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Optimal periodic strategies with dividends payable from gains only

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ABSTRACT

In this paper, we consider the compound Poisson insurance risk model and analyze the optimal dividend strategy (that maximizes the expected present value of dividend payments until ruin) when dividends can only be paid periodically as lump sums. If one makes the usual assumption that dividends can be paid from the available surplus, then the optimal strategies are often of band or barrier type, resulting in a ruin probability of one (e.g. Albrecher et al. (2011a)). As opposed to such an assumption, we propose that dividends can only be paid from a certain fraction of the gains (i.e. positive increment of the process between successive dividend decision times), and such a constraint allows the surplus process to have a positive survival probability. Some theoretical properties of the value function and the optimal strategy are derived in connection to the Bellman equation. These properties suggest that a bang-bang type of control can be a candidate for the optimal strategy, where dividend is paid at the highest possible amount as long as the surplus is high enough. The dividend function under the candidate strategy is subsequently derived under exponential inter-observation times and claims with a rational Laplace transform, and we also provide specific numerical examples with (mixed) exponential claims where the proposed strategy is optimal in such cases.

1. Introduction

Dividend payout strategies are a crucial component of insurance risk management, balancing the competing interests of policyholders and shareholders. While existing models often implicitly assume continuous dividend decisions due to their theoretical tractability under a continuous-time stochastic process, insurers operate under periodic decision schedules in the real world. Unlike previous models where dividends can be paid from the entire surplus, in this paper we introduce a novel constraint: dividends can only be paid from a fraction of the gains. This allows for a positive survival probability and aligns better with real-world insurance practices; otherwise, unrestricted dividend payments could lead to premature ruin.

To begin, we define the classical compound Poisson risk model $\{S_t\}_{t\geq 0}$, which describes the surplus evolution of an insurance company over time via

$$S_t = u + ct - \sum_{i=1}^{N_t} X_i, \qquad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus, and $c > 0$ is the premium rate per unit time. In addition, the claim number process $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$, and the claim amounts $\{X_i\}_{i=1}^{\infty}$ form a sequence of independent and identically distributed (i.i.d.) random variables with common density p and Laplace transform $\hat{p}$. It is further assumed that $\{N_t\}_{t\geq 0}$ and $\{X_i\}_{i=1}^{\infty}$ are mutually independent. The positive security loading condition of the model is given by $c > \lambda\mathbb{E}[X]$ with X being a generic claim amount. For later use, we will denote the set of real numbers by $\mathbb{R}$, the set of non-negative real numbers by $\mathbb{R}^+$, the set of negative real numbers by $\mathbb{R}^-$, the set of non-negative integers by $\mathbb{N}_0$, and the set of positive integers by $\mathbb{N}$.

The risk process (1.1) is often the baseline model where various modifications are considered by different researchers. In particular, the importance of dividend payout in insurance risk models was discussed by de Finetti (1957). See also Albrecher and Thonhauser (2009) and Avanzi (2009) for comprehensive reviews. The expected discounted dividends payable to the shareholders can indeed be regarded as the value of firm in corporate finance, and this present value can be a quantity that the company tries to maximize. In such an optimization problem, there is

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always a trade-off between (i) paying more dividends at earlier time points to take advantage of the time value of money (but risking earlier cessation of dividends due to possible early ruin) and (ii) paying dividends in a more sustainable manner over a longer time horizon. Much of the research concerning continuous-time risk processes with dividends was conducted under the assumption that the risk process is observed continuously and dividend is paid immediately once certain criteria are met (e.g. surplus reaching a certain level). Specifically, if dividend can be paid from the available surplus then the optimal dividend strategy is often a band strategy (e.g. Gerber (1969)). In particular, for a spectrally negative Lévy risk process, Loeffen (2008) showed that if the Lévy measure has a completely monotone density then this collapses to a barrier strategy. However, the ruin probability is one under the aforementioned band or barrier type strategy. To avoid such an undesirable consequence, a number of alternatives and related models have been proposed in the literature. For example, Avram et al. (2007) suggested a model with bail-out such that the beneficiaries of the dividend payments need to inject the necessary capital to keep the insurance company alive, and they showed that a double-barrier strategy (also known as a doubly reflected process) can be optimal in maximizing the difference between the expected present values of dividend payments and capital injections. See also e.g. Yao et al. (2011) for similar analysis in the dual risk model. Another way to allow for positive survival probability is to limit the amount of dividend paid by considering absolutely continuous dividend strategies and restricting the dividend rate (such that the surplus process still has a positive trend in the long run). In general, the optimal strategy under such assumptions has a band structure as well (see Azcue and Muler (2012)). When the Lévy measure has a completely monotone density, Kyprianou et al. (2012) proved that the optimal strategy is the one that pays dividend at the ceiling rate whenever the surplus process is above a certain level, and this is commonly known as a threshold strategy or a refracted process (e.g. Gerber and Shiu (2006), and Lin and Pavlova (2006)). As shown by Junca et al. (2019), the optimality of a threshold strategy remains valid when one further incorporates a terminal value at the ruin time (see also Thonhauser and Albrecher (2007)) or imposes a constraint on the Laplace transform of the ruin time.

However, in practice dividend decisions are made periodically (e.g. quarterly or semi-annually) rather than continuously, and this motivated Albrecher et al. (2011b) to propose a risk model with periodic observations. A periodic observation scheme also has the advantage of only having lump sum dividends paid at the dividend decision time points as opposed to the presence of unrealistic continuous payment streams in the theoretically optimal barrier or threshold strategy under continuous observations. Since then, periodic observations have become a popular feature in various risk models, and ruin-related quantities such as the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) and the expected discounted dividends until ruin are analyzed under more general aggregate claims process and/or modifications of the periodic observation scheme. See e.g. Avanzi et al. (2013, 2021), Choi and Cheung (2014), Zhang and Cheung (2016), and Noba et al. (2018). The case of Poisson observations often leads to particularly simple and insightful identities (e.g. Albrecher and Ivanovs (2013, 2017), Zhang et al. (2017), and Boxma and Mandjes (2023)).

In this paper, it is assumed that the insurer observes the surplus process periodically at a sequence of random time points $\{Z_n\}_{n=0}^{\infty}$ (with the definition $Z_0 = 0$) to decide whether or not to pay a dividend. The time lengths $T_n = Z_n - Z_{n-1}$ (for $n \in \mathbb{N}$) between observations are assumed to form an i.i.d. sequence that is independent of $\{N_t\}_{t\geq 0}$ and $\{X_i\}_{i=1}^{\infty}$ (and hence $\{S_t\}_{t\geq 0}$). All quantities are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \in \mathbb{N}$, the increment Y_n of the surplus process $\{S_t\}_{t\geq 0}$ is defined as the difference between the surplus levels at time Z_n and at time Z_{n-1} (and one can write $Y_n = S_{Z_n} - S_{Z_{n-1}}$). Note that $\{(T_n, Y_n)\}_{n=1}^{\infty}$ form a sequence of i.i.d. random vectors (with a generic pair denoted by (T, Y)). We propose that the insurer can choose to pay a dividend at the time point Z_n ($n \in \mathbb{N}$) if (i) the observed pre-dividend surplus level at time Z_n is positive; and (ii) the pre-dividend surplus level at time Z_n is larger than the post-dividend surplus level at time Z_{n-1} (i.e. there is a gain in the n-th observation period so that Y_n is positive). Like Albrecher et al. (2011b, 2013), the event of ruin is only checked at the time points $\{Z_n\}_{n=0}^{\infty}$. An objective is to find a strategy (among a suitably defined admissible set of dividend payment strategies) to maximize the expected present value of dividends paid until ruin.

At a first glance, one may think that a similar research problem has already been considered by Albrecher et al. (2011a). However, their work assumed that at a dividend decision time the insurer is allowed to pay out dividend from its entire surplus, i.e. the only constraint is that the dividend payment cannot exceed what the insurer has. They showed that the optimal strategy is in general a band strategy (and this collapses to a barrier strategy in both the Brownian motion risk model and the compound Poisson model with exponential claims), which leads to a ruin probability of one. See also Remark 1 in Section 2. Their findings can be regarded as a 'periodic analogue' of the previous results by Gerber (1969) and Loeffen (2008) who considered the case of continuous observations. Clearly, the situation of almost sure ruin is undesirable for the policyholders who expect the insurer to be able to pay claims. Motivated by this, we would like to maximize the expected discounted dividends until ruin (for the shareholders' interest) while taking into account the survival of the process (for the policyholders' interest). To this end, we shall impose a restriction such that the insurer can only pay a lump sum dividend out of (a fraction θ of) its realized gain from the previous observation period. Such a novel idea is somewhat a 'periodic analogue' of restricting the dividend rate to be bounded (and paid out from part of the premium income) among absolutely continuous strategies when the surplus is monitored continuously. It is worthwhile to point out that, apart from the interpretations in the insurance business, the techniques and stochastic models used for optimal dividend problems are often applicable to operations research as well. For example, concerning cost minimization in a continuously observed Lévy process, Baurdoux and Yamazaki (2015) showed that double reflection is optimal under singular control whereas Hernández-Hernández et al. (2016) proved that refraction is optimal among absolutely continuous strategies. Continuous-time models with periodic actions have also been recently introduced to inventory problems. In particular, Section 3.1 in Albrecher et al. (2017) considered periodic depletion of inventory while Pérez et al. (2020) analyzed periodic replenishment of inventory.

This paper is organized as follows. Section 2 first starts by formulating the research problem as a Markov decision model without imposing any distributional assumptions on the inter-observation times or the claim amounts. Subsequently, the Bellman equation is derived with the uniqueness of its solution proved, and various properties of the value function (such as bounds and monotonicity) and the optimal strategy are discussed. In particular, a derived property of the maximizer of the Bellman equation suggests that a bang-bang control may be optimal. Such a strategy resembles a periodic threshold type of dividend strategy considered by Cheung and Zhang (2019) apart from some modifications, and is described in Section 3.1. In Section 3.2, we focus on the case where the inter-observation times are exponentially distributed and the claim amounts have a rational Laplace transform, where an explicit expression for the expected present value of the dividends paid until ruin under the candidate strategy is derived. Section 4 is concerned with numerical illustrations using the derived explicit formulas, and some concrete examples are provided under (mixed) exponential claims to demonstrate that the proposed strategy can be optimal. Section 5 ends the paper with some concluding remarks. Proofs of various lemmas, theorems and proposition are collected in the Appendix.

2. Dividend payout model as a Markov decision process

2.1. Problem formulation

To optimize periodic dividend payouts under the proposed constraints, we now formulate a Markov decision process and follow closely the notation in Albrecher et al. (2011a) and the textbook by Bäuerle and Rieder (2011). Specifically, our Markov decision model is defined on the state space

$$E = \{(u,y) : u \in \mathbb{R}^+ \text{ and } y \in \mathbb{R}^-\} \cup \{(u,y) : u \geq y \text{ and } y \in \mathbb{R}^+\},$$

where $u \in \mathbb{R}^+$ denotes the current surplus level and $y \in \mathbb{R}$ denotes the latest increment. Here we adopt the convention that the increment y has already been included in the current surplus u: (i) if a loss is made most recently then the current surplus has to be non-negative in order for the insurer to survive and possibly pay dividends in the future (corresponding to $\{(u,y) : u \in \mathbb{R}^+ \text{ and } y \in \mathbb{R}^-\}$); and (ii) if the insurer has a gain most recently then the current surplus must be no less than the amount of the gain (corresponding to $\{(u,y) : u \geq y \text{ and } y \in \mathbb{R}^+\}$). At each observation time point, the insurer has to decide a dividend payout $a \in \mathbb{R}^+$ with the action space being $\mathbb{R}^+$. Precisely, given the state $(u,y) \in E$, the dividend payment a (i.e. the one-stage reward) is restricted to the admissible set $[0, \theta y^+]$ where $y^+ = \max(0, y)$ and $\theta \in (0,1)$, so that it is only possible to pay a dividend if the process has a positive increment and no dividend can be paid otherwise. The controlled surplus process $\{U_n\}_{n=0}^{\infty}$ at the observation time points $\{Z_n\}_{n=0}^{\infty}$ (prior to dividend payment) is then described by

$$U_n = U_{n-1} - {}_{n-1}f(U_{n-1}, Y_{n-1}) + Y_n, \qquad n \in \mathbb{N}, \tag{2.1}$$

where (U_0, Y_0) is the initial information that is known, and ${}_jf : E \to \mathbb{R}^+$ (for $j \in \mathbb{N}_0$) is a decision rule which is measurable and ${}_jf(u,y) \in [0, \theta y^+]$. For the controlled process, the number of observations before ruin is $\tau = \inf\{n \in \mathbb{N}_0 | U_n < 0\}$ and therefore the ruin time is Z_τ. Note that our above formulation ensures that ruin cannot be directly caused by a dividend payment because the payment of a positive dividend at time Z_n is only possible when $Y_n = y > 0$ (and ruin did not occur before) and in such a case the pre-dividend surplus level must satisfy $U_n = u \geq y$, and with the dividend payment capped at θy we observe that the post-dividend surplus level must be no less than $u - \theta y$ which is positive. Under the dividend policy $\pi = ({}_0f, {}_1f, \dots)$ that consists of the decision rules, the expected present value of dividends until ruin is given by

$$V(u,y;\pi) = \mathbb{E}_{u,y}\left[\sum_{n=0}^{\tau-1} e^{-\delta Z_n}\, {}_nf(U_n, Y_n)\right], \qquad (u,y) \in E, \tag{2.2}$$

where $\delta > 0$ is the force of interest and the expectation $\mathbb{E}_{u,y}$ is taken under the initial condition $(U_0, Y_0) = (u,y)$. The insurer aims to maximize the function $V(u,y;\pi)$ by choosing an admissible strategy π, and the optimization problem is given by

$$V(u,y) = \sup_{\pi} V(u,y;\pi), \qquad (u,y) \in E.$$

As in Albrecher et al. (2011a), in order to have a well-defined and non-trivial research problem, it is assumed that $\mathbb{P}(0 < T < \infty) = 1$, $\mathbb{P}(Y < 0) > 0$, and $\mathbb{E}[Y^+] < \infty$. While our general analysis regarding optimality and expected discounted dividends does not require the specification of θ as long as $\theta \in (0,1)$, the range of θ that will guarantee a positive survival probability is discussed follows. Clearly, in the absence of dividend payments, the discretely observed surplus process $\{S_{Z_n}\}_{n=0}^{\infty}$ constitutes a random walk with generic increment Y. For the controlled surplus process $\{U_n\}_{n=0}^{\infty}$ representing the pre-dividend surplus levels at the observation time points $\{Z_n\}_{n=0}^{\infty}$ defined via (2.1), the corresponding post-dividend surplus process $\{\widetilde{U}_n\}_{n=0}^{\infty}$ is given by

$$\widetilde{U}_n = U_n - {}_nf(U_n, Y_n) = \widetilde{U}_{n-1} + Y_n - {}_nf(U_n, Y_n), \qquad n \in \mathbb{N},$$

where the n-th increment is $Y_n - {}_nf(U_n, Y_n)$. Because the payment of a dividend itself cannot directly lead to ruin, the ruin times of $\{\widetilde{U}_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$ coincide. While $\{\widetilde{U}_n\}_{n=0}^{\infty}$ is generally not a random walk as this depends on the dividend policy $\pi = ({}_0f, {}_1f, \dots)$, the particular process (denoted by $\{\widetilde{U}_n^{\max}\}_{n=0}^{\infty}$) that implements a policy to always pay the maximum possible dividend θY_n when Y_n is positive can be regarded as a random walk with increment $Y_n - \theta Y_n 1_{\{Y_n > 0\}}$, where 1_A is the indicator function of the event A. Since $Y_n - {}_nf(U_n, Y_n) \geq Y_n - \theta Y_n 1_{\{Y_n > 0\}}$ (i.e. the increment of $\{\widetilde{U}_n\}_{n=0}^{\infty}$ is no less than that of $\{\widetilde{U}_n^{\max}\}_{n=0}^{\infty}$), the ruin probability of $\{\widetilde{U}_n\}_{n=0}^{\infty}$ (and hence $\{U_n\}_{n=0}^{\infty}$) must be upper bounded by that of $\{\widetilde{U}_n^{\max}\}_{n=0}^{\infty}$. The random walk $\{\widetilde{U}_n^{\max}\}_{n=0}^{\infty}$ has a positive trend if its increment $Y_n - \theta Y_n 1_{\{Y_n > 0\}}$ has a positive mean, i.e.

