(\alpha)^{r+1}$ are the images of f and g in the quotient rings $\mathcal{R}(\sigma_1)/I_{\sigma_1}(\alpha)^{r+1}$ and $\mathcal{R}(\sigma_2)/I_{\sigma_2}(\alpha)^{r+1}$, respectively.

Definition 2.11 (G^r-splines)**.** Let (Δ, Φ) be a G^r-domain for some integer $r \geqslant 0$. The space $G^r(\Delta, \Phi)$ of G^r-splines on (Δ, Φ) is the set of all piecewise polynomial functions $f = (f_\sigma)_{\sigma \in \Delta_n} \in \bigoplus_{\sigma \in \Delta_n} \mathcal{R}(\sigma)$ such that f_{σ_i} and f_{σ_j} join along the $(n-1)$-face $\tau = \sigma_i \cap \sigma_j$ with G^r-continuity with respect to Φ for every pair of adjacent n-faces $\sigma_i, \sigma_j \in \Delta_n$.

Remark 2.12. The notion of geometrically continuous splines was originally defined within differential geometry framework. The relation between this notion in these two frameworks is explained in Section 3.

2.2. **Chain complex for G^r-splines.** In this section we assume that Δ is an n-dimensional polyhedral complex. We write $\mathcal{C}_\bullet(\Delta, \partial\Delta) = \mathcal{C}_\bullet(\Delta)/\mathcal{C}_\bullet(\partial\Delta)$ for the chain complex of Δ relative to its boundary $\partial\Delta$. For readers who are not familiar with the notion, we recall in Appendix A the construction of this chain complex for a simplicial complex. The chain complex associated with an arbitrary polyhedral complex can be considered as a natural generalization of the simplicial case.

By an abuse of notation, we denote the k-faces of Δ relative to its boundary by $(\Delta, \partial\Delta)_n = \Delta_n$ and $(\Delta, \partial\Delta)_k = \Delta_k^\circ$, for $0 \leqslant k \leqslant n-1$. We call the faces in Δ_k° the *interior* k-faces of Δ. Recall that to each n-face $\sigma_i \in \Delta_n$, we associate a system of coordinates $X(\sigma_i) \cong \mathbb{R}^n$ and denote by $\mathcal{R}(\sigma_i)$ the polynomial ring in the coordinates $(u_{i1}, \ldots, u_{in})$ of $X(\sigma_i)$. In the following we extend this construction and associate a ring to each face of the polyhedral complex Δ.

If $0 \leqslant k \leqslant n-1$, and $\alpha \in \Delta_k$ is a k-face, we take $\mathcal{R}(\alpha)$ as the tensor product

$$\mathcal{R}(\alpha) = \bigotimes_{\substack{\sigma \supseteq \alpha \\ \sigma \in \Delta_n}} \mathcal{R}(\sigma). \tag{2.2}$$

For simplicity, we write $\mathcal{R}(\sigma_1, \dots, \sigma_m) = \mathcal{R}(\sigma_1) \otimes \cdots \otimes \mathcal{R}(\sigma_m)$, for n-faces $\sigma_i \in \Delta_n$.

For each pair $\alpha, \beta \in \Delta$ such that $\beta \subseteq \alpha$, we have a natural inclusion map:

$$\rho^\alpha_\beta \colon \mathcal{R}(\alpha) \to \mathcal{R}(\beta)\,. \tag{2.3}$$

These inclusion maps ρ^α_β are algebra homomorphisms that preserve the structure of the ambient rings. For notational convenience, we often apply these maps implicitly when the context is clear. For instance, given an ideal $I \subseteq \mathcal{R}(\alpha)$ and $\beta \subseteq \alpha$, the expression $I \cdot \mathcal{R}(\beta)$ should be interpreted as $\rho^\alpha_\beta(I) \cdot \mathcal{R}(\beta)$.

Definition 2.13. Let (Δ, Φ) be a G^r-domain and assume th coordinates $X(\sigma_i)$ of σ_i are $(u_{i1}, \dots, u_{in})$. If $\sigma_1, \sigma_2 \in \Delta_n$ are two adjacent n-faces such that $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$ and $\phi_{12} \in \Phi$ is a algebraic transition map from σ_1 to σ_2, we define $\mathcal{I}^r(\tau) \subseteq \mathcal{R}(\tau)$ as the ideal

$$\mathcal{I}^r(\tau) = \sum_{j=1}^{n} \big\langle u_{1j} - \widetilde{\phi}_{12}(u_{1j}) \big\rangle + \big(I_{\sigma_1}(\tau)^{r+1} \cdot \mathcal{R}(\tau) + I_{\sigma_2}(\tau)^{r+1} \cdot \mathcal{R}(\tau) \big)\,, \tag{2.4}$$

where $\widetilde{\phi}_{12}$ is a lift of ϕ_{12} in (2.1). If $\alpha \in \Delta_k$ for $0 \leqslant k \leqslant n-2$, we define

$$\mathcal{I}^r(\alpha) = \sum_{\substack{\tau \supseteq \alpha \\ \tau \in \Delta_{n-1}}} \mathcal{I}^r(\tau) \cdot \mathcal{R}(\alpha). \tag{2.5}$$

For $0 \leqslant k \leqslant n-1$, and $\alpha \in \Delta_k$, we put

$$\mathcal{F}^r(\alpha) = \mathcal{R}(\alpha)/\mathcal{I}^r(\alpha). \tag{2.6}$$

For $\sigma \in \Delta_n$, put $\mathcal{F}^r(\sigma) = \mathcal{R}(\sigma)$.

Remark 2.14. Notice that for an $(n-1)$-face $\tau = \sigma_1 \cap \sigma_2 \in \Delta_{n-1}$, the ideal $\mathcal{I}^r(\tau)$ in Definition 2.13 does not depend on the choice of the lift of the algebraic transition map ϕ_{12}.

For each pair $(\alpha, \beta) \in \Delta_k \times \Delta_{k-1}$ such that $\beta \subseteq \alpha$, we have the inclusion map ρ^α_β as defined in (2.3), for every $k = 1, \dots, n$. Suppose the k-th boundary map of the chain complex $\mathcal{C}_\bullet(\Delta, \partial\Delta)$ is given by

$$\partial_k = \sum_{(\alpha,\beta) \in (\Delta,\partial\Delta)_k \times (\Delta,\partial\Delta)_{k-1}} a_{\alpha,\beta} e_\beta \otimes e^*_\alpha, \tag{2.7}$$

where $\{e_\alpha\}$ and $\{e_\beta\}$ are generators of $\mathcal{C}_k(\Delta, \partial\Delta)$ and $\mathcal{C}_{k-1}(\Delta, \partial\Delta)$, respectively, and $a_{\alpha,\beta} \in \mathbb{Z}$. We denote by δ_k the $\mathbb{R}$-linear map

$$\delta_k \colon \bigoplus_{\alpha \in (\Delta,\partial\Delta)_k} \mathcal{R}(\alpha) \to \bigoplus_{\beta \in (\Delta,\partial\Delta)_{k-1}} \mathcal{R}(\beta) \tag{2.8}$$

defined as

$$\delta_k = \sum_{(\alpha,\beta) \in (\Delta,\partial\Delta)_k \times (\Delta,\partial\Delta)_{k-1}} a_{\alpha,\beta} \rho^\alpha_\beta e_\beta \otimes e^*_\alpha.$$

We first check that $(\mathcal{R}_\bullet, \delta_\bullet)$ is a chain complex.

Lemma 2.15. *If Δ is an n-dimensional polyhedral complex, and δ_k is the linear map in* (2.8)*, then*

$$\delta_{k+1} \circ \delta_{k+2} = 0,$$

for $k = 0, \dots, n-2$, and hence $(\mathcal{R}_\bullet, \delta_\bullet)$ is a chain complex of $\mathbb{R}$-vector spaces.

Proof. Note that for faces $\gamma \in \Delta_k$ and $\alpha \in \Delta_{k+2}$, if there are two faces $\beta, \beta' \in \Delta_{k+1}$ such that $\gamma \subseteq \beta \subseteq \alpha$ and $\gamma \subseteq \beta' \subseteq \alpha$, then

$$\rho_\gamma^\beta \circ \rho_\beta^\alpha = \rho_\gamma^{\beta'} \circ \rho_{\beta'}^\alpha,$$

for the homomorphisms in (2.3). Furthermore, for any $k = 0, \dots, n-2$ we have

$$\partial_{k+1} \circ \partial_{k+2} = 0.$$

Equivalently, this means if $\partial_{k+2} = \sum a_{\alpha,\beta} e_\alpha \otimes e_\beta^*$ and $\partial_{k+1} = \sum b_{\beta,\lambda} e_\beta \otimes e_\lambda^*$, then for faces α and λ, such that $\gamma \subseteq \alpha$ we have

$$\sum_{\alpha \subseteq \beta \subseteq \gamma} a_{\alpha,\beta} b_{\beta,\lambda} = 0.$$

Therefore,

$$\left(\delta_{k+1} \circ \delta_{k+2}\right)_{\alpha,\lambda} = \sum_{\alpha \subseteq \beta \subseteq \gamma} a_{\alpha,\beta} b_{\beta,\lambda} \rho_\gamma^\beta \circ \rho_\beta^\alpha = \left(\sum_{\alpha \subseteq \beta \subseteq \gamma} a_{\alpha,\beta} b_{\beta,\lambda} \right) \cdot \rho_\gamma^{\beta'} \circ \rho_{\beta'}^\alpha = 0\,,$$

where β' is a $(k+1)$-face between α and λ. Since

$$\delta_{k+1} \circ \delta_{k+2} = \sum_{(\alpha,\lambda) \in (\Delta, \partial\Delta)_{k+2} \times (\Delta, \partial\Delta)_k} (\delta_{k+1} \circ \delta_{k+2})_{\alpha,\lambda} e_\alpha \otimes e_\lambda^*,$$

then $\delta_{k+1} \circ \delta_{k+2} = 0$. □

Note that the restriction of ρ_β^α in (2.3) to $\mathcal{I}^r(\alpha)$ gives us a map $\mathcal{I}^r(\alpha) \to \mathcal{I}^r(\beta)$. Therefore, δ_k restricts to $\oplus_{\alpha \in (\Delta,\partial\Delta)_k} \mathcal{I}^r(\alpha) \to \oplus_{\beta \in (\Delta,\partial\Delta)_{k-1}} \mathcal{I}^r(\beta)$ and it induces a natural map $\oplus_{\alpha \in (\Delta,\partial\Delta)_k} \mathcal{F}^r(\alpha) \to \oplus_{\beta \in (\Delta,\partial\Delta)_{k-1}} \mathcal{F}^r(\beta)$. Thus, Lemma 2.15 implies that both $(\mathcal{I}_\bullet^r, \delta_\bullet)$ and $(\mathcal{F}_\bullet^r, \delta_\bullet)$ are chain complexes, and we have the following immediate consequence of our construction.

Theorem 2.16. *The space $G^r(\Delta, \Phi)$ of G^r-splines on a G^r-domain (Δ, Φ) is the top homology $H_n(\mathcal{F}_\bullet^r)$ of the chain complex $(\mathcal{F}_\bullet^r, \delta_\bullet)$.*

Proof. By definition, a tuple of polynomials $(f_\sigma)_{\sigma \in \Delta_n} \in G^r(\Delta, \Phi)$ if

$$\phi_{ij}(\bar{f}_{\sigma_i}) = \bar{f}_{\sigma_j}, \tag{2.9}$$

for each pair of adjacent n-faces $\sigma_i, \sigma_j \in \Delta_n$. Let $\tau = \sigma_i \cap \sigma_j \in \Delta_{n-1}$. We only need to show that (2.9) holds if and only if

$$f_{\sigma_i} - f_{\sigma_j} \in \mathcal{I}^r(\tau).$$

We know that

$$\widetilde{\phi}_{ij}(g_i) = g_j \;\Leftrightarrow\; g_i - g_j \in \left\langle u_{ik} - \widetilde{\phi}_{ij}(u_{ik}) \colon k = 1, \dots, n \right\rangle, \tag{2.10}$$

for any pair of polynomials $(g_i, g_j) \in \mathcal{R}(\sigma_i) \times \mathcal{R}(\sigma_j)$, where $\widetilde{\phi}_{ij}$ is a lifting of ϕ_{ij}. For $l = i, j$, denote by $\bar{f}_{\sigma_l}$ the quotient image of f_{σ_l} in $\mathcal{R}(\sigma_l)/I_{\sigma_l}(\tau)^{r+1}$. If (2.9) holds, then there exists $\widetilde{f}_i$ and $\widetilde{f}_j$ which are liftings of $\bar{f}_{\sigma_i}$ and $\bar{f}_{\sigma_j}$, respectively, such that $\widetilde{\phi}_{ij}(\widetilde{f}_i) = \widetilde{f}_j$. Hence, (2.10) implies $\widetilde{f}_i - \widetilde{f}_j \in \left\langle u_{ik} - \widetilde{\phi}_{ij}(u_{ik})\colon k = 1, \ldots, n \right\rangle$. Since $\widetilde{f}_i$ is a lifting of $\bar{f}_{\sigma_i}$, then $\widetilde{f}_i - f_{\sigma_i} \in I_{\sigma_i}(\tau)^{r+1}$. The same applies for $\widetilde{f}_j$ and f_{σ_j}, hence $\widetilde{f}_j - f_{\sigma_j} \in I_{\sigma_j}(\tau)^{r+1}$. Putting these together, we have $f_{\sigma_i} - f_{\sigma_j} \in \mathcal{I}^r(\tau)$.

Conversely, if $f_{\sigma_i} - f_{\sigma_j} \in \mathcal{I}^r(\tau)$, then there exist $\widetilde{f}_i$, and $\widetilde{f}_j$ such that $f_{\sigma_i} - \widetilde{f}_i \in I_{\sigma_i}(\tau)^{r+1}$, $f_{\sigma_j} - \widetilde{f}_j \in I_{\sigma_j}(\tau)^{r+1}$, and the difference $\widetilde{f}_i - \widetilde{f}_j \in \left\langle u_{ik} - \widetilde{\phi}_{ij}(u_{ik})\colon k = 1, \ldots, n \right\rangle$. Again, by (2.10), we have $\widetilde{\phi}_{ij}(\widetilde{f}_i) = \widetilde{f}_j$, which implies (2.9). □

The chain complex $(\mathcal{F}^r_\bullet, \delta_\bullet)$ specializes to the Billera-Schenck-Stillman spline chain complex when all transition maps are identities. This is a technical result which for completeness we have included in Appendix A.

2.3. **Chain complex for G^r-splines of bounded polynomial degree.** In the following we consider the subspace of G^r-splines over a given G^r-domain (Δ, Φ) which are of degree less than or equal to a fixed $d \geqslant 0$. For each d, we associate a chain complex to the subspace of $G^r(\Delta, \Phi)$ of splines of degree at most d and relate it to the chain complex $(\mathcal{F}^r_\bullet, \delta_\bullet)$ defined in Section 2.2.

Definition 2.17. Let $d \geqslant 0$ be an integer. The set of G^r-spline of degree at most d over a G^r-domain (Δ, Φ), denoted $G^r_d(\Delta, \Phi)$, is the subset of splines $f \in G^r(\Delta, \Phi)$ whose restriction to each n-face is a polynomial of degree less than or equal to d. Namely,

$$G^r_d(\Delta, \Phi) = \left\{ f \in G^r(\Delta, \Phi)\colon \deg(f_\sigma) \leqslant d \text{ for every } \sigma \in \Delta_n \right\},$$

where $f = (f_\sigma)_{\sigma \in \Delta_n}$, and $f|_\sigma = f_\sigma$.

It is clear that $G^r_d(\Delta, \Phi)$ is a finite dimensional $\mathbb{R}$-vector space. To find the dimension of this vector space, we construct a chain complex of finite dimensional vector spaces as follows.

If $\sigma \in \Delta_n$, denote by $\mathcal{R}_d(\sigma) = \mathcal{R}(\sigma)_{\leqslant d}$ the vector space of polynomials of degree at most d in the coordinates $X(\sigma) \cong \mathbb{R}^n$. For a k-face $\alpha \in \Delta_k$, for $0 \leqslant k \leqslant n$, let

$$\mathcal{T}_d(\alpha) = \left\{ \sum_{\alpha \subseteq \sigma \in \Delta_n} f_\sigma\colon f_\sigma \in \mathcal{R}_d(\sigma) \right\} \tag{2.11}$$

be the image of the inclusion

$$\bigoplus_{i=1}^{s} \mathcal{R}_d(\sigma_i) \hookrightarrow \mathcal{R}(\sigma_1, \ldots, \sigma_s) = \mathcal{R}(\alpha),$$

where $\sigma_1, \ldots, \sigma_s \in \Delta_n$ are the n-faces in Δ containing α. Notice that the set $\mathcal{T}_d(\alpha)$ is a subspace of the tensor product polynomial ring $\mathcal{R}(\alpha)$ defined in (2.2). The polynomials in $\mathcal{T}_d(\alpha)$ do not contain mixed terms i.e., each term is a monomial in one of the rings $\mathcal{R}(\sigma)$ for some n-face $\sigma \supseteq \alpha$. If σ is an n-face, then $\mathcal{T}_d(\sigma) = \mathcal{R}_d(\sigma)$.

Additionally, for each $\alpha \in \Delta_k$, for $0 \leqslant k \leqslant n-1$, we define

$$\mathcal{J}_d^r(\alpha) = \mathcal{T}_d(\alpha) \cap \mathcal{I}^r(\alpha), \tag{2.12}$$

and

$$\mathcal{Q}_d^r(\alpha) = \mathcal{T}_d(\alpha)/\mathcal{J}_d^r(\alpha), \tag{2.13}$$

where $\mathcal{I}^r(\alpha)$ is the ideal defined in (2.4) and (2.5) for $k = n-1$ and $0 \leqslant k \leqslant n-2$, respectively. If $\sigma \in \Delta_n$, we take $\mathcal{Q}_d^r(\sigma) = \mathcal{T}_d(\sigma)$.

Since the differential map δ in the chain complex $\big(\mathcal{R}_\bullet, \delta\big)$ restricts to

$$\delta_k \colon \bigoplus_{\alpha \in \Delta_k} \mathcal{T}_d(\alpha) \to \bigoplus_{\beta \in \Delta_{k-1}} \mathcal{T}_d(\beta),$$

then $(\mathcal{T}_{d,\bullet}, \delta)$ is a chain complex, and hence so are $(\mathcal{J}_{d,\bullet}^r, \delta)$ and $(\mathcal{Q}_{d,\bullet}^r, \delta)$.

As a corollary of Theorem 2.16, we have the following proposition.

Proposition 2.18. *Let $\big(\Delta, \Phi\big)$ be a G^r-domain and $d \geqslant 0$ an integer, then*

$$G_d^r(\Delta, \Phi) = H_n\big(\mathcal{Q}_{d,\bullet}^r\big), \tag{2.14}$$

where $H_n\big(\mathcal{Q}_{d,\bullet}^r\big)$ is the top homology of the chain complex $\big(\mathcal{Q}_{d,\bullet}^r, \delta\big)$.

