Generalized Segal–Bargmann transform for Poisson distribution revisited

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Abstract

For $\alpha > 0$ and $\sigma > 0$, we consider the following probability distribution on $\alpha\mathbb{N}_0$: $\pi_{\alpha,\sigma} = \exp\big(-\frac{\sigma}{\alpha^2}\big)\sum_{n=0}^{\infty}\frac{1}{n!}\big(\frac{\sigma}{\alpha^2}\big)^n\delta_{\alpha n}$, where δ_y denotes the Dirac measure with mass at y. For $\alpha = 1$, $\pi_{1,\sigma}$ is the Poisson distribution with parameter σ. Furthermore, the centered probability distribution $\tilde{\pi}_{\alpha,\sigma} = \exp\big(-\frac{\sigma}{\alpha^2}\big)\sum_{n=0}^{\infty}\frac{1}{n!}\big(\frac{\sigma}{\alpha^2}\big)^n\delta_{\alpha n-\sigma/\alpha}$ weakly converges to μ_σ as $\alpha \to 0$. Here μ_σ is the Gaussian distribution with mean zero and variance σ. Let $(c_n)_{n=0}^{\infty}$ be the monic polynomial sequence that is orthogonal with respect to the measure $\mu_{\alpha,\sigma}$. In particular, for $\alpha = 1$, $(c_n)_{n=0}^{\infty}$ is a sequence of Charlier polynomials. Let $\mathbb{F}_\sigma(\mathbb{C})$ denote the Bargmann space of all entire functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $f_n \in \mathbb{C}$ satisfying $\sum_{n=0}^{\infty}|f_n|^2\, n!\,\sigma^n < \infty$. The generalized Segal–Bargmann transform associated with the measure $\pi_{\alpha,\sigma}$ is a unitary operator $\mathcal{S} : L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma}) \to \mathbb{F}_\sigma(\mathbb{C})$ that satisfies $(\mathcal{S}c_n)(z) = z^n$ for $n \in \mathbb{N}_0$. We present some new results related to the operator $\mathcal{S}$. In particular, we observe how the study of $\mathcal{S}$ naturally leads to the normal ordering in the Weyl algebra.

Keywords: Segal–Bargmann transform, Poisson distribution, Charlier polynomials, Weyl algebra

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1 Introduction

The concept of the Segal–Bargmann transform was developed by I. E. Segal [22–24] and V. Bargmann [3–5]. In its simplest form, in the one-dimensional case, the Segal–Bargmann transform is the unitary operator $\mathbb{S}$ between the complex L^2-space of the standard Gaussian distribution μ on $\mathbb{R}$ and the Bargmann space $\mathbb{F}(\mathbb{C})$ of entire functions on $\mathbb{C}$.

Let us recall that $\mathbb{F}(\mathbb{C})$ is a Hilbert space that consists of all entire functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $f_n \in \mathbb{C}$ satisfying $\sum_{n=0}^{\infty}|f_n|^2 n! < \infty$, and the inner product of $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ in $\mathbb{F}(\mathbb{C})$ is given by $(f, g)_{\mathbb{F}(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\,\overline{g_n}\,n!$. The

Bargmann space $\mathbb{F}(\mathbb{C})$ is a proper subspace of the complex L^2-space $L^2(\mathbb{C},\nu)$, where $\nu(dz)=\pi^{-1}\exp(-|z|^2)\,dA(z)$ and $dA(z)=dx\,dy$ $(z=x+iy)$ is the Lebesgue measure on $\mathbb{C}$. The Bargmann space $\mathbb{F}(\mathbb{C})$ provides a realization of the Fock space over $\mathbb{C}$.

Let $(h_n)_{n=0}^\infty$ be the the sequence of monic Hermite polynomials that are orthogonal with respect to the measure μ. The Segal–Bargmann transform $\mathbb{S}: L^2(\mathbb{R},\mu)\to\mathbb{F}(\mathbb{C})$ satisfies $(\mathbb{S}h_n)(z)=z^n$ for $n\in\mathbb{N}_0$. The $\mathbb{S}$ is an integral operator with the integral kernel $\mathbb{E}(x,z)=\sum_{n=0}^\infty\frac{z^n}{n!}h_n(x)=\exp(-z^2/2-xz)$. For each $z\in\mathbb{C}$, the function $\mathbb{E}(\cdot,z)$ is called a coherent state. It is an eigenfunction for the annihilation operator a^- belonging to eigenvalue z. Here the operator a^- satisfies $a^-h_n=nh_{n-1}$ $(n\in\mathbb{N}_0)$. Under the Segal–Bargmann transform $\mathbb{S}$, the operator a^- goes over to the operator of differentiation in $\mathbb{F}(\mathbb{C})$. The adjoint of the annihilation operator a^- is the creation operator a^+, satisfying $a^+h_n=h_{n+1}$ $(n\in\mathbb{N}_0)$. Then $\mathbb{S}a^+\mathbb{S}^{-1}$ is the operator of multiplication by the variable z in $\mathbb{F}(\mathbb{C})$. For each function $f\in L^2(\mathbb{R},\mu)$, the restriction of the entire function $\mathbb{S}f$ to $\mathbb{R}$ can be written as $(\mathbb{S}f)(z)=\int_{\mathbb{R}}f(x+z)\mu(dx)$ $(z\in\mathbb{R})$.

The Segal–Bargmann transform has been similarly defined and studied in the multivariate case. In fact, in his original papers [22–24], Segal was already interested in the infinite dimensional case. However, when defining a counterpart of the Bargmann space, Segal used a sequence of Gaussian measures on $\mathbb{C}^n$ with $n\in\mathbb{N}$ increasing to ∞.

A study of the Segal–Bargmann transform associated with a Gaussian measure on a (complex) infinite dimensional space was carried out by Y. M. Berezansky and Y. G. Kondratiev in their monograph [6, Chapter 2], see also the earlier work [14]. Essentially at the same time, the Segal–Bargmann transform in the infinite dimensional setting was discussed within white noise analysis under the name of S-transform. See e.g. the monographs [10, 19] and the references therein. In fact, in the infinite dimensional setting, it was natural to study the Segal–Bargmann transform of spaces of test and generalized functions.

It is well known that the Poisson distribution and the Poisson point process possess quite a few properties that are similar to properties of Gaussian measures. For example, both the Poisson process and a Gaussian measure provide a natural unitary isomorphism between their L^2-spaces and the Fock space, see e.g. [25].

A counterpart of the Segal–Bargmann transform for the Poisson distribution was developed by N. Asai, I. Kubo, and H.-H. Kuo [2]. In fact, even earlier, Y.-J. Lee and H.-H. Shih [16] developed a generalized Segal–Bargmann transform for Lévy processes, which of course include the Poisson process.