$$\mathbb{E}[Y] - \theta \mathbb{E}[Y^+] > 0. \tag{2.3}$$

Using the results from e.g. Theorems 2 and 7 in Part I of Prabhu (1998), the above condition ensures that $\{\widetilde{U}_n^{\max}\}_{n=0}^{\infty}$ has a positive survival probability (under non-negative initial surplus) and will drift to infinity in the long run. Therefore, the equivalent condition $\theta < \mathbb{E}[Y]/\mathbb{E}[Y^+]$ is sufficient for ensuring that $\{U_n\}_{n=0}^{\infty}$ has a ruin probability that is strictly less than one. Intuitively, the condition (2.3) means that, on average, the increment Y between successive observations (before consideration of dividend) needs to be sufficient to pay out the maximal possible dividend θY^+ at an observation time point so that the insurer's surplus will grow over time.

Remark 1. It is important to point out our novel gain-based constraint must lead to a different optimal periodic strategy compared to the case under a traditional constraint. As mentioned before, when Albrecher et al. (2011a) assumed the traditional constraint of allowing the insurer to pay from the available surplus, they showed that a band strategy is optimal. In simple terms, in a band strategy, the set $\mathbb{R}^+$ of surplus levels is partitioned into a number of bands. There are two types of bands that alternate, representing 'habitable zone' and 'non-habitable zone' respectively. If the (pre-dividend) surplus level is observed to be in a habitable zone, then no dividend is paid. On the other hand, if the (pre-dividend) surplus level falls into a non-habitable zone, then a dividend is paid to reduce the surplus level to the upper boundary of the habitable zone below. See Definition 4.1 and Remark 4.2 in Albrecher et al. (2011a) for the formal mathematical definition. From Lemma 3.4 in Albrecher et al. (2011a), it is also known that the uppermost band in their optimal periodic strategy is non-habitable, and this implies ruin occurs with probability one. (A barrier strategy is a band strategy with a single non-habitable zone on top of a habitable zone.) Although such a band strategy is optimal under the traditional constraint, it can lead to the undesirable situation where a net loss occurring within an observation period brings the surplus level down to a non-habitable zone and a dividend needs to be paid despite the loss. With our proposed gain-based constraint, a dividend can only be paid from a fraction θ of the latest gain (i.e. if the latest increment is positive), but prior gains that have accumulated before (and have already become part of the surplus) cannot be used to pay a dividend. Therefore, the afore-mentioned undesirable situation cannot happen, and the resulting optimal periodic strategy must be of a different form. Note also that our decision rule is defined on two-dimensional state space consisting of the surplus level u and the latest increment y, and this presents a more challenging problem than the decision rule that only depends on u under the traditional constraint. Our proposed class of threshold-type strategies (see (3.1) in Section 3) as a candidate of the optimal strategy does not contain or belong to the class of band strategies. □

2.2. Bellman equation, and properties of value function and optimal policy

For a measurable function v: $E \to \mathbb{R}^+$, state (u,y) and action $a \in [0, \theta y^+]$, the transition law is given by

$$\int_0^\infty \int_{a-u}^\infty e^{-\delta t} v(u - a + x, x) Q(\mathrm{d}x, \mathrm{d}t),$$

where Q is the joint distribution of the increment Y and the time T between successive observations, the dummy x represents the next increment of the surplus process, and the lower limit of the inner integral ensures that dividend payments cease once ruin has occurred. We define the operator $\mathcal{T}_\circ$ which acts on the set $\mathbb{M} = \{v : E \to \mathbb{R}^+ \text{ measurable}\}$ by

$$\mathcal{T}_\circ v(u,y) = \sup_{a\in[0,\theta y^+]} \int_0^\infty \int_{a-u}^\infty e^{-\delta t} v(u-a+x,x) Q(\mathrm{d}x,\mathrm{d}t), \qquad (u,y)\in E. \tag{2.4}$$

We have the following lemma regarding some first properties of the value function and the operator $\mathcal{T}_\circ$, where the proof is provided in Appendix A.1.

Lemma 1.

(a) ***(Bounds for value function.)*** *The value function $V(u,y)$ of the optimal dividend problem satisfies the two-sided bounds*

$$\theta y^+ + \theta \frac{\mathbb{E}[e^{-\delta T} Y^+]}{1-\mathbb{E}[e^{-\delta T} 1_{\{Y\ge 0\}}]} \le V(u,y) \le \theta y^+ + \theta \frac{\mathbb{E}[e^{-\delta T} Y^+]}{1-\mathbb{E}[e^{-\delta T}]}, \qquad (u,y)\in E. \tag{2.5}$$

(b) ***(Convergence result for upper bounding function.)*** *For any $H_1, H_2 > 0$, define the upper bounding function*

$$\beta(u,y) = H_1 + H_2 y^+, \qquad (u,y)\in E. \tag{2.6}$$

The function β satisfies

$$\lim_{n\to\infty} \mathcal{T}_\circ^n \beta = 0. \tag{2.7}$$

Note that the terminology 'upper bounding function' follows from Bäuerle and Rieder (2011), as it is evident from (A.2) that the function (2.6) satisfies Definition 7.1.2 therein. Based on the upper bounding function (2.6), we define the set $\mathbb{M}_\beta = \{v \in \mathbb{M} : v \le \eta\beta \text{ for some } \eta \in \mathbb{R}^+\}$. Specification on how to select H_1 and H_2 will be made in Theorem 2. For later use, define the operator

$$\mathcal{T}_f v(u,y) = f(u,y) + \int_0^\infty \int_{f(u,y)-u}^\infty e^{-\delta t} v(u-f(u,y)+x,x) Q(\mathrm{d}x,\mathrm{d}t), \quad (u,y)\in E, \tag{2.8}$$

where $f : E \to \mathbb{R}^+$ is a decision rule and $v \in \mathbb{M}_\beta$. The maximal operator of this Markov decision model is given by

$$\mathcal{T} v(u,y) = \sup_f \mathcal{T}_f v(u,y), \qquad (u,y)\in E. \tag{2.9}$$

If a decision rule f is such that $\mathcal{T}_f v = \mathcal{T} v$, then f is a maximizer of v.

Then, we have the following theorem concerning the Bellman equation with the proof given in Appendix A.2.

Theorem 1. ***(Bellman equation and optimal stationary policy.)*** *The value function $V \in \mathbb{M}_\beta$ of the optimal dividend problem satisfies the Bellman equation*

$$V(u,y) = \sup_{a\in[0,\theta y^+]} \left\{ a + \int_0^\infty \int_{a-u}^\infty e^{-\delta t} V(u-a+x,x) Q(\mathrm{d}x,\mathrm{d}t) \right\}, \quad (u,y)\in E, \tag{2.10}$$

which is equivalent to $V = \mathcal{T} V$. There exists maximizer(s) of V, and every maximizer f^* defines an optimal stationary policy $\pi^* = (f^*, f^*, \ldots)$ that is also optimal among history-dependent dividend policies.

The Bellman equation (2.10) can also be conveniently rewritten as

$$V(u,y) = \sup_{a\in[0,\theta y^+]} \{a + G(u-a)\}, \qquad (u,y)\in E, \tag{2.11}$$

where

$$G(u) = \int_0^\infty \int_{-u}^\infty e^{-\delta t} V(u+x,x) Q(\mathrm{d}x,\mathrm{d}t), \qquad u\in\mathbb{R}^+. \tag{2.12}$$

Moreover, if f^* is a maximizer of V, then

$$V(u,y) = f^*(u,y) + G(u - f^*(u,y)), \qquad (u,y)\in E. \tag{2.13}$$

A natural question that arises is whether the Bellman equation (2.10) has a unique solution (i.e. whether the operator $\mathcal{T}$ has a unique fixed point). The answer is affirmative according to the next theorem where the proof is delayed to Appendix A.3.

Theorem 2. ***(Uniqueness of fixed point of the Bellman equation.)*** *With the positive constants H_1 and H_2 selected to satisfy*

$$\mathbb{E}[e^{-\delta T}] + \frac{H_2}{H_1}\mathbb{E}[e^{-\delta T} Y^+] < 1, \tag{2.14}$$

we define the set $\mathbb{M}_\beta^\star = \{v : E \to \mathbb{R}$ measurable and $|v| \le \eta\beta$ for some $\eta \in \mathbb{R}^+\}$ using the upper bounding function (2.6). Then, the operator $\mathcal{T}$ has a unique fixed point v^* in $\mathbb{M}_\beta^\star$ such that $v^* = \mathcal{T} v^*$.

Remark 2. Denote $V_f(u,y) = V(u,y;f^\infty)$ as the value of (2.2) with the policy π chosen to be the stationary one $f^\infty = (f,f,\ldots)$. Because each dividend payment is non-negative, our model belongs to the class of positive Markov decision models (see Chapter 7.4 of Bäuerle and Rieder (2011)). Therefore, it is known from Theorem 7.4.5 in Bäuerle and Rieder (2011) that V_f is a fixed point of $\mathcal{T}$ (i.e. $V_f = \mathcal{T} V_f$) if and only if the stationary policy f^∞ is optimal. In other words, if we are able to compute V_f under a stationary policy f^∞ and if V_f is such that $V_f = \mathcal{T} V_f$, then f^∞ is an optimal policy. □

The following lemma provides some further properties of the value function (see Appendix A.4 for its proof).

Lemma 2.

(a) ***(Monotonocity of value function.)*** *The value function $V(u,y)$ is increasing (i.e. non-decreasing) in both u and y for $(u,y)\in E$.*

(b) ***(Difference of value functions.)*** *The value function satisfies the property*

$$V(u_2,y_2) - V(u_1,y_1) \ge \theta(y_2 - y_1), \qquad y_2 \ge y_1 \ge 0; u_2 \ge u_1 + \theta(y_2 - y_1).$$

Some crucial properties of f^* are given in the following theorem where the proof can be found in Appendix A.5.

Theorem 3. ***(Properties of maximizer of Bellman equation.)*** *Each maximizer f^* of V satisfies*

$$V(u,y) - f^*(u,y) = V\Big(u - f^*(u,y), y - \frac{1}{\theta} f^*(u,y)\Big), \qquad (u,y)\in E. \tag{2.15}$$

Specifically, if f^* is the largest maximizer then it satisfies

$$f^*\Big(u - f^*(u,y), y - \frac{1}{\theta} f^*(u,y)\Big) = 0, \qquad (u,y)\in E. \tag{2.16}$$

The result (2.16) concerning the largest maximizer can be intuitively interpreted as follows. Given the initial state (u,y) such that $0 < y \le u$, suppose that the insurer looks at the possibility of paying a dividend of size $f^*(u,y) \in [0,\theta y]$. Doing so will cause the surplus level to fall to $u - f^*(u,y)$. Moreover, after paying out $f^*(u,y)$ the insurer is still eligible to pay a further dividend of $\theta y - f^*(u,y)$, and therefore the remaining 'unused increment' is equivalent to $y - (1/\theta) f^*(u,y)$ since the constraint $f^*(u,y) \in [0,\theta y]$ means that the insurer is allowed to pay θ unit of dividend for every unit of (positive) increment. Consequently, a dividend payment of $f^*(u,y)$ effectively moves the state from (u,y) to $(u - f^*(u,y), y - (1/\theta) f^*(u,y))$. If $f^*(u,y)$ is the largest maximizer, then no further should be paid from the state $(u - f^*(u,y), y - (1/\theta) f^*(u,y))$, which explains (2.16).

It is important to note that the results in this section are valid in general as no specific distributional assumptions on the inter-observation times or the claim amounts need to be made. Although determining an optimal dividend strategy that is applicable in general can be challenging in the present context (see Section 5 for future research), the theoretical results will be crucial for us in the numerical analysis in Section 4. In particular, in Section 3.1 we shall propose a candidate strategy which satisfies the necessary property of the largest maximizer f^* developed in Theorem 3. Explicit formula for the value of the candidate strategy is subsequently derived in Proposition 1 in Section 3.2 under specific distributional assumptions. This will be utilized in the specific examples in Section 4 to (numerically) verify that it is indeed a fixed point of the operator $\mathcal{T}$, and hence the proposed strategy is optimal in those cases according to Remark 2 concerning positive Markov decision models.