Proof. By definition, $G_d^r(\Delta, \Phi) = G^r(\Delta, \Phi) \cap \oplus_{\sigma \in \Delta_n} \mathcal{T}_d(\sigma)$. By Theorem 2.16, $G^r(\Delta, \Phi)$ is the kernel of $\delta_n : \oplus_{\sigma \in \Delta_n} \mathcal{R}(\sigma) \to \oplus_{\tau \in \Delta_{n-1}^\circ} \mathcal{F}^r(\tau)$. Hence, $G_d^r(\Delta, \Phi)$ is the kernel of δ_n restricted to $\oplus_{\sigma \in \Delta_n} \mathcal{T}_d(\sigma)$, which is exactly $H_n(\mathcal{Q}_{d,\bullet}^r)$. □

With Proposition 2.18, we estimate $\dim G_d^r(\Delta, \Phi)$ by comparing it to the *Euler characteristic* $\chi(\mathcal{Q}_{d,\bullet}^r)$ of $\mathcal{Q}_{d,\bullet}^r$, which is defined as

$$\chi(\mathcal{Q}_{d,\bullet}^r) = \sum_{k=0}^{n} (-1)^k \cdot \sum_{\alpha \in (\Delta, \partial\Delta)_k} \dim \mathcal{Q}_d^r(\alpha).$$

The dimension of G^r-spline spaces and Euler characteristic $\chi(\mathcal{Q}_{d,\bullet}^r)$ are related by the following corollary:

Corollary 2.19. *For any integer $d \geqslant 0$, and any G^r-domain (Δ, Φ), we have*

$$\dim G_d^r(\Delta, \Phi) = (-1)^n \chi(\mathcal{Q}_{d,\bullet}^r) + \sum_{k=0}^{n-1} (-1)^{n+k+1} \dim H_k(\mathcal{Q}_{d,\bullet}^r). \tag{2.15}$$

Proof. For any chain complex $C_\bullet$ of finite dimensional vector spaces such that $C_k = 0$ if $k > n$ or $k < 0$, we have

$$\chi(C_\bullet) = \sum_{k=0}^{n} (-1)^k \dim C_k = \sum_{k=0}^{n} (-1)^k \dim H_k(C_\bullet).$$

In particular, this holds for $C_\bullet = \mathcal{Q}^r_{d,\bullet}$. Hence, (2.14) implies (2.15). □

Specializing to the $n = 2$ case, the construction and results in this section can easily be adapted to the bidegree case.

More precisely, if Δ is a two dimensional complex, and $\sigma \in \Delta_2$ with coordinates (u, v), then a polynomial $f(u, v) \in \mathcal{R}(\sigma)$ is of *bidegree* (d, d) if $f(u, v) \in \mathcal{R}_{(d,d)}(\sigma)$, where

$$\mathcal{R}_{(d,d)}(\sigma) = \operatorname{span}_{\mathbb{R}} \left\{ u^i v^j \ : 0 \leqslant i, j \leqslant d \right\}.$$

The space of bidegree (d, d) splines is the set

$$G^r_{(d,d)}(\Delta, \Phi) = \left\{ f \in G^r(\Delta, \Phi) \colon \deg(f_\sigma) \in \mathcal{R}_{(d,d)}(\sigma) \text{ for every } \sigma \in \Delta_2 \right\},$$

where $f = (f_\sigma)_{\sigma \in \Delta_2}$, and $f|_\sigma = f_\sigma$.

Similarly, we can adapt (2.11), (2.12), and (2.13) and define $\mathcal{T}_{(d,d)}(\alpha)$, $\mathcal{J}^r_{(d,d)}(\alpha)$, and $\mathcal{Q}^r_{(d,d)}(\alpha)$, respectively, for $\alpha \in \Delta_k$ for any $0 \leqslant k \leqslant n$. We refer to $G^r_d(\Delta, \Phi)$, $\mathcal{T}_{d,\bullet}$, $\mathcal{J}^r_{d,\bullet}$, $\mathcal{Q}^r_{d,\bullet}$ as *total degree spaces*, and $G^r_{(d,d)}(\Delta, \Phi)$, $\mathcal{T}_{(d,d),\bullet}$, $\mathcal{J}^r_{(d,d),\bullet}$, $\mathcal{Q}^r_{(d,d),\bullet}$ as *bidegree spaces (or bigrading)*.

We study the terms of the splines dimension formula (2.15) in Sections 5 and 6, where we restrict the study to G^r-splines on 2-dimensional cell complexes Δ. The results will be presented for both $\dim G^r_d(\Delta, \Phi)$ and $\dim G^r_{(d,d)}(\Delta, \Phi)$.

2.4. **Constructing a basis for geometrically continuous spline spaces.** This section is devoted to develop a general method to generate a certain subspace of a quotient ring. In particular, it gives an algorithm to construct a basis for a G^r-spline space. The main result in this section is Lemma 2.21, which describes the method on generating certain subspaces of a quotient ring. The algorithm for the G^r-spline basis will be an immediate consequence of this result. Later in Section 5, Lemma 2.21 will also play an important role for computing the dimension of the terms in (2.15). Lemma 2.21 does not depend on our setting so we state it in general for a finitely generated real polynomial ring. We introduce the following notation that will be helpful to define the subspaces of the quotient ring we want to describe.

Definition 2.20. Let $f \in S$ be polynomial in a finitely generated polynomial ring S over $\mathbb{R}$, and let $I \subseteq S$ be an ideal. We define the *remainder of f with respect to I* to be the remainder of f by performing the division algorithm to f with respect to the reduced Gröbner basis of I for a fixed monomial order $\succ$ on S. We denote this by $\mathfrak{r}(f, I)$, or simply $\mathfrak{r}(f)$ if I is clear from the context.

For the reader not familiar with these concepts, we refer to [18] for an introduction to monomial orderings and Gröner bases.

Lemma 2.21. *Let S be a finitely generated polynomial ring over $\mathbb{R}$, with a fixed monomial order $\succ$, and let $W \subseteq S$ be a finite dimensional $\mathbb{R}$-vector space with basis $\mathcal{B}$. Assume $I \subseteq S$ is an ideal. Then, the linear span* $\operatorname{span}\{\mathfrak{r}(b) \colon b \in \mathcal{B}\} \cong W/(W \cap I)$*, where $\mathfrak{r}(b)$ is the remainder of $b \in \mathcal{B}$ with respect to I.*

Proof. Let $V = \operatorname{span}\{\mathfrak{r}(b) \colon b \in \mathcal{B}\} \subseteq S$. Then we can define a map $\mathfrak{r} \colon W \to V$ given by $f \mapsto \mathfrak{r}(f)$. If $W \to W/(W \cap I)$ is the quotient map $f \mapsto \overline{f}$, then there exists a map $h \colon V \to W/(W \cap I)$ for which the following diagram commutes

$$\begin{tikzcd} W \arrow[d, "\mathfrak{r}"] \arrow[dr] & \\ V \arrow[r, dashed, "h"] & W/(W \cap I)\,. \end{tikzcd}$$

Indeed, for any $g \in V$, we may take $g' \in W$ such that $\mathfrak{r}(g') = g$ and define $h(g) = \overline{g'}$. This map is well-defined, because if $\mathfrak{r}(g_1') = \mathfrak{r}(g_2') = g$ for $g_1', g_2' \in W$, then $g_1' - g_2' \in I$, which implies that $\overline{g_1'} = \overline{g_2'}$. It is easy to verify this map is linear, injective and surjective. Hence, h is an isomorphism of vector spaces. $\square$

By Proposition 2.18 we know that $G_d^r(\Delta, \Phi)$ is the kernel of the map $\delta_n \colon \mathcal{Q}_{d,n}^r \to \mathcal{Q}_{d,n-1}^r$, where $\mathcal{Q}_{d,k}^r$ is the k-th term in the chain complex $(\mathcal{Q}_{d,\bullet}^r, \delta_\bullet)$. Recall that by definition we take $\mathcal{Q}_{d,n}^r = \mathcal{T}_{d,n}$, see (2.11) and (2.13). By construction, for each $\tau \in \Delta_{n-1}^\circ$, the set $\mathcal{T}_d(\tau)$ is a linear subspace of the ring $\mathcal{R}_d(\tau)$, and $I = \mathcal{I}^r(\tau)$ is an ideal of $\mathcal{R}_d(\tau)$. Hence, Lemma 2.21 implies $\mathfrak{r}\big(\mathcal{T}_d(\tau)\big) \cong \mathcal{Q}_d^r(\tau)$.

This allows to see $G_d^r(\Delta, \Phi)$ as the kernel of the map $\mathfrak{r} \circ \delta_n \colon \mathcal{T}_{d,n} \to \mathfrak{r}(\mathcal{T}_{d,n-1})$, where (by an abuse of notation) we write $\delta_n \colon \mathcal{T}_{d,n} \to \mathcal{T}_{d,n-1}$ for the restriction of the differential map in $(\mathcal{R}_\bullet, \delta_\bullet)$ to $\mathcal{T}_{d,n}$.

We start the algorithm to compute a basis of $G_d^r(\Delta, \Phi)$ by fixing a maximal polynomial degree d. Taking $k = n$ in (2.7), the boundary map ∂_n can be written as a rectangular matrix $(a_{\sigma,\tau})_{\tau \in \Delta_{n-1}^\circ\ \sigma \in \Delta_n}$, whose rows are labeled by the interior $(n-1)$-faces and the columns by the maximal n-faces of Δ. We compute the reduced Gröbner basis of the ideal $\mathcal{I}^r(\tau)$, for each $\tau \in \Delta_{n-1}^\circ$. Then, for every n-face $\sigma \supseteq \tau$, if $\mathcal{B}(\sigma)$ is the monomial basis of $\mathcal{T}_d(\sigma)$, we use the division algorithm to find the remainder of each polynomial in $\mathcal{B}(\sigma)$ with respect to the Gröbner basis of $\mathcal{I}^r(\tau)$. By Lemma 2.21, we know that the collection of those remainders generate $\mathcal{Q}_{d,n-1}^r$. Next we construct a matrix A which will encode the map $\delta_n \colon \mathcal{T}_{d,n} \to \mathcal{Q}_{d,n-1}^r$, Explicitly, if

$$\mathfrak{r} \circ \delta_n = \sum_{\substack{\sigma \in \Delta_n, \\ m \in \mathcal{B}(\sigma)}} \left(\sum_{\substack{\tau \in \Delta_{n-1}^\circ, \\ q \in \mathfrak{r}(\mathcal{B}(\sigma))}} c_{\sigma,\tau,m,q} (q e_\tau) \otimes (m e_\sigma)^* \right),$$

where $c_{\sigma,\tau,m,q}$ is the product of $a_{\sigma,\tau}$ in the boundary map and the coefficient of q in $\mathfrak{r}(m)$. The entries of A in the row labeled by (τ, q) and the column labeled by (σ, m) is $c_{\sigma,m,\tau,q}$. Notice that by Proposition 2.18, the kernel of A is precisely the space $G^r_d(\Delta, \Phi)$. Therefore, a basis of this kernel give us a basis for the vector space $G^r_d(\Delta, \Phi)$, which is the list LB of tuples $\big(f(\sigma)\colon \sigma \in \Delta_n\big)$ of functions $f(\sigma) \in \mathcal{R}(\sigma)_{\leqslant d}$ in Algorithm 1.

Algorithm 1 Basis computation of the space $G^r_d(\Delta, \Phi)$

Fix a graded reverse lexicographic order $\succ$ for the polynomial ring $\otimes_{\sigma\in\Delta_n}\mathcal{R}(\sigma)$;
Compute the reduced Gröbner basis $\mathrm{GB}(\tau)$ of $\mathcal{I}^r(\tau)$ for each $\tau \in \Delta^\circ_{n-1}$;
for (σ, τ) in $\Delta_n \times \Delta^\circ_{n-1}$ with $\tau \subseteq \sigma$ **do**
  $\mathrm{sign}(\sigma, \tau) \leftarrow$ sign of ∂_n from σ to τ;
  for m in the list of all the monomials in $\mathcal{R}(\sigma)_{\leqslant d}$ **do**
    $\mathfrak{r}(m) \leftarrow$ remainder of m with respect to $\mathrm{GB}(\tau)$;
    for q in the list of all monomials in $\mathcal{R}(\tau)_{\leqslant d}$ **do**
      $c(\sigma, \tau, m, q) \leftarrow$ coefficient of q in $\mathfrak{r}(m)$ multiplied by $\mathrm{sign}(\sigma, \tau)$;
Define a matrix A by letting the entry of A in row (τ, q) and column (σ, m) to be $c(\sigma, \tau, m, q)$;
Calculate a list of basis LB of kernel of A, which is a basis of $G^r_d(\Delta, \Phi)$;

Remark 2.22. Algorithm 1, which is based on symbolic computation, has been implemented in `Macaulay2` [26]. The source code for computing bases of the G^r-spline spaces discussed in this paper is freely available online at `https://github.com/yuanbeihui05/demos/tree/GCDemos`. We use this implementation in Section 6 to obtain computational results, such as determining the dimension of a G^1-spline space on a cube and generating the data presented in Table 1.

3. Topological surfaces and G^r-domains

The notion of geometric continuity was originally defined by differential-geometric methods, see for example [31] and [49]. In [45], the space of G^1-spline functions is defined over a collection of 2-dimensional patches glued together by transition maps. In this section, we explain how the notion of G^1-geometric continuity in our framework and that in [45] are related.

First, recall from [45, Definition 2.1], a *topological surface* $\mathcal{M}$ is a collection $\mathcal{M}_2$ of polygons together with a set of homeomorphisms between pairs of polygonal edges with the condition that each polygonal edge can be paired with at most one other edge, and it cannot be glued with itself. Each polygon is assumed to be embedded in $\mathbb{R}^2$. By an abuse of notation, two polygons are said to share an edge if there is such a homeomorphism between two edges one in each of the two polygons. From this, it is clear that the collection of polygons and homeomorphisms between pairs of polygonal edges of a

topological surface $\mathcal{M}$ define a 2-dimensional cell complex Δ such that the set of 2-faces $\Delta_2 = \mathcal{M}_2$. The coordinates $X(\sigma)$ associated to each polygon $\sigma \in \Delta_2$ are given by the embedding of σ in $\mathbb{R}^2$. As in Section 2.1, we denote the corresponding homeomorphism by $\psi_\sigma \colon \sigma \to X(\sigma)$. Notice that these embeddings of the faces $\sigma \in \Delta_2$ as polygons in $\mathbb{R}^2$ mean that every edge of σ is locally algebraic (see Definition 2.2).

Suppose Δ is such a 2-dimensional cell complex obtained from a topological surface $\mathcal{M}$, with coordinates $\{X(\sigma) \colon \sigma \in \Delta_2\}$. The (geometric) transition maps associated to pairs of polygons in $\mathcal{M}$ as introduced in [45, Definition 2.2] can be interpreted in our setting as follows.

Definition 3.1 (Geometric transition maps)**.** Let σ_1 and σ_2 two faces in Δ_2, such that $\sigma_1 \cap \sigma_2 = \tau \in \Delta_1$. A *(geometric) transition map* associated to τ consists of the following:

- for each face $\sigma_i \in \Delta_2$ containing τ, an open set $U_{\tau,\sigma_i} \subseteq X(\sigma_i)$ containing $\psi_{\sigma_i}(\tau)$;
- a C^1-diffeomorphism between the open sets
$$\varphi_{21} \colon U_{\tau,\sigma_1} \to U_{\tau,\sigma_2} \tag{3.1}$$
such that $\varphi_{21}|_{\psi_{\sigma_1}(\tau)} = \psi_{\sigma_2} \circ \psi_{\sigma_1}^{-1}|_{\psi_{\sigma_1}(\tau)}$.

Remark 3.2. The map φ_{21} is called transition map in [45], we call it here a geometric transition map to distinguish it from the algebraic transition maps from Definition 2.4. Notice that the notion of geometric transition maps can be generalized to an n-dimensional cell complex Δ, for this we can simply replace Δ_2 with Δ_n, and Δ_1 with Δ_{n-1} in Definition 3.1.

Our aim is to obtain a G^r-domain from a given topological surface and a given collection of geometric transition maps. However, a geometric transition map associated to an edge τ, which is C^r-continuous along τ, is not necessarily polynomial and therefore it does not necessarily correspond to an algebraic transition map. To describe the type of geometric transition maps from which we are able to obtain algebraic ones, we introduce the property of being *almost algebraic*. We need some preparations before we can state such a notion in Definition 3.4.

Let $U \subseteq \mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$, and take $p \in U$. If $\boldsymbol{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$ is a multiindex, henceforth we follow the convention that $\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} \cdots x_n^{j_n}$ and $|\boldsymbol{j}| = \sum_{i=1}^n j_i$.

Recall that if $s \leqslant r$, then the s-th order *jet* $J_p^s(f)$ of a function f which is C^r-continuous at the point p is defined as the truncated Taylor expansion of f at p of degree s. Explicitly, if the coordinates of p are $(a_1, \dots, a_n) = \boldsymbol{a}$, then

$$\begin{aligned} J_p^s \colon C_p^r &\to \mathbb{R}[x_1, \dots, x_n]/I(p)^{s+1} \\ f &\mapsto \sum_{|\boldsymbol{j}| \leqslant s} \frac{\partial^{\boldsymbol{j}} f}{\partial \boldsymbol{x}^{\boldsymbol{j}}}(p)(\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{j}}, \end{aligned} \tag{3.2}$$

where C_p^s is the set of all functions that are C^s-continuous at p, and

$$I(p) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle \tag{3.3}$$

is the ideal of $\mathbb{R}[x_1, \ldots, x_n]$ generated by the polynomials $x_i - a_i$.

Denote by $C^1(U)$ the set of all functions $f\colon U \to \mathbb{R}$ which are continuously differentiable functions on $U \subseteq \mathbb{R}^n$.

Definition 3.3. If $f \in C^1(U)$, we say that f *vanishes to order s at a point* $p \in U$ if f is C^s-continuous at p and

$$J_p^s(f) = 0.$$

If $S \subseteq U$, then we say f *vanishes to order s on S* if f vanishes to order s at p for each $p \in S$.

Since a geometric transition map φ_{21} is a C^1-diffeomorphism, then it induces a homomorphism

$$\begin{aligned} \varphi_{21}^*\colon C^1(U_{\tau,\sigma_1}) &\to C^1(U_{\tau,\sigma_2}) \\ f &\mapsto f \circ \varphi_{21}. \end{aligned} \tag{3.4}$$

Definition 3.4 (Almost algebraic property)**.** Let φ_{21} be a geometric transition map across τ as introduced in (3.1). We say φ_{21} is *almost algebraic* (up to order r) if there exist a homomorphism

$$\phi_{21}\colon \mathcal{R}(\sigma_2)/I_{\sigma_2}(\tau)^{r+1} \to \mathcal{R}(\sigma_1)/I_{\sigma_1}(\tau)^{r+1} \tag{3.5}$$

such that $\big(\widetilde{\phi}_{21} - \varphi_{21}^*\big)(f)$ vanishes to order r along $\psi_{\sigma_1}(\tau)$, for any $f \in \mathcal{R}(\sigma_2)$, where φ_{21}^* is the homomorphism in (3.4), and $\widetilde{\phi}_{21}\colon \mathcal{R}(\sigma_2) \to \mathcal{R}(\sigma_1)$ is a lift of ϕ_{21}. If this is the case, we call ϕ_{21} the *G^r-algebraic transition map associated to φ_{21}*.

The following example shows that not all geometric transition maps are almost algebraic.