The present paper deals with a family of discrete probability distributions $\pi_{\alpha,\sigma}$ $(\alpha>0,\ \sigma>0)$ defined by

$$\pi_{\alpha,\sigma}=\exp\left(-\frac{\sigma}{\alpha^2}\right)\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{\sigma}{\alpha^2}\right)^n\delta_{\alpha n},\tag{1}$$

where δ_y denotes the Dirac measure with mass at y. In particular, for $\alpha=1$, $\pi_{1,\sigma}=\pi_\sigma$ is the Poisson distribution with parameter σ. Note that the measure $\pi_{\alpha,\sigma}$ is concentrated on the set $\alpha\mathbb{N}_0$. Furthermore, the centered probability distributions

$$\tilde{\pi}_{\alpha,\sigma} = \exp\bigg(-\frac{\sigma}{\alpha^2}\bigg)\sum_{n=0}^{\infty}\frac{1}{n!}\bigg(\frac{\sigma}{\alpha^2}\bigg)^n \delta_{\alpha n-\sigma/\alpha} \tag{2}$$

weakly converge to μ_σ as $\alpha \to 0$. Here μ_σ is the Gaussian distribution with mean zero and variance σ. From the viewpoint of quantum physics, the interpolating parameter α provides a connection between the particle density of an infinite free Bose gas at zero temperature and a free Bose field, see Remark 7 below for detail.

Let $(c_n)_{n=0}^{\infty}$ be the monic polynomial sequence that is orthogonal with respect to the measure $\mu_{\alpha,\sigma}$. In particular, for $\alpha = 1$, $(c_n)_{n=0}^{\infty}$ is a sequence of Charlier polynomials. The (exponential) generating function of the sequence $(c_n)_{n=0}^{\infty}$ has the form

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,c_n(z) = \exp\bigg(\frac{z}{\alpha}\,\log(1+t\alpha) - \frac{\sigma t}{\alpha}\bigg). \tag{3}$$

In particular, $(c_n)_{n=0}^{\infty}$ is a Sheffer sequence.

Let $\mathbb{F}_\sigma(\mathbb{C})$ denote the Bargmann space of all entire functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $f_n \in \mathbb{C}$ satisfying $\sum_{n=0}^{\infty}|f_n|^2\, n!\,\sigma^n < \infty$ and the inner product of $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ in $\mathbb{F}_\sigma(\mathbb{C})$ is given by $(f,g)_{\mathbb{F}_\sigma(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\,\overline{g_n}\,n!\,\sigma^n$. (Thus, the above defined Bargmann space $\mathbb{F}(\mathbb{C})$ corresponds to the choice of the parameter $\sigma = 1$ in $\mathbb{F}_\sigma(\mathbb{C})$.) We define a generalized Segal–Bargmann transform associated with the measure $\pi_{\alpha,\sigma}$ as a unitary operator $\mathcal{S} : L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma}) \to \mathbb{F}_\sigma(\mathbb{C})$ that satisfies $(\mathcal{S}c_n)(z) = z^n$ for $n \in \mathbb{N}_0$. In the case $\alpha = 1$, this is the operator studied in [2]. In this paper, we will discuss some new (together with some old) results related to the operator $\mathcal{S}$.

The paper is organized as follows. In Section 2, we will recall several key results of the umbral calculus, which is the theory of Sheffer polynomial sequences and Sheffer operators, e.g. [15, 20, 21]. We also discuss S. Grabiner's result [9] on an extension of a Sheffer operator (acting on polynomials) to a self-homeomorphism of the space $\mathcal{E}^1_{\min}(\mathbb{C})$ of entire functions of order at most 1 and minimal type.

For the reader's convenience, in Section 3, we will recall some well-known facts about the classical Segal–Bargmann transform (for a Gaussian measure) in the one-dimensional case. We will recall, in particular, how the Segal–Bargmann transform can be interpreted as (an extension of) a Sheffer operator.

In Section 4, we will define and study the generalized Segal–Bargmann transform $\mathcal{S}$ for the probability measure $\pi_{\alpha,\sigma}$. For $\alpha = 1$, part of the results in this section are due to Asai et al. [2]. Nevertheless, some results of this section are new even in the Poisson case. For example, we prove that, for $z \in \mathbb{R}$, $z > -\sigma/\alpha$, $(\mathcal{S}f)(z)$ can be written as $(\mathcal{S}f)(z) = \int f\, d\pi_{\alpha,\sigma+\alpha z}$. We also show that the operator $\mathcal{S}$, restricted to polynomials, can written as $\mathcal{S} = E_{\sigma/\alpha}\mathcal{T}_\alpha$. Here $E_{\sigma/\alpha}$ is the operator of shift by σ/α, and $\mathcal{T}_\alpha$ is the Sheffer operator associated with a sequence of Touchard polynomials.

Our studies in Section 4 will naturally lead us to a pair of operators, $\mathcal{U}$ and $\mathcal{V}$ that act on polynomials and satisfy the commutation relation $[\mathcal{V},\mathcal{U}] = \alpha$, hence they are generators of a Weyl algebra. The main connection with the operator $\mathcal{S}$ is that, under the action of $\mathcal{S}$, the operator of multiplication by the variable goes over to the operator $\mathcal{U}\mathcal{V}$. As a consequence of the normal ordering in the Weyl algebra [12], we derive explicit formulas for the polynomials c_n, as well as an explicit representation of a monomial though the polynomials c_n.

Finally, in Section 5, we use Grabiner's result [9] to treat the operator $\mathcal{S}$ as a self-homeomorphisim of $\mathcal{E}^1_{\min}(\mathbb{C})$. We also study the operators $U = \mathcal{S}^{-1}\mathcal{U}\mathcal{S}$ and $V = \mathcal{S}^{-1}\mathcal{V}\mathcal{S}$, acting in $\mathcal{E}^1_{\min}(\mathbb{C})$. We prove that these operators act as follows: $(Uf)(z) = zf(z-\alpha)$, $(Vf)(z) = f(z+\alpha)$.

We note that, in our recent paper [13], we dealt with the generalized Segal–Bargmann transform for the remaining Sheffer sequences of orthogonal polynomials in the classification of Meixner [18].

An extension of the results of this paper to an infinite dimensional setting will be a topic of our future research.

2 Elements of umbral calculus

2.1 Sheffer sequences

This subsection is based on [21] and Chapter IV, Sections 3 and 4 of [15].

Let $\mathbb{C}[z]$ denote vector space of polynomials over $\mathbb{C}$. We denote by $\mathcal{L}(\mathbb{C}[z])$ the vector space of linear operators acting in $\mathbb{C}[z]$.

For $h \in \mathbb{C}$, we define $E_h \in \mathcal{L}(\mathbb{C}[z])$ as the operator of shift by h: $(E_hp)(z) = p(z+h)$ for $p \in \mathbb{C}[z]$. Boole's formula states that $E_h = e^{hD} = \sum_{n=0}^{\infty} \frac{h^n}{n!} D^n$, where $D \in \mathcal{L}(\mathbb{C}[z])$ is the operator of differentiation. An operator $Q \in \mathcal{L}(\mathbb{C}[z])$ is called shift-invariant if $QE_h = E_hQ$ for each $h \in \mathbb{C}$. An operator $Q \in \mathcal{L}(\mathbb{C}[z])$ is called a delta operator if Q is shift-invariant and $Qz = 1$.

Let $(p_n)_{n=0}^{\infty}$ be a monic polynomial sequence, i.e., $p_n \in \mathbb{C}[z]$ ($n \in \mathbb{N}$), the degree of p_n is n, and the coefficient by z^n is 1. For $(p_n)_{n=0}^{\infty}$, its lowering operator $Q \in \mathcal{L}(\mathbb{C}[z])$ is defined by $Qp_n = np_{n-1}$ for $n \in \mathbb{N}_0$.