3. Threshold type strategy as a candidate optimal strategy

3.1. Proposed form of optimal strategy

We consider a stationary policy with decision rule of the form

$$f_b(u,y) = \begin{cases} 0, & 0 \le u \le b, \\ \min(u-b, \theta y), & b < u \le b+y, \\ \theta y, & u > b+y, \end{cases} \tag{3.1}$$

where $b \ge 0$ is a fixed threshold level. Here it is understood that we focus on the case $0 < y \le u$ because one must have $f_b(u,y) = 0$ outside this domain. The form of our proposed strategy (3.1) is motivated by a bang-bang type of control which can be optimal in dividend problems (e.g. Gerber and Shiu (2006)). In a bang-bang strategy, the insurer would pay out the maximum possible dividend amount as long as the surplus level is high enough; otherwise no dividend is paid. Therefore, there is a critical level b such that no dividend is paid if the current observed surplus level u is below b (first case in (3.1)). If the surplus level $u-y$ prior to adding the increment y was already greater than b, then the surplus process is deemed safe and the maximum possible amount θy is paid as a dividend since the dividend payment cannot cause the surplus to fall below b (third case in (3.1)). The trickiest situation is when the increment y has brought the surplus from below to above b. In this case, one still wants to pay as much dividend as possible provided that this does not make the surplus fall below b (second case in (3.1)). The strategy (3.1) can also be equivalently written as

$$f_b(u,y) = \begin{cases} 0, & 0 \le u \le b, \\ u-b, & b < u \le b+\theta y, \\ \theta y, & u > b+\theta y. \end{cases} \tag{3.2}$$

From the strategy (3.1) (or (3.2)), it is clear that if a positive dividend $f_b(u,y)$ is paid from a state (u,y), then it must be of size $u-b$ or θy. In the first case with $f_b(u,y) = u-b$, one has

$$f_b\Big(u - f_b(u,y), y - \frac{1}{\theta} f_b(u,y)\Big) = f_b\Big(b, y - \frac{1}{\theta}(u-b)\Big) = 0,$$

where the last equality follows from the first piece of (3.1). In the second case with $f_b(u,y) = \theta y$, we have

$$f_b\Big(u - f_b(u,y), y - \frac{1}{\theta} f_b(u,y)\Big) = f_b(u-\theta y, 0) = 0,$$

where the final equality is due to the fact that no dividend can be paid if the latest increment is non-positive. From the above two equalities, we observe that the proposed strategy (3.1) satisfies (2.16) provided in Theorem 3, which is a necessary condition that the largest maximizer f^* must satisfy. (Note also that (2.16) is obviously satisfied when no dividend is paid in the first piece of (3.1), i.e. $f_b(u,y) = 0$.) A candidate for the optimal strategy would be (3.1) implemented at an optimal threshold level b^*, denoted by f_{b^*}.

Fig. 1 shows a sample path of the surplus process under the proposed strategy (3.1) (or (3.2)), where the initial increment $Y_0 = y$ is negative (so that no dividend is paid at time 0) and the dividend payout fraction is $\theta = 0.5$.

Remark 3. At a first thought, the periodic threshold type dividend strategy analyzed by Cheung and Zhang (2019) may be used as a candidate strategy. Specifically, the decision rule f therein is given by

$$f(u,y) = \begin{cases} 0, & 0 \le u \le b, \\ \theta(u-b), & b < u \le b+y, \\ \theta y, & u > b+y, \end{cases} \tag{3.3}$$

for some fixed threshold level b. However, it can be easily checked that (2.16) is not satisfied by (3.3) due to the way the middle piece is defined. □

We would like to determine the expected discounted dividends until ruin under the proposed stationary strategy with decision rule (3.1) depending on an arbitrary threshold level $b \ge 0$ that we will later optimize

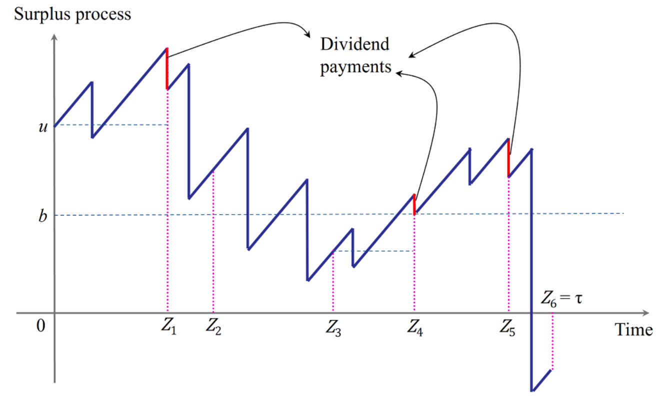


Fig. 1. Sample path under proposed threshold strategy at b (assuming $\theta = 0.5$ and $y < 0$).

with respect to. Such an expectation will be denoted by $V(u,y;b)$, since the strategy π in (2.2) can now be specified via the threshold level b. Like in Albrecher et al. (2011a), we further denote the corresponding expected discounted dividends until ruin by $\widetilde{V}(u;b)$ supposing that no dividend can be paid at time 0. It is easy to see that $\widetilde{V}(u;b)$ admits the representation

$$\widetilde{V}(u;b) = \int_0^\infty \int_{-u}^\infty e^{-\delta t} V(u+x, x; b) Q(\mathrm{d}x, \mathrm{d}t)$$

because one would follow the strategy (3.1) at the first observation time. Moreover, we have

$$V(u,y;b) = f_b(u,y) + \widetilde{V}(u - f_b(u,y); b). \tag{3.4}$$

If the proposed strategy with threshold b^* is an optimal strategy, then the Bellman equation (2.10) is satisfied so that

$$V(u,y;b^*) = \sup_{a \in [0,\theta y^+]} \{a + \widetilde{V}(u-a;b^*)\}, \tag{3.5}$$

where $f_{b^*}(u,y)$ is a maximizer of the right-hand side. Note that if one utilizes the operator (2.9) and applies it to $V(u,y;b)$, then

$$\mathcal{T} V(u,y;b) = \sup_{a \in [0,\theta y^+]} \{a + \widetilde{V}(u-a;b)\},$$

and therefore the condition (3.5) can be conveniently expressed as

$$V(u,y;b^*) = \mathcal{T} V(u,y;b^*), \tag{3.6}$$

or simply $V_{f_{b^*}} = \mathcal{T} V_{f_{b^*}}$ according to the notation in Remark 2.

3.2. Expected discounted dividends for exponential inter-observation times

In order to numerically verify the optimality of the proposed threshold strategy in Section 4, we shall utilize an explicit formula for $V(u,y;b)$. Thanks to (3.4), it is sufficient to determine the dividend function $\widetilde{V}(u;b)$. Note that an explicit formula will also allow us to perform optimization to find the optimal threshold level b^* (given the proposed strategy), so that the candidate value function of the Markov decision model will be $V(u,y;b^*)$. If $V(u,y;b^*)$ satisfies the fixed point property (3.6), then Remark 2 asserts that a stationary policy with decision rule f_{b^*} is optimal.

The derivation of $\widetilde{V}(u;b)$ requires specification of the joint distribution Q of (Y,T). Note that the quantity $\int_0^\infty e^{-\delta t} Q(\mathrm{d}x, \mathrm{d}t)$ can be in general written as

$$\int_0^\infty e^{-\delta t} Q(\mathrm{d}x, \mathrm{d}t) = \mathbb{E}[e^{-\delta T} 1_{\{Y>0; Y \in \mathrm{d}x\}}] + \mathbb{E}[e^{-\delta T} 1_{\{Y<0; Y \in \mathrm{d}x\}}].$$

In particular, when the inter-observation times are Erlang distributed, it is known from Section 3.2 in Albrecher et al. (2013) that we can write

$$\int_0^\infty e^{-\delta t} Q(\mathrm{d}x, \mathrm{d}t) = \{g_{\delta,-}(x) I_{\{x>0\}} + g_{\delta,+}(-x) I_{\{x<0\}}\} \mathrm{d}x,$$

where, following the notation therein, $g_{\delta,-}$ is the discounted density for the case where the increment is positive (i.e. there is gain between successive observations), and $g_{\delta,+}$ is the discounted density for the case where the increment is negative (i.e. there is loss between successive observations). General expressions for these densities are given by their (3.15) and (3.18). Then the Bellman equation (2.10) becomes

$$V(u,y)=\sup_{a\in[0,\theta y^+]}\left\{a+\int_0^\infty V(u-a+x,x)g_{\delta,-}(x)\mathrm{d}x+\int_0^{u-a}V(u-a-x,-x)g_{\delta,+}(x)\mathrm{d}x\right\},$$

where only single integrals appear on the right-hand side (instead of double integral as in (2.10)).

As one expects $\widetilde{V}(u;b)$ to be of different functional forms depending on whether $0\le u\le b$ or $u\ge b$, we shall write

$$\widetilde{V}(u;b)=\begin{cases}\widetilde{V}_1(u), & 0\le u\le b,\\ \widetilde{V}_2(u), & u\ge b,\end{cases}\tag{3.7}$$

and the dependence of $\widetilde{V}_1$ and $\widetilde{V}_2$ on b is suppressed for convenience. Then one has the integral equation

$$\begin{aligned}\widetilde{V}_1(u)=&\int_{\frac{b-u}{1-\theta}}^\infty\left[\theta x+\widetilde{V}_2(u+(1-\theta)x)\right]g_{\delta,-}(x)\mathrm{d}x\\&+\int_{b-u}^{\frac{b-u}{1-\theta}}\left[(u+x-b)+\widetilde{V}(b;b)\right]g_{\delta,-}(x)\mathrm{d}x\\&+\int_0^{b-u}\widetilde{V}_1(u+x)g_{\delta,-}(x)\mathrm{d}x+\int_0^u\widetilde{V}_1(u-x)g_{\delta,+}(x)\mathrm{d}x,\end{aligned}\tag{3.8}$$

for $0\le u\le b$ and

$$\widetilde{V}_2(u)=\int_0^\infty\left[\theta x+\widetilde{V}_2(u+(1-\theta)x)\right]g_{\delta,-}(x)\mathrm{d}x+\int_0^u\widetilde{V}(u-x;b)g_{\delta,+}(x)\mathrm{d}x,\tag{3.9}$$

for $u\ge b$. It is clear from the above two integral equations that $\widetilde{V}(u;b)$ is continuous at $u=b$ (and therefore $\widetilde{V}(b;b)$ may be taken as $\widetilde{V}_1(b)$ or $\widetilde{V}_2(b)$ in (3.7)).

In order to derive an explicit expression for $\widetilde{V}(u;b)$, one will need to solve the integral equations (3.8) and (3.9) for $\widetilde{V}_1$ and $\widetilde{V}_2$. This in turn requires explicit formulas for the densities $g_{\delta,-}$ and $g_{\delta,+}$. To this end, we assume that inter-observation times are exponential with mean $1/\gamma$ and the Laplace transform $\widehat{p}(s)$ of the claim amounts is rational of order m. The latter means that $\widehat{p}(s)=L_{2,m-1}(s)/L_{1,m}(s)$, where $L_{2,m-1}$ is polynomial of degree (at most) $m-1$ and $L_{1,m}$ is another polynomial of degree m. Without loss of generality, it is assumed that $L_{1,m}$ has leading coefficient 1 and the polynomials $L_{2,m-1}$ and $L_{1,m}$ have no common zeros. From Chapter 4 in Bladt and Nielsen (2017), it is known that the class of distributions with a rational Laplace transform is equivalent to the class of matrix exponential distributions, which is dense in the set of positive continuous distributions. Consider the Lundberg equation (in s)

$$cs-(\lambda+\gamma+\delta)+\lambda\widehat{p}(s)=0,\tag{3.10}$$

and define $\rho_\gamma>0$ to be the unique positive root with the remaining roots $\{-R_{\gamma,j}\}_{j=1}^m$ having negative real parts. From the special case of (3.15) and (4.2) in Albrecher et al. (2013) (with $n=1$ in their notation), it is known that

$$g_{\delta,-}(x)=Ae^{-\rho_\gamma x},\qquad x>0,\tag{3.11}$$

where

$$A=\frac{\gamma}{c}\frac{L_{1,m}(\rho_\gamma)}{\prod_{j=1}^m(\rho_\gamma+R_{\gamma,j})}.$$

Moreover, from Albrecher et al. (2013)'s (4.3) and the preceding equation therein (again with $n=1$), we have

$$g_{\delta,+}(x)=\sum_{j=1}^m B_j e^{-R_{\gamma,j}x},\qquad x\ge 0,\tag{3.12}$$

where

$$B_j=\frac{\gamma}{c}\frac{L_{1,m}(-R_{\gamma,j})}{(\rho_\gamma+R_{\gamma,j})\prod_{\ell=1,\ell\neq j}^m(R_{\gamma,\ell}-R_{\gamma,j})},\qquad j=1,\dots,m.$$

With such information, an explicit expression for computing the expected present value of dividends payable until ruin is provided in the next proposition, with the proof available in Appendix A.6.

Proposition 1. ***(Expected discounted dividends for threshold strategy.)*** *Suppose that the inter-observation times are exponentially distributed with mean $1/\gamma$ and the claim amounts have a rational Laplace transform of order m. The expected present value of dividends payable until ruin $\widetilde{V}(u;b)$ (defined in a piecewise manner in (3.7)) is given by*

$$\widetilde{V}_1(u)=\sum_{i=1}^{m+1}C_ie^{r_iu}+De^{\frac{\rho_\gamma}{1-\theta}u},\qquad 0\le u\le b,\tag{3.13}$$

and

$$\widetilde{V}_2(u)=w+\sum_{i=1}^m E_ie^{s_iu},\qquad u\ge b.\tag{3.14}$$

In (3.13), the quantities $\{r_i\}_{i=1}^{m+1}$ are the $m+1$ roots of a special case of the Lundberg equation (3.10) with $\gamma=0$. In (3.14), the quantity w is explicitly given by

$$w=\frac{\theta A/\rho_\gamma^2}{1-A/\rho_\gamma-\sum_{j=1}^m B_j/R_{\gamma,j}}=\frac{\theta A}{\rho_\gamma^2}\frac{\gamma+\delta}{\delta},\tag{3.15}$$

and $\{s_i\}_{i=1}^m$ satisfy the equation (in s)

$$1=\frac{A}{\rho_\gamma-(1-\theta)s}+\sum_{j=1}^m\frac{B_j}{s+R_{\gamma,j}},\tag{3.16}$$

and are those m roots with negative real parts. In addition, the constants $\{C_i\}_{i=1}^{m+1}$, D and $\{E_i\}_{i=1}^m$ can be solved from a system of $2m+2$ linear equations consisting of

$$\begin{aligned}&D\left(\sum_{j=1}^m\frac{B_j}{\frac{\rho_\gamma}{1-\theta}+R_{\gamma,j}}-\frac{(1-\theta)A}{\rho_\gamma\theta}-1\right)e^{\frac{\rho_\gamma}{1-\theta}b}\\&\quad+\sum_{i=1}^m E_i\left(\frac{1}{\rho_\gamma-(1-\theta)s_i}-\frac{1}{\rho_\gamma}\right)Ae^{s_ib}=\frac{(1-\theta)A}{\rho_\gamma^2},\end{aligned}\tag{3.17}$$

$$\sum_{i=1}^{m+1}\frac{C_ir_i}{\rho_\gamma-r_i}e^{r_ib}-\frac{D}{\theta}e^{\frac{\rho_\gamma}{1-\theta}b}=\frac{1}{\rho_\gamma},\tag{3.18}$$

$$\sum_{i=1}^{m+1}\frac{C_i}{R_{\gamma,j}+r_i}+\frac{D}{\frac{\rho_\gamma}{1-\theta}+R_{\gamma,j}}=0,\qquad j=1,\dots,m,\tag{3.19}$$

and

$$\sum_{i=1}^{m+1}\frac{C_i}{R_{\gamma,j}+r_i}e^{r_ib}+\frac{D}{\frac{\rho_\gamma}{1-\theta}+R_{\gamma,j}}e^{\frac{\rho_\gamma}{1-\theta}b}-\sum_{i=1}^m\frac{E_i}{R_{\gamma,j}+s_i}e^{s_ib}=\frac{w}{R_{\gamma,j}},\qquad j=1,\dots,m.\tag{3.20}$$