Example 3.5. Let Δ be the 2-dimensional polyhedral complex considered in Example 2.7, which is composed of two square faces σ_1 and σ_2 sharing a common edge τ. Assume that the coordinates $X(\sigma_i)$ for σ_1 and σ_2 are chosen as shown in Figure 1 in Section 4. Let U_{τ,σ_i} be an open neighborhood of $\psi_{\sigma_i}(\tau)$ in $X(\sigma_i)$, for $i = 1, 2$. Define φ_{21} as the C^1-diffeomorphism:

$$\begin{aligned} \varphi_{21}\colon U_{\tau,\sigma_1} &\to U_{\tau,\sigma_2}, \\ (u_1, v_1) &\mapsto (v_1 + 2^{v_1}, -u_1). \end{aligned}$$

Then φ_{21} is a geometric transition map associated to the edge τ. If φ_{21} were almost algebraic, then there would exist a polynomial $f(u_1, v_1)$, such that

$$v_1 + 2^{v_1} - f(u_1, v_1) \in o(u_1), \text{ for all } v_1 \in [0, 1].$$

However, no such polynomial f exists. Therefore, φ_{21} is not almost algebraic. $\diamond$

It is clear that Definition 3.4 does not depend on the choice of the lift $\widetilde{\phi}_{21}$. In Lemma 3.6 we prove that if there exists a G^r-algebraic transition map associated to a geometric transition map φ_{21}, then it is unique.

Lemma 3.6. *Let $\tau = \sigma_1 \cap \sigma_2 \in \Delta_1^\circ$, for $\sigma_1, \sigma_2 \in \Delta_2$. If φ_{21} is an almost algebraic transition map up to order r across τ, then the G^r-algebraic transition map ϕ_{21} associated to φ_{21} is unique.*

Proof. Suppose both ϕ_{21} and ϕ'_{21} are algebraic transition maps induced by φ_{21}. If $f \in \mathcal{R}(\sigma_2)$, then

$$\begin{aligned}\big(\widetilde{\phi}_{21} - \widetilde{\phi}'_{21}\big)(f) &= \big(\widetilde{\phi}_{21} - \varphi^*_{21} + \varphi^*_{21} - \widetilde{\phi}'_{21}\big)(f)\\ &= \big(\widetilde{\phi}_{21} - \varphi^*_{21}\big)(f) + \big(\varphi^*_{21} - \widetilde{\phi}'_{21}\big)(f).\end{aligned}$$

Both $\big(\widetilde{\phi}_{21} - \varphi^*_{21}\big)(f)$ and $\big(\varphi^*_{21} - \widetilde{\phi}'_{21}\big)(f)$ vanishes to order r along τ. Hence so does their sum. Because $\widetilde{\phi}_{21} - \widetilde{\phi}'_{21}$ is algebraic, then $\big(\widetilde{\phi}_{21} - \widetilde{\phi}'_{21}\big)(f) \in I_{\sigma_1}(\tau)^{r+1}$. This holds for any $f \in \mathcal{R}(\sigma_2)$, so this implies $\phi_{21} = \phi'_{21}$. □

It follows from Lemma 3.6 that given a collection Φ_G of geometric transition maps, which are almost algebraic up to order r, we obtain a collection of G^r-algebraic transition maps Φ_A associated to Φ_G. The compatibility conditions (C1)–(C3) in Definition 2.5 applied to Φ_A lead to a collection of algebraic transition maps $\Phi \supseteq \Phi_A$ which (besides the transition maps between adjacent faces) contains all the algebraic transition maps between pairs of n-faces whose intersection is nonempty, see Remark 2.9. In this case, we say that Φ is *associated to* Φ_G and the G^r-domain (Δ, Φ) is obtained from the topological surface $\mathcal{M}$ with geometric transition maps Φ_G.

In fact, for 2-dimensional cell complexes and $r = 1$, if the collection Φ of G^1-algebraic transition maps is compatible on Δ (i.e., it satisfies the compatibility conditions in Definition 2.5), then the maps in Φ_G satisfy the G^1-*compatibility conditions at the vertices* of Δ given in [45, Section 2.3]. In Section 4, we given several examples of Φ_G such that its associated Φ satisfies compatibility conditions, and thus defines a G^r-domain.

We now relate the definition of geometric continuity introduced in [45] and the one in our setting given in Section 2, Definition 2.10.

Definition 3.7. Let $\mathcal{M}$ be a topological surface and Δ the cell complex obtained from $\mathcal{M}$. The *space of differentiable functions* $\mathcal{S}^1(\mathcal{M})$ over $\mathcal{M}$ is defined as the subset of $\oplus_{\sigma\in\Delta_2}\mathcal{R}(\sigma)$ such that $f = (f_\sigma) \in \mathcal{S}^1(\mathcal{M})$ if for every pair $\sigma_1, \sigma_2 \in \Delta_2$ of adjacent faces, we have

$$J^1_p\big(f_{\sigma_1} \circ \varphi_{21}\big) = J^1_p(f_{\sigma_2}), \text{ for each } p \in \sigma_1 \cap \sigma_2,$$

where J^1_p is the 1-st order jet of f, as defined in (3.2). The set of all differentiable functions $f \in \mathcal{S}^1(\mathcal{M})$ such that $\deg f_\sigma \leqslant d$ for each $\sigma \in \Delta_2$ is denoted $\mathcal{S}^1_d(\mathcal{M})$.

For a topological surface $\mathcal{M}$, let

$$\Phi_G = \{\varphi_{21} \colon \sigma_1, \sigma_2 \in \mathcal{M}_2 \text{ share a common edge}\},$$

be a collection of geometric transition maps. The relation to the notation of geometric continuity is the following.

Theorem 3.8. *If the G^1-domain (Δ, Φ) is obtained from a topological surface $\mathcal{M}$ with geometric transition maps Φ_G, then*

$$G^1(\Delta, \Phi) = \mathcal{S}^1(\mathcal{M}), \text{ and } G^1_d(\Delta, \Phi) = \mathcal{S}^1_d(\mathcal{M}).$$

In particular, $\dim G^1_d(\Delta, \Phi) = \dim \mathcal{S}^1_d(\mathcal{M})$.

We prepare the proof of Proposition 3.8 with the following two lemmas. Similarly as before, we denote by $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\boldsymbol{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$ a multiindex, where $\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} \cdots x_n^{j_n}$ and $|\boldsymbol{j}| = \sum_{i=1}^n j_i$.

Lemma 3.9. *Let $p \in \mathbb{R}^n$ and $f, \ell \in \mathbb{R}[x_1, \ldots, x_n]$, with ℓ irreducible. Assume that $\ell = 0$ defines a hypersurface in $\mathbb{R}^n$. Then the following two conditions on f are equivalent:*

(a) the jet $J^r_p(f) = 0$ whenever $\ell(p) = 0$,
(b) $f \in \langle \ell^{r+1} \rangle$.

Proof. Since the ideal $\langle \ell^{r+1} \rangle \subseteq I(p)^{s+1}$ for all $s \leqslant r$ whenever $\ell(p) = 0$, where $I(p)$ is the ideal in (3.3), then clearly (b) implies (a). It remains to prove the converse. We prove this by induction on the order of smoothness r. Note that condition (a) is equivalent to saying that $\partial^{\boldsymbol{j}} f / \partial \boldsymbol{x}^{\boldsymbol{j}}(p) = 0$ for all $|\boldsymbol{j}| \leqslant r$ whenever $\ell(p) = 0$, which in turn, is equivalent to saying that $\partial^{\boldsymbol{j}} f / \partial \boldsymbol{x}^{\boldsymbol{j}} \in \langle \ell \rangle$ for all $|\boldsymbol{j}| \leqslant r$.

We first prove that for any $s \geqslant 1$, if $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n \in \langle \ell^s \rangle$, then $f \in \langle \ell^{s+1} \rangle$. Indeed, if $f = g\ell^s$ for some $g \in \mathbb{R}[x_1, \ldots, x_n]$, then $\partial f/\partial x_j = \big(\partial g/\partial x_i\big) \cdot \ell^s + g \cdot s\ell^{s-1}\big(\partial \ell/\partial x_j\big)$ for $j = 1, \ldots, n$. Since ℓ is non-constant, then there exists at least one variable x_j such that $\partial \ell/\partial x_j \neq 0$, so we have $\partial \ell/\partial x_j \notin \langle \ell \rangle$. Note that $\langle \ell \rangle$ is a prime ideal since ℓ is irreducible, so $\partial f/\partial x_j \in \langle \ell^s \rangle$ implies $g \in \langle \ell \rangle$. Hence $f = g \cdot \ell^s \in \langle \ell^{s+1} \rangle$. If follows that for $r = 1$, condition (a) implies (b).

Now assume that (a) implies (b) for any $r = 1, \ldots, s$. For $j = 1, \ldots, n$, put $h_j = \partial f/\partial x_j$. Note that for any point $p \in \mathbb{R}^n$, if $J^{s+1}_p(f) = 0$ then $J^s_p(h_j) = 0$. Hence by the induction hypothesis, $h_j \in \langle \ell^s \rangle$, for $j = 1, \ldots, n$. Since $J^{s+1}_p(f) = 0$ implies $J^s_p(f) = 0$, then applying (a) to f and $r = s+1$ yields $f \in \langle \ell^s \rangle$. Thus, we have $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n \in \langle \ell^s \rangle$. Therefore, $f \in \langle \ell^{s+1} \rangle$, which completes the proof. □

Lemma 3.10. *Let Δ be a cell complex obtained from a topological surface $\mathcal{M}$, with $\sigma_1, \sigma_2 \in \Delta_2$ two adjacent faces such that $\tau = \sigma_1 \cap \sigma_2$. Assume the transition map φ_{21} across τ is almost algebraic, with associated algebraic transition map ϕ_{12}, as given in* (3.5)*. Then, for polynomials $f \in \mathcal{R}(\sigma_1)$ and $g \in \mathcal{R}(\sigma_2)$, the following conditions are equivalent:*

(a) $J^r_p(f \circ \varphi_{21}) = J^r_p(g)$ for each $p \in \tau$;
(b) $\phi_{12}(\bar{f}) = \bar{g}$, where $\bar{f}$ and $\bar{g}$ are the classes of f and g in the quotient rings $\mathcal{R}(\sigma_1)/I_{\sigma_1}(\tau)^{r+1}$ and $\mathcal{R}(\sigma_2)/I_{\sigma_2}(\tau)^{r+1}$, respectively.

Proof. Let $\widetilde{\phi}_{12}\colon \mathcal{R}(\sigma_1) \to \mathcal{R}(\sigma_2)$ be a lift of ϕ_{12}. Then (b) holds if and only if

$$\widetilde{\phi}_{12}(f) - g \in I_{\sigma_2}(\tau)^{r+1}. \tag{3.6}$$

Since the edge τ is locally algebraic, then by Lemma 3.9, a polynomial $h \in \mathcal{R}(\sigma_2)$ vanishes along τ up to order r if and only if $h \in I_{\sigma_2}(\tau)^{r+1}$. Therefore, equation (3.6) is equivalent to saying that $\widetilde{\phi}_{12}(f) - g$ vanishes along τ up to order r. Because $J_p^k\big(\widetilde{\phi}_{12}(f) - g\big) = J_p^k\big(\widetilde{\phi}_{12}(f)\big) - J_p^k(g)$, then (3.6) is also equivalent to

$$J_p^r\big(\widetilde{\phi}_{12}(f)\big) = J_p^r(g), \text{ for each } p \in \tau. \tag{3.7}$$

Since $J_p^r\big(\widetilde{\phi}_{12}(f)\big) = J_p^r\big(f \circ \varphi_{21}\big)$ for any $p \in \tau$, then (3.7) holds if and only if condition (a) holds. □

We are now ready to prove Theorem 3.8.

Proof to Theorem 3.8. If $(f_\sigma) \in G^1(\Delta, \Phi)$, then for any pair of adjacent faces $\sigma_1, \sigma_2 \in \Delta_2$, the polynomials $f_{\sigma_1} \in \mathcal{R}(\sigma_1)$ and $f_{\sigma_2} \in \mathcal{R}(\sigma_2)$ satisfy condition (b) in Lemma 3.10, and hence also condition (a). This implies $(f_\sigma) \in \mathcal{S}^1(\mathcal{M})$. Conversely, a similar argument implies $\mathcal{S}^1(\mathcal{M}) \subseteq G^1\big(\Delta, \Phi\big)$, and so we have $G^1_d(\Delta, \Phi) = \mathcal{S}^1_d(\mathcal{M})$. □

4. Examples of G^r-domains

This section is devoted to examples of G^r-domains (Δ, Φ). Recall from Remark 2.9 in Section 2 that to describe a collection of algebraic transition maps Φ we only need to give the set of transition maps in Φ associated to each pair of adjacent n-faces in Δ.

Example 4.1. Let Δ be an n-dimensional polyhedral complex composed of any two adjacent n-faces σ_1, σ_2 sharing a common $(n-1)$-face τ, for some $n \geqslant 2$. We may assume up to a change of coordinates that the ideals $I_{\sigma_i}(\tau)$ of τ in $\mathcal{R}(\sigma_i)$ are given by

$$\begin{aligned} &\mathcal{R}(\sigma_1) = \mathbb{R}[u_{11}, \dots, u_{1n}], && \mathcal{R}(\sigma_2) = \mathbb{R}[u_{21}, \dots, u_{2n}], \\ &I_{\sigma_1}(\tau) = \langle u_{11} \rangle, && I_{\sigma_2}(\tau) = \langle u_{2n} \rangle. \end{aligned}$$

From this, we see that τ is a locally algebraic face of Δ. A G^r-algebraic transition map is of the form

$$\phi_{12}\colon \mathbb{R}[u_{11}, \dots, u_{1n}]/\langle u_{11}^{r+1} \rangle \to \mathbb{R}[u_{21}, \dots, u_{2n}]/\langle u_{2n}^{r+1} \rangle$$

$$\phi_{12}(u_{1i}) = \begin{cases} \sum_{j=1}^r u_{2n}^j p_j & \text{if } i = 1, \\ u_{2(i-1)} + \sum_{j=1}^r u_{2n}^j q_{2j} & \text{otherwise,} \end{cases}$$

where q_{2j}, p_j are polynomials in $\mathbb{R}[u_{21}, \dots, u_{2(n-1)}]$. ◇

4.1. **Algebraic transition maps associated to a topological surface.** In the following examples we obtain G^1-domains from a topological surface and geometric transition maps as described in Section 3.

Example 4.2. Let $\mathcal{M}$ be the topological surface in Figure 1 which is composed of two square patches σ_1 and σ_2 together with a homeomorphism between the identified edges $\tau_i \subseteq \sigma_i$, for $i = 1, 2$. This example was studied in [13], and we can considered it in our setting as follows. As we described in Section 3, a 2-dimensional polyhedral complex Δ is naturally defined by $\mathcal{M}$, it consists of two square faces σ_1 and σ_2 joining along a common edge τ which arises from the homeomorphism $\tau_1 \to \tau_2$.

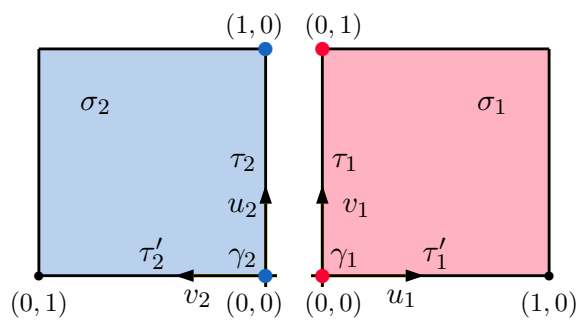


FIGURE 1. Topological surface $\mathcal{M}$ composed of the two squares together and a homeomorphism between the edges $\tau_2 \subseteq \sigma_2$ and $\tau_1 \subseteq \sigma_1$. In Example 4.2, the edge τ, and the vertex γ, denote the identification via this homeomorphism between the edges τ_i and vertices γ_i, respectively.

Up to a change of coordinates, we may assume that the ideals I_{σ_i} of polynomials vanishing at the edge τ in the coordinate rings $\mathcal{R}(\sigma_i)$ (in Definition 2.1) are given by:

$$\begin{aligned} \mathcal{R}(\sigma_1) &= \mathbb{R}[u_1, v_1], & \mathcal{R}(\sigma_2) &= \mathbb{R}[u_2, v_2], \\ I_{\sigma_1}(\tau) &= \langle u_1 \rangle, & I_{\sigma_2}(\tau) &= \langle v_2 \rangle. \end{aligned}$$

In particular, because both ideals I_{σ_i} are generated by an irreducible polynomials in $\mathcal{R}(\sigma_i)$ then τ is locally algebraic, see Definition 2.2. Following Definition 2.4, a G^1-algebraic transition map is an $\mathbb{R}$-algebra homomorphism which in this case can be written as

$$\begin{aligned} \phi_{12} \colon \mathbb{R}[u_1, v_1]/\langle u_1^2 \rangle &\to \mathbb{R}[u_2, v_2]/\langle v_2^2 \rangle \\ u_1 &\mapsto v_2 \mathfrak{b}(u_2) \\ v_1 &\mapsto u_2 + v_2 \mathfrak{a}(u_2), \end{aligned} \tag{4.1}$$

where $\mathfrak{a}(u_2)$ and $\mathfrak{b}(u_2)$ are univariate polynomials. $\diamond$

Remark 4.3. In Section 3 we described how to obtain a G^r-domain from a topological surface. If (Δ, Φ) is the G^1-domain obtained from the topological

surface $\mathcal{M}$ in Example 4.2, then the polynomials from (4.1) define the couple $[\mathfrak{a}, \mathfrak{b}]$ which in [45] is called the *gluing data* at the vertex γ along the edge τ. In this case, we say that an algebraic transition map ϕ_{12} on Δ is *defined from the gluing data*.

Definition 4.4 (G^1-gluing data)**.** Let Δ be a 2-dimensional polyhedral complex, and let σ_1, σ_2 be two adjacent 2-faces in Δ such that $\sigma_1 \cap \sigma_2 = \tau \in \Delta_1^\circ$. We write (u_i, v_i) for the coordinates of σ_i as in Figure 1. If the algebraic transition map ϕ_{12} is of the form in (4.1), then we say that ϕ_{12} is defined by the *gluing data* $[\mathfrak{a}_{12}^\tau, \mathfrak{b}_{12}^\tau]$, where $\mathfrak{a}_{12}^\tau, \mathfrak{b}_{12}^\tau \in \mathbb{R}[u_2]$. We refer to $[\mathfrak{a}_{12}^\tau, \mathfrak{b}_{12}^\tau]$, or simply $[\mathfrak{a}_{12}, \mathfrak{b}_{12}]$, as the gluing data from σ_1 to σ_2, or associated to τ if the order of the n-faces is clear from the context.

Remark 4.5. In Definition 4.4, for simplicity, we use (u_1, v_1) and (u_2, v_2) as relative coordinates to define the gluing data associated to an edge. This choice of coordinates are related to Figure 1 and they are independent of the notation $X(\sigma_1)$ and $X(\sigma_2)$ when Δ has more than one interior edge. In the cases we consider in this paper, the gluing data associated to different edges of a given partition Δ is given by the same formula once it is written in the relative coordinates (u_1, v_1), (u_2, v_2) for each $\tau \in \Delta_1^\circ$. If that is the case, we simply write $[\mathfrak{a}, \mathfrak{b}]$, and by an abuse of notation, we say that the collection of G^1-algebraic transition map Φ is given by the gluing data $[\mathfrak{a}, \mathfrak{b}]$.

For the next example, recall that the *valence* of a vertex γ in a 2-dimensional polyhedral cell complex Δ is the number of edges (or 1-faces) in Δ which contain γ as one of their vertices.