A monic polynomial sequence $(p_n)_{n=0}^{\infty}$ is said to be of binomial type if $p_n(z+\zeta) = \sum_{k=0}^{n} \binom{n}{k} p_k(z)p_{n-k}(\zeta)$ for all $n \in \mathbb{N}$ and $z, \zeta \in \mathbb{C}$.

Theorem 1. *Let $(p_n)_{n=0}^{\infty}$ be a monic polynomial sequence, and let Q be its lowering operator. The following statements are equivalent:*

(BT1) The sequence $(p_n)_{n=0}^{\infty}$ is of binomial type.

(BT2) The operator Q is a delta operator.

(BT3) The operator Q is of the form $Q = C(D) = \sum_{n=1}^{\infty} c_nD^n$, where $C(t) = \sum_{n=1}^{\infty} c_nt^n$ is a formal power series over $\mathbb{C}$ with $c_1 = 1$.

(BT4) The polynomial sequence $(p_n)_{n=0}^\infty$ has the (exponential) generating function of the form

$$\sum_{n=0}^\infty p_n(z)\frac{t^n}{n!}=\exp(zB(t)), \tag{4}$$

where $B(t)=\sum_{n=1}^\infty b_nt^n$ is a formal power series over $\mathbb{C}$ with $b_1=1$, and formula (4) is understood as an equality of formal power series in t.

Furthermore, the formal power series $B(t)$ and $C(t)$ above are inverse of each other, i.e., $B(C(t))=C(B(t))=t$.

By Theorem 1, for each delta operator Q, there exits a unique binomial sequence $(p_n)_{n=0}^\infty$ for which Q is its lowering operator. Then $(p_n)_{n=0}^\infty$ is called a basic sequence for Q.

The falling factorials are defined by $(z)_0=1$ and

$$(z)_n=z(z-1)\cdots(z-n+1),\quad n\in\mathbb{N}.$$

This is a polynomial sequence of binomial type with generating function of the form (4) in which $B(t)=\log(1+t)$.

Note that Stirling numbers of the first kind, $s(n,k)$, are the coefficient of the expansion

$$(z)_n=\sum_{k=1}^n s(n,k)z^k, \tag{5}$$

while Stirling numbers of the second kind, $S(n,k)$, are the coefficient of the expansion

$$z^n=\sum_{k=1}^n S(n,k)(z)_k. \tag{6}$$

One defines the monic polynomial sequence of Touchard (or exponential) polynomials by

$$T_n(z)=\sum_{k=1}^n S(n,k)\,z^k. \tag{7}$$

This is a polynomial sequence of binomial type with generating function of the form

$$\sum_{n=0}^\infty T_n(z)\frac{t^n}{n!}=\exp\big(z(e^t-1)\big). \tag{8}$$

Let Q be a delta operator. A monic polynomial sequence $(s_n)_{n=0}^\infty$ is called a Sheffer sequence for Q if Q is the lowering operator for $(s_n)_{n=0}^\infty$.

Theorem 2. *Let $Q = C(D)$ be a delta operator with basic sequence $(p_n)_{n=0}^{\infty}$ that has generating function* (4)*. Let $(s_n)_{n=0}^{\infty}$ be a monic polynomial sequence. Then the following statements are equivalent:*

(SS1) The polynomial sequence $(s_n)_{n=0}^{\infty}$ is a Sheffer sequence for Q.

(SS2) There exists a (unique, invertible) shift-invariant operator T that satisfies $s_n(z) = (Tp_n)(z)$ for all $n \in \mathbb{N}_0$.

(SS3) The polynomial sequence $(s_n)_{n=0}^{\infty}$ has the (exponential) generating function of the form

$$\sum_{n=0}^{\infty} s_n(z)\,\frac{t^n}{n!} = \exp(zB(t))A(t), \tag{9}$$

where $B(t)$ is as in (4) *and $A(t) = \sum_{n=0}^{\infty} a_n t^n$ is a formal power series over $\mathbb{C}$ with $a_0 = 1$.*

Furthermore, $T = \tau(D) = \sum_{k=0}^{\infty} \tau_k D^k$, where the formal power series $\tau(t) = \sum_{k=0}^{\infty} \tau_k t^k$ satisfies $\tau(t) = A(C(t))$, where $C(t)$ is the compositional inverse of $B(t)$.

A monic polynomial sequence $(s_n)_{n=0}^{\infty}$ is called an Appell sequence if it is a Sheffer sequence for the differential operator D. By Theorem 2, a monic polynomial sequence $(s_n)_{n=0}^{\infty}$ is an Appell sequence if and only if its generating function is of the form (9) with $B(t) = t$.

2.2 Sheffer operators

This section is based on [20, Chapter 3], [9] and [7].

Let $(s_n)_{n=0}^{\infty}$ be a (monic) Sheffer sequence. One defines the Sheffer operator associated with the sequence $(s_n)_{n=0}^{\infty}$ as the operator $S \in \mathcal{L}(\mathbb{C}[z])$ that satisfies $Sz^n = s_n(z)$ for all $n \in \mathbb{N}_0$. In the case where $(s_n)_{n=0}^{\infty}$ is an Appell sequence, the associated operator S is called an Appell operator. In the case where $(s_n)_{n=0}^{\infty} = (p_n)_{n=0}^{\infty}$ is a polynomial sequence of binomial type, the associated operator $P \in \mathcal{L}(\mathbb{C}[z])$ is called an umbral operator. We denote by $\mathfrak{S}$, $\mathfrak{A}$, and $\mathfrak{B}$ the sets of all Sheffer operators, Appell operators, and umbral operators, respectively.

Theorem 3. *The $\mathfrak{S}$ is a group for the product (composition) of linear operators in $\mathbb{C}[z]$. The $\mathfrak{A}$ is an abelian normal subgroup of $\mathfrak{S}$, $\mathfrak{B}$ is a subgroup of $\mathfrak{S}$, and $\mathfrak{S}$ is the semidirect product of $\mathfrak{A}$ and $\mathfrak{B}$, i.e., $\mathfrak{S} = \mathfrak{A} \rtimes \mathfrak{B}$.*

Theorem 3 implies that, if S is a Sheffer operator and $(r_n)_{n=0}^{\infty}$ is a Sheffer sequence, then the monic polynomial sequence $(Sr_n)_{n=0}^{\infty}$ is also a Sheffer sequence. A similar statement holds for Appell operators and Appell sequences, as well as umbral operators and sequences of binomial type.

We will now consider an extension of a class of Sheffer operators to a space of entire functions.

Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. One says that f is of order at most 1 and minimal type (when the order is equal to 1) if f satisfies the estimate

$$\|f\|_n=\sup_{z\in\mathbb{C}}|f(z)|\exp(-|z|/n)<\infty \quad \text{for all } n\in\mathbb{N}. \tag{10}$$

One denotes by $\mathcal{E}^1_{\min}(\mathbb{C})$ the vector space of all such functions. For each $n\in\mathbb{N}$, $\|\cdot\|_n$ is a norm on $\mathcal{E}^1_{\min}(\mathbb{C})$. Let $B_n(\mathbb{C})$ denote the Banach space obtained as the completion of $\mathcal{E}^1_{\min}(\mathbb{C})$ in this norm, i.e., $B_n(\mathbb{C})$ is the space of all entire functions f whose norm $\|f\|_n$ is finite. For any $n_1>n_2$, we have $B_{n_1}(\mathbb{C})\subset B_{n_2}(\mathbb{C})$, and the embedding of $B_{n_1}(\mathbb{C})$ into $B_{n_2}(\mathbb{C})$ is continuous. Note that, as a set, $\mathcal{E}^1_{\min}(\mathbb{C})=\bigcap_{n=1}^\infty B_n(\mathbb{C})$. Hence, one defines the projective limit topology on $\mathcal{E}^1_{\min}(\mathbb{C})$ given by the norms $\|\cdot\|_n$ ($n\in\mathbb{N}$). In particular, $\mathcal{E}^1_{\min}(\mathbb{C})$ is a Fréchet space.

Let us now discuss an equivalent description of the Fréchet space $\mathcal{E}^1_{\min}(\mathbb{C})$. For each $l\in\mathbb{N}$, denote by $E_l(\mathbb{C})$ the Hilbert space of all entire functions $f(z)=\sum_{n=0}^\infty f_n z^n$ that satisfy

$$\mathcal{N}_l(f)=\left(\sum_{n=0}^\infty |f_n|^2(n!)^2 2^{nl}\right)^{1/2}<\infty,$$

and the norm of $f\in E_l(\mathbb{C})$ is $\mathcal{N}_l(f)$. Then, as a set, $\mathcal{E}^1_{\min}(\mathbb{C})=\bigcap_{l=1}^\infty E_l(\mathbb{C})$, and the topology on $\mathcal{E}^1_{\min}(\mathbb{C})$ coincides with the projective limit of the $E_l(\mathbb{C})$ spaces.

Theorem 4 (Grabiner)**.** *Let $(s_n)_{n=1}^\infty$ be a Sheffer sequence with generating function* (9) *such that $A(t)$ and $B(t)$ are holomorphic functions in a neighborhood of zero. Then the Sheffer operator $S\in\mathcal{L}(\mathbb{C}[z])$ defined by $Sz^n=s_n(z)$ ($n\in\mathbb{N}_0$) extends by continuity to a linear self-homeomorphism of the space $\mathcal{E}^1_{\min}(\mathbb{C})$. In particular, each function $f\in\mathcal{E}^1_{\min}(\mathbb{C})$ admits a unique representation*

$$f(z)=\sum_{n=0}^\infty f_n\, s_n(z), \tag{11}$$

where the series on the right-hand side of formula (11) *converges in $\mathcal{E}^1_{\min}(\mathbb{C})$.*

Corollary 5. *Let $(s_n)_{n=0}^\infty$ be a Sheffer sequence satisfying the condition of Theorem 4. For each $l\in\mathbb{N}$, denote by $H_l(\mathbb{C})$ the Hilbert space of all entire functions $f(z)=\sum_{n=0}^\infty f_n s_n(z)$ that satisfy*

$$|||f|||_l=\left(\sum_{n=0}^\infty |f_n|(n!)^2 2^{nl}\right)^{1/2}<\infty,$$

and the norm of $f\in H_l(\mathbb{C})$ is $|||f|||_l$. Then, as a set, $\mathcal{E}^1_{\min}(\mathbb{C})=\bigcap_{l=1}^\infty H_l(\mathbb{C})$, and the topology on $\mathcal{E}^1_{\min}(\mathbb{C})$ coincides with the projective limit of the $H_l(\mathbb{C})$ spaces.

3 Segal–Bargmann transform in the one-dimensional case

In this section, we follow [3] and [6, Chapter 2, Section 5.2] (in a slightly generalized form).

Let $\sigma > 0$. Let ν_σ be the Gaussian measure on $\mathbb{C}$ given by

$$\nu_\sigma(dz) = \frac{1}{\pi\sigma} \exp\left(-\frac{|z|^2}{\sigma}\right) dA(z).$$

(Recall that $dA(z) = dx\,dy$ for $z = x + iy$.) It holds that

$$(z^m, z^n)_{L^2(\mathbb{C},\nu_\sigma)} = \int_{\mathbb{C}} z^m\, \overline{z^n}\, \nu_\sigma(dz) = \delta_{m,n}\, n!\, \sigma^n, \quad m, n \in \mathbb{N}_0,$$

where $\delta_{m,n}$ is the Kronecker delta.

Recall the Bargmann spce $\mathbb{F}_\sigma(\mathbb{C})$ that was defined in the Introduction. Note that, for $f, g \in \mathbb{F}_\sigma(\mathbb{C})$, we have $(f,g)_{\mathbb{F}_\sigma(\mathbb{C})} = (f,g)_{L^2(\mathbb{C},\nu_\sigma)}$. Thus, $\mathbb{F}_\sigma(\mathbb{C})$ is a proper subspace of the Hilbert space $L^2(\mathbb{C},\nu_\sigma)$. The $\mathbb{F}_\sigma(\mathbb{C})$ is called a Bargmann space.

Let $m \in \mathbb{R}$, and let $\mu_{m,\sigma}$ denote the Gaussian measure on $\mathbb{R}$ with mean m and variance σ. For $m = 0$, we denote $\mu_\sigma = \mu_{0,\sigma}$. Let $(h_n)_{n=0}^\infty$ be the sequence of monic Hermite polynomials that are orthogonal with respect to the measure μ_σ . The Hermite polynomials $(h_n)_{n=0}^\infty$ satisfy the recurrence relation

$$zh_n(z) = h_{n+1}(z) + \sigma n h_{n-1}(z), \quad n \in \mathbb{N}_0. \tag{12}$$

The (exponential) generating function of $(h_n)_{n=0}^\infty$ has the form

$$\sum_{n=0}^\infty \frac{t^n}{n!}\, h_n(z) = \exp\left(zt - \frac{1}{2}\sigma t^2\right). \tag{13}$$

In particular, $(h_n)_{n=0}^\infty$ is an Appell sequence. It follows from (12) that $\|h_n\|^2_{L^2(\mathbb{R},\mu_\sigma)} = n!\,\sigma^n$.