Remark 4. Note that $\{C_i\}_{i=1}^{m+1}$, D and $\{E_i\}_{i=1}^m$ all depend on the threshold level b via the coefficient matrix of the linear system, but the roots ρ_γ, $\{-R_{\gamma,j}\}_{j=1}^m$, $\{r_i\}_{i=1}^{m+1}$ and $\{s_i\}_{i=1}^{m+1}$ of various Lundberg equations do not depend on b. This is particularly important when inputting the formulas in software like *Mathematica*, as one would like to determine the optimal threshold level b^* that maximizes $\widetilde{V}(u;b)$ with respect to b (see numerical examples). □

Remark 5. Under exponential inter-observation times and claim amounts with a rational Laplace transform, the expected net gain is given by

$$\mathbb{E}[Y^+]=\int_0^\infty xg_{0,-}(x)\mathrm{d}x=\frac{A}{\rho_\gamma^2}\bigg|_{\delta=0}\tag{3.21}$$

and the expected increment is

$$\mathbb{E}[Y] = \frac{c - \lambda \mathbb{E}[X]}{\gamma}. \tag{3.22}$$

The condition (2.3) that guarantees a positive survival probability is thus reduced to

$$\frac{c - \lambda \mathbb{E}[X]}{\gamma} - \theta \frac{A}{\rho_\gamma^2}\bigg|_{\delta=0} > 0, \tag{3.23}$$

which is the same as Equation (12) in Cheung and Zhang (2019) with $n = 1$. □

Remark 6. In principle, the same methodology as in the proof of Proposition 1 is also applicable to obtain an explicit expression for $\widetilde{V}(u;b)$ when the inter-observation times follow an Erlang distribution (while retaining claim amounts with a rational Laplace transform). In such a case, $g_{\delta,-}$ and $g_{\delta,+}$ will further involve 'Erlang' terms rather than just exponential terms (see (3.15) and (4.3) in Albrecher et al. (2013)), and therefore the derivation will be far more tedious. Moreover, the approach typically involves solving of Lundberg-type equations that are equivalent to polynomial equations of higher order, where the presence of complex roots could possibly cause computational issues, a common problem in ruin theory. (The same problem also arises if the polynomial $L_{1,m}$ of the claim's rational Laplace transform is of high order m.) However, for claim amount distributions that do not possess a rational Laplace transform, the densities $g_{\delta,-}$ and $g_{\delta,+}$ do not generally admit nice analytic form, and this could make it difficult (if not impossible) to solve the integral equations (3.8) and (3.9) for $\widetilde{V}_1$ and $\widetilde{V}_2$. □

4. Numerical illustrations

In this section, it is assumed that the inter-observation times are exponentially distributed, and we demonstrate that the proposed threshold strategy is optimal via some specific examples. We shall consider a total of five claim amount distributions that have a rational Laplace transform and the same mean of one. Our calculations have been carried out using *Mathematica*. We assume in all numerical examples that the Poisson claim arrival rate is $\lambda = 1$, the incoming premium rate is $c = 1.5$, and the force of interest for discounting dividend payments is $\delta = 0.01$. To ensure the insurer has a positive survival probability, it is important to set the dividend payout fraction θ appropriately such that the condition (3.23) holds. From such a condition, we observe that the upper bound for θ is dependent on the choice of the Poissonian observation frequency γ. We have checked that for each of the claim amount distributions, the upper bound decreases from 1 to 1/3 as γ increases from 0 to ∞. With this in mind, we shall consider values θ up to 0.3.

4.1. Exponential claims

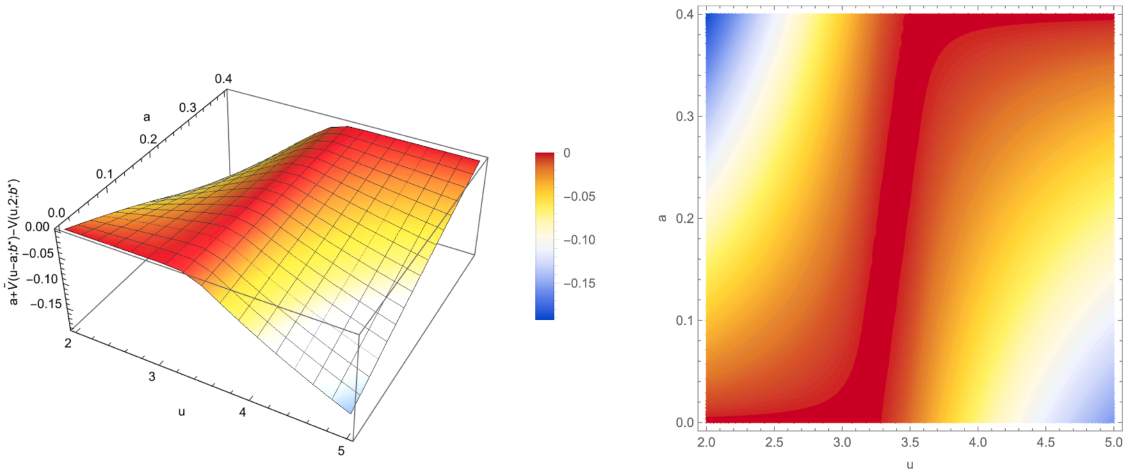
We start by studying the case where each claim amount follows an exponential distribution with mean one. First, we fix the mean of the exponential inter-observation times to be one (i.e. $\gamma = 1$) and $\theta = 0.2$. Since explicit formula for $\widetilde{V}(u;b)$ under the strategy (3.1) has been obtained in Section 3.2, numerical maximization can be performed with respect to b. While such an optimization should in principle be carried out for each initial surplus level u, it is found that the optimal threshold level $b^* = 3.17$ does not depend on u. To demonstrate that $V(u,y;b^*)$ under the proposed strategy f_{b^*} (i.e. strategy (3.1) implemented at b^*) satisfies the fixed point property (3.6) (or equivalently (3.5)) with $f_{b^*}(u,y)$ a maximizer on the right-hand side of (3.5), we fix $y = 2$ and analyze the quantity $a + \widetilde{V}(u-a;b^*) - V(u,2;b^*)$ as a function of (u,a) for $u \geq y = 2$ and $a \in [0, \theta y^+] = [0, 0.4]$. The 3D plot in Fig. 2a shows that $a + \widetilde{V}(u-a;b^*) - V(u,2;b^*)$ is non-positive with a maximum value of zero achieved at the reddest part of the plot. When viewed from above, we obtain the contour plot in Fig. 2b. The reddest part of Fig. 2b can be regarded as a plot of the maximizer $a \in [0, 0.4]$ of $a + \widetilde{V}(u-a;b^*)$ as a function of u, which is provided separately in Fig. 3. This exactly coincides with $f_{b^*}(u,2)$. Further numerical checks reveal that the fixed point property is also satisfied for other values of y. According to Remark 2, this is sufficient for us to conclude that our proposed stationary strategy $f_{b^*}^\infty$ with decision rule f_{b^*} is optimal and hence we write $f^*(u,y) = f_{b^*}(u,y)$. We have also checked that the fixed point property does not hold true if (i) the current strategy is implemented at another (non-optimal) threshold level; or (ii) Cheung and Zhang (2019)'s periodic threshold strategy (3.3) is implemented. Because the optimality of our proposed strategy has been verified, by utilizing (3.4) at $b = b^*$ we can compute the value function $V(u,y)$ of the optimal dividend problem by

$$V(u,y) = f_{b^*}(u,y) + \widetilde{V}(u - f_{b^*}(u,y); b^*). \tag{4.1}$$

Fig. 4 plots of $V(u,y)$ as a function of (u,y) for $0 \leq y \leq u$. As expected, $V(u,y)$ is increasing in both u and y (see Lemma 2(a)). It is important to point out that, all the above (and the subsequent) checks and plots are possible thanks to the explicit formulas from Section 3.2.

Next, we shall study the impact of a change in the observation frequency γ and the dividend payout fraction θ on the optimal threshold b^*. Like the previous case where $(\gamma,\theta) = (1, 0.2)$, for every other pair of (γ,θ) it is also found that b^* does not depend on u, and the proposed threshold strategy (3.1) at b^* is optimal. We observe from Fig. 5a that b^* increases with γ for each fixed θ. There are two intuitive reasons which could explain this phenomenon. First, suppose that we consider a process with a higher value of γ. If one maintains the same threshold level as before, then the insurer could be paying dividend too soon (due to frequent dividend decisions), possibly leading to inadequate surplus and hence early ruin (as ruin is also monitored more frequently). This could be a disadvantage in the long run because dividend payments could cease early. Therefore, as γ increases, a higher threshold level b^* is needed, meaning that the insurer is required to have achieved a higher surplus level before paying out dividends. This helps keep the process safe from potential early ruin so that more dividends can be paid in the long run in order to maximize dividend payout. On the other hand, we can also look into a process with a smaller value of γ to arrive at the same conclusion. Since dividend decisions are rarely made, dividends will hardly be paid if one does not alter the threshold level. To maximize dividend payout, the insurer will need to make sure a dividend can be paid as soon as possible when an opportunity (i.e. dividend decision) arises by implementing a lower threshold level b^*. Otherwise, the insurer will need to wait for a long period before a dividend can be paid which is a disadvantage due to the time value of money. Turning to Fig. 5b, we see that b^* increases with θ for each fixed γ. Recall that a higher θ means a larger fraction of the insurer's gains can be paid as dividends. If the same threshold level is retained, then too much dividend might be paid which can possibly increase the risk of ruin. Setting a higher threshold can help mitigate such risk by allowing the insurer to accumulate more surplus before dividend payout, ensuring more dividends can be paid in the long run for maximization purposes. This explains why the optimal threshold b^* should be higher as θ increases. It is also noted that $b^* = 0$ when γ or θ is low.

Fig. 6 plots the dividend function $\widetilde{V}(u;b^*)$ against u for different combinations of (γ,θ). Recall that $\widetilde{V}(u;b^*)$ is the expected discounted dividends until ruin when no dividend is paid at the start. From (4.1), one has $V(u,y) = \widetilde{V}(u;b^*)$ when $f_{b^*}(u,y) = 0$, which happens when the initial increment Y_0 is non-positive (i.e. $y \leq 0$) or when the initial surplus U_0 does not exceed the optimal threshold (i.e. $u \leq b^*$). The function $\widetilde{V}(u;b^*)$ is increasing in u, which is expected. For fixed $\gamma = 1$, Fig. 6a indicates that $\widetilde{V}(u;b^*)$ increases in θ. As θ is the maximum portion of the gain that can be paid out as dividend, a larger θ indeed represents less of a constraint in the optimization problem, leading to an increase in $\widetilde{V}(u;b^*)$. From Fig. 6b, it can be seen that for fixed $\theta = 0.2$ the quantity $\widetilde{V}(u;b^*)$ increases in γ. This suggests that more frequent dividend decisions would increase dividend payments. In this case, even the process is also frequently monitored for ruin, the resulting increase in the optimal thresh-



(a) 3D plot (b) Contours

Fig. 2. Plot of $a + \widetilde{V}(u - a; b^*) - V(u, 2; b^*)$ under exponential claims.

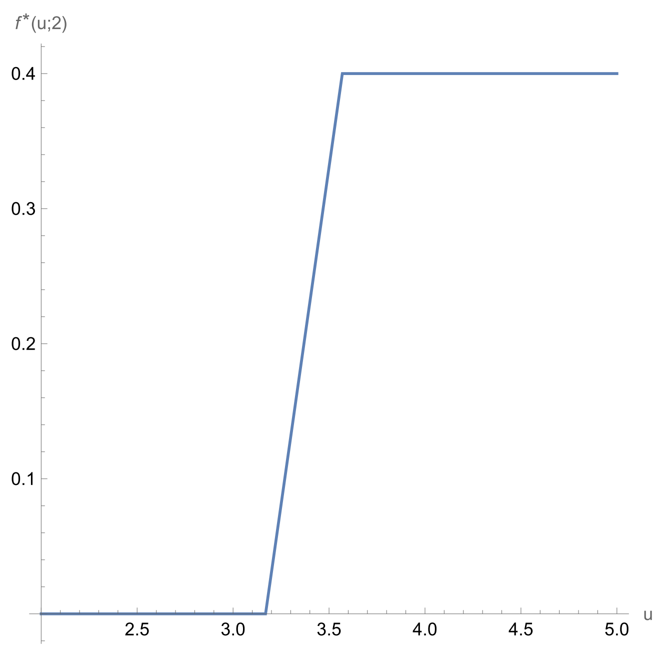


Fig. 3. Plot of $f^*(u, 2)$ against u under exponential claims.

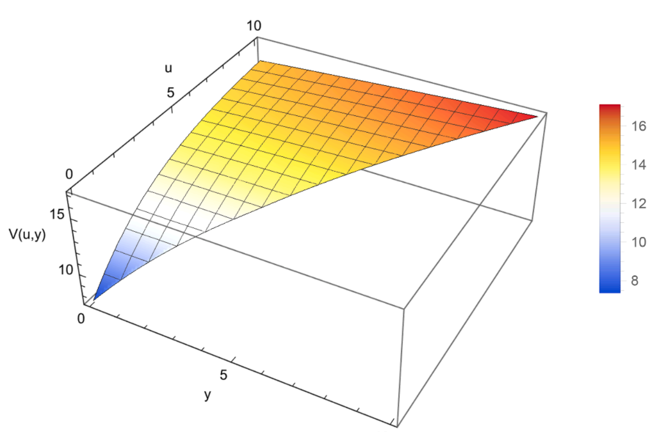


Fig. 4. Plot of $V(u, y)$ against u and y under exponential claims.

old b^* (see Fig. 5) does benefit the shareholders in the form of sustainable dividend payments in the long run.