Example 4.6 (Symmetric gluing data)**.** Here is an example of transition maps for G^1-splines between two adjacent faces of a surface using the so-called *symmetric gluing data* proposed in [31, §8.2], see also [45, Example 2.5] where this example is studied in the setting of a topological surface and (geometric) transition maps.

Let τ be the edge in Δ with vertices γ and γ' which is shared by the two adjacent 2-faces $\sigma_1, \sigma_2 \in \Delta$. As in Example 4.2, we denote by ϕ_{12} the transition map from σ_1 to σ_2, which is given by (4.1). Take

$$\begin{aligned} \mathfrak{a}(u_2) &= 2\cos\left(\frac{2\pi}{w}\right) h(u_2) - 2\cos\left(\frac{2\pi}{w'}\right) q(u_2), \text{ and} \\ \mathfrak{b}(u_2) &= -1, \end{aligned} \tag{4.2}$$

where w and w' are the valences of the vertices γ and γ', respectively, and h and q are univariate polynomials. Then, the algebraic transition map $\phi_{12}\colon \mathbb{R}[u_1, v_1]/\langle v_1^2\rangle \to \mathbb{R}[u_2, v_2]/\langle u_2^2\rangle$ with G^1-gluing data $[\mathfrak{a}, \mathfrak{b}]$ from σ_1 to σ_2 is given by

$$\begin{aligned} \phi_{12}(u_1) &= -v_2, \\ \phi_{12}(v_1) &= u_2 + 2v_2\left(\cos\left(\frac{2\pi}{w}\right) h(u_2) - \cos\left(\frac{2\pi}{w'}\right) q(u_2)\right). \end{aligned}$$

Assume without loss of generality that γ and γ' as vertices of σ_2 have coordinates $u_2 = 0$, $v_2 = 0$, and to $u_2 = 1$, $v_2 = 0$, respectively. To satisfy the compatibility conditions required in Definition 2.5, we will additionally require that the functions h and q interpolate 0 and 1. Namely, we require $h(0) = 1, h(1) = 0, q(0) = 0$ and $q(1) = 1$. A possible solution is to take

$$\mathfrak{a}(u_2) = 2\cos\left(\frac{2\pi}{w}\right)(1-u_2)^2 - 2\cos\left(\frac{2\pi}{w'}\right)u_2^2, \quad \text{and } \mathfrak{b} = -1. \tag{4.3}$$

Then, if ϕ_{12} is given by $[\mathfrak{a}, \mathfrak{b}]$ as in (4.3), a good property of this kind of transition maps is that the gluing data of the inverse map ϕ_{21} from σ_2 to σ_1 of ϕ_{12} is of the same form (4.3). Explicitly, let

$$u_1' = 1 - v_1,\ v_1' = u_1,\ u_2' = v_2,\ v_2' = 1 - u_2,$$

and

$$\mathfrak{a}(u_1') = 2\cos\left(\frac{2\pi}{w'}\right)(1-u_1')^2 - 2\cos\left(\frac{2\pi}{w}\right)u_1'^2, \text{ and } \mathfrak{b} = -1.$$

Then the map $\phi_{21}\colon \mathbb{R}[u_2, v_2]/\langle v_2^2\rangle \to \mathbb{R}[u_1, v_1]/\langle u_1^2\rangle$ given by

$$\phi_{21}(u_2') = v_1'\mathfrak{b},$$
$$\phi_{21}(v_2') = u_1' + v_1'\mathfrak{a}(u_1')$$

is the algebraic transition map from σ_2 to σ_1 which is the inverse of ϕ_{12}. $\diamond$

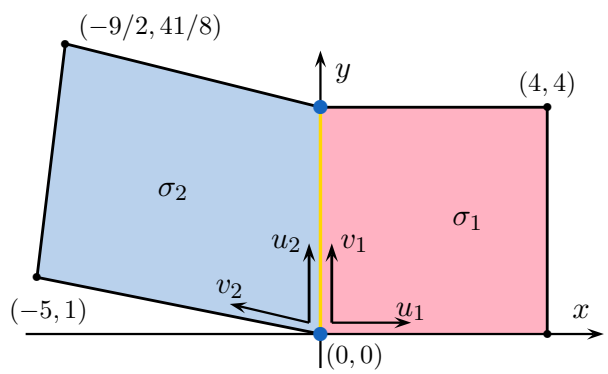


FIGURE 2. Two bilinearly parameterized patches considered in Example 4.7. We embed the polyhedral complex Δ consisting of these two patches σ_1, σ_2 in the (x, y)-plane, and write (u_i, v_i) for the coordinate system associated to σ_i.

Example 4.7 (Bilinearly parametrized patches)**.** In this example we look at the case of two quadrilateral patches, embedded in the (x, y)-plane (cf. Figure 2). The location of the patches in the (x, y)-plane is given by *bilinear parametrizations*, which means that the inverse of each map $\psi_i\colon \sigma_i \to X(\sigma_i) \cong \mathbb{R}^2$, defined by $(x, y) \mapsto (u_i, v_i)$ is described by two polynomials in $\mathbb{R}[u_i, v_i]_{\leqslant(1,1)}$. Since in this case we always assume the image of ψ_i is a

unit square, we shall refer to $\psi_i^{-1}\colon [0,1]^2 \to \sigma_i$ as the parametrization of the face σ_i. It is uniquely determined by four corner points $\psi_i^{-1}(0,0)$, $\psi_i^{-1}(0,1)$, $\psi_i^{-1}(1,0)$, $\psi_i^{-1}(1,1)$ of σ_i. Explicitly, assume ψ_1^{-1} and ψ_2^{-1} are defined by

$$\psi_1^{-1}(u_1,v_1) = (a_{11}u_1v_1 + a_{10}u_1 + a_{01}v_1 + a_{00},\ b_{11}u_1v_1 + b_{10}u_1 + b_{01}v_1 + b_{00}),$$

and

$$\psi_2^{-1}(u_2,v_2) = (c_{11}u_2v_2 + c_{10}u_2 + c_{01}v_2 + c_{00},\ e_{11}u_2v_2 + e_{10}u_2 + e_{01}v_2 + e_{00}).$$

Then

$$\begin{pmatrix} a_{11} & b_{11} & c_{11} & e_{11} \\ a_{10} & b_{10} & c_{10} & e_{10} \\ a_{01} & b_{01} & c_{01} & e_{01} \\ a_{00} & b_{00} & c_{00} & e_{00} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \psi_1^{-1}(0,0) & \psi_2^{-1}(0,0) \\ \psi_1^{-1}(1,0) & \psi_2^{-1}(1,0) \\ \psi_1^{-1}(0,1) & \psi_2^{-1}(0,1) \\ \psi_1^{-1}(1,1) & \psi_2^{-1}(1,1) \end{pmatrix}.$$

By choosing coordinates, we may always assume that in the (x,y)-plane, the edge τ shared by the two faces lies on the line defined by $x=0$, and that $\psi_1^{-1}(0,0) = \psi_2^{-1}(0,0) = (0,0)$. We may also assume $I_{\sigma_1}(\tau) = \langle u_1 \rangle$, and $I_{\sigma_2}(\tau) = \langle v_2 \rangle$. This choice of coordinates implies

$$a_{01} = a_{00} = b_{00} = c_{10} = c_{00} = e_{00} = 0.$$

Hence, ψ_1 defines a map

$$\begin{aligned} \eta_1\colon \mathbb{R}[x,y]/\langle x^2 \rangle &\to \mathcal{R}(\sigma_1)/\langle u_1^2 \rangle, \\ x &\mapsto a_{11}u_1v_1 + a_{10}u_1, \\ y &\mapsto b_{11}u_1v_1 + b_{10}u_1 + b_{01}v_1. \end{aligned}$$

Similarly, ψ_2 defines a map

$$\begin{aligned} \eta_2\colon \mathbb{R}[x,y]/\langle x^2 \rangle &\to \mathcal{R}(\sigma_2)/\langle v_2^2 \rangle \\ x &\mapsto c_{11}u_2v_2 + c_{01}v_2, \\ y &\mapsto e_{11}u_2v_2 + e_{10}u_2 + e_{01}v_2. \end{aligned}$$

If there is a map ϕ_{12} defined from gluing data $\mathfrak{a}(u_2)$ and $\mathfrak{b}(u_2)$ as in (4.1), and such that $\phi_{12} \circ \eta_1 = \eta_2$, then we say *the gluing data is determined by the parametrization of* σ_1 *and* σ_2. Explicitly, $\phi_{12} \circ \eta_1 = \eta_2$ means that the system of equations

$$\begin{cases} (a_{11}u_2 + a_{10})\mathfrak{b}(u_2) = c_{11}u_2 + c_{01} \\ (b_{11}u_2 + b_{10})\mathfrak{b}(u_2) + b_{01}\mathfrak{a}(u_2) = e_{11}u_2 + e_{01} \end{cases} \tag{4.4}$$

holds. Hence, the existence of such a ϕ_{12} is equivalent to say that (4.4) has a polynomial solution for both $\mathfrak{a}(u_2)$ and $\mathfrak{b}(u_2)$. In this case, it forces $a_{11} = b_{11} = 0$.

Using this gluing data, the parametric spline functions defined over these specific quadrilaterals on the (x,y)-plane are geometrically continuous over the G^1-domain (Δ, Φ), where Δ is the cell complex composed of the two patches sharing an edge and Φ is determined by ϕ_{12}.

This construction of gluing data has been extensively studied for G^1-splines and isogeometric applications on planar domains [36, 37, 38], and volumetric two-patch domains [11, 12]. There, the gluing data is defined using the determinants of the minors of the following 2×3 matrix

$$\begin{pmatrix} \partial_{u_1}\psi_1^{-1} & \partial_{v_1}\psi_1^{-1} & \partial_{v_2}\psi_2^{-1} \end{pmatrix}.$$

Explicitly,

$$\begin{aligned} \mathfrak{a}(u_2) &= \big(\det(\partial_{u_1}\psi_1^{-1},\ \partial_{v_2}\psi_2^{-1})/\det\nabla\psi_1^{-1}\big)|_{v_1=u_2,\ u_1=v_2=0}, \text{ and} \\ \mathfrak{b}(u_2) &= \big(-\det(\partial_{v_1}\psi_1^{-1},\ \partial_{v_2}\psi_2^{-1})/\det\nabla\psi_1^{-1}\big)|_{v_1=u_2,\ u_1=v_2=0}\,, \end{aligned}$$

and two polynomials g_1 and g_2 are said to join with G^1-continuity if and only if

$$\big(\mathfrak{a}(u_2)\,\partial_{v_1}g_1 + \mathfrak{b}(u_2)\,\partial_{u_1}g_1 - \partial_{v_2}g_2\big)|_{v_1=u_2,u_1=v_2=0} = 0\,.$$

One may check that this construction of gluing data is equivalent to solve the system of equations (4.4).

As a particular example, we consider the planar patches on the (x, y)-plane where σ_1 has vertices $(0, 0)$, $(0, 4)$, $(4, 0)$, and $(4, 4)$. The parametrization ψ_1^{-1} is given by the equation

$$\psi_1^{-1}(u_1, v_1) = (4u_1,\ 4v_1).$$

The patch σ_2 is a quadrilateral with vertices $(0, 0)$, $(0, 4)$, $(-5, 1)$, and $(-9/2, 41/8)$ on the (x, y)-plane. The parametrization ψ_2^{-1} is given by the bilinear map

$$\psi_2^{-1}(u_2, v_2) = \left(\frac{1}{2}u_2v_2 - 5v_2,\ \frac{1}{8}u_2v_2 + 4u_2 + v_2\right).$$

The common edge between the two patches has vertices $(0, 0)$ and $(0, 4)$. The gluing data take the form

$$\mathfrak{a}(u_2) = -\frac{1}{32}u_2 - \frac{1}{4}, \text{ and } \quad \mathfrak{b}(u_2) = -\frac{1}{8}u_2 + \frac{5}{4},$$

and the corresponding transition map is given by $\phi_{12}(u_1) = v_2\mathfrak{b}(u_2)$, and $\phi_{12}(v_1) = u_2 + v_2\mathfrak{a}(u_2)$. ◇

4.2. **G^r-domains obtained from differential manifolds with cellulation.** We denote by M a differential n-manifold with a given atlas. A *cellulation* of M is given by a homeomorphism $\textsc{cel}\colon \Delta \to M$, such that Δ is a cell complex. Assume Δ is such a cell complex, and suppose for each $\sigma \in \Delta_n$ there is a unique chart $U(\sigma)$ in the given atlas such that $\textsc{cel}(\sigma) \subseteq U(\sigma)$. Then a system of coordinates $X(\sigma)$ on each $\sigma \in \Delta_n$ is determined by the atlas in a natural way, and for every $(n-1)$-face τ of Δ which is common to two n-faces, a geometric transition map across τ is defined by the atlas. We denote this collection of geometric transition maps by Φ_G. We assume that every such $(n-1)$-face τ of Δ is locally algebraic (i.e., the ideal $I_\sigma(\tau)$ is generated by an irreducible polynomial for every n-face $\sigma \supseteq \tau$, see Definition 2.2), and each map in Φ_G is almost algebraic (see Definition 3.4). Then,

from Φ_G, we can construct a set of compatible algebraic transition maps Φ which lead us to a G^r-domain over Δ. In this case, we simply say that Φ is *obtained from the atlas* of the manifold M, and that the G^r-domain (Δ, Φ) is *obtained from the cellulation* of M.

In this section we give some examples in which we start with a differential manifold and construct a G^r-domain. In the first example, we consider an *affine manifold* i.e., a differential manifold admitting an atlas where all transition maps are affine [30].

Example 4.8. Let Δ be a 1-dimensional simplicial complex given by

$$\Delta = \big\{\emptyset, [1], [2], [3], [12], [23], [13]\big\}.$$

A geometric realization of Δ is homeomorphic to a circle, as illustrated in Figure 3a. We denote the vertices by $[i] = \gamma_i$, for $i = 1, 2, 3$, and the maximal faces are given by $\tau_1 = [12]$, $\tau_2 = [23]$, and $\tau_3 = [13]$. We choose coordinates $\psi_i \colon \tau_i \to X(\tau_i) \cong \mathbb{R}^1$, where each point $p \mapsto u_i(p)$ and such that $\psi_i(\tau_i) = [0, 1]$ on $X(\tau_i)$, for $i = 1, 2, 3$. Then, the maps

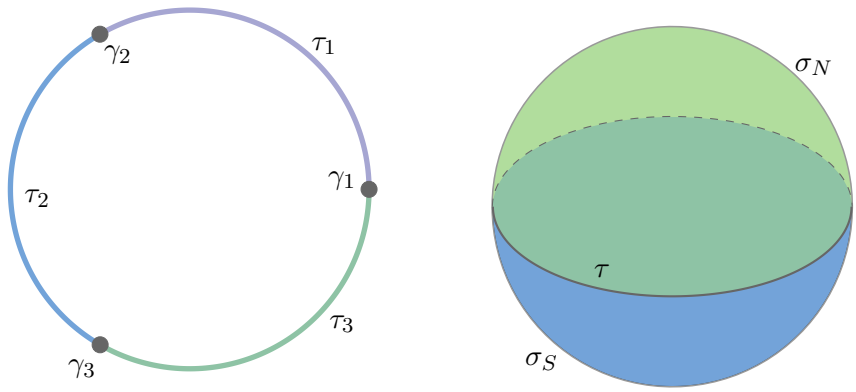
$$\begin{aligned} \phi_{12} \colon \mathbb{R}[u_1]/\langle (u_1-1)^2\rangle &\to \mathbb{R}[u_2]/\langle u_2^2\rangle \\ u_1 &\mapsto u_2 + 1, \\ \phi_{23} \colon \mathbb{R}[u_2]/\langle (u_2-1)^2\rangle &\to \mathbb{R}[u_3]/\langle u_3^2\rangle \\ u_2 &\mapsto u_3 + 1, \\ \phi_{31} \colon \mathbb{R}[u_3]/\langle (u_3-1)^2\rangle &\to \mathbb{R}[u_1]/\langle u_1^2\rangle \\ u_3 &\mapsto u_1 + 1. \end{aligned} \tag{4.5}$$

together with the identification map ϕ_{ii} on each τ_i form a collection Φ of algebraic transition maps satisfying the compatibility conditions in Definition 2.5. Thus we get a G^1-domain (Δ, Φ).

We can see this G^1-domain (Δ, Φ) as obtained from a cellulation of a differential 1-fold as follows. On $X(\tau_i)$, at each end point $\psi(\gamma_j)$ of the interval $\psi_i(\tau_i)$, we take U_{γ_j,τ_i} to be a neighborhood of $\psi(\gamma_j)$ of radius $1/4$. If $V_i = \psi_i(\tau_i) \cup_{\gamma_j \subseteq \tau_i} U_{\gamma_j,\tau_i}$ Then $\big\{(V_i, \psi_i) \colon i = 1, 2, 3\big\}$ is an atlas of the circle, with transition maps $\varphi_{12} \colon U_{\gamma_2,\tau_2} \to U_{\gamma_2,\tau_1}$, $\varphi_{23} \colon U_{\gamma_3,\tau_3} \to U_{\gamma_3,\tau_2}$, and $\varphi_{31} \colon U_{\gamma_1,\tau_1} \to U_{\gamma_1,\tau_3}$, all defined by $a \mapsto a - 1$. Each of these maps φ_{ij} is affine and therefore they are almost algebraic. Thus, the maps in (4.5) can be viewed as obtained from the atlas $\big\{(V_i, \psi_i) \colon i = 1, 2, 3\big\}$. $\diamond$

Since a torus is also an affine manifold, a construction similar to that in Example 4.8 applies to a torus. Next, we present an example in which the manifold is not affine.

Example 4.9. Let M be a unit sphere in $\mathbb{R}^3$. We denote the north and south poles of M by $N = (0, 0, 1)$ and $S = (0, 0, -1)$, respectively, and consider the atlas consisting of two charts $M \setminus \{N\}$, and $M \setminus \{S\}$. The coordinate system on each of these charts is given by the stereographic projection from N and S, respectively. We consider a cell complex Δ given by this sphere



(a) Cellulation of a circle by dividing it with three vertices. In Example 4.8, we use this cellulation to obtain a collection of algebraic transition maps and a G^1-domain over the circle.

(b) Cellulation of a sphere by dividing it with its equator. In Example 4.9, we use this cellulation to obtain a collection of algebraic transition maps and a G^r-domain over the sphere.