For each $z \in \mathbb{C}$, the corresponding coherent state is defined by

$$\mathbb{E}(x,z) = \sum_{n=0}^\infty \frac{z^n}{n!\,\sigma^n}\, h_n(x), \quad x \in \mathbb{R}. \tag{14}$$

Thus, for each $z \in \mathbb{C}$, we have $\mathbb{E}(\cdot, z) \in L^2(\mathbb{R},\mu_\sigma)$, and for each $x \in \mathbb{R}$, $\mathbb{E}(x,\cdot) \in \mathbb{F}_\sigma(\mathbb{C})$. By (13) and (14),

$$\mathbb{E}(x,z) = \exp\left(-\frac{z^2 - 2xz}{2\sigma}\right), \quad x \in \mathbb{R},\ z \in \mathbb{C}. \tag{15}$$

The Segal–Bargmann transform is the unitary operator $\mathbb{S} : L^2(\mathbb{R}, \mu_\sigma) \to \mathbb{F}_\sigma(\mathbb{C})$ satisfying $(\mathbb{S}h_n)(z) = z^n$. Thus, for $f \in L^2(\mathbb{R}, \mu_\sigma)$ and $\varphi \in \mathbb{F}_\sigma(\mathbb{C})$,

$$\begin{aligned} (\mathbb{S}f)(z) &= \int_{\mathbb{R}} f(x)\mathbb{E}(x,z)\,\mu_\sigma(dx), \quad z \in \mathbb{C}, \\ (\mathbb{S}^{-1}\varphi)(x) &= \int_{\mathbb{C}} \varphi(z)\mathbb{E}(x,z)\,\nu_\sigma(dz), \quad x \in \mathbb{R}. \end{aligned} \tag{16}$$

By (15) and (16),

$$\begin{aligned} (\mathbb{S}f)(z) &= \int_{\mathbb{R}} f(x) \exp\Big(-\frac{z^2 - 2xz}{2\sigma}\Big) \frac{1}{\sqrt{2\pi\sigma}} \exp\Big(-\frac{x^2}{2\sigma}\Big)\,dx \\ &= \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi\sigma}} \exp\Big(-\frac{(x-z)^2}{2\sigma}\Big)\,dx. \end{aligned} \tag{17}$$

Hence,

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(x+z)\,\mu_\sigma(dx) = \int_{\mathbb{R}} f(x)\,\mu_{z,\sigma}(dx), \quad z \in \mathbb{R}. \tag{18}$$

Note that, for $p \in \mathbb{C}[z] \subset L^2(\mathbb{R}, \mu_\sigma)$, formula (18) implies

$$(\mathbb{S}p)(z) = \int_{\mathbb{R}} p(x+z)\,\mu_\sigma(dx), \quad z \in \mathbb{C}. \tag{19}$$

It will be useful for us below to give another interpretation of formula (17). For each $z \in \mathbb{C} \setminus \mathbb{R}$, we define the complex-valued measure $\mu_{z,\sigma}(dx)$ on $\mathbb{R}$ by

$$\mu_{z,\sigma}(dx) = \frac{1}{\sqrt{2\pi\sigma}} \exp\Big(-\frac{(x-z)^2}{2\sigma}\Big)\,dx.$$

This measure $\mu_{z,\sigma}$ can be thought of as a complex-valued Gaussian measure on $\mathbb{R}$.

Then, by (17), for each $f \in L^2(\mathbb{R}, \mu_\sigma)$, we get

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f\,d\mu_{z,\sigma}, \quad z \in \mathbb{C}.$$

Note that $\mu_{z,\sigma}(\mathbb{R}) = \mathbb{S}1 = 1$.

Since both the Hermite sequence $(h_n)_{n=0}^\infty$ and the sequence of monomials $(z^n)_{n=0}^\infty$ are Appell systems, Theorem 3 implies that the restriction of $\mathbb{S}$ to $\mathbb{C}[z]$ is the Appell operator associated with the Appell sequence $(\tilde{h}_n)_{n=0}^\infty$, where $\tilde{h}_n(z) = \mathbb{S}z^n$. It follows from (19), the explicit form of the moments of the measure μ_σ, and the explicit form of the Hermite polynomials h_n that

$$\tilde{h}_n(z) = \int_{\mathbb{R}} (x+z)^k\,\mu_\sigma(dx) = \sum_{k=0}^n \binom{n}{k} z^{n-k} \int_{\mathbb{R}} x^k\,\mu_\sigma(dx)$$

$$= z^n + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} z^{n-2m} \int_{\mathbb{R}} x^{2m}\, \mu_\sigma(dx) = z^n + n! \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} z^{n-2m} \frac{\sigma^m}{2^m (n-2m)!\, m!}$$

$$= i^n \left((-iz)^n + n! \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} (-iz)^{n-2m} (-1)^m \frac{\sigma^m}{2^m (n-2m)!\, m!} \right) = i^n h_n(-iz). \qquad (20)$$

By (13) and (20),

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \tilde{h}_n(z) = \exp\left(zt + \frac{1}{2}\sigma t^2 \right).$$

Denote by a^+ and a^- the raising and lowering operators for the monic polynomial sequence $(h_n)_{n=0}^{\infty}$, i.e., $a^+, a^- \in \mathcal{L}(\mathbb{C}[z])$ and

$$(a^+ h_n)(z) = h_{n+1}(z), \quad (a^- h_n)(z) = n h_{n-1}(z), \qquad n \in \mathbb{N}_0. \qquad (21)$$

(These operators are often called the creation and annihilation operators, respectively.)

Let $Z \in \mathcal{L}(\mathbb{C}[z])$ denote the operator of multiplication by the variable z. By (12) and (21), $Z = a^+ + \sigma a^-$. The operator Z is essentially self-adjoint in $L^2(\mathbb{R}, \mu_\sigma)$ and its closure is the operator of multiplication by the variable in $L^2(\mathbb{R}, \mu_\sigma)$.

Next, it is easy to see that the operator a^- is closable in $L^2(\mathbb{R}, \mu_\sigma)$ and we keep the notation a^- for its closure. Then, for each $z \in \mathbb{C}$, the coherent state $\mathbb{E}(\cdot, z)$ is an eigenvector of the operator σa^- belonging to the eigenvalue z.

Obviously,

$$\mathbb{S} a^+ \mathbb{S}^{-1} = Z, \quad \mathbb{S} a^- \mathbb{S}^{-1} = D,$$

so that

$$\mathbb{S} Z \mathbb{S}^{-1} = Z + \sigma D.$$

The operator $Z + \sigma D \in \mathcal{L}(\mathbb{C}[z])$ is essentially self-adjoint in $\mathbb{F}_\sigma(\mathbb{C})$.

4 A generalized Segal–Bargmann transform for $\pi_{\alpha,\sigma}$

For $\alpha > 0$ and $\sigma > 0$, let the probability measure $\pi_{\alpha,\sigma}$ on $\alpha\mathbb{N}_0$ be defined by (1). Let $(c_n)_{n=0}^{\infty}$ denote the monic polynomial sequence that is orthogonal with respect to $\pi_{\alpha,\sigma}$. The $(c_n)_{n=0}^{\infty}$ is a Sheffer sequence having the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, c_n(z) = \exp\left(\frac{z}{\alpha} \log(1 + t\alpha) - \frac{\sigma t}{\alpha} \right). \qquad (22)$$

The sequence $(c_n)_{n=0}^{\infty}$ satisfies the recurrence formula

$$z s_n(z) = c_{n+1}(z) + (\alpha n + \sigma/\alpha) c_n(z) + \sigma n c_{n-1}(z), \quad n \in \mathbb{N}_0. \qquad (23)$$

It follows from (23) that $\|c_n\|^2_{L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})} = n!\, \sigma^n$.