Finally, in Fig. 7 we evaluate $\frac{\mathrm{d}}{\mathrm{d}u}\widetilde{V}(u; b^*)$ as a function of u for selected pairs of (γ, θ). In all cases, such a derivative is positive since $\widetilde{V}(u; b^*)$ increases in u. Moreover, the derivative decreases in u, suggesting that the marginal benefit of the shareholders decreases as the surplus level increases. In particular, when $(\gamma, \theta) = (1, 0.2)$, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}u}\widetilde{V}(u; b^*)\Big|_{u=b^*} = 1 \tag{4.2}$$

at the optimal threshold level $b^* = 3.17$, and also

$$\frac{\mathrm{d}}{\mathrm{d}u}\widetilde{V}(u; b^*) > 1 \;\text{ for }\; 0 \le u < b^* \;\text{ and }\; \frac{\mathrm{d}}{\mathrm{d}u}\widetilde{V}(u; b^*) < 1 \;\text{ for }\; u > b^*. \tag{4.3}$$

Numerical tests for other pairs of (γ, θ) (such as $(\gamma, \theta) = (1, 0.1)$ where $b^* = 0.86$) show that (4.2) and (4.3) hold true as long as $b^* > 0$. Financial interpretation of (4.3) in terms of connection between dividend payout and company efficiency can be found in Section 3 of Gerber and Shiu (2006). Specifically, with the optimal strategy being the proposed threshold strategy implemented at b^*, the insurance business is efficient when the observed surplus level x is such that $\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{V}(x; b^*) > 1$, and therefore it is better for the insurer to retain the funds and not to pay dividends. In contrast, an observed surplus level x such that $\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{V}(x; b^*) < 1$ indicates that the business is inefficient, and thus it is more advantageous for the insurer to pay as much dividend as possible rather than keeping the money.

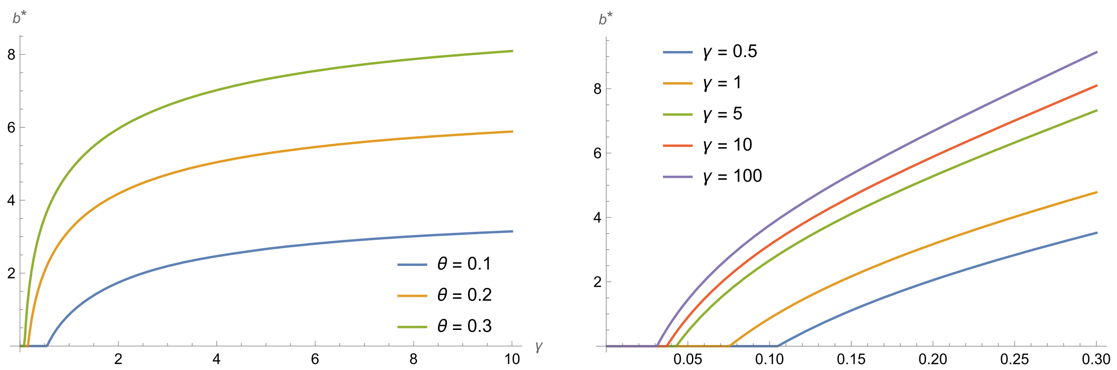
4.2. Mixed exponential and sum of exponentials claims

For now, we consider two more different claim size distributions, namely

- a sum of two independent exponential random variables (with respective means 1/3 and 2/3) so that $p(x) = 2\left(\frac{3}{2}e^{-\frac{3}{2}x}\right) + (-1)(3e^{-3x})$; and
- a mixture of two exponential distributions specified by the density

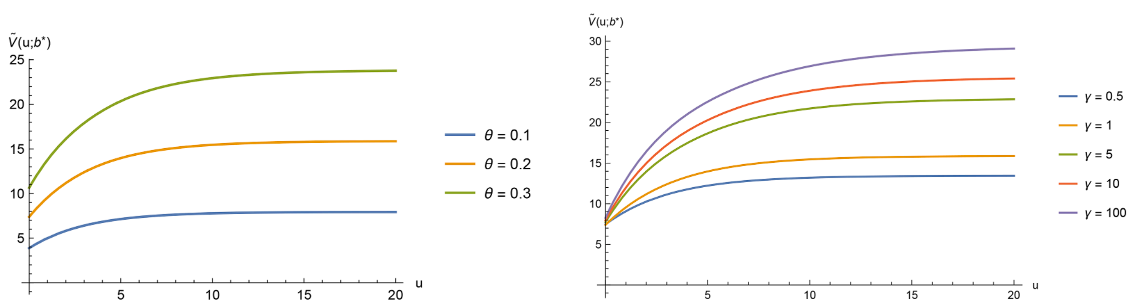
$$p(x) = \frac{1}{3}\left(\frac{1}{2}e^{-\frac{1}{2}x}\right) + \frac{2}{3}(2e^{-2x}). \tag{4.4}$$

Both distributions possess the same mean of 1 but they have variances of 0.5556 and 2 respectively (where the exponential claim distribution in Section 4.1 has mean 1 and variance 1). The same analyses as in the case of exponential claims have been performed. We can confirm that very similar results have been obtained, and the same interpretations are applicable. Instead of reproducing Figs. 2–7 for these two claim distributions, we summarize some major findings as follows.



(a) Plot against γ for fixed θ (b) Plot against θ for fixed γ

Fig. 5. Impact of γ and θ on b^* under exponential claims.



(a) $\gamma = 1$ (b) $\theta = 0.2$

Fig. 6. Plot of $\widetilde{V}(u; b^*)$ against u under exponential claims.

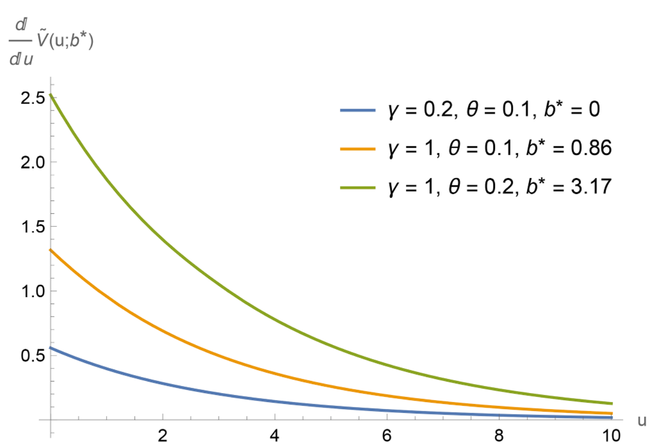


Fig. 7. Plot of $\frac{\mathrm{d}}{\mathrm{d}u}\widetilde{V}(u; b^*)$ against u under exponential claims.

(i) Maximizing $\widetilde{V}(u; b)$ with respect to the threshold level b leads to an optimal threshold b^* that does not depend on u.
(ii) Under the proposed strategy (3.1) implemented at b^* (written as f_{b^*}), the resulting dividend function $V(u, y; b^*)$ satisfies the fixed point property (3.6). Such a strategy is optimal so that the value function is $V(u, y) = V(u, y; b^*)$.
(iii) The value function $V(u, y)$, given by $V(u, y) = V(u, y; b^*)$ or (4.1), increases in u and y.
(iv) The optimal threshold b^* increases in γ and θ.
(v) The dividend function $\widetilde{V}(u; b^*)$ increases in γ and θ.
(vi) The conditions (4.2) and (4.3) hold true when $b^* > 0$.

To get further insights about the impact of claim variance on $\widetilde{V}(u; b^*)$, we would like to compare the values of $\widetilde{V}(u; b^*)$ across the three

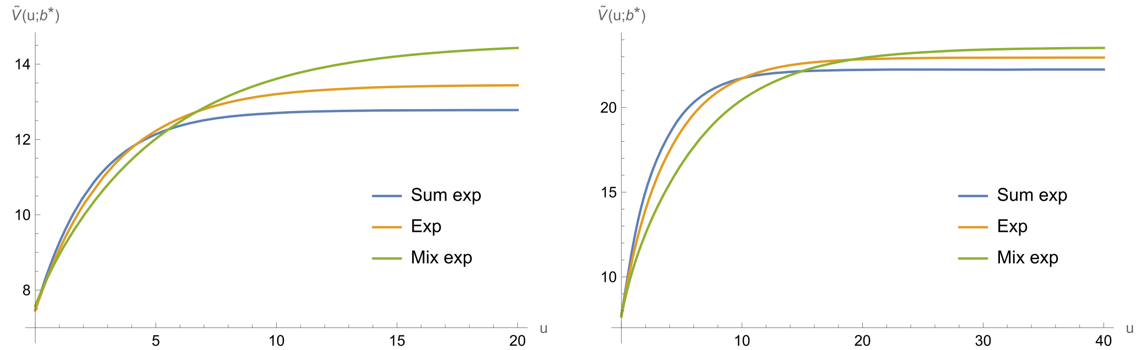
Table 1
Expected net gain $\mathbb{E}[Y^+]$ and expected net loss $\mathbb{E}[Y^-]$ for different claim variances when $\theta = 0.2$.

			$\gamma = 0.5$		$\gamma = 5$	
	Mean	Variance	$\mathbb{E}[Y^+]$	$\mathbb{E}[Y^-]$	$\mathbb{E}[Y^+]$	$\mathbb{E}[Y^-]$
Sum exp	1	0.5556	1.2986	0.2986	0.2228	0.1228
Exp	1	1	1.3660	0.3660	0.2299	0.1299
Mix exp	1	2	1.4794	0.4794	0.2358	0.1358

claim amount distributions. Throughout the analysis we fix $\theta = 0.2$, and $\widetilde{V}(u; b^*)$ is plotted as a function of u when $\gamma = 0.5$ (Fig. 8a) and when $\gamma = 5$ (Fig. 8b). The labels 'Sum exp', 'Exp' and 'Mix exp' in the figures correspond to a sum of exponentials, exponential and a mixture of exponentials, respectively, which have increasing variance. Although our intuition may suggest that $\widetilde{V}(u; b^*)$ shall decrease as the claim variance increases (which makes the surplus process riskier), we observe from Fig. 8 that this is generally not true. Indeed, for larger values of u the function $\widetilde{V}(u; b^*)$ increases with the claim variance. The same observation was also made by Cheung and Zhang (2019)'s Section 5.2 regarding their strategy (3.3), and we can apply similar reasoning to explain such a phenomenon. It is important to first recall that dividend is paid from the insurer's gain between successive observation time points. Therefore, we shall look at the values of the expected net gain $\mathbb{E}[Y^+]$ which can be computed via (3.21). These are provided in Table 1 for $\gamma = 0.5$ and $\gamma = 5$. The corresponding values of the expected net loss $\mathbb{E}[Y^-]$ (where $Y^- = \max(-Y, 0) = Y^+ - Y$) are also given, which can be evaluated as

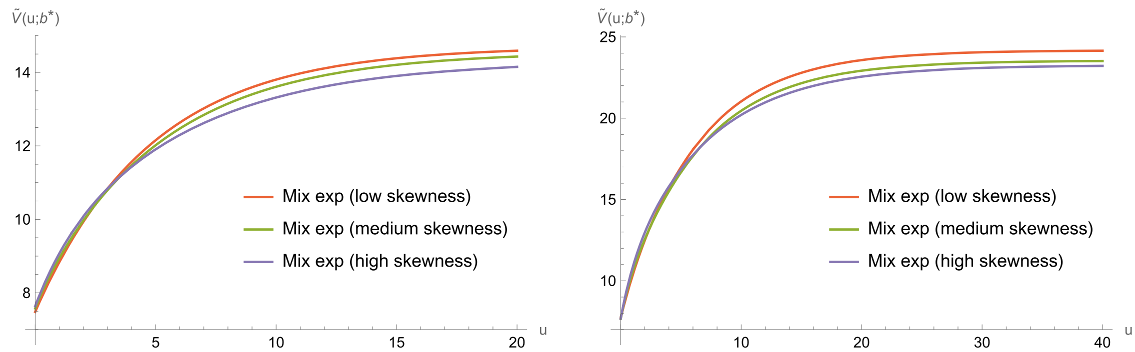
$$\mathbb{E}[Y^-] = \frac{A}{\rho_\gamma^2}\bigg|_{\delta=0} - \frac{c - \lambda\mathbb{E}[X]}{\gamma}$$

due to (3.22). We see from Table 1 that as the variance of the claim distribution increases, both $\mathbb{E}[Y^+]$ and $\mathbb{E}[Y^-]$ increase. While the increase in $\mathbb{E}[Y^-]$ agrees with our intuition of having a riskier business (which



(a) $\gamma = 0.5$ (b) $\gamma = 5$

Fig. 8. Plot of $\widetilde{V}(u; b^*)$ against u for three claim distributions with different variances when $\theta = 0.2$.



(a) $\gamma = 0.5$ (b) $\gamma = 5$

Fig. 9. Plot of $\widetilde{V}(u; b^*)$ against u for three claim distributions with different skewness when $\theta = 0.2$.

Table 2
Expected net gain $\mathbb{E}[Y^+]$ and expected net loss $\mathbb{E}[Y^-]$ for different skewness when $\theta = 0.2$.

				$\gamma = 0.5$		$\gamma = 5$	
	Mean	Variance	Skewness	$\mathbb{E}[Y^+]$	$\mathbb{E}[Y^-]$	$\mathbb{E}[Y^+]$	$\mathbb{E}[Y^-]$
Mix exp (low skewness)	1	2	2.5222	1.4928	0.4928	0.2421	0.1421
Mix exp (medium skewness)	1	2	3.3588	1.4794	0.4794	0.2358	0.1358
Mix exp (high skewness)	1	2	4.8284	1.4560	0.4560	0.2329	0.1329

may tend to decrease dividend payments due to earlier ruin), a higher value of $\mathbb{E}[Y^+]$ means that more dividends can possibly be paid from the gains. For large initial surplus u, the ruin probability is small anyway, and therefore the increase in dividends due to an increase in $\mathbb{E}[Y^+]$ outweighs the increased risk arising from an increase in $\mathbb{E}[Y^-]$. As a result, $\widetilde{V}(u; b^*)$ increases with the variance of the claim amounts for large u. In contrast, when the initial surplus u is small, the surplus process is more susceptible to increased risk and early ruin, thereby causing $\widetilde{V}(u; b^*)$ to decrease in the claim variance. Interestingly, when γ increases, the surplus level u after which $\widetilde{V}(u; b^*)$ increases with the claim variance becomes higher. In particular, when $\gamma = 0.5$ the switch happens at around $u = 7$ while for $\gamma = 5$ this occurs around $u = 20$. This can be attributed to the fact that an increase in γ leads to more frequent monitoring of the process for ruin, and consequently a larger value of u is needed for the process to stay away from ruin.