FIGURE 3. Examples of differential manifolds with cellulation.

divided by the 1-cell τ corresponding to the points in the plane $z = 0$, as in Figure 3b. Let σ_S and σ_N be the 2-faces of Δ which correspond to the points of M with $z \leqslant 0$ (the south hemisphere) and $z \geqslant 0$ (the north hemisphere), respectively. Notice that each of these two 2-faces falls on only one of the charts of the atlas. Explicitly, we may consider σ_S and σ_N in the coordinates $X(\sigma_N) \cong \mathbb{R}^2$ and $X(\sigma_S) \cong \mathbb{R}^2$ via the maps

$$\begin{aligned} \psi_S \colon \sigma_S &\to X(\sigma_S) \\ (x, y, z) &\mapsto (u_S, v_S) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right), \end{aligned}$$

and

$$\begin{aligned} \psi_N \colon \sigma_N &\to X(\sigma_N) \\ (x, y, z) &\mapsto (u_N, v_N) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right). \end{aligned}$$

The 1-face τ shared by σ_S and σ_N is locally algebraic (see Definition 2.2), because if $w_S = 1 - u_S^2 - v_S^2$ and $w_N = 1 - u_N^2 - v_N^2$, then $I_{\sigma_S}(\tau) = \langle w_S \rangle$ and $I_{\sigma_N}(\tau) = \langle w_N \rangle$, respectively. Then, the map

$$\begin{aligned} \phi_{SN} \colon \mathcal{R}(\sigma_S)/\langle w_S^{r+1} \rangle &\to \mathcal{R}(\sigma_N)/\langle w_N^{r+1} \rangle \\ u_S &\mapsto u_N(1 + w_N + \cdots + w_N^r), \\ v_S &\mapsto v_N(1 + w_N + \cdots + w_N^r) \end{aligned}$$

is an algebraic transition map, which can be viewed as obtained from the given atlas of M. The map ϕ_{SN}, its inverse and the two identity maps on

σ_S and σ_N form a collection of compatible algebraic transition maps Φ, and so we obtain a G^r-domain $\big(\Delta, \Phi\big)$ from a cellulation of a sphere covered by the stereographic charts. $\diamond$

5. Splines on 2-dimensional G^r-domains

Throughout this section we assume Δ is a 2-dimensional cell complex, and we take a collection Φ of compatible G^r-algebraic transition maps on Δ, for some $r \geqslant 0$. We will use the chain complex $(\mathcal{Q}^r_{d,\bullet}, \delta)$ introduced in (2.13) in Section 2.3 to study the dimension and construction of bases of the space $G^r_d(\Delta, \Phi)$ of splines on a G^r-domain (Δ, Φ), for a given degree $d \geqslant 0$.

Recall that by Proposition 2.18 we have that $G^r_d(\Delta, \Phi) = H_2(\mathcal{Q}^r_{d,\bullet})$. Thus, if Δ is a 2-dimensional cell complex, we have

$$\dim G^r_d(\Delta, \Phi) = \chi(\mathcal{Q}^r_{d,\bullet}) + \dim H_1(\mathcal{Q}^r_{d,\bullet}) - \dim H_0(\mathcal{Q}^r_{d,\bullet}), \tag{5.1}$$

where

$$\chi(\mathcal{Q}^r_{d,\bullet}) = \sum_{\sigma \in \Delta_2} \dim \mathcal{Q}^r_d(\sigma) - \sum_{\tau \in \Delta_1^\circ} \dim \mathcal{Q}^r_d(\tau) + \sum_{\gamma \in \Delta_0^\circ} \dim \mathcal{Q}^r_d(\gamma), \tag{5.2}$$

is the Euler characteristic of the complex $\big(\mathcal{Q}^r_{d,\bullet}, \delta\big)$.

A first estimate of $\dim G^r_d(\Delta, \Phi)$ arises from considering $\chi(\mathcal{Q}^r_{d,\bullet})$ in (5.2). In the following, we compute the terms of $\chi(\mathcal{Q}^r_{d,\bullet})$ when $r = 1$ and the algebraic transition maps on Δ are defined from some gluing data (see Remark 4.3). The homology terms $\dim H_1(\mathcal{Q}^r_{d,\bullet})$ and $\dim H_0(\mathcal{Q}^r_{d,\bullet})$ in (5.1) are not necessarily zero, we consider them in Section 6.

Similarly as above, we write (u_i, v_i) for the coordinates of $X(\sigma_i)$, for a maximal face of Δ, in this case a 2-face $\sigma_i \in \Delta_2$. If $\sigma_1 \cap \sigma_2 = \tau \in \Delta_1$ is an edge shared by two faces $\sigma_1, \sigma_2 \in \Delta_2$, the tensor product ring in (2.2) is simply $\mathcal{R}(\tau) = \mathcal{R}(\sigma_1) \otimes \mathcal{R}(\sigma_2)$, and the subset of polynomials without mixed terms (2.11) can be written as

$$\mathcal{T}_d(\tau) = \big\{ f_1 + f_2 \in \mathcal{R}(\tau) \colon f_i \in \mathcal{R}_d(\sigma_i) \big\}.$$

The quotient (2.13) is in this case $\mathcal{Q}^r_d(\tau) = \mathcal{T}_d(\tau)/\mathcal{J}^r_d(\tau)$, where $\mathcal{J}^r_d(\tau) = \mathcal{I}^r(\tau) \cap \mathcal{T}_d(\tau)$ and

$$\mathcal{I}^r(\tau) = \big\langle u_1 - \widetilde{\phi}_{12}(u_1),\ v_1 - \widetilde{\phi}_{12}(v_1) \big\rangle + I_{\sigma_1}(\tau)^{r+1} \cdot \mathcal{R}(\tau) + I_{\sigma_2}(\tau)^{r+1} \cdot \mathcal{R}(\tau),$$

for a lift $\widetilde{\phi}_{12}$ is of the G^r-algebraic transition map ϕ_{12} from σ_1 to σ_2. (See (2.4) and (2.12) for the general definition of $\mathcal{I}^r(\tau)$ and $\mathcal{J}^r_d(\tau)$, respectively.) Notice that to obtain a basis for $\mathcal{J}^r_d(\tau)$, one needs to compute the kernel of a natural map of vector spaces

$$\mathcal{T}_d(\tau) \oplus \mathcal{I}^r(\tau) \to \mathcal{R}(\tau).$$

Lemma 5.1. *Let $\tau = \sigma_1 \cap \sigma_2 \in \Delta_1$ be an edge shared by two faces $\sigma_1, \sigma_2 \in \Delta_2$, and let ϕ_{12} be a G^1-algebraic transition map on Δ given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$ as in (4.2). If $d_{\mathfrak{a}} = \deg \mathfrak{a} \geqslant 1$, then*

(a) in the total degree case, if $d \geqslant d_{\mathfrak{a}} + 1$, we have

$$\dim \mathcal{Q}_d^1(\tau) = 2d + d_{\mathfrak{a}} + 1, \text{ and } \dim \mathcal{J}_d^1(\tau) = d^2 + d - d_{\mathfrak{a}};$$

a basis of $\mathcal{J}_d^1(\tau)$ is given by

$$\begin{cases} u_1^i v_1^j,\ u_2^j v_2^i, & \text{for } 2 \leqslant i \leqslant d \text{ and } i + j \leqslant d, \\ u_1 v_1^i + u_2^i v_2, & \text{for } 0 \leqslant i \leqslant d - 1, \\ v_1^i - \big(u_2^i + i\, u_2^{i-1} v_2\, \mathfrak{a}(u_2)\big), & \text{for } 1 \leqslant i \leqslant d - d_{\mathfrak{a}}. \end{cases} \tag{5.3}$$

(b) For bidegree (d, d) with $d \geqslant d_{\mathfrak{a}}$, we have

$$\dim \mathcal{Q}_{(d,d)}^1(\tau) = 2d + d_{\mathfrak{a}} + 1, \text{ and } \dim \mathcal{J}_{(d,d)}^1(\tau) = 2d^2 + 2d - d_{\mathfrak{a}};$$

a basis of $\mathcal{J}_{(d,d)}^1(\tau)$ is given by

$$\begin{cases} u_1^i v_1^j,\ u_2^j v_2^i,\ u_1 v_1^j + u_2^j v_2, & \text{for } 2 \leqslant i \leqslant d \text{ and } 0 \leqslant j \leqslant d, \\ v_1^i - (u_2^i + i\, u_2^{i-1} v_2 \mathfrak{a}(u_2)), & \text{for } 1 \leqslant i \leqslant d + 1 - d_{\mathfrak{a}}. \end{cases} \tag{5.4}$$

Proof. Let $\succ$ be the elimination order $u_1 \succ v_1 \succ u_2 \succ v_2$. If we take a monomial basis $\mathcal{B}$ of $\mathcal{T}_d(\tau)$, then Lemma 2.21 implies that the space $V = \operatorname{span}\{\mathfrak{r}(b) \colon b \in \mathcal{B}\}$ is isomorphic to $\mathcal{Q}_d^1(\tau)$. Note that

$$\mathfrak{r}(u_1^i v_1^j) = \begin{cases} 0, & \text{if } i \geqslant 2, \\ u_2^j v_2, & \text{if } i = 1, \\ u_2^j + j\, u_2^{j-1} v_2 \mathfrak{a}(u_2), & \text{if } i = 0, \end{cases}$$

and,

$$\mathfrak{r}(u_2^i v_2^j) = \begin{cases} 0, & \text{for } j \geqslant 2, \\ u_2^i v_2^j, & \text{for } j = 0, 1. \end{cases}$$

Therefore, by an induction on $d_{\mathfrak{a}}$, one may verify that either for total degree $d \geqslant d_{\mathfrak{a}} + 1$, or for bidegree (d, d) with $d \geqslant d_{\mathfrak{a}}$, a basis for V is given by

$$\big\{u_2^i \colon i = 0, \ldots, d\big\} \cup \big\{u_2^i v_2 \colon i = 0, \ldots, d - 1 + d_{\mathfrak{a}}\big\}.$$

Hence $\dim V = 2d + d_{\mathfrak{a}} + 1$, and this proves the dimension formulas for $\mathcal{Q}_d^1(\tau)$ and $\mathcal{Q}_{(d,d)}^1(\tau)$ in (a) and (b), respectively. Since $\mathcal{Q}_d^1(\tau) = \mathcal{T}_d(\tau)/\mathcal{J}_d^1(\tau)$, then the formulas for $\dim \mathcal{J}_d^1(\tau)$ in (a) can be deduced from that of $\dim \mathcal{Q}_d^1(\tau)$. It is easy to verify that every element in (5.3) is in $\mathcal{J}_d^1(\tau)$. Because they have distinct leading terms with respect to the monomial order, so they are linearly independent. Hence, they form a basis for $\mathcal{J}_d^1(\tau)$. The same reasoning can be applied to the bidegree (d, d) case. □

We still have to deal with the case where $\mathfrak{a}$ is a constant. If the transition map is given by symmetric gluing data, this is equivalent to saying that the transition map can be lifted to an invertible affine map. The intuition is, if the transition map is affine, then the domain is "locally planar", in the sense of the following lemma.

Lemma 5.2. *For adjacent faces σ_1 and σ_2, if the G^r-algebraic transition map ϕ_{12} is invertible and can be lifted to an invertible affine map $\widetilde{\phi}_{12}\colon \mathcal{R}(\sigma_1) \to \mathcal{R}(\sigma_2)$, then its restriction to $\mathcal{R}_d(\sigma_1)$ is $\widetilde{\phi}_{12}\colon \mathcal{R}_d(\sigma_1) \to \mathcal{R}(\sigma_2)$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{R}_d(\sigma_1) \oplus \mathcal{R}_d(\sigma_1) & \xrightarrow{\partial_2} & \mathcal{R}_d(\sigma_1)/I_{\sigma_1}(\tau)^{r+1} \cap \mathcal{R}_d(\sigma_1) \\ \Big\downarrow{\scriptstyle \mathrm{id}\oplus\widetilde{\phi}_{12}} & & \Big\downarrow{\scriptstyle \eta} \\ \mathcal{R}_d(\sigma_1) \oplus \mathcal{R}_d(\sigma_2) & \xrightarrow{\quad\delta_2\quad} & \mathcal{Q}^r_d(\tau), \end{array} \tag{5.5}$$

where $\partial_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and η is induced by the natural inclusion $\mathcal{R}(\sigma_1) \to \mathcal{R}(\tau)$. Moreover, η is an isomorphism and $\ker \partial_2 \cong \ker \delta_2$.

Proof. First of all, since $I_{\sigma_1}(\tau)^{r+1} = \mathcal{I}^r(\tau) \cap \mathcal{R}(\sigma_1)$, then η is well-defined and injective. To check that η is also surjective, note that ϕ_{12} is invertible and its inverse ϕ_{21} is also affine. If $h(u_1, v_1, u_2, v_2) \in \mathcal{T}_d(\tau)$, then $h\big(u_1, v_1, \phi_{21}(u_2), \phi_{21}(v_2)\big) \in \mathcal{R}_d(\sigma_1)$ gives a preimage of $h(u_1, v_1, u_2, v_2)$. Thus, η is surjective.

Assume $(f, g) \in \mathcal{R}_d(\sigma_1) \oplus \mathcal{R}_d(\sigma_1)$. Denote by $\overline{f}$ the image of f in the quotient ring $\mathcal{R}(\sigma_1)/I_{\sigma_1}(\tau)^{r+1}$. Then $\delta_2 \circ (\mathrm{id} \oplus \widetilde{\phi}_{12})(f, g) = \overline{f} - \phi_{12}(\overline{g}) = \eta \circ \partial_2(f, g)$. Hence, diagram (5.5) commutes.

Both ∂_2 and δ_2 in (5.5) are surjective and all vertical maps are isomorphisms. Therefore, we conclude that $\ker \partial_2 = \ker \delta_2$. □

A dimension formula for $\mathcal{Q}^1_d(\tau)$, and hence for $\dim \mathcal{J}^1_d(\tau)$, when $\deg \mathfrak{a} < 1$ follows as a corollary of Lemma 5.2.

Corollary 5.3. *Assume the transition map ϕ_{12} from σ_1 to σ_2 is given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$ as in (4.2). If $\mathfrak{a}$ is constant, and $d \geqslant 0$, then*

(a) in the total degree case we have,

$$\dim \mathcal{Q}^1_d(\tau) = 2d + 1, \ \textit{and} \ \dim \mathcal{J}^1_d(\tau) = d^2 + d.$$

A basis of $\mathcal{J}^1_d(\tau)$ is given by

$$\begin{cases} v_1 - u_2, & \\ u_1 v_1^i + u_2^i v_2, & \textit{for } 0 \leqslant i \leqslant d-1, \\ u_1^i v_1^j,\ u_2^j v_2^i,\ v_1^i - u_2^i, & \textit{for } 2 \leqslant i \leqslant d \textit{ and } i + j \leqslant d, \end{cases}$$

(b) For bidegree (d, d) case, we have

$$\dim \mathcal{Q}^1_{(d,d)}(\tau) = 2d + 2, \ \textit{and} \ \dim \mathcal{J}^1_{(d,d)}(\tau) = 2d^2 + 2d - 1.$$

A basis of $\mathcal{J}^1_{(d,d)}(\tau)$ is given by

$$\begin{cases} v_1 - u_2, & \\ u_1^i v_1^j,\ u_2^j v_2^i,\ v_1^i - u_2^i,\ u_1 v_1^j + u_2^j v_2, & \textit{for } 2 \leqslant i \leqslant d \textit{ and } 0 \leqslant j \leqslant d. \end{cases}$$

Proof. The total degree case is an immediate consequence of Lemma 5.2. For the bidegree case, note that for symmetric gluing data, the only case where $\mathfrak{a}(u) = 0$ or $\deg \mathfrak{a}(u) = 0$ is that $(n_0, n_1) = (4, 4)$, which gives $\phi_{12}(u_1) = -v_2$ and $\phi_{12}(v_1) = u_2$. Hence, ϕ_{12} preserves bigrading and Lemma 5.2 can be applied. This proves the dimension formulas in both cases, (a) and (b). Since the transition maps are explicit, we can easily verify that the given polynomials constitute a basis for $\mathcal{J}_d^1(\tau)$ and $\mathcal{J}_{(d,d)}^1(\tau)$ in (a) and (b), respectively. □

Next, we calculate $\dim \mathcal{Q}_d^1(\gamma)$ for an interior vertex γ, when Φ is given by symmetric gluing data. Let $\sigma_1, \ldots, \sigma_s \in \Delta_2$ be the 2-faces with a common vertex γ, for some $s \geqslant 3$. We assume $\sigma_i \cap \sigma_{i+1} = \tau_i \in \Delta_1$, for $i = 1, \ldots, s$, where we identify $\sigma_{s+1} = \sigma_1$.

For each edge τ_i, let $\phi_{i,i+1} \colon \mathcal{R}(\sigma_i)/I_{\sigma_i}(\tau_i)^{r+1} \to \mathcal{R}(\sigma_{i+1})/I_{\sigma_i}(\tau_i)^{r+1}$ be the G^r-algebraic transition map from σ_i to σ_{i+1}, and write $\widetilde{\phi}_{i,i+1} \colon \mathcal{R}(\sigma_i) \to \mathcal{R}(\sigma_{i+1})$ for a lift of $\phi_{i,i+1}$. From (2.2), we have that in this case

$$\mathcal{R}(\tau_i) = \mathbb{R}[u_i, v_i, u_{i+1}, v_{i+1}], \text{ and } \mathcal{R}(\gamma) = \mathbb{R}[u_1, v_1, \ldots, u_s, v_s],$$

and (2.4) gives the ideals

$$\mathcal{I}^r(\tau_i) = \\ \big\langle u_i - \widetilde{\phi}_{i,i+1}(u_i),\ v_i - \widetilde{\phi}_{i,i+1}(v_i) \big\rangle + I_{\sigma_i}(\tau_i)^{r+1} \cdot \mathcal{R}(\tau_i) + I_{\sigma_{i+1}}(\tau_i)^{r+1} \cdot \mathcal{R}(\tau_i),$$

and

$$\mathcal{I}^r(\gamma) = \\ \sum_{i=1}^{s} \big\langle u_i - \widetilde{\phi}_{i,i+1}(u_i),\ v_i - \widetilde{\phi}_{i,i+1}(v_i) \big\rangle + \sum_{i=1}^{s} \big(I_{\sigma_i}(\tau_i)^{r+1} + I_{\sigma_{i+1}}(\tau_i)^{r+1} \big) \cdot \mathcal{R}(\gamma).$$

Lemma 5.4. *Let Φ be a collection of compatible G^1-algebraic transition maps given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$ as in (4.2). If $\gamma \in \Delta_0^\circ$ is an interior vertex which is common to $s \geqslant 3$ faces $\sigma_i \in \Delta_2$ and $s \neq 4$, then $\mathcal{I}^1(\gamma)$ is generated by*

$$\big\{ u_i + v_{i+1},\ v_i - u_{i+1} - a_0 v_{i+1},\ u_i^2,\ u_i v_i,\ v_i^2 \colon i = 1, \ldots, s \big\}, \tag{5.6}$$

where a_0 is the constant term in $\mathfrak{a}(u) = \sum_{k=0}^{d_\mathfrak{a}} a_k u^k$. Additionally, if in this case $d \geqslant 1$, then

$$\dim \mathcal{Q}_d^1(\gamma) = \dim \mathcal{Q}_{(d,d)}^1(\gamma) = 3. \tag{5.7}$$

Proof. We first show that every element in (5.6) is in $\mathcal{I}^1(\gamma)$. It is clear that u_i^2, v_i^2 and $u_i + v_{i+1}$ are in $\mathcal{I}^1(\gamma)$. To see $u_{i+1} v_{i+1} \in \mathcal{I}^1(\gamma)$, note that $v_i^2 \in \mathcal{I}^1(\gamma)$ and that

$$v_i^2 - 2a_0 u_{i+1} v_{i+1} \in \big\langle v_i - u_{i+1} - v_{i+1} \mathfrak{a}(u_{i+1}),\ u_{i+1}^2,\ v_{i+1}^2 \big\rangle \subseteq \mathcal{I}^1(\gamma).$$

Moreover, $s \neq 4$ implies $a_0 \neq 0$. Therefore, $u_{i+1}v_{i+1} \in \mathcal{I}^1(\gamma)$. Now note that since u_{i+1}^2, $u_{i+1}v_{i+1}$, and v_{i+1}^2 are polynomials in $\mathcal{I}^1(\gamma)$, then

$$\big(v_i - u_{i+1} - v_{i+1}\mathfrak{a}(u_{i+1})\big) - \big(v_i - u_{i+1} - a_0 v_{i+1}\big) \in \big\langle u_{i+1}^2,\ u_{i+1}v_{i+1},\ v_{i+1}^2\big\rangle \subseteq \mathcal{I}^1(\gamma).$$

This implies $v_i - u_{i+1} - a_0 v_{i+1} \in \mathcal{I}^1(\gamma)$.