Remark 6. Let $\tilde{\pi}_{\alpha,\sigma}$ be the centered measure $\pi_{\alpha,\sigma}$, see (2). The Fourier transform of $\tilde{\pi}_{\alpha,\sigma}$ is

$$\int_{\alpha\mathbb{N}_0} e^{ixy}\,\tilde{\pi}_{\alpha,\sigma}(dy) = \exp\bigg(\frac{\sigma}{\alpha^2}(e^{i\alpha y} - 1 - i\alpha y)\bigg).$$

Hence,

$$\lim_{\alpha\to 0}\int_{\alpha\mathbb{N}_0} e^{ixy}\,\tilde{\pi}_{\alpha,\sigma}(dy) = \exp\bigg(-\frac{\sigma}{2}\,y^2\bigg) = \int_{\mathbb{R}} e^{ixy}\,\mu_\sigma(dy). \tag{24}$$

Thus, $\tilde{\pi}_{\alpha,\sigma}$ converges weakly to μ_σ as $\alpha\to 0$.

Remark 7. Formula (24) admits the following interpretation from the viewpoint of quantum physics. Denote by Z the (unbounded) operator of multiplication by the variable in $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$: $(Zf)(z) = zf(z)$ for $f(z)$ from the domain of Z. Consider the complex space ℓ_2 with its standard orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$. Here $e_n = (0,\dots,0,1,0,0,\dots)$, where 1 is at the nth place. Define a unitary operator $I : L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma}) \to \ell_2$ satisfying $Ic_n = (n!\,\sigma^n)^{1/2}e_n$ ($n\in\mathbb{N}_0$). Next, define a self-adjoint (unbounded) linear operator $\rho_{\alpha,\sigma} = IZI^{-1}$ in ℓ_2. Formula (23) implies

$$\rho_{\alpha,\sigma}\,e_n = \sqrt{\sigma(n+1)}\,e_{n+1} + (\alpha n + \sigma/\alpha)e_n + \sqrt{\sigma n}\,e_{n-1}, \quad n\in\mathbb{N}_0. \tag{25}$$

Consider a creation operator a^+ and an annihilation operator a^- in ℓ_2 satisfying $a^+e_n = \sqrt{n+1}\,e_{n+1}$ and $a^-e_n = \sqrt{n}\,e_n$ ($n\in\mathbb{N}_0$). The operators a^+ and a^- are adjoint of each other and satisfy the commutation relation $[a^-, a^+] = 1$. By (25),

$$\rho_{\alpha,\sigma} = \alpha(a^+ + \sqrt{\sigma}/\alpha)(a^- + \sqrt{\sigma}/\alpha).$$

Note that the operators $A^+_{\alpha,\sigma} = a^+ + \sqrt{\sigma}/\alpha$ and $A^-_{\alpha,\sigma} = a^- + \sqrt{\sigma}/\alpha$ are also adjoint of each other and satisfy the commutation relation $[A^-_{\alpha,\sigma}, A^+_{\alpha,\sigma}] = 1$. It follows from [1] (see also [8]) that the operators $A^+_{\alpha,\sigma}$, $A^-_{\alpha,\sigma}$ form a representation of the canonical commutation relations (CCR) describing an infinite free Bose gas at zero temperature. Hence, the operator $A^+_{\alpha,\sigma}A^-_{\alpha,\sigma}$ is the particle density of this gas with average density σ/α^2. We observe that

$$\begin{aligned}\rho_{\alpha,\sigma} - \frac{\sigma}{\alpha} &= \alpha\bigg(A^+_{\alpha,\sigma}A^-_{\alpha,\sigma} - \frac{\sigma}{\alpha^2}\bigg)\\ &= \sqrt{\sigma}(a^+ + a^-) + \alpha a^+a^- \to \sqrt{\sigma}(a^+ + a^-) \quad \text{as } \alpha\to 0.\end{aligned}$$

The limiting operator $\sigma(a^+ + a^-)$ describes a free Bose field.

We define a generalized Segal–Bargmann transform $\mathcal{S} : L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma}) \to \mathbb{F}_\sigma(\mathbb{C})$ as a unitary operator that satisfies $(\mathcal{S}c_n)(z) = z^n$ for $n\in\mathbb{N}_0$.

For each $z\in\mathbb{C}$, the corresponding coherent state is given by

$$\mathcal{E}(x,z) = \sum_{n=0}^\infty \frac{z^n}{n!\,\sigma^n}\,c_n(x), \quad x\in\alpha\mathbb{N}_0,\ z\in\mathbb{C}. \tag{26}$$

For each $z \in \mathbb{C}$, we have $\mathcal{E}(\cdot, z) \in L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$, and for each $x \in \alpha\mathbb{N}_0$, $\mathcal{E}(x, \cdot) \in \mathbb{F}_\sigma(\mathbb{C})$. By (22) and (26),

$$\mathcal{E}(\alpha n, z) = \left(1 + \frac{\alpha z}{\sigma}\right)^n \exp\left(-\frac{z}{\alpha}\right), \quad n \in \mathbb{N}_0,\ z \in \mathbb{C}. \tag{27}$$

Thus, for $f \in L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$ and $\varphi \in \mathbb{F}_\sigma(\mathbb{C})$,

$$\begin{aligned}(\mathcal{S}f)(z) &= \int_{\alpha\mathbb{N}_0} f(x)\mathcal{E}(x, z)\, \pi_{\alpha,\sigma}(dx), \quad z \in \mathbb{C}, \\ (\mathcal{S}^{-1}\varphi)(x) &= \int_{\mathbb{C}} \varphi(z)\mathcal{E}(x, z)\, \nu_\sigma(dz), \quad x \in \alpha\mathbb{N}_0.\end{aligned} \tag{28}$$

For $\alpha > 0$ and $z \in \mathbb{C}$, we define a complex-valued measure $\pi_{\alpha,z}$ on $\alpha\mathbb{N}_0$ by

$$\pi_{\alpha,z} = \exp\left(-\frac{z}{\alpha^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{z}{\alpha^2}\right)^n \delta_{\alpha n}.$$

Thus, for $z > 0$, $\pi_{\alpha,z}$ is the probability distribution defined in (1). Note that, for $z \in (-\infty, 0)$, $\pi_{\alpha,z}$ is a signed (real-valued) measure.

Theorem 8. *For each $f \in L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$, we have*

$$(\mathcal{S}f)(z) = \int_{\alpha\mathbb{N}_0} f\, d\pi_{\alpha,\sigma+\alpha z}, \quad z \in \mathbb{C}. \tag{29}$$

In particular, for $z > -\frac{\sigma}{\alpha}$, the integration (summation) in (29) is with respect to the probability distribution $\pi_{\alpha,\sigma+\alpha z}$. The complex-valued series on the right-hand side of (29) converges absolutely and uniformly on compact sets in $\mathbb{C}$.

Remark 9. For $\alpha = 1$, formula (29) was mentioned (without proof) in [13].