The above discussion focuses on the impact of the variance of the claim distribution on the expected discounted dividends while keeping the mean of the claim amount fixed. In what follows, we would like to briefly investigate how the skewness of the claim amount may impact $\widetilde{V}(u; b^*)$. (The skewness of the claim amount X is given by $\mathbb{E}[(X - \mathbb{E}[X])^3]/(\mathrm{Var}(X))^{3/2}$.) Three different mixtures of exponentials will be considered, and they have the same mean 1 and the same variance 2 but different degree of skewness. In addition to the mixture of exponentials with density (4.4) possessing a skewness of 3.3588, we additionally utilize the mixture $p(x) = \frac{3}{5}(0.6340e^{-0.6340x}) + \frac{2}{5}(7.4641e^{-7.4641x})$ with a lower skewness of 2.5222 and the mixture $p(x) = \frac{1}{10}(0.3204e^{-0.3204x}) + \frac{9}{10}(1.3084e^{-1.3084x})$ with a higher skewness of 4.8284. The results (i)-(vi) stated at the beginning of Section 4.2 are also valid for these two new claim distributions. Fixing $\theta = 0.2$, we plot $\widetilde{V}(u; b^*)$ against the surplus level u when $\gamma = 0.5$ (Fig. 9a) and when $\gamma = 5$ (Fig. 9b). For small values of u, it is observed that $\widetilde{V}(u; b^*)$ increases as the claim distribution becomes more positively skewed. On the other hand, for larger values of u the dividend function $\widetilde{V}(u; b^*)$ decreases in the skewness of the claim amount. Such a phenomenon can be interpreted in the same manner as we did for Fig. 8 by calculating the expected net gain $\mathbb{E}[Y^+]$ and the expected net loss $\mathbb{E}[Y^-]$ in Table 2 and noting that an increase in skewness in our case results in lower values of both $\mathbb{E}[Y^+]$ and $\mathbb{E}[Y^-]$.

5. Concluding remarks

One of the aims of the paper is to formulate a dividend problem with periodic observations such that the insurer has a positive survival probability when the optimal dividend strategy is implemented. To this end, we propose a novel gain-based constraint where dividend can only be paid from a fraction θ of the gain between successive observations (rather than implementing the traditional constraint of allowing to pay from the entire available surplus). As mentioned in Section 2.1, it is sufficient to set $\theta < \mathbb{E}[Y]/\mathbb{E}[Y^+]$. Through specific numerical examples, it is demonstrated that a threshold type periodic strategy (i.e. one that pays the highest possible dividend when the surplus is above a certain threshold) is optimal.

Naturally, one would wonder whether general sufficient conditions under which our proposed threshold strategy is optimal can be established. This will indeed be very difficult for a number of reasons. First, we recall that our novel gain-based constraint restricts dividend payment to be paid from the latest increment (if it is positive), and the decision rule is based on a two-dimensional state space (i.e. the surplus level u and the latest increment y). This is more complex than the optimal periodic dividend problem considered by the main reference Albrecher et al. (2011a), which is concerned with the traditional constraint where the state space consists of u only. Even Albrecher et al. (2011a) were able to show that the optimal strategy is generally in the form of a band strategy in their setting (see also Remark 1), it does not appear possible to determine the number of bands in the optimal strategy at the outset. As in their Section 7, they need to first determine the dividend function under specific distributional assumptions (e.g. exponential inter-observation times and Erlang(2) claim amounts) by fixing the number of bands before verifying the fixed point property numerically with all the model parameters specified. Indeed, our numerical approach in Section 4 was motivated by the one in Albrecher et al. (2011a). Second, we note that, assuming the simplest situation of exponential inter-observation times and exponential claims, Albrecher et al. (2011a) showed in their Section 5 that the optimal periodic strategy is a barrier strategy under the traditional constraint (i.e. the band strategy collapses to a barrier strategy). However, their method to analytically prove that the value of a barrier strategy is a fixed point of their Bellman equation relies on the availability of an explicit expression for the optimal dividend barrier. The determination of the optimal barrier in their case is possible because the expected present value of dividend payments under a barrier strategy is a simple function of the barrier level, thanks to the fact that one only needs to determine the dividend function in a single layer (i.e. below the barrier). However, in our proposed threshold strategy, there are two interconnecting layers (see the integral equations (3.8) and (3.9)) from which the respective dividend functions need to be solved for. Even for exponential claims ($m = 1$ in Proposition 1), the coefficients C_1, C_2, D and E_1 appearing in the dividend functions (3.13) and (3.14) need to be solved from a system of four linear equations, and they all depend on the threshold level b (see Remark 4 in Section 3.2). This makes it impossible to get an explicit formula for the optimal threshold b^* by taking derivative for maximization, making any analytic attempt to prove that the fixed point property is satisfied extremely hard.

For future research, it will be an interesting topic to develop different methodologies to derive sufficient conditions for the optimality of our proposed threshold strategy under a gain-based constraint. Such conditions may be related to the complete monotonicity of the claim distribution as shown by Kyprianou et al. (2012) in the case of continuous observations under the constraint of a bounded dividend rate. However, this is far from trivial in the present context of periodic observations because the observation scheme (i.e. the distribution of the inter-observation times) may also play a role in specifying the sufficient conditions. Moreover, the verification of optimality in Kyprianou et al. (2012) is largely based on (i) an expression of the value of their proposed continuously observed threshold strategy in terms of the scale function; and (ii) the properties of the scale function in Lévy processes where the Lévy measure has a completely monotone density. But in our case of a periodically observed process, it is unclear how the scale function can be applied to obtain the expected discounted dividends under our proposed threshold strategy. Another related research question is the form of the optimal strategy in cases where the proposed threshold strategy is not optimal, and this is not easy either. Nevertheless, our work provides a starting point as the largest maximizer of V must satisfy (2.16).

Concerning numerical procedure to approximate the value function and hence an optimal periodic strategy under our gain-based constraint, the existence of a unique fixed point of $\mathcal{T}$ due to it being a contraction mapping (see Theorem 2) may suggest that one could iteratively apply the operator $\mathcal{T}$ to an initial guess function to obtain the solution to the Bellman equation. However, since the state space and the action space are both continuous, discretization or grid-based methods will be required and these will be subject to the curse of dimensionality. The design of efficient numerical algorithms can be important for future research as well.

We also wish to point out that it is not our primary objective here to maximize the expected present value of dividends paid until ruin such that the resulting ruin probability is no larger than a given tolerance level. Indeed, dividend optimization under a ruin probability constraint presents a very challenging research problem. While this has been considered in models with continuous observations, exact solutions are not available in the literature. The existing results include numerical schemes and approximation procedures (see Grandits (2015), Hipp (2018, 2019), and Albrecher et al. (2025)), and the optimal strategy is no longer a threshold strategy. Added complexity is anticipated if the observations are periodic. As an alternative, one may restrict the set of admissible strategies to be periodic threshold strategies and optimize dividends subject to a ruin probability constraint. Since the expected discounted dividends and the ruin probability can both be determined under a periodic threshold strategy, numerical optimization can be performed with respect to threshold level b and the dividend fraction θ. See Dickson and Drekic (2006) for similar ideas in the context of continuous observations. Another interesting research problem will be to take into account the risk-sensitivity of the insurance company via maximizing the expected utility of the discounted dividends (see Bäuerle and Jaśkiewicz (2015)). We leave these as open questions for future research.

CRediT authorship contribution statement

Eric C.K. Cheung: Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Formal analysis, Conceptualization; **Guo Liu:** Writing – review & editing, Software; **Jae-Kyung Woo:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization; **Jiannan Zhang:** Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis, Conceptualization; **Dan Zhu:** Writing – review & editing, Supervision.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this submission.

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Appendix A.

A.1. Proof of Lemma 1

(a) If each increment Y_n (for $n \in \mathbb{N}_0$) is replaced by Y_n^+, then the value function for the modified process clearly increases and the modified process will never experience ruin even when dividend is paid at the maximum possible amount θY_n^+ at each observation time Z_n. Due to the time value of money, it is optimal to pay such maximum amount at the earliest opportunity, and therefore we have

$$V(u,y) \le \theta y^+ + \theta \mathbb{E}\left[\sum_{n=1}^{\infty} e^{-\delta Z_n} Y_n^+\right] = \theta y^+ + \theta \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\prod_{k=1}^{n-1} e^{-\delta T_k}\right)\left(e^{-\delta T_n} Y_n^+\right)\right].$$

The upper bound in (2.5) then follows by the mutual independence of $T_1,\dots,T_{n-1}$ and (T_n, Y_n).

Concerning the lower bound, one can consider a specific admissible strategy that pays the maximal dividend at each observation time until ruin occurs. This leads to

$$V(u,y) \ge \theta y^+ + \theta \mathbb{E}_{u,y}\left[\sum_{n=1}^{\infty} e^{-\delta Z_n} Y_n^+ 1_{\{U_1,\dots,U_n \ge 0\}}\right],$$

where it is understood that the surplus levels $\{U_n\}_{n=1}^{\infty}$ are specific to the afore-mentioned strategy, i.e. $U_n = U_{n-1} - \theta Y_{n-1}^+ + Y_n$ for $n \in \mathbb{N}$. It is clear that if $Y_1,\dots,Y_n$ are all non-negative then $U_1,\dots,U_n$ are all non-negative as well. Therefore, one can lower bound the above expectation by replacing $1_{\{U_1,\dots,U_n\ge 0\}}$ by $1_{\{Y_1,\dots,Y_n \ge 0\}}$ so that

$$V(u,y) \ge \theta y^+ + \theta \mathbb{E}\left[\sum_{n=1}^{\infty} e^{-\delta Z_n} Y_n^+ 1_{\{Y_1,\dots,Y_n\ge 0\}}\right] = \theta y^+ + \theta \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\prod_{k=1}^{n-1} e^{-\delta T_k} 1_{\{Y_k \ge 0\}}\right)\left(e^{-\delta T_n} Y_n^+\right)\right],$$

from which the lower bound in (2.5) follows.

(b) From the upper bound in (2.5), it is clear that

$$V \le C\beta, \tag{A.1}$$

for some positive constant C, where β is the upper bounding function defined in (2.6). For $a \in [0,\theta y^+]$ and $(u,y)\in E$, it is noted that

$$\begin{aligned}\int_0^\infty \int_{a-u}^\infty e^{-\delta t}\beta(u-a+x,x)Q(\mathrm{d}x,\mathrm{d}t) &= \int_0^\infty \int_{a-u}^\infty e^{-\delta t}(H_1+H_2x^+)Q(\mathrm{d}x,\mathrm{d}t)\\ &\le \int_0^\infty \int_{-\infty}^\infty e^{-\delta t}(H_1+H_2x^+)Q(\mathrm{d}x,\mathrm{d}t)\\ &= H_1\mathbb{E}[e^{-\delta T}] + H_2\mathbb{E}[e^{-\delta T}Y^+], \end{aligned} \tag{A.2}$$

where the inequality follows from the non-negativity of the integrand. Therefore, taking supremum and using the definition of the operator (2.4) yields

$$\mathcal{T}_\circ \beta(u,y) \le H_1\mathbb{E}[e^{-\delta T}] + H_2\mathbb{E}[e^{-\delta T}Y^+]. \tag{A.3}$$

Recursively, it can be seen that

$$\mathcal{T}_\circ^n \beta \le H_1(\mathbb{E}[e^{-\delta T}])^n + H_2(\mathbb{E}[e^{-\delta T}])^{n-1}\mathbb{E}[e^{-\delta T}Y^+], \qquad n\in\mathbb{N}.$$

Letting $n\to\infty$ gives rise to (2.7).

A.2. Proof of Theorem 1

The results are a direct consequence of Theorems 7.1.8 and 7.2.1 in Bäuerle and Rieder (2011) (see also Theorem 3.2 in Albrecher et al. (2011a)). In particular, the properties (2.7) and (A.1) from Lemma 1 imply that the integrability assumption (A) and the convergence assumption (C) on pp.195–196 as well as the structure assumption (SA) on p.199 of Bäuerle and Rieder (2011) are satisfied. Therefore, $\pi^* = (f^*, f^*, \dots)$ is an optimal stationary policy, where f^* is a maximizer of V. Moreover, history-dependent policies do not improve the expected present value of dividend payments until ruin (see Remark 7.1.3 in Bäuerle and Rieder (2011)).

A.3. Proof of Theorem 2

The proof follows closely that of Lemma 7.3.3 in Bäuerle and Rieder (2011). First, from (A.3) we can get, for $(u,y)\in E$,

$$\mathcal{T}_\circ\beta(u,y) \le \left(\mathbb{E}[e^{-\delta T}] + \frac{H_2}{H_1}\mathbb{E}[e^{-\delta T}Y^+]\right)(H_1 + H_2y^+).$$

Therefore, if H_1 and H_2 satisfy (2.14), then we have

$$\mathcal{T}_\circ\beta(u,y) \le \alpha\beta(u,y), \tag{A.4}$$

where

$$\alpha = \mathbb{E}[e^{-\delta T}] + \frac{H_2}{H_1}\mathbb{E}[e^{-\delta T}Y^+]$$

is such that $\alpha\in(0,1)$. It is important to note that, with $\mathbb{E}[e^{-\delta T}]<1$ and $\mathbb{E}[e^{-\delta T}Y^+]>0$, we can always choose a large value of H_1 and/or a small positive value of H_2 to ensure that (2.14) is satisfied.

Next, defining the weighted supremum norm

$$||v||_\beta = \sup_{(u,y)\in E}\frac{|v(u,y)|}{\beta(u,y)}, \qquad v\in\mathbb{M}_\beta^\star, \tag{A.5}$$

it is known that $(\mathbb{M}_\beta^\star, ||\cdot||_\beta)$ is a Banach space. For the remainder of this proof, we shall extend the domain on which the operators $\mathcal{T}_f$ (in (2.8)) and $\mathcal{T}$ (in (2.9)) can act on from $\mathbb{M}_\beta$ to $\mathbb{M}_\beta^\star$. Suppose that $v,w\in\mathbb{M}_\beta^\star$. With f a decision rule satisfying $f(u,y)\in[0,\theta y^+]$ for $(u,y)\in E$, we have

$$\sup_f \mathcal{T}_f v - \sup_f \mathcal{T}_f w \le \sup_f(\mathcal{T}_f v - \mathcal{T}_f w).$$

Utilizing the definitions (2.9) and (2.8) on the left-hand side and the right-hand side respectively leads to, for $(u,y)\in E$,

$$\begin{aligned}&\mathcal{T}v(u,y) - \mathcal{T}w(u,y)\\ &\le \sup_{a\in[0,\theta y^+]}\int_0^\infty\int_{a-u}^\infty e^{-\delta t}\{v(u-a+x,x) - w(u-a+x,x)\}Q(\mathrm{d}x,\mathrm{d}t)\\ &\le ||v-w||_\beta \sup_{a\in[0,\theta y^+]}\int_0^\infty\int_{a-u}^\infty e^{-\delta t}\beta(u-a+x,x)Q(\mathrm{d}x,\mathrm{d}t),\end{aligned}$$

where the last inequality follows from the definition (A.5). Noting from the definition (2.4) that the supremum of the integral above is simply $\mathcal{T}_\circ\beta(u,y)$, further use of the bound (A.4) gives rise to

$$\mathcal{T}v(u,y) - \mathcal{T}w(u,y) \le \alpha||v-w||_\beta\,\beta(u,y).$$

Similarly, reversing the roles of v and w in the above argument yields

$$\mathcal{T}w(u,y) - \mathcal{T}v(u,y) \le \alpha||v-w||_\beta\,\beta(u,y).$$

Combining the above two inequalities, one can take the weighted supremum norm on $\mathcal{T}v - \mathcal{T}w$ to see that

$$||\mathcal{T}v - \mathcal{T}w||_\beta \le \alpha||v-w||_\beta.$$

Since $\alpha\in(0,1)$, the operator $\mathcal{T}$ is contracting on $(\mathbb{M}_\beta^\star, ||\cdot||_\beta)$, and the statement of the theorem follows from the Banach's fixed point theorem (e.g. Theorem A.3.5 in Bäuerle and Rieder (2011)).