Next we prove that (5.6) generates $\mathcal{I}^1(\gamma)$. Let I be the ideal generated by (5.6). Then $I \subseteq \mathcal{I}^1(\gamma)$. Suppose there exists $f \in \mathcal{I}^1(\gamma)$ such that $f \notin I$. Let $\succ$ be the elimination order $u_1 \succ v_1 \succ \cdots \succ u_s \succ v_s$ on the polynomial ring $\mathcal{R}(\gamma)$. Then the leading term of the remainder $\mathfrak{r}(f)$ with respect to I is less than v_s^2, i.e., $\mathfrak{r}(f) = c_1 u_s + c_2 v_s + c_3$ for some constants c_1, c_2, and c_3. Hence, $\mathfrak{r}(f) \in \mathcal{I}^1(\gamma)$, and this implies $c_1 = c_2 = c_3 = 0$, which means $f \in I$, which is a contradiction.

Note that for any subspace $W \subseteq \mathcal{R}(\gamma)$ such that $\{u_s, v_s, 1\} \subseteq W$, the set of remainders $\mathfrak{r}(W)$ with respect to $\mathcal{I}^1(\gamma)$ is the subspace of $\mathcal{R}(\gamma)$ spanned by $\{u_s, v_s, 1\}$. Since for $d \geqslant 1$, in the total degree case we have $\{u_s, v_s, 1\} \subseteq \mathcal{T}_d(\gamma)$, and in the bidegree case $\{u_s, v_s, 1\} \subseteq \mathcal{T}_{(d,d)}(\gamma)$, then (5.7) holds, which concludes the proof. □

Lemma 5.5. *Let Φ be a collection of compatible G^1-algebraic transition maps given by gluing data $[\mathfrak{a}, \mathfrak{b}]$ as in (4.2). If $\gamma \in \Delta_0^\circ$ is an interior vertex which is common to 4 faces $\sigma_i \in \Delta_2$, then $\mathcal{I}^1(\gamma)$ is generated by*

$$\big\{u_i + v_{i+1},\ v_i - u_{i+1},\ u_i^2,\ v_i^2 \colon i = 1, 2, 3, 4\big\}. \tag{5.8}$$

Additionally, if $d \geqslant 1$ then $\dim \mathcal{Q}^1_{(d,d)}(\gamma) = 4$, and if $d \geqslant 2$ then $\dim \mathcal{Q}^1_d(\gamma) = 4$.

Proof. It is clear that $u_i^2,\ v_i^2,\ u_i + v_{i+1} \in \mathcal{I}^1(\gamma)$. Not that if the number of faces around γ is 4, then $\mathfrak{a}(u_i) = a_2 u_t^2$. Thus $v_i - u_{i+1} = v_i - u_{i+1} - v_{i+1}\mathfrak{a}(u_{i+1}) - a_2 v_{i+1} u_{i+1}^2 \in \mathcal{I}^1(\gamma)$. It is easy to verify by an explicit computation that (5.8) generate $\mathcal{I}^1(\gamma)$.

Let $\succ$ be the elimination order $u_1 \succ v_1 \succ \cdots \succ u_4 \succ v_4$ on the polynomial ring $\mathcal{R}(\gamma)$. Note that for any subspace $W \subseteq \mathcal{R}(\gamma)$ such that $\{u_4v_4, u_4, v_4, 1\} \subseteq W$, the set $\mathfrak{r}(W)$ of reminders of W with respect to $\mathcal{I}^1(\gamma)$ is the subspace of $\mathcal{R}(\gamma)$ spanned by $\big\{u_4v_4, u_4, v_4, 1\big\}$. Since $\big\{u_4v_4, u_4, v_4, 1\big\} \subseteq \mathcal{T}_d(\gamma)$ for total degree $d \geqslant 2$ and for bidegree $d \geqslant 1$ we have $\{u_4v_4, u_4, v_4, 1\} \subseteq \mathcal{T}_{(d,d)}(\gamma)$, then the dimension of $\mathcal{Q}^1_d(\gamma)$ and $\mathcal{Q}^1_{(d,d)}(\gamma)$ is 4 for the correspondent values of d, which proves the statement. □

6. Explicit computations of the dimension of G^r-spline spaces

Following the notation in Section 5, in this section we also assume Δ is a 2-dimensional cell complex. We take a collection Φ of compatible G^r-algebraic transition maps on Δ, for some $r \geqslant 0$. We will focus on the homology terms $H_1(\mathcal{Q}^r_{d,\bullet})$ and $H_0(\mathcal{Q}^r_{d,\bullet})$, which appear in the dimension formula for $\dim G^r_d(\Delta, \Phi)$ given in (5.1). We compute $\dim H_1(\mathcal{Q}^r_{d,\bullet})$ and

$\dim H_0(\mathcal{Q}^r_{d,\bullet})$ in two particular cases of G^1-domains (Δ, Φ), namely when Δ consists of two faces in Section 6.1 and when Δ is a star of vertex in Section 6.2. The dimension formulas for the homology terms together with the Euler characteristic $\chi(\mathcal{Q}^r_{d,\bullet})$ computed in Section 5 lead us to to explicit dimension formulas of G^1-spline spaces in these two cases in Proposition 6.3 and Theorem 6.8, respectively.

First, notice that when Δ is a 2-dimensional cell complex, the following diagram commutes

$$\begin{array}{lccccc}
\mathcal{J}^r_{d,\bullet}\colon & 0 & \longrightarrow & \bigoplus_{\tau\in\Delta_1^\circ} \mathcal{J}^r_d(\tau) & \xrightarrow{\delta_1} & \bigoplus_{\gamma\in\Delta_0^\circ} \mathcal{J}^r_d(\gamma) \\
 & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}_{d,\bullet}\colon & \bigoplus_{\sigma\in\Delta_2} \mathcal{T}_d(\sigma) & \xrightarrow{\delta_2} & \bigoplus_{\sigma\supseteq\tau,\,\tau\in\Delta_1^\circ} \mathcal{T}_d(\sigma) & \xrightarrow{\delta_1} & \bigoplus_{\sigma\supseteq\gamma,\,\gamma\in\Delta_0^\circ} \mathcal{T}_d(\sigma) \\
 & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Q}^r_{d,\bullet}\colon & \bigoplus_{\sigma\in\Delta_2} \mathcal{Q}^r_d(\sigma) & \xrightarrow{\delta_2} & \bigoplus_{\tau\in\Delta_1^\circ} \mathcal{Q}^r_d(\tau) & \longrightarrow & \bigoplus_{\gamma\in\Delta_0^\circ} \mathcal{Q}^r_d(\gamma),
\end{array} \tag{6.1}$$

where

$$\delta_1\colon \bigoplus_{\tau\in\Delta_1^\circ} \mathcal{R}(\sigma) \to \bigoplus_{\gamma\in\Delta_0^\circ} \mathcal{R}(\gamma), \quad \text{and} \quad \delta_2\colon \bigoplus_{\sigma\in\Delta_2} \mathcal{R}(\sigma) \to \bigoplus_{\tau\in\Delta_1^\circ} \mathcal{R}(\tau)$$

are the differential maps defined in (2.8). By an abuse of notation, we write δ_i for its restriction to the subspaces in $\mathcal{T}_{d,\bullet}$ and $\mathcal{J}^r_{d,\bullet}$, and for the map it induces on the quotient spaces $\mathcal{Q}^r_{d,\bullet}$.

The short exact sequence of chain complexes of vector spaces

$$0 \to \mathcal{J}^r_{d,\bullet} \to \mathcal{T}_{d,\bullet} \to \mathcal{Q}^r_{d,\bullet} \to 0$$

leads us to the long exact sequence

$$\begin{aligned} 0 \to H_2(\mathcal{J}^r_{d,\bullet}) \to H_2(\mathcal{T}_{d,\bullet}) \to H_2(\mathcal{Q}^r_{d,\bullet}) \to H_1(\mathcal{J}^r_{d,\bullet}) \to H_1(\mathcal{T}_{d,\bullet}) \to \\ H_1(\mathcal{Q}^r_{d,\bullet}) \to H_0(\mathcal{J}^r_{d,\bullet}) \to H_0(\mathcal{T}_{d,\bullet}) \to H_0(\mathcal{Q}^r_{d,\bullet}) \to 0. \end{aligned} \tag{6.2}$$

Thus, we can study $\dim H_1(\mathcal{Q}^r_{d,\bullet})$ and $\dim H_0(\mathcal{Q}^r_{d,\bullet})$ by analyzing the terms in the sequence (6.2). We start with the homologies of $\mathcal{T}_{d,\bullet}$.

Remark 6.1. Notice that the commutativity of the diagram (6.1) and the exactness of the long sequence (6.2) hold if we replace the chain complexes $\mathcal{T}_{d,\bullet}$, $\mathcal{J}^r_{d,\bullet}$, $\mathcal{Q}^r_{d,\bullet}$ by their bidegree counterparts $\mathcal{T}_{(d,d),\bullet}$, $\mathcal{J}^r_{(d,d),\bullet}$, and $\mathcal{Q}^r_{(d,d),\bullet}$, for any $d \geqslant 0$. To simplify the exposition, we only introduce the notation in terms of the total degree case $G^r_d(\Delta, \Phi)$, but we apply the construction and give explicit formulas for each case $G^r_d(\Delta, \Phi)$ and $G^r_{(d,d)}(\Delta, \Phi)$ in the results and examples below.

For an integer $k \geqslant 0$ and a 2-face $\sigma \in \Delta_2$, we denote by $\mathcal{R}(\sigma)_k$ the set of homogeneous polynomials in $\mathcal{R}(\sigma)$ of degree k. We write $(\mathcal{R}_\bullet)_k$ for the chain complex of the k-graded pieces of $\mathcal{R}_\bullet$, and take $(\mathcal{T}_{d,\bullet})_k = \mathcal{T}_{d,\bullet} \cap (\mathcal{R}_\bullet)_k$. Using this notation we can rewrite $\mathcal{T}_{d,\bullet} = \bigoplus_{0\leqslant k\leqslant d} (\mathcal{T}_{d,\bullet})_k$.

Lemma 6.2. *If Δ is a 2-dimensional polyhedral complex with only one connected component and the number of 2-faces of Δ is strictly greater than 1, then*

$$H_i(\mathcal{T}_{d,\bullet})_0 = H_i\big(\Delta, \partial\Delta; \mathbb{R}\big),$$

for $i = 0, 1, 2$. Moreover, for any $k \geqslant 1$, we have

$$H_0(\mathcal{T}_{d,\bullet})_k = \bigoplus_{\sigma\in\Delta_2'} \mathcal{R}(\sigma)_k,$$

$$H_1(\mathcal{T}_{d,\bullet})_k = \bigoplus_{\sigma\in\Delta_2\setminus\Delta_2'} \mathcal{R}(\sigma)_k^{m(\sigma)-1}, \text{ and } H_2(\mathcal{T}_{d,\bullet})_k = 0,$$

where $\Delta_2' = \big\{\sigma \in \Delta_2 \colon \sigma \cap \partial\Delta = \emptyset\big\}$, and $m(\sigma)$ is the number of connected components of $\partial\Delta \cap \sigma$ for any $\sigma \in \Delta_2$. (We always assume that a 2-face $\sigma \in \Delta$ contains its edges and vertices.) In the bidegree case, the corresponding formulas for $H_i\big(\mathcal{T}_{(d,d),\bullet}\big)_k$ hold.

Proof. We only prove the formulas in the total degree case, the bidegree case follows by a similar reasoning. It is clear that

$$(\mathcal{T}_{d,\bullet})_0 \colon \bigoplus_{\sigma\in\Delta_2} \mathbb{R} \xrightarrow{\delta_2} \bigoplus_{\tau\in\Delta_1^\circ} \mathbb{R} \xrightarrow{\delta_1} \bigoplus_{\gamma\in\Delta_0^\circ} \mathbb{R},$$

and hence $H_i(\mathcal{T}_{d,\bullet})_0 = H_i(\Delta, \partial\Delta; \mathbb{R})$, for $i = 0, 1, 2$.

If $k \geqslant 1$, then we can rewrite $\mathcal{T}_{d,\bullet}$ as a direct sum $(\mathcal{T}_d(\tau))_k = \bigoplus_{\sigma\supseteq\tau} \mathcal{R}(\sigma)_k$ for every edge $\tau \in \Delta_1^\circ$, and $(\mathcal{T}_d(\gamma))_k = \bigoplus_{\sigma\supseteq\gamma} \mathcal{R}(\sigma)_k$ for every vertex $\gamma \in \Delta_0^\circ$. Thus, if $(\delta_i)_k$ denotes the boundary map in the chain complex $\big(\mathcal{T}_{d,\bullet}\big)_k$ then, we can rewrite it as the direct sum

$$(\delta_i)_k = \bigoplus_{\sigma\in\Delta_2} \big(\delta_i(\sigma)\big)_k,$$

where for each 2-face $\sigma \in \Delta_2$, the map $(\delta_i(\sigma))_k$ is the boundary map in the chain complex $\mathcal{R}_\bullet(\sigma)_k$ given by

$$\mathcal{R}_\bullet(\sigma)_k \colon \mathcal{R}(\sigma)_k \xrightarrow{\delta_2(\sigma)_k} \bigoplus_{\tau\subseteq\sigma\ \tau\in\Delta_1^\circ} \mathcal{R}(\sigma)_k \xrightarrow{\delta_1(\sigma)_k} \bigoplus_{\gamma\subseteq\sigma,\ \gamma\in\Delta_0^\circ} \mathcal{R}(\sigma)_k.$$

Therefore, we can write the chain complex $(\mathcal{T}_{d,\bullet})_k = \bigoplus_{\sigma\in\Delta_2} \mathcal{R}_\bullet(\sigma)_k$. Since

$$\mathcal{R}_\bullet(\sigma)_k = \mathcal{R}(\sigma)_k \otimes C_\bullet(\sigma, \partial\Delta \cap \sigma),$$

where $C_\bullet(\sigma, \partial\Delta \cap \sigma)$ is the quotient complex $C_\bullet(\sigma)/C_\bullet(\partial\Delta \cap \sigma)$ of the chain complexes of σ and $\partial\Delta \cap \sigma$, respectively. In particular, $H_i(\mathcal{T}_{d,\bullet})_k = \bigoplus_{\sigma\in\Delta_2} \mathcal{R}(\sigma)_k \otimes H_i(\sigma, \partial\Delta \cap \sigma)$. Thus, to compute $H_i(\mathcal{T}_{d,\bullet})_k$, we first consider $H_i(\sigma, \partial\Delta \cap \sigma)$. Notice that the sequence $0 \to C_\bullet(\partial\Delta \cap \sigma) \to C_\bullet(\sigma) \to C_\bullet(\sigma, \partial\Delta \cap \sigma) \to 0$ yields the long exact sequence of homologies

$$\begin{aligned} &0 \to H_2(\partial\Delta \cap \sigma) \to H_2(\sigma) \to H_2(\sigma, \partial\Delta \cap \sigma) \to H_1(\partial\Delta \cap \sigma) \to \\ &H_1(\sigma) \to H_1(\sigma, \partial\Delta \cap \sigma) \to H_0(\partial\Delta \cap \sigma) \to H_0(\sigma) \to H_0(\sigma, \partial\Delta \cap \sigma) \to 0\,. \end{aligned}$$

By hypothesis $\sigma \neq \Delta$, so

$$H_2(\sigma) = H_1(\sigma) = 0, \qquad H_0(\sigma) = \mathbb{Z},$$
$$H_2(\partial\Delta \cap \sigma) = H_1(\partial\Delta \cap \sigma) = 0, \qquad H_0(\partial\Delta \cap \sigma) = \mathbb{Z}^{m(\sigma)}.$$

From this we see that $H_2(\sigma, \partial\Delta \cap \sigma) = 0$, which implies $H_2(\mathcal{T}_{d,\bullet})_k = 0$. Furthermore, we can rewrite the last part of the sequence as

$$0 \to H_1(\sigma, \partial\Delta \cap \sigma) \to \mathbb{Z}^{m(\sigma)} \to \mathbb{Z} \to H_0(\sigma, \partial\Delta \cap \sigma) \to 0.$$

Note that $\sigma \in \Delta_2'$ if and only if $m(\sigma) = 0$, in which case $H_1(\sigma, \partial\Delta \cap \sigma) = 0$ and $H_0(\sigma, \partial\Delta \cap \sigma) = \mathbb{Z}$. On the other hand, if $\sigma \notin \Delta'$, or equivalently if $m(\sigma) \geqslant 1$, then the map $\mathbb{Z}^{m(\sigma)} \to \mathbb{Z}$ (defined by multiplication with $(1, \ldots, 1) \in \mathbb{Z}^{m(\sigma)}$) is surjective. Hence, for $\sigma \notin \Delta'$ we have $H_1(\sigma, \partial\Delta \cap \sigma) = \mathbb{Z}^{m(\sigma)-1}$ and $H_0(\sigma, \partial\Delta \cap \sigma) = 0$. Putting these two cases together leads to the formulas for $H_i(\mathcal{T}_{d,\bullet})_k$ for $i = 0, 1$, which concludes the proof. □

6.1. **G^r-splines on a cell complex composed of two patches.** If Δ is a 2-dimensional cell complex, and $\tau \in \Delta_1^\circ$ is an interior edge of Δ, then we define $\Delta_\tau \subseteq \Delta$ as the subcomplex composed of the two 2-faces $\sigma_1, \sigma_2 \in \Delta$ containing τ. As before, we write $\phi_{12} \colon \mathcal{R}(\sigma_1)/I_{\sigma_1}(\tau)^{r+1} \to \mathcal{R}(\sigma_2)/I_{\sigma_2}(\tau)^{r+1}$ for the algebraic transition map of τ from σ_1 to σ_2. The cell complex and ϕ_{12} define a 2-dimensional G^r-domain, which we denote by $\big(\Delta_\tau, \Phi\big)$. Since Δ_τ does not have interior vertices, the term associated to the 0-faces in (6.1) is zero i.e.,

$$\mathcal{T}_{d,0} = \mathcal{J}_{d,0}^r = \mathcal{Q}_{d,0}^r = 0,$$

and $H_0(\mathcal{T}_{d,\bullet}) = 0$. By Lemma 6.2, we have $H_1(\mathcal{T}_{d,\bullet})_0 = H_1\big(\Delta_\tau, \partial\Delta_\tau; \mathbb{R}\big) = 0$, and since the number of connected components $m(\sigma_i)$ of $\partial\Delta_\tau \cap \sigma_i$ is 1 for $i = 1, 2$, then $H_1(\mathcal{T}_{d,\bullet})_k = 0$ for every degree $k \geqslant 0$. Thus,

$$H_1(\mathcal{T}_{d,\bullet}) = \bigoplus_{k=0}^{\infty} H_1(\mathcal{T}_{d,\bullet})_k = 0,$$

and the long exact sequence (6.2) reduces to

$$H_1(\mathcal{Q}_{d,\bullet}^r) = H_0(\mathcal{J}_{d,\bullet}^r) = H_0(\mathcal{Q}_{d,\bullet}^r) = 0.$$

Therefore the dimension formula (5.1) of the G^r-spline space on Δ_τ reduces to

$$\dim G_d^r\big(\Delta_\tau, \Phi\big) = \dim \mathcal{T}_d(\sigma_1) + \dim \mathcal{T}_d(\sigma_2) - \dim \mathcal{Q}_d^r(\tau). \tag{6.3}$$

Notice that a similar reasoning yields the corresponding formulas in the bidegree case.