Proof of Theotrem 8. By (27) and (28), for $f \in L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$,

$$\begin{aligned}(\mathcal{S}f)(z) &= \sum_{n=0}^{\infty} f(\alpha n)\left(1 + \frac{\alpha z}{\sigma}\right)^n \exp\left(-\frac{z}{\alpha}\right) \exp\left(-\frac{\sigma}{\alpha^2}\right) \frac{1}{n!}\left(\frac{\sigma}{\alpha^2}\right)^n \\ &= \exp\left(-\frac{\sigma + \alpha z}{\alpha^2}\right) \sum_{n=0}^{\infty} f(\alpha n)\, \frac{1}{n!}\left(\frac{\sigma + \alpha z}{\alpha^2}\right)^n = \int_{\alpha\mathbb{N}_0} f\, d\pi_{\alpha,\sigma+\alpha z}.\end{aligned}$$

Next, by the Cauchy inequality,

$$\sum_{n=0}^{\infty} |f(\alpha n)|\, \frac{1}{n!}\left|\frac{\sigma + \alpha z}{\alpha^2}\right|^n \le \sum_{n=0}^{\infty} |f(\alpha n)|\, \frac{1}{n!}\left(\frac{\sigma + |\alpha z|}{\alpha^2}\right)^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} |f(\alpha n)| \left(\frac{1}{n!}\right)^{\frac{1}{2}} \left(\frac{\sigma}{\alpha^2}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{n!}\right)^{\frac{1}{2}} \left(\frac{\sigma + |\alpha z|}{\alpha^2}\right)^{n} \left(\frac{\sigma}{\alpha^2}\right)^{-\frac{n}{2}} \\
&\le \left(\sum_{n=0}^{\infty} |f(\alpha n)|^2 \, \frac{1}{n!} \left(\frac{\sigma}{\alpha^2}\right)^{n}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(\sigma + |\alpha z|)^2}{\sigma}\right)^{n}\right)^{\frac{1}{2}} \\
&= \exp\left(\frac{\sigma}{\alpha^2}\right) \|f\|_{L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(\sigma + |\alpha z|)^2}{\sigma}\right)^{n}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence, the complex-valued series on the right-hand side of (29) converges absolutely and uniformly on compact sets in $\mathbb{C}$. □

Define the monic polynomial sequence $(T_{\alpha,n})_{n=0}^{\infty}$ by

$$
T_{\alpha,n}(z) = \sum_{k=1}^{n} S(n,k)\alpha^{n-k} z^k = \alpha^n T_n\left(\frac{z}{\alpha}\right), \quad n \in \mathbb{N}, \tag{30}
$$

where $(T_n)_{n=0}^{\infty}$ is the sequence of Touchard polynomials, see (7). By (8), $(T_{\alpha,n})_{n=0}^{\infty}$ is a polynomial sequence of binomial type with generating function

$$
\sum_{n=0}^{\infty} \frac{t^n}{n!} \, T_{\alpha,n}(z) = \exp\left(z \, \frac{e^{t\alpha} - 1}{\alpha}\right).
$$

Let $\mathcal{T}_\alpha$ denote the umbral operator associated with the binomial sequence $(T_{\alpha,n})_{n=0}^{\infty}$, i.e., $\mathcal{T}_\alpha z^n = T_{\alpha,n}(z)$ $(n \in \mathbb{N}_0)$.

Let $((\cdot \mid \alpha)_n)_{n=0}^{\infty}$ denote the sequence of generalized factorials with increment α, i.e., for $z \in \mathbb{C}$, $(z \mid \alpha)_0 = 1$ and

$$
(z \mid \alpha)_n = z(z-\alpha)(z-2\alpha)\cdots(z-(n-1)\alpha), \quad n \in \mathbb{N}. \tag{31}
$$

In particular, $(z \mid 1)_n = (z)_n$ is a falling factorial. The $((\cdot \mid \alpha)_n)_{n=0}^{\infty}$ is a binomial sequence with generating function

$$
\sum_{n=0}^{\infty} \frac{t^n}{n!} \, (z \mid \alpha)_n = \exp\left(\frac{z}{\alpha} \log(1+\alpha t)\right).
$$

By (5) and (31),

$$
(z \mid \alpha)_n = \sum_{k=1}^{n} s(n,k)\alpha^{n-k} z^k, \quad n \in \mathbb{N}. \tag{32}
$$

Let $\mathcal{F}_\alpha$ denote the umbral operator associated with $((\cdot \mid \alpha)_n)_{n=0}^{\infty}$, i.e.,

$$
\mathcal{F}_\alpha z^n = (z \mid \alpha)_n, \quad n \in \mathbb{N}_0.
$$

Using formulas (5), (6), (30) and (32), one can easily show that $\mathcal{F}_\alpha$ is the inverse of the operator $\mathcal{T}_\alpha$, i.e., $\mathcal{F}_\alpha = \mathcal{T}_\alpha^{-1}$.

Note that the restriction of $\mathcal{S}$ to $\mathbb{C}[z]$ is a bijective operator in $\mathbb{C}[z]$, for which we keep the notation $\mathcal{S}$,

Proposition 10. *We have the following equalities of linear operators in* $\mathbb{C}[z]$*:*

$$\mathcal{S} = E_{\sigma/\alpha}\mathcal{T}_\alpha, \quad \mathcal{S}^{-1} = \mathcal{F}_\alpha E_{-\sigma/\alpha}.$$

Here E_h *denotes the operator of shift by* h*.*

Proof. We use ideas similar to those in [13, Section 4]. We define linear operator $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{C}[z]$ by

$$\mathcal{U} = Z + \frac{\sigma}{\alpha}, \quad \mathcal{V} = \alpha D + 1. \tag{33}$$

As easily seen, the operators $\mathcal{U}$, $\mathcal{V}$ satisfy the commutation relation $[\mathcal{V}, \mathcal{U}] = \alpha$. Hence, they are generators of a Weyl algebra, see e.g. [17, Section 5.6].

Let

$$\rho = \mathcal{U}\mathcal{V} = Z + \alpha Z D + \frac{\sigma}{\alpha} + \sigma D.$$

Hence,

$$\rho z^n = z^{n+1} + \left(\alpha n + \frac{\sigma}{\alpha}\right) z^n + \sigma n z^{n-1}. \tag{34}$$

By (23) and (34), we have the following equality of linear operators in $\mathbb{C}[z]$:

$$\rho = \mathcal{S} Z \mathcal{S}^{-1}. \tag{35}$$

Therefore,

$$\mathcal{S} z^n = (\rho^n 1)(z). \tag{36}$$

By Katriel's theorem [12],

$$\rho^n = (\mathcal{U}\mathcal{V})^n = \sum_{k=1}^{n} S(n,k)\, \alpha^{n-k} \mathcal{U}^k\, \mathcal{V}^k. \tag{37}$$

For each $k \in \mathbb{N}$,

$$(\mathcal{U}^k \mathcal{V}^k 1)(z) = (\mathcal{U}^k 1)(z) = \left(z + \frac{\sigma}{\alpha}\right)^k. \tag{38}$$

By (30) and (36)–(38), we get

$$\mathcal{S} z^n = \sum_{k=1}^{n} S(n,k)\, \alpha^{n-k} \left(z + \frac{\sigma}{\alpha}\right)^k = T_{\alpha,n}\left(z + \frac{\sigma}{\alpha}\right), \tag{39}$$

which implies $\mathcal{S} = E_{\sigma/\alpha}\mathcal{T}_\alpha$. Since $\mathcal{F}_\alpha = \mathcal{T}_\alpha^{-1}$, we obtain $\mathcal{S}^{-1} = \mathcal{F}_\alpha E_{-\sigma/\alpha}$. □

Corollary 11. *We have, for each $n \in \mathbb{N}$,*

$$z^n = T_{\alpha,n}(\sigma/\alpha) + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \binom{n}{k} T_{\alpha,n-k}(\sigma/\alpha) S(k,i) \alpha^{k-i} \right) c_i(z), \tag{40}$$

and

$$c_n(z) = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{\sigma}{\alpha} \right)^{n-k} (z \mid \alpha)_k \tag{41}$$

$$= \left(-\frac{\sigma}{\alpha} \right)^{n} + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-i} \binom{n}{k} s(n-k,i)\, \alpha^{n-2k-i}\, (-1)^k \sigma^k \right) z^i. \tag{42}$$

Proof. We have, for $n \in \mathbb{N}$,

$$\begin{aligned} c_n(z) &= \mathcal{S}^{-1} z^n = \mathcal{F}_\alpha E_{-\sigma/\alpha} z^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{\sigma}{\alpha} \right)^{n-k} \mathcal{F}_\alpha z^k = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{\sigma}{\alpha} \right)^{n-k} (z \mid \alpha)_k\,, \end{aligned}$$

which implies (41). Formula (42) follows from (32) and (41).