A.4. Proof of Lemma 2

(a) First, for fixed $y\in\mathbb{R}$, we consider $u_2\ge u_1$ such that $(u_1,y)\in E$ and $(u_2,y)\in E$. For a surplus process starting with the initial condition (u_2,y), one can implement a strategy that is optimal for the initial condition (u_1,y), resulting in expected discounted dividend payments of $V(u_1,y)$. Note that the ruin event of this process (if it ever happens) will not happen earlier than the process starting with (u_1,y) as it always has u_2-u_1 units of surplus in excess of the latter process. The optimal strategy under the initial condition (u_2,y) cannot be inferior to the afore-mentioned strategy, implying $V(u_2,y)\ge V(u_1,y)$.

Second, for fixed $u\in\mathbb{R}^+$, we suppose $y_2\ge y_1$ with $(u,y_1)\in E$ and $(u,y_2)\in E$. Starting with (u,y_2), the optimization on the right-hand side of (2.10) is performed over the interval $[0,\theta y_2^+]$ which is wider than the interval $[0,\theta y_1^+]$ if one has instead started with (u,y_1). Therefore, one must have $V(u,y_2)\ge V(u,y_1)$.

(b) For later use, we start by observing that the function G defined in (2.12) is increasing. Specifically, for $u_2 \geq u_1 \geq 0$ we have

$$\begin{aligned} G(u_2) &= \int_0^\infty \int_{-u_2}^\infty e^{-\delta t} V(u_2+x,x) Q(\mathrm{d}x,\mathrm{d}t) \\ &\geq \int_0^\infty \int_{-u_1}^\infty e^{-\delta t} V(u_2+x,x) Q(\mathrm{d}x,\mathrm{d}t) \\ &\geq \int_0^\infty \int_{-u_1}^\infty e^{-\delta t} V(u_1+x,x) Q(\mathrm{d}x,\mathrm{d}t) \\ &= G(u_1), \end{aligned}$$

where the second line follows by the non-negativity of the integrand and the third line is due to the fact that V is increasing (see part (a)). Next, we note that $V(u,y)=V(u,0)$ for $u \in \mathbb{R}^+$ and $y \in \mathbb{R}^-$ because no dividend can be paid at time 0 if the initial increment is negative. So we focus on comparing $V(u_1,y_1)$ and $V(u_2,y_2)$ when $y_2 \geq y_1 \geq 0$, and we set the constraint $u_2 \geq u_1 + \theta(y_2-y_1)$. Then, one has from (2.11) that

$$\begin{aligned} V(u_2,y_2) &= \sup_{a\in[0,\theta y_2]} \{a + G(u_2-a)\} \\ &\geq \sup_{\overline{a}\in[0,\theta y_1]} \{\theta(y_2-y_1) + \overline{a} + G(u_2-\theta(y_2-y_1)-\overline{a})\} \quad \text{(A.6)} \\ &\geq \sup_{\overline{a}\in[0,\theta y_1]} \{\theta(y_2-y_1) + \overline{a} + G(u_1-\overline{a})\} \quad \text{(A.7)} \\ &= \theta(y_2-y_1) + \sup_{\overline{a}\in[0,\theta y_1]} \{\overline{a} + G(u_1-\overline{a})\} \\ &= \theta(y_2-y_1) + V(u_1,y_1). \end{aligned}$$

The inequality (A.6) can be explained as follows. For any $\overline{a} \in [0,\theta y_1]$, one has $\theta(y_2-y_1)+\overline{a} \in [\theta(y_2-y_1),\theta y_2] \subset [0,\theta y_2]$. In other words, $\theta(y_2-y_1)+\overline{a}$ is an admissible action at position (u_2,y_2), and such an action cannot be better than an optimal action $a \in [0,\theta y_2]$. The inequality (A.7) holds because $u_2-\theta(y_2-y_1) \geq u_1$ according to the assumption and G is an increasing function.

A.5. Proof of Theorem 3

When $u \in \mathbb{R}^+$ and $y \in \mathbb{R}^-$, a decision rule f must satisfy $f(u,y)=0$ (and the same is true for a maximizer), and therefore (2.15) and (2.16) become trivial. It is sufficient to consider the case $u \geq y \geq 0$ in this proof.

For now we consider the case where we start with the initial condition $(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ which must belong to the state space E. We proceed to utilize the Bellman equation (2.11) under such initial condition and choose the action $a=0$ which cannot outperform a maximizer. This results in

$$V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \geq G(u-f^*(u,y)). \quad \text{(A.8)}$$

From (2.13), the expression on the right-hand side of (A.8) equals $V(u,y)-f^*(u,y)$ and thus

$$V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \geq V(u,y)-f^*(u,y).$$

On the other hand, application of Lemma 2(b) with $(u_1,y_1)=(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ and $(u_2,y_2)=(u,y)$ leads to

$$V(u,y) - V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \geq f^*(u,y).$$

Combining the above two inequalities gives rise to the desired result (2.15).

Next, because of (2.13), the result (2.15) can be written as

$$V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) = G(u-f^*(u,y)).$$

Comparison with (2.11) reveals that under the initial state $(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ the action $a=0$ is optimal. However, as the uniqueness of f^* may not be guaranteed, one cannot conclude that (2.16) must hold true. Hence, we shall focus on the largest maximizer of V.

Suppose that f^* is the largest maximizer of V. It remains to show that f^* satisfies (2.16). Suppose on the contrary that $f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y)) > 0$. Using (2.13) under the initial state $(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ yields

$$\begin{aligned} & V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \\ &= f^*\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \\ &\quad + G\Big(u-f^*(u,y)-f^*\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big)\Big). \end{aligned} \quad \text{(A.9)}$$

Note that if one starts with an initial state of (u,y), then a dividend payment of $f^*(u,y)+f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ is admissible because $f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y)) \leq \theta y - f^*(u,y)$ and hence $f^*(u,y)+f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y)) \leq \theta y$. Then we study the expression

$$\begin{aligned} & f^*(u,y) + f^*\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \\ &\quad + G\Big(u-f^*(u,y)-f^*\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big)\Big) \\ &= f^*(u,y) + V\Big(u-f^*(u,y), y-\frac{1}{\theta}f^*(u,y)\Big) \\ &= V(u,y), \end{aligned}$$

where the two equalities follow from (A.9) and (2.15) respectively. The above result implies that $f^*(u,y)+f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ is also a maximizer of $V(u,y)$ (see (2.11)). Consequently, with $f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y)) > 0$ we have $f^*(u,y)+f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y)) > f^*(u,y)$, meaning that $f^*(u,y)$ is not the largest maximizer of $V(u,y)$ and leading to a contradiction. Therefore, $f^*(u-f^*(u,y), y-(1/\theta)f^*(u,y))$ cannot be positive (and the dividend payout cannot be negative in our model) and one concludes that (2.16) holds true for the largest maximizer f^*.

A.6. Proof of Proposition 1

We start by handling the integral equation (3.8). Substitution of (3.11) into the first two integrals in (3.8) followed by straightforward algebra gives rise to

$$\begin{aligned} & \int_{\frac{b-u}{1-\theta}}^\infty \Big[\theta x + \widetilde{V}_2(u+(1-\theta)x)\Big] g_{\delta,-}(x)\mathrm{d}x \\ &\quad + \int_{b-u}^{\frac{b-u}{1-\theta}} \Big[(u+x-b) + \widetilde{V}(b;b)\Big] g_{\delta,-}(x)\mathrm{d}x \\ &= \frac{A}{1-\theta}\Big(\int_b^\infty \widetilde{V}_2(x) e^{-\frac{\rho_\gamma}{1-\theta}x}\mathrm{d}x\Big) e^{\frac{\rho_\gamma}{1-\theta}u} + \frac{A}{\rho_\gamma^2} e^{-\rho_\gamma b} e^{\rho_\gamma u} - \frac{(1-\theta)A}{\rho_\gamma^2} e^{-\frac{\rho_\gamma}{1-\theta}b} e^{\frac{\rho_\gamma}{1-\theta}u} \\ &\quad + \widetilde{V}(b;b)\frac{A}{\rho_\gamma} e^{-\rho_\gamma b} e^{\rho_\gamma u} - \widetilde{V}(b;b)\frac{A}{\rho_\gamma} e^{-\frac{\rho_\gamma}{1-\theta}b} e^{\frac{\rho_\gamma}{1-\theta}u}, \end{aligned} \quad \text{(A.10)}$$

which is a sum of two exponential terms in u. Therefore, we have

$$\begin{aligned} & \Big(\frac{\mathrm{d}}{\mathrm{d}u} - \rho_\gamma\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}u} - \frac{\rho_\gamma}{1-\theta}\Big)\Big(\int_{\frac{b-u}{1-\theta}}^\infty \Big[\theta x + \widetilde{V}_2(u+(1-\theta)x)\Big] g_{\delta,-}(x)\mathrm{d}x \\ &\quad + \int_{b-u}^{\frac{b-u}{1-\theta}} \Big[(u+x-b) + \widetilde{V}_1(b)\Big] g_{\delta,-}(x)\mathrm{d}x\Big) \\ &= 0. \end{aligned} \quad \text{(A.11)}$$

Regarding the third integral in (3.8), with the use of (3.11) it can readily be shown that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}u} - \rho_\gamma\Big) \int_0^{b-u} \widetilde{V}_1(u+x) g_{\delta,-}(x)\mathrm{d}x = -A\widetilde{V}_1(u). \quad \text{(A.12)}$$

Following the analysis leading to Equation (49) in Cheung and Zhang (2019), the fourth integral in (3.8) satisfies

$$\Big[\prod_{\ell=1}^m \Big(\frac{\mathrm{d}}{\mathrm{d}u} + R_{\gamma,\ell}\Big)\Big] \int_0^u \widetilde{V}_1(u-x) g_{\delta,+}(x)\mathrm{d}x$$

$$= \sum_{j=1}^{m} B_j \left[\prod_{\ell=1,\ell\neq j}^{m} \left(\frac{\mathrm{d}}{\mathrm{d}u} + R_{\gamma,\ell} \right) \right] \widetilde{V}_1(u). \tag{A.13}$$

Utilizing (A.11)-(A.13), we apply the operator $(\mathrm{d}/\mathrm{d}u - \rho_\gamma)(\mathrm{d}/\mathrm{d}u - \rho_\gamma/(1-\theta)) \prod_{\ell=1}^{m} (\mathrm{d}/\mathrm{d}u + R_{\gamma,\ell})$ to (3.8) to arrive at the $(m+2)$-th order homogeneous ordinary differential equation

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}u} - \rho_\gamma \right) \left(\frac{\mathrm{d}}{\mathrm{d}u} - \frac{\rho_\gamma}{1-\theta} \right) \left[\prod_{\ell=1}^{m} \left(\frac{\mathrm{d}}{\mathrm{d}u} + R_{\gamma,\ell} \right) \right] \widetilde{V}_1(u) \\ &= - A \left(\frac{\mathrm{d}}{\mathrm{d}u} - \frac{\rho_\gamma}{1-\theta} \right) \left[\prod_{\ell=1}^{m} \left(\frac{\mathrm{d}}{\mathrm{d}u} + R_{\gamma,\ell} \right) \right] \widetilde{V}_1(u) \\ &\quad + \sum_{j=1}^{m} B_j \left(\frac{\mathrm{d}}{\mathrm{d}u} - \rho_\gamma \right) \left(\frac{\mathrm{d}}{\mathrm{d}u} - \frac{\rho_\gamma}{1-\theta} \right) \left[\prod_{\ell=1,\ell\neq j}^{m} \left(\frac{\mathrm{d}}{\mathrm{d}u} + R_{\gamma,\ell} \right) \right] \widetilde{V}_1(u). \end{aligned} \tag{A.14}$$

The characteristic equation (in s) is given by

$$\begin{aligned} &(s - \rho_\gamma) \left(s - \frac{\rho_\gamma}{1-\theta} \right) \prod_{\ell=1}^{m} (s + R_{\gamma,\ell}) \\ &= - A \left(s - \frac{\rho_\gamma}{1-\theta} \right) \prod_{\ell=1}^{m} (s + R_{\gamma,\ell}) \\ &\quad + \sum_{j=1}^{m} B_j (s - \rho_\gamma) \left(s - \frac{\rho_\gamma}{1-\theta} \right) \prod_{\ell=1,\ell\neq j}^{m} (s + R_{\gamma,\ell}), \end{aligned}$$

which consists of the root $\rho_\gamma/(1-\theta)$ and the $m+1$ roots (namely $\{r_i\}_{i=1}^{m+1}$) of the equation (in s)

$$1 = \frac{A}{\rho_\gamma - s} + \sum_{j=1}^{m} \frac{B_j}{s + R_{\gamma,j}}. \tag{A.15}$$

From Equations (3.2) and (4.1) in Albrecher et al. (2013), the right-hand side can be written as

$$\begin{aligned} \frac{A}{\rho_\gamma - s} + \sum_{j=1}^{m} \frac{B_j}{s + R_{\gamma,j}} &= \mathbb{E}\left[e^{-\delta T_1 - s\left(\sum_{i=1}^{N_{T_1}} X_i - c T_1 \right)} \right] \\ &= \frac{\gamma}{\gamma + \delta - \{cs - \lambda[1 - \widehat{p}(s)]\}}. \end{aligned}$$

Consequently, (A.15) is equivalent to a special case of (3.10) with $\gamma = 0$ (and therefore $\{r_i\}_{i=1}^{m+1}$ is equivalent to the set consisting of ρ_0 and $\{-R_{0,j}\}_{j=1}^{m}$). The solution to (A.14) is thus in the form of (3.13), where $\{C_i\}_{i=1}^{m+1}$ and D are constants to be determined.

Next, we consider the integral equation (3.9) which is structurally identical to Equation (21) in Cheung and Zhang (2019). Following the procedure leading to Equation (57) therein, one has the result (3.14), where w is given by (3.15), $\{s_i\}_{i=1}^{m}$ are those m roots with negative real parts of the equation (3.16) in s, and $\{E_i\}_{i=1}^{m}$ are constants to be determined.