Proposition 6.3. *Let Δ_τ be as above, and suppose the algebraic transition map ϕ_{12} is given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$ as in (4.2). If $d_\mathfrak{a} = \deg \mathfrak{a} \geqslant 1$ then*

$$\dim G_d^1\big(\Delta_\tau, \Phi\big) = d^2 + d + 1 - d_\mathfrak{a}, \qquad \text{if } d \geqslant d_\mathfrak{a} + 1,$$

and

$$\dim G^1_{(d,d)}\big(\Delta_\tau,\Phi\big) = 2d^2+2d+1-d_{\mathfrak{a}}, \quad \textit{if } d \geqslant d_{\mathfrak{a}}.$$

If $\mathfrak{a}$ *is constant, and* $d \geqslant 0$*, then*

$$\dim G^1_d\big(\Delta_\tau,\Phi\big) = d^2+d+1, \ \textit{and} \ \ \dim G^1_{(d,d)}\big(\Delta_\tau,\Phi\big) = 2d^2+2d.$$

Proof. The case $\deg \mathfrak{a} \geqslant 1$ follows by replacing $\dim \mathcal{Q}^1_d(\tau)$ obtained in Lemma 5.1 in the dimension formula for $G^1_d(\Delta_\tau,\Phi)$ in (6.3). If $\mathfrak{a}$ is constant, the dimension formulas follow from Corollary 5.3. □

Remark 6.4. It is worth mentioning that the dimension formulas in Proposition 6.3 for $G^1_{(d,d)}(\Delta_\tau,\Phi)$ coincide with those in [45], where it was proved that

$$\dim G^1_{(d,d)}\big(\Delta_\tau,\Phi\big) = 2d(d+1) - \max\{\deg \mathfrak{b}, \deg \mathfrak{a} - 1\}.$$

In this paper, we only consider the case of symmetric gluing data $[\mathfrak{a},\mathfrak{b}]$, where $\mathfrak{b}$ is a constant polynomial as in Example 4.6.

Example 6.5 (G^1-splines on Δ_τ)**.** Let Δ be a 2-dimensional cell complex, and take $\Delta_\tau \subseteq \Delta$ a subcomplex composed of only two adjacent 2-faces $\sigma_1,\sigma_2 \in \Delta$ as above. Suppose the edge $\tau = \sigma_1 \cap \sigma_2$ has vertices $\gamma,\gamma' \in \Delta$ with valencies $s=3$ and $s=4$, respectively. We define an algebraic transition map ϕ_{12} from the symmetric gluing data $[\mathfrak{a},\mathfrak{b}]$ associated to the vertex valences at τ following the construction (4.2) in Example 4.6. We get

$$\begin{aligned}\phi_{12}\colon \mathbb{R}[u_1,v_1]/\langle u_1^2\rangle &\to \mathbb{R}[u_2,v_2]/\langle v_2^2\rangle\\ u_1 &\mapsto -v_2,\\ v_1 &\mapsto u_2 + v_2(-u_2^2+2u_2-1).\end{aligned}$$

Therefore, $\mathcal{I}^1(\tau) = \big\langle u_1+v_2,\ v_1-u_2+v_2(u_2^2-2u_2+1),\ u_1^2,\ v_2^2\big\rangle$. Hence, we take $\mathfrak{a}(u_2) = -u_2^2+2u_2-1$, and in particular $d_{\mathfrak{a}}=2$. By Lemma 5.1(a), replacing $\mathfrak{a}(u_2)$ in (5.3) we get a basis for $\mathcal{J}^1_d(\tau)$, and for $d \geqslant 3$, the dimension of the quotient space is $\dim \mathcal{Q}^1_d(\tau) = 2d+3$. If we replace $\mathcal{Q}^1_d(\tau)$ in (6.3), we get that for any $d \geqslant 3$, the dimension

$$\dim G^1_d\big(\Delta_\tau,\Phi\big) = 2\binom{d+2}{2} - (2d+3) = d^2+d-1.$$

In the bidegree (d,d) case, by Lemma 5.1(b), a basis of $\mathcal{J}^1_{(d,d)}(\tau)$ can be obtained by replacing $\mathfrak{a}(u_2)$ into the polynomials in (5.4), and if $d \geqslant 2$ we have $\dim \mathcal{Q}^1_{(d,d)}(\tau) = 2d+3$. Hence, the dimension formula for the G^1-spline space is

$$\begin{aligned}\dim G^1_{(d,d)}\big(\Delta_\tau,\Phi\big) &= \dim \mathcal{T}_{(d,d)}(\sigma_1) + \dim \mathcal{T}_{(d,d)}(\sigma_2) - \dim \mathcal{Q}^1_{(d,d)}(\tau)\\ &= 2(d+1)^2 - (2d+3) = 2d^2+2d-1,\end{aligned}$$

whenever $d \geqslant 2$. ◇

6.2. **G^r-splines on the star of a vertex.** If Δ is a 2-dimensional cell complex, and $\gamma \in \Delta_0^\circ$ is an interior vertex of Δ, then we define $\Delta_\gamma \subseteq \Delta$ as the subcomplex composed of all the faces of Δ containing γ. Recall that we always assume that a face of Δ contains all its faces. We assume that the number of 2-faces σ_t in Δ_γ is $s \geqslant 3$, and that $\sigma_t \cap \sigma_{t+1} = \tau_t \in \big(\Delta_\gamma\big)_1^\circ$ is an edge for every $t = 1, \dots, s$, where we take $\sigma_{s+1} = \sigma_1$.

Lemma 6.6. *If $\big(\Delta_\gamma, \Phi\big)$ is a G^r-domain for some $r \geqslant 0$, then the homology terms of the chain complexes $\mathcal{J}^r_{d,\bullet}$ and $\mathcal{Q}^r_{d,\bullet}$ of $\big(\Delta_\gamma, \Phi\big)$ in (6.1) satisfy $H_0(\mathcal{Q}^r_{d,\bullet}) = 0$ and $H_1(\mathcal{Q}^r_{d,\bullet}) = H_0(\mathcal{J}^r_{d,\bullet})$, for both total degree and bidegree case, for any degree $d \geqslant 0$.*

Proof. Since Δ_γ has only one component, then $\Delta'_\gamma = \Delta_\gamma \setminus \partial\Delta_\gamma = \emptyset$ and by Lemma 6.2, we have that $H_0(\mathcal{T}_{d,\bullet}) = H_1(\mathcal{T}_{d,\bullet}) = 0$ and $H_2(\mathcal{T}_{d,\bullet}) \cong \mathbb{R}$. Then, by the long exact sequence of homologies (6.2) we deduce that $H_0(\mathcal{Q}^r_{d,\bullet}) = 0$, and $H_1(\mathcal{Q}^r_{d,\bullet}) \cong H_0(\mathcal{J}^r_{d,\bullet})$. □

Recall that taking $r = 1$ in (2.12), we have that $\mathcal{J}^1_d(\gamma) = \mathcal{T}_d(\gamma) \cap \mathcal{I}^1(\gamma)$, where $\mathcal{T}_d(\gamma) = \big\{\sum_{t=1}^s f_t \colon f_t = f_{\sigma_t},\ \text{for } \sigma_i \in \big(\Delta_\gamma\big)_2\big\}$ as defined in (2.11).

Lemma 6.7. *Following the notation above, let (Δ_γ, Φ) be a G^1-domain, where Φ is a collection of G^1-algebraic transition maps given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$ (as in (4.2) in Example 4.6). If $\mathcal{J}^1_{d,\bullet}$ and $\mathcal{J}^1_{(d,d),\bullet}$ are the chain complexes of Δ_γ in total and bidegree, respectively, then $H_0\big(\mathcal{J}^1_{d,\bullet}\big) = 0$ for $d \geqslant 4$, and $H_0\big(\mathcal{J}^1_{(d,d),\bullet}\big) = 0$ for $d \geqslant 3$.*

Proof. Taking $r = 1$ and Δ_γ in (6.1), we have that $H_0(\mathcal{J}^1_{d,\bullet}) = \mathcal{J}^1_d(\gamma)/\mathrm{Im}\,(\delta_1)$. If s is the number of edges τ_i containing γ as above, then we need to show that δ_1 is surjective, where

$$\delta_1 \colon \bigoplus_{t=1}^{s} \mathcal{J}^1_d(\tau_t) \to \mathcal{J}^1_d(\gamma).$$

From Lemma 5.1(a) we know that if $d_{\mathfrak{a}} = \deg \mathfrak{a} \geqslant 1$ then

$$\mathcal{J}^1_d(\tau_t) = \mathrm{span}\begin{cases} u_t^i v_t^j,\ u_{t+1}^j v_{t+1}^i, & \text{for } 2 \leqslant i \leqslant d \text{ and } i + j \leqslant d, \\ u_t v_t^i + u_{t+1}^i v_{t+1}, & \text{for } 0 \leqslant i \leqslant d-1, \qquad (6.4a) \\ v_t^i - \big(u_{t+1}^i + i\, u_{t+1}^{i-1} v_{t+1}\, \mathfrak{a}(u_{t+1})\big), & \text{for } 1 \leqslant i \leqslant d - d_{\mathfrak{a}}. \qquad (6.4b) \end{cases}$$

On the other hand, $\mathcal{J}^1_d(\gamma) = \mathcal{T}_d(\gamma) \cap \mathcal{I}^1(\gamma)$ and we have two cases depending on the number s of edges at γ. If $s \neq 4$, by Lemma 5.4 we have that

$$\mathcal{I}^1(\gamma) = \mathrm{span}\big\{u_t + v_{t+1},\ v_t - u_{t+1} - a_0 v_{t+1},\ u_t^2,\ u_t v_t,\ v_t^2 \colon t = 1, \dots, s\big\},$$

where a_0 is the constant term in $\mathfrak{a}(u) = \sum_{k=0}^{d_{\mathfrak{a}}} a_k u^k$. If $s = 4$, Lemma 5.5 implies

$$\mathcal{I}^1(\gamma) = \mathrm{span}\big\{u_t + v_{t+1},\ v_t - u_{t+1},\ u_t^2,\ v_t^2 \colon t = 1, 2, 3, 4\big\}.$$

Notice the generators of $\mathcal{I}^1(\gamma)$ are polynomials in $\mathcal{T}_d(\gamma)$, hence they generate $\mathcal{J}^1_d(\gamma)$, and it is enough to check that they all are in Im (δ_1).

Notice that $u_t^i v_t^j \in \text{Im }(\delta_1)$, for every $t = 1, \dots, s$ whenever $i \geqslant 2$ or $j \geqslant 2$; taking $i = 0$ in (6.4a), we see that $u_t + v_{t+1} \in \text{Im }(\delta_1)$.

By hypothesis $d \geqslant 4 \geqslant d_{\mathfrak{a}} + 2$, hence $d - d_{\mathfrak{a}} \geqslant 2$. Thus, if $s \neq 4$, we have that $v_t^2 - \big(u_{t+1}^2 + 2u_{t+1}v_{t+1}\mathfrak{a}(u_{t+1})\big) \in \text{Im }(\delta_1)$. But $v_t^2,\ u_{t+1}^2 \in \text{Im }(\delta_1)$, so $2u_{t+1}v_{t+1}\mathfrak{a}(u_{t+1}) = 2u_{t+1}v_{t+1}(a_0 + a_1u_{t+1} + a_2u_{t+1}^2) \in \text{Im }(\delta_1)$. Since $d \geqslant 4$, then $u_t^2v_t,\ u_t^3v_t \in \text{Im }(\delta_1)$. This implies $u_tv_t \in \text{Im }(\delta_1)$, and taking $i = 1$ in (6.4b) we see that also $v_t - u_{t+1} - a_0v_{t+1} \in \text{Im }(\delta_1)$. It remains to verify that $v_t - u_{t+1} \in \text{Im }(\delta_1)$ if $s = 4$. But in that case $\mathfrak{a}(u_{t+1}) = a_2u_{t+1}^2$. Hence, $u_tv_t^2 \in \text{Im }(\delta_1)$ implies $v_t - u_{t+1} \in \text{Im }(\delta_1)$. This proves that $H_0\big(\mathcal{J}^1_{d,\bullet}\big) = 0$.

By using Lemma 5.1(b) and the corresponding results for the bidegree case (d, d) in Lemma 5.4 and 5.5, an analogous argument implies $H_0\big(\mathcal{J}^1_{(d,d),\bullet}\big) = 0$ for any $d \geqslant 3$. □

Theorem 6.8. *For a G^r-domain $\big(\Delta_\gamma, \Phi\big)$, the dimension of the G^r spline space is given by*

$$\dim G^r_d\big(\Delta_\gamma, \Phi\big) = \chi(\mathcal{Q}^r_{d,\bullet}) + \dim H_1(\mathcal{Q}^r_{d,\bullet}). \tag{6.5}$$

In particular, $\chi(\mathcal{Q}^r_{d,\bullet})$ is a lower bound of $\dim G^r_d((\Delta, \Phi)_\gamma)$. The corresponding formula follows for $G^r_{(d,d)}\big(\Delta_\gamma, \Phi\big)$ in the bidegree case. If in addition $r = 1$ and Φ is given by symmetric gluing data $[\mathfrak{a}, \mathfrak{b}]$, then

$$\begin{aligned} \dim G^1_d\big(\Delta_\gamma, \Phi\big) &= \chi(\mathcal{Q}^1_{d,\bullet}), \quad \textit{for } d \geqslant 4 \quad \textit{and}, \\ \dim G^1_{(d,d)}\big(\Delta_\gamma, \Phi\big) &= \chi(\mathcal{Q}^1_{(d,d),\bullet}), \quad \textit{for } d \geqslant 3. \end{aligned} \tag{6.6}$$

If $d_{\mathfrak{a}} = \deg \mathfrak{a}$, then

$$\chi(\mathcal{Q}^1_{d,\bullet}) = \begin{cases} s\binom{d+2}{2} - s(2d + d_{\mathfrak{a}} + 1) + 3, & \textit{if } s \neq 4, \textit{ and } d \geqslant d_{\mathfrak{a}} + 1, \quad (6.7a) \\ 2(d^2 - d - 2), & \textit{if } s = 4,\ \mathfrak{a} \neq 0, \textit{ and } d \geqslant 2, \quad (6.7b) \\ 2(d^2 - d + 2), & \textit{if } s = 4,\ \mathfrak{a} = 0, \textit{ and } d \geqslant 2, \quad (6.7c) \end{cases}$$

and

$$\chi\big(\mathcal{Q}^1_{(d,d),\bullet}\big) = \begin{cases} s(d+1)^2 - s(2d + d_{\mathfrak{a}} + 1) + 3, & \textit{if } s \neq 4, \textit{ and } d \geqslant d_{\mathfrak{a}}, \\ 4(d^2 - 1), & \textit{if } s = 4,\ \mathfrak{a} \neq 0, \textit{ and } d \geqslant 1, \\ 4d^2, & \textit{if } s = 4,\ \mathfrak{a} = 0, \textit{ and } d \geqslant 1, \end{cases}$$

where s is the number of 2-faces in Δ_γ.

Proof. From (5.1) we know that $\dim G^r_d(\Delta, \Phi) = \chi(\mathcal{Q}^r_{d,\bullet}) + \dim H_1(\mathcal{Q}^r_{d,\bullet}) - \dim H_0(\mathcal{Q}^r_{d,\bullet})$. By Lemma 6.6, $\dim H_0(\mathcal{Q}^r_{d,\bullet}) = 0$. This implies (6.5). In addition, if $r = 1$ and Φ is given by gluing data, then by Lemma 6.7 in the total degree case gives $\dim H_1(\mathcal{Q}^r_{d,\bullet}) = 0$ for every $d \geqslant 4$, which implies (6.6). The bidegree case follows similarly.

The explicit formulas for $\chi\big(\mathcal{Q}^1_{d,\bullet}\big)$ follow from (5.2). In the case of a vertex star with s faces $\sigma_t \in (\Delta_\gamma)_2$ and s interior edges τ_t, (5.2) can be rewritten

as

$$\chi\big(\mathcal{Q}^1_{d,\bullet}\big) = \sum_{t=1}^{s} \dim \mathcal{Q}^1_d(\sigma_t) - \sum_{t=1}^{s} \dim \mathcal{Q}^1_d(\tau_t) + \dim \mathcal{Q}^1_d(\gamma).$$

By construction, $\dim \mathcal{Q}^1_d(\sigma_t) = \binom{d+2}{2}$ for each $t = 1, \dots, s$. By Lemma 5.1, if $d \geqslant d_{\mathfrak{a}} + 1 \geqslant 2$ then $\dim \mathcal{Q}^1_d(\tau) = 2d + d_{\mathfrak{a}} + 1$ for each interior edge τ in Δ_γ, and therefore $\sum_{t=1}^{s} \mathcal{Q}^1_d(\tau_t) = s(2d + d_{\mathfrak{a}} + 1)$. Furthermore, from Lemma 5.4, if $s \neq 4$ and $d \geqslant 1$, then by (5.7) we have $\dim \mathcal{Q}^1_d(\gamma) = \dim \mathcal{Q}^1_{(d,d)}(\gamma) = 3$. Note that $s \neq 4$ implies that $d_{\mathfrak{a}} \geqslant 1$. This proves (6.7a).

Assume now $s = 4$. If $d \geqslant 2$, Lemma 5.5 implies $\dim \mathcal{Q}^1_d(\gamma) = 4$. Additionally, for any edge $\tau_t = \sigma_t \cap \sigma_{t+1} \in (\Delta_\gamma)^\circ_1$, we can write the gluing data as in (4.3), and we get $\mathfrak{a}(u_{t+1}) = -2\cos\left(\frac{2\pi}{s'_t}\right) u^2_{t+1}$, where s'_t is the valence of the vertex $\gamma'_t \neq \gamma$ of τ_t in Δ.