Since $(T_{\alpha,n})_{n=0}^{\infty}$ is a polynomial sequence of binomial type, formulas (30) and (39) imply, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{S} z^n &= \sum_{k=0}^{n} \binom{n}{k} T_{\alpha,n-k}(\sigma/\alpha) T_{\alpha,k}(z) \\ &= T_{\alpha,n}(\sigma/\alpha) + \sum_{k=1}^{n} \binom{n}{k} T_{\alpha,n-k}(\sigma/\alpha) \sum_{i=1}^{k} S(k,i) \alpha^{k-i} z^i \\ &= T_{\alpha,n}(\sigma/\alpha) + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \binom{n}{k} T_{\alpha,n-k}(\sigma/\alpha) S(k,i) \alpha^{k-i} \right) z^i. \end{aligned} \tag{43}$$

Applying the operator $\mathcal{S}^{-1}$ to (43) gives (40). □

Remark 12. In the case $\alpha = 1$, the explicit form of the Charlier polynomials is well known. The reader is advised to compare formula (40) in the case $\alpha = 1$ with an expansion of $\binom{z}{n}$ in the Charlier polynomials discussed in Example 1 on page 479 of [11]. Corollary 11 can also be derived from [13, Theorem 4.6] by taking the limit $\beta \to 0$.

Denote by ∂^+ and ∂^- the raising and lowering operators for the monic polynomial sequence $(c_n)_{n=0}^{\infty}$, i.e., $\partial^+, \partial^- \in \mathcal{L}(\mathbb{C}[z])$ and

$$(\partial^+ c_n)(z) = c_{n+1}(z), \quad (\partial^- c_n)(z) = n c_{n-1}(z), \qquad n \in \mathbb{N}_0.$$

The following proposition can be easily shown.

Proposition 13. *The operator* ∂^- *is closable in* $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$*, and let us keep the notation* ∂^- *for its closure. Then, for each* $z \in \mathbb{C}$*, the coherent state* $\mathcal{E}(\cdot, z)$ *is eigenfunction of the operator* ∂^- *belonging to the eigenvalue* z*.*

5 The images of the operators $\mathcal{U}$ and $\mathcal{V}$ under $\mathcal{S}^{-1}$

Recall that a function $f : \alpha\mathbb{N}_0 \to \mathbb{C}$ belongs to $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$ if and only if it can be uniquely represented in the form $f(\alpha k) = \sum_{n=0}^\infty f_n c_n(\alpha k)$, where $k \in \mathbb{N}$ and $f_n \in \mathbb{C}$ ($n \in \mathbb{N}_0$) satisfy $\|f\|^2_{L^2(\alpha\mathbb{N}_0,\pi_{\alpha,\sigma})} = \sum_{n=0}^\infty |f_n|^2 n!\, \sigma^n < \infty$.

By (22), the Sheffer sequence $(c_n)_{n=0}^\infty$ satisfies the condition of Theorem 4. Hence, the statement of Corollary 5 holds for $s_n(z) = c_n(z)$ ($n \in \mathbb{N}_0$).

For each $l \in \mathbb{N}$, there exists $C > 0$ such that, for each $f(z) = \sum_{n=0}^\infty f_n c_n(z) \in \mathcal{E}^1_{\min}(\mathbb{C})$, we have $\|f\|_{L^2(\alpha\mathbb{N}_0,\pi_{\alpha,\sigma})} \le C|||f|||_l$. Hence, for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$, the restriction of f to $\alpha\mathbb{N}_0$ determines a function from $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$, and furthermore, if the restriction of f to $\alpha\mathbb{N}_0$ is identically equal to zero, then the function f is identically equal to zero on $\mathbb{C}$. Thus, we obtain an embedding of $\mathcal{E}^1_{\min}(\mathbb{C})$ into $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$ and this embedding is continuous.

Theorem 4 implies

Proposition 14. *The Fréchet space* $\mathcal{E}^1_{\min}(\mathbb{C})$ *is continuously embedded into* $L^2(\alpha\mathbb{N}_0, \pi_{\alpha,\sigma})$ *(in the above explained sense). Furthermore, the operator* $\mathcal{S}$ *restricted to* $\mathcal{E}^1_{\min}(\mathbb{C})$ *is a self-homeomorphism of* $\mathcal{E}^1_{\min}(\mathbb{C})$*.*

Recall the linear operators $\mathcal{U}$ and $\mathcal{V}$ acting in $\mathbb{C}[z]$, see (33). Define linear operators $U, V \in \mathcal{L}(\mathbb{C}[z])$ by $U = \mathcal{S}^{-1}\mathcal{U}\mathcal{S}$ and $V = \mathcal{S}^{-1}\mathcal{V}\mathcal{S}$. Thus,

$$U = \partial^+ + \frac{\sigma}{\alpha}, \quad V = \alpha\partial^- + 1.$$

Since $\rho = \mathcal{U}\mathcal{V}$, formula (35) implies

$$Z = UV. \tag{44}$$

It is easy to see that the operators U, V, Z and E_h for $h \in \mathbb{C}$ admit (unique) extensions to continuous linear operators in $\mathcal{E}^1_{\min}(\mathbb{C})$, for which we preserve their original notation. In particular, formula (44) can be understood as an equality of continuous linear operators in $\mathcal{E}^1_{\min}(\mathbb{C})$.

Proposition 15. *We have the following two equalities of continuous linear operators in* $\mathcal{E}^1_{\min}(\mathbb{C})$*:*

$$U = ZE_{-\alpha}, \quad V = E_\alpha.$$

Proof. By Theorems 1, 2 and formula (22),

$$\partial^- = C(D), \tag{45}$$

where $C(t)$ is the compositional inverse of $B(t) = \frac{1}{\alpha}\log(1+t\alpha)$. Hence,

$$C(t) = \frac{1}{\alpha}(e^{\alpha t} - 1). \tag{46}$$

By (45), (46), and Boole's formula,

$$\partial^- = \frac{1}{\alpha}(e^{\alpha D} - 1) = \frac{1}{\alpha}(E^\alpha - 1).$$

Hence,

$$V = \alpha\partial^- + 1 = E^\alpha.$$

By (44), $Z = UE^\alpha$, which implies $U = ZE^{-\alpha}$. □

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