To find the unknown constants, we proceed by substituting the solution forms (3.13) and (3.14) (along with (3.11) and (3.12)) into the integral equations (3.8) and (3.9) and evaluating various integrals. We begin by considering (3.8). First, (A.10) becomes

$$\begin{aligned} &\int_{\frac{b-u}{1-\theta}}^{\infty} \left[\theta x + \widetilde{V}_2(u + (1-\theta)x) \right] g_{\delta,-}(x)\mathrm{d}x \\ &\quad + \int_{b-u}^{\frac{b-u}{1-\theta}} \left[(u + x - b) + \widetilde{V}_2(b) \right] g_{\delta,-}(x)\mathrm{d}x \\ &= A \left(\frac{w}{\rho_\gamma} e^{-\frac{\rho_\gamma}{1-\theta} b} + \sum_{i=1}^{m} \frac{E_i}{\rho_\gamma - (1-\theta)s_i} e^{-\left(\frac{\rho_\gamma}{1-\theta} - s_i \right) b} \right) e^{\frac{\rho_\gamma}{1-\theta} u} + \frac{A}{\rho_\gamma^2} e^{-\rho_\gamma b} e^{\rho_\gamma u} \\ &\quad - \frac{(1-\theta)A}{\rho_\gamma^2} e^{-\frac{\rho_\gamma}{1-\theta} b} e^{\frac{\rho_\gamma}{1-\theta} u} + \left(\sum_{i=1}^{m+1} C_i e^{r_i b} + D e^{\frac{\rho_\gamma}{1-\theta} b} \right) \frac{A}{\rho_\gamma} e^{-\rho_\gamma b} e^{\rho_\gamma u} \\ &\quad - \left(w + \sum_{i=1}^{m} E_i e^{s_i b} \right) \frac{A}{\rho_\gamma} e^{-\frac{\rho_\gamma}{1-\theta} b} e^{\frac{\rho_\gamma}{1-\theta} u}. \end{aligned} \tag{A.16}$$

(In the second last term, we can replace $\widetilde{V}(b; b)$ by $\widetilde{V}_1(b)$ or $\widetilde{V}_2(b)$ thanks to the continuity of $\widetilde{V}(u; b)$ at $u = b$.) Second, the third integral on the right-hand side of (3.8) is evaluated as

$$\begin{aligned} \int_0^{b-u} \widetilde{V}_1(u + x) g_{\delta,-}(x)\mathrm{d}x &= A \sum_{i=1}^{m+1} \frac{C_i}{\rho_\gamma - r_i} e^{r_i u} - A \sum_{i=1}^{m+1} \frac{C_i}{\rho_\gamma - r_i} e^{-(\rho_\gamma - r_i)b} e^{\rho_\gamma u} \\ &\quad - AD \frac{1-\theta}{\rho_\gamma \theta} e^{\frac{\rho_\gamma}{1-\theta} u} + AD \frac{1-\theta}{\rho_\gamma \theta} e^{\frac{\rho_\gamma \theta}{1-\theta} b} e^{\rho_\gamma u}. \end{aligned} \tag{A.17}$$

Third, the fourth integral in (3.8) is

$$\begin{aligned} \int_0^{u} \widetilde{V}_1(u - x) g_{\delta,+}(x)\mathrm{d}x &= \sum_{i=1}^{m+1} C_i \left(\sum_{j=1}^{m} \frac{B_j}{R_{\gamma,j} + r_i} \right) e^{r_i u} \\ &\quad + D \sum_{j=1}^{m} \frac{B_j}{\frac{\rho_\gamma}{1-\theta} + R_{\gamma,j}} e^{\frac{\rho_\gamma}{1-\theta} u} \\ &\quad - \sum_{j=1}^{m} B_j \left(\sum_{i=1}^{m+1} \frac{C_i}{R_{\gamma,j} + r_i} + \frac{D}{\frac{\rho_\gamma}{1-\theta} + R_{\gamma,j}} \right) e^{-R_{\gamma,j} u}. \end{aligned} \tag{A.18}$$

Recall from (3.8) that (3.13) is equal to the sum of (A.16)-(A.18). We start by equating the coefficients of $e^{r_i u}$ (for $i = 1, \ldots, m+1$). But this does not provide us with useful information because each r_i satisfies (A.15). Now, we equate the coefficients of $e^{\frac{\rho_\gamma}{1-\theta} u}$ to arrive at

$$\begin{aligned} D &= A \left(\frac{w}{\rho_\gamma} e^{-\frac{\rho_\gamma}{1-\theta} b} + \sum_{i=1}^{m} \frac{E_i}{\rho_\gamma - (1-\theta)s_i} e^{-\left(\frac{\rho_\gamma}{1-\theta} - s_i \right) b} \right) - \frac{(1-\theta)A}{\rho_\gamma^2} e^{-\frac{\rho_\gamma}{1-\theta} b} \\ &\quad - \left(w + \sum_{i=1}^{m} E_i e^{s_i b} \right) \frac{A}{\rho_\gamma} e^{-\frac{\rho_\gamma}{1-\theta} b} - AD \frac{1-\theta}{\rho_\gamma \theta} + D \sum_{j=1}^{m} \frac{B_j}{\frac{\rho_\gamma}{1-\theta} + R_{\gamma,j}}, \end{aligned}$$

where simplifications lead to (3.17). Similarly, equating the coefficients of $e^{\rho_\gamma u}$ gives

$$\begin{aligned} 0 &= \frac{A}{\rho_\gamma^2} e^{-\rho_\gamma b} + \left(\sum_{i=1}^{m+1} C_i e^{r_i b} + D e^{\frac{\rho_\gamma}{1-\theta} b} \right) \frac{A}{\rho_\gamma} e^{-\rho_\gamma b} - A \sum_{i=1}^{m+1} \frac{C_i}{\rho_\gamma - r_i} e^{-(\rho_\gamma - r_i)b} \\ &\quad + AD \frac{1-\theta}{\rho_\gamma \theta} e^{\frac{\rho_\gamma \theta}{1-\theta} b}, \end{aligned}$$

which simplifies to (3.18). Finally, the coefficients of $e^{-R_{\gamma,j} u}$ imply (3.19).

Next, we look at (3.9) and note that we shall first decompose the second integral as

$$\begin{aligned} &\int_0^{u} \widetilde{V}(u - x; b) g_{\delta,+}(x)\mathrm{d}x \\ &= \int_0^{u-b} \widetilde{V}_2(u - x) g_{\delta,+}(x)\mathrm{d}x + \int_{u-b}^{u} \widetilde{V}_1(u - x) g_{\delta,+}(x)\mathrm{d}x, \end{aligned}$$

so as to substitute the solution forms (3.13) and (3.14). Omitting the details, (3.9) becomes

$$\begin{aligned} w + \sum_{i=1}^{m} E_i e^{s_i u} &= \frac{\theta A}{\rho_\gamma^2} + w \left(\frac{A}{\rho_\gamma} + \sum_{j=1}^{m} \frac{B_j}{R_{\gamma,j}} \right) \\ &\quad + \sum_{i=1}^{m} E_i \left(\frac{A}{\rho_\gamma - (1-\theta)s_i} + \sum_{j=1}^{m} \frac{B_j}{R_{\gamma,j} + s_i} \right) e^{s_i u} \\ &\quad + \sum_{j=1}^{m} B_j \left(\sum_{i=1}^{m+1} \frac{C_i}{R_{\gamma,j} + r_i} (e^{(R_{\gamma,j} + r_i)b} - 1) \right. \\ &\quad + \frac{D}{\frac{\rho_\gamma}{1-\theta} + R_{\gamma,j}} \left(e^{\left(\frac{\rho_\gamma}{1-\theta} + R_{\gamma,j} \right) b} - 1 \right) \\ &\quad \left. - \frac{w}{R_{\gamma,j}} e^{R_{\gamma,j} b} - \sum_{i=1}^{m} \frac{E_i}{R_{\gamma,j} + s_i} e^{(R_{\gamma,j} + s_i)b} \right) e^{-R_{\gamma,j} u}. \end{aligned}$$

Equating the constant term on both sides does not yield any information because of the first equality in (3.15). Equating the coefficients of

$e^{s_i u}$ (for $i = 1, \ldots, m$) does not reveal additional information either because each s_i satisfies (3.16). Finally, with the help of (3.19), one obtains (3.20) from the coefficients of $e^{-R_{\gamma,j} u}$.

References

Albrecher, H., Bäuerle, N., Thonhauser, S., 2011a. Optimal dividend-payout in random discrete time. Stat. Risk Model. 28 (3), 251–276.

Albrecher, H., Boxma, O., Essifi, R., Kuijstermans, R., 2017. A queueing model with randomized depletion of inventory. Probab. Eng. Inf. Sci. 31 (1), 43–59.

Albrecher, H., Cheung, E. C.K., Thonhauser, S., 2011b. Randomized observation periods for the compound poisson risk model: dividend. ASTIN Bull. 41 (2), 645–672.

Albrecher, H., Cheung, E. C.K., Thonhauser, S., 2013. Randomized observation periods for the compound poisson risk model: the discounted penalty function. Scand. Actuar. J. 2013 (6), 424–452.

Albrecher, H., Flores, B.G., Hipp, C., 2025. Dividend corridors and a ruin constraint. Insurance: Math. Econ. 121, 1–25.

Albrecher, H., Ivanovs, J., 2013. A risk model with an observer in a markov environment. Risks 1 (3), 148–161.

Albrecher, H., Ivanovs, J., 2017. Strikingly simple identities relating exit problems for lévy processes under continuous and poisson observations. Stoch Process. Their Appl. 127, 643–656.

Albrecher, H., Thonhauser, S., 2009. Optimality results for dividend problems in insurance. Revista de la Real Academia de Ciencias Exactas 103, 295–320.

Avanzi, B., 2009. Strategies for dividend distribution: a review. N. Am. Actuar. J. 13 (2), 217–251.

Avanzi, B., Cheung, E. C.K., Wong, B., Woo, J.-K., 2013. On a periodic dividend barrier strategy in the dual model with continuous monitoring of solvency. Insurance: Math. Econ. 52 (1), 98–113.

Avanzi, B., Lau, H., Wong, B., 2021. On the optimality of joint periodic and extraordinary dividend strategies. Eur. J. Oper. Res. 295 (3), 1189–1210.

Avram, F., Palmowski, Z., Pistorius, M.R., 2007. On the optimal dividend problem for a spectrally negative lévy process. Ann. Appl. Probab. 17 (1), 156–180.

Azcue, P., Muler, N., 2012. Optimal dividend policies for compound poisson processes: the case of bounded dividend rates. Insurance: Math. Econ. 51 (1), 26–42.

Bäuerle, N., Jaśkiewicz, A., 2015. Risk-sensitive dividend problems. Eur. J. Oper. Res. 242 (1), 161–171.

Bäuerle, N., Rieder, U., 2011. Markov Decision Processes with Applications to Finance. Berlin Heidelberg.

Baurdoux, E.J., Yamazaki, K., 2015. Optimality of doubly reflected lévy processes in singular control. Stoch. Process. Their. Appl. 125, 2727–2751.

Bladt, M., Nielsen, B.F., 2017. Vol. 81 of *New York*. Springer.

Boxma, O., Mandjes, M., 2023. A decomposition for lévy processes inspected at poisson moments. J. Appl. Probab. 60 (2), 557–569.

Cheung, E. C.K., Zhang, Z., 2019. Periodic threshold-type dividend strategy in the compound poisson risk model. Scand. Actuar. J. 2019 (1), 1–31.

Choi, M. C.H., Cheung, E. C.K., 2014. On the expected discounted dividends in the cramérlundberg risk model with more frequent ruin monitoring than dividend decisions. Insurance: Math. Econ. 59, 121–132.

de Finetti, B., 1957. Su un' impostazione alternativa dell teoria collettiva del rischio. Trans. XVth Int. Congr. Actuar. 2, 433–443.

Dickson, D. C.M., Drekic, S., 2006. Optimal dividends under a ruin probability constraint. Ann. Actuar. Sci. 1 (2), 291–306.

Gerber, H.U., 1969. Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker. Vol. 69.

Gerber, H.U., Shiu, E. S.W., 1998. On the time value of ruin. N. Am. Actuar. J. 2 (1), 48–72.

Gerber, H.U., Shiu, E. S.W., 2006. On optimal dividend strategies in the compound poisson model. N. Am. Actuar. J. 10 (2), 76–93.

Grandits, P., 2015. An optimal consumption problem in finite time with a constraint on the ruin probability. Finance Stochast. 19 (4), 791–847.

Hernández-Hernández, D., Pérez, J.-L., Yamazaki, K., 2016. Optimality of refraction strategies for spectrally negative lévy processes. SIAM J. Control Optim. 54 (3), 1126–1156.

Hipp, C., 2018. Company value with ruin constraint in lundberg models. Risks 6 (3), 73.

Hipp, C., 2019. Dividend payment with ruin constraint. In: Barrieu, P. (Ed.), Risk and Stochastics: Festschrift in Honor of Ragnar Norberg, pp. 1–22.

Junca, M., Moreno-Franco, H.A., Pérez, J.-L., Yamazaki, K., 2019. Optimality of refraction strategies for a constrained dividend problem. Adv. Appl. Probab. 51 (3), 633–666.

Kyprianou, A.E., Loeffen, R.L., Pérez, J.-L., 2012. Optimal control with absolutely continuous strategies for spectrally negative lévy processes. J. Appl. Probab. 49 (1), 150–166.

Lin, X.S., Pavlova, K.P., 2006. The compound Poisson risk model with a threshold dividend strategy 38, 57–80.

Loeffen, R.L., 2008. On optimality of the barrier strategy in de finetti's dividend problem for spectrally negative lévy processes. Annal. Appl. Probab. 18 (5), 1669–1680.

Noba, K., Pérez, J.-L., Yamazaki, K., Yano, K., 2018. On optimal periodic dividend strategies for lévy risk processes. Insurance: Math. Econ. 80, 29–44.

Pérez, J.-L., Yamazaki, K., Bensoussan, A., 2020. Optimal periodic replenishment policies for spectrally positive lévy demand processes. SIAM J. Control Optim. 58 (6), 3019–3867.

Prabhu, N.U., 1998. Stochastic Storage Processes: Queues, Insurance Risk, Dams, and Data Communication. New York, Springer.

Thonhauser, S., Albrecher, H., 2007. Dividend maximization under consideration of the time value of ruin. Insurance: Math. Econ. 41 (1), 163–184.

Yao, D., Yang, H., Wang, R., 2011. Optimal dividend and capital injection problem in the dual model with proportional and fixed transaction costs. Eur. J. Oper. Res. 211 (3), 568–576.

Zhang, Z., Cheung, E. C.K., 2016. The markov additive risk process under an erlangized dividend barrier strategy. Methodol. Comput. Appl. Probab. 18 (2), 275–306.

Zhang, Z., Cheung, E. C.K., Yang, H., 2017. Lévy insurance risk process with poissonian taxation. Scand. Actuar. J. 2017 (1), 51–87.