Then we have two cases, either $s' \neq 4$ and $d_{\mathfrak{a}} = 2$, or $s' = 4$ and $\mathfrak{a} = 0$. If $d_{\mathfrak{a}} = 2$, then Lemma 5.1 implies $\dim \mathcal{Q}^1_d(\tau_t) = 2d + 3$ for every $d \geqslant 3$. In the second case $\mathfrak{a} = 0$, by Corollary 5.3 we know that $\dim \mathcal{Q}^1_d(\tau_t) = 2d + 1$ for every $d \geqslant 0$. These two cases imply (6.7b) and (6.7c), respectively. The corresponding results in the bidegree case lead to the formulas for $\chi\big(\mathcal{Q}^1_{(d,d)}, \bullet\big)$. □

Example 6.9. Take Δ_γ as the 2-dimensional cell complex in Figure 4, which is composed of $s = 3$ quadrilaterals σ_i sharing a common vertex γ. Let Φ be the collection of algebraic transition maps obtain from taking symmetric gluing data along each of the edges $\tau_i = [\gamma, \delta_i]$ of Δ_γ. In this case, the valence of the vertex γ is 3, and we will assume the valence at each vertex δ_i is 4. If we take the gluing data $\mathfrak{a}$ as the gluing data in (4.3), we see that $\mathfrak{a} \neq 0$ and it is a polynomial of degree $d_{\mathfrak{a}} = 2$. Then, Theorem 6.8 implies

$$\dim G^1_d\big(\Delta_\gamma, \Phi\big) = \frac{3}{2}(d^2 - d - 2),$$

for every $d \geqslant 4$, and in the bidegree case

$$\dim G^1_{(d,d)}\big(\Delta_\gamma, \Phi\big) = 3d^2 - 3,$$

for every $d \geqslant 3$. ◇

6.3. **A general 2-dimensional G^r-domain.** For a general 2-dimensional cell complex Δ, we can use (5.1) and the formulas for the Euler characteristic equations from Theorem 6.8 to estimate the dimension of a $G^r_d(\Delta, \Phi)$ spline space. Using formula (5.1), we may estimate the dimension of $G^r_d(\Delta, \Phi)$ by the Euler characteristic $\chi(\mathcal{Q}^r_{d,\bullet})$. Controlling the homology terms in (5.1) is not straightforward, as shown by the following example where the G^r-domain cannot be topologically embedded into the plane. We call it a *cube*, because the polyhedral cell complex Δ we consider is homeomorphic to the boundary of a cube: it has 6 square faces, 12 edges and 8 vertices. The formula (5.1) still works in this case. However, it is not necessary that

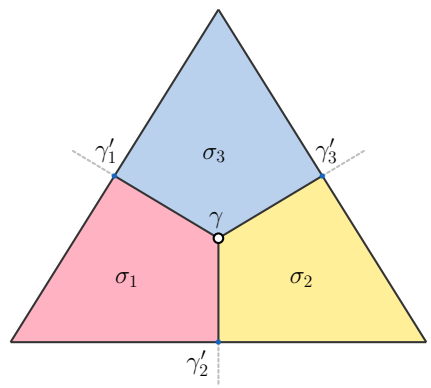


FIGURE 4. 2-dimensional cell complex considered in Example 6.9, which is the star of the vertex γ. The valence of γ is 3, and we consider valence 4 at each vertex γ_i, for $i = 1, 2, 3$.

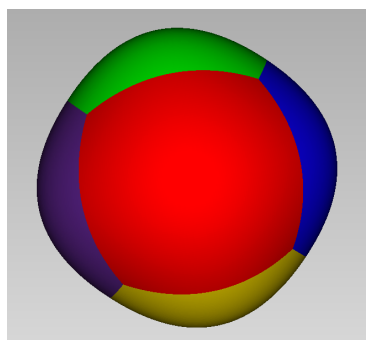


FIGURE 5. A surface parametrized by degree $(2, 2)$ G^1-splines over the cube for the gluing data $\mathfrak{a}(u) = 2u - 1, \mathfrak{b} = -1$. See Example 6.10. The surface is interpolating the points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ at the center of 6 faces.

$\dim G_d^r(\Delta, \Phi) = \chi(\mathcal{Q}_{d,\bullet}^r)$, even for $r = 1$ and Φ given by symmetric gluing data.

Example 6.10. We take Δ as a cell complex homeomorphic to the surface of a cube, so Δ is composed of six 2-dimensional faces, 12 interior edges, and 8 vertices, each of valence $s = 3$. Following the notation in (4.3), the G^1-symmetric gluing data at every edge of Δ is given by

$$\mathfrak{a}(u) = -2u + 1, \ \mathfrak{b} = -1.$$

In particular, the degree of $\mathfrak{a}$ is $d_{\mathfrak{a}} = 1$. From (5.1) we know that the dimension of the space is $\dim G_d^1(\Delta, \Phi) = \chi(\mathcal{Q}_{d,\bullet}^1) + \dim H_1(\mathcal{Q}_{d,\bullet}^1) - \dim H_0(\mathcal{Q}_{d,\bullet}^1)$,

and (5.2) gives

$$\chi(\mathcal{Q}^1_{d,\bullet}) = \sum_{\sigma\in\Delta_2} \dim \mathcal{Q}^1_d(\sigma) - \sum_{\tau\in\Delta_1^\circ} \dim \mathcal{Q}^1_d(\tau) + \sum_{\gamma\in\Delta_0^\circ} \dim \mathcal{Q}^1_d(\gamma).$$

By Lemma 5.1, for each $\tau \in \Delta_1^\circ$ we have

$$\begin{cases} \dim \mathcal{Q}^1_d(\tau) = 2d+2, & \text{for } d \geqslant 2, \\ \dim \mathcal{Q}^1_{(d,d)}(\tau) = 2d+2, & \text{for } d \geqslant 1, \end{cases}$$

and for each vertex $\gamma \in \Delta_0^\circ$,Lemma 5.4 implies

$$\begin{cases} \dim \mathcal{Q}^1_d(\gamma) = 3, & \text{for } d \geqslant 3, \\ \dim \mathcal{Q}^1_{(d,d)}(\gamma) = 3, & \text{for } d \geqslant 2. \end{cases}$$

Hence,

$$\begin{cases} \chi(\mathcal{Q}^1_{d,\bullet}) = 3(d^2-5d+2), & \text{for } d \geqslant 3, \\ \chi(\mathcal{Q}^1_{(d,d),\bullet}) = 6(d-1)^2, & \text{for } d \geqslant 2. \end{cases}$$

In Table 1, we present the exact dimension of the space $G^1_d(\Delta,\Phi)$, which we independently compute using two methods: the implementation of Algorithm 1[1] in `Macaulay2` [26], and the JULIA package `G1Splines.jl`[2]. The computations in `Macaulay2` [26] follow the basis construction described in Algorithm 1, while `G1Splines.jl` uses the Bernstein representation of splines and the linear relations deduced from sample points on the edges of the partition.

These computations show that the difference between the exact dimension of the spline space and the Euler characteristic remains constant. In the total degree case we have

$$\dim G^1_d(\Delta,\Phi) - \chi(\mathcal{Q}^1_{d,\bullet}) = 12, \quad \text{for } 4 \leqslant d \leqslant 10,$$

and in the bidegree case

$$\dim G^1_{(d,d)}(\Delta,\Phi) = \chi(\mathcal{Q}^1_{(d,d),\bullet}), \quad \text{for } 2 \leqslant d \leqslant 10.$$

This dimension formula in the bidegree case was proved in [45, Theorem 6.3] for any $d \geqslant 1$.

We include a parametric surface of degree $(2,2)$ on the cube in Figure 5. The coordinate functions are in the spline space $G^1_{(2,2)}(\Delta,\Phi)$ of dimension 6. They are computed so that the parametric surface interpolates the points $(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)$ at the center of the faces of the cube. $\diamond$

[1] `https://github.com/yuanbeihui05/demos/tree/GCDemos`

[2] `http://github.com/AlgebraicGeometricModeling/GSplines.jl`

d	1	2	3	4	5	6	7	8	9	10
$\dim G^1_d(\Delta,\Phi)$	1	1	1	6	18	36	60	90	126	168
$\chi(\mathcal{Q}^1_{d,\bullet})$	-6	-12	-12	-6	6	24	48	78	114	156
$\dim G^1_{(d,d)}(\Delta,\Phi)$	1	6	24	54	96	150	216	294	384	486
$\chi(\mathcal{Q}^1_{(d,d),\bullet})$	0	6	24	54	96	150	216	294	384	486

TABLE 1. Dimension of the G^1-spline spaces from Example 6.10, where the 2-dimensional cell complex Δ is homeomorphic to the surface of a cube, and the collection Φ of G^1-algebraic transition maps is defined from symmetric gluing data. For a given d, the space $G^1_d(\Delta,\Phi)$ is the linear space of G^1-splines on Δ of total degree $\leqslant d$, and $G^1_{(d,d)}(\Delta,\Phi)$ is the space of splines of bidegree $\leqslant (d,d)$. For small d, the Euler characteristics $\chi(\mathcal{Q}^1_{d,\bullet})$ and $\chi(\mathcal{Q}^1_{(d,d),\bullet})$ are computed in `Macaulay2` [26].

7. Concluding remarks

We have made a theoretical and algorithmic contribution to the study of geometrically continuous splines by developing an algebraic framework that extends Billera's algebraic methods for parametric splines. This framework has two advantages. First, we can use algebraic tools to study geometrically continuous splines. In fact, in this paper, we demonstrate this by applying chain complexes and the division algorithm to construct bases and compute dimensions for G^r-spline spaces. Second, G^r-spline spaces have rich algebraic structures that can be further explored using this framework. Thus, this framework makes it easy for algebraists to explore the exciting research topic of geometrically continuous splines.

Extending the study from parametric splines to geometrically continuous splines enables us to construct shapes with more complex topology. For example, Example 6.10 in Section 6.3 is one such shape that cannot be constructed within the scope of parametric splines. However, this flexibility comes at a cost: we need a more complicated algebraic model to describe the spline space. While parametric spline spaces have only one coordinate system for the whole domain, G^r-spline require several coordinate systems, one for each face. As a result, transition maps are needed to define the smoothness condition between different coordinate systems. In Section 4, we have several examples of G^r-domains having non-trivial transition maps which fit in our framework. Hence, unlike in the case of parametric splines, the G^r-spline space $G^r(\Delta,\Phi)$ is not always a finitely generated module over a known polynomial ring. In fact, it is not even clear if $G^r(\Delta,\Phi)$ is finitely generated as an $\mathbb{R}$-algebra in general. This will be of interest for future research.

We have obtained explicit dimension formulas for G^r-spline spaces over a two-patches configuration and a star of vertex in Section 6. To obtain these formulas, we have written the dimension of $G^r_d(\Delta, \Phi)$ into the sum of Euler characteristic $\chi(\mathcal{Q}^r_{d,\bullet})$ and homologies of the chain complex $\mathcal{Q}^r_{d,\bullet}$, as defined in Section 2.3. Because the transition map does not always preserve degrees between polynomial rings, finding $\chi(\mathcal{Q}^r_{d,\bullet})$ and homologies of $\mathcal{Q}^r_{d,\bullet}$ is difficult. In fact, the whole Section 5 is devoted in computing $\chi(\mathcal{Q}^r_{d,\bullet})$ for certain kinds of transition maps, where the division algorithm plays an important role in the computation. It may be of interest for future studies to find dimension formulas which work more generally.

The method used for the construction of the G^r-basis in Section 2.4, particularly Lemma 2.21, applies to a more general setting and provides a general method to generate certain subspaces of a quotient ring. We have implemented Algorithm 1 in `Macaulay2` [26] computation in Example 6.10. It is of our interest to develop a software package based on this Algorithm 1, which computes a basis for G^r-spline spaces over any given G^r-domain.

One issue that remains to be solved is the gluing data in this paper are assumed to be polynomials, while they may be in the form of rational functions as defined in the context of topological surfaces in [45]. We are confident that our framework can be extended in the future to cover those cases where transition maps are given by rational gluing data.

Appendix A. The chain complex for G^r-splines

For the reader's convenience, we begin by recalling some key concepts from algebraic topology, which can be found in [32, Chapter 2]. By a *chain complex*, we mean a sequence of homomorphisms of Abelian groups:

$$C_\bullet\colon \cdots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \to \dots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for each $i \in \mathbb{Z}$. The group homomorphisms ∂_i of $C_\bullet$ are called the *boundary maps*, or *differentials*, of the chain complex $C_\bullet$. The *homologies* of the chain complex $C_\bullet$ are the quotient groups

$$H_i(C_\bullet) = \ker \partial_i / \mathrm{Im}\ \partial_{i+1}.$$

If Δ is an n-dimensional simplicial complex, we can define a chain complex as follows. For $i = 0, \dots, n$, we define $C_i(\Delta)$ as the free Abelian group generated by the i-dimensional faces $\alpha \in \Delta_i$. We take $C_i(\Delta) = 0$, for any other index i. If $\alpha \in \Delta_i$ is a simplex with vertices $v_0, \dots, v_i$, we write $\alpha = [v_0, \dots, v_i]$, and define

$$\partial_i(\alpha) = \sum_{j=0}^{i} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i].$$

We extend the map ∂_i linearly to $C_i(\Delta)$, which gives us the boundary map $\partial_i\colon C_i(\Delta) \to C_{i-1}(\Delta)$. One may check that the sequence of boundary maps satisfies $\partial_i \circ \partial_{i+1} = 0$, for $i = 0, \dots, n-1$, and therefore, the collection

$C_i(\Delta)$ together with the maps ∂_i forms a chain complex, which we denote by $C_\bullet(\Delta)$. Clearly, the maps ∂_i can be restricted to the boundary $\partial\Delta$ of Δ, and so $C_\bullet(\partial\Delta)$ is also a chain complex, and a subcomplex of $C_\bullet(\Delta)$. The quotient $C_\bullet(\Delta)/C_\bullet(\partial\Delta)$ is called the complex of Δ relative to its boundary, and it is denoted $C_\bullet(\Delta, \partial\Delta)$.

We now define a chain complex associated to a spline space over a simplicial complex. This complex was introduced by Billera in [8] and refined by Schenck and Stillman [57]. We call it the Billera-Schenck-Stillman spline complex. We follow the notation in Section 2.2. Let Δ be an n-dimensional simplicial complex, take $S = \mathbb{R}[x_1, \dots, x_n]$, and write

$$S_\bullet = C_\bullet(\Delta, \partial\Delta) \otimes_{\mathbb{R}} S,$$

where $C_\bullet(\Delta, \partial\Delta)$ is the chain complex of Δ relative to its boundary $\partial\Delta$. For each interior face $\tau \in \Delta^\circ_{n-1}$, we denote by $\ell_\tau \in S$ a choice of a degree one polynomial vanishing on τ.

We define $J(\tau) = \langle \ell_\tau^{r+1} \rangle \subseteq S$, the ideal generated by ℓ_τ^{r+1}. If $\beta \in \Delta_k$, for $k = 0, \dots, n-2$, we define

$$J(\beta) = \sum_{\tau \supseteq \beta} J(\tau) = \sum_{\tau \supseteq \beta} \langle \ell_\tau^{r+1} \rangle.$$

If $\sigma \in \Delta_n$ we take $J(\sigma) = 0$. The ideals $J(\beta)$ define a sub-complex $I_\bullet$ of $S_\bullet$, where the boundary maps ∂_i are the restriction of the differentials of $S_\bullet$:

$$I_\bullet \colon 0 \to \bigoplus_{\sigma \in \Delta_n} J(\sigma) \xrightarrow{\partial_n} \bigoplus_{\tau \in \Delta^\circ_{n-1}} J(\tau) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta^\circ_0} J(\gamma) \to 0.$$

The *Billera-Schenck-Stillman spline complex* is the quotient $S_\bullet/I_\bullet$, namely

$$S_\bullet/I_\bullet \colon 0 \to \bigoplus_{\sigma \in \Delta_n} S(\sigma) \xrightarrow{\bar\partial_n} \bigoplus_{\tau \in \Delta^\circ_{n-1}} S/J(\tau) \xrightarrow{\bar\partial_{n-1}} \dots \xrightarrow{\bar\partial_1} \bigoplus_{\gamma \in \Delta^\circ_0} S/J(\gamma) \to 0,$$

where $S(\beta) = S$ for each face $\beta \in \Delta$.

Given a G^r-domain (Δ, Φ), for n-faces $\sigma_1, \sigma_2 \in \Delta_n$ such that $\tau = \sigma_1 \cap \sigma_2 \in \Delta_{n-1}$, a transition map $\phi_{12} \in \Phi$ from σ_1 to σ_2 as defined in (2.1) is an *identity* if $u_{1j} \mapsto u_{2j}$ for every $j = 1, \dots, n$, where $X(\sigma_i)$ is the system of coordinates of σ_i and $\mathcal{R}(\sigma_i)$ is the polynomial ring in the coordinates $(u_{i1}, \dots, u_{in})$ of $X(\sigma_i)$.

Proposition A.1. *Let (Δ, Φ) be a G^r-domain, where Δ is an n-dimensional simplicial complex and all transition maps in Φ are identities, then the chain complex $(\mathcal{F}^r_\bullet, \delta_\bullet)$ defined by* (2.6) *and* (2.8) *in Section 2.2 is isomorphic to the Billera-Schenck-Stillman spline complex $S_\bullet/I_\bullet$.*

Proof. If for all interior faces $\tau \in \Delta^\circ_{n-1}$ the transition maps in the (Δ, Φ) are identities, then the polynomial ring

$$\mathcal{R}(\sigma) \cong S,$$

for every $\sigma \in \Delta_n$. In particular, if $\tau = \sigma_1 \cap \sigma_2$ for $\sigma_i \in \Delta_n$, then (2.2) takes the form

$$\mathcal{R}(\tau) \cong S \otimes_{\mathbb{R}} S,$$

and (2.4) can be rewritten as

$$\mathcal{I}^r(\tau) = \big\langle u_{11} - u_{21}, \dots,\ u_{1n} - u_{2n} \big\rangle + I_{\sigma_1}(\tau)^{r+1} \cdot \mathcal{R}(\tau) + I_{\sigma_2}(\tau)^{r+1} \cdot \mathcal{R}(\tau),$$

where $I_{\sigma_i}(\tau) = \big\langle \ell_\tau(u_{i1}, \dots, u_{in}) \big\rangle \subseteq \mathcal{R}(\sigma_i)$, for $i = 1, 2$. Take $J(\tau) = \big\langle \ell_\tau(x_1, \dots, x_n)^{r+1} \big\rangle \subseteq S$. It is clear that the homomorphism $\mathcal{R}(\tau) \to S$ given by $u_{ij} \mapsto x_j$, for $i = 1, 2$ and $j = 1, \dots, n$, induces an isomorphism $\mathcal{F}^r(\tau) \cong S/J(\tau)$. It is also easy to check that the diagram

$$\begin{array}{ccc} \bigoplus_{\sigma \in \Delta_n} \mathcal{F}^{r+1}(\sigma) & \xrightarrow{\delta_n} & \bigoplus_{\tau \in \Delta^\circ_{n-1}} \mathcal{F}^{r+1}(\tau) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{\sigma \in \Delta_n} S & \xrightarrow{\partial_n} & \bigoplus_{\tau \in \Delta^\circ_{n-1}} S/J(\tau) \end{array}$$

commutes. Similarly, for every $\alpha \in \Delta^\circ_k$, for $k = 0, \dots, n-2$, the homomorphism $\mathcal{R}(\alpha) \to S$, given by $u_{\sigma j} \mapsto x_j$ where $(u_{\sigma 1}, \dots, u_{\sigma n})$ are the coordinates of $X(\sigma)$ for every $\sigma \supseteq \alpha$, induces an isomorphism $\mathcal{F}^r(\alpha) \cong S/J(\alpha)$. Hence, also in this case the diagram

$$\begin{array}{ccc} \bigoplus_{\beta \in \Delta^\circ_{k+1}} \mathcal{F}^{r+1}(\beta) & \xrightarrow{\delta_{k+1}} & \bigoplus_{\alpha \in \Delta^\circ_k} \mathcal{F}^{r+1}(\alpha) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{\beta \in \Delta^\circ_{k+1}} S & \xrightarrow{\partial_{k+1}} & \bigoplus_{\alpha \in \Delta^\circ_k} S/J(\alpha) \end{array}$$

commutes. It follows that $\mathcal{F}^r_\bullet \cong S_\bullet / I_\bullet$ as chain complexes. □

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